

# Towards Optimal Demand-Side Bidding in Parallel Auctions for Time-Shiftable Electrical Loads

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**Abstract**—Increasing electricity production from renewable energy sources has, by its fluctuating nature, created the need for more flexible demand side management. How to integrate flexible demand in the electricity system is an open research question. We consider the case of procuring the energy needs of a time-shiftable load through a set of simultaneous second price auctions. We derive a required condition for optimal bidding strategies. We then show the following results and bidding strategies under different market assumptions. For identical uniform auctions and multiple units of demand, we show that the global optimal strategy is to bid uniformly across all auctions. For non-identical auctions and multiple units, we provide a way to find solutions through a recursive approach and a non-linear solver. We show that our approach outperforms the literature under higher uncertainty conditions.

**Index Terms**—simultaneous auctions, demand side bidding, time-shiftable loads

## I. INTRODUCTION

Climate change is a defining challenge of the 21st century. To reduce CO2 emissions, massive investments in renewable energies have been made and will still be required. Since renewable energy source are often not controlled generators, electricity production as well as electricity prices are subject to greater fluctuation than before.

Given the increased fluctuation on the production side, there has been a greater interest in time shift-able loads on the demand side [8, 14, 16, 19]. Time-shiftable and other flexible loads can be found in various domains, such as: data centres [7], heating systems [10], water distribution systems [15], and household energy consumption [1]. Attention should be given to industrial processes, which by their high energy intensity can achieve a significant impact on the electricity system [3].

In order to take advantage of fluctuating prices on day-ahead electricity markets, bidding strategies for time shift-able loads have been investigated. The approach of Mohsenian-Rad [14] suggests to bid only in a single time slot – the cheapest one in expectation. We consider a slightly more abstract version of the problem described in [14], to drive the point that under the assumption of free disposal participation in all instead of a single auction provides better results.

This work is part of the research programme Heat and Power Systems at Industrial Sites and Harbours (HaPSISH) with project number OND1363719, which is partly financed by the Dutch Research Council (NWO).

We extend the literature as follows. In the general case, we extend the optimality requirement derived in Gerding et al. [6] from single unit to multi-unit demand (see Lemma 1 in Section IV). For the case of identical auctions, we show that uniform bidding, i.e., not just bidding in a single auction but participating in all auctions, is the optimal bidding strategy for uniform distributions (see Theorem 2). For the non-identical auction setting, in Section IV-C we: 1) provide a way to guide a non-linear solver to converge more often to a non-trivial solution; 2) provide a dynamic programming approach to make the problem computationally feasible; and 3) show that our approach outperforms solutions found in literature ([14]) under higher uncertainty conditions.

## II. RELATED WORK

Given our abstract procurement problem of a time shift-able load, we will consider both literature particular to demand side bidding and literature on bidding strategies more generally.

The problem of demand side bidding in electricity markets with fixed demand in a single time slot is considered in Liu and Guan [13], Philpott and Pettersen [17], while Herranz et al. [11] considers multiple time slots and stochastic demand. Neither, however, consider any control over the demand they have to satisfy. In contrast, an electric vehicle aggregator, as considered in Bessa et al. [4], Vagropoulos and Bakirtzis [20], can control the charging rate of the electric vehicles in its fleet. However, this flexibility is in magnitude only and their bidding strategies, therefore, ignore the flexibility to shift demand in time. Demand side bidding for time shift-able loads is considered in Mohsenian-Rad [14]. However, we will show that their assumption of no free disposal is causing their solution to be sub-optimal.

The general problem of bidding in simultaneous auctions is considered in the Trading Agent Competition [9, 21]. The goal of their agent, however, was to construct bundles of non-identical items, which were strong complements. Our agent on the other hand values all electricity equally up to a particular demand. Bidding in simultaneous auctions with identical items has been considered in Rothkopf [18]. However, in their work winning any auction yields some valuation independently of the other auctions and what connects the auctions is a shared budget that should not be exceeded. A shift-able load, as

considered by our work, on the other hand values electricity only up to its demand and has zero valuation beyond that. Closest to our work, Gerding et al. [6] develops optimal bidding strategies for simultaneous auctions of identical items with a unit demand agent. We extend their work from single unit to multi unit demand.

### III. MODEL

We consider the electricity acquisition problem of a deferrable load, which within a bounded discretized time horizon has to run for a total of  $s$  time units – possibly non-consecutive. We assume that this deferrable load requires the same amount of energy for every time step in which it is switched on. Since, the energy part of the bid does not change, we will, going forward, only focus on the price bid.

For the given time horizon, there exists a set of electricity auctions  $T$ , which are held in parallel and ahead of time. For readability reasons, we will use the terms unit(s) and auction(s) interchangeably, where clear. To win the necessary  $s$  units/auctions to run the load, the agent submits a bid vector  $B = (b_1, b_2, \dots, b_{|T|})$  over auctions  $T$ .  $f_t(\tau)$ , supported on the interval  $[0, \lambda_{el}]$ , describes the clearing price distribution of auction  $t \in T$ , while  $F_t(\tau)$  is its corresponding cumulative distribution, with  $F_t(0) = 0$  and  $F_t(\lambda_{el}) = 1$ . The agent wins auction  $t \in T$  if the submitted bid  $b_t$  is at least the clearing price of auction  $t$ .

The expected cost the agent incurs when submitting bid  $B$  given a demand of  $s$  units and a set of auctions  $T$  is composed of a market cost and a backup cost, see (1).

$$Cost(B|T, s) = Cost_M(B|T, s) + Cost_B(B|T, s) \quad (1)$$

The expected market cost is the sum of payments to auctions  $T$ . We assume that the agent is a price taker.

*Assumption 1 (Price Taker):* The agent has no effect on the clearing price.

The payment per auction  $t \in T$  is the clearing price of that auction, conditionalized on the agent winning auction  $t$ .

$$Cost_M(B|T, s) = \sum_{t \in T} \int_0^{b_t} \tau f_t(\tau) d\tau \quad (2)$$

The expected backup cost is the payment made to a backup generator in case of shortfall. If fewer than the desired  $s$  units are acquired, the agent incurs the maximum price of  $\lambda_{el}$ . We are making the assumption of free disposal.

