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OPTIMIZING HYPERGRAPH-BASED POLYNOMIALS MODELING JOB-OCCUPANCY IN QUEUING WITH REDUNDANCY SCHEDULING*

D. BROSCH†, M. LAURENT‡, AND A. STEENKAMP§

Abstract. We investigate two classes of multivariate polynomials with variables indexed by the edges of a uniform hypergraph and coefficients depending on certain patterns of unions of edges. These polynomials arise naturally to model job-occupancy in some queuing problems with redundancy scheduling policies. The question, posed by Cardinaels, Borst, and van Leeuwaarden in [*Redundancy Scheduling with Locally Stable Compatibility Graphs*, arXiv preprint, 2020], is to decide whether their global minimum over the standard simplex is attained at the uniform probability distribution. By exploiting symmetry properties of these polynomials we can give a positive answer for the first class and partial results for the second one, where we in fact show a stronger convexity property of these polynomials over the simplex.

Key words. Terwilliger algebra, polynomial optimization, symmetry, convex polynomial

AMS subject classifications. 90C23, 90C26, 05E30, 32E20

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1. Introduction.

We consider the minimization of two classes of polynomials over the standard simplex. These polynomials have their variables labeled by the edges of a complete uniform hypergraph, and their coefficients are defined in terms of some cardinality patterns of unions of edges. They arise naturally within the modeling of job-occupancy in some queuing problems with redundancy scheduling policies [3]. The question is whether these polynomials attain their minimum value at the barycenter of the standard simplex, which corresponds to showing optimality of the uniform distribution for the underlying queuing problem. This paper is devoted to this question.

We now introduce the classes of polynomials of interest. Given integers $n, L \geq 2$, we set $V = [n] = \{1, \ldots, n\}$ and $E = \{e \subseteq V : |e| = L\}$, so that (V, E) can be seen as the complete L-uniform hypergraph on n elements. We set $m := |E| = \binom{n}{L}$, where we omit the explicit dependence on n, L to simplify notation, and we let

$$\Delta_m = \left\{ x = (x_e)_{e \in E} \in \mathbb{R}^m : x \geq 0, \sum_{e \in E} x_e = 1 \right\}$$

denote the standard simplex in $\mathbb{R}^m$. The elements of Δ_m correspond to probability

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†Department of Econometrics and Operations Research, Tilburg University, 5000 LE, Tilburg, The Netherlands (d.brosch@tilburguniversity.edu).

‡Centrum Wiskunde & Informatica (CWI), 1090 GB, Amsterdam, The Netherlands, and Department of Econometrics and Operations Research, Tilburg University, 5000 LE, Tilburg, The Netherlands (M.Laurent@cwi.nl).

§Centrum Wiskunde & Informatica (CWI), 1090 GB, Amsterdam, The Netherlands (andries.steenkamp@cwi.nl).

vectors on m items, and the barycenter $x^* = \frac{1}{m}(1, \ldots, 1)$ of Δ_m corresponds to the uniform probability vector.

Given an integer $d \geq 2$ we consider the following m-variate polynomial in the variables $x = (x_e : e \in E)$, which is a main player in the paper:

$$f_d(x) = \sum_{(e_1,\ldots,e_d)\in E^d} \prod_{i=1}^{d} \frac{x_{e_i}}{|e_1 \cup \cdots \cup e_i|}. \tag{1.1}$$

So f_d is a homogeneous polynomial with degree d. We are interested in the optimization problem

$$f_d^* := \min_{x\in\Delta_m} f_d(x),$$

asking to minimize the polynomial f_d over the simplex Δ_m. The main question, which is posed in [3] (in the case $L = 2$), is whether the minimum is attained at the uniform probability.

QUESTION 1. *Given integers $n, d, L \geq 2$, is it true that the polynomial $f_d(x)$ in* (1.1) *attains its minimum over Δ_m at the barycenter x^* of Δ_m?*

As explained in [3], the motivation for this question comes from its relevance to a problem in queuing theory, which we will briefly describe in the next section. In this paper we are only able to give a partial positive answer to this question, namely, in the case $d = 2$ (which follows from Theorem 1.1 below) and in the case $d = 3$ and $L = 2$ (Theorem 1.2 below). As a first step toward understanding the polynomials f_d we investigate a related, easier-to-analyze class of polynomials.

Given an integer $d \geq 2$, we consider the related class of polynomials

$$p_d(x) = \sum_{(e_1,\ldots,e_d)\in E^d} \frac{1}{|e_1 \cup \cdots \cup e_d|} x_{e_1} \cdots x_{e_d}, \tag{1.2}$$

which are also homogeneous with degree d. Note that, for degree $d = 2$, we have $f_2 = \frac{1}{L} p_2$. For degree $d \geq 3$ the structure of the polynomials f_d is related to, but more complicated than, that of the polynomials p_d (see section 4 for more details on the links between both classes). Here too we may ask whether the minimum of p_d over the standard simplex Δ_m is attained at the uniform probability vector x^*. For the polynomials p_d we are able to give a positive answer in the general case. The following is the first main result of the paper.

THEOREM 1.1. *For any integers $n, L, d \geq 2$, the global minimum of the polynomial p_d from* (1.2) *over the standard simplex Δ_m is attained at the barycenter $x^* = \frac{1}{m}(1, \ldots, 1)$ of Δ_m.*

As noted above, f_2 and p_2 coincide up to positive scaling, and hence it follows directly that Question 1 has a positive answer in the case $d = 2$. As a further partial result we give a positive answer for the case of degree $d = 3$ and edge size $L = 2$. The following is the second main result.

THEOREM 1.2. *For $n \geq 2$, $d = 3$, and $L = 2$, the global minimum of the polynomial f_d from* (1.1) *over the standard simplex Δ_m is attained at the barycenter $x^* = \frac{1}{m}(1, \ldots, 1)$ of Δ_m.*

As we will see, the analysis of the polynomials f_d is technically much more involved than for the polynomials p_d, and we have only partial results so far. In both cases the key ingredient is showing that the polynomials are convex on the simplex, i.e., that they have positive semidefinite Hessians at any vector in Δ_m. It turns out that the Hessian of the polynomial p_d enters in some way as a component of the Hessian of the polynomial f_d. So this forms a natural motivation for the study of the polynomials p_d, though they form a natural class of symmetric polynomials that are interesting in their own right.

Exploiting symmetry plays a central role in our proofs. Indeed the key idea is to show that the polynomials are convex, which, combined with their symmetry properties, implies that the global minimum is attained at the barycenter of the simplex. For this we show that their Hessian matrices are positive semidefinite at each point of the simplex, which we do through exploiting again their symmetry structure and links to Terwilliger algebras.

Symmetry is a widely used ingredient in optimization, in particular in semidefinite optimization and algebraic questions involving polynomials. We mention a few landmark examples as background information. Symmetry can indeed be used to formulate equivalent, more compact reformulations for semidefinite programs. The underlying mathematical fact is Artin–Wedderburn theory, which shows that matrix $*$-algebras can be block-diagonalized (see Theorem 2.3 below). An early well-known example is the linear programming reformulation from [21] for the Lovász theta number of Hamming graphs, showing the link to the Delsarte bound and Bose–Mesner algebras of Hamming schemes [5, 6]. Symmetry is used more generally to give tractable reformulations for the semidefinite bounds arising from the next levels of Lasserre's hierarchy in [22] (which gives the explicit block-diagonalization for the Terwilliger algebra of Hamming schemes; see Theorem 2.4 below) and, e.g., in [9, 10, 12, 13]. For more examples and a broad exposition about the use of symmetry in semidefinite programming, we refer the reader to, e.g., [1, 4] and further references therein. Symmetry is also a crucial ingredient in the study of algebraic questions about polynomials, like representations in terms of sums of squares, and in polynomial optimization. We refer the reader to [8] for a broad exposition and, e.g., to [20] (for compact reformulations of Lasserre relaxations of symmetric polynomial optimization problems), [19] (for methods to reduce the number of variables in programs involving symmetric polynomials), and the recent works [16, 17] (which consider symmetric polynomials with variables indexed by the k-subsets hypercube (as in our case) and uncover links with the theory of flag algebras by Razborov [18]).

Example 1. As an illustration let us consider the polynomial p_d for edge size $L = 2$. Given a sequence $\underline{e} = (e_1, \dots, e_d) \in E^d$ set $c_{\underline{e}} = 1/|e_1 \cup \cdots \cup e_d|$ as shorthand for the coefficients in the definition (1.2) of the polynomial p_d. So we need to enumerate the possible configurations of d-tuples of edges, i.e., the distinct multigraphs with d edges. Note that their number is given by the OEIS sequence A050535 [14], which takes the values $1, 3, 8, 23, 66, 212, 686$ for $d = 1, 2, 3, 4, 5, 6, 7$.

For $d = 1$, we have $p_1(x) = \frac{1}{2}\sum_{e\in E} x_e$. For $d = 2$ we have

$$p_2(x) = \frac{1}{2}\sum_{e\in E} x_e^2 + \frac{1}{3}\sum_{\substack{(e_1,e_2)\in E^2:\\ |e_1\cup e_2|=3}} x_{e_1}x_{e_2} + \frac{1}{4}\sum_{\substack{(e_1,e_2)\in E^2:\\ |e_1\cup e_2|=4}} x_{e_1}x_{e_2}.$$

We show in Figure 1 the three possible patterns for pairs of edges $\underline{e} = (e_1, e_2)$ and the corresponding coefficients $c_{\underline{e}}$.

In the same way, for $d \geq 3$, $p_d(x) = \sum_{k=2}^{2d} \frac{1}{k} q_{d,k}(x)$, where the summand $q_{d,k}(x)$ is a summation over all d-tuples of edges with a given pattern, depending on the

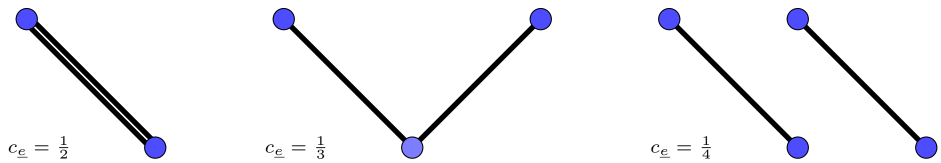

FIG. 1. *The three patterns of pairs of edges in case $(d = 2, L = 2)$.*

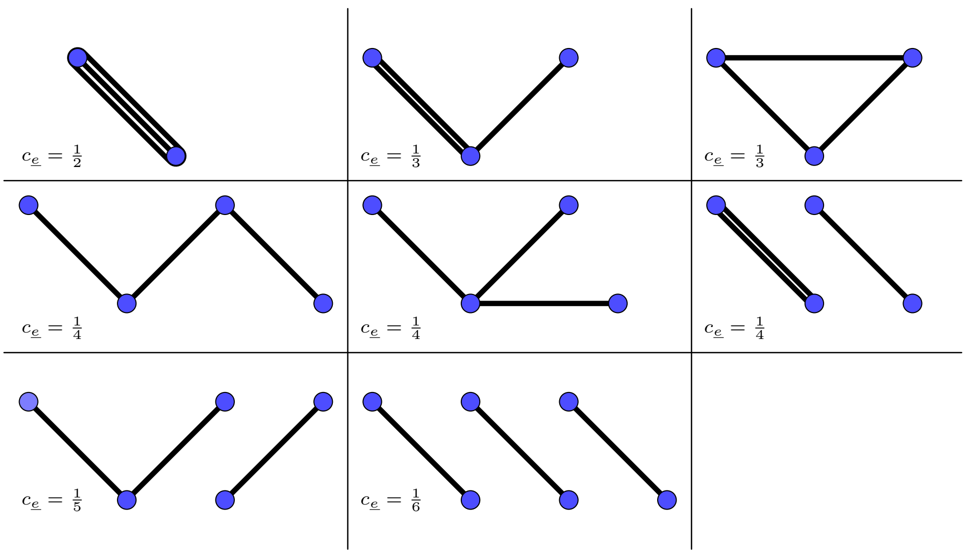

FIG. 2. *The eight patterns of triplets of edges in case $(d = 3, L = 2)$.*

cardinality of their union:

$$q_{d,k}(x) = \sum_{\substack{(e_1,\ldots,e_d)\in E^d:\\ |e_1\cup\cdots\cup e_d|=k}} x_{e_1}\cdots x_{e_d}.$$

For the case $d = 3$ we need to consider the values $k = 2, 3, 4, 5, 6$; as an illustration we show in Figure 2 all eight possible patterns of triplets of edges $\underline{e} = (e_1, e_2, e_3)$ and the corresponding coefficients $c_{\underline{e}}$ that contribute to the summands $q_{3,k}$.

Organization of the paper. In the rest of this section we first indicate in section 2.1 how the polynomials f_d naturally arise within a problem of queuing theory with redundancy scheduling policies. After that we present in section 2.2 the main ideas of the proofs, which highly rely on exploiting symmetry properties of the polynomials. This involves in particular using the Terwilliger algebra of the binary Hamming cube, so we include some preliminaries about these Terwilliger algebras in section 2.3.

