Applied Mathematics and Computation 406 (2021) 125999

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Corrigendum to "Total value adjustment for a stochastic volatility model. A comparison with the Black–Scholes model"

Beatriz Salvador[a,*], Cornelis W. Oosterlee[a,b]

[a] CWI - Centrum Wiskunde & Informatica, Amsterdam, the Netherlands
[b] DIAM, Delft University of Technology, Delft, the Netherlands

ARTICLE INFO

Article history:
Available online 27 April 2021

Keywords:
(non)linear PDEs
Heston model
Expected Exposure
Potential Future Exposure
credit value adjustment
finite element method

ABSTRACT

Since the 2007/2008 financial crisis, the total value adjustment (XVA) should be included when pricing financial derivatives. In the present paper, the derivative values of European and American options have been priced where we take into account counterparty risk. Whereas European and American options considering counterparty risk have already been priced under Black-Scholes dynamics in [2], here the novel contribution is the introduction of stochastic volatility resulting in a Heston stochastic volatility type partial differential equation to be solved. We derive the partial differential equation modeling the XVA when stochastic volatility is assumed. For both European and American options, a linear and a nonlinear problem have been deduced. In order to obtain a numerical solution, suitable and appropriate boundary conditions have been considered. In addition, a method of characteristics for the time discretization combined with a finite element method in the spatial discretization has been implemented. The expected exposure and potential future exposure are also computed to compare the current model with the associated Black–Scholes model.

1. Introduction

Since the 2007/2008 financial crisis, counterparty credit risk has become an important aspect of the Basel regulations, which should be considered when pricing and risk managing financial derivatives. Different value adjustments are nowadays included to the (risk-free) financial derivative values, such as, credit value adjustment, CVA [21,25], debit value adjustment, DVA [10], funding value adjustment, FVA [18,19], collateral value adjustment, CollVA, or capital value adjustment, KVA [15]. The set of adjustments is collectively known as total value adjustment or XVA

The PDE model for pricing the XVA associated with European options was derived in Burgard and Kjaer [5], with more discussion on funding costs found in [4,6]. A way to extend the PDE model to American options is presented in [2]. Linear and nonlinear Black–Scholes-type equations, connected to the mark-to-market value, which model the total value adjustment, were derived. As a basic principle behind these Black–Scholes XVA models constant asset volatility was assumed. However, these equations cannot take into account the implied volatility surface features observed in the market, such as the implied volatility smile and skew phenomena. These indicate that implied volatility tends to vary with respect to the strike price and expiration time. In this work, we model the volatility of the underlying price as a stochastic process, and

DOI of original article: 10.1016/j.amc.2020.125489

* Corresponding author.
E-mail address: beatriz.salvador@udc.es (B. Salvador).

https://doi.org/10.1016/j.amc.2021.125999
0096-3003/© 2021 Published by Elsevier Inc.

consider the Heston model [16] to value derivatives with XVA more accurately. As a result, we will deal with a model depending on two stochastic factors, the asset price and the volatility/variance of the asset process.

Particularly, we will value European and American options traded between two risky counterparties. As in [2], we derive the partial differential equation (PDE) modeling the risky derivative value, when the risk–free value, v, is modeled by the Heston PDE. The initial value problem for the XVA is based on a similar procedure as for the Black–Scholes model, and the risky derivative value $\widehat{v}$ is decomposed as the sum $\widehat{v} = v + u$, where u denotes the total value adjustment component.

We consider two possible values for the mark–to–market close–out value, the risk–free value which leads to a linear valuation problem, and the risky derivative value where a nonlinear model results.

Numerical European option pricing PDE valuation based on the Heston model, has been addressed by many authors, like in [1,17]. American option pricing under stochastic volatility is also presented in different papers, see, for example, [20,26]. Some authors do not only compute the option values but also counterparty risk has been included, in the form of CVA. In [12], the CVA for cliquet options under stochastic volatility is addressed. In [11,13], a comparison of CVA under the Heston and Black Scholes models is made to analyze the relevance of incorporating stochastic volatility in this context, and in [11,14] the exposures associated with CVA under Heston model were also computed.

The goal of the present paper is to compute the total value adjustment that should be incorporated under Heston's stochastic volatility. In particular, we incorporate the total value adjustments formed by CVA, DVA and FVA. We compare the constant and stochastic volatility models, to show the relevance of considering the stochastic volatility behaviour. For this, the expected exposure, EE, and the potential future exposure, PFE, will also be computed.

To obtain a numerical solution, first of all, appropriate boundary conditions are imposed on a truncated computational domain. We will work with a Lagrange–Galerkin discrete method for the time and spatial discretizations. Moreover, a fixed-point iterative scheme is implemented to solve the nonlinear XVA problem when considering the mark–to–market value being equal to the risky value. Various numerical results are presented and compared to show the relevance of incorporating counterparty risk in combination with stochastic volatility.

The paper is organized as follows: in Section 2, the PDE problem modeling the XVA, is derived. In Section 3, the numerical methods are explained. Section 4 presents some numerical results to show the relevance of including stochastic volatility and some conclusions are drawn in Section 5.

2. Mathematical model

We consider a contract between two counterparties, a seller X and a buyer Y, and we assume the following assets associated to the trade:

- Counterparty X's zero recovery bond price, B_X, with yield r_{B_X};
- Counterparty Y's zero recovery bond price, B_Y, with yield r_{B_Y};
- An underlying asset without default risk.

As a novel aspect compared to the work in [2], we introduce stochastic volatility. The assets are modeled as stochastic processes which here satisfy the following stochastic differential equations (SDEs):

$$\begin{aligned} dB_{X_t} &= r_{B_X}(t)B_{X_t}dt - B_{X_t}dJ_t^X, \\ dB_{Y_t} &= r_{B_Y}(t)B_{Y_t}dt - B_{Y_t}dJ_t^Y, \\ dS_t &= r_R(t)S_t dt + \sqrt{\nu_t}S_t dW_t^S, \end{aligned} \tag{1}$$

where ν_t, the instantaneous variance, follows a CIR process [8],

$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t}dW_t^\nu. \tag{2}$$

Here, W_t^S and W_t^ν are two Wiener processes with correlation coefficient ρ, so the instantaneous covariance equals ρdt. Moreover, J_t^X and J_t^Y denote two independent jump processes which take value 1 in the case X and Y go into default before the maturity time, respectively, otherwise their value equals zero. The other parameters in the above equations are as follows. r_R is the rate of return of the asset; θ is the mean variance. When t tends to infinity the expected value of ν_t tends to θ; κ is the speed at which ν_t reverts to θ; σ is the volatility of the variance, or "vol of vol" (in short) and determines the volatility of ν_t.

When the variance process parameters satisfy the Feller condition, given by

$$\frac{2\kappa\theta}{\sigma^2} > 1,$$

then the process ν_t is strictly positive, otherwise, the variance process may reach zero.

Next, assume a derivative trade between two defaultable counterparties. We denote by $\widehat{v}_t = \widehat{v}(t, S_t, \nu_t, J_t^X, J_t^Y)$ the derivative value at time t, considering counterparty risk from the point of view of the seller X, which depends on the asset's spot value, S_t, variance, ν_t, and the default states, J_t^X and J_t^Y, associated to the seller X and buyer Y, respectively. When a transaction takes place between two risk–free counterparties, the derivative value is denoted by $v_t = v(t, S_t, \nu_t)$.

If we deal with a derivatives contract between two defaultable counterparties, some technical issues related to the close–out of such a contract should be mentioned. Two possible values are assumed to be assigned to the close–out mark–to–market value, M, [5], i.e. the risk–free or the defaultable derivative value. The derivative value including the probability of default, $\widehat{v}(t, S_t, \nu_t, J_t^X, J_t^Y)$, includes several adjustments due to counterparty risk, whereas the derivative value based on risk–free counterparties, $v(t, S_t, \nu_t)$, does not consider any adjustment associated with the counterparty risk and equals the well-known Heston PDE derivative value.

Moreover, in a derivative trade between risky counterparties, we need to model the following two conditions regarding the default of the issuers of the contract:

- If the seller X would go into default first, then

$$\widehat{v}(t, S_t, \nu_t, 1, 0) = M^+(t, S_t, \nu_t) + R_X M^-(t, S_t, \nu_t); \tag{3}$$

- If the buyer Y would go into default first, then

$$\widehat{v}(t, S_t, \nu_t, 0, 1) = R_Y M^+(t, S_t, \nu_t) + M^-(t, S_t, \nu_t). \tag{4}$$

In the above equations, $R_X \in [0, 1]$ and $R_Y \in [0, 1]$ denote the recovery rates on the derivatives position of the counterparties X and Y, respectively. In addition, the following notation has been employed:

$$z^+ = \max(z, 0) \qquad z^- = \min(z, 0)\,.$$

Depending on the, positive or negative, value of M, an amount is received or paid, respectively, by counterparty X. This amount may, or may not, be affected by the corresponding recovery rate.

Note that according to the choice of mark–to–market value, based on a risk–free or risky derivative value, a linear or nonlinear problem will be derived.

2.1. Heston model with counterparty risk

With the aim of computing the value of a derivative under the existence of counterparty risk in the contract, the derivative is totally hedged by a self–financing portfolio Π, covering the underlying risk factors in the model. We value the derivative from the point of view of the seller X, so that, as in [5], the following hedge equation is built.

$$-\widehat{v}_t = \Pi_t.$$

Assuming portfolio Π_t, at time t, consists of the following products,

- $\Phi(t)$ units of another risky option $\widetilde{v}_t = \widetilde{v}(t, S, \nu, J^X, J^Y)$;
- $\Delta(t)$ units of the underlying asset S_t;
- $\alpha_X(t)$ units of B_{X_t};
- $\alpha_Y(t)$ units of B_{Y_t};
- $\gamma(t)$ units of cash, made up of a financing amount, to buy Y's bonds, and an amount which guarantees that the value of the portfolio at time t hedges the derivative contract value, such amount is called a repo amount.

We will denote by γ_P the cost of the portfolio and by γ_{B_X} the cost of a position in X's bonds, which means the necessary amount to buy or sell X's bonds. We define the funding account, γ_F, as the difference between the cost of the hedge portfolio and the price of the position in counterparty X's bonds, $\gamma_{\widetilde{v}}$ is added to the equation, which refers to the cost of Φ options. So,

$$\gamma_F = \gamma_P - (\gamma_{B_X} + \gamma_{\widetilde{v}}).$$

In addition, similarly to γ_{B_X}, γ_{B_Y} denotes the cost of a position in Y's bonds. Moreover, the repo account, γ_R, is formed by the cash amount invested or borrowed, with the aim of funding the asset position $\Delta(t)S_t$ through a repurchase agreement. Although γ_P, γ_{B_X}, $\gamma_{\widetilde{v}}$, γ_F and Φ all depend on t, for simplicity we omit this dependence in the next expressions.

