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GROUP DIVISIBLE DESIGNS WITH BLOCK-SIZE FOUR

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It is proved that the obvious necessary conditions for the existence of a group divisible design with $k = 4$ are sufficient, except for the cases corresponding to the non-existing transversal designs $T[4,1;2]$ and $T[4,1;6]$.

Introduction

This paper has been written as Section 6.3 of the paper [0] by H. Hanani. All the theorems and lemmas referred to, as well as all the relevant definitions, may be found in [0].

Concerning the notation of groups and blocks of a design, we use the convention that if a block is enclosed in brackets ⟨ ⟩, the points are denoted by exponents of some primitive element, as explained in Section 1.5, while if a block is enclosed in braces { }, the points themselves are written.

We first give a few additions and improvements to earlier sections.

In the same way as Lemma 2.16 it can be proved:

Lemma 2.27. *If* $v \in \mathrm{GD}(K, \lambda, M)$ *and* $mK \subset \mathrm{GD}(K', \lambda', m)$, *then* $mv \in \mathrm{GD}(K', \lambda\lambda', mM)$.

It can be easily seen, that if $\lambda' = \lambda$ in Lemma 2.21, we get:

Lemma 2.28. *If* $v \in \mathrm{GD}(K, \lambda, M')$ *and* $M' \subset \mathrm{GD}(K, \lambda, M)$, *then* $v \in \mathrm{GD}(K, \lambda, M)$.

In the same way as Lemma 2.28 can be proved:

Lemma 2.29. *If* $v \in \mathrm{GD}(K, \lambda, M)$ *and* $M + m \subset \mathrm{GD}(K, \lambda, m)$, *then* $v + m \in \mathrm{GD}(K, \lambda, m)$.

Further we prove:

Lemma 3.24. $4n \in \mathrm{T}(5,1)$ *holds for every integer* $n > 0$.

Proof. For $n \geq 11$ the lemma follows from Theorem 3.10; for $n \in \{1,2,4,5,7,8,9,10\}$ — from Theorem 3.1; for $n = 3$ — from Lemma 3.21 and for $n = 6$ — from Lemma 4.11 and Theorem 3.2.

Lemma 5.10 may be improved as follows:

Lemma 5.10*. *For every integer* $v \geq 4$, $v \in B(K_4, 1)$ *holds, where* $K_4 = \{4,5,\ldots,12,14,15,18,19,23\}$.

Proof. We have to show that $27 \in B(K_4, 1)$. Consider the design $B[6,1;31]$, which exists by Theorem 2.1, and delete 4 points, no 3 of which are in a block. The obtained design is $B[\{4,5,6\},1;27]$.

Remark. It may be verified that K_4 is minimal in the sense that for every $v \in K_4$, $v \notin B(K_4 \setminus \{v\}, 1)$ holds.

6.3. Group divisible designs with block-size 4

Lemma 6.8. *Let* $h \leq t$, $t \in \mathrm{T}(5,\lambda)$ *and* $\{3h+m, 3t+m\} \subset \mathrm{GD}(4,\lambda,m)$; *then* $3(4t+h)+m \in \mathrm{GD}(4,\lambda,m)$.

Proof. By deleting $t-h$ points from one group of a transversal design $\mathrm{T}[5,\lambda;t]$ we obtain a pairwise group divisible design $\mathrm{GD}[\{4,5\},\lambda,\{h,t\};4t+h]$. Since by Lemma 4.10, $\{12,15\} \subset \mathrm{GD}(4,1,3)$ it follows by Lemma 2.27 that $3(4t+h) \in \mathrm{GD}(4,\lambda,\{3h,3t\})$ and by Lemma 2.29 our lemma follows.

Lemma 6.9. *If* $v \in \mathrm{GD}(4,\lambda,m)$ *and* $v \in \mathrm{T}(4,\lambda)$, *then* $4v \in \mathrm{GD}(4,\lambda,m)$.

Proof. $v \in \mathrm{T}(4,\lambda)$ means $4v \in \mathrm{GD}(4,\lambda,v)$. Further apply Lemma 2.28.

Lemma 6.10. *If* $q \equiv 1 \pmod 6$ *is a prime-power, then* $2q \in \mathrm{GD}(4,1,2)$.

Proof. $X = Z(2) \times \mathrm{GF}(q, f(x)=0)$. Let $d = (q-1)/6$.

$$P = \langle(\emptyset;\emptyset),(0,\alpha),(0;\alpha+2d),(0;\alpha+4d)\rangle \bmod (2;q), \qquad \alpha = 0,1,\ldots,d-1.$$

Lemma 6.11. *If* $n \equiv 1 \pmod 3$ *and* $n \neq 4$, *then* $v = 2n \in \mathrm{GD}(4,1,2)$ *holds.*

Proof. Let $v = 2n = 6s+2$, $s \neq 1$. For $s = 0$ the lemma is trivial. Considering Lemma 6.8 with $t \equiv 0 \pmod 4$, $h \equiv 0 \pmod 2$, $\lambda = 1$ and $m = 2$ and applying Lemma 3.24 it suffices to prove the lemma for $s \in S = \{2,3,\ldots,15,17,21,22,23,25,31,33,41\}$. By Lemma 6.9 and Theorem 3.5 it is sufficient to prove the lemma for $s \not\equiv 1 \pmod 4$ and for $s = 5$ and Lemma 6.10 proves for $s \in S$ all the cases $s \equiv 0 \pmod 2$. Consequently it remains to prove the lemma for $s \in \{3, 5, 7, 11, 15, 23, 31\}$ which is done presently.

