CONFERENTIE VAN NUMERIEK WISKUNDIGEN

15, 16 en 17 oktober 1979

CONFERENTIEOORD WOUDSCHOTEN
ZEIST

Werkgemeenschap Numerieke Wiskunde

VIERDE CONFERENTIE NUMERIEKE WISKUNDE

Uitgave verzorgd door het
MATHEMATISCH CENTRUM
AMSTERDAM

DOEL VAN DE CONFERENTIE

Het doel van de conferentie is hen die betrokken zijn bij onderzoek en/of onderwijs in de numerieke wiskunde de gelegenheid te geven kennis te nemen van de stand van zaken op enkele gebieden van de numerieke wiskunde en de gelegenheid te geven met elkaar en met buitenlandse sprekers intensief van gedachte te wisselen.

THEMA

Het conferentiethema is "Numerieke behandeling van slecht gestelde problemen in analyse en algebra".

ORGANISATIE

De organisatie is in handen van de voorbereidingscommissie bestaande uit de heren Dekker (UvA), Slagt (MC), van de Vooren (RUG) en Wesseling (THD), en van het Mathematisch Centrum.

SPREKERS

H. BRUNNER, Dalhousie University, Halifax, Canada en MC.

L. ELDÉN, Linköping University, Linköping, Sweden.

K.P. HADELER, Universität Tübingen, Tübingen, BRD.

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J.H. WILKINSON, National Physical Laboratory, Teddington, UK.

Bovendien hebben 3 deelnemers zich bereid verklaard in een korte bijdrage een schets van hun onderzoek op het gebied van het conferentiethema te geven, t.w:

J.J.M. CUPPEN, Universiteit van Amsterdam, Amsterdam.

R. VAN DER HOUT, AKZO Research, Arnhem.

A.I. VAN DE VOOREN, Rijksuniversiteit, Groningen.

PROGRAMMA

maandag 15 oktober

10.00-11.15	aankomst en koffie	15.15-15.45	thee
11.15-12.30	opening, Wilkinson	15.45-16.45	Eldén
12.45	lunch	17.00-18.00	borrel
14.15-15.15	Natterer	18.15	diner

dinsdag 16 oktober

8.00	ontbijt	12.45	lunch
9.00-10.00	Hadeler	15-15-15-45	thee
10.00-10.30	koffie	15.45-16.45	Eldén
10.30-11.30	Brunner	16.45-17.45	Natterer
11.30-12.00	van de Vooren	18.15	diner
12.00-12.30	Cuppen		

woensdag 17 oktober

8.00	ontbijt	12.30	lunch
9.00-10.00	Wilkinson	13.45-14.45	Hadeler
10.00-10.30	koffie	14.45	sluiting, thee
10.30-11.30	Brunner		vertrek
11.30-12.00	van der Hout		

TITELS EN SAMENVATTINGEN VOORDRACHTEN

Monday, 15 october

11.15 opening

J.H. Wilkinson: The generalized eigenvalue problem $Ax = \lambda Bx$ when the pencil $A - \lambda B$ is (almost) singular I.

14.15 F. Natterer: Regularization of ill-posed problems by optimal discretization.

15.45 L. Eldén: The numerical solution of parabolic problems backwards in time (theory and methods).

Tuesday, 16 october

9.00 K.P. Hadeler: Numerical approaches to non-linear delay equations I.

10.30 H. Brunner: Some basic aspects of numerical methods for Volterra and Abel integral equations of the first kind.

11.30 A.I. van de Vooren: The numerical solution of the Orr-Sommerfeld equation.

12.00 J.J.M. Cuppen: Some methods for the estimation of an optimal regularization parameter.

15.45 L. Eldén: The numerical solution of parabolic problems backwards in time (algorithms for problems in two space dimensions).

16.45 F. Natterer: The ill-posedness of Radon's integral equation.

Wednesday, 17 october

9.00 J.H. Wilkinson: The generalized eigenvalue problem $Ax = \lambda Bx$ when the pencil $A - \lambda B$ is (almost) singular II.

