Large Deviations for Semi-exponential Distributions: Theory and Applications

Mihail Bazhba

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Mihail Bazhba

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Contents

Chapter 1

Introduction

The understanding of rare events is extremely important to our society. Rare events like the 2020 pandemic, the nancial crisis in 2008, and the tsunami in Japan have a major impact on the economy and the individual. It can be challenging to obtain insights into rare events; usually there is lack of data from real life-measurements and large scale simulation experiments can be expensive. For this reason, it is essential that we develop models and theoretical foundations that can give intuition and accurate estimates for the rare events under consideration.

A prominent eld of research within probability theory is large deviations theory. Large deviations theory provides physical insights that can be extremely useful for design and performance evaluation; see [90]. Other application areas include climatology ([37, 82]), engineering ([75, 90]), nance/insurance ([35, 77]), communication networks ([74]), and logistics ([91]). Large deviations have been extremely successful in providing systematic tools for the understanding of rare events that arise in stochastic systems; large deviations theory can be considered as the bridge between rare events, optimization theory, and probability theory. The theory also provides intuition about the most likely realization of a rare event.

Nevertheless, not all rare events can be modeled in the same way. There are two di erent classes of probability distributions that model certain types of rare events: light-tailed distributions and heavy-tailed distributions. The distinguishing feature is that some phenomena are \less extreme" | the probability of associated extreme values is relatively small (light-tailed) | whereas other events are \more extreme" | that is to say, the probability of associated extreme values is relatively big (heavy-tailed). Whether distributions are light or heavy makes a huge difference. There is a structural difference in the way rare events manifest themselves when the underlying uncertainties are heavy-tailed or light-tailed. In light-tailed settings, the system-wide rare events arise because of small deviations of every component in the system (conspiracy principle), whereas, in heavy-tailed settings, the system-wide rare events arise because of extreme deviations of a few components which shock the system (catastrophe principle). The large deviations theory has been very successful when the underlying uncertainties are light-tailed. To illustrate how rare events manifest themselves, suppose that the average height of a group of people is more than two meters. Then it is highly probable that a considerable number of them have a height exceeding two meters. On the other hand, if the average wealth of a group of people is in the billions, then we would expect to have an extremely wealthy individual in the group. The former example corresponds to the light-tailed (normal) distribution of height, while the latter one refers to the heavy-tailed (Pareto) distribution of wealth. Some examples where heavy tails occur are file sizes stored on a server [83], transmission rates of files [83], social networks [1, 26, 93], and financial models [62].

While the research line on rare events for heavy-tails is not as mature as its light-tailed counterpart, there has been a lot of progress regarding the theory of heavy tails [43, 62, 16]. More specifically, in [71], it is shown that extreme behaviors of the sum of heavy-tailed random variables are determined by a single large summand. This phenomenon has been documented as the principle of a single big jump and relates (intuitively) to the wealth distribution paradigm where the wealth distribution exhibits a heavy-tailed behavior. Nevertheless, not all applications can be explained by the principle of a single big jump. Recent applications in insurance/finance ([6]), communication ([16]), and social networks ([93]) led to problems that cannot be dealt using the single big jump phenomenon. It can be that multiple big jumps are necessary to cause a rare event. For example, it may require the simultaneous download of several big files to saturate a link in a communication network ([83]).

This thesis aims to contribute to the foundation of heavy-tailed large deviations, allowing scenarios of multiple big jumps. In the next section, we introduce the reader to the basic definitions and results related to heavy-tails and the theory of large deviations along with their interactions.

1.1 Heavy tails

Heavy-tailed distributions (probability measures) play a major role in the analysis of many stochastic systems. In this section, we present some basic definitions regarding heavy-tailed random variables and their probability measures. Let $\{X_n\}_{n\geq 1}$ be a sequence of identically distributed and independent random variables. Denote by F the distribution of X_1. Let $\bar{F}$ be the tail function so that $\bar{F}(x) = \mathbf{P}(X \in (x, \infty))$. Let $M_{X_1}(t) \triangleq \int_{\mathbb{R}_+} e^{tx}F(dx)$ be the moment generating function of the random variable X_1. The precise definition of a heavy-tailed distribution on $\mathbb{R}_+$ is as follows.

Definition 1.1.1. A distribution F is heavy-tailed if and only if

$$\int_{\mathbb{R}_+} e^{tx}F(dx) = \infty, \text{ for all } t > 0.$$

On the contrary, a distribution F is light-tailed if there exists a neighborhood around zero where the moment generating function is finite. The above definition implies that the tail probability of a heavy-tailed distribution decreases with a slower rate than any exponential rate. There are many examples of heavy-tailed distributions on $\mathbb{R}_+$: the Pareto distributions, the lognormal distributions, and the Weibull distributions.

An important subclass of heavy-tailed distributions is the class of subexponential distributions, denoted by $\mathcal{S}$. Let F^{n*} denote the n-fold convolution of F i.e,

$$F^{n*}(x) = \int_{t=0}^{x} F^{(n-1)*}(x-t)dF(t).$$

Definition 1.1.2. F is a subexponential distribution if

$$\frac{\bar{F}^{n*}(x)}{n\bar{F}(x)} = \frac{\mathbf{P}(\sum_{i=1}^{n} X_i \geq x)}{n\mathbf{P}(X_1 \geq x)} \to 1, \quad \text{for some } n \geq 2, \quad \text{as } x \to \infty.$$

Another characterization of subexponential distributions, which can be informative of their properties, is the following:

Definition 1.1.3. A distribution F is in $\mathcal{S}$, if for some $n \geq 2$,

$$\mathbf{P}\left(X_1 + \dots + X_n \geq x\right) \sim \mathbf{P}(\max\{X_1, \dots, X_n\}), \quad \text{as} \quad x \to \infty.$$

Intuitively, with regard to subexponential distributions ($\mathcal{S}$), the most likely way a random walk displays an extreme behavior is through a single big jump. This phenomenon has been documented as the principle of one big jump [33].

The framework of subexponential distribution functions was introduced in [19]. In this paper, the framework of subexponential distributions was used to derive asymptotic properties of branching processes. One of the first papers where the importance of subexponential distributions is recognized is [92]. We list some important subexponential distributions:

1. The Pareto distribution,

$$\mathbf{P}(X > x) = \left(\frac{x}{x_{min}}\right)^{-k}, \quad x_{min}, k > 0, \; x > x_{min};$$

2. the lognormal distribution,

$$\mathbf{P}(X > x) = \mathbf{P}(e^{\mu + \sigma Z} > x), \quad \mu \in \mathbb{R}, \quad \sigma \in \mathbb{R}_+,$$

   and Z is a standard normal random variable;

3. the Weibull distribution,

$$\mathbf{P}(X > x) = e^{-kx^{\alpha}}, \; k > 0, \; x > 0, \quad \alpha \in (0, 1).$$

An important class of distributions which serves as a generalization of the Pareto distribution is the class of regularly varying distributions.

Definition 1.1.4. A non-negative random variable X and its distribution are said to be regularly varying with index $-a$, $a > 0$, if the right tail $\bar{F}(\cdot)$ satisfies the limit

$$\lim_{x \to \infty} \frac{\bar{F}(tx)}{\bar{F}(x)} \to t^{-a}.$$

If $a = 0$, then F is a slowly varying function.

In this thesis, we study large deviations with semi-exponential distributions (which include the heavy-tailed Weibull distributions). We give a precise definition of these distributions:

Definition 1.1.5. A distribution F is semi-exponential if

$$\mathbf{P}(X > x) = e^{-L(x)x^{\alpha}}, \quad \alpha \in (0, 1),$$

and L is a slowly varying function.

Semi-exponential distributions appear in many applications e.g. [58], and [52]. To see how semi-exponential distributions appear even in a light-tailed setting, consider the following example by [33]. Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with a light-tailed distributions $F(x) = 1 - e^{-ax^k}$, $k \ge 1$. Then, for $n > k$, the tail distribution of the product $Y_1 \cdots Y_n$ is heavy tailed:

$$\mathbf{P}(Y_1 \cdots Y_n > x) \ge \prod_{i=1}^{n} \mathbf{P}(Y_i > x^{1/n}) = e^{-cnx^{k/n}}.$$

1.2 The large deviation principle

In this section, we give an introduction and an intuitive interpretation of the basic definitions of large deviations theory. We present the definition of the large deviation principle (LDP).

The scaled random walk with Gaussian increments

Consider a sequence of independent, identically distributed Gaussian random variables $\{X_n\}_{n \ge 1}$ with mean 0 and unit variance. Now, define the random walk $S_n = \sum_{i=1}^{n} X_i$, and subsequently, the scaled random walk $\bar{S}_n \triangleq \frac{1}{n} S_n$. The weak law of large numbers dictates that the scaled random walk $\bar{S}_n$ converges in probability to $\mathbf{E}(X_1) = 0$ as $n \to \infty$. That is,

$$\text{for any } \epsilon > 0, \ \mathbf{P}(|\bar{S}_n| > \epsilon) \to 0 \text{ as } n \to \infty.$$

However, one would like to have more information with regard to the fluctuations of $\bar{S}_n$ around 0; this can be achieved by the use of the central limit theorem (CLT). In particular, we have that

$$\mathbf{P}\left(\bar{S}_n > \frac{\alpha}{\sqrt{n}}\right) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\frac{x^2}{2}} dx.$$

In view of the above result, the central limit theorem gives information on the fluctuations of $\bar{S}_n$ from 0 of size $O(1/\sqrt{n})$. Furthermore, the CLT implies that the probability of $O(1/\sqrt{n})$ fluctuations is $O(1)$. On the other hand, what about larger fluctuations; namely of size $O(1)$, and what about their probability? Let us draw some intuition from the following elementary calculations:

$$\mathbf{P}\left(\bar{S}_n > \epsilon\right) = \frac{1}{\sqrt{2\pi}} \int_{\epsilon}^{\infty} e^{-\frac{nx^2}{2}} dx \le \frac{1}{\sqrt{2\pi}} \int_{\epsilon}^{\infty} \frac{x}{\epsilon} e^{-\frac{nx^2}{2}} dx = \frac{1}{n\epsilon\sqrt{2\pi}} e^{-\frac{n\epsilon^2}{2}};$$

while

$$\mathbf{P}\left(S_n > \delta\right) \geq \frac{1}{\sqrt{2\pi}} \int_{\delta}^{\delta+1/n} e^{-\frac{nx^2}{2}} dx \geq \frac{1}{n\sqrt{2\pi}} e^{-\frac{n}{2}(\delta+1/n)^2}.$$

In conclusion, $\frac{1}{n}\log \mathbf{P}\left(S_n > \delta\right) \to -\frac{\delta^2}{2}$ as $n \to \infty$. Utilizing the symmetry of the normal distribution, we also have that $\frac{1}{n}\log \mathbf{P}\left(S_n < -\delta\right) \to -\frac{\delta^2}{2}$ as $n \to \infty$. Therefore,

$$\mathbf{P}\left(|S_n| > \delta\right) \approx e^{-n\delta^2/2}, \text{ for large enough } n.$$

In this example, we see that the probability of large fluctuations ($O(1)$) decreases exponentially in n as n goes to infinity. To handle more general cases, we need to introduce a suitable asymptotic framework, involving a scaling parameter, which was n in the example.

Definition 1.2.1. Let (S, d) be a metric space with its topology induced by the metric d, and let X_n be a sequence of S-valued random variables. The probability measures of X_n satisfy the LDP in (S, d) with speed a_n and the rate function I if

$$-\inf_{x\in A^\circ} I(x) \leq \liminf_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq \limsup_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq -\inf_{x\in \bar{A}} I(x)$$

for any measurable set A.

Here, A° and $\bar{A}$ are respectively the interior and the closure of the set A; I is a non-negative lower semi-continuous function on S, and (a_n) is a sequence of positive real numbers that tends to infinity as $n \to \infty$. In the literature is it also said that the process X_n satisfies the LDP instead of its probability measures. If the large deviation principle's upper bound holds for all compact sets instead of all closed sets, then we say that X_n satisfies the weak large deviation principle (WLDP).

A rough interpretation of the above definition is as follows: for a rare event A, we can have an estimate of its probability on an exponential scale i.e;

$$\mathbf{P}(X_n \in A) \approx e^{-a_n \inf_{\alpha\in A} I(\alpha)} \tag{1.1}$$

where $\inf_{\alpha\in A} I(\alpha)$ is the decay rate of the associated rare event A. The decay rate leads to a deterministic optimization problem, and it can provide useful insights about the rare event under consideration.