$s = 3; 20 \in \mathrm{GD}(4,1,2)$. $X = Z(2) \times (Z(2) \times Z(5,2))$.
$\mathcal{P} = \langle(\emptyset;\emptyset,\emptyset),(\emptyset;\emptyset,0),(\emptyset;0,1),(\emptyset;0,2)\rangle \bmod (-;-,5)$,
$\langle(\emptyset;\emptyset,\emptyset),(\emptyset;\emptyset,1),(0;0,2),(0;0,3)\rangle \bmod (-;-,5)$,
$\langle(\emptyset;\emptyset,\emptyset),(\emptyset;0,\emptyset),(0;\emptyset,0),(0;\emptyset,2)\rangle \bmod (-;-,5)$,
$\langle(\emptyset;\emptyset,\emptyset),(0;0,\emptyset),(0;\emptyset,1),(0;\emptyset,3)\rangle \bmod (-;-,5)$,
$\langle(\emptyset;0,\emptyset),(0;\emptyset,\emptyset),(0;0,0),(0;0,2)\rangle \bmod (-;-,5)$,
$\langle(0;\emptyset,\emptyset),(0;0,\emptyset),(\emptyset;0,1),(\emptyset;0,3)\rangle \bmod (-;-,5)$.

$s = 5; 32 \in \mathrm{GD}(4,1,2)$. $X = Z(2) \times Z(16)$.
$\mathcal{G} = \{(0,j),(0,j+8)\} \bmod (2,-)$, $\quad j = 0,1,\ldots,7$.
$\mathcal{P} = \{(0,0),(0,6),(1,0),(1,2)\} \bmod (-,16)$,
$\{(0,0),(0,4),(0,11),(1,15)\} \bmod (-,16)$,
$\{(0,0),(0,1),(0,3),(1,6)\} \bmod (-,16)$,
$\{(0,8),(1,0),(1,1),(1,5)\} \bmod (-,16)$,
$\{(0,2),(1,0),(1,3),(1,9)\} \bmod (-,16)$.

$s = 7; 44 \in \mathrm{GD}(4,1,2)$. $X = Z(2) \times Z(22)$.
$\mathcal{G} = \{(0,j),(0,j+11)\} \bmod (2,-)$, $\quad j = 0,1,\ldots,10$.
$\mathcal{P} = \{(0,0),(0,5),(1,0),(1,15)\} \bmod (-,22)$,
$\{(0,0),(0,8),(0,9),(1,6)\} \bmod (-,22)$,
$\{(0,0),(0,2),(0,6),(1,9)\} \bmod (-,22)$,
$\{(0,0),(0,7),(0,10),(1,11)\} \bmod (-,22)$,
$\{(0,0),(1,16),(1,18),(1,21)\} \bmod (-,22)$,
$\{(0,0),(1,2),(1,8),(1,12)\} \bmod (-,22)$,
$\{(0,0),(1,5),(1,13),(1,14)\} \bmod (-,22)$.

$s = 11; 68 \in \mathrm{GD}(4,1,2)$. $X = Z(2) \times (Z(3) \times Z(3) \times Z(3)) \cup \{(\infty_i): i = 0,1,\ldots,13\}$.
$\mathcal{P} = \mathrm{GD}[4,1,2;14]$ on $\{(\infty_i): i = 0,1,\ldots,13\}$,
$\{(\infty_j),(0;0,0,0),(0;1,j,2j+1),(0;2,2j,j+2)\} \bmod (2;-,3,3)$, $\quad j = 0,1,2$,
$\{(\infty_{h+3}),(0;0,0,0),(0;1,2h+2,h),(0;2,h+1,2h)\} \bmod (2;-,3,3)$, $\quad h = 0,1$,
$\{(\infty_{j+5});(0;0,j,0),(1;0,j+1,0),(1;0,j+2,2)\} \bmod (2;3,-,3)$, $\quad j = 0,1,2$,
$\{(\infty_{j+8}),(0;j+h,0,h),(1;j+h,0,h+1),(1;j+h+1,1,h)\} \bmod (2;-,3,-)$,
$j = 0,1,2, \quad h = 0,1,2,$
$\{(\infty_{j+11}),(0;j+h,h,0),(1;j+h,h+1,2),(1;j+h+1,h,0)\} \bmod (2;-,-,3)$,
$j = 0,1,2, \quad h = 0,1,2,$
$\{(0;0,0,0),(0;0,0,1),(0;1,0,0),(1;2,1,1)\} \bmod (2;3,3,3)$,
$\{(0;0,0,0),(0;0,1,1),(1;1,0,2),(1;1,1,2)\} \bmod (2;3,3,3)$.

$s = 15; 92 \in \mathrm{GD}(4,1,2)$. $X = I(23) \times I(4)$.

We start with a construction of a pairwise group divisible design $\mathrm{GD}[4,1,\{2,5\};23]$

with exactly one group of size 5, on the set $Y' = Z(2)\times(Z(3)\times Z(3))\cup\{(\infty_i)\colon i = 0,1,2,3,4\}$.

$$\begin{aligned}\mathcal{P} = &\{(\infty_j),(0;0,j),(1;0,j+1),(1;1,j+2)\} \bmod (2;3,-),\quad j=0,1,2,\\ &\{(\infty_3),(0;0,0),(0;1,0),(0;2,0)\} \bmod (2;-,3),\\ &\{(\infty_4),(0;0,2),(0;1,1),(0;2,0)\} \bmod (-;3,-),\\ &\{(\infty_4),(1;0,0),(1;0,1),(1;0,2)\} \bmod (-;3,-),\\ &\{(0;0,0),(0;0,1),(1;1,1),(1;2,0)\} \bmod (-;3,3).\end{aligned}$$

As the next step we prove the existence of a transversal design T[4, 1; 23] with a subdesign T[4, 1; 5]. Consider a design T[5, 1; 5] and delete two points from one of the groups; the result is a design GD[{4, 5}, 1, {3, 5}; 23]. By Theorem 3.2 it follows that $23 \in$ T(4, 1), while the proof of this theorem proves the existence of a subdesign T[4, 1; 5].