10.30 H. Brunner: Collocation methods for first-kind integral equations of Volterra type.

11.30 R. van der Hout: Extension of the Gauss-Newton method to under-determined systems of non-linear equations.

13.45 K.P. Hadeler: Numerical approaches to non-linear delay equations II.

SOME BASIC ASPECTS OF THE NUMERICAL METHODS FOR VOLTERRA AND ABEL INTEGRAL EQUATIONS OF THE FIRST KIND

H. Brunner,
Dalhousie University, Halifax (Canada) and
MC, Amsterdam

ABSTRACT

First-kind integral equations of the form

$$(1) \qquad (Ty)(t) = g(t), \quad t \in I := [a,b]$$

where the integral operator T is given by

$$(1a) \qquad (Tf)(t) := \int_a^t \frac{K(t,s)}{(t-s)^{\alpha}} f(s)ds \quad (0 \le \alpha < 1),$$

or by

$$(1b) \qquad (Tf)(t) := \int_a^t \frac{f(s)ds}{(t^2-s^2)^{\frac{1}{2}}},$$

arise in many physical problems (compare, for example [1]). A large variety of methods for the numerical solution of (1) exists ([3]; [1], [2], [4], [6] - [13]); our discussion will concentrate on a number of aspects (continuous versus weakly singular kernels; convergence and accuracy; change of stepsize; noisy data; computational efficiency) common to such methods, in order to exhibit their respective advantage and weakness. Numerical examples will serve to illustrate the analysis.

COLLOCATION METHODS FOR FIRST-KIND INTEGRAL EQUATIONS OF VOLTERRA TYPE

H. Brunner,
Dalhousie University, Halifax (Canada) and
MC, Amsterdam

ABSTRACT

Collocation methods which are based on piecewise polynomial spaces (of a given degree of continuity) have proved to be rather powerful and flexible tools for the numerical solution of (1), both for continuous and for weakly singular kernels. Among the topics discussed in this talk will be: (i) choice of collocation points and discretization of the collocation equation; (ii) degree of continuity of the approximating space versus convergence, (iii) ill-posedness of the discrete problem; (iv) implementation and open problems.

REFERENCES (for both talks)

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SOME METHODS FOR THE ESTIMATION OF AN OPTIMAL REGULARIZATION PARAMETER

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ABSTRACT

Consider the linear Fredholm integral equation of the first kind

$$\int_0^1 k(s,t)f(t)dt = g(s), \quad s \in [0,1],$$

where the kernel k is smooth and the right hand side g is subject to noise. We shall consider regularised solutions to this problem obtained by three methods:

1) Tychonov's method,
2) Filtered least squares minimal C-norm approximation (based on truncation of the S.V.D.),
3) Regularisation by coarse discretisation in cooperation with method 2.

For all these methods the selection of the amount of regularisation, usually governed by some regularisation parameter, in crucial. Three methods of estimating the optimal choice of the regularisation parameter shall be given:

a) regularising as much as possible while keeping the residue below a prescribed bound.
b) regularising as little as possible while holding the regularised solution to a prescribed level of smoothness,
c) minimizing an error bound for the singularised solution.

The results of a number of numerical experiments show that especially b) and c) lead to rather good results if a good estimate is known of, respectively, the smoothness of the exact solution or its ratio to the residue of

the exact solution in the perturbed equation. The experiments also show that the regularisation method 3) can achieve results competitive with those of method 1) while working with a much lower order discretisation.