In section 3 we give the full proof for Theorem 1.1, showing that the polynomials p_d attain their global minimum at the barycenter of the simplex, and in section 4 we investigate the second class of polynomials f_d. We prove several properties of these polynomials, which we use to show Theorem 1.2. We also present a range of values of (n, d, L) for which the polynomials f_d are indeed convex, and thus Question 1 has a positive answer.

Some notation. Throughout we let I, J denote the identity matrix and the all-ones matrix, whose size should be clear from the context. When we want to specify the size we let I_n (resp., J_n) denote the $n \times n$ identity matrix (resp., all-ones matrix) and, given two integers $n, m \geq 1$, $J_{m,n}$ denotes the $m \times n$ all-ones matrix. For a symmetric matrix A the notation $A \succeq 0$ means that A is positive semidefinite. Given two matrices $A, B \in \mathbb{R}^{n\times n}$ we let $A \circ B \in \mathbb{R}^{n\times n}$ denote their Hadamard product, with entries $(A \circ B)_{ij} = A_{ij}B_{ij}$ for $i, j \in [n]$. It is known that $A \succeq 0$ and $B \succeq 0$ imply $A \circ B \succeq 0$, which follows from the fact that the matrix $A \circ B$ is a principal submatrix of the Kronecker product of A and B.

For a sequence $\alpha \in \mathbb{N}^n$ we set $|\alpha| = \sum_{i=1}^n \alpha_i$ and, for an integer $d \in \mathbb{N}$, we set $\mathbb{N}^n_d = \{\alpha \in \mathbb{N}^n : |\alpha| = d\}$. Given a vector $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Throughout we let $u_1, \ldots, u_m$ denote the standard basis of $\mathbb{R}^m$, where all entries of u_i are 0 except its ith entry, which is equal to 1. We let $\mathrm{Sym}(n)$ denote the set of permutations of the set $V = [n]$.

2. Preliminaries.

In this section we first explain the relevance of the polynomials f_d and p_d for the problem from queuing theory considered in [3]. Then we present a sketch of proof for our main results and conclude with some preliminaries about Terwilliger algebras that we will use in the symmetry reduction.

2.1. Motivation.

Our motivation for the study of the polynomials p_d and f_d comes from their relevance to a problem in queuing theory. The question of whether they attain their minimum at the uniform probability distribution was posed to us by the authors of [3], who use a positive answer to this question to establish a result about the asymptotic behavior of the job-occupancy in a parallel-server system with redundancy scheduling in the light-traffic regime. In what follows we will give only a high level sketch of this connection and refer the reader to the paper [3] for a detailed exposition. An extended review of the relevant literature is also available in [3].

A crucial mechanism that has been considered to improve the performance of parallel-server systems in queuing theory is redundancy scheduling. The key feature of this policy is that several replicas are created for each arriving job, which are then assigned to distinct servers (and then, as soon as the first of these replicas completes (or enters) service on a server, the remaining ones are stopped). The underlying idea is that sending replicas of the same job to several servers will increase the chance of having shorter queuing times. This, however, must be weighted against the risk of wastage of capacity. An important question is thus to assess the impact of redundancy scheduling policies. While most papers in the literature of redundant scheduling assume that the set of servers to which the replicas are sent is selected uniformly at random, the paper [3] considers the case when the set of servers is selected according to a given probability distribution, and it investigates the impact of this probability distribution on the performance of the system. It is shown there that while the impact remains relatively limited in the heavy-traffic regime, the system occupancy is much more sensitive to the selected probability distribution in the light-traffic regime.

We will now only introduce a few elements of the model considered in [3], so that we can make the link to the polynomials studied in this paper. We keep our presentation high level and refer the reader to [3] for details. The setting is as follows. There are n parallel servers, with average speed μ. Jobs arrive as a Poisson process at rate $n\lambda$ for some $\lambda > 0$. When a job arrives, L replicas of it are created that are sent—with probability x_e—to a subset $e \subseteq [n]$ of L servers. Here, $L \geq 2$ is an integer and $x = (x_e)_{e\in E}$ is a probability distribution on the set $E = \{e \subseteq [n] : |e| = L\}$ of possible collections of L servers. As noted in [3] this can be seen as selecting an edge

$e \in E$ with probability x_e in the uniform hypergraph $(V = [n], E)$ (with edge size L).

An important performance parameter is the system occupancy at time t, which is represented by a vector $(e_1, \ldots, e_M) \in E^M$, where $M = M(t)$ is the total number of jobs present in the system and $e_i \in E$ is the collection of servers to which the replicas of the ith longest job in the system have been assigned. We need three modeling assumptions. First, one needs to assume suitable stability conditions. Second, all servers should have the same speed μ, and, third, the service requirements of the jobs are assumed to be independent and exponentially distributed with unit mean. Under these assumptions, the stationary distribution of the occupancy of the above edge selection is given by

$$\pi(e_1, \ldots, e_M) = C \prod_{i=1}^{M} \frac{n\lambda x_{e_i}}{\mu |e_1 \cup \cdots \cup e_i|}$$

for some constant $C > 0$ ([7]; see relation (3) in [3]). Following [3], let $Q_\lambda(x)$ be a random variable with the stationary distribution of the system occupancy when the edge selection is given by the probability vector $x = (x_e)_{e \in E}$. It then follows that, for any integer $d \geq 1$, the probability that d jobs are present in the system is given by

$$\mathbb{P}\{Q_\lambda(x) = d\} = \sum_{(e_1, \ldots, e_d) \in E^d} \pi(e_1, \ldots, e_d).$$

Hence, $\mathbb{P}\{Q_\lambda(x) = 0\} = C$ and

$$\mathbb{P}\{Q_\lambda(x) = d\} = \mathbb{P}\{Q_\lambda(x) = 0\} \Big(\frac{n\lambda}{\mu}\Big)^d \sum_{(e_1, \ldots, e_d) \in E^d} \prod_{i=1}^{d} \frac{x_{e_i}}{|e_1 \cup \cdots \cup e_i|}.$$

(See relation (11) in [3].) Therefore, $\mathbb{P}\{Q_\lambda(x) = d\}$ is the polynomial $f_d(x)$ (up to a scalar multiple). In [3] the light-traffic regime is considered, i.e., when $\lambda \downarrow 0$, in the case $L = 2$. By doing a Taylor expansion one can see that

$$\mathbb{P}\{Q_\lambda(x) = 0\} = 1 + o(1), \qquad \mathbb{P}\{Q_\lambda(x) \geq d\} = \Big(\frac{n\lambda}{\mu}\Big)^d f_d(x) + o(\lambda^d)$$

(see relation (13) in [3]). Therefore, with $x^* = (1, \ldots, 1)/|E|$ denoting the uniform probability vector, we have

$$\lim_{\lambda \downarrow 0} \frac{\mathbb{P}\{Q_\lambda(x^*) \geq d\}}{\mathbb{P}\{Q_\lambda(x) \geq d\}} = \lim_{\lambda \downarrow 0} \frac{f_d(x^*) + o(1)}{f_d(x) + o(1)}.$$

Hence, if the polynomial f_d attains its minimum at the uniform distribution x^*, then one has

$$\lim_{\lambda \downarrow 0} \frac{\mathbb{P}\{Q_\lambda(x^*) \geq d\}}{\mathbb{P}\{Q_\lambda(x) \geq d\}} \leq 1.$$

This indicates that in the light-traffic regime the system occupancy is minimized when selecting uniformly at random the assignments to the servers of the job replicas. This thus motivates Question 1 of showing that the polynomial f_d attains its minimum over the probability simplex at the uniform point x^*.

2.2. Sketch of proof. Here we give a sketch of proof for our main results. We start by indicating the main steps for proving Theorem 1.1, dealing with the (simpler) class of polynomials p_d, and after that briefly indicate how to deal with the polynomials f_d.

A first easy observation is that in order to show that the polynomial p_d attains its minimum at the barycenter of the standard simplex Δ_m it suffices to show that p_d is convex over Δ_m. This follows from a symmetry argument; namely, we exploit the fact that the polynomial p_d is invariant under the permutations of the edge set E that are induced by permutations of $[n]$.

LEMMA 2.1. *Assume the polynomial p_d is convex on the simplex Δ_m. Then the point $x^* = (1/m)(1,\dots,1) \in \Delta_m$ is a global minimizer of p_d over Δ_m.*

Proof. The key fact we use is that the polynomial p_d enjoys some symmetry property; namely, for any tuple $(e_1,\dots,e_d) \in E^d$, the coefficient of the monomial $x_{e_1}\cdots x_{e_d}$ in p_d is $1/|e_1\cup\cdots\cup e_d|$, which depends only on the cardinality of the set $e_1\cup\cdots\cup e_d$. Recall that $E = \{e \subseteq V = [n] : |e| = L\}$. Any permutation $\sigma \in \mathrm{Sym}(n)$ of $[n]$ induces a permutation of E (still denoted σ) by setting $\sigma(e) = \{j_{\sigma(1)},\dots,j_{\sigma(L)}\}$ for $e = \{j_1,\dots,j_L\} \in E$. In turn, σ acts on Δ_m by setting $\sigma(x) = (x_{\sigma(e)})_{e\in\Delta_m}$ for $x = (x_e)_{e\in E} \in \Delta_m$. We now observe that p_d is invariant under this action of permutations $\sigma \in \mathrm{Sym}(n)$. Indeed, for any $\sigma \in \mathrm{Sym}(n)$, we have

$$\begin{aligned}\sigma(p_d)(x) = p_d(\sigma(x)) &= \sum_{(e_1,\dots,e_d)\in E^d} \frac{1}{|e_1\cup\cdots\cup e_d|} x_{\sigma(e_1)}\cdots x_{\sigma(e_d)}\\ &= \sum_{(f_1,\dots,f_d)\in E^d} \frac{1}{|\sigma^{-1}(f_1)\cup\cdots\cup\sigma^{-1}(f_d)|} x_{f_1}\cdots x_{f_d}\\ &= \sum_{(f_1,\dots,f_d)\in E^d} \frac{1}{|f_1\cup\cdots\cup f_d|} x_{f_1}\cdots x_{f_d}\\ &= p_d(x).\end{aligned}$$

Let $x \in \Delta_m$ be a global minimizer of p_d. For any permutation $\sigma \in \mathrm{Sym}(n)$ the permuted point $\sigma(x)$ belongs to Δ_m and $p_d(x) = p_d(\sigma(x))$ holds. Hence, for the full symmetrization of x,

$$x^* := \frac{1}{n!} \sum_{\sigma\in\mathrm{Sym}(n)} \sigma(x),$$

we have $x^* \in \Delta_m$ and all its entries are equal, so that $x^* = (1/m)(1,\dots,1)$. Moreover, as the polynomial p_d is convex over Δ_m, we have

$$p_d(x^*) \le \frac{1}{n!} \sum_{\sigma\in\mathrm{Sym}(n)} p_d(\sigma(x)) = p_d(x).$$

This shows that x^* is again a global minimizer of p_d in Δ_m. As $x^* = (1/m)(1,\dots,1)$, the proof is complete. ∎

Therefore we are left with the task of showing that the polynomial p_d is convex over the simplex Δ_m or, equivalently, that its Hessian matrix

$$H(p_d)(x) = (\partial^2 p_d(x)/\partial x_e \partial x_f)_{e,f\in E}$$

is positive semidefinite over Δ_m. This forms the core technical part of the proof. Here is a rough sketch of our proof technique.

A first step is to express the Hessian matrix of p_d as a matrix polynomial, involving a collection of matrices M_γ, which (up to positive scaling) are the coefficients of the Hessian $H(p_d)$ in the monomial basis; see Lemma 3.3. The next step is to show that each of the matrices M_γ appearing in this decomposition of the Hessian is positive semidefinite. For this, one first reduces to the task of showing that certain well-structured matrices are positive semidefinite; see Lemmas 3.4 and 3.5. After that, the final task is done by showing that these matrices lie in the Terwilliger algebra of the Hamming cube, which enables us to exploit its explicitly known block-diagonalization. The proof is then concluded by using an integral representation argument; see section 3.3.

The treatment for the polynomials f_d has the same starting point: the polynomial f_d is invariant under any permutation of the edge set E induced by permutations of $[n]$, and thus it suffices to show that f_d is convex in order to conclude that it attains its global minimum at the barycenter of the simplex (i.e., the analogue of Lemma 2.1 holds for f_d). After that we again express the Hessian matrix $H(f_d)$ as a matrix polynomial, involving a collection of matrices Q_γ that occur as its coefficients in the monomial basis; see Lemma 4.1. Hence, here too the task boils down to showing that each of these matrices Q_γ is positive semidefinite. This task turns out to be considerably more difficult than for the matrices M_γ which occurred in the analysis of the polynomial p_d. As a first step toward the analysis of the matrices Q_γ we give a recursive reformulation for them, which also makes apparent how the matrices M_γ enter their definition (namely, as a factor of a Hadamard product definition of Q_γ); see Lemma 4.4. Based on this we can show that the matrices Q_γ are indeed positive semidefinite in the case $d = 3$ and $L = 2$, thus showing Theorem 1.2; see section 4.2.