We are now able to construct a design GD[4, 1, 2; 92]. Denote by $\mathcal{T}$ a transversal design $\mathcal{T} =$ T[4, 1; 23] on $X = I(23)\times I(4)$ with a subdesign $\mathcal{T}_0 =$ T[4, 1; 5] on $Y = I(5)\times I(4)$. Further, let $\mathcal{B}_i =$ GD[4, 1, {2, 5}; 23] on $I(23)\times\{i\}$ with exactly one group of size 5, namely $I(5)\times\{i\}$, $i\in I(4)$. Finally let $\mathcal{D} =$ GD[4, 1, 2; 20] on Y. We take as groups of our design the groups of size 2 of $\mathcal{B}_i$, $i \in I(4)$ and the groups of $\mathcal{D}$, and as blocks — the blocks of $\mathcal{B}_i$, $i\in I(4)$, of $\mathcal{D}$ and of $\mathcal{T}\setminus\mathcal{T}_0$.

$$s \in \{23, 31\};\ \{140, 188\} \subset \mathrm{GD}(4,1,2).$$

We prove moreover, that for every $m \geq 3$, $v = 48m - 4 \in$ GD(4, 1, 2). For every integer m there exists, by Theorem 7 of [36], a resolvable BIBD RB[4, 1; $12m+4$]. Form a partial completion of this design by adjoining $4m-6$ points p_i, $i = 0, 1, \ldots, 4m-7$, where the point p_i is adjoined to each of the blocks of the i^{th} parallel class. In this way we get a design $B[\{4, 5, 4m-6\}, 1; 16m-2]$. Since we did not use all the parallel classes of the original design, we can pick the blocks of one of the remaining parallel classes together with the block of size $4m-6$ and call them groups. It follows that $16m-2 \in \mathrm{GD}(\{4,5\},1,\{4,4m-6\})$. By Lemma 2.27 with $m = 3$ and Lemma 4.10 we get $48m-6\in\mathrm{GD}(4,1,\{12,12m-18\})$ and considering $\{14, 12m-16\}\subset\mathrm{GD}(4,1,2)$ our assertion follows from Lemma 2.29 with $m = 2$.

Lemma 6.12. *If* $n \equiv 0$ *or* $1 \pmod 4$, *then* $v = 3n \in \mathrm{GD}(4,1,3)$.

Proof. Follows from Lemmas 5.11 and 2.12.

Lemma 6.13. *If* $n \equiv 1 \pmod 3$, *then* $v = 4n \in \mathrm{GD}(4,1,4)$.

Proof. By Theorem 7 of [36] there exists a resolvable BIBD RB[4, 1; v]. Consider one of the parallel classes of blocks in this design as groups of a group divisible design.

Lemma 6.14. *If q is a power of an odd prime and $q \neq 3$, then* $6q \in \mathrm{GD}(4,1,6)$.

Proof. The proof is given separately for $q \equiv 1 \pmod 4$ and for $q \equiv 3 \pmod 4$. $q \equiv 1 \pmod 4$. $X = (Z(2) \times Z(3,2)) \times \mathrm{GF}(q, f(x) = 0)$. Let $d = (q-1)/2$ and $x^u = (x+1)/(x-1)$.

$$\mathcal{P} = \langle(\emptyset,\emptyset;\alpha),(\emptyset,\emptyset;\alpha+d),(\emptyset,\alpha;\emptyset),(0,\alpha;\alpha+u)\rangle \bmod(-,3;q), \quad \alpha = 0,1,\ldots,d-1,$$
$$\langle(0,\emptyset;\alpha),(0,\emptyset;\alpha+d),(0,\alpha+1;\emptyset),(\emptyset,\alpha+1;\alpha+u)\rangle \bmod(-,3;q), \quad \alpha = 0,1,\ldots,d-1.$$

$q \equiv 3 \pmod 4$. $X = (Z(2) \times Z(3,2)) \times \mathrm{GF}(q, f(x) = 0)$.

Let $d = \frac{1}{2}(q-1)$ and let u be chosen in such a way that $x^{2u} - 1$ is an odd power of x. (From the identity $x^2 - 1 = x^{2+d}(x^{-2} - 1)$ it is seen that either $u = 1$ or $u = -1$ satisfies this requirement.)

$$\mathcal{P} = \langle(\emptyset,\emptyset;2\alpha),(\emptyset,\emptyset;2\alpha+d),(\emptyset,0;\emptyset),(0,0;2\alpha+u)\rangle \bmod(2,3;q), \quad \alpha = 0,1,\ldots,d-1.$$

Lemma 6.15. *If $n > 4$, then* $v = 6n \in \mathrm{GD}(4,1,6)$.

Proof. By Lemmas 2.16 and 5.18 it suffices to prove our lemma for $n \in K_5 = \{5,6,\ldots,20,22,23,24,27,28,29,32,33,34,39\}$. Lemmas 6.11, 2.27 and 4.10 take care of the cases $n \equiv 1 \pmod 3$; and Lemma 6.14 — of the cases that n is a power of an odd prime. Further, by Lemma 6.9 and Theorem 3.5 we do not have to prove the lemma for $n \equiv 0 \pmod 4$, $n \geq 20$. It leaves us with the cases $n \in \{6,8,12,14,15,18,33,39\}$, the proof of which is given presently.