*Assumption 2 (Free Disposal):* We assume free disposal: any amount of electricity acquired beyond the agent's need can be disposed of at zero cost.

This assumption can be justified by assuming that there exists an intra-day market at which excess electricity can be sold at a non-negative price. Any excess energy that has been obtained from the day-ahead market can be sold on the intra-day market. We therefore only have to assume that prices on the intra-day market are not negative. The backup cost is zero if the agent wins more than  $s$  units. For any number of won auctions  $j \in [0, s - 1]$  there is a set of subsets  $w \subset T$  such

that  $|w| = j$ . For every subset of auctions  $w$ , the probability that the agent wins the subset and loses all other auctions is

$$\prod_{t \in w} F_t(b_t) \prod_{t \in T \setminus w} [1 - F_t(b_t)]$$

By summing over all possible sets  $w \subset T$  s.t.  $|w| = j$ , we obtain the probability of the agent winning  $j$  out of auctions  $T$  and can therefore calculate the expected backup cost as follows

$$Cost_B(B|T, s) = \sum_{j < s} \sum_{\substack{w \subset T \\ |w|=j}} \underbrace{\prod_{t \in w} F_t(b_t) \prod_{t \in T \setminus w} [1 - F_t(b_t)]}_{Prob(x=j|T, B)} (s-j) \lambda_{el}. \quad (3)$$

Here  $Prob(x = j|T, B)$  represents the probability of winning  $j$  auctions out of the set of  $T$  when submitting bid  $B$ .

### IV. OPTIMAL BIDDING STRATEGIES

We begin by introducing a requirement that any optimal bid has to satisfy, see Lemma 1, and provide a strategy that always meets this requirement, see Theorem 1.

Lemma 1 introduces the first optimality condition. We define  $B_{-k}$  as the bid vector  $B$  but with the bid for auction  $k$  removed.  $x$  is the random variable indicating the number of auctions the agent won. Intuitively speaking (4) states that the bid submitted to any particular auction  $k \in T$  is proportional to the probability of falling short in the remaining auctions.

*Lemma 1:* Any optimal bid for  $s$  out of  $T$  auctions has to satisfy the requirement in (4).

$$b_k = Prob(x < s | T \setminus \{k\}, B_{-k}) \lambda_{el} \quad \forall k \in T. \quad (4)$$

Proofs for Lemmas and Theorems are given in Appendix A.

The most straight-forward bidding strategy is to bid  $\lambda_{el}$  in  $s$  auctions and zero in all other auctions. We define this as a special strategy (Definition 1). Among the  $\Lambda_s$ -bidding strategies there exists one that submits  $\lambda_{el}$  to the auctions with the lowest clearing price in expectation. Whenever we compare any strategy to  $\Lambda_s$ -bidding strategies, we mean this lowest price version of it.

*Definition 1:* Let  $\Lambda_s$  be the set of bid vectors which consist of  $s$  bids of  $\lambda_{el}$  and  $(|T| - s)$  bids of zero value.

Next, we will show that any strategy that meets the requirement set out in Lemma 1 and wins or loses an auction with certainty, i.e.,  $F_t(b_t) = 1$  or  $F_t(b_t) = 0$ , is a  $\Lambda_s$ -bidding strategy (Theorem 1). This means that an optimal bidding strategy is either a  $\Lambda_s$ -bidding strategy or participates in all auctions with some non-zero bid.

*Theorem 1:* Any bid vector  $B$  that satisfies condition (4) and contains a bid of value  $\lambda_{el}$  or 0 is an element of  $\Lambda_s$ .

#### A. Comparison to the Literature

Having established our model, condition for optimality and first bidding strategy, we connect our work to that of [14]. Mohsenian-Rad addresses the energy acquisition problem of a time-shiftable load that can be run within a single time step. His approach considers submitting energy-price bid pairs to a

set of day ahead auctions with a secondary intra-day market for recourse. Our bids on the other hand only consider the price and we consider a backup generator instead of a second stage intra-day market.

However, Theorem 3 in [14] states that the optimal bidding strategy on the day ahead market is to acquire the entire energy need from a single auction, i.e., submitting a single non-zero bid for the entire energy need with a price equal to the expected price on the intra-day market. In our setting the intra-day market is replaced with a backup generator of deterministic cost  $\lambda_{el}$ . The expected price on the intra-day market in [14] can be viewed as our backup generator cost  $\lambda_{el}$ . Translated to our setting, Mohsenian-Rad suggests bidding  $\lambda_{el}$  in a single auction and 0 in all others. We will therefore associate the strategy suggested by [14] with  $\Lambda_s$ -bidding as defined in Definition 1.

### B. Identical Auctions

We first consider an identical auctions setting. We will show that uniform bidding satisfies (4) and provide an algorithm that can quickly find the correct uniform bid value.

*Assumption 3 (Identical Auctions):* All auctions are identical:  $f(b) = f_t(b) \quad \forall t \in T$ .

We define the uniform bidding strategy as a bid  $B$  that submits the same value to all auctions. For this setting, uniform bidding satisfies (4).

*Definition 2 (Uniform Bid):* A uniform bid vector  $B$  is a bid vector such that  $b_k = b_l \quad \forall k, l \in T$

*Lemma 2:* Under the assumption 3, there exists a unique uniform bid  $B_u = (b_u, b_u, \dots, b_u)$  s.t. condition (4) holds.

This uniform bid value,  $b_u$ , can be found via interval halving, as the right side of (4) strictly increases while the left side decreases with increasing  $b_u$ .  $b_u$  can quickly be found by interval halving.

1) *Identical Uniform Auctions:* Next, we consider the case where every auction has an identical uniform price distribution.