2.3. Preliminaries on the Terwilliger algebra. As mentioned above we need to exploit the symmetry structure of the polynomial p_d in order to show that its Hessian matrix is positive semidefinite. A crucial ingredient will be that the Hessian matrix can be decomposed into matrices that (after some reduction steps) all lie in the Terwilliger algebra of the binary Hamming cube. We begin with introducing the definition of the Terwilliger algebra $\mathcal{A}_n$ of the binary Hamming cube on n elements.

DEFINITION 2.2 (Terwilliger algebra of the binary Hamming cube). *Let $\mathcal{P}_n$ denote the collection of all subsets of the set $V = [n]$. For every triple of nonnegative integers i, j, t we define the $2^n \times 2^n$ matrix $D^t_{i,j}$, indexed by $\mathcal{P}_n$, with entries*

$$\left(D^t_{i,j}\right)_{S,T} = \begin{cases} 1 & \text{if } |S| = i, |T| = j, |S \cap T| = t, \\ 0 & \text{else} \end{cases}$$

for sets $S, T \in \mathcal{P}_n$. Then the Terwilliger algebra of the binary Hamming cube*, denoted by $\mathcal{A}_n$, is defined as the (real) span of all these matrices:*

$$\mathcal{A}_n = \left\{ \sum_{i,j,t \geq 0} x^t_{i,j} D^t_{i,j} : x^t_{i,j} \in \mathbb{R} \right\}.$$

It is easy to see that $\mathcal{A}_n$ is a *matrix $*$-algebra*, i.e., $\mathcal{A}_n$ is closed under taking linear combinations, matrix multiplications, and transposition. One way to see this is by realizing that the matrices $D^t_{i,j}$ are exactly the indicator matrices of the orbits of pairs in $\mathcal{P}_n \times \mathcal{P}_n$ under the elementwise action of the symmetric group $\mathrm{Sym}(n)$.

All matrix $*$-algebras can be block-diagonalized by the Artin–Wedderburn theory (see [23]; see also [2] for a proof).

THEOREM 2.3 (Artin–Wedderburn). *Let $\mathcal{A}$ be a matrix $*$-algebra. Then there exist nonnegative integers d and $m_1, \ldots, m_d$ and a $*$-algebra isomorphism*

$$\varphi\colon \mathcal{A} \to \bigoplus_{k=1}^{d} \mathbb{C}^{m_k \times m_k}.$$

The important property here is that φ is an algebra isomorphism. Hence we know that this isomorphism maintains positive semidefiniteness: for any matrix $A \in \mathcal{A}$, we have $A \succeq 0 \iff \varphi(A) \succeq 0$. Moreover, the matrix $\varphi(A)$ is block-diagonal, with d diagonal blocks of sizes $m_1, \ldots, m_d$. This is a crucial property which can be exploited in order to get a more efficient way of encoding positive semidefiniteness of matrices in $\mathcal{A}$.

The explicit block-diagonalization of the Terwilliger algebra $\mathcal{A}_n$ was given by Schrijver [22].

THEOREM 2.4 (Schrijver [22]). *The Terwilliger algebra $\mathcal{A}_n$ can be block-diagonalized into $\lfloor \frac{n}{2} \rfloor + 1$ blocks of sizes $m_k = n - 2k + 1$ for $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$. The algebra isomorphism φ sends the matrix*

$$A = \sum_{i,j,t=0}^{n} x_{i,j}^t D_{i,j}^t$$

to the block-matrix $\varphi(A) = \oplus_{k=0}^{\lceil n/2 \rceil} B_k$, where the matrix $B_k \in \mathbb{R}^{m_k \times m_k}$ is given by

$$B_k := \left(\binom{n-2k}{i-k}^{-\frac{1}{2}} \binom{n-2k}{j-k}^{-\frac{1}{2}} \sum_t \beta_{i,j,k}^t x_{i,j}^t \right)_{i,j=k}^{n-k} \tag{2.1}$$

for $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$. Here, for any nonnegative integers i, j, t, k, we set

$$\beta_{i,j,k}^t := \sum_{\ell=0}^{n} (-1)^{\ell - t} \binom{\ell}{t} \binom{n-2k}{n-k-\ell} \binom{n-k-\ell}{i-\ell} \binom{n-k-\ell}{j-\ell}. \tag{2.2}$$

In particular we have

$$\sum_{i,j,t=0}^{n} x_{i,j}^t D_{i,j}^t \succeq 0 \iff B_k \succeq 0 \quad \text{for } k = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{2.3}$$

3. Proof of Theorem 1.1. In this section we give the proof of Theorem 1.1. As a warmup we start with the special case when the degree is $d = 2$ and the edge size is $L = 2$, where we can easily show that the polynomial p_2 is convex.

After that we proceed to the general case. We follow the steps as sketched earlier. First, we express the Hessian matrix of p_d as a matrix polynomial, so that it suffices to show that a set of matrices are positive semidefinite, namely, the matrices M_γ in (3.6) for any $\gamma \in \mathbb{N}_{d-2}^m$, which are (up to scaling) the coefficients of $H(p_d)$ in the monomial basis. After that we indicate some reductions that lead to the task of showing that another set of smaller, well-structured matrices are positive semidefinite, namely, the matrices M_p in (3.10) for any integer $p \leq L(d-2)$. Finally we show the positive semidefiniteness of these matrices M_p by exploiting a link to the Terwilliger algebra of the Boolean Hamming cube.

3.1. The case $d = 2$ and $L = 2$. Here we consider the polynomial

$$p_2(x) = \sum_{e,f \in E} \frac{1}{|e \cup f|} x_e x_f,$$

where $E = \{e \subseteq [n] : |e| = 2\}$. We show that the polynomial p_2 is convex over the standard simplex or, equivalently, that its Hessian matrix is positive semidefinite over Δ_m. Here, the Hessian matrix of p_2 is given by $H(p_2) = 2M$, where M is the matrix indexed by E with entries

$$M_{e,f} = \frac{1}{|e \cup f|} \quad \text{for } e, f \in E. \tag{3.1}$$

Consider the matrices A_2, A_3, A_4 indexed by E, with entries

$$(A_s)_{e,f} = 1 \text{ if } |e \cup f| = s, \quad (A_s)_{e,f} = 0 \text{ otherwise}, \qquad \text{for } s = 2, 3, 4.$$

Then we have $A_2 = I$ and $A_2 + A_3 + A_4 = J$. Clearly we can express the matrix M as a linear combination of these matrices:

$$M = \frac{1}{2}I + \frac{1}{3}A_3 + \frac{1}{4}A_4 = \frac{1}{4}I + \frac{1}{12}A_3 + \frac{1}{4}J = \frac{1}{12}I + \frac{1}{4}J + \frac{1}{12}(A_3 + 2I). \tag{3.2}$$

We can now conclude that $M \succeq 0$ (and thus the polynomial p_2 is convex) in view of the next lemma, which claims that $A_3 + 2I \succeq 0$.

LEMMA 3.1. *Consider the $\binom{n}{2} \times n$ matrix Γ_n, with entries $(\Gamma_n)_{e,i} = |e \cap \{i\}|$ for $e \in E$ and $i \in [n]$. Then $A_3 + 2I = \Gamma_n \Gamma_n^T \succeq 0$.*

Proof. The proof is obtained by direct verification. ∎

Note that the matrices $A_2 = I, A_3, A_4$ generate the Bose–Mesner algebra of the Johnson scheme J_2^n, with length n and weight 2, and thus the matrix M belongs to this Bose–Mesner algebra (see [6] for details on the Johnson scheme). For arbitrary degree $d \geq 3$ and edge size $L = 2$ one could proceed to show that the Hessian matrix of p_d is convex by using a similar symmetry reduction based on the Bose–Mesner algebra of the Johnson scheme J_2^p for suitable values of p. However, for general edge size $L \geq 3$ we will need to use a richer algebra, namely, the Terwilliger algebra of the Hamming cube. Hence we will treat in the rest of the section the general case $d \geq 2$ and $L \geq 2$.

3.2. Computing the Hessian matrix of p_d. In this section we indicate how to compute the Hessian matrix of the polynomial

$$p_d(x) = \sum_{(e_{i_1},\ldots,e_{i_d}) \in E^d} \frac{1}{|e_{i_1} \cup \cdots \cup e_{i_d}|} x_{e_{i_1}} \cdots x_{e_{i_d}}, \tag{3.3}$$

where as before $E = \{e \subseteq V = [n] : |e| = L\}$ with $L \geq 2$. We begin with getting the explicit coefficients of the polynomial p_d expressed in the standard monomial basis. The basic fact we will now use is that the coefficients depend only on the set of distinct edges that are present in the tuple $(e_{i_1}, \ldots, e_{i_d}) \in E^d$ and not on their multiplicities.

To formalize this, recall that $m = |E|$, and let us label the edges as $e_1, \ldots, e_m$ so that $E = \{e_1, \ldots, e_m\}$. For a d-tuple $\underline{e} := (e_{i_1}, \ldots, e_{i_d}) \in E^d$ with $i_1, \ldots, i_d \in [m]$, define the sequence $\alpha(\underline{e}) \in \mathbb{N}^m$, where, for $\ell \in [m]$, $\alpha(\underline{e})_\ell$ is the number of indices among $i_1, \ldots, i_d$ that are equal to ℓ. Then we have

$$x_{e_{i_1}} \cdots x_{e_{i_d}} = x_{e_1}^{\alpha(\underline{e})_1} \cdots x_{e_m}^{\alpha(\underline{e})_m} = x^{\alpha(\underline{e})}$$

and $|\alpha(\underline{e})| = d$ so that $\alpha(\underline{e}) \in \mathbb{N}^m_d$. This justifies the following definition. For $\alpha \in \mathbb{N}^m_d$, consider a d-tuple $\underline{e} = (e_{i_1}, \ldots, e_{i_d}) \in E^d$ such that $\alpha(\underline{e}) = \alpha$ and define

$$c_\alpha := \frac{1}{|e_{i_1} \cup \cdots \cup e_{i_d}|}. \tag{3.4}$$

As an example, for $d = n = m = 3$, if $\alpha = (1, 0, 2)$, then $c_\alpha = \frac{1}{|e_1 \cup e_3|}$, and if $\alpha = (2, 0, 1)$, then we also have $c_\alpha = \frac{1}{|e_1 \cup e_3|}$.

We can now reformulate the polynomial p_d in the (usual) monomial basis.

Lemma 3.2. *The polynomial p_d from* (3.3) *can be reformulated as follows:*

$$p_d(x) = \sum_{\alpha \in \mathbb{N}^m_d} c_\alpha \frac{d!}{\alpha!} x^\alpha, \tag{3.5}$$

setting $\alpha! = \alpha_1! \cdots \alpha_m!$ and where c_α is as defined in (3.4).

Proof. Using the definition of the coefficients c_α, we can rewrite p_d as

$$p_d(x) = \sum_{\alpha \in \mathbb{N}^m_d} \Bigg(\sum_{\substack{\underline{e}=(e_{i_1},\ldots,e_{i_d}) \in E^d: \\ \alpha(\underline{e}) = \alpha}} \frac{1}{|e_{i_1} \cup \cdots \cup e_{i_d}|} \Bigg) x^\alpha = \sum_{\alpha \in \mathbb{N}^m_d} \Bigg(\sum_{\underline{e} \in E^d : \alpha(\underline{e}) = \alpha} c_\alpha \Bigg) x^\alpha,$$

which is equal to $\sum_{\alpha \in \mathbb{N}^m_d} c_\alpha \frac{d!}{\alpha!} x^\alpha$. Here, for this last equality, we use the monomial theorem, which claims the identity

$$\left(\sum_{i=1}^m x_i \right)^d = \sum_{\alpha \in \mathbb{N}^m_d} \frac{d!}{\alpha!} x^\alpha,$$

or, equivalently, that the number of d-tuples $\underline{e} \in E^d$ for which $\alpha(\underline{e}) = \alpha$ is equal to $d!/\alpha!$. ∎

We now proceed to compute the Hessian matrix of p_d.

Lemma 3.3. *The Hessian of the polynomial p_d is the matrix*

$$H(p_d)(x) = \left(\frac{\partial^2 p_d(x)}{\partial x_{e_i} \partial x_{e_j}} \right)_{i,j=1}^m = \sum_{\gamma \in \mathbb{N}^m_{d-2}} \frac{d!}{\gamma!} x^\gamma M_\gamma,$$

where, for any $\gamma \in \mathbb{N}^m_{d-2}$, we set

$$M_\gamma = (c_{\gamma + u_i + u_j})_{i,j=1}^m \tag{3.6}$$

and where the vectors $u_1, \ldots, u_m \in \mathbb{R}^m$ denote the standard basis of $\mathbb{R}^m$.