$n = 6$; $36 \in \mathrm{GD}(4,1,6)$. $X = (Z(2) \times Z(3,2)) \times (Z(2) \times Z(3,2))$.

$$\mathcal{P} = \langle(\emptyset,\emptyset;\emptyset,\emptyset),(0,\emptyset;\emptyset,0),(\emptyset,\emptyset;0,\emptyset),(0,0;0,0)\rangle \bmod(2,3;-,3),$$
$$\langle(\emptyset,\emptyset;\emptyset,\emptyset),(0,0;\emptyset,0),(\emptyset,\emptyset;0,0),(0,1;0,1)\rangle \bmod(2,3;-,3),$$
$$\langle(\emptyset,\emptyset;\emptyset,\emptyset),(0,1;\emptyset,0),(0,0;0,\emptyset),(\emptyset,0;0,1)\rangle \bmod(2,3;-,3),$$
$$\langle(\emptyset,\emptyset;\emptyset,\emptyset),(\emptyset,1;\emptyset,0),(\emptyset,\emptyset;\emptyset,1),(0,\emptyset;0,\emptyset)\rangle \bmod(2,3;-,3),$$
$$\langle(\emptyset,\emptyset;\emptyset,\emptyset),(\emptyset,1;0,\emptyset),(\emptyset,1;0,0),(\emptyset,\emptyset;0,1)\rangle \bmod(2,3;-,3)$$

$n = 8$; $48 \in \mathrm{GD}(4,1,6)$. $X = (Z(2) \times Z(3,2)) \times (Z(7,3) \cup \{\infty\})$.

$$\mathcal{P} = \langle(\emptyset,\emptyset;\infty),(\emptyset,\emptyset;\emptyset),(\emptyset,0;5),(0,0;1)\rangle \bmod(-,3;7),$$
$$\langle(\emptyset,\emptyset;\infty),(\emptyset,1;\emptyset),(0,1;3),(0,\emptyset;5)\rangle \bmod(-,3;7),$$
$$\langle(0,\emptyset;\infty),(\emptyset,\emptyset;\emptyset),(\emptyset,0;2),(0,1;0)\rangle \bmod(-,3;7),$$
$$\langle(0,\emptyset;\infty),(0,\emptyset;\emptyset),(0,0;4),(\emptyset,1;5)\rangle \bmod(-,3;7),$$
$$\langle(0,\emptyset;2\alpha+2),(0,\emptyset;2\alpha+5),(\emptyset,0;\emptyset),(0,1;2\alpha+4)\rangle \bmod(-,3;7), \alpha = 0,1,$$
$$\langle(\emptyset,\emptyset;4),(\emptyset,\emptyset;5),(0,\emptyset;0),(0,\emptyset;3)\rangle \bmod(-,3;7),$$
$$\langle(\emptyset,\emptyset;0),(\emptyset,\emptyset;3),(\emptyset,0;2),(\emptyset,0;5)\rangle \bmod(-,3;7).$$

$n = 12$; $72 \in \mathrm{GD}(4,1,6)$. $X = Z(6) \times (Z(11) \cup \{\infty\})$.

$$\mathcal{P} = \{(0;\infty),(0;0),(1;10),(2;2)\} \bmod(6,11),$$

$\{(0;\infty),(3;0),(4;7),(5;5)\} \bmod (6,11)$,
$\{(0;2),(0;9),(1;10),(3;5)\} \bmod (6,11)$,
$\{(0;4),(0;7),(1;9),(3;5)\} \bmod (6,11)$,
$\{(0;3),(0;8),(2;0),(2;1)\} \bmod (6,11)$,
$\{(0;0),(2;1),(2;10),(3;5)\} \bmod (6,11)$.

$n = 14; 84 \in GD(4,1,6)$. $X = Z(6) \times (Z(13,2) \cup \{\infty\})$.
$\mathcal{P} = \langle(\alpha',\infty),(0';\alpha),(2';\alpha+4),(4';\alpha+8)\rangle \bmod (6;13), \quad \alpha = 0,1,$
$\langle(0';\alpha),(0';\alpha+4),(0';\alpha+8),(1';\emptyset)\rangle \bmod (6;13), \quad \alpha = 0,1,$
$\langle(0';\emptyset),(1';4\beta),(3';4\beta+2),(4';4\beta+5)\rangle \bmod (6;13), \quad \beta = 0,1,2.$
$n = 15; 90 \in GD(4,1;6)$, $X = Z(6) \times Z(15)$.
$\mathcal{P} = \{(0;0),(4;1),(4;2),(1;8)\} \bmod (6;15)$,
$\{(4;1),(1;2),(1;4),(3;5)\} \bmod (6;15)$,
$\{(4;2),(1;4),(4;8),(0;10)\} \bmod (6;15)$,
$\{(0;0),(0;4),(5;5),(4;8)\} \bmod (6;15)$,
$\{(4;1),(0;5),(4;8),(3;10)\} \bmod (6;15)$,
$\{(0;0),(1;2),(2;5),(0;10)\} \bmod (6;15)$,
$\{(0;0),(1;1),(1;4),(5;10)\} \bmod (6;15)$.