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THE NUMERICAL SOLUTION OF PARABOLIC PROBLEMS BACKWARDS IN TIME

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ABSTRACT

We consider the partial differential equation

$$\begin{cases} u_t = -Lu & \text{in } \Omega \times [0,1], \\ u = 0 & \text{on the boundary } \partial\Omega \times [0,1], \\ u(x,1) = \omega(x), \end{cases}$$

where L is a selfadjoint, elliptic operator with smooth coefficients independent of t. This is the problem of solving a parabolic equation backward in time and it is improperly posed in the sense that the solution (if it exists) does not depend continuously on the data. Using logarithmic convexity (see e.g. [7]) a stability estimate can be derived for the following constrainted problem

$$\begin{cases} u_t = -Lu & \text{in } \Omega \times [0,1], \\ u = 0 & \text{on } \partial\Omega \times [0,1], \\ \|u(\cdot,1) - \omega(\cdot)\| \le \delta, \\ \|u(\cdot,0)\| \le M, \end{cases} \tag{1}$$

where δ and M are some known positive constants. Suppose that u_1 and u_2 are two arbitrary solutions of (1). Then

$$\|u_1(\cdot,t) - u_2(\cdot,t)\| \le 2M^{1-t}\delta^t \tag{2}$$

It is difficult to compute a solution of (1) (in general there is no unique solution), and various numerical methods have been devised, which produce solutions close to any solution of (1). A number of such methods are considered:

a) Fourier methods;

b) the regularization method [8], [4], [6];

c) the backward beam method [1];

d) quasi-reversibility methods [5];

e) the pseudo-parabolic equation method [3];

f) iterative methods.

These methods are discussed from the point of view of efficient numerical computations. When a problem in two (or more) space dimensions is solved by any of the above methods, it is necessary to take advantage of the structure of the problem in order to save arithmetic operations and computer storage. By structure we here mean the sparseness of discretizations of the elliptic operator.

In the methods e) and f) above sparseness can easily be utilized, since these methods can be implemented by marching procedures (as in *forward* parabolic equations). In the discrete versions of b) - d) the solutions can be written essentially in the form [2], [5],

$$u = (Q(C))^{-1}P(C)\omega,$$

where Q and P are polynomials, and C is a discretization of L. In two-dimensional problems the matrix Q(C) is a huge, dense matrix, and therefore it should never be formed explicitly. Instead the solution can be computed by factorizing Q in quadratic factors, and by solving a sequence of linear systems [2]

$$(\alpha C^2 + \beta C + \gamma I)x_i = y_i \qquad (3)$$

Even if the matrix C^2 is not as sparse as C, its structure can be utilized (solving (3) is about as complicated as solving a fourth order elliptic equation). Algorithm for solving (3) are discussed, especially in the case when separation of variables can be applied.

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NUMERICAL APPROACHES TO NON-LINEAR DELAY EQUATIONS

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ABSTRACT

A delay equation (retarded differential equation, differential difference equation) is a differential equation, where the derivative $x(t)$ at time t is a function not only of t and $x(t)$, but also of values $x(s)$ for $s < t$. Such equations occur quite naturally, whenever in a system the present change of the state variable depends on the history of the system or if state variables act with delays upon each other.

A typical example is the scalar equation

$$\dot{x}(t) = f(t,x(t), x(t-1)).$$

More general equations may contain several retarded arguments or even distributed delays.

In particular Biology provides many examples of delay equations describing e.g. the evolution of single or interacting populations, the production of blood cells, the action of enzymes, or the transmission of sensory inputs.

Most models from Biology describe oscillatory phenomena. It is easy to conceive that sufficiently long delays may cause oscillations in an otherwise non-oscillatory system ("bicycle rider with slow reaction"). In many cases oscillations attenuate to a stable periodic oscillation.

For this reason the existence and the stability properties of periodic solutions have been studied recently with great intensity. In the first lecture we shall give an outline of some basic approaches and results: The stability of the non-linear equation via linearization, characteristic equations,

shift operator, periodic solutions as fixed points of the shift operator, existence of fixed points, stability of periodic solutions. Bifurcation of periodic solutions.

In the second part of the lecture we shall concentrate on the numerical solution of delay equations. In a way every method for ordinary differential equations gives rise to a class of methods for delay equations, and the major part of the convergence theory for discrete methods can be formally carried over to delay equations.