Proof. The partial derivatives of p_d are

$$\frac{\partial p_d(x)}{\partial x_{e_i}} = \sum_{\alpha \in \mathbb{N}^m_d : \alpha_i \geq 1} \frac{d!}{(\alpha - u_i)!} c_\alpha x^{\alpha - u_i} = \sum_{\beta \in \mathbb{N}^m_{d-1}} \frac{d!}{\beta!} c_{\beta + u_i} x^\beta.$$

Similarly we see that

$$\frac{\partial^2 p(x)}{\partial x_{e_j} \partial x_{e_i}} = \sum_{\beta \in \mathbb{N}^m_{d-1} : \beta_j \geq 1} \frac{d!}{(\beta - u_j)!} c_{\beta + u_i} x^{\beta - u_j} = \sum_{\gamma \in \mathbb{N}^m_{d-2}} c_{\gamma + u_i + u_j} \frac{d!}{\gamma!} x^\gamma.$$

This concludes the proof. ∎

Hence, if we can show that the matrices M_γ in (3.6) are all positive semidefinite, then it follows directly that the Hessian matrix of p_d is positive semidefinite on the standard simplex. In the rest of this section we indicate two successive simplifications that reduce the task of checking positive semidefiniteness of the matrices M_γ (for $\gamma \in \mathbb{N}^m_{d-2}$) to the same task for a smaller set of simpler matrices: first for the matrices M_W (for $W \subseteq V$), and second for the matrices M_p (for $0 \le p \le n$ integer). In section 3.3 we will make a final reduction to show that the matrices M_p are positive semidefinite, exploiting the fact that they belong to Terwilliger algebras.

We begin with the first reduction. For $\gamma \in \mathbb{N}^m$, define its *support* as the set $S_\gamma = \{e \in E : \gamma_e \ge 1\}$ and let

$$W_\gamma = \bigcup_{e \in S_\gamma} e$$

denote the subset of elements of $V = [n]$ that are covered by some edge in the support of γ. Then, for any $i, j \in [m]$, the support of $\gamma + u_i + u_j$ is the set $S_\gamma \cup \{e_i, e_j\}$, and we have

$$(M_\gamma)_{e_i,e_j} = \; c_{\gamma+u_i+u_j} = \frac{1}{|W_\gamma \cup e_i \cup e_j|}.$$

Hence the matrix M_γ depends only on the set W_γ (and not on the specific choice of the sequence γ). This justifies defining the matrices

$$M_W = \left(\frac{1}{|W \cup e \cup f|}\right)_{e,f \in E} \tag{3.7}$$

for any set $W \subseteq V = [n]$. Hence, for any $\gamma \in \mathbb{N}^m_{d-2}$, we have

$$M_\gamma = M_{W_\gamma}. \tag{3.8}$$

Summarizing, we have shown the following.

Lemma 3.4. *Assume that the matrices M_W from* (3.7) *are positive semidefinite for all $W \subseteq V$ with $|W| \ge L$ (if $d \ge 3$) and $|W| \le L(d-2)$. Then the polynomial p_d is convex over the standard simplex.*

If $d = 2$, then there is only one matrix to check, namely, the matrix $M_\emptyset$ (for $W = \emptyset$). Note that the matrix $M_\emptyset$ coincides with the matrix in (3.1), so we already know that it is positive semidefinite when $L = 2$. However, if $d \ge 3$, then one needs to check all the matrices of the form M_W in (3.7).

Now comes the second reduction, which will be useful to link these matrices M_W to the Terwilliger algebra. We observe that in the matrix M_W there are identical rows and columns and the reduction consists simply in removing duplicate rows and columns in M_W and keeping just one copy. For this, set $p := |W|$ and $U := V \setminus W$, so that $|U| = n - p$. In addition set

$$F := \{e \setminus W : e \in E\} = \{e \subseteq U : L - p \le |e| \le L\}, \tag{3.9}$$

which consists of the intersections with U of the edges in E. Then $F = E$ when $p = 0$ and the condition $|e| \ge L - p$ is redundant when $p \ge L$. Now we consider the following matrix M_p, which is indexed by F, with entries

$$(M_p)_{e,f} = \frac{1}{p + |e \cup f|} \quad \text{for } e, f \in F. \tag{3.10}$$

Note that for $p = 0$ the matrix M_0 coincides with the matrix $M_\emptyset$ in (3.7) (and with the matrix in (3.1)). The next lemma links the matrices M_W and M_p and relies on showing that M_p is obtained from M_W by deleting duplicate rows and columns.

LEMMA 3.5. *Let $L \geq 2$ and $d \geq 2$. Consider the matrices M_W in* (3.7) *and M_p in* (3.10). *The following assertions are equivalent:*

(i) *$M_W \succeq 0$ for all $W = e_1 \cup \cdots \cup e_{d-2}$ with $e_1, \ldots, e_{d-2} \in E$.*

(ii) *$M_p \succeq 0$ for all $p \leq L(d-2)$ such that $p \geq L$ if $d \geq 3$.*

Proof. If $d = 2$, then the result holds since $M_0 = M_\emptyset$ as observed above. So assume now that $d \geq 3$. Let $W = e_1 \cup \cdots \cup e_{d-2}$, where $e_1, \ldots, e_{d-2} \in E$, and set $p = |W|$. Consider the partition of the set E into $E = \cup_{i=0}^{L} E_i$, where $E_i = \{e \in E : |e \setminus W| = i\}$. With respect to this partition of its index set, the matrix M_W has the following block-form:

$$M_W = \left(\begin{array}{c|c|c|c} M_W^{0,0} & M_W^{0,1} & \cdots & M_W^{0,L} \\ \hline M_W^{1,0} & M_W^{1,1} & \cdots & M_W^{1,L} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline M_W^{L,0} & M_W^{L,1} & \cdots & M_W^{L,L} \end{array}\right),$$

where the block $M_W^{i,j}$ has its rows indexed by E_i and its columns by E_j. Note that if two edges $e, e' \in E$ satisfy $e \setminus W = e' \setminus W$, then the two rows of M_W indexed by e and e' coincide: for any $f \in E$ we have

$$(M_W^{i,j})_{e,f} = \frac{1}{|W| + |(e \cup f) \setminus W|} = \frac{1}{|W| + |(e' \cup f) \setminus W|} = (M_W^{i,j})_{e',f}.$$

In fact, after removing these duplicate rows (and columns) and keeping only one copy for each subset of $U = V \setminus W$, we obtain the matrix

$$\left(\begin{array}{c|c|c|c} M_p^{0,0} & M_p^{0,1} & \cdots & M_p^{0,L} \\ \hline M_p^{1,0} & M_p^{1,1} & \cdots & M_p^{1,L} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline M_p^{L,0} & M_p^{L,1} & \cdots & M_p^{L,L} \end{array}\right),$$

which coincides with the matrix M_p in (3.10). Indeed, the above matrix is indexed by the set F in (3.9), and its block-form is with respect to the partition $F = \cup_{i=0}^{L} F_i$, where $F_i = \{e \subseteq U : |e| = i\}$. So the block $M_p^{i,j}$ has its rows indexed by F_i, its columns indexed by F_j, and its entries are

$$(M_p^{i,j})_{e,f} = \frac{1}{p + |e \cup f|} = \frac{1}{p + i + j - |e \cap f|} \quad \text{for } e \in F_i, f \in F_j. \tag{3.11}$$

As the matrices M_p arise from M_W by removing its duplicate rows and columns, it is clear that the matrices M_W are positive semidefinite if and only if the same holds for the matrices M_p. This concludes the proof. ∎

In the next section we show that the matrices M_p are positive semidefinite for all $0 \leq p \leq n$ by exploiting their link to Terwilliger algebras.

3.3. The general case $d \geq 2$ and $L \geq 2$. In section 2.3 we gave preliminary results on the Terwilliger algebra, which we will now use to prove that the matrices M_p in (3.10) are positive semidefinite. Fix an integer $0 \leq p \leq n$ and consider the matrix M_p in (3.10), which has a block-form with blocks as in (3.11). We start by observing that M_p belongs to the Terwilliger algebra $\mathcal{A}_{n-p}$. This is clear since relation (3.11) provides the explicit correspondence between the blocks $M_p^{i,j}$ of M_p and the generating matrices $D_{i,j}^t$ of the algebra $\mathcal{A}_{n-p}$:

$$M_p = \sum_{i=0}^{L}\sum_{j=0}^{L}\sum_{t=0}^{\min\{i,j\}} \frac{1}{p+i+j-t} D_{i,j}^t = \sum_{i=0}^{L}\sum_{j=0}^{L}\sum_{t=0}^{\min\{i,j\}} x_{i,j}^t D_{i,j}^t,$$

after setting

$$x_{i,j}^t = \frac{1}{p+i+j-t}. \tag{3.12}$$

Let B_k be the corresponding matrices from (2.1) (replacing n by $n-p$). Then, in view of Theorem 2.4, we know that $M_p \succeq 0$ if and only if $B_k \succeq 0$ for all $0 \leq k \leq \lfloor (n-p)/2 \rfloor$.

In what follows p, k are fixed integers. We now proceed to show that $B_k \succeq 0$.

To simplify the notation we introduce the following parameters:

$$a(i) := \binom{n-p-2k}{i-k}^{-\frac{1}{2}}, \quad b(\ell, i) := \binom{n-p-k-\ell}{i-\ell}, \quad c(\ell) := \binom{n-p-2k}{n-p-k-\ell}$$

for any integers i, ℓ. Note that we may omit the obvious bounding conditions on i and ℓ since the corresponding parameters are zero if these conditions are not satisfied; for instance, $a(i) = 0$ if $i < k$ and $b(\ell, i) = 0$ if $\ell > i$. Then we have

$$B_k = \left(a(i)a(j) \sum_{t=0}^{\min\{i,j\}} \beta_{i,j,k}^t x_{i,j}^t \right)_{i,j=k}^{n-p-k} \tag{3.13}$$

and

$$\beta_{i,j,k}^t := \sum_{\ell=0}^{n-p} (-1)^{\ell-t} \binom{\ell}{t} c(\ell) b(\ell, i) b(\ell, j). \tag{3.14}$$

We now give an integral reformulation for the entries of the matrix B_k from (3.13). It is based on the fact that

$$\frac{1}{i} = \int_0^1 z^{i-1} dz \quad \text{for any integer } i \geq 1, \tag{3.15}$$

which permits us to give an integral reformulation for the scalars $x_{i,j}^t$ in (3.12). This simple but powerful fact will be very useful to show $B_k \succeq 0$. Note that this is similar to the classical argument used by Hilbert [11] to show that the Hilbert matrix $(\frac{1}{i+j-1})_{i,j=1}^n$ is positive semidefinite for any $n \in \mathbb{N}$.

LEMMA 3.6. *We have*

$$\sum_{t=0}^{\min\{i,j\}} \beta_{i,j,k}^t x_{i,j}^t = \sum_{\ell=0}^{\min\{i,j\}} c(\ell) b(\ell, i) b(\ell, j) \int_0^1 g(\ell, z) z^{i+j} dz,$$

where we define the function $g(\ell, z) = z^{p-1}(\frac{1-z}{z})^\ell$ *for* $z \in (0, 1]$.

Proof. First we use the expressions of $\beta^t_{i,j,k}$ in (3.14) and of $x^t_{i,j}$ in (3.12) and exchange the summations in t and ℓ to obtain

$$\sum_{t=0}^{\min\{i,j\}} \beta^t_{i,j,k} x^t_{i,j} = \sum_{\ell=0}^{\min\{i,j\}} \left(\sum_{t=0}^{\ell} \frac{1}{p+i+j-t} (-1)^{\ell-t} \binom{\ell}{t} \right) c(\ell) b(\ell,i) b(\ell,j). \tag{3.16}$$

Now we use (3.15), which gives the following integral representation:

$$\frac{1}{p+i+j-t} = \int_0^1 z^{p+i+j-t-1} dz.$$

Using this integral representation (and the binomial theorem for the equality marked (*) below), we can reformulate the inner summation appearing in (3.16) as follows:

$$\begin{aligned}
\sum_{t=0}^{\ell} \frac{1}{p+i+j-t} (-1)^{\ell-t} \binom{\ell}{t} &= \sum_{t=0}^{\ell} (-1)^{\ell-t} \binom{\ell}{t} \int_0^1 z^{p+i+j-t-1} dz \\
&= \int_0^1 z^{p+i+j-1} (-1)^{\ell} \left(\sum_{t=0}^{\ell} \left(-\frac{1}{z}\right)^t \binom{\ell}{t} \right) dz \\
&\overset{(*)}{=} \int_0^1 z^{p+i+j-1} (-1)^{\ell} \left(1 - \frac{1}{z}\right)^{\ell} dz \\
&= \int_0^1 z^{p+i+j-1} (-1)^{\ell} \left(\frac{z-1}{z}\right)^{\ell} dz \\
&= \int_0^1 z^{p-1} \left(\frac{1-z}{z}\right)^{\ell} z^{i+j} dz.
\end{aligned}$$

This concludes the proof. ∎

We can now proceed to show that the matrices B_k in (3.13) are positive semidefinite.