$n = 18; 108 \in GD(4,1,6)$. $X = Z(6) \times Z(13,2)\ \{(\infty_i): \cup\ i = 0,1,\ldots,29\}$.
$\mathcal{P}$ = blocks of $GD[4,1,6;30]$ on $\{(\infty_i): i = 0,1,\ldots,29\}$,
blocks of $B[4,1;13]$ on $\{j\} \times Z(13)$, $j \in Z(6)$,
$\langle(\infty_{\alpha+3\beta}),(\beta';4\alpha),((\beta+1)';4\alpha+4),((\beta+3)';4\alpha+8)\rangle \bmod (6;13),$
$\alpha = 0,1,2, \quad \beta = 0,1,\ldots,5,$
$\langle(\infty_{\alpha+3\beta'}),((\beta+2)';4\alpha+1),((\beta+4)';4\alpha+5),((\beta+5)';4\alpha+9)\rangle \bmod (6;13),$
$\alpha = 0,1,2, \quad \beta = 0,1,\ldots,5,$
$\langle(\infty_{\alpha+3\gamma+18}),((\gamma+3\delta)';4\alpha+5),((\gamma+3\delta+1)';4\alpha+8), ((\gamma+3\delta+2)';4\alpha+11)\rangle$
$\bmod (6;13), \quad \alpha = 0,1,2, \quad \gamma = 0,1,2, \quad \delta = 0,1,$
$\langle(\infty_{\alpha+27}),(\delta';4\alpha+2),((\delta+2)';4\alpha+6),((\delta+4)';4\alpha+10)\rangle$
$\bmod (6;13), \quad \alpha = 0,1,2, \quad \delta = 0,1.$

$n \in \{33,39\}$; $\{198,234\} \subset GD(4,1,6)$.

Follows from Lemma 6.8 with $\lambda = 1$, $m = 6$, $t = 16$ and $h \in \{0,12\}$.

Lemma 6.16. *For every $n, n \geq 4$, $v = 12n \in GD(4,1,12)$ holds.*

Proof. By Lemmas 2.16 and 5.10* it suffices to prove our lemma for $n \in K_4 = \{4,5,\ldots,12,14,15,18,19,23\}$. Considering Lemmas 6.12, 6.13 and 2.27 it remains to prove the lemma for $n \in K^* = \{6,11,14,15,18,23\}$. By Lemmas 2.27 and 4.10 it suffices to prove that $4K^* \subset GD(\{4,5\},1,4)$. For $n \in \{6,11,15\}$, $4n \in GD(5,1,4)$ by Lemmas 5.19 and 2.12. For $n \in \{14,23\}$ form a resolvable BIBD $RB[4,1;4(n-1)]$ and partially complete it with points p_i, $i = 0,1,2,3$; further, consider as groups a parallel class of blocks of the resolvable BIBD and the set $\{p_i : i = 0,1,2,3\}$, this yields a $GD[\{4,5\},1,4;4n]$. For $n = 18$ let $X = I(4) \times Z(17,3) \cup \{(\infty_i): i = 0,1,2,3\}$.

$\mathcal{P} = \langle(\infty_\alpha), (0'; \alpha), (1'; \alpha+1), (2'; \alpha+2), (3'; \alpha+3)\rangle \bmod (-; 17), \quad \alpha = 0, 1, 2, 3,$
$\langle(0'; \beta+4), (1'; \beta+5), (2'; \beta+6), (3'; \beta+7)\rangle \bmod (-; 17), \quad \beta = 0, 1, \ldots, 11,$
$B[\{4,5\}, 1; 17]$ on $\{i\} \times Z(17)$, $i = 0, 1, 2, 3$:
The existence of $B[\{4,5\}, 1; 17]$ follows from Lemma 3.12, considering $4 \in \mathrm{T}(4, 1)$.

Lemma 6.17. *If q is a power of an odd prime, then $3q \in \mathrm{GD}(4, 2, 3)$.*

Proof. $X = Z(3,2) \times \mathrm{GF}(q, f(x) = 0)$. Put $d = (q-1)/2$.

$$\mathcal{P} = \langle(0; \alpha), (0; \alpha+d), (1; \alpha+1), (1; \alpha+d+1)\rangle \bmod (3; q), \quad \alpha = 0, 1, \ldots, d-1.$$

Lemma 6.18. *For every $n \geq 4$, $v = 3n \in \mathrm{GD}(4, 2, 3)$ holds.*

Proof. By Lemmas 2.16 and 5.10* it suffices to prove our lemma for $n \in K_4$. For $n \equiv 0$ or $1 \pmod 4$ the lemma follows from Lemma 6.12 and for $n \equiv 1 \pmod 3$ — from Lemmas 2.16 and 5.12. Further if $n = q$ is a power of an odd prime, the proof is given in Lemma 6.17, and it remains to prove the lemma for $n \in \{6, 14, 15, 18\}$ which is done presently.

$n = 6$; $18 \in \mathrm{GD}(4,2,3)$. $X = Z(3,2) \times (Z(5,2) \cup \{\infty\})$.
$\mathcal{P} = \langle(\emptyset; \infty), (\emptyset; \emptyset), (\alpha; \alpha), (\alpha; \alpha+2)\rangle \bmod (3; 5), \quad \alpha = 0, 1,$
$\langle(0; 0), (0; 2), (1; 1), (1; 3)\rangle \bmod (3; 5).$