As in [2], the values of the different bonds satisfy the following relations for $s \geq t$:

$$dB_F(t, s) = \begin{cases} r_F(s)B_F(s)ds & \text{if} \quad \gamma_F \leq 0, \\ r(s)B_F(s)ds & \text{if} \quad \gamma_F > 0, \end{cases}$$

and

$$\begin{aligned} dB_{B_Y} &= r(s)B_{B_Y}(s)ds, \\ dB_R(t, s) &= r_R(s)B_R(s)ds, \end{aligned}$$

jointly with $B_F(t, t) = B_{B_Y}(t, t) = B_R(t, t) = 1$. B_F and B_R are two bonds with different interest rates, in which the amount of cash is invested. Moreover, r denotes the risk–free interest rate, r_F the funding rate from the issuer and r_R is the rate paid for the underlying in a repurchase agreement. According to the previous details, the portfolio value can be written as:

$$\Pi_t = \Delta(t)S_t + \alpha_X(t)B_{X_t} + \alpha_Y(t)B_{Y_t} + \gamma(t) + \Phi(t)\widetilde{v}_t, \tag{5}$$

and, with the purpose of preventing arbitrage opportunities,

$$d\Pi_t = -d\widehat{v}_t.$$

Taking into account the relation among the intensity of default, the spread and the recovery rate, as B_X and B_Y are zero recovery bonds, their spreads and their default intensities λ_X and λ_Y, are given by, respectively:

$$\lambda_X = r_{B_X} - r, \qquad \lambda_Y = r_{B_Y} - r. \tag{6}$$

Based on the self–financing feature of the portfolio, we obtain:

$$d\Pi_t = \Delta(t)dS_t + \alpha_X(t)dB_{X_t} + \alpha_Y(t)dB_{Y_t} + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{B_Y} - r_R\gamma_R)(t)dt + \Phi(t)d\widetilde{v}_t\,. \tag{7}$$

We apply Ito's lemma for jump–diffusion processes to obtain the equation for the variation of the derivative value $\widehat{v}_t$

$$\begin{aligned} d\widehat{v}_t &= \left(\frac{\partial\widehat{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widehat{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widehat{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widehat{v}}{\partial S\partial\nu}\right)dt + \frac{\partial\widehat{v}}{\partial S}dS + \frac{\partial\widehat{v}}{\partial\nu}d\nu \\ &\quad + \Delta\widehat{v}_{X_t}dJ^X + \Delta\widehat{v}_{Y_t}dJ^Y \\ &= \left(\frac{\partial\widehat{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widehat{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widehat{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widehat{v}}{\partial S\partial\nu} + r_R S\frac{\partial\widehat{v}}{\partial S} + \kappa(\theta-\nu)\frac{\partial\widehat{v}}{\partial\nu}\right)dt \\ &\quad + \sqrt{\nu}S\frac{\partial\widehat{v}}{\partial S}dW^S + \sigma\sqrt{\nu}\frac{\partial\widehat{v}}{\partial\nu}dW^\nu + \Delta\widehat{v}_{X_t}dJ^X + \Delta\widehat{v}_{Y_t}dJ^Y, \end{aligned} \tag{8}$$

where $\widehat{v}$ and all partial derivatives of $\widehat{v}$ are evaluated at $(t, S_t, \nu_t, J_t^X, J_t^Y)$. Moreover, we have used the notations,

$$\begin{aligned} \Delta\widehat{v}_{X_t} &= \widehat{v}(t, S_t, \nu_t, 1, 0) - \widehat{v}(t, S_t, \nu_t, 0, 0), \\ \Delta\widehat{v}_{Y_t} &= \widehat{v}(t, S_t, \nu_t, 0, 1) - \widehat{v}(t, S_t, \nu_t, 0, 0). \end{aligned}$$

These quantities can be computed using the default conditions given by (3) and (4). A similar result is obtained for the option $\widetilde{v}$.

Considering the expressions (7) and (8), the following hedging equation is obtained,

$$\begin{aligned} &\Delta(t)dS + \alpha_X(t)dB_X + \alpha_Y(t)dB_Y + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{B_Y} - r_R\gamma_R)(t)dt \\ &\quad + \Phi\left(\left(\frac{\partial\widetilde{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widetilde{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widetilde{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widetilde{v}}{\partial S\partial\nu} + r_R S\frac{\partial\widetilde{v}}{\partial S} + \kappa(\theta-\nu)\frac{\partial\widetilde{v}}{\partial\nu}\right)dt\right. \\ &\quad \left. + \sqrt{\nu}S\frac{\partial\widetilde{v}}{\partial S}dW^S + \sigma\sqrt{\nu}\frac{\partial\widetilde{v}}{\partial\nu}dW_t^\nu + \Delta\widetilde{v}_X dJ^X + \Delta\widetilde{v}_Y dJ^Y\right) \\ &= -\left(\left(\frac{\partial\widehat{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widehat{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widehat{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widehat{v}}{\partial S\partial\nu} + r_R S\frac{\partial\widehat{v}}{\partial S} + \kappa(\theta-\nu)\frac{\partial\widehat{v}}{\partial\nu}\right)dt\right. \\ &\quad \left. + \sqrt{\nu}S\frac{\partial\widehat{v}}{\partial S}dW^S + \sigma\sqrt{\nu}\frac{\partial\widehat{v}}{\partial\nu}dW^\nu + \Delta\widehat{v}_X dJ^X + \Delta\widehat{v}_Y dJ^Y\right). \end{aligned}$$

Connected to the SDEs in (1), we deduce

$$\begin{aligned} &\Delta(t)dS + \alpha_X(t)(r_{B_X}(t)B_X dt - B_X dJ^X) + \alpha_Y(t)(r_{B_Y}(t)B_Y dt - B_Y dJ^Y) \\ &\quad + (r\gamma_F^+ r_F\gamma_F^- - r\gamma_{B_Y} - r_R\gamma_R)(t)dt + \Phi\left(\left(\frac{\partial\widetilde{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widetilde{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widetilde{v}}{\partial\nu^2}\right.\right. \\ &\quad \left. + \rho\sigma\nu S\frac{\partial^2\widetilde{v}}{\partial S\partial\nu} + r_R S\frac{\partial\widetilde{v}}{\partial S} + \kappa(\theta-\nu)\frac{\partial\widetilde{v}}{\partial\nu}\right)dt + \sqrt{\nu_t}S\frac{\partial\widetilde{v}}{\partial S}dW^S + \sigma\sqrt{\nu}\frac{\partial\widetilde{v}}{\partial\nu}dW^\nu \\ &\quad \left. + \Delta\widetilde{v}_X dJ^X + \Delta\widetilde{v}_Y dJ^Y\right) \\ &= -\left(\left(\frac{\partial\widehat{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widehat{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widehat{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widehat{v}}{\partial S\partial\nu} + r_R S\frac{\partial\widehat{v}}{\partial S} + \kappa(\theta-\nu)\frac{\partial\widehat{v}}{\partial\nu}\right)dt\right. \\ &\quad \left. + \sqrt{\nu}S\frac{\partial\widehat{v}}{\partial S}dW^S + \sigma\sqrt{\nu}\frac{\partial\widehat{v}}{\partial\nu}dW^\nu + \Delta\widehat{v}_X dJ^X + \Delta\widehat{v}_Y dJ^Y\right). \end{aligned} \tag{9}$$

When removing all risk from the portfolio Π_t, the following weights need to be chosen:

$$\Phi(t) = -\frac{\partial\widehat{v}/\partial\nu}{\partial\widetilde{v}/\partial\nu},$$

$$
\begin{aligned}
\Delta(t) &= -\frac{\partial\widehat{v}}{\partial S} - \Phi\frac{\partial\widetilde{v}}{\partial S} = -\frac{\partial\widehat{v}}{\partial S} + \frac{\partial\widehat{v}/\partial\nu}{\partial\widetilde{v}/\partial\nu}\frac{\partial\widetilde{v}}{\partial S},\\
\alpha_X(t) &= \frac{\Delta\widehat{v}_X + \Phi\Delta\widetilde{v}_X}{B_X} = \frac{-\widehat{v} + (M^+ + R_X M^-) + \Phi(\widetilde{v} - (M^+ + R_X M^-))}{B_X},\\
\alpha_Y(t) &= \frac{\Delta\widehat{v}_Y + \Phi\Delta\widetilde{v}_Y}{B_Y} = \frac{-\widehat{v} + (M^- + R_Y M^+) + \Phi(\widetilde{v} - (M^- + R_Y M^+))}{B_Y}.
\end{aligned}
$$

As a result, the hedging Eq. (9) leads to:

$$
\begin{aligned}
&\alpha_X(t) r_{B_X}(t) B_{X_t} dt + \alpha_Y(t) r_{B_Y}(t) B_{Y_t} dt + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{B_Y} - r_R\gamma_R)(t)dt\\
&+ \Phi\left(\frac{\partial\widetilde{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widetilde{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widetilde{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widetilde{v}}{\partial S\partial\nu}\right)dt\\
&= -\left(\frac{\partial\widehat{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widehat{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widehat{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widehat{v}}{\partial S\partial\nu}\right)dt.
\end{aligned}
\tag{10}
$$

Taking into account the following equivalence:

$$
\begin{aligned}
&\alpha_X r_{B_X} B_X dt + \alpha_Y r_{B_Y} B_Y dt + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{B_Y} - r_R\gamma_R)dt\\
&= r\gamma_P - r\gamma_{\widetilde{v}} + s_F\gamma_F^- - r_R\Delta S + (r_{B_Y} - r)\alpha_Y B_Y + (r_{B_Y} - r)\alpha_X B_X,
\end{aligned}
$$

and, similarly to the derivations in [5], following (6), replacing the values of γ_F, γ_R, $\gamma_{\widetilde{v}}$ and γ_{B_Y}, and rewriting $\alpha_X B_X$ and $\alpha_Y B_Y$ in terms of the mark–to–market value, the previous equivalence can be written as follows,

$$
\begin{aligned}
&\alpha_X r_{B_X} B_X dt + \alpha_Y r_{B_Y} B_Y dt + (r\gamma_F^+ + r_F\gamma_F^- - r\gamma_{B_Y} - r_R\gamma_R)dt\\
&= -r\widehat{v} + r\Phi\widetilde{v} + s_F(-\widehat{v} - \alpha_X B_X - \Phi\widetilde{v})^- + \lambda_X\alpha_X B_X + \lambda_Y\alpha_Y B_Y - r_R\Delta S\\
&= -r\widehat{v} + r\Phi\widetilde{v} + s_F(-\widehat{v} - \alpha_X B_X - \Phi\widetilde{v})^- + \lambda_X\big(-\widehat{v} + (M^+ + R_X M^-)\\
&\quad + \Phi(\widetilde{v} - (M^+ + R_X M^-))\big) - r_R\Delta S + \lambda_Y\big(-\widehat{v} + (M^- + R_Y M^+)\\
&\quad + \Phi(\widetilde{v} - (M^- + R_Y M^+))\big)\\
&= -(r + \lambda_X + \lambda_Y)\widehat{v} + (r + \lambda_X + \lambda_Y)\Phi\widetilde{v} + s_F(-\widehat{v} - \alpha_X B_X - \Phi\widetilde{v})^- - r_R\Delta S\\
&\quad + \lambda_X(M^+ + R_X M^-) + \lambda_X\Phi(M^+ + R_X M^-)\\
&\quad + \lambda_Y(M^- + R_Y M^+) + \lambda_Y\Phi(M^- + R_Y M^+).
\end{aligned}
$$

By introducing this into the hedging Eq. (10), we obtain:

$$
\begin{aligned}
&-(r + \lambda_X + \lambda_Y)\widehat{v} + (r + \lambda_X + \lambda_Y)\Phi\widetilde{v} + s_F(-\widehat{v} - \alpha_X B_X - \Phi\widetilde{v})^- - r_R\Delta S\\
&+ \lambda_X(M^+ + R_X M^-) + \lambda_X\Phi(M^+ + R_X M^-) + \lambda_Y(M^- + R_Y M^+)\\
&+ \lambda_Y\Phi(M^- + R_Y M^+) + \Phi\left(\frac{\partial\widetilde{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widetilde{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widetilde{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widetilde{v}}{\partial S\partial\nu}\right)dt\\
&= -\left(\frac{\partial\widehat{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widehat{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widehat{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widehat{v}}{\partial S\partial\nu}\right)dt.
\end{aligned}
$$

After rearranging and replacing the value of Φ, we deduce:

$$
\begin{aligned}
&\frac{1}{\partial\widehat{v}/\partial\nu}\left(\frac{\partial\widehat{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widehat{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widehat{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widehat{v}}{\partial S\partial\nu} - (r + \lambda_X + \lambda_Y)\widehat{v} + r_R S\frac{\partial\widehat{v}}{\partial S}\right.\\
&\left. + \lambda_X(M^+ + R_X M^-) + \lambda_Y(M^- + R_Y M^+) - s_F M^+\right)\\
&= \frac{1}{\partial\widetilde{v}/\partial\nu}\left(\frac{\partial\widetilde{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widetilde{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widetilde{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widetilde{v}}{\partial S\partial\nu} + r_R S\frac{\partial\widetilde{v}}{\partial S}\right.\\
&\left. - (r + \lambda_X + \lambda_Y)\widetilde{v} + \lambda_X(M^+ + R_X M^-) + \lambda_Y(M^- + R_Y M^+) - s_F M^+\right).
\end{aligned}
$$

As this equation must hold for all derivatives of $\widehat{v}$ and $\widetilde{v}$, the right-hand side should be equal to some general function $\lambda(t, S, \nu) - \kappa(\theta - \nu)$.