A simple example that demonstrates the importance of the decay rate is the following. Let A, B be two disjoint rare events i.e; $\mathbf{P}(X_n \in A) \to 0$, and

$\mathbf{P}(X_n \in B) \to 0$. Obviously, $A \cup B$ is also a rare event. To make the example technically easier let us assume that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{P}(X_n \in A) = -a; \quad \lim_{n\to\infty} \frac{1}{n} \log \mathbf{P}(X_n \in B) = -b; \quad \text{and } a \neq b.$$

Then,

$$\mathbf{P}(X_n \in A | X_n \in A \cup B) = \frac{\mathbf{P}(X_n \in A)}{\mathbf{P}(X_n \in A \cup B)} \approx \frac{e^{-na}}{e^{-na} + e^{-nb}}, \text{ for large enough } n.$$

Therefore,

$$\mathbf{P}(X_n \in A | X_n \in A \cup B) \to \begin{cases} 1, & \text{if } a < b \\ 0, & \text{if } a > b. \end{cases}$$

That is, rare events manifest themselves through the most probable way. With regard to Equation (1.1), every scenario that is associated with the rare event A is measured by the rate function I; consequently, the large deviation principle reveals the most dominant — in an asymptotic sense — realization of the rare event A. For a rigorous treatment of the way rare events occur we refer to Lemma 4.2 of [39].

We examine the assumptions on the rate function I; in the definition of the large deviation principle we saw that I is a lower semi-continuous function mapping a space S to $[0, \infty]$. In addition, if the level sets of I are compact, then I is a good rate function. These are not merely technical terms. For every rare event A, its estimation is strongly related to optimization theory through its associated optimization problem $\inf_{\theta\in A} I(\theta)$. The following result gives insights on the assumptions made on I.

Result 1.2.1 (Lemma 4.1 of [39]). *For any rate function I, if A is a compact set then the infimum* $\inf_{\theta\in A} I(\theta)$ *is attained at some $\theta \in A$. If I is a good rate function then the infimum is attained on any closed set.*

To summarize, the lemma above formalizes the conditions upon which a solution to optimization problem $\inf_{\theta\in A} I(\theta)$ exists in the set A. We end this section with a key tool which is used extensively in the asymptotic evaluation of probabilities.

Result 1.2.2 (Lemma 1.2.15 of [22]). *Let N be a fixed integer. Then, for every $c_n^{(i)} > 0$, and $a_n \to \infty$,*

$$\limsup_{n \to \infty} \frac{1}{a_n} \log \left(\sum_{i=1}^{N} c_n^{(i)} \right) = \max_{i=1,\dots,N} \limsup_{n \to \infty} \frac{1}{a_n} \log c_n^{(i)}.$$

1.3 Further background on large deviations

Large deviations can be traced back to the 19th century with the introduction of the Laplace principle. The Laplace principle gives an asymptotic evaluation for $\int_{x \in \mathbb{R}} \exp(-nf(x))dx$, where f is continuous, and n tends to infinity. In [20], Cramer, driven by insurance and actuarial applications, determined estimates for i.i.d. sequences of random variables with finite moment generating function in a neighborhood of zero. This is now known as Cramer exponential moment condition.

At the mid 20th century a Russian school of mathematics centered on the asymptotic estimates of tail probabilities. In particular, a lot of attention was placed on asymptotic expansions for tail probabilities associated with the random walk measuring its deviations from the central limit theorem ([69, 68, 36, 49, 76]) examining cases where the Cramer exponential moment condition does not hold. Seminal works (cf. [67, 8, 7, 9, 13]) provide large deviation results in function spaces. The formal definition of the large deviation principle was introduced by Varadhan who was awarded the Abel prize (2007) for his contributions.

In this section, we list some pivotal results on large deviations, which are also used in this thesis, and we portray the subtle differences that heavy-tailed and light-tailed distributions induce to their respective LDPs.

1.3.1 One-dimensional large deviation results

We start with Cramer's theorem.

Result 1.3.1 (Cramer). *Let $\{X_n\}_{n \geq 1}$ be a sequence of i.i.d. random variables. Consider the scaled random walk $S_n = \frac{1}{n} \sum_{i=1}^{n} X_i$; $i \geq 1$. Let $\Lambda(\theta) = \log E e^{\theta X}$, and let Λ^* be the convex conjugate of Λ where $\Lambda^*(y) \triangleq \sup_{\theta \in \mathbb{R}} \{\theta y - \Lambda(\theta)\}$. Suppose that Λ is finite in a neighborhood of zero. Then the sequence of random variables $\{S_n\}_{n \geq 1}$ satisfies an LDP in $\mathbb{R}$ with the good convex rate function Λ^*. That is,*

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbf{P}(S_n \in F) \le -\inf_{y \in F} \Lambda^*(y) \quad \textit{for every closed set } F \subset \mathbb{R}; \textit{ and}$$
$$\liminf_{n \to \infty} \frac{1}{n} \log \mathbf{P}(S_n \in G) \ge -\inf_{x \in G} \Lambda^*(x) \quad \textit{for every open set } G \subset \mathbb{R}.$$

There exists a multivariate version of Cramer's theorem where the X_i's are considered to be *i.i.d.* vector valued random variables in $\mathbb{R}^k$ for every $k \ge 1$. Another generalization worth mentioning is the Gärtner-Ellis theorem (see [22]) where the LDP for weakly dependent sequences is obtained under a mild limiting assumption on the moment generating function of S_n.

Large deviation principles for light-tailed probability distributions can, usually, be derived applying Chernoff upper bounds, and an exponential change of measure. Both of these techniques involve the use of moment generating functions. On the contrary, moment generating functions are vacuous in the heavy-tailed case. A pioneering study in large deviations for heavy-tailed distributions can be seen in [71].

Result 1.3.2 ([71]). *Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with tail distribution $\bar{F}(y) = l(y)y^{-t}$ as $y \to \infty$, where $l(\cdot)$ is a slowly varying function and $t > 2$. If, in addition, $EY_1 = 0$, $\mathbf{var}(Y_1) = 1$, then*

$$\mathbf{P}\left(\sum_{i=1}^{n} Y_i > y\right) = n(\bar{F}(y))(1 + o(1)) \tag{1.2}$$

for $n \to \infty$ and $y > \sqrt{(t-2)n \log n}$.

If we evaluate the above result, we can see that extreme behaviors of the random walk are due to large values of one of the summands. Furthermore, we obtain a relationship on how fast y should grow in relationship with n so that (1.2) holds.

The investigation of tail estimates of the one-dimensional distributions of random walks with heavy-tailed step size distribution was initiated in [68, 69]. The state of the art of such results is well summarized in [12], [23], [27], [33]. In particular, [23] describes in detail how fast y needs to grow with n for the asymptotic relation $\mathbf{P}(S_n > y) = n\mathbf{P}(X_1 > y)(1 + o(1))$ to hold as $n \to \infty$, and for a sufficiently large class of subexponential distributions. With regard to the asymptotics in the previous equation, (1.2) does not hold for all subexponential distributions; in particular, when X_1 has a a Weibull tail $e^{-x^{\alpha}}$, $\alpha \in (1/2, 1)$;

the deviation of S_n from the mean described by the CLT makes a non-negligible contribution to the tail of S_n. This phenomenon, which is referred to as square root insensitivity, is due to deviations of order $O(\sqrt{n})$ which are induced by the central limit theorem, see [48, 3, 30, 70]. Therefore, exact asymptotics of Weibull and more general semi-exponential distributions is not an easy task in full generality.

1.3.2 Functional large deviations

The finite dimensional LDP considered in the previous subsection allows us to make estimates of the rare event probabilities associated with the tail behavior of empirical means. Although these estimates are useful in many cases, this is not always the most efficient approach. In many situations, we are interested in the probability that the whole path of the random process belongs to a set. An example would be the probability a random process is enclosed between two curves. In this section, we review basic results on sample-path large deviations (functional large deviations). We start with the random walk. Let

$$\bar{S}_n(t) \triangleq \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} X_i, \quad t \in [0,1],$$

where $X_i, i \geq 1,$ are i.i.d. random variables. Let $\mathbb{D}[0,1]$ denote the Skorokhod space, the space of real-valued cadlag functions, with its topology induced by the usual Skorokhod J_1 metric (d_{J_1}); the precise definition of the Skorokhod (J_1) topology is presented in the next section.

A sample-path LDP for light tails

Result 1.3.3 (Mogulskii [22])**.** *Let* $\log \mathbf{E}(e^{\theta X_1}) < \infty$ *for every* $\theta \in \mathbb{R}$*. Then the probability measures of* $\bar{S}_n$ *satisfy the large deviation principle in* $(\mathbb{D}[0,1], \mathcal{T}_{J_1})$ *with the good convex rate function* $I_0 : \mathbb{D}[0,1] \mapsto [0,\infty]$ *where*

$$I_0(\xi) \triangleq \begin{cases} \int_0^1 \Lambda^*(\dot{\xi}(t))\mathrm{d}t, & \text{if } \xi \in AC[0,1], \text{ and } \xi(0) = 0 \\ \infty, & \text{otherwise,} \end{cases}$$

where $AC[0,1]$ *denotes the subspace of absolutely continuous functions.*

In large deviations for light tails, one can often find a convex function $I(x)$ such that $\frac{1}{n}\log \mathbf{P}(\bar{S}_n \in A) \approx -\inf_{\xi \in A} I(\xi)$, and solve $\inf_{\xi \in A} I(\xi)$ using optimization techniques tailored for functional optimization, see [56, 42]. As an illustration, let us examine the random walk, $\bar{S}_n(t),\ t \in [0,1],$ with i.i.d. light-tailed increments so that $\mathbf{E}(X_1) = \mu > 0$. We consider the event that the all time supremum of the random walk is bigger than $C > \mu > 0$ i.e, $\{\sup_{t\in[0,1]} \bar{S}_n(t) \geq C\}$. We can rewrite the event as $\{\bar{S}_n \in E\}$ where

$$E = \{\xi \in \mathbb{D}[0,1] : \sup_{t\in[0,1]} \xi(t) \geq C\}.$$

Since I_0 is a good rate function and E is a closed set, Result 1.2.1 implies that the variational problem

$$\inf_{\xi\in E} I_0(\xi) \quad s.t. \quad \xi \in AC[0,1], \quad \xi(0) = 0$$

has an optimal solution $\xi^* \in E$. In addition, it is not difficult to show that $\xi^*(t) = C\cdot t,\ t \in [0,1],$ is an optimal path to the above optimization problem. To do so, we mainly utilise Jensen's inequality. For a rigorous treatment of the above example, we refer to Section 6.3 in [39], specifically, on the linear geodesics property.

A large deviation result for heavy tails

In contrast to the above result, we can expect a different outcome under a heavy-tailed setting. The following theorem is a simplified version of a result in [84] regarding large deviations for the random walk with regularly varying increments. Recall the random walk $S_n(\cdot)$, and the scaled random walk $\bar{S}_n(t),\ t \in [0,1]$.

Result 1.3.4. *Let X_1 be a non-negative r.v. such that $\mu = \mathbf{E}(X_1) > 0$, and $\mathbf{P}(Y_1 > y) = y^{-\alpha}L(y)$ for some regularly varying function L. Let*

$$\mathbb{D}_{=k}[0,1] \triangleq \left\{\xi \in \mathbb{D}[0,1] : \xi(t) = \mu t + \sum_{i=1}^{k} x_i \mathbb{1}_{[u_i,1]}(t), x_i \in (0,\infty), u_i\text{'s distinct in } (0,1)\right\}$$

and let $\mathbb{D}_{<l}[0,1] = \cup_{i=0}^{l-1}\mathbb{D}_{=k}[0,1]$. For a set $A \subseteq D[0,T]$, if

i) $k^* = \min\{k \geq 0 : \mathbb{D}_{=k}[0,1] \cap A \neq \emptyset\}$; and

ii) $\inf_{\{(\xi,\zeta):\ \xi\in\mathbb{D}_{<k^*}[0,1],\ \zeta\in A\}} d_{J_1}(\xi,\zeta) > 0$, *then there exists a measure* $\mathbf{M}$ *on* $D[0,T]$ *so that* $(\mathbb{D}_{=k}[0,T])^c$ *is a* $\mathbf{M}$*-null set, and*

$$\mathbf{M}(A^\circ) \le \liminf_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^k\,\mathbf{P}(X_1 \ge n)^k} \le \limsup_{n\to\infty} \frac{\mathbf{P}(\bar{S}_n \in A)}{n^k\,\mathbf{P}(X_1 \ge n)^k} \le \mathbf{M}(A^-).$$

A direct application of Result 1.3.4 implies that the most likely path associated with the rare event $E = \{\xi \in \mathbb{D}[0,1] : \sup_{t\in[0,1]} \xi(t) \ge C\}$ is a one-step function with step size at least equal to C. The above result solidifies the previously mentioned difference on how rare events manifest themselves in the heavy-tailed case.