*Assumption 4 (Identical Uniform Auctions):* The price of every auction is uniformly distributed on the interval  $[0, \lambda_{el}]$ :  $f_t(b) = \frac{1}{\lambda_{el}} \quad \forall t \in T$ .

We will establish uniform bidding as described in Lemma 2 as the optimal bidding strategy for settings of identical uniform distributions (Theorem 2). To do so, we first show that uniform bidding is the only non- $\Lambda_s$  bidding strategy that satisfies (4) (Lemma 3) and that the uniform bid is a local minimum (Lemma 4). Finally, we compare our results to a strategy suggested by Mohsenian-Rad [14] and discuss the difference in modelling assumptions we make.

*Lemma 3:* Under Assumption 4, any bid  $B$  that satisfies condition (4) is either a uniform bid across all auctions or  $B \in \Lambda_{el}$

*Lemma 4:* Under Assumption 4, the uniform bid as defined in Lemma 2 is a local minimum of the cost function.

*Theorem 2:* Under the assumption of identical uniform distributions, uniform bidding is a global optimum, i.e., for any  $n \geq 2$  and  $s \in [1, n - 1]$ , the following holds

$$Cost_u(B_{n,s}^u | n, s) \leq Cost_u(B | n, s) \quad \forall B \in [0, \lambda_{el}]^n,$$

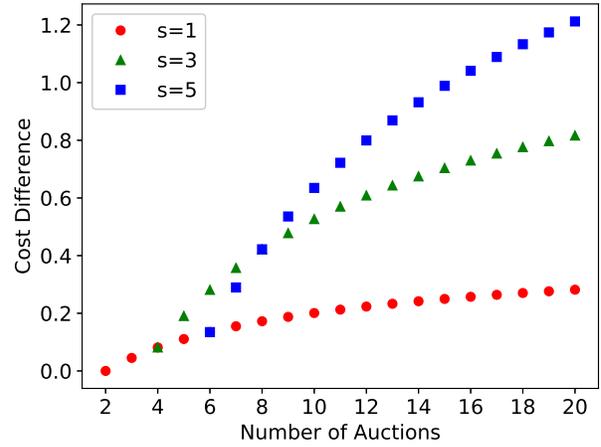


Fig. 1. Cost Difference to  $\Lambda_s$ -Bidding

where  $B_{n,s}^u$  is the optimal uniform Bid and  $Cost_u(B | n, s)$  is the cost of submitting bid  $B$  in  $n$  auctions with uniform price distributions when requiring  $s$  units.

In Theorem 2, we only establish optimality of uniform bidding, but say nothing yet about the effect of the type of load on performance.

Figure 1 shows numerical results of uniform bidding for loads of different demand  $s$ . The graph displays the cost difference between  $\Lambda_s$ -bidding and optimal uniform bidding. For all numerical results we assume  $\lambda_{el} = 1$ . As was expected uniform bidding outperforms  $\Lambda_s$ -bidding for all cases. As the number of auctions increases uniform bidding further improves. This positive effect of adding auctions appears to be stronger for loads that have higher demand, as indicated by the steeper slope for  $s = 5$  compared to  $s = 1$ .

It is noteworthy here that Theorem 2 and Figure 1 seem to directly contradict some of the results by Mohsenian-Rad [14]. In particular Theorem 3 in [14] states that the optimal bidding strategy is to acquire all energy from a single auction with a bid equal the expected cost on the intra-day market. In section IV-A we associated this bidding strategy with  $\Lambda_s$ -bidding in our setting. To resolve this seeming contradiction, we need to look towards the assumptions made by Mohsenian-Rad. In particular the constraint of equation (6) in [14] restricts the bidding strategy to those that never run the risk of obtaining more energy than needed, while we assume free disposal of additional energy, see Assumption 2. Our uniform bidding strategy runs the risk of – in the worst case – winning all auctions. Furthermore, Figure 2 shows the expected number of units the uniform bidding strategy wins. Figure 2 indicates that we not only run the risk of winning more units than necessary but that for most cases we expect to obtain more units than demanded. This over-consumption can be explained by the backup cost being weakly larger than market clearing prices. Therefore obtaining too few units causes a relatively large penalty of  $\lambda_{el}$ , while an agent winning too many auctions pays an often much lower market clearing price.

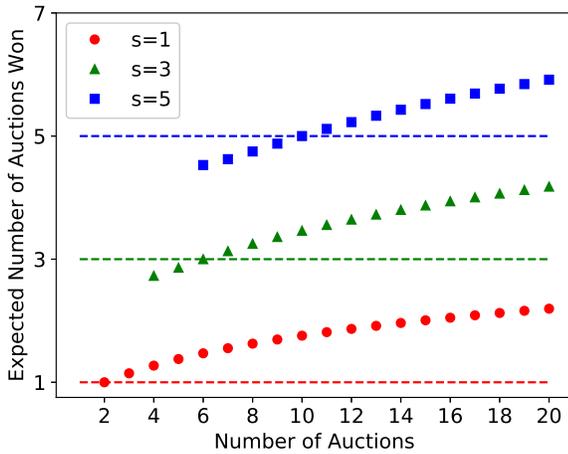


Fig. 2. Expected # of units won

### C. Non-Identical Auctions

We consider the problem of acquiring  $s$  units from a set of non-identical auctions  $T$ , which follow truncated normal distributions. We assume that the clearing prices for auctions  $t \in T$  are distributed according to  $\mathcal{N}_{0,\lambda_{el}}(\mu_t, \sigma^2)$ , i.e., the un-truncated distributions differ in mean but not standard deviation. Let the set of mean values be  $\boldsymbol{\mu} = \{\mu_t | t \in T\}$ . Without loss of generality, we assume that  $\mu_i \leq \mu_j$  for  $i < j$ .