For simplicity, we set $\lambda = 0$, resulting in the following Heston PDE

$$
\begin{aligned}
&\frac{\partial\widehat{v}}{\partial t} + \frac{1}{2}\nu S^2\frac{\partial^2\widehat{v}}{\partial S^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2\widehat{v}}{\partial\nu^2} + \rho\sigma\nu S\frac{\partial^2\widehat{v}}{\partial S\partial\nu} + r_R S\frac{\partial\widehat{v}}{\partial S} + \kappa(\theta - \nu)\frac{\partial\widehat{v}}{\partial\nu} - (r + \lambda_X + \lambda_Y)\widehat{v}\\
&= s_F M^+ - \lambda_X(M^+ + R_X M^-) - \lambda_Y(M^- + R_Y M^+).
\end{aligned}
$$

As a result, the derivative value considering counterparty risk and stochastic volatility is modeled by the PDE problem

$$\begin{cases}\partial_t\widehat{v}+\mathcal{A}\widehat{v}-(r+\lambda_X+\lambda_Y)\widehat{v}=s_FM^+-\lambda_X(M^++R_XM^-)-\lambda_Y(M^-+R_YM^+)\,,\\ \widehat{v}(T,S,\nu)=g(S),\end{cases}\tag{11}$$

where s_F represents the funding costs of the entity, M the mark–to–market value, the differential operator $\mathcal{A}$ is given by,

$$\mathcal{A}v=\frac{1}{2}\nu S^2\frac{\partial^2 v}{\partial S^2}+\frac{1}{2}\sigma^2\nu\frac{\partial^2 v}{\partial\nu^2}+\rho\sigma\nu S\frac{\partial^2 v}{\partial S\partial\nu}+r_RS\frac{\partial v}{\partial S}+\kappa(\theta-\nu)\frac{\partial v}{\partial\nu}\,,\tag{12}$$

and $g(S)$ denotes the payoff function.

As we introduced previously, according to the choices of the mark–to–market value at default, based on the two common scenarios, two different PDEs models are deduced. When $M=\widehat{v}$, a nonlinear problem is posed,

$$\begin{cases}\partial_t\widehat{v}+\mathcal{A}\widehat{v}-r\widehat{v}=s_F\widehat{v}^++\lambda_X(1-R_X)\widehat{v}^-+\lambda_Y(1-R_Y)\widehat{v}^+,\\ \widehat{v}(T,S,\nu)=g(S)\,.\end{cases}$$

With $M=v$, a linear problem is deduced,

$$\begin{cases}\partial_t\widehat{v}+\mathcal{A}\widehat{v}-(r+\lambda_X+\lambda_Y)\widehat{v}=s_Fv^+-(\lambda_X+R_Y\lambda_Y)v^+-(\lambda_Y+R_X\lambda_X)v^-,\\ \widehat{v}(T,S,\nu)=g(S)\,.\end{cases}$$

To obtain the equation which models the XVA, we split up the risky derivative value, $\widehat{v}$, as the sum of the risk–free value, and the total value adjustment, u:

$$\widehat{v}=v+u.$$

Note that function v is the solution of the classical Heston PDE model [24] given by

$$\begin{cases}\partial_tv+\mathcal{A}v-rv=0,\\ v(T,S,\nu)=g(S),\end{cases}\tag{13}$$

where the operator $\mathcal{A}$ has been previously introduced in (12).

With the separation of $\hat{v}$ in (11), noting that v is the solution of (13), the problem which models the XVA reads,

$$\begin{cases}\partial_tu+\mathcal{A}u-(r+\lambda_X+\lambda_Y)u=s_FM^+-\lambda_X(M^++R_XM^-)\\ \qquad\qquad\qquad\qquad\qquad\qquad\quad-\lambda_Y(M^-+R_YM^+)+(\lambda_X+\lambda_Y)v\\ u(T,S,\nu)=0\,.\end{cases}\tag{14}$$

Like in the case of the risky derivative problem, depending on the choice of the mark–to–market close–out value, two models are obtained. When $M=\widehat{v}$, a nonlinear problem is to be solved,

$$\begin{cases}\partial_tu+\mathcal{A}u-ru=s_F(v+u)^++\lambda_X(1-R_X)(v+u)^-+\lambda_Y(1-R_Y)(v+u)^+,\\ u(T,S,\nu)=0\,.\end{cases}\tag{15}$$

For $M=v$, the linear problem to be solved reads,

$$\begin{cases}\partial_tu+\mathcal{A}u-(r+\lambda_X+\lambda_Y)u=s_Fv^++\lambda_X(1-R_X)v^-+\lambda_Y(1-R_Y)v^+,\\ u(T,S,\nu)=0\,.\end{cases}\tag{16}$$

Based on the PDE problems, we can write the total value adjustment in terms of conditional expectations. With this aim, following [11], we consider an arbitrage-free economy with the risk-neutral pricing measure denoted by $\mathbb{Q}$, such that a price associated to any attainable claim is computed as the expectation of a discounted value under this probability measure. We choose the numeraire of $\mathbb{Q}$ to be, $C_t=\exp(rt)$, where r is the risk-free rate. Such numeraire refers to the bank account, with $C_0=1$.

In order to compute the expected exposure and potential future exposure, we introduce generally the exposure. Let's assume that $\{X_t,t\in[0,T]\}$ is a Markov process, and let T be the maturity time for a position. The associated discounted and added cash flows at time $t\le T$ are denoted by $\Pi(t,T)$. The exposure at time t is now defined as

$$E_t=\left(\mathbb{E}^{\mathbb{Q}}[\Pi(t,T)\mid X_t]\right)^+,$$

where $E_t:\Omega\to\mathbb{R}^+\cup\{0\}$ is a random variable on the sample space Ω. The potential future exposure, *PFE*, measures the exposure to occur at a future date at a high confidence level. It is defined as a quantile of the exposure distribution, often under the real-world probability measure $\mathbb{P}$. At a fixed time $t\in[0,T]$, the value of the *PFE* is thus defined by

$$PFE_\alpha(t)=\inf\{y\mid\mathbb{P}(\{\omega:E_t(\omega)<y\})\le\alpha\}\,,$$

where α is the confidence level and $\omega\in\Omega$.

The expected exposure, *EE*, is the probability-weighted average exposure at a future date. At a fixed time $t\in[0,T]$, $EE(t)$ is given by

$$EE(t)=\mathbb{E}^{\mathbb{P}}[E_t]=\mathbb{E}^{\mathbb{P}}\left[\left(\mathbb{E}^{\mathbb{Q}}[\Pi(t,T)\mid X_t]\right)^+\right]=\int_\Omega E_t(\omega)d\mathbb{P}(\omega)\,,$$

with $\omega \in \Omega$.

Finally, we define the risk-neutral discounted expected exposure in terms of the expected exposure as follows

$$EE^*(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{E_t}{C_t}\right] = \int_\Omega \frac{E_t(\omega)}{C_t(\omega)} d\mathbb{Q}(\omega),$$

which is independent of the counterparty default. Thanks to the Feynman–Kac theorem, the following expressions at initial time can be derived. For $M = \widehat{v}$,

$$\begin{aligned} u(0,s,\nu) = \mathbb{E}^{\mathbb{Q}}\Bigg[\int_0^T e^{\int_0^\omega r d\tau}\Big(&-s_F(v(\omega,S(\omega),\nu(\omega)) + u(\omega,S(\omega),\nu(\omega)))^+ \\ &-\lambda_X(1-R_X)(v(\omega,S(\omega),\nu(\omega)) + u(\omega,S(\omega),\nu(\omega)))^- \\ &-\lambda_Y(1-R_Y)(v(\omega,S(\omega),\nu(\omega)) + u(\omega,S(\omega),\nu(\omega)))^+\Big)d\omega \mid S_0 = s, \nu_0 = \nu\Bigg]. \end{aligned}$$

Due to the constant parameters, this expression can be rewritten:

$$\begin{aligned} u(0,s,\nu) = &-\mathbb{E}^{\mathbb{Q}}\left[s_F\int_0^T e^{r\omega}(v(\omega,S(\omega),\nu(\omega)) + u(\omega,S(\omega),\nu(\omega)))^+ d\omega \mid S_0 = s, \nu_0 = \nu\right] \\ &-\mathbb{E}^{\mathbb{Q}}\left[\lambda_X(1-R_X)\int_0^T e^{r\omega}(v(\omega,S(\omega),\nu(\omega)) + u(\omega,S(\omega),\nu(\omega)))^- d\omega \mid S_0 = s, \nu_0 = \nu\right] \\ &-\mathbb{E}^{\mathbb{Q}}\left[\lambda_Y(1-R_Y)\int_0^T e^{r\omega}(v(\omega,S(\omega),\nu(\omega)) + u(\omega,S(\omega),\nu(\omega)))^+ d\omega \mid S_0 = s, \nu_0 = \nu\right]. \end{aligned} \tag{17}$$

In the case of $M = v$,

$$\begin{aligned} u(0,s,\nu) = \mathbb{E}^{\mathbb{Q}}\Bigg[\int_0^T e^{\int_0^\omega (r+\lambda_X+\lambda_Y)d\tau}\Big(&-s_F v(\omega,S(\omega),\nu(\omega))^+ - \lambda_X(1-R_X)v(\omega,S(\omega),\nu(\omega))^- \\ &-\lambda_Y(1-R_Y)v(\omega,S(\omega),\nu(\omega))^+\Big)d\omega \mid S_0 = s, \nu_0 = \nu\Bigg]. \end{aligned}$$

We can rewrite this expression, as,

$$\begin{aligned} u(0,s,\nu) = &-\mathbb{E}^{\mathbb{Q}}\left[s_F\int_0^T e^{(r+\lambda_X+\lambda_Y)\omega}v(\omega,S(\omega),\nu(\omega))^+ d\omega \mid S_0 = s, \nu_0 = \nu\right] \\ &-\mathbb{E}^{\mathbb{Q}}\left[\lambda_X(1-R_X)\int_0^T e^{(r+\lambda_X+\lambda_Y)\omega}v(\omega,S(\omega),\nu(\omega))^- d\omega \mid S_0 = s, \nu_0 = \nu\right] \\ &-\mathbb{E}^{\mathbb{Q}}\left[\lambda_Y(1-R_Y)\int_0^T e^{(r+\lambda_X+\lambda_Y)\omega}v(\omega,S(\omega),\nu(\omega))^+ d\omega \mid S_0 = s, \nu_0 = \nu\right]. \end{aligned} \tag{18}$$

In the previous expression, S_0 and ν_0 have been used to denote the values of S and ν at the time instance $t = 0$