For the extreme behavior of S_n, the so-called principle of one big jump holds. The first functional version of this insight has been derived in [44] — in the regularly varying case. A significant number of studies investigate the question of whether and how the principle of a single big jump is influenced by the structural properties of various random processes. This includes dependence of the increments, autoregressive processes, and stochastic differential equations (cf. [18], [29], [45], [53], [64], [65] [66], [88]).

Ideally, we want a framework to study rare events which are caused by multiple jumps. In [6], [32], and [60], the authors used ad-hoc approaches to study rare events, in specific models, which can be characterized by the principle of multiple big jumps. In [84], the first systematic principle of multiple big jumps was provided. The authors proved functional limit theorems for Levy processes and random walks allowing them to study rare events where the principle of multiple big jumps is said to hold.

1.3.3 Some basic large deviations tools

Although we would prefer to obtain functional LDPs, this is not always an easy task. An arsenal bridging large deviations in a finite-dimensional setting, functional large deviations, and applications is presented in this section.

The contraction principle

One particularly useful result in the toolbox of large deviations theory is the contraction principle. The contraction principle can be used to infer an LDP for continuous transformations of processes which satisfy a large a deviation principle. The contraction principle can be used either to infer an LDP in a space of interest or for applications. The idea is to utilize the representation $Y_n = f(X_n)$ where f models the interaction of the uncertainties (X_n) with the desired output (Y_n). If the map f is continuous, then the contraction principle enables one to understand rare events for Y_n from the LDP for X_n. The importance of the contraction principle is evident: a result for X_n can be reused in many other applications that require a different function f.

Result 1.3.5 (Contraction principle; see [22]). *Let X and Y be Hausdorff topological spaces and $f : X \to Y$ a continuous function. Consider a good rate function $I : X \to [0, \infty]$:*

(a) For each $y \in Y$, define

$$I'(y) \triangleq \inf\{I(x) : x \in X, y = f(x)\}.$$

Then I' is a good rate function on Y, where as usual the infimum over the empty set is taken as ∞.

(b) If I controls the LDP associated with a family of probability measures $\mathbf{P}_n$ on X, then I' controls the LDP for the family of probability measures $\{\mathbf{P}_n \circ f^{-1}\}$ on Y.

The result above holds under the weaker condition that f is continuous over the effective domain of the rate function I — i.e., on $D_I \triangleq \{x \in X : I(x) < \infty\}$. This particular extension of the contraction principle is called the extended contraction principle (p. 367 of [80]; Theorem 2.1 of [79]). Other sophisticated extensions of the contraction principle can be found in [22] and [39].

LDP for product spaces

Many applications require a multidimensional setting, for example, the multiple server queue. The next result can be used to derive LDPs for product spaces.

Result 1.3.6 (Theorem 4.14 of [39]). *Let X_n satisfy an LDP in X with good rate function I and speed a_n, let Y_n satisfy an LDP in Y with good rate function J and speed a_n, and suppose that X_n is independent of Y_n for each n. Assume that X and Y are separable spaces. Then the pair (X_n, Y_n) satisfies an LDP in $X \times Y$ with good rate function $K(x, y) = I(x) + J(y)$ and with speed a_n.*

Intuitively, if

$$\mathbf{P}(X_n \in A) \approx e^{-a_n \inf_{x \in A} I(x)}, \quad \text{and} \quad \mathbf{P}(Y_n \in B) \approx e^{-a_n \inf_{y \in B} J(y)},$$

then, due to, independence $\mathbf{P}((X_n, Y_n) \in (A \times B)) \approx e^{-a_n \inf_{(x,y) \in A \times B}(I(x)+J(y))}$.

The projective limit approach

In the previous section we presented some tools that allow us to infer probabilistic estimates in a finite dimensional setting. In some cases one would like to use these estimates to infer an LDP on the process level. Towards this end, one of the most important tools in large deviations theory is the projective limit approach of Dawson and Gärtner. It enables us to make probabilistic estimates in a finite dimensional setting and use these estimates to deduce an LDP in bigger spaces: that is, we transport a collection of LDPs in "small" spaces into the LDP in the bigger space S, which is their projective limit. The idea is to identify S with the projective limit of a collection of spaces $\{\mathsf{S}_j\}_{j\in J}$ with the intention that the LDP, for any given family $\mathbf{P}_n$ of probability measures on S, is the result of the LDP of $\mathbf{P}_n$ to S_j — for any $j \in J$.

Result 1.3.7 ([22]). *Let $\mathbf{P}_n$ be a sequence of probability measures on S, such that for any $j \in J$ the probability measures $\mathbf{P}_n \circ p_j^{-1}$ on S_j satisfy the large deviation with the good rate function $I_j(\cdot)$. Then, $\mathbf{P}_n$ satisfies the LDP with the good rate function*

$$I(\mathbf{x}) = \sup_{j\in J}\{I_j(p_j(\mathbf{x}))\}$$

Let $p_j(\mathsf{S})$ be equipped with the standard topology on R^j. The projective limit topology is the weakest topology which makes every p_j continuous, that is the topology of pointwise convergence. If $p_j(\mathsf{S})$ is equipped with the uniform convergence topology, the projective limit topology is the weakest topology which makes every p_j continuous, which, in this case, is the topology of uniform convergence on compact sets.

Intuitively, we can relate the projective limit approach to the established framework used in weak convergence theory. That is, convergence of finite dimensional distributions and tightness of the distributions implies convergence of infinite dimensional distributions.

1.4 Large deviations and topology

In addition to measure-theoretic probability, topology is a central concept in the large deviation theory of stochastic processes. The implementation of basic large deviation tools like the contraction principle is contingent on the topology of the space in which processes are defined. Intuitively, a topology of a metrizable space portrays how elements of this space relate spatially to each other with respect to the metric (a measure which induces distance in some sense). Intuitively, a topology is a collection of sets $\mathcal{T}$ with a certain structure; it enables us to separate two distinct elements of X by two distinct elements from the collection $\mathcal{T}$. For a formal definition of a topology, let X be a set and let $\mathcal{T}$ be a family of subsets of X. Then, $\mathcal{T}$ is called a topology on X if

i) both the empty set and X are elements of $\mathcal{T}$;

ii) any union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$;

iii) any intersection of finitely many elements of $\mathcal{T}$ is an element of $\mathcal{T}$.

If $\mathcal{T}$ is a topology on X, then the pair $(X, \mathcal{T})$ is called a topological space. The notation $(X, \mathcal{T})$ is used to denote a set X endowed with the topology $\mathcal{T}$.

We can define many topologies over a space X, however, not all topologies provide equal information. For example, we can equip X with the discrete topology $\mathcal{T}_d = \{\emptyset, X\}$; nevertheless, this topology provides little to no information over the spatial relationships between the elements of X, that is to say, we cannot separate two distinct elements of X by two distinct sets. Therefore, one would like to have a bigger collection of sets, and hence, a finer topology but also a well-regulated topology for the large deviation principle.

With regards to sample path large deviations, we interpret a stochastic process as a random element in a function space equipped with a topology. Under this interpretation, a sequence of probability measures on a function space is strongly related to the convergence of their associated stochastic processes. The space X, and the topology $\mathcal{T}$ should be chosen accordingly so that

i) the space X should contain elements that correspond to the irregularities of our stochastic processes;

ii) the notion of convergence should be meaningful with regard to applications.

Let us give some background on the function space we mainly work with, the topologies considered in this thesis, and let us discuss their applicability. Let $\mathbb{D}[0, T]$ denote the Skorokhod space — the space of cadlag functions from $[0, T]$ to $\mathbb{R}$. We can endow $\mathbb{D}[0, T]$ with an appropriate topology. The widely used supremum metric has been extremely useful in the study of continuous stochastic processes. Let us define the supremum metric. For a function $\xi \in \mathbb{D}[0, T]$, denote the supremum metric with $\|\xi\|_\infty = \sup_{t \in [0,T]} |\xi(t)|$. That is, the distance between two functions $\xi, \zeta \in \mathbb{D}[0, T]$ is equal to $\|\xi - \zeta\|_\infty$. On the subspace of continuous functions $\mathbb{C}[0, T]$, the supremum metric is a sufficiently good measure of distance but it does not perform well when we consider discontinuous functions in D[0; T]. When functions have discontinuities, it is not necessary that the respective discontinuity times of the converging sequence ($_n$) and the respective discontinuity times of the limit process ( ) are the same: consider the jump functions $_n( ) = \mathbb{1}_{[+1=n;1]}( )$ and the limit function $( ) = \mathbb{1}_{[;1]}( )$, > 0. The discontinuity of the limit function is not synchronized with the discontinuities of the sequential functions $_n$. If we consider the supremum metric, then $k \; _n \quad k_1 = 1$ for every $n \quad 1$.

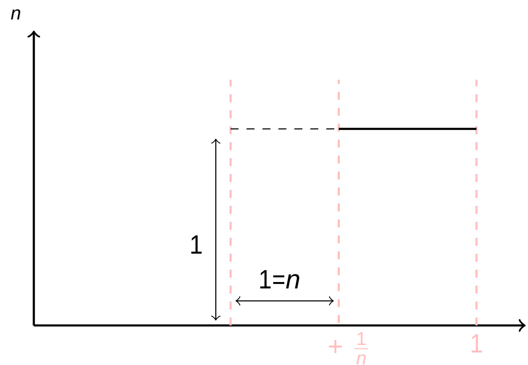

Figure 1.1: The gure displays the sequence $_n = \mathbb{1}_{[+ \frac{1}{n};1]}$ which we want converging to $_n = \mathbb{1}_{[;1]}$. We can see that for every $n \quad 1$ the supremum metric of $\quad _n$ is equal to 1, hence, convergence with respect to the supremum metric is not possible.

However, the two processes are close to each other | with respect to time deformations | therefore, we need a di erent topology on D[0; 1] that incorporates small time deformations. Let T_{J_1} denote the J_1 Skorokhod topology on D[0; 1]. That is, D[0; 1] is metrized by the Skorokhod J_1 metric.

Definition 1.4.1. Let d_{J_1} denote the Skorokhod J_1 metric,

$$d_{J_1}(\; ; \;) \; , \; \inf_{2} \; f \max f k \quad (\;) k_1 ; k \quad e k_1 g g ; \tag{1.3}$$

where $;\; 2$ D[0; 1], is a non-decreasing homeomorphism of [0; 1] onto itself, is the set of such homeomorphisms, and $e(t) = t$ is the identity map.

That is, w.r.t. the J_1 metric, functions are close if they are uniformly close over [0; 1] under time perturbations. Remember our previous example, where $_n( ) = \mathbb{1}_{[+1=n;1]}( )$ and the limit function is $( ) = \mathbb{1}_{[;1]}( )$. The use of the

Skorokhod J_1 metric implies $d_{J_1}(\xi_n, \xi) \le 1/n \to 0$ as $n \to \infty$. However, the J_1 topology is too strong for certain applications. What if we want to allow continuous functions to be arbitrarily close to a discontinuous function or merging of the jumps?

In particular, a discontinuous element of $D([0, T])$ cannot be approximated in the J_1 topology by a sequence of continuous functions, which makes the J_1 topology unsuitable for some applications. For example, consider a sequence of continuous functions which are linear interpolations of a pure jump function (between the jump points). These continuous functions have parts with steep slope and are close enough to the limit function (see Figure 1.2).

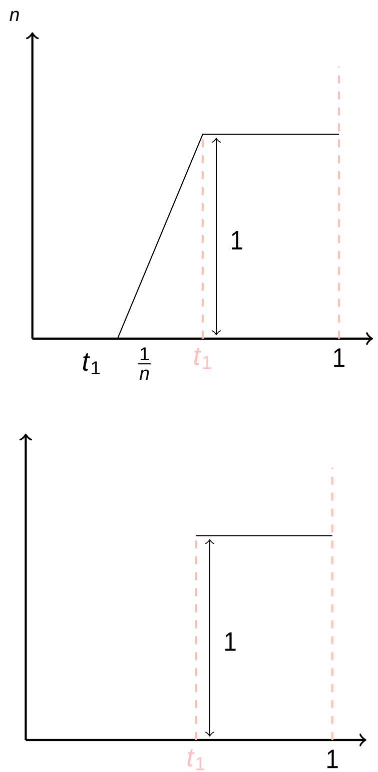

Figure 1.2: The top graph displays the continuous function $\xi_n(t) = n(t - t_1 + \frac{1}{n})\mathbb{1}_{[t_1 - \frac{1}{n}, t_1)}(t) + \mathbb{1}_{[t_1, 1]}(t)$ which we want converging to $\xi = \mathbb{1}_{[t_1, 1]}$ (graph in the bottom figure).