However, the actual application of such methods causes great difficulties, since usually additional interpolations are necessary.

Of course the periodic solution mentioned earlier should be computed numerically. Here serious problems arise from the infinite-dimensional state space and from the fact that the trajectories of delay equations cannot be followed backward. Possible approaches are discussed in detail.

EXTENSION OF THE GAUSS-NEWTON METHOD TO UNDERDETERMINED SYSTEMS OF NONLINEAR EQUATIONS

R. van der Hout
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Let $f: \mathbb{R}^n \to \mathbb{R}^k$ be continuously differentiable and let $K = \{x \in \mathbb{R}^n \mid f(x) = 0\}$ be nonempty. We adopt the following conventions and notations:

(i) The Jacobian of f at $x \in \mathbb{R}^n$ is denoted by J_x.

(ii) $x \in \mathbb{R}^n$ is called f-regular if the rank of J_y is constant over a neighbourhood of x.

(iii) For $x \in \mathbb{R}^n$, $\bar{x}$ is a point $\in K$ such that $|x - \bar{x}|$ is minimal; we also write $d(x,K)$ instead of $|x - \bar{x}|$. Note that $\bar{x}$ is not necessarily unique.

Let $x_0 \in K$ be f-regular and suppose that x_0 has a neighbourhood W together with constants $\gamma > 0$ and $\tau > 0$ such that $|f(x) - f(y) - J_x(x-y)| \leq \gamma |x-y|^{1+\tau}$ for all $x,y \in W$. Let P be a positive definite, symmetric $n \times n$ matrix. Define an iterative process by $x_{k+1} = x_k - \lim_{\alpha \downarrow 0} (J_{x_k}^T K_{x_k} + \alpha P)^{-1} J_{x_k}^T f(x_k)$.
We shall prove the following results:

(i) The process is well-define.

(2) There exists an $\varepsilon > 0$ such that the process, if started in x_1 satisfying $|x_1 - x_0| < \varepsilon$, converges to some $a \in K$. There are constants $B > 0$ and $C > 0$ such that $B^{\frac{1}{\tau}} \varepsilon < 1$ and

(a) $d(x_{k+1},K) \leq B d(x_k,K)^{\tau+1}$

(b) $|x_k - a| \leq C(B^{\frac{1}{\tau}} \varepsilon)^{((\tau+1)^{(k-1)})}$.

As a corollary we have quadratic convergence if J satisfies a Lipschits-condition in W. Our method is more general than the one contained in [1], which is restricted to the case where the Jacobians J_{x_k} have full rank.

We plan to incorporate inequalities in our method as well as a modification for stepsizes in "far" points, such as to enlarge the range of convergence

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REGULARIZATION OF ILL-POSED PROBLEMS BY OPTIMAL DISCRETIZATION

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ABSTRACT

Consider the problem $Ax = y$ where $A: X \to Y$ is a linear bounded operator and X, Y are normed linear spaces. Assume that A has an inverse A^{-1} which is unbounded. Then our problem is ill-posed in the X-Y-setting.

Consider a discrete version $A_h x_h = y_h$ of our problem where h is the discretization parameter. For h small A_h approximates A closely, and we expect the condition number $\kappa(A_h)$ to be large. For h large A_h approximates A poorly, but $\kappa(A_h)$ need not be large. Thus we see that discretization has a regularizing side-effect and h plays the role of the regularization parameter which has to be chosen in an optimal way. The question is whether or not the accuracy obtained by choosing an optimal h is comparable to the accuracy of other regularization techniques such as the Tikhonov-Phillips method. The answer is affirmative for some discretization methods of the projective type.