So, the XVA is modeled by means of a PDE and it is rewritten in terms of expectations by the Feynman–Kac theorem. Moreover, the expressions for XVA have been split into three individual terms, each of which represents a specific adjustment, the FVA, DVA and CVA, respectively. In order to analyze the impact of stochastic volatility, we also compute the *EE* and the *PFE* values. For this, the XVA is re-written in terms of the loss–given default and the probability of default quantities. The value of these adjustments is written in terms of the expected exposure, as follows:

$$\begin{aligned} \text{CVA}(0) &= LGD_Y\int_0^T EE^*(t)dPD_Y(t) \\ &\simeq LGD_Y\sum_{j=0}^{M-1} EE^*\big(t_j\big)\big(PD_Y\big(t_{j+1}\big) - PD_Y\big(t_j\big)\big), \\ \text{DVA}(0) &= LGD_X\int_0^T EE^*(t)dPD_X(t) \\ &\simeq LGD_X\sum_{j=0}^{M-1} EE^*\big(t_j\big)\big(PD_X\big(t_{j+1}\big) - PD_X\big(t_j\big)\big), \\ \text{FVA}(0) &= FS\int_0^T EE^*(t)dt, \end{aligned}$$

where $\{0 = t_0 < t_1, \ldots < t_{M+T}\}$ is a fixed time grid. Moreover LGD denotes the loss given default, which describes the percentage of loss when a bank's counterparty goes into default and the amount of funds that are lost by an investor when a borrower fails to make the payments. The LGD is computed as $LGD = 1 - R$, with R the recovery rate. The indices X and Y refer to the counterparty and the institution, respectively. PD defines the risk-neutral probability of counterparty default between the times zero and t. Finally, FS refers to the funding spread. Note that in the DVA formula, the quantity EE^* is related to the expected negative exposure, which then leads to negative values. Taking into account the expressions for the individual adjustments, XVA can be written as follows:

$$
\begin{aligned}
\mathrm{XVA}(0) &= \mathrm{CVA}(0) + \mathrm{DVA}(0) + \mathrm{FVA}(0) = LGD_Y \int_0^T EE^*(t) dPD_Y(t) + LGD_X \int_0^T EE^*(t) dPD_X(t) + FS \int_0^T EE^*(t) dt \\
&\simeq LGD_Y \sum_{j=0}^{M-1} EE^*(t_j)(PD_Y(t_{j+1}) - PD_Y(t_j)) + LGD_X \sum_{j=0}^{M-1} EE^*(t_j)(PD_X(t_{j+1}) - PD_X(t_j)) + FS \sum_{j=0}^{M-1} EE^*(t_j). \quad (19)
\end{aligned}
$$

Note that the EE and PFE for the Heston model depend on both the asset price and the stochastic variance process. In order to compute these values throughout the life of the option, the asset price and stochastic volatility are fixed to specific values. Moreover, with the aim of comparing with the Black–Scholes model solutions, where EE and PFE only depend on the asset price, the exposures in this case will be computed considering the derivative value to be evaluated at the same asset prices, and for the particular volatility which is then fixed for the Heston exposure profiles. Finally, as commonly found in finance, the computation of the exposures will be addressed by Monte Carlo techniques.

2.2. Heston model for American options considering counterparty risk

In the previous section, the XVA model for European options has been derived, next, we derive the American option valuation adjustments, including counterparty risk. Based on hedging arguments, a self-financing portfolio is defined and no-arbitrage properties are imposed. In particular, we build the same portfolio as in (5) for the European options. However, in the American case, with the purpose of avoiding arbitrage opportunities, the following hedging inequality is found,

$$
d\Pi_t + d\widehat{v}_t \leq 0 \,.
$$

Following the same reasoning as in the European option case, the following linear complementarity problem results,

$$
\begin{cases}
\mathcal{N}(\widehat{v}) = \partial_t \widehat{v} + \mathcal{A}\widehat{v} - (r + \lambda_X + \lambda_Y)\widehat{v} - s_F M^+ + \lambda_X (M^+ + R_X M^-) + \lambda_Y (M^- + R_Y M^+) \leq 0, \\
\widehat{v}(t, S, \nu) \geq g(S), \\
\mathcal{N}(\widehat{v})(\widehat{v} - g) = 0, \\
\widehat{v}(T, S, \nu) = g(S) \,.
\end{cases} \quad (20)
$$

The above formulation is also written in terms of the mark–to–market value. According to the mark–to–market close–out value, we thus find also two complementarity problems. When $M = \widehat{v}$, a nonlinear operator is obtained:

$$
\begin{cases}
\mathcal{N}_1(\widehat{v}) = \partial_t \widehat{v} + \mathcal{A}\widehat{v} - r\widehat{v} - s_F \widehat{v}^+ - \lambda_X (1 - R_X)\widehat{v}^- - \lambda_Y (1 - R_Y)\widehat{v}^+ \leq 0, \\
\widehat{v}(t, S, \nu) \geq g(S), \\
\mathcal{N}_1(\widehat{v})(\widehat{v} - g) = 0, \\
\widehat{v}(T, S, \nu) = g(S) \,.
\end{cases}
$$

For $M = v$, a linear operator results,

$$
\begin{cases}
\mathcal{N}_2(\widehat{v}) = \partial_t \widehat{v} + \mathcal{A}\widehat{v} - (r + \lambda_X + \lambda_Y)\widehat{v} - s_F v^+ + (\lambda_X + R_Y \lambda_Y) v^+ + (\lambda_Y + R_X \lambda_X) v^- \leq 0, \\
\widehat{v}(t, S, \nu) \geq g(S), \\
\mathcal{N}_2(\widehat{v})(\widehat{v} - g) = 0, \\
\widehat{v}(T, S, \nu) = g(S) \,.
\end{cases}
$$

As in [2], in the case of American options, to obtain the XVA value, we compute the value of the risky derivative and the risk–free value and define the total value adjustment as the difference between them, i.e. $XVA = \widehat{v} - v$. Note that the risk–free value is the solution of the following classical complementarity problem,

$$
\begin{cases}
\widetilde{\mathcal{N}}(v) = \partial_t v + \mathcal{A}v - rv \leq 0, \\
v(t, S, \nu) \geq g(S), \\
\widetilde{\mathcal{N}}(v)(v - g) = 0, \\
v(T, S, \nu) = g(S) \,.
\end{cases} \quad (21)
$$

The expected exposure, EE, and the potential future exposure, PFE, are computed as for the European options, where v and $\widehat{v}$ are solutions of the corresponding complementarity problems.

3. Numerical methods

In order to solve the previously defined PDE problems numerically, various numerical techniques are employed. We focus on the nonlinear problem, as the linear version can be addressed in a very similar way. First of all, we need to apply a localization procedure to define a suitable finite domain with appropriate boundary conditions. The time discretization is based on the Lagrangian method, which is combined with a piecewise linear finite element spatial discretization. Comparing with the literature, finite difference methods, in particular ADI-type methods [17], have also been successfully applied to solve the classical Heston PDE. Non–uniform meshes in both spatial directions, S and ν, are then used. In particular, a mesh with clustered grid points at $(S,\nu)=(K,0)$ is built. Here, we will work with a finite element method to solve the PDE, and employ a uniform mesh, for simplicity.

Numerical methods are proposed to solve the nonlinear problem (15), the solution of which will be the discrete adjustment value, including credit value adjustment, debit value adjustment and funding value adjustment. For the linear problem (16) the same numerical techniques will be applied. Note that the fixed-point iterative scheme is not needed in that case.

A change of time direction, using $\tau = T-t$, is introduced, so that the following initial value problem is solved forward in "time" τ:.

$$\begin{cases} \frac{\partial u}{\partial \tau} - \frac{1}{2}\nu_t S_t^2 \frac{\partial^2 u}{\partial S^2} - \frac{1}{2}\sigma^2 \nu_t \frac{\partial^2 u}{\partial \nu^2} - \rho\sigma\nu_t S_t \frac{\partial^2 u}{\partial S \partial \nu} - r_R S \frac{\partial u}{\partial S} - \kappa(\theta-\nu)\frac{\partial u}{\partial \nu} + ru, \\ \qquad\qquad = -s_F(v+u)^+ - \lambda_X(1-R_X)(v+u)^- - \lambda_Y(1-R_Y)(v+u)^+ \\ u(0,S,\nu) = 0\,. \end{cases} \tag{22}$$

To solve the problem by finite elements, the PDE is written in terms of a divergence operator. With matrix A and vector b as follows,

$$A = \frac{1}{2}\begin{pmatrix} \nu S^2 & \rho\nu S\sigma \\ \rho\nu S\sigma & \nu\sigma^2 \end{pmatrix} \quad b = \begin{pmatrix} \left(\nu + \frac{\rho\sigma}{2} - r_R\right)S \\ \frac{\sigma^2}{2} + \frac{\rho\sigma}{2}\nu - \kappa(\theta-\nu) \end{pmatrix}, \tag{23}$$

the PDE in (22) is written as,

$$\frac{\partial u}{\partial \tau} - \operatorname{div}(A\nabla u) + b\cdot\nabla u + ru = -f(v+u), \tag{24}$$

where function f is defined as:

$$f(v) = s_F v^+ + \lambda_X(1-R_X)v^- + \lambda_Y(1-R_Y)v^+\,. \tag{25}$$

3.1. Method of characteristics

For the time discretization, we use a characteristics method (also known as semi-Lagrangian scheme), which is based on the material derivative of u, i.e

$$\frac{Du}{D\tau} := \frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial S}\frac{\partial S}{\partial \tau} + \frac{\partial u}{\partial \nu}\frac{\partial \nu}{\partial \tau}, \tag{26}$$

for a given function $S=S(\tau)$ and $\nu=\nu(\tau)$. In our particular model, the material derivative term corresponds with:

$$\frac{Du}{D\tau} = \frac{\partial u}{\partial \tau} + \left(\nu + \frac{\rho\sigma}{2} - r_R\right)S\frac{\partial u}{\partial S} + \left(\frac{\sigma^2}{2} + \frac{\rho\sigma}{2}\nu - \kappa(\theta-\nu)\right)\frac{\partial u}{\partial \nu}. \tag{27}$$

Equation (28) then reads:

$$\frac{Du}{D\tau} - \operatorname{div}(A\nabla u) + ru = -f(v+u)\,. \tag{28}$$

Including the advective term in (27), we use the number of time points $N_T>0$, a constant time step $\Delta\tau = T/N_T > 0$, and the time instants $\tau^n = n\Delta\tau$ $(n=0,1,\ldots,N_T)$. Approximating the material derivative in (28) by the method of characteristics, the following semi-discrete problem is posed:

$$\begin{cases} \frac{u^{n+1}-u^n\circ\chi^n}{\Delta\tau} - \operatorname{div}(A\nabla u^{n+1}) + ru^{n+1} = -f(v^{n+1}+u^{n+1})\,, \\ u^0(S,\nu) = 0\,, \end{cases} \tag{29}$$

where $\chi^n = \chi(\tau^n) = \chi((S,\nu),\tau^{n+1};\tau^n)$ and $u^n(\cdot) \approx u(\tau^n,\cdot)$. Moreover, the characteristic curve, $\chi(\tau) = \chi((S,\nu),\tau^{n+1};\tau)$, associated to the vector b passing through the point (S,ν) at time τ^{n+1}, is the solution of the following ODE problems:

$$\begin{cases} \frac{d\chi_1}{d\tau} = \left(\nu + \frac{\rho\sigma}{2} - r_R\right)\chi_1, \\ \chi_1(\tau^{n+1}) = S, \end{cases} \qquad \begin{cases} \frac{d\chi_2}{d\tau} = \frac{\sigma^2}{2} + \frac{\rho\sigma}{2}\chi_2 - \kappa(\theta-\chi_2), \\ \chi_2(\tau^{n+1}) = \nu\,. \end{cases} \tag{30}$$

The components of this characteristic curve, can be computed as:

$$\chi_1(\tau) = S\exp\left(-\left(\nu+\frac{\rho\sigma}{2}-r_R\right)(\tau^{n+1}-\tau)\right),$$
$$\chi_2(\tau) = -\frac{1}{\frac{\rho\sigma}{2}+\kappa}\left(\frac{\sigma^2}{2}-\kappa\theta\right)+\left(\nu+\frac{1}{\frac{\rho\sigma}{2}+\kappa}\left(\frac{\sigma^2}{2}-\kappa\theta\right)\right)\exp\left(-\left(\frac{\rho\sigma}{2}+\kappa\right)(\tau^{n+1}-\tau)\right).$$

Finally, to evaluate $u^n\circ\chi^n$ in (29) at the quadrature nodes used to compute the integrals in the finite element method, we apply a piecewise bilinear interpolation method.

3.2. Fixed-point algorithm

To deal with the nonlinearity of the problem (29), a fixed-point algorithm is proposed at each time step n. Then, at each iteration k of the fixed-point scheme, the function $u^{n+1,k+1}$ is obtained. As a result, the global numerical iterative scheme, represented by Algorithm 1 below, is implemented.