Another phenomenon which the J_1 topology fails to resolve is when jumps in the limit arise as accumulation of small jumps. That is, a single big jump of the limit process may correspond to the accumulation of many small jumps of the converging process which occur close (with respect to time) to each other.

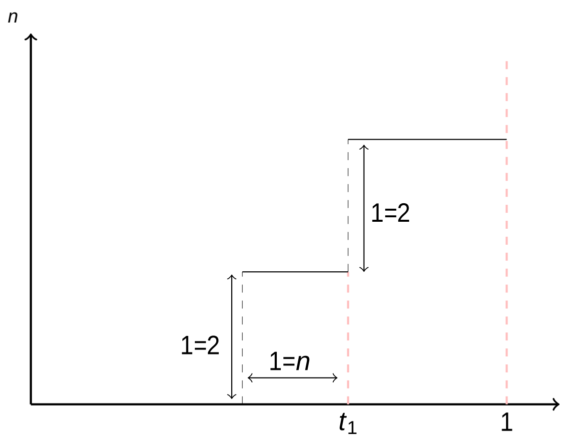

Figure 1.3: A display of the two-jump function $_n = \frac{1}{2}\mathbb{1}_{[t_1 \ \ 1=n;1]} + \frac{1}{2}\mathbb{1}_{[t_1;1]}$ which we want to converge to $\mathbb{1}_{[t_1;1]}$.

To establish a large deviation principle with merged jumps in the limit process, we use the M_1^0 topology. We denote the M_1^0 Skorokhod topology on D[0; T] with $T_{M_1^0}$. The M_1^0 topology is generated by the metric $d_{M_1^0}$ which is de ned in terms of the extended completed graphs of the functions in D[0; T].

Definition 1.4.2. For 2 D[0; T], de ne the extended completed graph $^0(\ )$ of as

$$^0(\ ) \ , \ f(u;t) \ 2 \ \mathrm{R} \quad [0;T] : u \ 2 \ [\ (t\ \ ) \wedge \ (t); \ \ (t\ \ ) \vee \ (t)]g$$

where (0) , 0. De ne an order on the graph $^0(\ )$ by setting $(u_1; t_1) < (u_2; t_2)$, for every $(u_1; t_1); (u_2; t_2)$ 2 $^0(\ )$, if either

- $t_1 < t_2$; or
- $t_1 = t_2$ and $j\ (t_1\ \ )\ \ u_1 j < j\ (t_2\ \ )\ \ u_2 j$.

We call a continuous non-decreasing function $(u, t) = (u(s), t(s)), s \in [0, T]$ from $[0, T]$ to $\mathbb{R} \times [0, T]$ an M_1' parametrization of $\Gamma'(\xi)$ if $\Gamma'(\xi) = \{(u(s), t(s)) : s \in [0, T]\}$. We also just call it a parametrization of ξ.

The extended completed graph is a connected subset of the plane $\mathbb{R}^2$ containing the segment $[(t, \xi(t-)), (t, \xi(t))]$ for all $t \in [0, T]$. The M_1' topology was introduced by Whitt and Puhalskii (cf. [80]) for paths defined on the positive half axis. The M_1' topology is an extension of the M_1 topology (essentially we add the vertical segment $[(0, 0), (0, \xi(0))]$ in the completed graph of a path ξ), hence, allowing us to treat functions that have discontinuities at time zero. This property is useful if we want to study inverses of stochastic processes (see Chapter 4). In Chapter 3 we study large deviations for the Lindley process with light-tailed increments where a cluster of small jumps of the converging process correspond to a big jump of the limit process.

Definition 1.4.3. Define the M_1' metric on $\mathbb{D}[0, T]$ as follows

$$d_{M_1'}(\xi, \zeta) \triangleq \inf_{\substack{(u,t) \in \Pi_{M_1'}(\xi) \\ (v,r) \in \Pi_{M_1'}(\zeta)}} \{\|u - v\|_\infty + \|t - r\|_\infty\},$$

where $\Pi_{M_1'}(\xi)$ is the set of all M_1' parametrizations of $\Gamma'(\xi)$.

Let us demonstrate the applicability of the M_1' topology. Consider the sequence $\xi_n \in \mathbb{D}[0, 1]$,

$$\xi_n(t) = \begin{cases} 0, & t \in \left[0, \frac{1}{2} - \frac{1}{n}\right), \\ n(t - \frac{1}{2} + \frac{1}{n}), & t \in \left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right), \\ 1, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Observe that $\xi_n \to \xi$ with the pointwise convergence topology where

$$\xi(t) = \begin{cases} 0, & t \in \left[0, \frac{1}{2}\right), \\ 1, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

However, ξ_n is continuous for each $n = 1, \ldots,$ and ξ has a jump at $1/2$, and hence, ξ_n cannot converge to ξ with respect to the J_1 topology. On the other hand, the M_1' distance of ξ_n, ξ i.e; $d_{M_1'}(\xi_n, \xi)$ is bounded by $1/n$. Thus, ξ_n converges to ξ with respect to the M_1' topology.

Product topology

Let $T_{(\ )}$ be a topology of the Skorokhod space $\mathbb{D}[0,T]$ generated by a metric $d_{(\ )}$. We consider $\prod_{i=1}^{k}\mathbb{D}[0,T]$ the product space equipped with the product topology $\prod_{i=1}^{k}T_{(\ )}$ which is induced by the product metric d_p. More precisely, for $\ ,\ \in \prod_{i=1}^{k}\mathbb{D}[0,T]$ such that $\ = (\ ^{(1)},\dots,\ ^{(k)})$ and $\ = (\ ^{(1)},\dots,\ ^{(k)})$ we have that

$$d_p(\ ,\ ) \triangleq \sum_{i=1}^{k} d_{(\ )}(\ ^{(i)},\ ^{(i)}).$$

The following definition formally states the convergence of functions with respect to the $T_{(\ )}$ topology.

Definition 1.4.4. Let $\ _n \in \left(\prod_{i=1}^{k}\mathbb{D}[0,T], \prod_{i=1}^{k}T_{(\ )}\right)$. Then, $d_p(\ _n,\ ) \to 0$ if $\ _n^{(i)} \to \ ^{(i)}$ w.r.t. the $d_{(\ )}$ metric for every $i = 1,\dots,k$.

We use the component-wise partial order on $\mathbb{D}[0,T]$ and $\mathbb{R}^k$. That is,

$$x_1 \triangleq (x_1^{(1)},\dots,x_1^{(k)}) \le x_2 \triangleq (x_2^{(1)},\dots,x_2^{(k)}) \text{ in } \mathbb{R}^k$$

$$\text{if } x_1^{(i)} \le x_2^{(i)} \text{ in } \mathbb{R} \text{ for all } i \in \{1,\dots,k\}.$$

In this regard, $\ \le\ $ in $\mathbb{D}[0,T]$ if $\ (t) \le\ (t)$ in $\mathbb{R}^k$ for all $t \in [0,T]$.

A large deviation principle with semi-exponential increments

In the literature on heavy-tailed distributions (cf. [2]) the three most important examples of heavy-tailed distributions are the lognormal, regularly varying, and Weibull distributions. The functional LDP for the lognormal case have not been obtained; the regularly varying case is studied [84]. In this section, we review the existing functional LDP for random walks with semi-exponential increments ([40]).

To explain the topology for which this LDP has been derived, let $L[0,1] \triangleq \{\ \in L^1[0,1] : \ (0) = 0\}$ denote the space of integrable functions which vanish at the origin. Let T_{L_1} denote the topology induced by the L_1 metric d_{L_1}; the L_1 distance of two integrable functions $\ ,\ $ is $d_{L_1}(\ ,\ ) = \int_0^1 |\ (s) - \ (s)|ds$. Lastly, let $\{Y_i\}_{i\ge 1}$ be i.i.d increments. The following result establishes an LDP in $L[0,1]$, w.r.t. the L_1 topology, for $Z_n = Z_n(t),\ t \in [0,1],$ where $Z_n(t) = \frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor} Y_i + (t - \frac{\lfloor nt\rfloor}{n})Y_{\lfloor nt\rfloor+1}$.

Result 1.4.1. *Let Y_1 satisfy the following conditions:*

i) $\mathbf{E}(Y_1) = 0$ *and* $\mathbf{E}(e^{\lambda Y_1}) < \infty$ *for all* $\lambda \geq 0$, *and*

ii) there exist a slowly varying function b *with the property that* $b(t)/t^{1-\alpha}$ *is non-increasing, and* $\mathbf{P}(Y_1 \geq t) = e^{-b(t)t^{\alpha}}$, $\alpha \in (0,1)$.

Then, the probability measures of Z_n *satisfy the extended large deviation principle in* $(L[0,1], \mathcal{T}_{L_1})$ *with speed* $b(n)n^{\alpha}$ *and the good rate function*

$$I_g(\phi) = \begin{cases} \sum_{t:\,\phi(t)\neq\phi(t-)} (\phi(t) - \phi(t-))^{\alpha} & \text{if } \phi \text{ is a non-decreasing pure jump function,} \\ \infty, & \text{otherwise.} \end{cases} \tag{1.4}$$

That is, for any measurable A,

$$-\inf_{\phi \in A^{\circ}} I_g(\phi) \leq \liminf_{n\to\infty} \frac{\log \mathbf{P}(Z_n \in A)}{b(n)n^{\alpha}} \leq \limsup_{n\to\infty} \frac{\log \mathbf{P}(Z_n \in A)}{b(n)n^{\alpha}} \leq -\inf_{\phi \in \bar{A}} I_g(\phi);$$

The above large deviation principle has been the first result concerning asymptotics for semi-exponential increments. Although Result 1.4.1 is sufficient for the applications examined in [40], it does not provide good estimates for all potential applications. To illustrate this, let us examine the event

$$E_l = \{\phi \in L[0,1] : \sup_{t\in[0,1]} \phi(t) \geq C\}.$$

With the use of the L_1 metric the zero function is a limit point of the set E_l. If we let $\phi_n = C\mathbb{1}_{[1-1/n,1]}$, then $\phi_n \in E_l$ for all n. Since $d_{L_1}(\phi_n, 0) \leq C/n \to 0$ as $n \to \infty$, the zero function belongs to the closure of E_l. Consequently, $\inf_{\phi\in\bar{E}_l} I_g(\phi) = 0$ resulting in a trivial upper bound of the LDP.

The extended LDP

Not in all cases a random process satisfies an LDP. Therefore, it is desirable to have more tools and establish a connection with the framework of the standard LDP. With this intention, we present the concept of the extended LDP.

Let $(\mathbb{S}, d)$ be a metric space, and $\mathcal{T}$ denote the topology induced by the metric d. Let X_n be a sequence of $\mathbb{S}$-valued random variables. Let I be a non-negative lower semi-continuous function on $\mathbb{S}$, and (a_n) be a sequence of positive real numbers that tends to infinity as $n \to \infty$.

Definition 1.4.5. The probability measures of X_n satisfy the *extended* LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and rate function I if

$$\inf_{x \in A^\circ} I(x) \leq \liminf_{n \to \infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq \limsup_{n \to \infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq \lim_{\epsilon \to 0} \inf_{x \in A^\epsilon} I(x)$$

for any measurable set A.

Here we denote $A^\epsilon \triangleq \{\xi \in \mathbb{S} : d(\xi; A) \leq \epsilon\}$ where $d(\xi; A) = \inf_{\zeta \in A} d(\xi; \zeta)$. The notion of the extended LDP was introduced in [14]. This concept of an LDP was developed to treat cases where the standard large deviation principle is difficult or impossible to obtain. In the above definition, if we assume that I has compact level sets, then I is a good rate function and the extended LDP implies the standard LDP in Definition 1.2.1. Note that lower semi-continuity and compactness of the level sets depend on the topology of the space $\mathbb{S}$. In particular, in [14] the authors have examined conditions such that

$$\lim_{\epsilon \to 0} \inf_{x \in A^\epsilon} I(x) = \inf_{x \in \bar{A}} I(x).$$

Moreover, the definition of the extended LDP can be less strict so that it can cover cases where I is not a lower semi-continuous function; we do not consider these alternative definitions in this thesis.