Given this setting we can rewrite the condition in (4) as a root finding problem of a system of  $|T|$  non-linear equations:

$$b_k - \text{Prob}(x < s | T \setminus \{k\}, B_{-k}) \lambda_{el} = 0 \quad k \in T$$

For finding a solution, we use the Minpack's [5] 'hybrj' method, which is an adaptation of Powell hybrid method. This method requires us to provide a Jacobian. The diagonal elements of the Jacobian are  $J_{k,k} = 1$  and the off-diagonal elements are  $J_{k,j} = f_j(b_j) \text{Prob}(x = s - 1 | T \setminus \{j, k\}) \lambda_{el}$ . The entries for the Jacobian only differ by a factor of  $f_k(b_k)$  from the second derivative and the reader is therefore referred to Appendix C for a derivation.

This approach exhibits two challenges. First, evaluating the function and Jacobian requires repeated calculation of a term of the form  $\text{Prob}(x = i | T \setminus \{k\}, B_{-k})$ , which if done explicitly is composed of  $\binom{|T|-1}{i}$  terms. This makes finding a solution for larger systems computationally infeasible. The second challenge is convergence to non-trivial solutions. Since any bid  $B_s \in \Lambda_s$ , which can be constructed without the need for any solver, satisfies the system of equations, a badly initialized solver often converges to this solution.

The first challenge is solved by recursion. Let  $D$  be a set of auctions and let  $B_D$  be a corresponding bid vector. Let  $d = |D|$  be the corresponding auction for bid  $b_{|D|}$  – the last bid in bid vector  $B_D$ . We calculate  $\text{Prob}(x = i | D, B_D)$  recursively as follows

$$\begin{aligned} \text{Prob}(x = i | D, B_D) &= F_d(b_d) \text{Prob}(x = i - 1 | D \setminus \{d\}, B_{D,-d}) \\ &+ [1 - F_d(b_d)] \text{Prob}(x = i | D \setminus \{d\}, B_{D,-d}), \end{aligned}$$

where  $B_{D,-d}$  is bid vector  $B_D$  with the last entry removed. Note that  $F_d$  and  $[1 - F_d(b_d)]$  are the probabilities of winning and losing auction  $d$ , respectively. Going through the recursion, we can calculate  $\text{Prob}(x = i | D, B_D)$  by calculating the entries in a table of size  $|D| \times i$ .

The second problem, creating an appropriate initial guess, is addressed by iteratively solving approximate versions of the problem. To do so we partition  $T = \uplus G$ . For each group  $g \in G$ , a group auction price distribution  $\mathcal{N}(\bar{\mu}_g, \sigma^2)$ , where  $\bar{\mu}_g$  is the average mean of the auctions in  $g$ , is defined.

To create the approximate problem, we replace the clearing price distribution  $f_t(\tau)$  for each auction  $t \in T$  with the corresponding group auction price distribution  $\mathcal{N}(\bar{\mu}_g, \sigma^2)$ . The partitioning starts as a singleton  $G = \{T\}$ , i.e., all individual price distributions are replaced with the same group price distribution. Therefore, the first approximate problem is an identical auction setting for which we find the uniform bid as in Section IV-B by interval halving. The grouping is then refined and a new approximate problem is constructed. The solution to the prior solution is sorted (to ensure that higher bids go to cheaper auctions) and used as the initial guess for the new approximate problem. We refine the grouping of auctions until every group consists of a single auction at which point we reached the original problem.

The refinement of the partitioning can be done in one of two ways. Either, we split every group in two approximately equal sized subgroups. This means that we require  $\log(|T|)$  iterations of refinement to reach the original problem and therefore add a factor of  $\log(|T|)$  to the computational complexity. Alternatively, we increase the number of groups by one every time we refine the grouping, adding a factor of  $|T|$  to the computational complexity. When increasing the number of groups, we use k-means clustering on the set of mean values  $\boldsymbol{\mu}$  to find the grouping.

1) *Results and Discussion:* We consider the problem of acquiring a certain number of units from a set of non-identical auctions, where the clearing prices for auctions  $t \in T$  are distributed according to  $\mathcal{N}_{0,\lambda_{el}}(\mu_t, \sigma^2)$ . Let  $\mu_t$  be drawn from a uniform distribution on the interval  $[0, \lambda_{el}]$ .

We first look at the convergence rate. We consider the solver to have successfully converged, when it finds a non-trivial solution, i.e., a bid  $B \notin \Lambda_s$  that satisfies (4). Figure 3 compares the convergence success rate of different initializations. The demand of the agent is set to  $s = 5$ . When initialized with a uniform bid based on the identical auction approximation the solver virtually never converges. Note that we do not mean that the solver did not converge but rather that the solver converged to a  $\Lambda_s$ -bidding strategy, which can be found easily without the need of a solver. When increasing the number of groups by one (k-means Grouping) the convergence rate is similar to when groups are split in half at every iteration (Grouping). This more refined increase of groups appears to add little to the chance of converging to a non-trivial solution, while adding a factor of  $|T|$  instead of  $\log(|T|)$  to the computational complexity. Increasing the standard deviation  $\sigma$  from 0.1 to 0.3 and 0.5 improve the chances of the solver converging to a non-