$n = 14$; $42 \in \mathrm{GD}(4,2,3)$. $X = Z(3,2) \times (Z(13,2) \cup \{\infty\})$.
$\mathcal{P} = \langle(\emptyset; \infty), (\emptyset; \emptyset), (\alpha; 3\alpha), (\alpha; 3\alpha+6)\rangle \bmod (3; 13), \quad \alpha = 0, 1,$
$\langle(0; 3\alpha+1), (0; 3\alpha+7), (1; 3\alpha+2), (1; 3\alpha+8)\rangle \bmod (3; 13), \quad \alpha = 0, 1,$
$\langle(0; \beta), (0; \beta+6), (1; \beta+3), (1; \beta+9)\rangle \bmod (3; 13), \quad \beta = 0, 1, 2.$

$n = 15$; $45 \in \mathrm{GD}(4,2,3)$. $X = Z(3,2) \times (Z(3,2) \times Z(5,2))$.
$\mathcal{P} = \langle(\emptyset; 0, 0), (\emptyset; 0, 2), (\emptyset; 1, 1), (\emptyset; 1, 3)\rangle \bmod (3; 3, 5),$
$\langle(\emptyset; \emptyset, \alpha), (\emptyset; \emptyset, \alpha+2), (0; 0, \emptyset), (0; 1, \emptyset)\rangle \bmod (3; 3, 5), \quad \alpha = 0, 1,$
$\langle(\emptyset; 0, \emptyset), (\alpha; 0, \alpha+2\beta), (\alpha; 1, \alpha+2\beta+1), (\alpha+1; 1, \emptyset)\rangle \bmod (3; 3, 5),$
$\alpha = 0, 1, \quad \beta = 0, 1.$

$n = 18$; $54 \in \mathrm{GD}(4,2,3)$. $X = Z(3,2) \times (Z(17,3) \cup \{\infty\})$.
$\mathcal{P} = \langle(\emptyset; \infty), (\emptyset; \emptyset), (\alpha; 4\alpha), (\alpha; 4\alpha+8)\rangle \bmod (3; 17), \quad \alpha = 0, 1,$
$\langle(0; 1), (0; 9), (1; 5), (1; 13)\rangle \bmod (3; 17),$
$\langle(0; 4\alpha+2), (0; 4\alpha+10), (1; 4\alpha+3), (1; 4\alpha+11)\rangle \bmod (3; 17), \quad \alpha = 0, 1,$
$\langle(0; 2\beta+1), (0; 2\beta+9), (1; 2\beta+2), (1; 2\beta+10)\rangle \bmod (3; 17), \quad \beta = 0, 1, 2, 3.$

Lemma 6.19. *If q is a power of an odd prime, then $2q \in \mathrm{GD}(4, 3, 2)$.*

Proof. $X = Z(2) \times \mathrm{GF}(q, f(x) = 0)$. Let $d = \frac{1}{2}(q-1)$.

$$\mathcal{P} = \langle(\emptyset; \emptyset), (0; \alpha), (0; \alpha+1), (0; \alpha+2)\rangle \bmod (2; q), \quad \alpha = 0, 1, \ldots, d-1.$$

Lemma 6.20. *For every* $n \geq 4$, $v = 2n \in \mathrm{GD}(4,3,2)$ *holds.*

Proof. By Lemma 2.16 and 5.10* it suffices to prove our lemma for $n \in K_4$. For $n \equiv 1 \pmod 3$ and $n \neq 4$ the lemma follows from Lemma 6.11 and for $n = 4$ — from Theorem 3.11. If $n = q$ is a power of an odd prime, the proof is given in Lemma 6.19. It remains to prove the lemma for $n \in \{6, 8, 12, 14, 15, 18\}$ which is done herewith.

$n = 6$; $12 \in \mathrm{GD}(4,3,2)$. $X = Z(2) \times (Z(5,2) \cup \{\infty\})$.
$\mathcal{P} = \langle(\emptyset;\infty), (\emptyset;\emptyset), (\emptyset;0), (\emptyset;2)\rangle \bmod (-;5)$,
$\langle(\emptyset;\infty), (0;\emptyset), (0;1), (0;3)\rangle \bmod (-;5)$,
$\langle(0;\infty), (\emptyset;\emptyset), (0;1), (0;3)\rangle \bmod (-;5)$,
$\langle(0;\infty), (0;\emptyset), (\emptyset;0), (\emptyset;2)\rangle \bmod (-;5)$,
$\langle(\emptyset;0), (\emptyset;2), (0;1), (0;3)\rangle \bmod (2;5)$.

$n = 8$; $16 \in \mathrm{GD}(4,3,2)$. $X = Z(2) \times (Z(7,3) \cup \{\infty\})$.
$\mathcal{P} = \langle(\emptyset;\infty), (\emptyset;0), (\emptyset;1), (\emptyset;2)\rangle \bmod (-;7)$,
$\langle(\emptyset;\infty), (0;0), (0;1), (0;5)\rangle \bmod (-;7)$,
$\langle(0;\infty), (0;\emptyset), (\emptyset;2), (\emptyset;5)\rangle \bmod (-;7)$,
$\langle(0;\infty), (\emptyset;\emptyset), (0;1), (0;4)\rangle \bmod (-;7)$,
$\langle(\emptyset;2\alpha), (\emptyset;2\alpha+3), (0;2\alpha+2), (0;2\alpha+5)\rangle \bmod (-;7)$, $\alpha = 0,1$,
$\langle(\emptyset;\emptyset), (0;0), (0;2), (0;4)\rangle \bmod (2;7)$.