1.5 Contribution

One of the most fundamental contributions of this thesis, developed in Chapter 2, is the sample path large deviation principle for Levy processes and random walks with heavy-tailed Weibull (semi-exponential) increments. This result holds in the Skorokhod space with respect to the M_1' topology. In addition, we prove the extended sample path LDP in the Skorokhod space with the finer J_1 topology. Furthermore, we develop theoretical tools for extended LDP and show that the standard LDP cannot be satisfied for the Levy processes with heavy-tailed Weibull increments. This suggests that the extended LDP is the optimal result one can achieve with respect to the J_1 topology. We illustrate this by constructing a counterexample; showing that the LDP in the J_1 topology is not possible. These large deviations results have been extended to multidimensional settings in the case of independent Levy processes and random walks.

To enhance the applicability of the extended LDP, we have also developed a form of contraction principle. In particular, we study ruin probabilities in a reinsurance example. That is, we consider level crossing probabilities of Levy processes where the jump sizes are conditioned to be moderate. These types of events appear in actuarial models | in case excessively large insurance claims are reinsured, and therefore, do not play a role in the ruin of an insurance company. In conclusion, for the random processes treated in this chapter, our large deviation analysis demonstrates that associated rare events are caused by big discontinuities of their sample paths; this phenomenon has been characterized as the principle of multiple big jumps.

The third chapter centers on sample path large deviations for Markov additive processes. More precisely, we prove the sample path LDP for unbounded additive functionals of processes with light-tailed increments that are induced by the Lindley recursion. The LDP holds in the Skorokhod space equipped with the M_1' topology and with sub-linear speed. Although the process under consideration is constructed by light-tailed increments, rare events are caused by \big jumps". This result establishes that the structure of light-tailed random processes can induce (asymptotically) a heavy-tailed behavior. Our technique hinges on a suitable decomposition of the Markov chain in terms of regeneration cycles. At each regeneration cycle we study the accumulated area of the Lindley process. Consequently, the area displays heavy-tailed behavior, and it satis es an LDP. To derive tail asymptotics for the area, we use sample path analysis; as a by-product of our LDP we show that large areas are caused by concave trajectories of our process.

In the fourth chapter, we focus on one of the most celebrated models in queueing theory namely, the multiple server queue. The multiple server queue model ($G=G=d$) is a fundamental model and serves as a key model-component in many occasions, for example, performance analysis of web servers and databases [15]. An important question is the likelihood of a large queue length or waiting time in such systems. Logarithmic asymptotics in the case of light-tailed service times have been studied in [79], and [87]. The case of heavy-tailed Weibull service times has been an open problem that dates back to Whitt (2000) ([95]) and it was also mentioned by Sergey Foss in the 2009 Erlang centennial conference. To exemplify, consider d parallel servers, each working at a certain speed and suppose that the service time distribution is heavy-tailed. If k large jobs appear in the system simultaneously, then they reduce the capacity of the system, which is not detrimental if the remaining service capacity exceeds the system load . One expects that k large jobs are required to make the system behave poorly, where k is the minimum number of big jobs needed to cause instability | in the sense of congestion | in the system. The main results in Chapter 4 provide an estimate for the probability of large queue lengths as well as the detailed answers on how large queue lengths occur. For the latter part, we determine the number of big jobs and their sizes that lead to congestion; since the Weibull case is near the boundary of the light-tailed and heavy-tailed cases, our results show qualitative and quantitative differences in comparison to both the power law case (cf. [32]) and the light-tailed cases.

In Chapter 5, we apply our fundamental results of Chapter 2 to study a stochastic fluid network model with heavy-tailed input (compound Poisson processes with semiexponential increments). This stochastic network model is an important framework within applied probability and has many applications in industry. Our results include the continuity of the multidimensional reflection map on certain subspaces of the Skorokhod space under the product J_1 topology. Based on the continuity of the multidimensional reflection map we prove large deviation bounds for the multidimensional buffer content process of the stochastic fluid network. Furthermore, we use the large deviation bounds of the buffer content process to estimate overflow probabilities for a subset of nodes of the stochastic fluid network. We associate the overflow probabilities with a simplified optimization problem. Lastly, we perform explicit computations in the case of a certain network which relates to — w.r.t. its network topology — the multiple on-off sources model.

Chapter 2

Limit laws with semi-exponential increments

2.1 Introduction

In this chapter, we develop sample path large deviations for Levy processes and random walks, assuming that the jump sizes have a semi-exponential distribution. Specifically, let $X(t), t \geq 0$, be a centered Levy process with positive jumps and Levy measure ν which has non-negative support. Assume that $-\log \nu[x, \infty)$ is regularly varying of index $\alpha \in (0, 1)$ and define $\bar{X}_n = \{\bar{X}_n(t), t \in [0, 1]\}$, with $\bar{X}_n(t) = X(nt)/n$. We are interested in large deviations of $\bar{X}_n$.

The study of large deviations of sample paths of processes with Weibullian increments is relatively limited. Let us now present our contributions. We first develop an extended LDP (large deviations principle) in the J_1 topology, i.e. we show that there exists a rate function $I(\cdot)$ such that

$$\liminf_{n \to \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{L(n)n^{\alpha}} \geq -\inf_{x \in A} I(x) \tag{2.1}$$

if A is open, and

$$\limsup_{n \to \infty} \frac{\log \mathbf{P}(\bar{X}_n \in A)}{L(n)n^{\alpha}} \leq -\lim_{\epsilon \downarrow 0} \inf_{x \in A^{\epsilon}} I(x) \tag{2.2}$$

if A is closed. Here $A^{\epsilon} = \{x : d(x; A) \le \epsilon\}$. The rate function I is given by

$$I(\xi) = \begin{cases} \sum_{t: \xi(t) \neq \xi(t-)} (\xi(t) - \xi(t-))^{\alpha} & \text{if } \xi \in \mathbb{D}_{\uparrow}[0,1]; \\ \infty & \text{otherwise;} \end{cases}$$

where $\mathbb{D}_{\uparrow}[0,1]$ is the subspace of $\mathbb{D}[0,1]$ consisting of non-decreasing pure jump functions vanishing at the origin and continuous at 1. (As usual, $\mathbb{D}[0,1]$ is the space of cadlag functions from $[0,1]$ to $\mathbb{R}$.)

We derive this result as follows: We use a suitable representation for the Levy process in terms of Poisson random measures, allowing us to decompose the process into the contribution generated by the k largest jumps, and the remainder. The contribution generated by the k largest jumps is a step function for which we obtain the large deviations behavior by Bryc's inverse Varadhan lemma (see e.g. Theorem 4.4.13 of [22]). The remainder term is controlled by modifying a concentration bound due to [47].

To combine both estimates we need to consider the ϵ-fattening A^{ϵ} of the set A, which precludes us from obtaining a full LDP. To show that our approach cannot be improved, we construct a set A that is closed in the Skorokhod J_1 topology for which the large deviation upper bound does not hold. In this sense, our extended large deviations principle can be seen as optimal. This is in line with the observation made for the regularly varying Levy processes and random walks [84], for which the full LDP w.r.t. J_1 topology in a classical sense is shown to be unobtainable as well.

Following a similar proof strategy, we also derive an extended sample path LDP for random walks in $\mathbb{D}[0,1]$. However, there are some differences. The distributional representation of our random walk is different from the continuous-time case. More importantly, the resulting rate function differs at time 1, since the rescaled random walk always has a jump at time 1.

We derive several implications of our extended LDP that facilitate its use in applications. First of all, if a Lipschitz functional ϕ of X_n is chosen for which the function $I'(y) = \inf_{x: \phi(x)=y} I(x)$ is a good rate function, then $\phi(X_n)$ satisfies an LDP.

A second implication of the extended LDP is an application to a reinsurance example in actuarial science. Moreover, we derive the sample path LDP for Levy processes and random walks in the M_1' topology. We show that the rate function I is good in this topology, allowing us to conclude that $\lim_{\epsilon \downarrow 0} \inf_{x \in A^{\epsilon}} I(x) = \inf_{x \in A} I(x)$, if A is closed in the M_1' topology. We extend our previous large deviation results (extended LDP, and LDP) to multidimensional function spaces endowed with the product topology.

The rest of the chapter is organized as follows. In Section 2.2, we present our results regarding the extended LDP for Levy processes and random walks. Section 2.3 includes implications of the extended LDP while Section 2.4 contains the counterexample for the standard LDP with the J_1 topology, and our LDP results with respect to the M_1' topology. Lastly, we include mainly technical proofs in Section 2.5.

2.2 Extended LDP for Lévy processes and random walks

2.2.1 Useful results on the extended LDP

In this section, we present and prove some abstract results used in our large deviation analysis. Before displaying our auxiliary results, we remind the reader of the notion the extended LDP. Let $(\mathbb{S}, d)$ be a metric space, and $\mathcal{T}$ denote the topology induced by the metric d. Let X_n be a sequence of $\mathbb{S}$-valued random variables. Let $A^\delta \triangleq \{\xi \in \mathbb{S} : d(\xi, A) \leq \delta\}$ where $d(\xi, A) = \inf_{\zeta \in A} d(\xi, \zeta)$, and A° denotes the interior of A. Let I be a non-negative lower semi-continuous function on $\mathbb{S}$, and (a_n) be a sequence of positive real numbers that tends to infinity as $n \to \infty$.

We say that X_n satisfies the *extended* LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and rate function I if

$$-\inf_{x \in A^\circ} I(x) \leq \liminf_{n \to \infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq \limsup_{n \to \infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \leq -\lim_{\delta \to 0} \inf_{x \in A^\delta} I(x)$$

for any measurable set A. The next proposition provides the necessary framework for proving the extended LDP. Let $A_{-\delta} \triangleq \{\xi \in \mathbb{S} : d(\xi, \zeta) \leq \delta \text{ implies } \zeta \in A\}$.

Proposition 2.2.1. *Let I and I_k, $k \geq 1$ be rate functions. Suppose that for each n, X_n has a sequence of approximations $\{Y_n^k\}_{k=1,\ldots}$ such that*

(i) For each k, Y_n^k satisfies the extended LDP in $(\mathbb{S}, \mathcal{T})$ with speed a_n and rate function I_k;

(ii) For each closed set F,

$$\lim_{k \to \infty} \inf_{x \in F} I_k(x) \geq \inf_{x \in F} I(x);$$

(iii) *For each* $\delta > 0$ *and each open set* G*, there exist* $\epsilon > 0$ *and* $K \geq 0$ *such that* $k \geq K$ *implies*

$$\inf_{x\in G} I_k(x) \leq \inf_{x\in G} I(x) + \delta;$$

(iv) *For every* $\epsilon > 0$ *it holds that*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(d(X_n, Y_n^k) > \epsilon\right) = -\infty. \tag{2.3}$$

Then, X_n *satisfies the extended LDP in* $(\mathbb{S}, \mathcal{T})$ *with speed* a_n *and rate function* I*.*

Proof. We start with the extended large deviation upper bound. For any measurable set A,

$$\begin{aligned}\mathbf{P}(X_n \in A) &= \mathbf{P}\left(X_n \in A, d(X_n, Y_n^k) \leq \epsilon\right) + \mathbf{P}\left(X_n \in A, d(X_n, Y_n^k) > \epsilon\right) \\ &\leq \underbrace{\mathbf{P}\left(Y_n^k \in A^{\epsilon}\right)}_{(\mathrm{I})} + \underbrace{\mathbf{P}\left(d(X_n, Y_n^k) > \epsilon\right)}_{(\mathrm{II})}.\end{aligned} \tag{2.4}$$

From the principle of the largest term and (i),

$$\begin{aligned}&\limsup_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{a_n} \\ &\leq \max\left\{-\inf_{x\in A^{\epsilon}} I_k(x), \limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(d(X_n, Y_n^k) > \epsilon\right)\right\}.\end{aligned}$$

Now letting $k \to \infty$ and then $\epsilon \to 0$, (ii) and (iv) lead to

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}(X_n \in A) \leq -\lim_{\epsilon\to 0} \inf_{x\in A^{\epsilon}} I(x);$$

which is the upper bound of the extended LDP.