Algorithm 1

Let $N>1$, $n=0$, $\varepsilon>0$ and u^0 given
While $n\le N$:

(1) Let $u^{n+1,0}=u^n$, $k=0$, $e=\varepsilon+1$
(2) While $e\ge\varepsilon$:
 (2.a) Search $u^{n+1,k+1}$ as the solution of:

$$(1+r\Delta\tau)u^{n+1,k+1}-\Delta\tau\,\mathrm{div}(A\nabla u^{n+1,k}) = u^n\circ\chi^n-\Delta\tau\, f(v^n+u^{n+1,k}) \tag{31}$$

 (2.b) $e=\dfrac{\|u^{n+1,k+1}-u^{n+1,k}\|}{\|u^{n+1,k+1}\|}$

3.3. Boundary conditions

As is common for vanilla options, the definition of the computational domain is based on a maximum value for the underlying price coordinate, which is set to four times the strike prices, $S_{\max}=4K$. Similarly, we consider the interval $[0,\nu_{\max}]$ for the admissible variance values, we set $\nu_{\max}=1$, however, higher values can also be employed. The following boundary conditions are proposed. First of all, we introduce the notation $x_0=\tau$, $x_1=S$, $x_2=\nu$, and the domain $\Omega^*=(0,x_0^\infty)\times(0,x_1^\infty)\times(0,x_2^\infty)$, where $x_0^\infty=T$, $x_1^\infty=S_{\max}$ and $x_2^\infty=\nu_{\max}$. The boundary of Ω^* is:

$$\partial\Omega^*=\bigcup_{i=0}^{2}(\Gamma_i^{*,-}\cup\Gamma_i^{*,+}),$$

where we use the notation:

$$\Gamma_i^{*,-}=\{(x_0,x_1,x_2)\in\partial\Omega^*(x_i=0)\},$$

$$\Gamma_i^{*,+}=\{(x_0,x_1,x_2)\in\partial\Omega^*(x_i=x_i^\infty)\}.$$

The PDE in (22) can then be written in the form:

$$\sum_{i,j=0}^{2} b_{i,j}\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{j=0}^{2}p_j\frac{\partial u}{\partial x_j}+c_0u=g_0,$$

where the coefficients are defined as follows:

$$B(x_0,x_1,x_2)=(b_{ij})=\begin{pmatrix}0 & 0 & 0\\ 0 & \frac{1}{2}x_2x_1^2 & \frac{\rho\sigma x_1x_2}{2}\\ 0 & \frac{\rho\sigma x_1x_2}{2} & \frac{1}{2}\sigma_2^2x_2\end{pmatrix},\qquad c_0=r,$$

$$p(x_0,x_1,x_2)=(p_j)=\begin{pmatrix}-1\\ r_Rx_1\\ \kappa(\theta-x_2)\end{pmatrix},$$

$$g(x_0,x_1,x_2)=s_F(v+u)^+ +\lambda_X(1-R_X)(v+u)^- +\lambda_Y(1-R_Y)(v+u)^+.$$

Next, we introduce the following subset of Γ^*, in terms of the normal vector to the boundary pointing inwards the domain Ω^*, $\vec{m} = (m_0, m_1, m_2)$

$$\Sigma^0 = \left\{ x \in \partial\Omega^* / \sum_{i,j=0}^{2} b_{ij} m_i m_j = 0 \right\}, \qquad \Sigma^1 = \partial\Omega^0 - \Sigma^0,$$

$$\Sigma^2 = \left\{ x \in \Sigma^0 / \sum_{i=0}^{2} \left(p_i - \sum_{j=0}^{2} \frac{\partial b_{ij}}{\partial x_j} \right) m_i \le 0 \right\}.$$

For our particular problem, we have: $\Sigma^0 = \Gamma_0^{*,-} \cup \Gamma_0^{*,+} \cup \Gamma_1^{*,-} \cup \Gamma_2^{*,-}$, $\Sigma^1 = \Gamma_1^{*,+} \cup \Gamma_2^{*,+}$ and $\Sigma^2 = \Gamma_0^{*,+}$ or $\Sigma^2 = \Gamma_0^{*,+} \cup \Gamma_2^{*,-}$, if the Feller condition is satisfied or not, respectively. Thus, following [23], the boundary conditions must be imposed over the subset $\Sigma^1 \cup \Sigma^2$, which matches with the set $\Gamma_0^{*,-} \cup \Gamma_1^{*,+} \cup \Gamma_2^{*,+}$ or $\Gamma_0^{*,-} \cup \Gamma_1^{*,+} \cup \Gamma_2^{*,+} \cup \Gamma_2^{*,-}$, according to the Feller condition. Note that at the boundary $S = 0$ is not necessary to impose any condition. Similar procedure is made for the risk–free value problem, obtaining the same boundaries to impose a condition.

Like the boundary conditions in [2] and the solution obtained for the Black–Scholes model, the following boundary conditions are imposed (we also explain the boundary conditions for the classical Heston PDE and not only for the XVA problem).

3.3.1. Boundary conditions, the risk–free derivative value

To obtain a numerical solution for the classical Heston PDE, different boundary conditions have been proposed in [17],[26]. We use a formulation, based on the properties of the equation at the boundaries, which are linked with the Black–Scholes equation obtained in [2].

At the boundary Γ_2^+, where ν takes its maximum value, different Dirichlet boundary conditions are proposed in the literature, see, for example, [17]. Following Forsyth [26], the following condition is imposed in this work:

$$\lim_{\nu \to \nu_{\max}} \frac{\partial v}{\partial \nu}(\tau, S, \nu) = 0\,.$$

Assuming $\lim_{\nu \to \nu_{\max}} \nu \frac{\partial v}{\partial \nu}(\tau, S, \nu) = 0$, for $\nu_{\max}$ large enough, the equation on Γ_2^+ turns into:

$$\frac{\partial v}{\partial \tau} - \frac{1}{2}\nu S^2 \frac{\partial^2 v}{\partial S^2} - r_R S \frac{\partial v}{\partial S} + r v = 0,$$

which corresponds to prescribing the Black–Scholes equation at this boundary. So, at the boundary, where the volatility takes the value $\nu_{\max}$, we impose the classical Black–Scholes formula, given by the expressions:

$$\begin{aligned} v(\tau, S, \nu_{\max}) &= S\exp(-D_0\tau)N(d_1) - K\exp(-r\tau)N(d_2), && \text{call option,} \\ v(\tau, S, \nu_{\max}) &= K\exp(-r\tau)N(-d_2) - S\exp(-D_0\tau)N(-d_1), && \text{put option,} \end{aligned}$$

with

$$D_0 = r - r_R,$$

$$d_1 = \frac{\log(S/K) + (r - D_0 + 0.5\nu_{\max})\tau}{\sqrt{\nu_{\max}}\sqrt{\tau}}, \quad d_2 = \frac{\log(S/K) + (r - D_0 - 0.5\nu_{\max})\tau}{\sqrt{\nu_{\max}}\sqrt{\tau}}\,.$$

At the boundary Γ_1^+, where the asset price S takes its maximum value, different Dirichlet and Neumann conditions have been proposed in the literature, see [26],[17]. Following [9] and [7], dividing the equation by S^2, we employ the condition

$$\lim_{S \to S_{\max}} \frac{\partial^2 v}{\partial S^2} = 0\,,$$

where $S_{\max}$ is large enough.

As in [3] we consider the solution to be of the form,

$$v(\tau, S, \nu) = H_1(\tau)S + H_2(\tau)\nu + H_3(\tau)\nu^2 + H_4(\tau)\nu S + H_5(\tau)\nu^2 S + H_6(\tau), \tag{32}$$

where the coefficients $H_1(\tau)$, $H_2(\tau)$, $H_3(\tau)$, $H_4(\tau)$, $H_5(\tau)$, and $H_6(\tau)$ are all independent of S and ν. These coefficients can be obtained by substituting the expression (32) in the discretized equations with $S \to \infty$. Choosing, at each fixed-point iteration, the particular values $H_1^{n+1,k+1}(\tau) = H_2^{n+1,k+1}(\tau) = H_3^{n+1,k+1}(\tau) = H_4^{n+1,k+1}(\tau) = H_5^{n+1,k+1}(\tau) = 0$, the following nonhomogenous Dirichlet condition is deduced

$$v^{n+1,k+1}(S_\infty, \nu) = H_6^{n+1,k+1}(\tau) = \frac{v^n \circ \chi^n}{1 + r\Delta\tau}.$$

However, to derive a condition which matches the condition in the corner of the domain, we consider the analytic solution for the Black–Scholes problem with varying volatility ν on the boundary. Moreover, such Dirichlet condition also satisfies, $\lim_{S \to S_{\max}} \frac{\partial^2 v}{\partial S^2} = 0$.

In the case the Feller condition is violated, we impose the classical condition from the literature at boundary Γ_2^-. By substituting $\nu = 0$ in the equation, the following equation is obtained:

$$\frac{\partial v}{\partial \tau} - r_R S\frac{\partial v}{\partial S} - \kappa\theta\frac{\partial v}{\partial \nu} + rv = 0. \tag{33}$$

We solve such equation by finite differences and then the numerical solution is imposed as a boundary condition.

3.3.2. Boundary conditions for the XVA problem

To obtain the XVA values, we will reason similarly as for the classical Heston option pricing PDE.

At the boundary Γ_2^+, we also assume $\lim\limits_{\nu\to\nu_{\max}} \frac{\partial u}{\partial \nu}(\tau, S, \nu) = 0$, and the resulting boundary condition reads,

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\nu S^2\frac{\partial^2 u}{\partial S^2} - r_R S\frac{\partial u}{\partial S} + ru = -f(v+u),$$

where function f has been defined in (25). Note that this equation corresponds to the XVA equation obtained for the Black–Scholes model, see also [2]. So, we impose a non-homogeneous Dirichlet boundary condition given by the XVA solution of the Black–Scholes problem.

At the boundary Γ_1^+, we follow essentially the same procedure as for the risk–free value v. So, after division by S^2, we obtain $\lim\limits_{S\to S_{\max}} \frac{\partial^2 u}{\partial S^2} = 0.$

To determine a condition which works well at the intersection point with Γ_2^+, we impose the solution of the one-dimensional XVA problem, modeled by the Black–Scholes equation in [2], which also satisfies $\lim\limits_{S\to S_{\max}} \frac{\partial^2 u}{\partial S^2} = 0.$

In the case the Feller condition is not satisfied, based on a similar reasoning as for the risk–free value at boundary Γ_2^-, by replacing the parameter ν by the value 0, we find the following condition,

$$\frac{\partial u}{\partial \tau} - r_R S\frac{\partial u}{\partial S} - \kappa\theta\frac{\partial u}{\partial \nu} + ru = -f(u+v), \tag{34}$$

The value of u on this boundary can be computed by solving the PDE in $[0, S_{\max}]$, applying numerical methods. Then, the numerical solution is imposed as boundary condition.

3.4. Finite element method

For the spatial discretization the finite element method is employed. A mesh with triangular cells is placed in Ω, and the associated piecewise linear Lagrange finite element space is considered. At each time step, $n = 0, 1, 2, \dots, N_T - 1$, and for each iteration of the fixed-point scheme, $k = 0, 1, \dots,$ after applying Green's formula, the following variational formulation is obtained:

Find $u^{n+1,k+1} \in \{\varphi \in H^1(\Omega)/\varphi = u_{BS}^{n+1,k+1} \text{on } \Gamma_2^+, \varphi = u_{BS}^{n+1,k+1} \text{on } \Gamma_1^+\}$, such that

$$\begin{aligned}
&\int_\Omega (1 + r\Delta\tau)u^{n+1,k+1}\varphi dSd\nu + \Delta\tau\int_\Omega A\nabla u^{n+1,k+1}\nabla\varphi dSd\nu \\
&\quad = \int_\Omega (u^n \circ \chi^n)\varphi dSd\nu - \Delta\tau\int_\Omega f(v^{n+1} + u^{n+1,k})\varphi dSd\nu, \quad \forall\varphi \in H_*^1(\Omega),
\end{aligned} \tag{35}$$

where $H_*^1(\Omega) = \{\varphi \in H^1(\Omega)/\varphi = 0 \text{ on } \Gamma_2^+ \cup \Gamma_1^+\}$ and u_{BS} denotes the solution of the Black–Scholes equation, as obtained in [2].