LEMMA 3.7. *We have $B_k \succeq 0$.*

Proof. We use Lemma 3.6 to reformulate the matrix B_k. First, note that in the result of Lemma 3.6, since $b(\ell,i)b(\ell,j) = 0$ if $\ell > \min\{i,j\}$, we may replace the summation on ℓ from $0 \le \ell \le \min\{i,j\}$ to $0 \le \ell \le n-p$. This implies

$$\begin{aligned}
B_k &= \left(a(i)a(j) \sum_{t=0}^{n-p} \beta^t_{i,j,k} x^t_{i,j} \right)_{i,j=k}^{n-p-k} \\
&= \int_0^1 \left(\sum_{\ell=0}^{n-p} g(\ell,z) c(\ell) \underbrace{(z^i a(i) b(\ell,i))}_{=:h(\ell,z,i)} \underbrace{(z^j a(j) b(\ell,j))}_{=:h(\ell,z,j)} \right)_{i,j=k}^{n-p-k} dz \\
&= \sum_{\ell=0}^{n-p} \int_0^1 g(\ell,z) c(\ell) \underbrace{\left(h(\ell,z,i) h(\ell,z,j) \right)_{i,j=k}^{n-p-k}}_{=:H(\ell,z,k)} dz \\
&= \sum_{\ell \ge 0} \int_0^1 \underbrace{g(\ell,z) c(\ell)}_{\ge 0} \underbrace{H(\ell,z,k)}_{\succeq 0} dz \succeq 0.
\end{aligned}$$

Here we used the fact that, for any $\ell \in [0, n-p]$, the function $g(\ell, z)$ is nonnegative on $(0,1]$ and that the matrix $H(\ell, z, k)$ is positive semidefinite for any $z \in [0,1]$ since it is the outer product of the vector $(h(\ell, z, i))_{i=k}^{n-p-k}$ with itself. ∎

Therefore we have shown that the matrices B_k are positive semidefinite and thus that the following result holds.

COROLLARY 3.8. *The matrices M_p from* (3.10) *are positive semidefinite for all $0 \le p \le n$.*

In view of Lemmas 3.4 and 3.5 we can conclude that the polynomial p_d is convex on Δ_m, which concludes the proof of Theorem 1.1.

4. Investigating the polynomials f_d. Here we consider the second class of polynomials f_d from (1.1), namely,

$$f_d(x) = \sum_{(e_1,\ldots,e_d)\in E^d} \prod_{i=1}^{d} \frac{x_{e_i}}{|e_1 \cup \cdots \cup e_i|}.$$

We address Question 1, which asks whether f_d attains its minimum value on the simplex Δ_m at the barycenter of Δ_m. Here too this question has a positive answer if one can show that f_d is convex over Δ_m. This follows since the analogue of Lemma 2.1 extends easily for the polynomial f_d. We conjecture that convexity holds in general.

CONJECTURE 1. *For any integers $n, L, d \ge 2$ the polynomial f_d is convex over the simplex Δ_m.*

For degree $d = 2$, we have $f_2 = \frac{1}{L} p_2$, and thus we know from Theorem 1.1 that f_2 is convex. We will prove in section 4.2 that Conjecture 1 holds for degree $d = 3$ and edge size $L = 2$, and in section 4.3 and Appendix A we will give a range of values for (n, L, d) that were numerically tested and support Conjecture 1.

In what follows we begin in section 4.1 by giving a polynomial matrix decomposition for the Hessian of f_d. Hence, convexity of f_d over Δ_m follows if we can show that certain well-structured matrices Q_γ, arising as the coefficients of $H(f_d)$ in the monomial basis, are positive semidefinite (see Lemmas 4.1). Then we give a recursive reformulation for the matrices Q_γ, which makes apparent some links to the matrices M_γ arising in the Hessian of p_d (see Lemma 4.4). Using this reformulation we can show positive semidefiniteness of the matrices Q_γ in the case $d = 3$ and $L = 2$ (see section 4.2). However, understanding the general case is technically involved and would require developing new tools for exploiting the symmetry structure present in the matrices Q_γ (which is now not captured by the Terwilliger algebra). This goes beyond the scope of this paper, and we leave it for further research. In very recent work Polak [15] carried out this symmetry reduction, which enabled him to show that Conjecture 1 holds in the case when $d \le 8$ and $L = 2$.

4.1. Computing the Hessian of f_d. We begin by expressing the polynomial f_d in the standard monomial basis:

$$f_d(x) = \sum_{\alpha\in\mathbb{N}_d^m} x^\alpha \sum_{\substack{\underline{e}=(e_1,\ldots,e_d)\in E^d \\ \alpha(\underline{e})=\alpha}} \prod_{i=1}^{d} \frac{1}{|e_1 \cup \cdots \cup e_i|} = \sum_{\alpha\in\mathbb{N}_d^m} b_\alpha x^\alpha, \tag{4.1}$$

where we set

$$b_\alpha = \sum_{\substack{\underline{e}=(e_1,\ldots,e_d)\in E^d \\ \alpha(\underline{e})=\alpha}} \prod_{i=1}^{d} \frac{1}{|e_1 \cup \cdots \cup e_i|}. \tag{4.2}$$

Next we compute the Hessian of f_d and give a matrix polynomial reformulation for it.

LEMMA 4.1. *The Hessian of the polynomial f_d is given by*

$$\frac{\partial^2 f(x)}{\partial x_{e_i} \partial x_{e_j}} = \begin{cases} \sum_{\gamma\in\mathbb{N}^m_{d-2}} (\gamma_i+1)(\gamma_j+1)x^\gamma b_{\gamma+u_i+u_j} & \text{if } i \neq j, \\ \sum_{\gamma\in\mathbb{N}^m_{d-2}} (\gamma_i+1)(\gamma_i+2)x^\gamma b_{\gamma+2u_i} & \text{if } i = j, \end{cases}$$

where, as before, $u_1,\ldots,u_m$ denote the standard basis of $\mathbb{R}^m$. In other words,

$$H(f_d)(x) = \sum_{\gamma\in\mathbb{N}^m_{d-2}} x^\gamma Q_\gamma,$$

where, for $\gamma \in \mathbb{N}^m_{d-2}$, we define the symmetric $m \times m$ matrix Q_γ with entries

$$(Q_\gamma)_{ij} = (\gamma_i+1)(\gamma_j+1)b_{\gamma+u_i+u_j} \ \text{ if } i \neq j, \quad (Q_\gamma)_{ii} = (\gamma_i+1)(\gamma_i+2)b_{\gamma+2u_i} \tag{4.3}$$

for $i,j \in [m]$. Hence, $H(f_d)(x) \succeq 0$ for all $x \in \Delta_m$ if $Q_\gamma \succeq 0$ for all $\gamma \in \mathbb{N}^m_{d-2}$.

Proof. The proof is obtained by direct verification. ∎

We now give a recursive reformulation for the coefficients of the polynomial f_d and for its Hessian matrix, which may possibly be helpful for a proof by induction. Recall the definition of the coefficients b_α of f_d in (4.2). Fix $\alpha \in \mathbb{N}^m_d$. There are $\frac{d!}{\alpha!}$ distinct tuples $\underline{e}$ such that $\alpha(\underline{e}) = \alpha$. For any such sequence $\underline{e} = (e_{i_1},\ldots,e_{i_d})$ with $i_1,\ldots,i_d \in [m]$, $\alpha = \alpha(\underline{e})$ means that, for any $\ell \in [m]$, α_ℓ is the number of occurrences of ℓ within the multiset $\{i_1,\ldots,i_d\}$; so $\alpha_\ell \geq 1$ if $\ell \in \{i_1,\ldots,i_d\}$ and $\alpha_\ell = 0$ if $\ell \notin \{i_1,\ldots,i_d\}$. For instance, for $\underline{e} = (e_1,e_2,e_3,e_2,e_1)$, $d = 5$, $m = 4$, we have $(i_1,\ldots,i_5) = (1,2,3,2,1)$ and $\alpha(\underline{e}) = (2,2,1,0)$.

To reformulate b_α we exploit the fact that b_α enjoys some invariance property under permutations of $[d]$, namely,

$$b_\alpha = \sum_{\substack{\underline{e}=(e_{i_1},\ldots,e_{i_d})\in E^d: \\ \alpha(\underline{e})=\alpha}} \prod_{k=1}^{d} \frac{1}{|e_{i_1} \cup \cdots \cup e_{i_k}|} \tag{4.4}$$

$$= \frac{1}{d!} \sum_{\sigma\in\mathrm{Sym}(d)} \sum_{\substack{\underline{e}=(e_{i_1},\ldots,e_{i_d})\in E^d: \\ \alpha(\underline{e})=\alpha}} \prod_{k=1}^{d} \frac{1}{|e_{i_{\sigma(1)}} \cup \cdots \cup e_{i_{\sigma(k)}}|} \tag{4.5}$$

$$= \frac{1}{d!} \sum_{\substack{\underline{e}=(e_{i_1},\ldots,e_{i_d})\in E^d \\ \alpha(\underline{e})=\alpha}} \underbrace{\sum_{\sigma\in\mathrm{Sym}(d)} \prod_{k=1}^{d} \frac{1}{|e_{i_{\sigma(1)}} \cup \cdots \cup e_{i_{\sigma(k)}}|}}_{=:S}. \tag{4.6}$$

Observe that the inner summation S in (4.6) does not depend on the choice of the sequence $\underline{e}$ such that $\alpha(\underline{e}) = \alpha$; thus we may consider it fixed, denoted by $(e_{i_1},\ldots,e_{i_d})$.

Since there are $\frac{d!}{\alpha!}$ possible choices for selecting this sequence, using relation (4.6) we can reformulate b_α as follows:

$$b_\alpha = \frac{1}{d!}\frac{d!}{\alpha!} \sum_{\sigma\in\mathrm{Sym}(d)} \prod_{k=1}^{d} \frac{1}{|e_{i_{\sigma(1)}} \cup \cdots \cup e_{i_{\sigma(k)}}|} = \frac{1}{\alpha!} \sum_{\sigma\in\mathrm{Sym}(d)} \prod_{k=1}^{d} \frac{1}{|e_{i_{\sigma(1)}} \cup \cdots \cup e_{i_{\sigma(k)}}|}.$$

Next we pull out the factor $\frac{1}{|e_{i_1}\cup\cdots\cup e_{i_d}|} = c_\alpha$ which occurs for $k = d$ and get

$$\begin{aligned} b_\alpha &= \frac{c_\alpha}{\alpha!} \sum_{r=1}^{d} \sum_{\sigma\in\mathrm{Sym}(d):\sigma(d)=r} \prod_{k=1}^{d-1} \frac{1}{|e_{i_{\sigma(1)}} \cup \cdots \cup e_{i_{\sigma(k)}}|} \\ &= \frac{c_\alpha}{\alpha!} \sum_{r=1}^{d} b_{\alpha-u_{i_r}} (\alpha - u_{i_r})! \\ &= c_\alpha \sum_{r=1}^{d} \frac{b_{\alpha-u_{i_r}}}{\alpha_{i_r}} \\ &\overset{(*)}{=} c_\alpha \sum_{k\in[m]:\alpha_k\geq 1} b_{\alpha-u_k}. \end{aligned}$$

Here, in the last equality marked (*), we use the fact that α_k of the elements in the multiset $\{i_1,\ldots,i_d\}$ are equal to k. Summarizing we have shown the following.

LEMMA 4.2. *For any* $\alpha\in\mathbb{N}^m_d$ *we have*

$$b_\alpha = c_\alpha \sum_{k\in[m]:\alpha_k\geq 1} b_{\alpha-u_k}.$$

We now proceed to give a recursive reformulation for the matrices Q_γ in (4.3). First we reformulate them using the scaled parameters

$$\widehat{b}_\alpha := \alpha!\ b_\alpha, \tag{4.7}$$

which satisfy the recursive relation:

$$\widehat{b}_\alpha = c_\alpha \sum_{k:\alpha_k\geq 1} \alpha_k \widehat{b}_{\alpha-u_k}. \tag{4.8}$$

Indeed, by Lemma 4.2 we have

$$\widehat{b}_\alpha = \alpha!\ b_\alpha = \alpha!\ c_\alpha \sum_{k:\alpha_k\geq 1} b_{\alpha-u_k} = \alpha!\ c_\alpha \sum_{k:\alpha_k\geq 1} \frac{\widehat{b}_{\alpha-u_k}}{\alpha-u_k!} = c_\alpha \sum_{k:\alpha_k\geq 1} \alpha_k \widehat{b}_{\alpha-u_k}.$$

LEMMA 4.3. *For any* $\gamma\in\mathbb{N}^m_{d-2}$ *we have* $Q_\gamma = \frac{1}{\gamma!}(\widehat{b}_{\gamma+u_i+u_j})_{i,j=1}^m$.