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THE ILL-POSEDNESS OF RADON'S INTEGRAL EQUATION

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ABSTRACT

The Radon transform

$$(Rf(s,\omega) = \int f(s\omega + t\omega^{\perp})dt$$

($s \in \mathbb{R}^1$, $\omega \in \mathbb{R}^2$, $|\omega| = 1$, $\omega \cdot \omega^{\perp} = 0$) is considered as an operator from $L_2(\Omega)$, Ω the unit disk, into $L_2(Z)$, Z the unit cylinder in $\mathbb{R}^3$. Using the equivalence of the norms

$$\|Rf\|_{L_2(Z)}, \quad \|f\|_{H^{-1/2}(\Omega)} \qquad (H^{\alpha} \text{ Sobolev-space})$$

(which shows that the equation Rf = g is ill-posed in the $L_2(\Omega)$-$L_2(Z)$-setting), several theorems concerning the solution of Rf = g with incomplete and noisy data are proved.

Another series of theorems makes use of the fact that the range of R and related operators is rather pecular. This has applications to the solution of Rf = g if g is known only on a section of Z and to the inversion of the attenuated Radon transform R_{μ} without knowing the attenuation μ.

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THE NUMERICAL SOLUTION OF THE ORR-SOMMERFELD EQUATION

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ABSTRACT

Usual methods applied to the fourth-order Orr-Sommerfeld ordinary differential equation give use to exponentially growing solutions (in both directions) which are physically incorrect. We shall give an outline of a newly developed method which does not suffer from this phenomenon.

THE GENERALIZED EIGENVALUE PROBLEM $Ax = \lambda Bx$ WHEN THE PENCIL $A-\lambda B$ IS (ALMOST) SINGULAR

J.H. WILKINSON

The material in the two lectures is closely related and the break-point will be made as proves convenient.

The solution of the *generalized* eigenvalue problem

$$Ax = \lambda Bx \tag{1}$$

arises in many branches of applied mathematics, using that term in its broadest sense. The problem is of importance primarily because its close relationship with that of solving the differential system

$$B\frac{dx}{dt} = Ax + f, \tag{2}$$

where f is a vector of (forcing) functions of t. When B is I equation (1) reduces to

$$Ax = \lambda x \tag{3}$$

usually referred to as the *standard* eigenvalue problem, while (2) reduces to

$$\frac{dx}{dt} = Ax + f \tag{4}$$

usually referred to as an *explicit* system of D.E.'s. The theoretical background to these problems is provided by similarity theory and an understanding of the structure of the general solution of (4) is provided by the Schur canonical form and the Jordan canonical form. The former can be obtained by stable (unitary) transformations but the latter is inherently unstable and may be regarded as ill-posed.

In problem (4) we may investigate the existence of solutions and, when they exist, the general structure of such solutions even when A and B are $m \times n$ matrices. Even when the given system involves square matrices, related

reduced systems may be rectangular and hence the rectangular case is of general relevance.

When A and B are square Stewart has shown that there are unitary transformations such that QAZ and QBZ are both upper-triangular. (This may be regarded as a generalization of Schur's canonical form). This theorem is the basis of the elegant and powerful QZ algorithm of Moler and Stewart [4]. This algorithm is backward stable in practice and this desirable feature has led to a widespread misunderstanding of the significance of the results given by the QZ algorithm. The misunderstanding arises chiefly from a lack of appreciation of the potentially ill-posed nature of the generalized eigenvalue problem.

The classical background [1] for the general problem is provided by the Weierstrass and Kronecker canonical forms under strict equivalence of the linear pencil $A-\lambda B$. Although the Jordan canonical form (J.c.f.) is widely referred to by numerical analysts discussion of the W.c.f. and K.c.f. is comparatively rare. The Weierstrass canonical form covers the case of *regular* pencils, ie when A and B are square and $\det(A-\lambda B) \not\equiv 0$. There are then nonsingular P and Q, independent of λ, such that $\tilde{A}-\lambda\tilde{B} \equiv P(A-\lambda B)Q$ is block diagonal each diagonal block being of one or other of the forms $J_{k_i}(\lambda_i) - \lambda I_{k_i}$ or $I_{m_i} - \lambda J_{m_i}(0)$; here $J_s(\alpha)$ denotes an elementary Jordan block of order s associated with α. Diagonal blocks of the first form are associated with finite eigenvalues of $Ax = \lambda Bx$, while blocks of the second form are associated with 'infinite' eigenvalues, ie zero eigenvalues of $Bx = \mu Ax$. Assuming a knowledge of the J.c.f. the existence of the W.c.f. is comparatively straightforward.