In order to describe the finite element discretization, we fix $N_S > 0$ and $N_\nu > 0$, to build a uniform mesh $(S_i = i\Delta S, \nu_j = j\Delta\nu)$ for $i = 0, \dots, N_S + 1$ and $j = 0, \dots, N_\nu + 1$, in the computational domain Ω. Associated with this uniform mesh, the following finite element spaces are introduced

$$\begin{aligned}
W_h &= \{\varphi_h \in \mathcal{C}(\Omega) \,/\, \varphi_h|_{T_j} \in \mathcal{P}_1,\ \forall T_j \in \mathcal{T}\}, \\
W_{h,*} &= \{\varphi_h \in W_h \,/\, \varphi_h = 0 \text{ on } \Gamma_2^+ \cup \Gamma_1^+\},
\end{aligned}$$

to determine $u^{n+1,k+1} \in W_h$, satisfying the boundary conditions and such that:

$$\begin{aligned}
&\int_\Omega (1 + r\Delta\tau)u_h^{n+1,k+1}\varphi_h dSd\nu + \Delta\tau\int_\Omega A\nabla u_h^{n+1,k+1}\nabla\varphi_h dSd\nu \\
&\quad = \int_\Omega (u_h^n \circ \chi^n)\varphi dSd\nu - \Delta\tau\int_\Omega f(v^{n+1} + u_h^{n+1,k})\varphi_h dSd\nu \quad \forall\varphi_h \in W_{h,*}(\Omega)\,.
\end{aligned} \tag{36}$$

The elements of the matrix and the right-hand side vector are found by means of a quadrature formula, based on the midpoints of the edges of the triangles. The resulting discrete system is solved using a partial pivoting LU factorization method. For the American case, the discrete complementarity problem is solved by the augmented Lagrangian active set algorithm, as introduced in [22].

The risk-free option value is computed solving the classical Heston PDE, using the same numerical methods. For this particular case, due to the linearity of the problem, the fixed-point scheme is not required.

Table 1
Parameter values of Tests A and B.

	Test A	Test B
Spot (S)	10	10
Strike (K)	10	10
Interest (r)	0.04	0.04
Initial vol. ($\sqrt{\nu_0}$)	0.2, 0.4, 0.6, 0.8	0.2, 0.4, 0.6, 0.8
Maturity time (T)	0.25	0.25
Mean reversion (κ)	1.4	0.4
Mean var (θ)	0.33	0.33
Vol of var (σ)	0.7	0.7
Correlation (ρ)	0.1	0.1

3.5. Augmented Lagrangian active set for American options

The numerical methods have been explained so far for European options, where early-exercise is not allowed. The American option value has been modeled by the linear and nonlinear complementarity problems introduced in (21) and (20). For the explanation of the numerical methods for the American option problems, let us focus on the nonlinear problem for $\widehat{\nu}^{n+1,k+1}$. After applying a time discretization by the method of characteristics and the spatial discretization by the finite element method, the discrete problem at each fixed-point iteration can be written as:

$$\begin{cases} A_h\widehat{\nu}^{n+1,k+1} \geq b_h^{n+1,k+1}, \\ \widehat{\nu}^{n+1,k+1} \geq \Psi_h, \\ (A_h\widehat{\nu}^{n+1,k+1} - b_h^{n+1,k+1})(\widehat{\nu}^{n+1,k+1} - \Psi_h), \end{cases} \tag{37}$$

where Ψ_h denotes the discretized exercise value, $g(S)$. The discrete problem (37) has been solved by the Augmented Lagrangian active set (ALAS) algorithm proposed by Kärkkäinen et al. [22].

4. Numerical results

In this section, we discuss several numerical results explaining the impact of incorporating total value adjustment in the pricing of financial options. We include counterparty risk and the XVA for some examples. To study the impact of the stochastic volatility to the exposure, the expected exposure, $EE(t)$, associated with the XVA is computed. In addition, we compute the $PFE(t)$ corresponding with the total value adjustment. Both quantities, under the Heston and Black–Scholes models, are compared.

For this purpose and in order to see if the quantities are affected by the Feller condition, we define two tests: Test A, where the Feller condition is satisfied, and Test B, where Feller condition is violated. The parameter data for both tests are found in Table 1. We compute the risk measures for European call and put options. In addition, we also study the American option XVA value and the associated exposures for a put option.

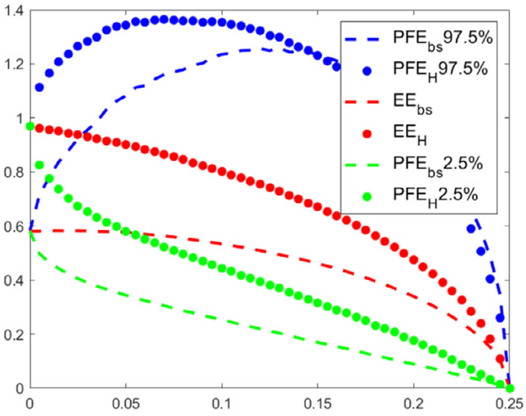
In Figs. 1, 2 and 3, the EE, 97.5% PFE and 2.5% PFE XVA profiles related to different European and American options have been computed.

In Fig. 1 the different profiles with respect to changes in the initial volatility are plotted for the nonlinear problem. Clearly, the EE and PFE profiles increase as the volatility increases. Moreover, it can be seen that the Black–Scholes model exhibits an observable sensitivity regarding changes in the volatility. A similar behaviour is observed in the case of the Test A parameter set (which is not shown here). In addition, comparing Fig. 1 for $\sqrt{\nu_0} = 0.6$ with Fig. 2 (b), we can observe that the exposures are not affected by the Feller condition.

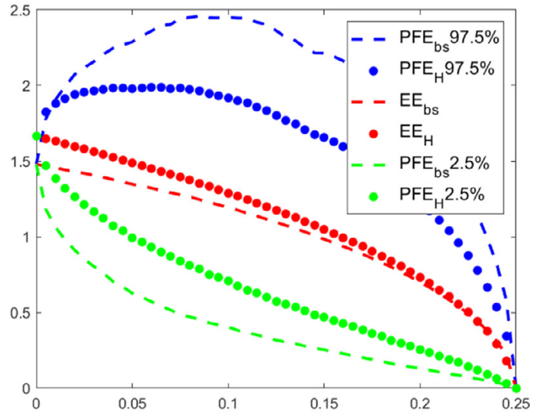
In Fig. 2, the volatility value is fixed, and we show how the exposure profiles are affected by the decision of the mark–to–market value. Comparing the obtained values for both models, a clearly different behaviour is seen when comparing the mark–to–market being equal to the risk–free value (the linear problem) with the mark–to–market being the risky derivative value (nonlinear problem). While in the linear case, the EE and the PFE profiles are higher for the Heston model, for the nonlinear problem, this is observed only for low volatility values. Such pattern can be explained by the sensitivity in Fig. 1. In the nonlinear problem, the exposure formulas are given in terms of the risk–free value and of the XVA, which makes that this sensitivity is doubled. This is different for the linear problem where the exposure profiles under the Black–Scholes model are lower than under the Heston model which is not affected by the mark–to–market value, but the opposite situation is observed in the nonlinear case.

In Fig. 3, we compute the EE and PFE profiles for an American put option with the parameters in Test A. We show the exposures for different values of the volatility, like in Fig. 1 for the European options. The Heston exposure profiles remain almost constant in the presence of changes in the volatility, in particular, there are only minor differences between the EE and PFE profiles.

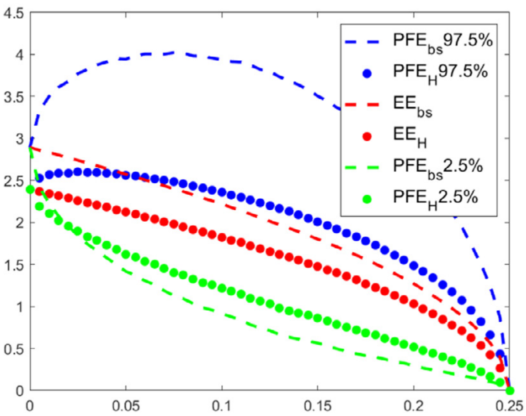
Concluding, based on the previous results, it is clear that for the European case, where the options can only be exercised at maturity time, the exposures are higher in the middle of the time interval for a low volatility, while for a higher volatility the exposures decrease with the life of the option, having their maximum value at the beginning. However, in the American



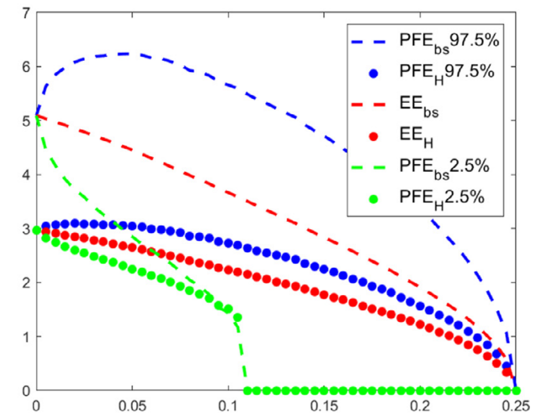
(a) $\sqrt{\nu_0} = 0.2$



(b) $\sqrt{\nu_0} = 0.4$

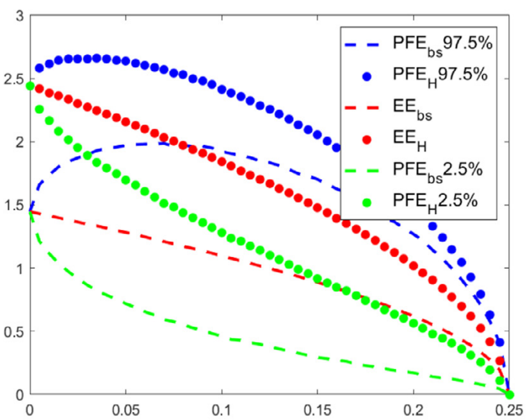


(c) $\sqrt{\nu_0} = 0.6$

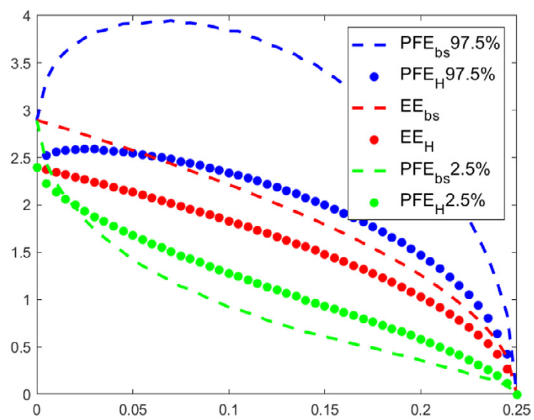


(d) $\sqrt{\nu_0} = 0.8$

Fig. 1. *EE* and *PFE* profiles for different volatility values $\sqrt{\nu_0}$. Test B (Feller condition is not satisfied). Nonlinear problem, European call option.



(a) $M = v$



(b) $M = \widehat{v}$

Fig. 2. Linear and nonlinear problem: Test A, European call option, Feller condition satisfied. Volatility value $\sqrt{\nu_0} = 0.6$.