Proof. We obtain the proof by direct verification: for $i\neq j$ we have $(Q_\gamma)_{ij} = (\gamma_i+1)(\gamma_j+1)b_{\gamma+u_i+u_j} = \widehat{b}_{\gamma+u_i+u_j}(\gamma_i+1)(\gamma_j+1)/(\gamma+u_i+u_j)! = \widehat{b}_{\gamma+u_i+u_j}/\gamma!$ and, for $i=j$, we have $(Q_\gamma)_{ii} = (\gamma_i+1)(\gamma_i+2)b_{\gamma+2u_i} = \widehat{b}_{\gamma+2u_i}(\gamma_i+1)(\gamma_i+2)/(\gamma+2u_i)! = \widehat{b}_{\gamma+2u_i}/\gamma!$. ∎

LEMMA 4.4. *For $d \geq 3$ and $\gamma \in \mathbb{N}^m_{d-2}$ we have*

$$Q_\gamma = \underbrace{(c_{\gamma+u_i+u_j})_{i,j=1}^m}_{M_\gamma} \circ \Big(\sum_{k\in[m]:\gamma_k\geq 1} Q_{\gamma-u_k} + \underbrace{\frac{1}{\gamma!}(\widehat{b}_{\gamma+u_i} + \widehat{b}_{\gamma+u_j})_{i,j=1}^m}_{=:R_\gamma} \Big)$$

$$= M_\gamma \circ \Big(\sum_{k\in[m]:\gamma_k\geq 1} Q_{\gamma-u_k} + R_\gamma \Big),$$

where the matrices M_γ were introduced in (3.6).

Proof. Combining Lemmas 4.2 and 4.4 we obtain

$$\begin{aligned}
(Q_\gamma)_{ij} &= \frac{1}{\gamma!}\widehat{b}_{\gamma+u_i+u_j} = \frac{1}{\gamma!}c_{\gamma+u_i+u_j} \sum_{k:(\gamma+u_i+u_j)_k\geq 1} \widehat{b}_{\gamma+u_i+u_j-u_k}(\gamma+u_i+u_j)_k \\
&= \frac{1}{\gamma!}c_{\gamma+u_i+u_j}\Big(\sum_{k\neq i,j:\gamma_k\geq 1} \widehat{b}_{\gamma+u_i+u_j-u_k}\gamma_k + \widehat{b}_{\gamma+u_j}(\gamma_i+1) + \widehat{b}_{\gamma+u_j}(\gamma_i+1)\Big) \\
&= \frac{1}{\gamma!}c_{\gamma+u_i+u_j}\Big(\sum_{k:\gamma_k\geq 1} \widehat{b}_{\gamma-u_k+u_i+u_j}\gamma_k + \widehat{b}_{\gamma+u_i} + \widehat{b}_{\gamma+u_j}\Big) \\
&= c_{\gamma+u_i+u_j}\Big(\sum_{k:\gamma_k\geq 1} \frac{\widehat{b}_{\gamma-u_k+u_i+u_j}}{(\gamma-u_k)!} + \frac{1}{\gamma!}(\widehat{b}_{\gamma+u_i} + \widehat{b}_{\gamma+u_j})\Big) \\
&= c_{\gamma+u_i+u_j}\Big(\sum_{k:\gamma_k\geq 1} (Q_{\gamma-u_k})_{ij} + \frac{1}{\gamma!}(\widehat{b}_{\gamma+u_i} + \widehat{b}_{\gamma+u_j})\Big),
\end{aligned}$$

which shows the claim. ∎

4.2. The polynomial f_d in the case $d = 3$, $L = 2$. Here we show that the polynomial f_d is convex in the case $d = 3$ and $L = 2$. In view of Lemma 4.1 it suffices to show that the matrix Q_γ is positive semidefinite for any $\gamma \in \mathbb{N}^m_1$. Up to symmetry it suffices to show that $Q_\gamma \succeq 0$ for $\gamma = u_1$. In view of Lemma 4.4 we have

$$Q_{u_1} = \underbrace{(c_{u_1+u_i+u_j})_{i,j=1}^m}_{=M_{u_1}} \circ \Big(Q_0 + \underbrace{(\widehat{b}_{u_1+u_i} + \widehat{b}_{u_1+u_j})_{i,j=1}^m}_{=R_{u_1}} \Big).$$

We have shown earlier that the matrix M_{u_1} is positive semidefinite, which follows from the fact that the matrix M_2 is positive semidefinite (in view of relation (3.8), Lemma 3.5, and Corollary 3.8). Hence, if we can show that $Q_0 + R_{u_1} \succeq 0$, then this will imply that $Q_{u_1} \succeq 0$ and conclude the proof of Theorem 1.2. In the rest of this section we show that $Q_0 + R_{u_1} \succeq 0$.

We begin by describing the entries of the matrix $Q_0 + R_{u_1}$. By definition, the entries of Q_0 (case $\gamma = 0$) are

$$(Q_0)_{ii} = 2b_{2u_i} = \frac{2}{L}, \quad (Q_0)_{ij} = b_{u_i+u_j} = \frac{2}{|e_i \cup e_j|} \quad \text{for } i \neq j \in [m].$$

Moreover, $\widehat{b}_{2u_1} = 2b_{2u_1} = \frac{2}{L}$ and $\widehat{b}_{u_1+u_i} = b_{u_1+u_i} = \frac{2}{|e_1\cup e_i|}$ for $i \geq 2$. Using this, we obtain that

$$Q_0 + R_{u_1} = 2 \cdot \Big(\frac{1}{|e_1 \cup e_j|} + \frac{1}{|e_i \cup e_j|} + \frac{1}{|e_1 \cup e_i|} \Big)_{i,j=1}^m =: 2B,$$

where we define the matrix B as

$$B := \Big(\frac{1}{|e\cup f|}+\frac{1}{|e_1\cup e|}+\frac{1}{|e_1\cup f|}\Big)_{e,f\in E}. \tag{4.9}$$

The main result of this section is the next lemma, which shows that the matrix B (and thus $Q_0 + R_{u_1}$) is positive semidefinite. As observed above, this implies that the polynomial f_3 is convex for $L = 2$ and thus settles Conjecture 1 for the case $d = 3, L = 2$.

PROPOSITION 4.5. *Assume $L = 2$. The matrix B in* (4.9) *is positive semidefinite.*

Before proceeding to the proof, let us make a few observations. Note that the matrix B from (4.9) can be decomposed as

$$B = \underbrace{\Big(\frac{1}{|e\cup f|}\Big)_{e,f\in E}}_{=M_0} + \underbrace{\Big(\frac{1}{|e_1\cup e|}+\frac{1}{|e_1\cup f|}\Big)_{e,f\in E}}_{=:R}.$$

So, $B = M_0 + R$, where $M_0 = M_\emptyset$ has been shown earlier to be positive semidefinite (recall Corollary 3.8, or note that M_0 is the matrix M from (3.1) as we are in the case $L = 2$). On the other hand, the matrix R is not positive semidefinite. In fact, R has rank 2 and has a negative eigenvalue. One can infer from the results in section 3.1 that $\lambda_{\min}(M_0) = 1/12$, while one can check that $\lambda_{\min}(R) < -1/12 = -0.0833...$ when $n \geq 6$ (see Table 1 below). Hence in general one cannot argue that $B \succeq 0$ by simply looking at the smallest eigenvalues of its summands M_0 and R. On a very high level, we will show positive semidefiniteness of the matrix B by observing that it has a simple block structure, which we can exploit by taking several successive Schur complements; in this way we obtain well-structured matrices that can be directly shown to be positive semidefinite. The exact details are not difficult, but a bit technically involved.

In the rest of the section we prove Proposition 4.5. To fix ideas we let e_1 be the edge $e_1 = \{1, 2\}$, and to simplify notation we set $p = n-2$ and $q = \binom{n-2}{2}$. Then the index set of B can be partitioned into $\{e_1\}\cup I_1\cup I_2\cup I_0$, setting $I_k = \{\{k, i\} : 3 \leq i \leq n\}$ for $k = 1, 2$, and $I_0 = \{\{i, j\} : 3 \leq i < j \leq n\}$. So $|I_1| = |I_2| = p$ and $|I_0| = q$. With respect to this partition one can verify that the matrix B has the following block-form:

$$B = \begin{array}{c} \\ e_1 \\ I_1 \\ I_2 \\ I_0 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} e_1 & I_1 & I_2 & I_0 \end{array} \\ \left(\begin{array}{c|c|c|c} \frac{3}{2} & \frac{7}{6}J_{1,p} & \frac{7}{6}J_{1,p} & J_{1,q} \\ \hline \frac{7}{6}J_{p,1} & J_p+\frac{1}{6}I_p & \frac{11}{12}J_p+\frac{1}{12}I_p & \frac{5}{6}J_{p,q}+\frac{1}{12}\Gamma^T \\ \hline \frac{7}{6}J_{p,1} & \frac{11}{12}J_p+\frac{1}{12}I_p & J_p+\frac{1}{6}I_p & \frac{5}{6}J_{p,q}+\frac{1}{12}\Gamma^T \\ \hline J_{q,1} & \frac{5}{6}J_{q,p}+\frac{1}{12}\Gamma & \frac{5}{6}J_{q,p}+\frac{1}{12}\Gamma & M+\frac{1}{2}J_q \end{array}\right) \end{array}.$$

Here M is the matrix from (3.1) (replacing n with $p = n - 2$). We have shown in section 3.1 (see relation (3.2) and Lemma 3.1) that M can be decomposed as

$$M = \frac{1}{12}I_q + \frac{1}{4}J_q + \frac{1}{12}\Gamma\Gamma^T,$$

where $\Gamma = \Gamma_p$ is the $\binom{p}{2} \times p$ matrix whose (f, i)th entry is $|\{i\} \cap f|$.

We now proceed to show that the matrix B is positive semidefinite. Note that its lower right diagonal block indexed by the set I_0 is positive semidefinite (since $M \succeq 0$).

Our strategy is now to "eliminate" the three borders indexed by the sets $\{e_1\}$, I_1, and I_2 successively, one by one, by taking Schur complements, until reaching a final matrix (indexed by I_0) whose positive semidefiniteness can be seen directly. To do the Schur complement operations we will need to invert matrices of the form $aI+bJ$. The next lemma indicates how to do that; its proof is straightforward and thus omitted.

LEMMA 4.6. *For $a, b \in \mathbb{R}$ such that $a+pb \neq 0$, the matrix $aI_p + bJ_p$ is nonsingular with inverse*

$$(aI_p + bJ_p)^{-1} = \frac{1}{a}\Big(I_p - \frac{b}{pb+a}\Big)J_p.$$

Now come three steps where we successively eliminate the three borders of B indexed by $\{e_1\}$, I_1, and I_2, by taking successive Schur complements with respect to the upper left corner.

Step 1: We take a first Schur complement with respect to the upper left corner of B (indexed by e_1). We call $\widetilde{B}_1$ the resulting matrix, which reads

$$\left(\begin{array}{c|c|c} J_p + \frac{1}{6}I_p & \frac{11}{12}J_p + \frac{1}{12}I_p & \frac{5}{6}J_{p,q} + \frac{1}{12}\Gamma^T \\ \hline \frac{11}{12}J_p + \frac{1}{12}I_p & J_p + \frac{1}{6}I_p & \frac{5}{6}J_{p,q} + \frac{1}{12}\Gamma^T \\ \hline \frac{5}{6}J_{q,p} + \frac{1}{12}\Gamma & \frac{5}{6}J_{q,p} + \frac{1}{12}\Gamma & \frac{1}{12}I_q + \frac{1}{12}\Gamma\Gamma^T + \frac{3}{4}J_q \end{array}\right)$$

$$-\frac{2}{3}\begin{pmatrix} \frac{7}{6}J_{p,1} \\ \frac{7}{6}J_{p,1} \\ J_{q,1} \end{pmatrix}\begin{pmatrix} \frac{7}{6}J_{1,p} & \frac{7}{6}J_{1,p} & J_{1,q} \end{pmatrix}$$

$$= \left(\begin{array}{c|c|c} \frac{5}{54}J_p + \frac{1}{6}I_p & \frac{1}{108}J_p + \frac{1}{12}I_p & \frac{1}{18}J_{p,q} + \frac{1}{12}\Gamma^T \\ \hline \frac{1}{108}J_p + \frac{1}{12}I_p & \frac{5}{54}J_p + \frac{1}{6}I_p & \frac{1}{18}J_{p,q} + \frac{1}{12}\Gamma^T \\ \hline \frac{1}{18}J_{q,p} + \frac{1}{12}\Gamma & \frac{1}{18}J_{q,p} + \frac{1}{12}\Gamma & \frac{1}{12}I_q + \frac{1}{12}\Gamma\Gamma^T + \frac{1}{12}J_q \end{array}\right).$$