When A and B are rectangular or are square with $\det(A-\lambda B) \equiv 0$ the pencil is said to be *singular*. Kronecker showed that in this case there still exist nonsingular P and Q (independent of λ) such that $\tilde{A}-\lambda\tilde{B}$ is block diagonal but additional blocks to those existing in the W.c.f. are required. These additional blocks are denoted by L_k and L_k^T where eg

$$L_3 = \begin{bmatrix} 0 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{bmatrix}, \qquad L_2^T = \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}.$$

In general L_k is a $k \times (k+1)$ matrix and L_k^T is a $(k+1) \times k$ matrix. Obviously when $m > n$ there must be m-n more blocks of type L_k^T than of type L_k in order to give the right dimension and conversely when $m < n$. When $m = n$ there must be the same number of each. The dimensions of the L_k and L_k^T in the K.c.f. are characteristic of the pencil and are known as the minimal indices of $A-\lambda B$. The K.c.f. reduces to the J.c.f. when $A-\lambda B$ is $A-\lambda I$. The existence or non-existence of solutions of (4) and their structure is completely exposed by the K.c.f. The classical proofs of the K.c.f. are based on considerations of the left-hand and righ-hand null vectors (with polynomial elements in λ) of $A-\lambda B$ but the techniques used do not provide a basis for practical algorithms.

When the QZ is used the final product is two upper triangular matrices $\tilde{A}$ and $\tilde{B}$ and the eigenvalues are $\tilde{a}_{ii}/\tilde{b}_{ii}$. When both $\tilde{a}_{ii}$ and $\tilde{b}_{ii}$ are 'small' then the corresponding eigenvalue is obviously poorly determined. However it is not generally realised that when for any one value of i both $\tilde{a}_{ii}$ and $\tilde{b}_{ii}$ are small then in general even the λ_j given by ratios of other $\tilde{a}_{jj}$ and $\tilde{b}_{jj}$, *neither of which is small*, are poorly determined. This will be illustrated by a constructive proof of Stewart's theorem which shows that when the pencil $A-\lambda B$ is exactly singular, then in general there are no true eigenvalues (or rather, elementary divisors of $A-\lambda B$). (This is discussed in references [5]. I shall bring preprints.) An understanding of this phenomenon is best provided by the K.c.f.

A proof of the K.c.f. which lends itself to an algorithmic development is provided quite naturally by consideration of the differential system (2) when m and n are possibly different. The attempts to determine compatibility conditions which are necessary for the existence of the solution and also the number of arbitrary functions in the solutions when it exists lead immediately to the concept of the minimal indices. (This is discussed in reference [3]. Again I shall bring copies.) These considerations lead naturally to a quasi-K.c.f. which is simpler, gives the minimal indices and can be achieved by unitary transformations and is therefore at least backward stable.