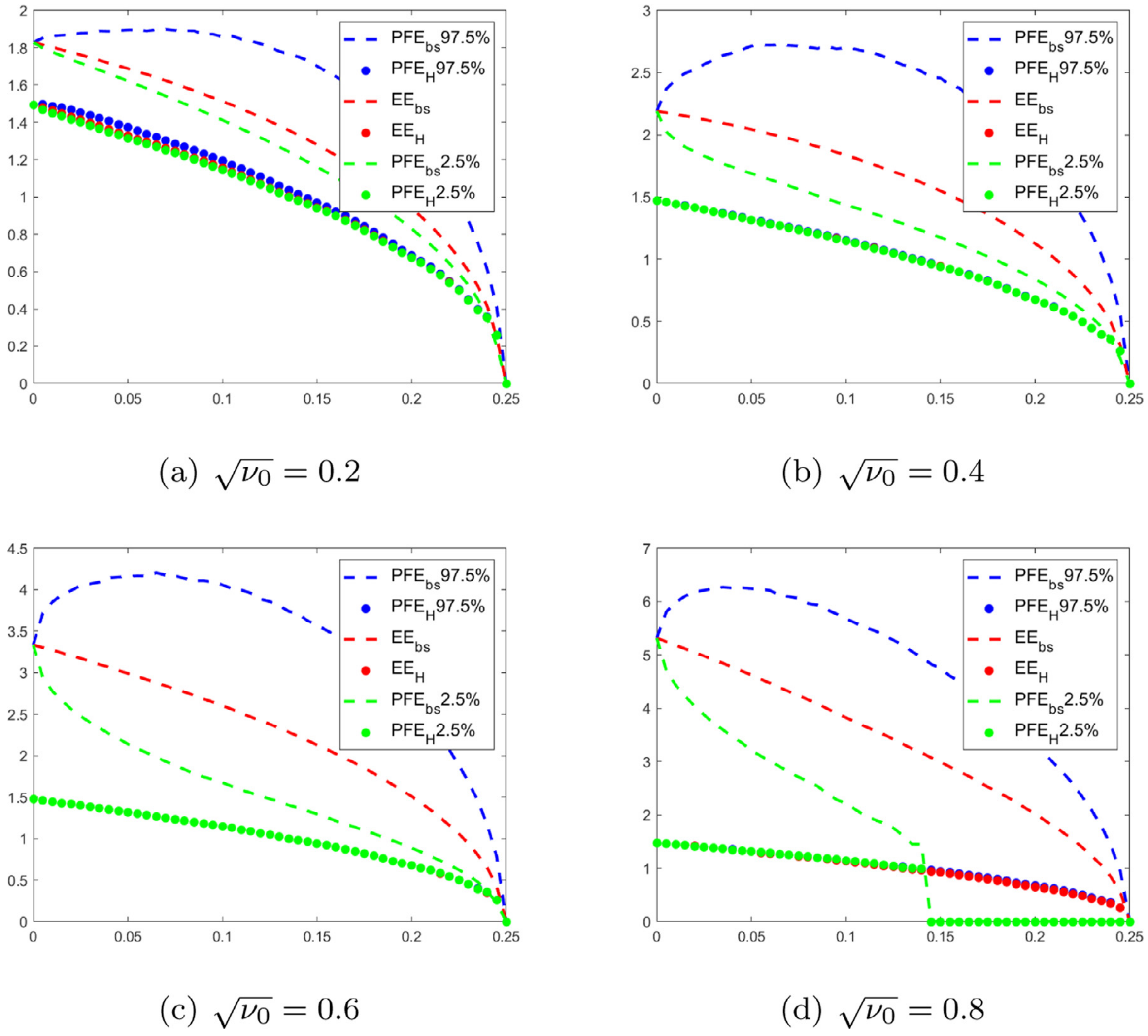


Fig. 3. Nonlinear problem, American put option, Test A (Feller condition satisfied).

case, where the option can be exercised at any moment until maturity, the exposures take their maximum value at the beginning of the option's lifetime and then decrease until the maturity time.

The results obtained show us the relevance of incorporating stochastic volatility when pricing derivatives with XVA. For the constant volatility Black–Scholes model, the exposures are sensitives with the volatility values. Moreover, the Feller condition does not affect the exposure profiles in the parameter settings studied here.

In addition, so that the exposure will increase.

In the case of an American put option, presented in Fig. 5, the XVA is equal to zero in the exercise region, where the derivative should be exercised (and terminates). In this case, as we can deduce from Fig. 3, the XVA is not much affected by variation in the volatility. However, similar to European options, the main variation of the XVA comes from the asset price variation, obtaining a similar behaviour as for Black–Scholes model [2].

Next, we present the empirical convergence of the numerical methods in Table 2. As mentioned, we use a uniform mesh and the empirical convergence ratio is defined as follows:

$$CR = \frac{\|u_{h/2} - u_{h/4}\|_\infty}{\|u_h - u_{h/2}\|_\infty}.$$

From this quantity we compute the experimental order of convergence, i.e. $p = \log_2(CR)$. Table 2 shows that we obtain first-order convergence, which is the expected order of convergence (using piecewise linear finite elements and a first-order time discretization).

In Fig. 6, the influence of the maturity time to the total value adjustment has been represented. We have computed the XVA for a European call option, with the same parameter values as in the example represented in Fig. 4, where the maturity

Table 2
Empirical order of convergence for an European call option with the following parameter data:$\theta = 0.3$, $\kappa = 0.8$, $\sigma = 0.7$, $r = 0.04$, $\delta = 0.0$, $k = 10$, $\rho = -0.3$ and $T = 1$.

$\Delta S = \Delta \nu$	$\Delta \tau$	CR	p
2^{-2}	1/12		
2^{-3}	1/24		
2^{-4}	1/48	2.0578	1.0411
2^{-5}	1/96	2.5246	1.3361
2^{-6}	1/192	2.0029	1.0021
2^{-7}	1/384	1.9593	0.9703

Table 3
XVA value for a put option with the following parameters: $\theta = 0.33$, $\kappa = 1$, $\sigma = 0.5$, $r = 0.04$, $\delta = 0.0$, $k = 15$, $\rho = -0.3$, $\lambda_B = \lambda_C = 0.04$, $R_B = R_C = 0.3$ and $T = 0.25$.

	American options		European options	
(S, ν)	$M = \nu$	$M = \hat{\nu}$	$M = \nu$	$M = \hat{\nu}$
(9,0.25)	0.00000000	0.00000000	−0.08120621	−0.08145264
(9,0.5)	0.00000000	0.00000000	−0.08246912	−0.08254409
(9,0.75)	0.00000000	0.00000000	−0.08434795	−0.08285474
(15,0.25)	−0.00022315	−0.00000626	−0.01982075	−0.01988082
(15,0.5)	−0.00022318	−0.00000626	−0.02736400	−0.02734565
(15,0.75)	−0.00022321	−0.00000626	−0.03329995	−0.03247313
(18,0.25)	−0.00016440	−0.00000461	−0.00794203	−0.00796562
(18,0.5)	−0.00016445	−0.00000462	−0.01450847	−0.01443643
(18,0.75)	−0.00016450	−0.00000462	−0.02016455	−0.01914779

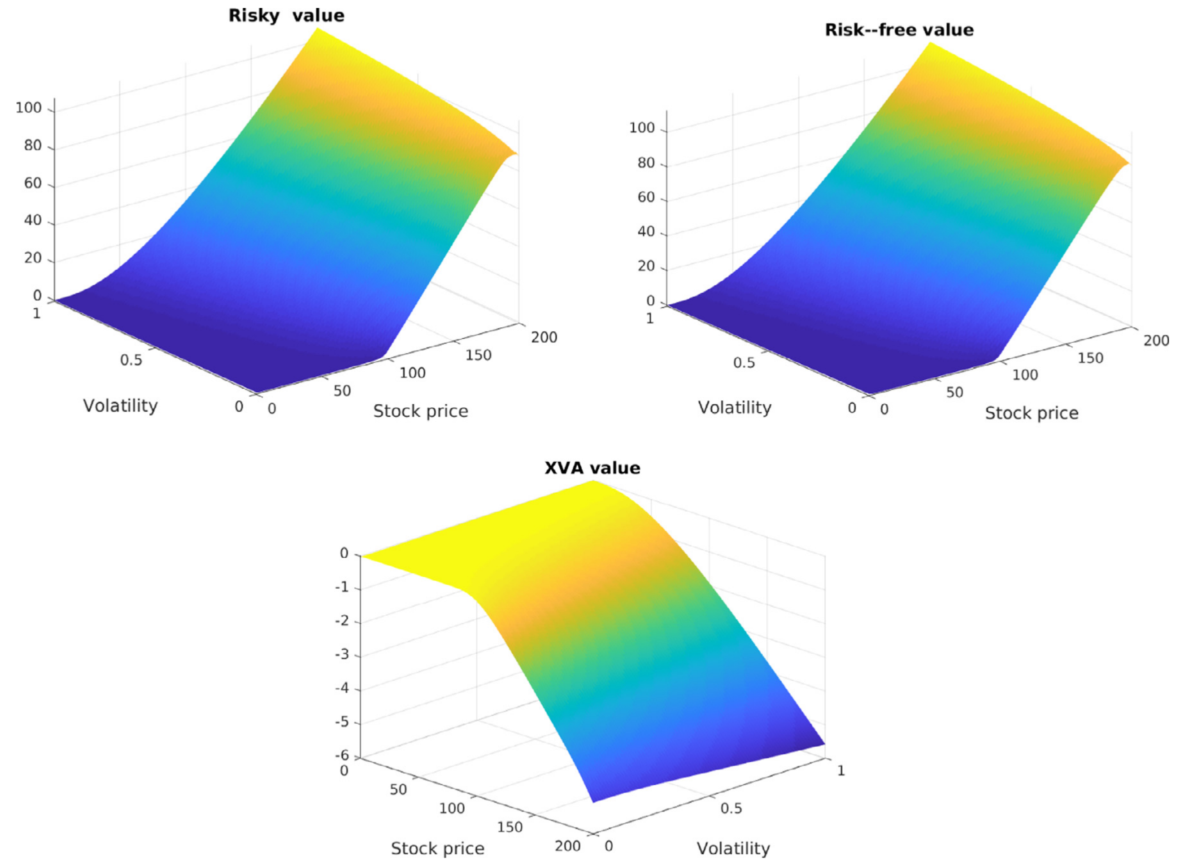


Fig. 4. Option values decomposed to visualize the XVA component, with parameter values, $\kappa = 0.1$, $\theta = 0.12$, $\sigma = 0.8$, $\rho = -0.3$, $T = 1$ and $K = 100$.