Setting $B_1 = 6\widetilde{B}_1$, we obtain $B \succeq 0 \Longleftrightarrow \widetilde{B}_1 \succeq 0 \Longleftrightarrow B_1 \succeq 0$, where

$$B_1 = \left(\begin{array}{c|c|c} \frac{5}{9}J_p + I_p & \frac{1}{18}J_p + \frac{1}{2}I_p & \frac{1}{3}J_{p,q} + \frac{1}{2}\Gamma^T \\ \hline \frac{1}{18}J_p + \frac{1}{2}I_p & \frac{5}{9}J_p + I_p & \frac{1}{3}J_{p,q} + \frac{1}{2}\Gamma^T \\ \hline \frac{1}{3}J_{q,p} + \frac{1}{2}\Gamma & \frac{1}{3}J_{q,p} + \frac{1}{2}\Gamma & \frac{1}{2}I_q + \frac{1}{2}\Gamma\Gamma^T + \frac{1}{2}J_q \end{array}\right).$$

Step 2: We now take the Schur complement with respect to the upper left corner of B_1 (indexed by I_1), where we use Lemma 4.6 to invert it:

$$(I_p + 5/9J_p)^{-1} = I_p - 5/(5p+9)J_p.$$

After taking this Schur complement the resulting matrix $\widetilde{B}_2$ reads

$$\widetilde{B}_2 = \left(\begin{array}{c|c} \frac{5}{9}J_p + I_q & \frac{1}{3}J_{p,q} + \frac{1}{2}\Gamma^T \\ \hline \frac{1}{3}J_{q,p} + \frac{1}{2}\Gamma & \frac{1}{2}I_q + \frac{1}{2}\Gamma\Gamma^T + \frac{1}{2}J_q \end{array}\right)$$

$$- \begin{pmatrix} \frac{1}{18}J_p + \frac{1}{2}I_p \\ \frac{1}{3}J_{q,p} + \frac{1}{2}\Gamma \end{pmatrix} \left(I_p - \frac{5}{(5p+9)}J_p\right) \left(\frac{1}{18}J_p + \frac{1}{2}I_p \quad \frac{1}{3}J_{p,q} + \frac{1}{2}\Gamma^T\right)$$

$$= \left(\begin{array}{c|c} \frac{3}{4}I_p + \frac{11p+23}{4(5p+9)}J_p & \frac{1}{4}\Gamma^T + \frac{3p+7}{2(5p+9)}J_{p,q} \\ \hline \frac{1}{4}\Gamma + \frac{3p+7}{2(5p+9)}J_{q,p} & \frac{1}{2}I_q + \frac{1}{4}\Gamma\Gamma^T + \frac{3p+7}{2(5p+9)}J_q \end{array}\right).$$

Setting $B_2 = 4\widetilde{B}_2$ we obtain $B \succeq 0 \Longleftrightarrow B_1 \succeq 0 \Longleftrightarrow B_2 \succeq 0$, where

$$B_2 = \left(\begin{array}{c|c} 3I_p + \frac{11p+23}{5p+9}J_p & \Gamma^T + \frac{2(3p+7)}{5p+9}J_{p,q} \\ \hline \Gamma + \frac{2(3p+7)}{5p+9}J_{q,p} & 2I_q + \Gamma\Gamma^T + \frac{2(3p+7)}{5p+9}J_q \end{array}\right).$$

Step 3: Inverting the top left block of B_2 via Lemma 4.6 gives

$$\left(3I_p + \frac{11p+23}{5p+9}J_p\right)^{-1} = \frac{1}{3}I_p - \frac{(11p+23)}{3(11p^2+38p+27)}J_p.$$

Taking the third and final Schur complement with respect to this block in B_2 we get the matrix

$$B_3 := 2I_q + \Gamma\Gamma^T + \frac{2(3p+7)}{5p+9}J_q$$

$$-\left(\Gamma^T + \frac{2(3p+7)}{5p+9}J_{q,p}\right)\left(\frac{1}{3}I_p - \frac{(11p+23)}{3(11p^2+38p+27)}J_p\right)\left(\Gamma^T + \frac{2(3p+7)}{5p+9}J_{p,q}\right)$$

$$= 2I_q + \frac{2}{3}\Gamma\Gamma^T + \frac{2(9p+25)}{3(11p+27)}J_q.$$

It is now clear that $B_3 \succeq 0$. In turn, this implies that $B_2 \succeq 0$ and thus $B \succeq 0$, which concludes the proof of Proposition 4.5.

We conclude with an indication as to why the above proof seems difficult to extend to the general case $L \geq 3$. The biggest hurdle lies in the richness of the possible intersections between edges of large size. More concretely, recall that the (e, f)th entry of the matrix B in (4.9) depends on $|e \cup f|$, $|e \cup e_1|$, and $|f \cup e_1|$. So one has to take into account how the two edges e, f pairwise interact within e_1 and outside of it, which becomes technically involved when $|e_1| = L$ is large. So the matrix B has an increasingly involved block structure when L grows. In addition some of the blocks in B may have a form that requires an additional symmetry reduction to become amenable.

TABLE 1
Case $d = 3, L = 2$.

d	L	n	γ	$\lambda_{min}(Q_\gamma)$	$\lambda_{min}(B_\gamma)$	$\lambda_{min}(R_\gamma)$
3	2	3	[[1, 2]]	0.0556	0.1667	-0.0236
3	2	4	[[1, 2]]	0.0347	0.0833	-0.0478
3	2	5	[[1, 2]]	0.0347	0.0833	-0.0729
3	2	6	[[1, 2]]	0.0347	0.0833	-0.0987
3	2	7	[[1, 2]]	0.0347	0.0833	-0.1249
3	2	8	[[1, 2]]	0.0347	0.0833	-0.1514

4.3. Some numerical justification for convexity of f_d. We have carried out some numerical experiments for a range of values of d, L, n and verified that the matrices Q_γ are positive semidefinite for all $\gamma \in \mathbb{N}^n_{d-2}$ in these cases. Hence for these values the polynomial f_d is convex and Conjecture 1 holds. Recall from Lemma 4.4 that the matrix Q_γ can be decomposed as

$$Q_\gamma = M_\gamma \circ \Bigg(\underbrace{\sum_{k\in[m]:\gamma_k\geq 1} Q_{\gamma-u_k}}_{=:B_\gamma} + R_\gamma \Bigg) = M_\gamma \circ (B_\gamma + R_\gamma).$$

By the results in section 3 we already know that the matrix M_γ is positive semidefinite. Hence it now suffices to show that the matrix $B_\gamma + R_\gamma$ is positive semidefinite for each $\gamma \in \mathbb{N}^n_{d-2}$. We did this in the previous section for the case $d = 3$ (and $L = 2$). We have computed the minimum eigenvalues of the matrices Q_γ, B_γ, and R_γ for different values of n, d, and L and give this information for the case $L = 2$ in Table 1 below (for $d = 3$) and in Tables 2–6 in Appendix A (for $d \geq 4$). (Further numerical results for $L \geq 3$ can be found in the arXiv version of this paper.) In each case we consider the possible different cases for selecting $\gamma \in \mathbb{N}^n_{d-2}$ up to symmetry; the different instances of γ are indicated in the column labeled γ. For instance, for $d = 3, L = 2$ there is only one possibility, say $\gamma = u_1$ corresponding to edge $e_1 = \{1, 2\}$ (see Table 1). For $d = 4, L = 2$ there are three possibilities: $\gamma = 2e_1$ with $e_1 = \{1, 2\}$; $\gamma = u_1 + u_2$ with $e_1 = \{1, 2\}$ and $e_2 = \{1, 3\}$; and $\gamma = u_1 + u_2$ with $e_1 = \{1, 2\}$ and $e_2 = \{3, 4\}$ (see Table 2).

In all cases we find that Q_γ is positive semidefinite (in fact, positive definite). As already mentioned in section 4.2 for the case $d = 3$, we see that in general this cannot be deduced by considering its summands separately, since R_γ has a negative smallest eigenvalue and $\lambda_{min}(B_\gamma)+\lambda_{\min}(R_\gamma) < 0$ from a certain n (which depends on d and L). In addition we observe that $\lambda_{\min}(B_\gamma)$ stays constant from a certain n while $\lambda_{\min}(R_\gamma)$ keeps decreasing. It remains an open problem to show that the property $Q_\gamma \succeq 0$ holds in general.

For recent progress on this problem we refer the reader to Polak [15], who proved that all the matrices Q_γ are positive semidefinite in the case $d \leq 8$ and $L = 2$. One of the difficulties is that one needs to enumerate the distinct patterns for $\gamma \in \mathbb{N}^m_{d-2}$, i.e., the number of multigraphs with $d - 2$ edges. As mentioned earlier in Example 1 this number is given by the OEIS sequence A050535 [14], and it grows quickly with d.

Appendix A. Numerical results for the polynomials f_d.

We group here Table 2 through Table 6, which show the eigenvalues of the matrices Q_γ, B_γ, and R_γ for $L = 2$ and small values of n, d. To see that these are indeed

TABLE 2
Case $d = 4, L = 2$.

d	L	n	γ	$\lambda_{min}(Q_\gamma)$	$\lambda_{min}(B_\gamma)$	$\lambda_{min}(R_\gamma)$
4	2	3	[[1, 2], [1, 2]]	0.0185	0.0556	-0.0415
4	2	4	[[1, 2], [1, 2]]	0.0133	0.0347	-0.0805
4	2	5	[[1, 2], [1, 2]]	0.0133	0.0347	-0.1189
4	2	6	[[1, 2], [1, 2]]	0.0133	0.0347	-0.1572
4	2	3	[[1, 3], [1, 2]]	0.0593	0.1778	-0.0028
4	2	4	[[1, 3], [1, 2]]	0.0238	0.0802	-0.0478
4	2	5	[[1, 3], [1, 2]]	0.0214	0.0743	-0.092
4	2	6	[[1, 3], [1, 2]]	0.0214	0.0741	-0.1359
4	2	7	[[1, 3], [1, 2]]	0.0214	0.074	-0.1798
4	2	4	[[3, 4], [1, 2]]	0.0174	0.0694	-0.0012
4	2	5	[[3, 4], [1, 2]]	0.0174	0.0694	-0.029
4	2	6	[[3, 4], [1, 2]]	0.0174	0.0694	-0.0565
4	2	7	[[3, 4], [1, 2]]	0.0174	0.0694	-0.084
4	2	8	[[3, 4], [1, 2]]	0.0174	0.0694	-0.1115
4	2	9	[[3, 4], [1, 2]]	0.0174	0.0694	-0.139

TABLE 3
Case $d = 5, L = 2$.

d	L	n	γ	$\lambda_{min}(Q_\gamma)$	$\lambda_{min}(B_\gamma)$	$\lambda_{min}(R_\gamma)$
5	2	3	[[1, 2], [1, 2], [1, 2]]	0.0062	0.0185	-0.0425
5	2	4	[[1, 2], [1, 2], [1, 2]]	0.0049	0.0133	-0.0804
5	2	5	[[1, 2], [1, 2], [1, 2]]	0.0049	0.0133	-0.1163
5	2	3	[[1, 3], [1, 2], [1, 2]]	0.0298	0.0894	-0.0062
5	2	4	[[1, 3], [1, 2], [1, 2]]	0.0111	0.0396	-0.0605
5	2	5	[[1, 3], [1, 2], [1, 2]]	0.0098	0.0358	-0.112
5	2	6	[[1, 3], [1, 2], [1, 2]]	0.0098	0.0358	-0.162
5	2	4	[[3, 4], [1, 2], [1, 2]]	0.0077	0.0307	-0.0085
5	2	5	[[3, 4], [1, 2], [1, 2]]	0.0072	0.0307	-0.038
5	2	6	[[3, 4], [1, 2], [1, 2]]	0.0067	0.0307	-0.0667
5	2	7	[[3, 4], [1, 2], [1, 2]]	0.0067	0.0307	-0.0948
5	2	4	[[1, 4], [1, 3], [1, 2]]	0.0263	0.1052	-0.009
5	2	5	[[1, 4], [1, 3], [1, 2]]	0.0162	0.0716	-0.0681
5	2	6	[[1, 4], [1, 3], [1, 2]]	0.0151	0.0676	-0.1255
5	2	7	[[1, 4], [1, 3], [1, 2]]	0.015	0.0675	-0.1819
5	2	8	[[1, 4], [1, 3], [1, 2]]	0.015	0.0675	-0.2374
5	2	4	[[2, 4], [1, 3], [1, 2]]	0.0188	0.0753	-0.0063
5	2	5	[[2, 4], [1, 3], [1, 2]]	0.0151	0.0678	-0.0613
5	2	6	[[2, 4], [1, 3], [1, 2]]	0.0139	0.0635	-0.1147
5	2	7	[[2, 4], [1, 3], [1, 2]]	0.0139	0.0635	-0.167
5	2	8	[[2, 4], [1, 3], [1, 2]]	0.0139	0.0635	-0.2186
5	2	5	[[2, 3], [1, 5], [1, 4]]	0.0114	0.0571	-0.0053
5	2	6	[[2, 3], [1, 5], [1, 4]]	0.0113	0.0569	-0.0395
5	2	7	[[2, 3], [1, 5], [1, 4]]	0.0107	0.0569	-0.0731
5	2	8	[[2, 3], [1, 5], [1, 4]]	0.0107	0.0569	-0.1062
5	2	9	[[2, 3], [1, 5], [1, 4]]	0.0107	0.0569	-0.1391
5	2	3	[[2, 3], [1, 3], [1, 2]]	0.0926	0.2778	-0.0
5	2	4	[[2, 3], [1, 3], [1, 2]]	0.0237	0.085	-0.0967
5	2	5	[[2, 3], [1, 3], [1, 2]]	0.0212	0.0764	-0.1882
5	2	6	[[2, 3], [1, 3], [1, 2]]	0.0212	0.0764	-0.2766
5	2	6	[[5, 6], [3, 4], [1, 2]]	0.0087	0.0521	-0.0011
5	2	7	[[5, 6], [3, 4], [1, 2]]	0.0087	0.0521	-0.0233
5	2	8	[[5, 6], [3, 4], [1, 2]]	0.0087	0.0521	-0.0452
5	2	9	[[5, 6], [3, 4], [1, 2]]	0.0087	0.0521	-0.067
5	2	10	[[5, 6], [3, 4], [1, 2]]	0.0087	0.0521	-0.0885
5	2	11	[[5, 6], [3, 4], [1, 2]]	0.0087	0.0521	-0.11