Turning to the lower bound; the lower bound is trivial if $\inf_{x\in A^\circ} I(x) = \infty$; therefore, we focus on the case $\inf_{x\in A^\circ} I(x) < \infty$. Consider an arbitrary but fixed $\delta \in (0, 1)$. In view of (iii) and (iv), one can pick $\epsilon > 0$ and $k \geq 1$ in such a way that

$$-\inf_{x\in A^\circ} I(x) \leq -\inf_{x\in A^{-\epsilon}} I_k(x) + \delta; \text{ and} \tag{2.5}$$

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}\left(d(X_n, Y_n^k) > \epsilon\right)}{a_n} \leq -\inf_{x\in A^\circ} I(x) - 1.$$

Hence

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}\left(d(X_n, Y_n^k) > \delta \right)}{a_n} \le -\inf_{x \in A} I_k(x) + 1. \tag{2.6}$$

Now, from (2.6) and the lower bound of the assumed extended LDP for Y_n^k, one can easily verify that

$$\frac{\mathbf{P}\left(d(X_n, Y_n^k) > \delta \right)}{\mathbf{P}\left(Y_n^k \in A \right)} \to 0 \tag{2.7}$$

as $n \to \infty$. Using (2.7) and the inequality in (2.5),

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}(X_n \in A) &\ge \liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(Y_n^k \in A, d(X_n, Y_n^k) \le \delta \right) \\
&\ge \liminf_{n\to\infty} \frac{1}{a_n} \log \left(\mathbf{P}\left(Y_n^k \in A \right) - \mathbf{P}(d(X_n, Y_n^k) > \delta) \right) \\
&= \liminf_{n\to\infty} \frac{1}{a_n} \log \left(\mathbf{P}\left(Y_n^k \in A \right) \left(1 - \frac{\mathbf{P}\left(d(X_n, Y_n^k) > \delta \right)}{\mathbf{P}\left(Y_n^k \in A \right)} \right) \right) \\
&= \liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(Y_n^k \in A \right) \ge -\inf_{x \in A} I_k(x) \ge -\left(\inf_{x \in A} I(x) - \epsilon \right).
\end{aligned}$$

Since ϵ was arbitrary in $(0, 1)$, the lower bound is proved by letting $\epsilon \to 0$. □

Corollary 2.2.2. *Suppose that Y_n satisfies the extended LDP in $(\mathsf{S}, \mathcal{T})$ with speed a_n and rate function I. If for each $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}(d(X_n, Y_n) > \delta) = -\infty;$$

then X_n satisfies the extended LDP in $(\mathsf{S}, \mathcal{T})$ with speed a_n and rate function I.

Proof. Let $Y_n^k \equiv Y_n$ and $I_k \equiv I$ for $k = 1, 2, \ldots$. Then, (i) and (ii) of Proposition 2.2.1 are trivially satisfied. For (iii), we note that by the definition of G^δ, for each $\epsilon > 0$ and G an open set, there exists $\delta > 0$ such that

$$\inf_{x \in G^\delta} I(x) \ge \inf_{x \in G} I(x) + \epsilon;$$

and hence, (iii) are satisfied for $I_k = I$. Since (iv) is also satisfied by the assumption, all the conditions of Proposition 2.2.1 are satisfied and the conclusion of the corollary follows. □

2.2.2 Extended LDP for Lévy processes

We make two assumptions regarding the Levy processes:

A1. X is a real-valued Levy process with Levy measure ν which has non-negative support satisfying $\nu[x,\infty) = \exp(-L(x)x^{\alpha})$ where $\alpha \in (0,1)$ and $L(\cdot)$ is slowly varying at infinity.

A2. The mapping $x \mapsto L(x)x^{\alpha-1}$ is non-increasing for sufficiently large x.

Let $X_n(t), t \in [0,1],$ denote the centered and scaled process:

$$X_n(t) \triangleq \frac{1}{n}X(nt) - t\mathbf{E}X(1).$$

The following representation of the above Levy process is an important feature of our proof: Recall that $X_n(\cdot) \triangleq X(n\cdot)$ has Itô representation

$$X_n(s) = nsa + B(ns) + \int_{x<1} x[\hat{N}([0,ns]\times dx) - ns\nu(dx)] + \int_{x\ge 1} x\hat{N}([0,ns]\times dx), \tag{2.8}$$

with a a drift parameter, B a Brownian motion, and $\hat{N}$ a Poisson random measure with mean measure Leb $\times\,\nu$ on $[0,n]\times(0,\infty)$; Leb here denotes the Lebesgue measure. All terms in (2.8) are independent. We will see that the large deviation behavior is dominated by the last term of (2.8). It turns out to be convenient to consider the following distributional representation of the centered and scaled version of the last term:

$$\begin{aligned} Y_n(\cdot) &\triangleq \frac{1}{n}\sum_{l=1}^{N(n\cdot)} (Z_l - \mathbf{E}Z) \\ &\stackrel{\mathcal{D}}{=} \frac{1}{n}\int_{x\ge 1} x\hat{N}([0,n\cdot]\times dx) - \frac{1}{n}(\mathbf{E}Z)\,\hat{N}([0,n\cdot]\times[1,\infty)), \end{aligned}$$

where $N(t) \triangleq \hat{N}([0,t]\times[1,\infty))$ is a Poisson process with arrival rate $\nu_1 \triangleq \nu[1,\infty)$, and the Z_i's are i.i.d. copies of Z such that $\mathbf{P}(Z \ge x) = \nu[x\vee 1,\infty)/\nu_1$, independent of N. We consider a further decomposition of Y_n into two pieces, one of which consists of the big increments, and the other one keeps the residual fluctuations. To be more specific, we introduce an extra notation for the rank of the increments. Given $N(n)$, define $\mathbf{S}_{N(n)}$ to be the set of all permutations of $\{1,\ldots,N(n)\}$. Let $R_n : \{1,\ldots,N(n)\} \to \{1,\ldots,N(n)\}$ be a random permutation of $\{1,\ldots,N(n)\}$ sampled uniformly from $\Sigma_n \triangleq \{\sigma \in \mathbf{S}_{N(n)} : Z_{\sigma^{-1}(1)} \ge \cdots \ge Z_{\sigma^{-1}(N(n))}\}$. In words, $R_n(i)$ is the rank of Z_i among $\{Z_1,\ldots,Z_{N(n)}\}$ when sorted in decreasing order with the ties broken uniformly. Now,

- let $\frac{1}{n}\sum_{i=1}^{N(nt)} Z_i \mathbb{1}_{\{R_n(i)\ \ k\}} \triangleq J_n^k(t)$;
- let $\frac{1}{n}\sum_{i=1}^{N(nt)} (Z_i \mathbb{1}_{\{R_n(i)>k\}} \ \ \mathbf{E}Z) \triangleq H_n^k(t)$;
- and see that $Y_n(t) = J_n^k(t) + H_n^k(t)$.

The extended large deviation principle for Levy processes is straightforward given the following technical lemmas; their proofs are provided in Section 2.5. Let $\mathbb{D}[0,1]$ denote the Skorokhod space — space of cadlag functions over the domain $[0,1]$ — and let $\mathcal{T}_{J_1}$ denote the J_1 topology induced by the J_1 metric. Let $\mathbb{D}_{\uparrow\infty}[0,1]$ be the subspace of $\mathbb{D}[0,1]$ consisting of non-decreasing pure jump functions vanishing at the origin and continuous at 1. Let $\mathbb{D}_{\uparrow k}[0,1]$ denote the subspace of $\mathbb{D}_{\uparrow\infty}[0,1]$ consisting of paths that have less than or equal to k discontinuities. Let

$$I_k(\ ) = \begin{cases} \sum_{t\in[0,1]} (\ (t) \quad (t\ \ )) & \text{if } \ \in \mathbb{D}_{\uparrow k}[0,1]; \\ \infty & \text{otherwise:} \end{cases} \quad ;\text{and} \tag{2.9}$$

$$I(\ ) = \begin{cases} \sum_{t:\ (t)\neq\ (t\ \ )} \ (t) \quad (t\ \ ) & \text{if } \ \in \mathbb{D}_{\uparrow\infty}[0,1]; \\ \infty; & \text{otherwise:} \end{cases}$$

Lemma 2.2.3. *I and I_k are lower semi-continuous, and hence, rate functions.*

Lemma 2.2.4. *For each xed k, J_n^k satis es the LDP in $(\mathbb{D}[0,1], \mathcal{T}_{J_1})$ with speed $L(n)n$ and rate function I_k.*

Recall that $A\ \ \triangleq \{\ \in \mathbb{D} : d_{J_1}(\ ,\ )\ \ \text{implies}\ \ \in A\}$.

Lemma 2.2.5. *For each $\ >0$ and each open set G, there exist $\ >0$ and $K\ \ 0$ such that for any $k\ \ K$*

$$\inf_{\ \in G} I_k(\ )\ \ \inf_{\ \in G} I(\ ) + \ : \tag{2.10}$$

Let $B_{J_1}(\ ;\ )$ be the open ball w.r.t. the J_1 Skorokhod metric centered at with radius and $B\ \ \triangleq B_{J_1}(0;\ )$.

Lemma 2.2.6. *For every $\ >0$ it holds that*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\ \ \|H_n^k\|_\infty >\ \ \ = \ \ \infty \tag{2.11}$$

where $H_n^k(t) = \frac{1}{n}\sum_{i=1}^{N(nt)} (Z_i \mathbb{1}_{\{R_n(i)>k\}} \ \ \mathbf{E}Z)$*:*

Now, we are ready to state and prove our main result.

Theorem 2.2.7. *The process X_n satisfies the extended large deviation principle in $(\mathbb{D}[0,1], \mathcal{T}_{J_1})$ with speed $L(n)n^\alpha$ and rate function*

$$I(\xi) = \begin{cases} \sum_{t:\, \xi(t)\neq\xi(t-)} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \mathbb{D}_{<\infty}[0,1]; \\ \infty; & \text{otherwise.} \end{cases} \tag{2.12}$$

That is, for any measurable A,

$$-\inf_{\xi\in A^\circ} I(\xi) \le \liminf_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{L(n)n^\alpha} \le \limsup_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{L(n)n^\alpha} \le -\lim_{\epsilon\to 0} \inf_{\xi\in A^\epsilon} I(\xi); \tag{2.13}$$

where $A^\epsilon \triangleq \{\xi \in \mathbb{D}[0,1] : d_{J_1}(\xi,\zeta) \le \epsilon \text{ for some } \zeta \in A\}$.

Proof. For this proof, we use the following representation of X_n:

$$X_n \stackrel{\mathcal{D}}{=} Y_n + R_n = J_n^k + H_n^k + R_n, \tag{2.14}$$

where $R_n(s) = \frac{1}{n}B(ns) + \frac{1}{n}\int_{|x|\le 1} x[N([0,ns]\times dx) - ns\nu(dx)] + \frac{1}{n}(\mathbf{E}Z)\hat{N}([0,n]\times[1,\infty)) - \gamma_1 s$. Next, we verify the conditions of Proposition 2.2.1. Lemma 2.2.3 confirms that I is lower semi-continuous. Lemma 2.2.4 verifies (i). To see that (ii) is satisfied, note that $I_k(\xi) \le I(\xi)$ for any $\xi \in \mathbb{D}$. Lemma 2.2.5 verifies (iii). Since $d_{J_1}(X_n, J_n^k) \le \|H_n^k\|_\infty + \|R_n\|_\infty$, the condition (iv) is implied by Lemma 2.2.6 and $\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\|R_n\|_\infty > \delta) = -\infty$. Note that R_n is a Levy process whose moment generating function is finite everywhere, and hence, the LDP upper bound in Theorem 2.5 of [67] applies to $\mathbf{P}(d_{J_1}(0, R_n) > \delta)$. This, in turn, implies that $\limsup_{n\to\infty} \frac{1}{L(n)n^\alpha} \log \mathbf{P}(\|R_n\|_\infty > \delta) = -\infty$. Now, the conclusion of the theorem follows from Proposition 2.2.1. □

Remark 1. Note that it is straightforward to extend Theorem 2.2.7 to spectrally two-sided Levy processes. For instance, suppose that the Levy measure ν of the process X has Weibull tail $\nu[x,\infty) = \exp(-L(x)x^\alpha)$ where $\alpha \in (0,1)$, $L(x)x^{\alpha-1}$ satisfies Assumption $A2$, and $\nu(-\infty, x]$ is light-tailed. We can decompose X_n into a centered spectrally positive part Y_n and a centered spectrally negative part $X_n - Y_n$. Then, Y_n satisfies the extended LDP in Theorem 2.2.7. On the other hand, observe that

$$\mathbf{P}(d(X_n, Y_n) > \epsilon) \le \mathbf{P}(\|X_n - Y_n\|_\infty > \epsilon) \le 3\mathbf{P}(|X_n(1) - Y_n(1)| > \epsilon/3);$$

where we used Etemadi's inequality for Levy processes (see e.g. [84], Lemma A.4) in the last step. Since $X_n - Y_n$ is light-tailed, the latter probability vanishes at exponential rate due to Cramers theorem. This allows one to apply Corollary 2.2.2 with Y_n and conclude that X_n satisfies the same LDP as the one in Theorem 2.2.7.