$n = 12$; $24 \in \mathrm{GD}(4,3,2)$. $X = Z(2) \times Z(11,2) \cup \{(\infty_i)\colon i = 0,1\}$.
$\mathcal{P} = \langle(\infty_\alpha), (\emptyset;\emptyset), (0;2\alpha+1), (0;2\alpha+7)\rangle \bmod (2;11)$, $\alpha = 0,1$,
$\langle(\emptyset;\emptyset), (0;0), (0;1), (0;4)\rangle \bmod (2;11)$,
$\langle(\emptyset;\beta+2), (\emptyset;\beta+7), (0;\beta+3), (0;\beta+8)\rangle \bmod (-;11)$, $\beta = 0,1,2,3$,
$\langle(\emptyset;0), (\emptyset;1), (\emptyset;2), (\emptyset;3)\rangle \bmod (-;11)$.
$\langle(0;0), (0;1), (0;2), (0;6)\rangle \bmod (-;11)$.

$n = 14$; $28 \in \mathrm{GD}(4,3,2)$. $X = Z(2) \times Z(13,2) \cup \{(\infty_i)\colon i = 0,1\}$.
$\mathcal{P} = \langle(\infty_\alpha), (\emptyset;\emptyset), (0;3\alpha), (0;3\alpha+1)\rangle \bmod (2;13)$, $\alpha = 0,1$,
$\langle(\emptyset;\alpha), (\emptyset;\alpha+6), (0;4-\alpha), (0;10-\alpha)\rangle \bmod (-;13)$, $\alpha = 0,1$,
$\langle(\emptyset;\alpha+3), (\emptyset;\alpha+9), (0;\alpha), (0;\alpha+6)\rangle \bmod (-;13)$, $\alpha = 0,1$,
$\langle(\emptyset;\emptyset), (0;\alpha), (0;\alpha+4), (0;\alpha+8)\rangle \bmod (2;13)$, $\alpha = 0,1$,
$\langle(\emptyset;\emptyset), (\emptyset;0), (\emptyset;4), (\emptyset,8)\rangle \bmod (2;13)$.

$n = 15$; $30 \in \mathrm{GD}(4,3,2)$. $X = Z(2) \times (Z(3,2) \times Z(5,2))$.
$\mathcal{P} = \langle(\emptyset;\emptyset,\emptyset), (0;0,\emptyset), (0;1,\alpha), (0;1,\alpha+2)\rangle \bmod (2;3,5)$, $\alpha = 0,1,2$,
$\langle(\emptyset;\emptyset,\emptyset), (0;\emptyset,\alpha), (0;0,\alpha+3), (0;1,\alpha+3)\rangle \bmod (2;3,5)$, $\alpha = 0,1,2$,
$\langle(\emptyset;\emptyset,0), (0;\emptyset,0), (0;\emptyset,1), (0;\emptyset,3)\rangle \bmod (2;3,5)$.

$n = 18$; $36 \in \mathrm{GD}(4,3,2)$. $X = Z(2) \times Z(17,3) \cup \{(\infty_i)\colon i = 0,1\}$.
$\mathcal{P} = \langle(\infty_0), (\emptyset;\emptyset), (0;0), (0;2)\rangle \bmod (2;17)$,
$\langle(\infty_1), (\emptyset;\emptyset), (0;5), (0;14)\rangle \bmod (2;17)$,
$\langle(\emptyset;0), (\emptyset;10), (0;7), (0;13)\rangle \bmod (2;17)$,

$\langle(\emptyset; 2\alpha), (\emptyset; 2\alpha+1), (\emptyset; 2\alpha+2), (\emptyset; 2\alpha+4)\rangle \bmod (2; 17), \quad \alpha = 0, 1,$
$\langle(\emptyset; \beta), (\emptyset; \beta+8), (0; \beta+4), (0; \beta+12)\rangle \bmod (2; 17), \quad \beta = 0, 1, 2, 3.$

Lemma 6.21. *If q is a power of an odd prime, then* $4q \in \mathrm{GD}(4, 3, 4)$.

Proof. $X = (Z(3,2) \cup \{\infty\}) \times \mathrm{GF}(q, f(x) = 0)$. Let $d = (q-1)/2$.

$\mathcal{P} = \langle(\infty; \alpha), (\infty; \alpha+d), (\emptyset; \alpha+1), (\emptyset; \alpha+d+1)\rangle \bmod (3; q), \quad \alpha = 0, 1, \ldots, d-1,$
$\langle(0; \alpha), (0; \alpha+d), (1; \alpha+1), (1; \alpha+d+1)\rangle \bmod (3; q), \quad \alpha = 0, 1, \ldots, d-1,$
$\langle(\infty; \emptyset), (\emptyset; \beta), (0; \beta+1), (1; \beta+2)\rangle \bmod (-; q), \quad \beta = 0, 1, \ldots, q-2.$

Lemma 6.22. *For every $n \geqslant 4$, $v = 4n \in \mathrm{GD}(4, 3, 4)$ holds.*

Proof. By Lemmas 2.16 and 5.10* it suffices to prove our lemma for $n \in K_4$. For $n \equiv 1 \pmod 3$ the lemma follows from Lemma 6.13 and for $n \equiv 0$ or $1 \pmod 4$ — from Lemmas 5.13 and 2.16. If $n \equiv 0$ or $1 \pmod 5$, then by Lemmas 5.19 and 2.12, $4n \in \mathrm{GD}(5, 1, 4)$ and considering $5 \in B(4, 3)$ by Lemma 5.13, our lemma holds. If $n = q$ is a power of an odd prime, the proof is given in Lemma 6.21. It remains to prove the lemma for $n \in \{14, 18\}$ which is done presently.