TABLE 4
Case $d = 6, L = 2$ (first part).

d	L	n	γ	$\lambda_{min}(Q_\gamma)$	$\lambda_{min}(B_\gamma)$	$\lambda_{min}(R_\gamma)$
6	2	3	[[1, 2], [1, 2], [1, 2], [1, 2]]	0.0021	0.0062	-0.0349
6	2	4	[[1, 2], [1, 2], [1, 2], [1, 2]]	0.0017	0.0049	-0.0652
6	2	5	[[1, 2], [1, 2], [1, 2], [1, 2]]	0.0017	0.0049	-0.0931
6	2	3	[[1, 3], [1, 2], [1, 2], [1, 2]]	0.0124	0.0371	-0.0094
6	2	4	[[1, 3], [1, 2], [1, 2], [1, 2]]	0.0044	0.0165	-0.0579
6	2	5	[[1, 3], [1, 2], [1, 2], [1, 2]]	0.004	0.0148	-0.1029
6	2	6	[[1, 3], [1, 2], [1, 2], [1, 2]]	0.004	0.0148	-0.1457
6	2	3	[[1, 3], [1, 3], [1, 2], [1, 2]]	0.0261	0.0785	-0.0016
6	2	4	[[1, 3], [1, 3], [1, 2], [1, 2]]	0.0064	0.0237	-0.0626
6	2	5	[[1, 3], [1, 3], [1, 2], [1, 2]]	0.0057	0.0211	-0.1193
6	2	6	[[1, 3], [1, 3], [1, 2], [1, 2]]	0.0057	0.0211	-0.1732
6	2	4	[[3, 4], [1, 2], [1, 2], [1, 2]]	0.0031	0.0125	-0.0141
6	2	5	[[3, 4], [1, 2], [1, 2], [1, 2]]	0.0026	0.0124	-0.0384
6	2	6	[[3, 4], [1, 2], [1, 2], [1, 2]]	0.0024	0.0115	-0.0616
6	2	4	[[3, 4], [3, 4], [1, 2], [1, 2]]	0.0038	0.0153	-0.0007
6	2	5	[[3, 4], [3, 4], [1, 2], [1, 2]]	0.0036	0.0153	-0.0255
6	2	6	[[3, 4], [3, 4], [1, 2], [1, 2]]	0.0033	0.0153	-0.0492
6	2	7	[[3, 4], [3, 4], [1, 2], [1, 2]]	0.0033	0.0153	-0.0721
6	2	4	[[1, 4], [1, 3], [1, 2], [1, 2]]	0.0147	0.0589	-0.0142
6	2	5	[[1, 4], [1, 3], [1, 2], [1, 2]]	0.0084	0.0386	-0.0771
6	2	6	[[1, 4], [1, 3], [1, 2], [1, 2]]	0.0078	0.036	-0.1372
6	2	7	[[1, 4], [1, 3], [1, 2], [1, 2]]	0.0078	0.036	-0.1952
6	2	4	[[2, 4], [1, 3], [1, 2], [1, 2]]	0.0129	0.0514	-0.0151
6	2	5	[[2, 4], [1, 3], [1, 2], [1, 2]]	0.0079	0.037	-0.0766
6	2	6	[[2, 4], [1, 3], [1, 2], [1, 2]]	0.0073	0.0344	-0.1352
6	2	7	[[2, 4], [1, 3], [1, 2], [1, 2]]	0.0073	0.0344	-0.1919
6	2	4	[[2, 4], [1, 3], [1, 3], [1, 2]]	0.0102	0.0407	-0.0089
6	2	5	[[2, 4], [1, 3], [1, 3], [1, 2]]	0.0074	0.0343	-0.064
6	2	6	[[2, 4], [1, 3], [1, 3], [1, 2]]	0.0068	0.0318	-0.1167
6	2	7	[[2, 4], [1, 3], [1, 3], [1, 2]]	0.0068	0.0318	-0.1675
6	2	5	[[2, 3], [1, 5], [1, 4], [1, 4]]	0.0059	0.0294	-0.0148
6	2	6	[[2, 3], [1, 5], [1, 4], [1, 4]]	0.0052	0.0293	-0.0485
6	2	7	[[2, 3], [1, 5], [1, 4], [1, 4]]	0.0049	0.0278	-0.0813
6	2	8	[[2, 3], [1, 5], [1, 4], [1, 4]]	0.0049	0.0278	-0.1132
6	2	5	[[2, 3], [2, 3], [1, 5], [1, 4]]	0.0053	0.0266	-0.0055
6	2	6	[[2, 3], [2, 3], [1, 5], [1, 4]]	0.0047	0.0259	-0.0344
6	2	7	[[2, 3], [2, 3], [1, 5], [1, 4]]	0.0044	0.0249	-0.0623
6	2	8	[[2, 3], [2, 3], [1, 5], [1, 4]]	0.0044	0.0249	-0.0897

exhaustive, we again refer to the OEIS sequence A050535 [14] giving the number of multigraphs with up to four edges. Note that some of the multigraphs can only appear if the number n of vertices is large enough.

TABLE 5
Case $d = 6, L = 2$ (second part).

d	L	n	γ	$\lambda_{min}(Q_\gamma)$	$\lambda_{min}(B_\gamma)$	$\lambda_{min}(R_\gamma)$
6	2	3	[[2, 3], [1, 3], [1, 2], [1, 2]]	0.0525	0.1574	-0.0017
6	2	4	[[2, 3], [1, 3], [1, 2], [1, 2]]	0.0125	0.0467	-0.1176
6	2	5	[[2, 3], [1, 3], [1, 2], [1, 2]]	0.0112	0.0416	-0.2252
6	2	6	[[2, 3], [1, 3], [1, 2], [1, 2]]	0.0112	0.0416	-0.3274
6	2	6	[[5, 6], [3, 4], [1, 2], [1, 2]]	0.0038	0.023	-0.0086
6	2	7	[[5, 6], [3, 4], [1, 2], [1, 2]]	0.0035	0.0229	-0.0278
6	2	8	[[5, 6], [3, 4], [1, 2], [1, 2]]	0.0033	0.022	-0.0466
6	2	9	[[5, 6], [3, 4], [1, 2], [1, 2]]	0.0033	0.022	-0.065
6	2	5	[[1, 5], [1, 4], [1, 3], [1, 2]]	0.0208	0.104	-0.0167
6	2	6	[[1, 5], [1, 4], [1, 3], [1, 2]]	0.0121	0.066	-0.0852
6	2	7	[[1, 5], [1, 4], [1, 3], [1, 2]]	0.0115	0.063	-0.1515
6	2	8	[[1, 5], [1, 4], [1, 3], [1, 2]]	0.0115	0.0629	-0.2164
6	2	5	[[2, 3], [1, 5], [1, 4], [1, 2]]	0.0141	0.0706	-0.0132
6	2	6	[[2, 3], [1, 5], [1, 4], [1, 2]]	0.0107	0.0592	-0.0743
6	2	7	[[2, 3], [1, 5], [1, 4], [1, 2]]	0.0101	0.0562	-0.1336
6	2	8	[[2, 3], [1, 5], [1, 4], [1, 2]]	0.0101	0.0562	-0.1915
6	2	6	[[2, 3], [1, 6], [1, 5], [1, 4]]	0.0084	0.0505	-0.0119
6	2	7	[[2, 3], [1, 6], [1, 5], [1, 4]]	0.0079	0.0503	-0.0503
6	2	8	[[2, 3], [1, 6], [1, 5], [1, 4]]	0.0075	0.049	-0.0879
6	2	9	[[2, 3], [1, 6], [1, 5], [1, 4]]	0.0075	0.0489	-0.1248
6	2	4	[[2, 3], [1, 4], [1, 3], [1, 2]]	0.0246	0.0985	-0.0122
6	2	5	[[2, 3], [1, 4], [1, 3], [1, 2]]	0.0162	0.0746	-0.1185
6	2	6	[[2, 3], [1, 4], [1, 3], [1, 2]]	0.0151	0.0695	-0.2198
6	2	7	[[2, 3], [1, 4], [1, 3], [1, 2]]	0.0151	0.0695	-0.3177
6	2	4	[[3, 4], [2, 4], [1, 3], [1, 2]]	0.0204	0.0815	-0.0003
6	2	5	[[3, 4], [2, 4], [1, 3], [1, 2]]	0.014	0.0644	-0.0946
6	2	6	[[3, 4], [2, 4], [1, 3], [1, 2]]	0.0129	0.0594	-0.1846
6	2	7	[[3, 4], [2, 4], [1, 3], [1, 2]]	0.0129	0.0594	-0.2716
6	2	6	[[2, 6], [2, 3], [1, 5], [1, 4]]	0.0077	0.046	-0.005
6	2	7	[[2, 6], [2, 3], [1, 5], [1, 4]]	0.0074	0.0456	-0.0392
6	2	8	[[2, 6], [2, 3], [1, 5], [1, 4]]	0.0071	0.0456	-0.0727
6	2	9	[[2, 6], [2, 3], [1, 5], [1, 4]]	0.0071	0.0456	-0.1055
6	2	5	[[2, 5], [2, 3], [1, 4], [1, 3]]	0.0121	0.0603	-0.0084
6	2	6	[[2, 5], [2, 3], [1, 4], [1, 3]]	0.01	0.055	-0.0643
6	2	7	[[2, 5], [2, 3], [1, 4], [1, 3]]	0.0094	0.0523	-0.1184
6	2	8	[[2, 5], [2, 3], [1, 4], [1, 3]]	0.0094	0.0523	-0.1712
6	2	6	[[5, 6], [2, 4], [1, 3], [1, 2]]	0.0077	0.0461	-0.0096
6	2	7	[[5, 6], [2, 4], [1, 3], [1, 2]]	0.0073	0.0461	-0.0451
6	2	8	[[5, 6], [2, 4], [1, 3], [1, 2]]	0.007	0.0457	-0.0799
6	2	9	[[5, 6], [2, 4], [1, 3], [1, 2]]	0.007	0.0457	-0.114

TABLE 6
Case $d = 6, L = 2$ (last part).

d	L	n	γ	$\lambda_{min}(Q_\gamma)$	$\lambda_{min}(B_\gamma)$	$\lambda_{min}(R_\gamma)$
6	2	7	[[4, 5], [2, 3], [1, 7], [1, 6]]	0.0057	0.0399	-0.0047
6	2	8	[[4, 5], [2, 3], [1, 7], [1, 6]]	0.0056	0.0399	-0.0276
6	2	9	[[4, 5], [2, 3], [1, 7], [1, 6]]	0.0053	0.0398	-0.0501
6	2	10	[[4, 5], [2, 3], [1, 7], [1, 6]]	0.0053	0.0398	-0.0723
6	2	5	[[4, 5], [2, 3], [1, 3], [1, 2]]	0.0121	0.0606	-0.0192
6	2	6	[[4, 5], [2, 3], [1, 3], [1, 2]]	0.0112	0.0606	-0.0784
6	2	7	[[4, 5], [2, 3], [1, 3], [1, 2]]	0.0106	0.0594	-0.1359
6	2	8	[[4, 5], [2, 3], [1, 3], [1, 2]]	0.0106	0.0594	-0.1919
6	2	8	[[7, 8], [5, 6], [3, 4], [1, 2]]	0.0043	0.0347	-0.0008
6	2	9	[[7, 8], [5, 6], [3, 4], [1, 2]]	0.0043	0.0347	-0.0161
6	2	10	[[7, 8], [5, 6], [3, 4], [1, 2]]	0.0043	0.0347	-0.0312
6	2	11	[[7, 8], [5, 6], [3, 4], [1, 2]]	0.0043	0.0347	-0.0462

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