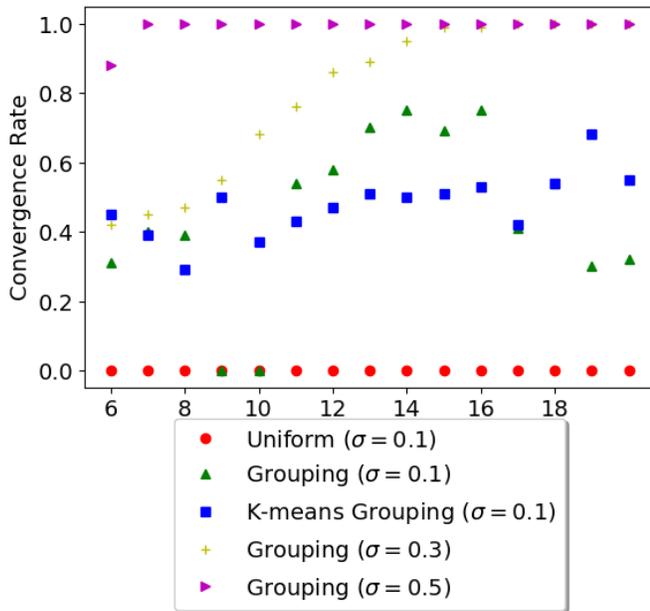


Fig. 3. Convergence-Rate

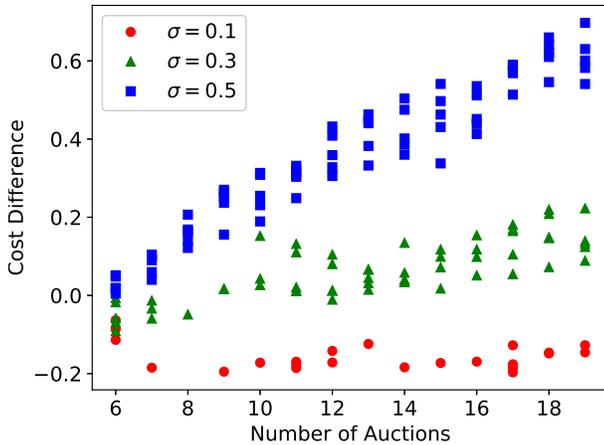


Fig. 4. Cost Difference

trivial solution. Similarly, as the number of auctions increases the success rate of the solver increases. For large number of auctions and for high uncertainty settings, i.e.,  $\sigma = 0.3$  or  $\sigma = 0.5$ , we reach a near 100% success rate.

Next, we compare the performance of our approach to the  $\Lambda_S$ -bidding, which we associated in Section IV-A to the work of Mohsenian-Rad. Figure 4 shows the cost difference between  $\Lambda_S$ -bidding and the solution provided by our solver approach. For visual clarity results where the solver did not converge to  $B \notin \Lambda_S$  were omitted. While, under low uncertainty,  $\sigma = 0.1$ ,  $\Lambda_S$  is the better choice, as the uncertainty in the prices grows ( $\sigma = 0.3$ ,  $\sigma = 0.5$ ) our approach appears to improve. The solution provided by us further improves as the number of auctions increases.

## V. CONCLUSION AND FUTURE WORK

This paper considered the problem of an interruptible time-shiftable electric load in acquiring multiple units of electricity from a set of parallel auctions. We derived a condition for optimality and established a direct comparison to Mohsenian-Rad [14]. First, for the identical auction setting, we show that uniform bidding is a solution that satisfies our optimality condition. For identical uniform price distributions, we show that uniform bidding, i.e., participating in all rather than a single auction, is optimal. In comparing our results to Mohsenian-Rad [14], we show that assuming free disposal yields significantly reduced cost.

Second, for non-identical auctions, we use approximate problem formulations to guide a non-linear-solver and provide a dynamic programming approach to make solving the set of non-linear equations computationally feasible. Numerical results show that under high price uncertainty our approach again outperforms the literature.

Regarding future work, we would like to extend this line of research to include budget constraints of the agent. A second line of extension can be to consider a wider range of valuation functions that take into account potential secondary uses of electricity as heat or temporal constraints of industrial processes.

## ACKNOWLEDGMENT

R.S. thanks Daniël Willemsen for his feedback.

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## APPENDIX

### A. Proofs

Proof of Lemma 1.

*Proof:* We set the first derivative of the cost function (1) equal to zero.

$$\frac{\partial Cost(B|T, s)}{\partial b_k} = \frac{\partial Cost_M(B|T, s)}{\partial b_k} + \frac{\partial Cost_B(B|T, s)}{\partial b_k} = 0 \quad (5)$$

The derivative of the market cost is simply

$$\frac{\partial Cost_M(B|T, s)}{\partial b_k} = f_k(b_k)b_k \quad (6)$$

For the backup cost, we have to consider the inner sum over subsets  $w$  in (3). We split the sum into the terms for which  $k \in w$ , see second line in (7) and into those for which  $k \notin w$ , see third line in 7.

$$\begin{aligned} \frac{\partial Cost_B(B|T, s)}{\partial b_k} = f_k(b_k) & \left[ \sum_{j \in [1, s-1]} \sum_{\substack{w \subset T \setminus \{k\} \\ s.t. |w|=j-1}} \prod_{t \in w} F_t(b_t) \prod_{t \in T \setminus (w \cup \{k\})} [1 - F_t(b_t)](s-j)\lambda_{el} \right. \\ & \left. - \underbrace{\sum_{j \in [0, s-1]} \sum_{\substack{w \subset T \setminus \{k\} \\ s.t. |w|=j}} \prod_{t \in w} F_t(b_t) \prod_{t \in T \setminus (w \cup \{k\})} [1 - F_t(b_t)](s-j)\lambda_{el}}_{\text{Prob}(x < s | T \setminus \{k\}, B_{-k})} \right] \quad (7) \end{aligned}$$

When  $k \in w$ , second line in (7), we know that  $|w| \geq 1$  and therefore the outer sum is over  $j \in [1, s-1]$ . The derivative of  $F_k(b_k)$  is  $f_k(b_k)$ . Since  $k \in w$ , the inner sum then sums over the subsets of  $w$  of size  $(j-1)$ . When  $k \notin w$ , third line in (7), we take the derivative of  $[1 - F_k(b_k)]$  resulting in  $-f_k(b_k)$ , while the outer and inner sum stay unchanged.