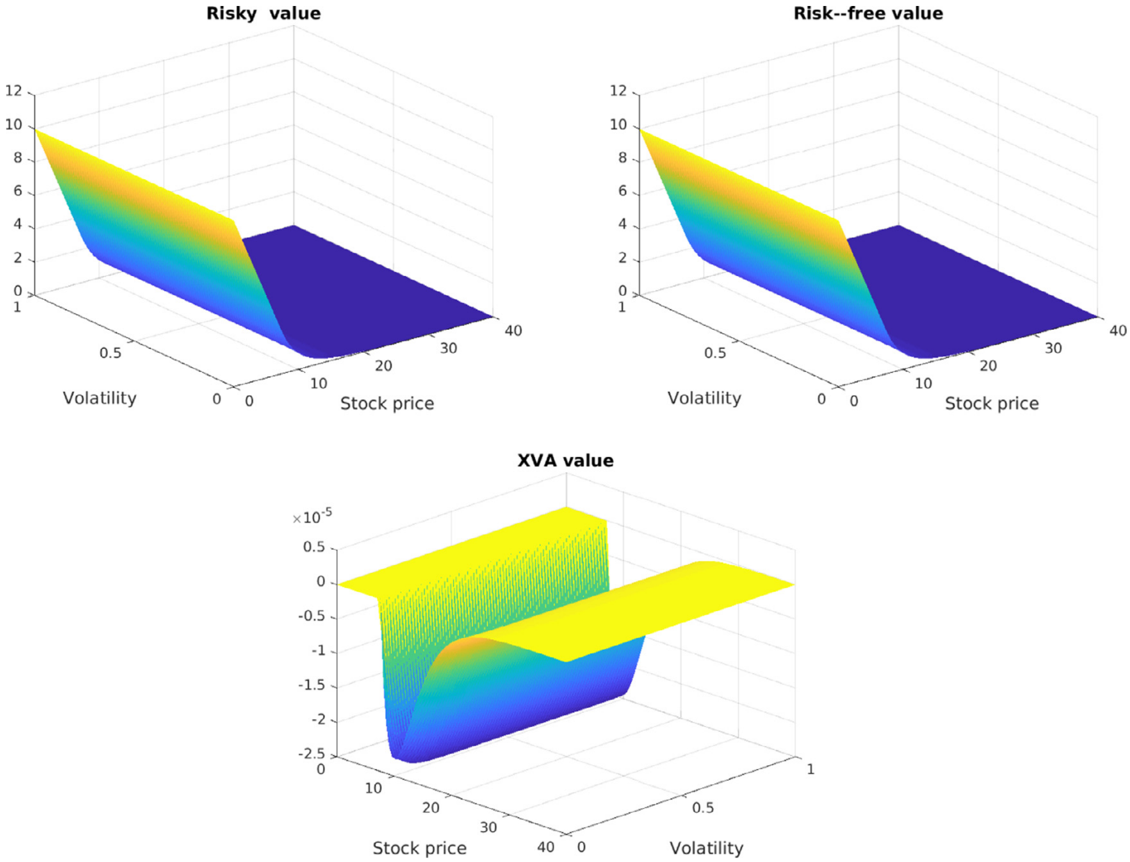
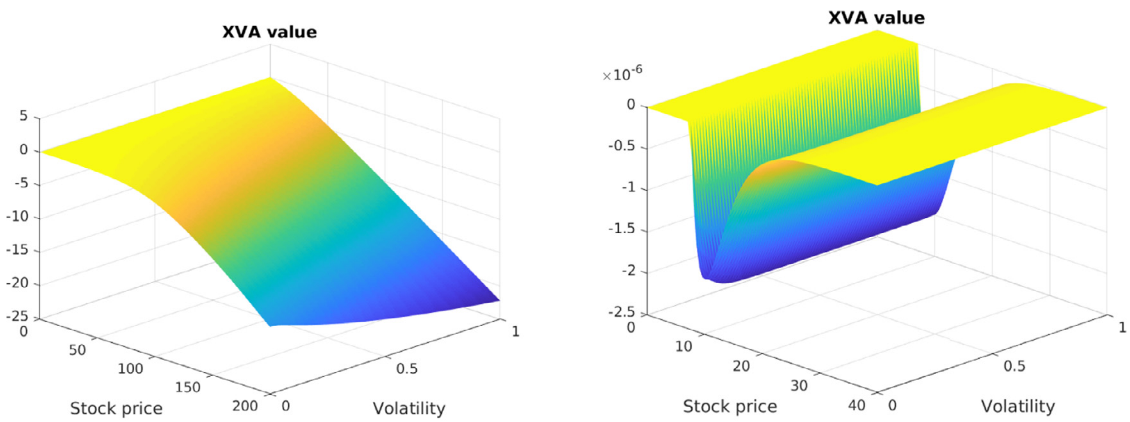


Fig. 5. American put option values decomposed; Parameter values from Test A.



(a) European option (b) American option

Fig. 6. XVA for a long maturity time.

time is set to 4 years. Moreover, the equivalent American put option is found in Fig. 5, with a maturity time of 3 years. For the European option, as in [2], the XVA increases when the maturity time is longer. However, in the case of an American option, the maturity time does not seem to affect the total value adjustment. In addition, comparing with the European options, the American option buyer is less exposed to the risk, and the XVA is lower in absolute terms.

Finally, we compare the European and American option values and the impact of the choice of the mark–to–market value. Tables 3 and 4 present some XVA values for some specific price and volatility values. Moreover, in Table 5, the value for constant volatility is shown. The XVA is higher in absolute terms for the European option, where no early-exercise is

Table 4
XVA value for a call option with the following parameters: $\theta = 0.33$, $\kappa = 1$, $\sigma = 0.5$, $r = 0.04$, $\delta = 0.0$, $k = 15$, $\rho = -0.3$, $\lambda_B = \lambda_C = 0.04$, $R_B = R_C = 0.3$ and $T = 0.25$.

	American options		European options	
(S, ν)	$M = v$	$M = \hat{v}$	$M = v$	$M = \hat{v}$
(9,0.25)	−0.00009147	−0.00000257	−0.00025381	−0.00025365
(9,0.5)	−0.00009145	−0.00000257	−0.00166034	−0.00153987
(9,0.75)	−0.00009142	−0.00000257	−0.00369311	−0.00249308
(15,0.25)	−0.00022321	−0.00000626	−0.02207651	−0.02214413
(15,0.5)	−0.00022318	−0.00000626	−0.02985921	−0.02985922
(15,0.75)	−0.00022316	−0.00000626	−0.03605155	−0.03247313
(18,0.25)	−0.00000000	−0.00000000	−0.05180187	−0.05196003
(18,0.5)	−0.00000000	−0.00000000	−0.05865562	−0.05870790
(18,0.75)	−0.00000000	−0.00000000	−0.06461869	−0.06359331

Table 5
XVA value for call and put options with constant volatility (Black–Scholes model) and $M = \hat{v}$ and the following parameters: $r = 0.04$, $\delta = 0.0$, $k = 15$, $\lambda_B = \lambda_C = 0.04$, $R_B = R_C = 0.3$ and $T = 0.25$.

	American options		European options	
(S, σ^2)	Call	Put	Call	Put
(9,0.25)	−0.00085907	−0.00460660	−0.00061468	−0.08177000
(9,0.5)	−0.00323710	−0.02928685	−0.00255262	−0.08354573
(9,0.75)	−0.00572234	−0.03942949	−0.00494266	−0.08576836
(15,0.25)	−0.02393488	−0.016110151	−0.02299540	−0.02065941
(15,0.5)	−0.03204289	−0.027638443	−0.03143356	−0.02882724
(15,0.75)	−0.03776497	−0.02254091	−0.03799388	−0.0351089521
(18,0.25)	−0.051917344	−0.00729616	−0.05198151	−0.00789987
(18,0.5)	−0.05817008	−0.01342870	−0.05972470	−0.01531877
(18,0.75)	−0.06316036	−0.01860614	−0.06627720	−0.02147804

allowed. Moreover, the XVA value is not affected by the choice of the mark-to-market value. Finally, comparing puts and calls, we observe that the XVA value varies with the asset price, according to whether the option is in the money or out of the money. Furtheremore, the fixed-point scheme, converges in 3 or 4 iterations for the previous examples.

5. Conclusions

In this paper, we have studied the XVA components that are added to the risk–free option derivative values under the Heston stochastic volatility model. As a result, linear and nonlinear valuation PDE models depending on two stochastic factors, the asset price and the variance, have been derived.

After presenting the mathematical model, we have presented the chosen boundary conditions, based on those in [2]. Moreover, the method of characteristics, combined with a piecewise finite element method, is used to obtain the numerical solution. To deal with the nonlinearity, a fixed-point scheme has been employed.

The relevance of incorporating stochastic volatility is shown by comparing the XVA, *EE* and *PFE* profiles under the Heston and Black–Scholes models, for different parameter sets. The results show the impact of the stochastic volatility, where the variations of the XVA and the exposure profiles are sometimes significantly lower in the presence of changes in the volatility values. The stochastic volatility results in this paper are not much affected by the Feller condition. Finally, the numerical examples present the values of the XVA and the risky derivative values in terms of the asset price and the volatility values. According to these results, the XVA for American options is hardly affected by volatility in this test case.

The study of the XVA presented in this work can be extended for a portfolio with multiples options and depending of several underlying actives or with different payoff functions. In addition, these options can have different maturities, resulting in discontinuities in time, [13]. However, when multi-assets options are included, numerical methods, as multigrid methods to solve the PDE, can be more desirable. Furthermore, the methodology followed in this work, introduced previously in [5] is already applied in the industry.

Acknowledgments

B.Salvador was funded by European Research Consortium for Informatics and Mathematics (ERCIM) fellowship.

References

[1] H. Albrecher, P.A. Mayer, W. Schoutens, J. Tistaert, The little Heston trap, Wilmott Mag. 1 (2007) 83–92.
[2] I. Arregui, B. Salvador, C. Vázquez, PDE models and numerical methods for total value adjustment in European and American options with counterparty risk, Appl. Math. Comput. 308 (2017) 31–53.

[3] I. Arregui, B. Salvador, D. Ševčovič, C. Vázquez, Total value adjustment for European options with two stochastic factors. Mathematical model, analysis and numerical simulation, Comput. Math. Appl. 76 (2018) 725–740.
[4] C. Burgard, M. Kjaer, In the balance, Risk (2011), doi:10.2139/ssrn.1785262. Available at SSRN
[5] C. Burgard, M. Kjaer, PDE representations of derivatives with bilateral counterparty risk and funding costs, J. Credit Risk 7 (3) (2011) 1–19.
[6] C. Burgard, M. Kjaer, Funding costs, funding strategies, Risk (2013) 82–87.
[7] D. Castillo, A.M. Ferreiro, J.A. García-Rodríguez, C. Vázquez, Numerical methods to solve PDE models for pricing business companies in different regimes and implementation in GPUs, Appl. Math. Comput. 219 (2013) 11233–11257.
[8] J.C. Cox, J.E. Ingersoll, S.A. Ross, A theory of the term structure of interest rates, Ecnonometrica 53 (1985) 385–408.
[9] Y. D'Halluin, P.A. Forsyth, G. Labahn, A semi–Lagrangian approach for American Asian options under jump-diffusion, SIAM J. Sci. Comput. 27 (2005) 315–345.
[10] D. Duffie, M. Huang, Swap rates and credit quality, J. Financ. 51 (1996) 921–950.
[11] Q. Feng, C. Oosterlee, Monte Carlo calculation of exposure profiles and greeks for bermudan and barrier options under the Heston Hull-White model, Recent Progress and Modern Challenges in Applied Mathematics, Modeling and Computational Science, 79, 2017.
[12] Y. Feng, M. Wang, Y. Zhang, CVA for Cliquet options under Heston model, North Am. J. Econ. Financ. 48 (2019) 272–282.
[13] C.S.L. de Graaf, Q. Feng, B.D. Kandhai, C.W. Oosterlee, Efficient computation of exposure profiles for counterparty credit risk, Int. J. Theor. Appl. Financ. 17 (04) (2014) 1450024.
[14] C.S.L. de Graaf, D. Kandhai, C. Reisinger, Efficient exposure computation by risk factor decomposition, Int. J. Theor. Appl. Financ. 18 (10) (2018).
[15] A. Green, C. Kenyon, C. Dennis, KVA: Capital Valuation Adjustment, 2014, Available at http://ssrn.com/abstract=2400324.
[16] S.L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, Rev. Financ. Stud. 6 (2) (1993) 327–343.
[17] K.J. In 't Hout, S. Foulon, ADI finite difference schemes for option pricing in the Heston model with correlation, Int. J. Numer. Anal. Model. 7 (2014) 1–27.
[18] J. Hull, A. White, The FVA debate, Risk 25 (8) (2012) 83–85.
[19] J. Hull, A. White, Valuing derivatives: funding value adjustment and fair value, Financ. Anal. J. 70 (3) (2014) 46–56.
[20] S. Ikonenl, J. Toivanen, Efficient Numerical Methods for Pricing American Options under Stochastic Volatility, Wiley InterScience10.1002/num.20239.
[21] R.A. Jarrow, S.M. Turnbull, Pricing options on financial securities subject to credit risk, J. Financ. 50 (1) (1995) 53–85.
[22] T. Kärkkäinen, K. Kunisch, P. Tarvainen, Augmented Lagrangian active set methods for obstacle problems, J. Optim. Theory Appl. 119 (2003) 499–533.
[23] O. Oleinik, E. Radkevic, Second Order Equations with Nonnegative Characteristics Form, AMS, Plenum Press, New York-London, 1973.
[24] F.D. Rouah, The Heston Model and Its Extensions in Matlab and C#, John Willey & Sons, 2013.
[25] E.H. Sorensen, T.F. Bollier, Pricing swap default risk, Financ. Anal. J. 50 (1994) 23–33.
[26] R. Zvan, P.A. Forsyth, K.R. Vetzal, Penalty methods for American options with stochastic volatility, J. Comput. Appl. Math. 91 (1998) 199–218.