2.2.3 Extended LDP for random walks

Let $S_n \triangleq Z_1 + \cdots + Z_n$ where the Z_i's are non-negative random variables. Consider the centered and scaled process $\bar S_n = \{\bar S_n(t), t \in [0,1]\}$ where $\bar S_n(t) \triangleq \frac{1}{n}\sum_{i=1}^{[nt]}(Z_i - \mathbf{E}Z)$, $t \in [0,1]$. We assume that $\mathbf{P}(Z \ge x) = \exp(-L(x)x^\alpha)$ where $\alpha \in (0,1)$ and $L(\cdot)$ is a slowly-varying function. We also assume A2 is in force i.e., the mapping $x \mapsto L(x)x^{\alpha-1}$ is non-increasing for sufficiently large x. The goal of this section is to prove an extended LDP for $\bar S_n$. Towards this goal, we construct an auxilliary process $\hat S_n$. Let $Q^{\leftarrow}(x) = \inf\{y \ge 0 : \mathbf{P}(Z \ge y) < x\}$, and set $\hat S_n \triangleq \hat J_n^k + \hat H_n^k$ where

$$\hat J_n^k(t) \triangleq \frac{1}{n}\sum_{i=1}^{k} Q^{\leftarrow}(V_{(i)})\mathbb{1}_{[U_i,1]}(t) + \frac{1}{n} Z \mathbb{1}_{\{1\}}(t)$$

and

$$\hat H_n^k(t) \triangleq \frac{1}{n}\sum_{i=k+1}^{n-1} Q^{\leftarrow}(V_{(i)})\mathbb{1}_{[U_i,1]}(t) - \frac{1}{n}\mathbf{E}Z\sum_{i=1}^{n-1}\mathbb{1}_{[U_i,1]}(t) - \frac{1}{n}\mathbf{E}Z\mathbb{1}_{\{1\}}(t).$$

Note that $V_{(1)}, V_{(2)}, \ldots, V_{(n-1)}$ are the order statistics (in ascending order) of $V_1, V_2, \ldots, V_{n-1}$, which are i.i.d. Uniform$[0,1]$ and independent of Z. Similarly to the case of Levy processes, the extended LDP of $\bar S_n$ hinges on the following technical lemmas; their proofs are deferred to a technical section. Let $\mathbb{D}_{\leqslant\infty}[0,1]$ denote the subspace of $\mathbb{D}[0,1]$ consisting of non-decreasing pure jump functions vanishing at the origin (not necessarily continuous at 1, though). Let $\mathbb{D}_{\leqslant k}[0,1]$ denote the subspace of $\mathbb{D}_{\leqslant\infty}[0,1]$ consisting of paths that have at most k discontinuities. Define I

$$I(\xi) = \begin{cases} \sum_{t:\,\xi(t)\neq\xi(t-)} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \mathbb{D}_{\leqslant\infty}[0,1]; \\ \infty & \text{otherwise.} \end{cases}$$

Let I_k be defined as

$$I_k(\xi) = \begin{cases} \sum_{t:\,\xi(t)\neq\xi(t-)} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \mathbb{D}_{\leqslant k}[0,1]; \\ \infty & \text{otherwise.} \end{cases} \tag{2.15}$$

Lemma 2.2.8. *For each fixed k, $\tilde{J}_n^k$ satisfies the LDP in* $(\mathbb{D}[0,1], \mathcal{T}_{J_1})$ *with speed* $L(n)n^{\alpha}$ *and rate function* $\tilde{I}_k$.

Lemma 2.2.9. *For each* $\delta > 0$ *and each open set* G, *there exist* $\epsilon > 0$ *and* $K \geq 0$ *such that for any* $k \geq K$

$$\inf_{\xi \in G} \tilde{I}_k(\xi) \leq \inf_{\xi \in G} \tilde{I}(\xi) + \delta.$$

The next lemma shows that $\tilde{H}_n^k$ is asymptotically negligible.

Lemma 2.2.10. *For every* $\epsilon > 0$ *it holds that*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{L(n)n^{\alpha}} \log \mathbf{P}(\|\tilde{H}_n^k\|_{\infty} > \epsilon) = -\infty.$$

With the above lemmas we are ready to prove the extended large deviation principle for $\bar{S}_n$.

Theorem 2.2.11. *The scaled random walk* $\bar{S}_n$ *satisfies the extended large deviation principle in* $(\mathbb{D}[0,1], \mathcal{T}_{J_1})$ *with speed* $L(n)n^{\alpha}$ *and rate function* $\tilde{I}$.

Proof. We show that $\tilde{S}_n$ satisfies the extended LDP with speed $L(n)n^{\alpha}$ and is exponentially equivalent to $\bar{S}_n$ so that Corollary 2.2.2 applies, and hence, in turn, $\bar{S}_n$ satisfies the same extended LDP. With regard to the exponential equivalence, let $R_i \triangleq \#\{j \in \mathbb{N} : U_j \leq U_i,\ 1 \leq j \leq n-1\}$. Then, we claim that

$$\bar{S}_n \overset{\mathcal{D}}{=} \frac{1}{n}\sum_{i=1}^{n-1} \big(Q_n^{\leftarrow}(V_{(i)}) - \mathbf{E}Z\big) \mathbb{1}_{\{[R_i/n,1]\}} + \frac{1}{n}(Z - \mathbf{E}Z)\mathbb{1}_{\{1\}}.$$

To see why this distributional equality holds, note that $\{R_1, \ldots, R_{n-1}\}$ is a uniformly random permutation of $\{1, \ldots, n-1\}$, and $\{Q_n^{\leftarrow}(V_{(1)}), \ldots, Q_n^{\leftarrow}(V_{(n-1)})\}$ has the same distribution as the order statistics (in descending order) of $Z_1, \ldots, Z_{n-1}$ since $Q_n^{\leftarrow}(V_i)$ has the same distribution as Z for each i. Now, we move on to showing that $\bar{S}_n$ is close to $\tilde{S}_n$ — i.e., $\mathbf{P}(d_{J_1}(\bar{S}_n, \tilde{S}_n) > \epsilon)$ is asymptotically negligible. Recall that

$$\tilde{S}_n = \frac{1}{n}\sum_{i=1}^{n-1} \big(Q_n^{\leftarrow}(V_{(i)}) - \mathbf{E}Z\big) \mathbb{1}_{\{[U_i,1]\}} + \frac{1}{n}(Z - \mathbf{E}Z)\mathbb{1}_{\{1\}}.$$

First, observe that R_i is the rank of U_i among $U_1, \ldots, U_{n-1}$, and hence, the R_ith earliest jump of both $\bar{S}_n$ and $\tilde{S}_n$ equals $Q_n^{\leftarrow}(V_{(i)})$. Therefore, the jumps associated with $Q^{\leftarrow}(V_{(1)}),\dots,Q^{\leftarrow}(V_{(n-1)}),Z$ are arranged in the same order for S_n and $\hat{S}_n$ with probability 1. Moreover, the jump times of S_n and $\hat{S}_n$ are $\frac{1}{n},\frac{2}{n},\dots,\frac{n-1}{n},\frac{n}{n}$ and $U_{(1)},U_{(2)},\dots,U_{(n-1)},1$, respectively. Since $0<U_{(1)}<\cdots<U_{(n-1)}<1$ with probability 1, the piecewise linear time change $\lambda:[0,1]\to[0,1]$ defined by the linear interpolation of $\lambda(0)=0$, $\lambda(1)=1$, and $\lambda(i/n)=U_{(i)}$ for $i=1,\dots,n-1$ is a homeomorphism with probability 1. Therefore, the J_1 distance between $\hat{S}_n$ and S_n is bounded by

$$\sup_{1\le i\le n-1}|i/n-U_{(i)}|$$

with probability 1. The latter supremum can be bounded in terms of the Kolmogorov-Smirnov statistic, and from the inequality (1.5) in Corollary 1 of [61], we obtain

$$\begin{aligned}\mathbf{P}\Big(\sup_{1\le i\le n-1}|i/n-U_{(i)}|>\delta\Big) &\le \mathbf{P}\Big(\sqrt{n}\sup_{x\in[0,1]}\Big|\frac{1}{n}\sum_{i=1}^{n}I(U_i\le x)-x\Big|>\delta\sqrt{n}\Big)\\ &\le 2e^{-2\delta^2 n}.\end{aligned}$$

Hence,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\mathbf{P}(d_{J_1}(\hat{S}_n,S_n)>\delta)\\ &\le\limsup_{n\to\infty}\frac{1}{L(n)n^\alpha}\log\mathbf{P}\Big(\sup_{1\le i\le n-1}|i/n-U_{(i)}|>\delta\Big)=-\infty.\end{aligned}$$

In view of Corollary 2.2.2, the proof is done if we show that $\hat{S}_n$ satisfies the extended LDP with speed $L(n)n^\alpha$ and rate function $\hat{I}$. To do so, we apply Proposition 2.2.1. Note that Lemma 2.2.8 verifies condition (i) of Proposition 2.2.1; (ii) is trivially satisfied since $\hat{I}_k\equiv\hat{I}$; Lemma 2.2.9 verifies (iii); Lemma 2.2.10 verifies (iv). Therefore Proposition 2.2.1 applies to $\hat{J}_n^k+\hat{H}_n^k$, and the proof of Theorem 2.2.11 is complete. □

2.2.4 Extension to multidimensional processes

Let $X^{(1)},\dots,X^{(d)}$ be independent processes satisfying assumptions $a1$ and $a2$.

- $a1$. For each i, $X^{(i)}$ is a real-valued Levy process with Levy measure $\nu^{(i)}$ which has non-negative support satisfying $\nu^{(i)}[x,\infty)=\exp(-L(x)x^\alpha)$ where $\alpha\in(0,1)$ and $L(\cdot)$ is slowly varying at infinity.

a2. The mapping $x \mapsto L(x)x^{-1}$ is non-increasing for sufficiently large x.

Let $X_n^{(i)}(t)$ denote the centered and scaled processes:

$$X_n^{(i)}(t) \triangleq \frac{1}{n}X^{(i)}(nt) - t\mathbf{E}X^{(i)}(1).$$

The next theorem establishes the extended LDP for $(X_n^{(1)},\dots,X_n^{(d)})$.

Theorem 2.2.12. *$\left(X_n^{(1)}, X_n^{(2)},\dots,X_n^{(d)}\right)$ satisfies the extended LDP in the product space $\left(\prod_{i=1}^d \mathbb{D}([0,1],\mathbb{R}_+), \prod_{i=1}^d \mathcal{T}_{J_1}\right)$ with speed $L(n)n$ and rate function*

$$I^d(\xi_1,\dots,\xi_d) = \begin{cases} \sum_{j=1}^d \sum_{t\in[0,1]} \left(\xi_j(t) - \xi_j(t-)\right)^\alpha & \text{if } \xi_j \in \mathbb{D}_{<\infty}[0,1] \\ & \text{for each } j = 1,\dots,d; \\ \infty & \text{otherwise.} \end{cases} \tag{2.16}$$

For each i, we consider the same distributional decomposition of $X_n^{(i)}$ as in Section 2.2.2:

$$X_n^{(i)} \stackrel{\mathcal{D}}{=} J_n^{k\,(i)} + H_n^{k\,(i)} + R_n^{(i)}.$$

The proof of the theorem follows the same lines as in the one-dimensional case, from Proposition 2.2.1, Lemma 2.2.6, and the following lemmas that parallel Lemma 2.2.4 and Lemma 2.2.5.

Lemma 2.2.13. *For each fixed $k > 0$, $\left(J_n^{k\,(1)},\dots,J_n^{k\,(d)}\right)$ satisfies the LDP in $\left(\prod_{i=1}^d \mathbb{D}[0,1], \prod_{i=1}^d \mathcal{T}_{J_1}\right)$ with speed $L(n)n$ and rate function $I_k^d : \prod_{i=1}^d \mathbb{D}[0,1] \to [0,\infty]$*

$$I_k^d(\xi_1,\dots,\xi_d) \triangleq \begin{cases} \sum_{i=1}^d \sum_{t\in[0,1]} \left(\xi_i(t) - \xi_i(t-)\right)^\alpha & \text{if } \xi_i \in \mathbb{D}_{\leqslant k}[0,1] \\ & \text{for each } i = 1,\dots,d; \\ \infty & \text{otherwise.} \end{cases} \tag{2.17}$$

Lemma 2.2.14. *For each $\delta > 0$ and each open set G, there exist $\epsilon > 0$ and $K \geq 0$ such that for any $k \geq K$*

$$\inf_{(\xi_1,\dots,\xi_d)\in G} I_k^d(\xi_1,\dots,\xi_d) \leq \inf_{(\xi_1,\dots,\xi_d)\in G} I^d(\xi_1,\dots,\xi_d) + \delta. \tag{2.18}$$

We conclude this section with the extended LDP for multidimensional random walks. Let $S_n^{(i)} = Z_1^{(i)} + \cdots + Z_n^{(i)}$ be a random walk with non-negative increments for each $i = 1,\dots,d$. Consider $\bar S_n^{(i)} = \{\bar S_n^{(i)}(t), t \in [0,1]\}$ where $\bar S_n^{(i)}(t) = \frac{1}{n}\sum_{j=1}^{[nt]}\big(Z_j^{(i)} - \mathbf{E}Z^{(i)}\big)$. We assume that $\mathbf{P}(Z_j^{(i)} \geq x) = \exp(-L(x)x^{\alpha})$ where $\alpha \in (0,1)$ and $L(\cdot)$ is a slowly varying function, and a2 is in force. The following theorem can be obtained by adjusting Lemma 2.2.8 and Lemma 2.2.9 to the multi-dimensional context — in the same way as Lemma 2.2.4 and Lemma 2.2.5 were adjusted to the multi-dimensional counterparts in the proof of Theorem 2.2.12 — and then applying Proposition 2.2.1.