Next we combine the terms of the second and third line in (7) into one summation. To do so, we first extend and shift the summation in the second line of (7):

$$\sum_{j \in [1, s-1]} \sum_{\substack{w \subset T \setminus \{k\} \\ s.t. |w|=j-1}} \prod_{t \in w} F_t(b_t) \prod_{t \in T \setminus (w \cup \{k\})} [1 - F_t(b_t)](s-j)\lambda_{el}.$$

We extended the summation to iterate over the range  $[1, s]$ , which can be done as  $(s-s)\lambda_{el} = 0$ . We then shift the index  $j$  by one so that both sums iterate over  $j \in [0, s-1]$ . This only changes the tail of the second line from  $(s-j)\lambda_{el}$  to  $(s-j-1)\lambda_{el}$ . These operations alter the second line in (7) to

$$\sum_{j \in [0, s-1]} \sum_{\substack{w \subset T \setminus \{k\} \\ s.t. |w|=j}} \prod_{t \in w} F_t(b_t) \prod_{t \in T \setminus (w \cup \{k\})} [1 - F_t(b_t)](s-j-1)\lambda_{el}$$

$\underbrace{\hspace{15em}}_{\text{Prob}(x < s | T \setminus \{k\}, B_{-k})}$

Next, we combine the two terms in (7) to obtain

$$\frac{\partial Cost_B(B|T, s)}{\partial b_k} = -f_k(b_k)\text{Prob}(x < s | T \setminus \{k\}, B_{-k})\lambda_{el} \quad (8)$$

Given (6) and (8) we can rewrite (5) in the following way:

$$\frac{\partial Cost(b)}{\partial b_k} = f_k(b_k) [b_k - \text{Prob}(x < s | T \setminus \{k\}, B_{-k})\lambda_{el}]$$

Proof of Theorem 1. ■

*Proof:* Let  $B = (b_1, b_2, \dots, b_{|T|})$  be a bid vector which satisfies condition (4). Let  $b_i = 0$  for some  $i \in T$ . Since  $B$  satisfies (4), we know that  $Prob(x < s|T \setminus \{i\}, B_{-i}) = 0$ . This implies that it is certain that the agent wins  $s$  auctions from  $T \setminus \{i\}$ , which can only be true if we submit  $\lambda_{el}$  to at least  $s$  auctions. However, submitting  $\lambda_{el}$  in more than  $s$  auctions cannot satisfy (4) as it would follow that for every auction  $k \in T$  the agent wins  $s$  auctions from the remaining  $T \setminus \{k\}$  auctions with certainty. Therefore  $Prob(x < s|T \setminus \{k\}, B_{-k}) = 0$  holds, causing a bid of 0, for every auction  $k \in T$  contradicting the previous statement of bidding  $\lambda_{el}$  in more than  $s$  auctions. Thus the agent bids  $\lambda_{el}$  in exactly  $s$  auctions. By (4) the agent bids 0 in the remaining  $(|T| - s)$ . The proof works similarly for  $b_i = \lambda_{el}$ . ■

Proof of Lemma 2.

*Proof:* Given a uniform bid  $B_u = (b_u, b_u, \dots, b_u)$ , the set of equations in (4) becomes a set of the same equation and  $Prob(x < s|T \setminus \{k\}, B_{-k})\lambda_{el}$  becomes a function of  $b_u$ . Therefore, we only need to find the value of  $b_u$  for which this one equation holds:

$$b_u = Prob(x < s|T \setminus \{k\}, B_{-k})\lambda_{el}.$$

As  $Prob(x < s|T \setminus \{k\}, B_{-k})\lambda_{el}$  is  $\lambda_{el}$  for  $b_u = 0$ , 0 for  $b_u = \lambda_{el}$  and strictly decreasing and continuous in between, there can only exist one crossover point where  $b_u = Prob(x < s|T \setminus \{k\}, B_{-k})\lambda_{el}$ . ■

Proof of Lemma 2

*Proof:* Let  $B$  be a bid vector that satisfies (4) and let  $b_l$  and  $b_m$  be the bids on auctions  $l$  and  $m$ , respectively. The proof will show that either  $B \in \Lambda_s$  or  $b_l = b_m$ . Let  $B_{-l,m}$  be the bid vector  $B$  without the bids  $b_l$  and  $b_m$ . We assume  $B_{-l,m}$  to be fixed and solve (4) for  $b_l$  and  $b_m$ , respectively (derivation can be found in Appendix B):

$$\begin{aligned} b_l &= Prob(x < s|T \setminus \{l, m\}, B_{-l,m})\lambda_{el} \\ &\quad - Prob(x = s - 1|T \setminus \{l, m\}, B_{-l,m})\lambda_{el}F(b_m) \\ b_m &= \underbrace{Prob(x < s|T \setminus \{l, m\}, B_{-l,m})\lambda_{el}}_a \\ &\quad - \underbrace{Prob(x = s - 1|T \setminus \{l, m\}, B_{-l,m})\lambda_{el}F(b_l)}_c \end{aligned} \quad (9)$$

Given the constants  $a$  and  $c$  and the fact that  $F(\tau) = \frac{\tau}{\lambda_{el}}$ , (9) can be rewritten as follows:

$$\begin{aligned} b_l &= a - cb_m \\ b_m &= a - cb_l \end{aligned}$$

Subtracting the second from the first equation leaves us with  $b_l - b_m = c(b_l - b_m)$ , which can only be true if  $b_l = b_m$  or  $c = 1$ . However,  $c = 1$ , i.e.,  $Prob(x = s - 1|T \setminus \{l, m\}, B_{-l,m}) = 1$ , means winning  $s - 1$  units from auctions  $T \setminus \{l, m\}$  with certainty. This can only be true when  $B \in \Lambda_s$  and ( $b_l = 0, b_m = \lambda_{el}$ ) or ( $b_l = \lambda_{el}, b_m = 0$ ). Therefore either  $B \in \Lambda_s$  or  $B$  is a uniform bid over all auctions. ■

Proof of Lemma 4

*Proof:* The condition for a minimum is that the Hessian matrix  $M$  is positive definite[12], i.e.,  $x^T M x > 0 \quad \forall x \in \mathbb{R}^{|T|}$ . Given are the uniform bid  $B_u$  and identical uniform price distributions. The Diagonal elements of the Hessian  $M$ , are identical  $d$ , with  $d = \frac{\partial^2 Cost(b)}{(\partial b_k)^2} \Big|_{B_u} = f_k(b_u)$ , while all off-diagonal elements are

$$p = \frac{\partial^2 Cost(b)}{\partial b_l \partial b_k} \Big|_{B_u} = f_k(b_u) f_l(b_u) Prob(x = s - 1|T \setminus \{k, l\}) \lambda_{el}.$$