$n = 14$; $56 \in \mathrm{GD}(4, 3, 4)$. $X = Z(4) \times Z(13, 2) \cup \{(\infty_i)\colon i = 0, 1, 2, 3\}$.
$\mathcal{P} = \langle(\infty_\alpha), (0'; \emptyset), (1'; 3\alpha), (1'; 3\alpha+6)\rangle \bmod (4; 13), \quad \alpha = 0, 1,$
$\langle(\infty_2), (0'; \emptyset), (2'; 0), (2'; 2)\rangle \bmod (4; 13),$
$\langle(\infty_3), (0'; \emptyset), (0'; 1), (0'; 5)\rangle \bmod (4; 13),$
$\langle(0'; \beta+2), (0'; \beta+8), (1'; \beta+5), (1'; \beta+11)\rangle \bmod (4; 13), \quad \beta = 0, 1, 2, 3,$
$\langle(0'; \emptyset), (0'; 2), (2'; 6), (2'; 9)\rangle \bmod (4; 13),$
$\langle(0'; \emptyset), (0'; 0), (2'; 3), (2'; 4)\rangle \bmod (4; 13),$
$\langle(0'; \emptyset), (1'; 3\beta+2), (2'; 3\beta+8), (3'; 3\beta+5)\rangle \bmod (4; 13), \quad \beta = 0, 1, 2, 3.$

$n = 18$; $72 \in \mathrm{GD}(4, 3, 4)$. $X = Z(4) \times Z(17, 3) \cup \{(\infty_i)\colon i = 0, 1, 2, 3\}$.
$\mathcal{P} = \langle(\infty_\alpha), (0'; \emptyset), (0'; 4\alpha), (0'; 4\alpha+1)\rangle \bmod (4; 17), \quad \alpha = 0, 1,$
$\langle(\infty_{\alpha+2}), (0'; \emptyset), (0'; 4\alpha+3), (0'; 4\alpha+4)\rangle \bmod (4; 17), \quad \alpha = 0, 1,$
$\langle(0'; \beta), (0'; \beta+8), (1'; \beta+4), (1'; \beta+12)\rangle \bmod (4; 17), \quad \beta = 0, 1, 2, 3,$
$\langle(0'; \emptyset), (0'; 4\alpha+14), (2'; 4\alpha+1), (2'; 4\alpha+2)\rangle \bmod (4; 17), \quad \alpha = 0, 1,$
$\langle(0'; \emptyset), (1'; 2\gamma), (2'; 2\gamma+6), (3'; 2\gamma+11)\rangle \bmod (4; 17), \quad \gamma = 0, 1, \ldots, 7.$

Theorem 6.3. *Let m, λ and v be positive integers. A necessary and sufficient condition for the existence of a group divisible design* $\mathrm{GD}[4, \lambda, m; v]$ *is that the design is not* $\mathrm{GD}[4, 1, 2; 8]$ *and not* $\mathrm{GD}[4, 1, 6; 24]$ *and that*

$$v \equiv 0 \pmod m, \quad \lambda(v-m) \equiv 0 \pmod 3, \quad \lambda v(v-m) \equiv 0 \pmod{12},$$
$$\text{and } v \geqslant 4m \text{ or } v = m.$$

Proof. $8 \notin \mathrm{GD}(4, 1, 2)$ follows from Lemma 3.14 and $24 \notin \mathrm{GD}(4, 1, 6)$ has been proved by Tarry [32]. The necessity of the other conditions follows from Theorem 6.1. Regarding the sufficiency it follows from Lemma 2.7 that we may limit

Table 6.2

λ	m	v	$n = v/m$	Proof
1	1	1 or 4 (mod 12)	1 or 4 (mod 12)	Lemma 5.11.
1	2	2 (mod 6), $v \neq 8$	1 (mod 3), $n \neq 4$	Lemma 6.11.
1	3	0 or 3 (mod 12)	0 or 1 (mod 4)	Lemma 6.12.
1	4	4 (mod 12)	1 (mod 3)	Lemma 6.13.
1	6	0 (mod 6), $v \geq 30$	every $n \geq 5$	Lemma 6.15.
1	0 (mod 2), $m \geq 8$	m (mod $3m$)	1 (mod 3)	Lemma 6.11 or 6.13 and Theorem 3.5.
1	12	0 (mod 12), $v \geq 48$	every $n \geq 4$	Lemma 6.16.
1	0 (mod 6), $m \geq 18$	0(mod m), $v \geq 4m$	every $n \geq 4$	Lemma 6.15 or 6.16 and Theorem 3.5
2	1	1 (mod 3)	1 (mod 3)	Lemma 5.12.
2	2	2 (mod 6)	1 (mod 3)	Lemma 6.11 and Theorem 3.11.
2	3	0 (mod 3), $v \geq 12$	every $n \geq 4$	Lemma 6.18.
2	6	0 (mod 6), $v \geq 24$	every $n \geq 4$	Lemma 6.15 and Theorem 3.11.
3	1	0 or 1 (mod 4)	0 or 1 (mod 4)	Lemma 5.13.
3	2	0 (mod 2), $v \geq 8$	every $n \geq 4$	Lemma 6.20.
3	4	0 (mod 4), $v \geq 16$	every $n \geq 4$	Lemma 6.22.
3	6	0 (mod 6), $v \geq 24$	every $n \geq 4$	Lemma 6.15 and Theorem 3.11.
6	1	every $v \geq 4$	every $n \geq 4$	Lemma 5.14.

ourselves to the values of λ which are factors of 12. Further, it follows from Theorems 3.1 and 3.5 that for $r \notin \{2, 6\}$, if $v \in \mathrm{GD}(4, \lambda, m)$, then $rv \in \mathrm{GD}(4, \lambda, rm)$. Consequently, the sufficiency is proved completely in Table 6.2.

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