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3.	Baker, P.M.	Rijksuniversiteit	Utrecht
4.	Bollen, J.A.M.	Technische Hoogschool	Eindhoven
5.	Bylsma, S.J.	K.N.M.I.	de Bilt
6.	Botta, E.F.F.	Rijksuniversiteit	Groningen
7.	Brunner, H.	Dalhousie University en MC	Amsterdam
8.	Bus, J.C.P.	Mathematisch Centrum	Amsterdam
9.	Cuppen, J.J.M.	Universiteit van A'dam	Amsterdam
10.	Dekker, K.	Technische Hogeschool	Eindhoven
11.	Dekker, Th.J.	Universiteit van A'dam	Amsterdam
12.	Dijkstra, D.	Technische Hogeschool Twente	Eindhoven
13.	Eldén, L.	Linköping University	Linköping
14.	Gee, M. de	Rijksuniversiteit	Utrecht
15.	Geurts, A.J.	Technische Hogeschool	Eindhoven
16.	Ginneken, C.J.J.M van	Technische Hogeschool	Eindhoven
17.	Gragert, P.K.H.	Technische Hogeschool Twente	Enschede
18.	Griend, J.A. van de	Rijksuniversiteit	Leiden
19.	Groen, P.P.N. de	Vrije Universiteit	Brussel
20.	Groot, J. de	Philips	Eindhoven
21.	Gustafsson, E.I.	Kath. Universiteit	Nijmegen
22.	Hadeler, K.P.	Universität Tübingen	Tübingen
23.	Hagebeuk, H.J.L.	Technische Hogeschool	Eindhoven
24.	Hemker, P.W.	Mathematisch Centrum	Amsterdam
25.	Hilhorst, D.	Mathematisch Centrum	Amsterdam
26.	Hoffmann, W.	Universiteit van A'dam	Amsterdam
27.	Hollenberg, J.P.	Rijksuniversiteit (RC)	Groningen
28.	Hout, R. van der	Akzo	Arnhem
29.	Houwen, P.J. van der	Mathematisch Centrum	Amsterdam
30.	Jansen, J.K.M.	Technische Hogeschool	Eindhoven
31.	Jong, L.S. de	Technische Hogeschool	Eindhoven
32.	Kok, J.	Mathematisch Centrum	Amsterdam
33.	Laan, C.G. van der	Rijksuniversiteit (RC)	Groningen

34. Laan, R.C.	I.T.C.	Enschede
35. Laan-de Klerk P.	Technische Hogeschool Twente	Enschede
36. Leenderstse, G.P.	E.C.N.	Petten
37. Matén, E.J.W. ter	Rijksuniversiteit	Utrecht
38. Mattheij, R.M.M.	Kath. Universiteit	Nijmegen
39. Mol, W.J.A.	Mathematisch Centrum	Amsterdam
40. Natterer, F.	Universität des Saarlandes	Saarbrücken
41. Ouden, A.C.B. den	E.C.N.	Petten
42. Piers, W.J.	Nat. Lucht. en Ruimtevaart Lab.	Amsterdam
43. Riele, H.J.J. te	Mathematisch Centrum	Amsterdam
44. Schippers, H.	Mathematisch Centrum	Amsterdam
45. Slagt, E.	Mathematisch Centrum	Amsterdam
46. Sleijpen, G.L.G.	Rijksuniversiteit	Utrecht
47. Sluis, A. van der	Rijksuniversiteit	Utrecht
48. Spijker, M.N.	Rijksuniversiteit	Leiden
49. Stijn, Th. L. van	Rijksuniversiteit	Groningen
50. Stroeker, R.J.	Erasmusuniversiteit	Rotterdam
51. Traas, C.R.	Technische Hogeschool Twente	Enschede
52. Tusscher, A.B.G.M. ten	Kon. Shell. Lab.	Rijswijk
53. Veldhuizen, M. van	Vrije Universiteit	Amsterdam
54. Veltkamp, G.W.	Technische Hogeschool	Eindhoven
55. Verwer, J.G.	Mathematisch Centrum	Amsterdam
56. Vooren, A.I. van de	Rijksuniversiteit	Groningen
57. Vries, H.B. de	Mathematisch Centrum	Amsterdam
58. Weber, C.	Philips	Eindhoven
59. Wesseling, P.	Technische Hogeschool	Delft
60. Wetterling, W.W.E.	Technische Hogeschool Twente	Enschede
61. Wilkinson, J.	National Physical Lab.	Teddington
62. Wolkenfelt, P.H.M.	Mathematisch Centrum	Amsterdam
63. Woude, M. van de	Technische Hogeschool	Eindhoven
64. Wuytack, L.	Universitaire Instelling	Antwerpen