A derivation for the second order derivative can be found in Appendix C. Note that for uniform price distributions  $f_i(b_u)\lambda_{el} = 1$  and that for uniform bidding  $Prob(x = s - 1|T \setminus \{k, l\}) < 1$ . Therefore  $d > p$ .

$$\begin{aligned} x^T M x &= \left[ \sum_{t_1 \in T} \left( x_{t_1} \sum_{t_2 \in T \setminus \{t_1\}} x_{t_2} \right) \right] p + \sum_{t_1 \in T} x_{t_1}^2 d \\ &> \left[ \sum_{t_1 \in T} \left( x_{t_1} \sum_{t_2 \in T \setminus \{t_1\}} x_{t_2} \right) \right] p + \sum_{t_1 \in T} x_{t_1}^2 p = (x^T A x) p. \end{aligned}$$

Matrix  $A$  is 1 at every entry, making all rows linearly dependent and therefore  $A$  has rank 1 and only one non-zero eigenvalue[2]. Any eigenvalue, eigenvector  $(\lambda, x)$  pair has to satisfy  $Ax = \lambda x$ , which holds for  $x = (x_1, x_1, \dots, x_1)$  and  $\lambda = |T| > 0$ . Since all eigenvalues of  $A$  are non-negative,  $A$  is positive semi-definite[2], i.e.,  $x^T A x \geq 0 \quad \forall x \in \mathbb{R}^{|T|}$ , and therefore  $x^T M x > 0 \quad \forall x \in \mathbb{R}^{|T|}$ . ■

Proof of Theorem 2

*Proof:* Given Lemma 3 we know that uniform bidding and  $\Lambda_s$ -bidding are the only two strategies satisfying (4). Given these two options, we only need to show that uniform bidding is always at least as good as  $\Lambda_s$ -bidding. We start by showing that the theorem holds for a more restricted case of demanding  $n - 1$  units out of  $n$  auctions.

**Base Case:** For  $n = 2$  and  $s = 1$  the probability in (4) is the probability of losing one of two auctions. The uniform bid value  $b_{2,1}^u$  therefore has to satisfy  $b_{2,1}^u = [1 - F(b_{2,1}^u)] \lambda_{el} = [1 - b_{2,1}^u / \lambda_{el}] \lambda_{el}$ , which resolves to  $b_{2,1}^u = 0.5\lambda_{el}$ . The expected cost associated with the uniform bid  $B_{2,1}^u = (b_{2,1}^u, b_{2,1}^u)$  is  $Cost_u(B_{2,1}^u|2, 1) = 0.5\lambda_{el}$ . When bidding  $B \in \Lambda_s$ , the expected cost per auction won is  $0.5\lambda_{el}$ , while the expected backup cost is zero, since winning  $s$  auctions is guaranteed. Therefore,  $Cost_u(B_{n,n-1}^\lambda|n, n - 1) = 0.5(n - 1)\lambda_{el}$ . For the case of  $n = 2$  and  $s = 1$ , the cost associated with the  $\Lambda_{el}$ -Bid  $B_{2,1}^\lambda$  is  $Cost_u(B_{2,1}^\lambda|2, 1) = 0.5\lambda_{el}$  - equivalent to the cost of submitting  $B_{2,1}^u$ .

**Inductive Step:** Assuming  $Cost_u(B_{n,n-1}^u|n, n - 1) \leq Cost_u(B_{n,n-1}^\lambda|n, n - 1)$ , we show that  $Cost_u(B_{n+1,n}^u|n + 1, n) \leq Cost_u(B_{n+1,n}^\lambda|n + 1, n)$  by constructing  $\tilde{B}_{n+1,n} = (B_{n,n-1}^u, \lambda_{el})$ , i.e., bidding in  $n$  auctions according to the uniform bid  $B_{n,n-1}^u$  and submitting  $\lambda_{el}$  in the last auction. Since  $\tilde{B}_{n+1,n}$  wins the last auction with certainty,  $F(\lambda_{el}) = 1$ ,  $Cost(\tilde{B}_{n+1,n}|n + 1, n)$  can be expressed as  $Cost(\tilde{B}_{n+1,n}|n + 1, n) = Cost_u(B_{n,n-1}^u|n, n - 1) +$

$0.5\lambda_{el}$ . For an equivalent reason,  $Cost_u(B_{n+1,n}^\lambda|n+1,n) = Cost_u(B_{n,n-1}^\lambda|n,n-1) + 0.5\lambda_{el}$ . Using the inductive assumption, we get

$$Cost_u(\tilde{B}_{n+1,n}|n+1,n) \leq Cost_u(B_{n+1,n}^\lambda|n+1,n). \quad (10)$$

Since by Lemma 3  $\tilde{B}_{n+1,n}$  is not a local minimum, its associated cost has to be strictly larger than the cost of at least one local minimum, which by (10) cannot be at bid  $B_{n+1,n}^\lambda$ . The only alternative according to Lemma 3 is the uniform bid  $B_{n+1,n}^u$ , which by Lemma 4 is always a local minimum. Therefore,

$$\begin{aligned} Cost_u(B_{n+1,n}^u|n+1,n) &< Cost_u(\tilde{B}_{n+1,n}|n+1,n) \\ &\leq Cost_u(B_{n+1,n}^\lambda|n+1,n). \end{aligned}$$

Having shown that the theorem holds for  $n-1$  out of  $n$  auctions, we next generalize this result to the setting of  $s$  units out of  $n$  auctions. To do so, we introduce  $\tilde{B}_{n,s} = (B_{s+1,s}^u, 0)$ , i.e., bidding uniformly as if there only existed  $s+1$  auctions and submitting 0 to all other auctions. Note that adding auctions in which the agent bids zero has no effect on the cost, because a bid of zero incurs no market cost and also does not change the probability of needing the backup generator. Therefore,  $Cost_u(\tilde{B}_{n,s}|n,s) = Cost_u(B_{s+1,s}^u|s+1,s)$  as well as  $Cost_u(B_{n,s}^\lambda|n,s) = Cost_u(B_{s+1,s}^\lambda|s+1,s)$ . Since we already established the validity of the theorem for  $n-1$  out of  $n$  auctions, we know  $Cost_u(B_{s+1,s}^u|s+1,s) \leq Cost_u(B_{s+1,s}^\lambda|s+1,s)$  and using this with the previous two equalities we get  $Cost_u(\tilde{B}_{n,s}|n,s) \leq Cost_u(B_{n,s}^\lambda|n,s)$ . Since,  $\tilde{B}_{n,s}$  is not a local minimum, its associated cost has to be larger than at least one local minimum and therefore with the help of the previous equation we know that

$$Cost_u(B_{n,s}^u|n,s) < Cost_u(\tilde{B}_{n,s}|n,s) \leq Cost_u(B_{n,s}^\lambda|n,s).$$

## B. Solving for a Pair of Bids

We assume a fixed bid vector  $B_{-l,m}$  and try to find the bids  $b_l$  and  $b_m$  such that (4) is satisfied. The procedure is the same for  $b_l$  and  $b_m$ , we will therefore only do it for  $b_l$ . Recall (4).

$$b_l = Prob(x < s|T \setminus \{l\}, B_{-l})\lambda_{el}$$

. Winning fewer than  $s$  units from  $T \setminus \{l\}$  is the probability of winning fewer than  $s$  units when losing auction  $m$  plus the probability of winning fewer than  $s-1$  units when winning auction  $m$ .

$$\begin{aligned} b_l &= F(b_m)Prob(x < s|T \setminus \{l, m\}, B_{-l,m})\lambda_{el} \\ &+ [1 - F(b_m)]Prob(x < s-1|T \setminus \{l, m\}, B_{-l,m})\lambda_{el} \end{aligned}$$

This can be rewritten as

$$\begin{aligned} b_l &= Prob(x < s|T \setminus \{l, m\}, B_{-l,m})\lambda_{el} \\ &- Prob(x = s-1|T \setminus \{l, m\}, B_{-l,m})\lambda_{el}F(b_m) \end{aligned}$$

## C. Second Derivatives

We derive the second derivative for when submitting a uniform bid that satisfies (4). Recall that the first order derivative is

$$\frac{\partial Cost(b)}{\partial b_k} = f_k(b_k) [b_k - Prob(x < s|T \setminus \{k\}, B_{-k})\lambda_{el}]$$

Therefore,

$$\begin{aligned} \frac{\partial^2 Cost}{(\partial b_k)^2} &= \frac{\partial f_k(b_k)}{\partial b_k} \overbrace{[b_k - Prob(x < s|T \setminus \{k\}, B_{-k})\lambda_{el}]}^{=0} \\ &+ f_k(b_k) \frac{\partial}{\partial b_k} [b_k - Prob(x < s|T \setminus \{k\}, B_{-k})\lambda_{el}] \end{aligned}$$

Note, that the first term in the above equation is zero because  $B_u$  satisfies (4) and that  $Prob(x < s|T \setminus \{k\}, B_{-k})$  is not a function of  $b_k$ . Therefore,  $\left. \frac{\partial^2 Cost}{(\partial b_k)^2} \right|_{B_u} = f_k(b_k)$ . Similarly,

$$\frac{\partial^2 Cost}{\partial b_k \partial b_l} = -f_k(b_k) \frac{\partial}{\partial b_l} Prob(x < s|T \setminus \{k\}, B_{-k})\lambda_{el} \quad (11)$$

in which  $Prob(x < s|T \setminus \{k\}, B_{-k})$  can be expressed more verbosely as

$$\sum_{j \in [0, s-1]} \sum_{\substack{w \subset T \setminus \{k\} \\ s.t. |w|=j}} \prod_{t \in w} F_t(b_t) \prod_{t \in T \setminus (w \cup \{k\})} [1 - F_t(b_t)]$$

The following derivation works similar to what has been done in the proof for Lemma 1. For the derivative of the above mentioned term we need to separately consider when  $l \in w$  and when  $l \notin w$ . When  $l \in w$ , we know that  $|w| > 1$ , causing the outer sum to run over the interval  $[1, s-1]$ , take the derivative of  $F_l(b_l)$  and obtain the following

$$f_l(b_l) \sum_{j \in [1, s-1]} \sum_{\substack{w \subset T \setminus \{k, l\} \\ s.t. |w|=j-1}} \prod_{t \in w} F_t(b_t) \prod_{t \in T \setminus (w \cup \{k, l\})} [1 - F_t(b_t)].$$

We shift the outer sum by one resulting in

$$\underbrace{f_l(b_l) \sum_{j \in [0, s-2]} \sum_{\substack{w \subset T \setminus \{k, l\} \\ s.t. |w|=j}} \prod_{t \in w} F_t(b_t) \prod_{t \in T \setminus (w \cup \{k, l\})} [1 - F_t(b_t)]}_{Prob(x < s-1|T \setminus \{k, l\}, B_{-k, l})} \quad (12)$$

When  $l \notin w$ , the outer sum is not affected and we take the derivative of  $[1 - F_l(b_l)]$  and therefore obtain

$$\underbrace{-f_l(b_l) \sum_{j \in [0, s-1]} \sum_{\substack{w \subset T \setminus \{k, l\} \\ s.t. |w|=j}} \prod_{t \in w} F_t(b_t) \prod_{t \in T \setminus (w \cup \{k, l\})} [1 - F_t(b_t)]}_{Prob(x < s|T \setminus \{k, l\}, B_{-k, l})} \quad (13)$$

Combining (12) and (13) and substituting the term in (11) we obtain:

$$\frac{\partial^2 Cost}{\partial b_k \partial b_l} = f_k(b_k) f_l(b_l) Prob(x = s-1|T \setminus \{k, l\}, B_{-k, l})\lambda_{el}.$$