Let

$$\bar I^d(\xi_1,\dots,\xi_d) = \begin{cases} \sum_{j=1}^{d}\sum_{t\in[0,1]}\big(\xi_j(t) - \xi_j(t-)\big)^{\alpha} & \text{if } \xi_j \in \mathbb{D}_{\leqslant\infty}[0,1] \\ & \text{for each } j = 1,\dots,d; \\ \infty & \text{otherwise.} \end{cases} \tag{2.19}$$

Theorem 2.2.15. *$\bar S_n^{(1)}, \bar S_n^{(2)}, \dots, \bar S_n^{(d)}$ satisfies the extended LDP in the product space $\big(\prod_{i=1}^{d}\mathbb{D}[0,1], \prod_{i=1}^{d}\mathcal{T}_{J_1}\big)$ with speed $L(n)n^{\alpha}$ and rate function $\bar I^d$.*

Remark 2. Note that Theorem 2.2.12 and Theorem 2.2.15 can be extended to heterogeneous processes. For example, if the Levy measure $\nu^{(i)}$ of the process $X^{(i)}$ has Weibull tail distribution $\nu^{(i)}[x,\infty) = \exp(-c_i L(x)x^{\alpha})$ where $c_i \in (0,\infty)$ for each $i \leq d_0 \leq d$, and all the other processes have lighter tails — i.e., $L(x)x^{\alpha} = o(L_i(x)x^{\alpha_i})$ for $i > d_0$ — then it is straightforward to check that $(\bar X_n^{(1)},\dots,\bar X_n^{(d)})$ satisfies the extended LDP with rate function

$$I^d(\xi_1,\dots,\xi_d) = \begin{cases} \sum_{j=1}^{d_0} c_j \sum_{t\in[0,1]}\big(\xi_j(t) - \xi_j(t-)\big)^{\alpha} & \text{if } \xi_j \in \mathbb{D}_{\leqslant\infty}[0,1] \\ & \text{for } j = 1,\dots,d_0; \\ & \text{and } \xi_j \equiv 0 \text{ for } j > d_0; \\ \infty & \text{otherwise.} \end{cases}$$

Similarly, $(\bar S_n^{(1)},\dots,\bar S_n^{(d)})$ satisfies the extended LDP with rate function $\bar I^d$ defined by replacing $\mathbb{D}_{\leqslant\infty}[0,1]$ with $\mathbb{D}_{\leqslant\infty}[0,1]$ in the definition of I^d above under the corresponding conditions on the tail distribution of the $Z_1^{(i)}$'s.

2.3 Implications of the extended LDP

This section consists of two parts. In the first part, we develop a large deviation principle for Lipschitz functions of Levy processes and random walks. In the second part, we derive the large deviation principle, for the same processes, in the Skorokhod space equipped with the M_1' topology.

2.3.1 A contraction principle

Let $\mathbf{X}_n$ denote the scaled Levy processes $(X_n^{(1)}, \dots, X_n^{(d)})$, and let $\mathbf{S}_n$ denote the scaled random walks $(S_n^{(1)}, \dots, S_n^{(d)})$ as defined in Section 2.2.2. Recall also the rate functions I^d defined in (2.16) and $\tilde{I}^d$ defined in (2.19).

Corollary 2.3.1. *Let $(\mathbb{S}, d)$ be a metric space and $\phi : \prod_{i=1}^{d} \mathbb{D}[0,1] \to \mathbb{S}$ be a Lipschitz continuous mapping w.r.t. the J_1 Skorokhod metric. Set*

$$I'(x) \triangleq \inf_{\phi(\xi)=x} I^d(\xi) \quad \text{and} \quad \tilde{I}'(x) \triangleq \inf_{\phi(\xi)=x} \tilde{I}^d(\xi)$$

and suppose that I' (or $\tilde{I}'$) is a good rate function — i.e., $\Psi_{I'}(a) \triangleq \{x \in \mathbb{S} : I'(s) \le a\}$ (or $\Psi_{\tilde{I}'}(a) \triangleq \{x \in \mathbb{S} : \tilde{I}'(s) \le a\}$) is compact for each $a \in [0, \infty)$. Then, $\phi(\mathbf{X}_n)$ (or $\phi(\mathbf{S}_n)$) satisfies the large deviation principle in $(\mathbb{S}, d)$ with speed $L(n)n$ and good rate function I' (or $\tilde{I}'$).

Proof. Since the argument for $\phi(\mathbf{S}_n)$ is essentially identical, we only prove the LDP for $\phi(\mathbf{X}_n)$. We start with the upper bound. Suppose that the Lipschitz constant of ϕ is $\|\phi\|_{\text{Lip}}$. Note that the J_1 distance is dominated by the supremum distance therefore, $\|\mathbf{H}_n^k\|_\infty \le \delta$ and $\|\mathbf{R}_n\|_\infty \le \delta$ implies that $d_{J_1}(\phi(\mathbf{J}_n^k), \phi(\mathbf{X}_n)) \le 2\|\phi\|_{\text{Lip}}\delta$, where $\mathbf{J}_n^k \triangleq (J_n^{k\,(1)}, \dots, J_n^{k\,(d)})$, $\mathbf{H}_n^k \triangleq (H_n^{k\,(1)}, \dots, H_n^{k\,(d)})$, and $\mathbf{R}_n \triangleq (R_n^{(1)}, \dots, R_n^{(d)})$. Thus, for any closed set F,

$$\begin{aligned}
\mathbf{P}\big(\phi(\mathbf{X}_n) \in F\big) &\le \mathbf{P}\Big(\phi(\mathbf{X}_n) \in F, d_{J_1}\big(\phi(\mathbf{J}_n^k), \phi(\mathbf{X}_n)\big) \le 2\|\phi\|_{\text{Lip}}\delta\Big) \\
&\quad + \mathbf{P}\Big(d_{J_1}\big(\phi(\mathbf{J}_n^k), \phi(\mathbf{X}_n)\big) > 2\|\phi\|_{\text{Lip}}\delta\Big) \\
&\le \mathbf{P}\Big(\phi(\mathbf{J}_n^k) \in F^{2\|\phi\|_{\text{Lip}}\delta}\Big) + \mathbf{P}\Big(d_{J_1}\big(\phi(\mathbf{J}_n^k), \phi(\mathbf{X}_n)\big) > 2\|\phi\|_{\text{Lip}}\delta\Big) \\
&\le \mathbf{P}\Big(\mathbf{J}_n^k \in \phi^{-1}\big(F^{2\|\phi\|_{\text{Lip}}\delta}\big)\Big) \\
&\quad + \mathbf{P}\big(\|\mathbf{H}_n^k\|_\infty > \delta\big) + \mathbf{P}\big(\|\mathbf{R}_n\|_\infty > \delta\big).
\end{aligned} \tag{2.20}$$

Since $\mathbf{P}\left(\|\mathbf{H}_n^k\|_\infty > \delta\right) \le \sum_{i=1}^d \mathbf{P}\left(\|H_n^{k(i)}\|_\infty > \delta/d\right)$, and $\mathbf{P}\left(\|\mathbf{R}_n\|_\infty > \delta\right)$ decays at an exponential rate, we get the following bound by applying the principle of the maximum term and Theorem 2.2.12:

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\left(\mathbf{X}_n \in F\right) \\
&= \limsup_{n\to\infty} \frac{\log \mathbf{P}\left(\mathbf{J}_n^k \in \Phi^{-1} F^{2\delta\|\Phi\|_{\mathrm{Lip}}}\right)}{L(n)n} \vee \limsup_{n\to\infty} \frac{\log \mathbf{P}\left(\|\mathbf{H}_n^k\|_\infty > \delta\right)}{L(n)n} \\
&\le \limsup_{n\to\infty} \frac{\log \mathbf{P}\left(\mathbf{J}_n^k \in \Phi^{-1} F^{2\delta\|\Phi\|_{\mathrm{Lip}}}\right)}{L(n)n} \vee \limsup_{n\to\infty} \frac{\log \sum_{i=1}^d \mathbf{P}\left(\|H_n^{k(i)}\|_\infty > \delta/d\right)}{L(n)n} \\
&\le -\inf_{(\xi_1,\dots,\xi_d)\in\Phi^{-1}F^{2\delta\|\Phi\|_{\mathrm{Lip}}}} I^d(\xi_1,\dots,\xi_d) \vee \limsup_{n\to\infty} \frac{\log \mathbf{P}\left(\|H_n^{k(1)}\|_\infty > \delta/d\right)}{L(n)n}.
\end{aligned}$$

From Lemma 2.2.6, we can take $k\to\infty$ to get

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\log \mathbf{P}\left(\mathbf{X}_n \in F\right)}{L(n)n} &\le -\inf_{(\xi_1,\dots,\xi_d)\in\Phi^{-1}\left(F^{2\delta\|\Phi\|_{\mathrm{Lip}}}\right)} I^d(\xi_1,\dots,\xi_d) \\
&= -\inf_{x\in F^{2\delta\|\Phi\|_{\mathrm{Lip}}}} I'(x). \qquad (2.21)
\end{aligned}$$

From Lemma 4.1.6 of [22], $\lim_{\delta\to 0} \inf_{x\in F^{\delta\|\Phi\|_{\mathrm{Lip}}}} I'(x) = \inf_{x\in F} I'(x)$. Letting $\delta\to 0$ in (2.21), we arrive at the desired large deviation upper bound.

Turning to the lower bound, consider an open set G. Since $\Phi^{-1}(G)$ is open, from Theorem 2.2.12,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\left(\Phi(\mathbf{X}_n) \in G\right) &= \liminf_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\left(\mathbf{X}_n \in \Phi^{-1}(G)\right) \\
&\ge -\inf_{(\xi_1,\dots,\xi_d)\in\Phi^{-1}(G)} I(\xi) = -\inf_{x\in G} I'(x).
\end{aligned}$$

□

2.3.2 An application to actuarial science

In this section, we illustrate the value of Corollary 2.3.1. We consider level crossing probabilities of Levy processes where the jump sizes are conditioned to be moderate. Specifically, we apply Corollary 2.3.1 in order to provide large deviations estimates for

$$\mathbf{P}\left(\sup_{t\in[0,1]} X_n(t) \ge c;\ \sup_{t\in[0,1]} \left(X_n(t) - X_n(t-)\right) \le b\right). \qquad (2.22)$$

We emphasize that this type of rare events are difficult to analyze with the tools developed previously. In particular, the sample path LDP proved in [40] is w.r.t. the L_1 topology. Since the closure of the sets in (2.22) w.r.t. the L_1 topology contains the zero function, the LDP upper bound would not provide any information. On the other hand, we will see that our extended LDP w.r.t. the J_1 topology can successfully provide a useful asymptotics.

Functionals like (2.22) appear in actuarial models, in case excessively large insurance claims are reinsured and therefore do not play a role in the ruin of an insurance company. In [84], the authors studied the finite-time ruin probability, using probabilistic techniques in case of regularly varying Levy measures and confirmed that the conventional wisdom "the principle of a single big jump" can be extended to "the principle of the minimal number of big jumps" in such a context. Here we show that a similar result — with subtle differences — can be obtained in case the Levy measure has a Weibull tail.

Define the function $\varphi : \mathbb{D}[0,1] \to \mathbb{R}^2$ as

$$\varphi(\xi) = (\varphi_1(\xi), \varphi_2(\xi)) \triangleq \left(\sup_{t\in[0,1]} \xi(t); \sup_{t\in[0,1]} |\xi(t) - \xi(t-)| \right).$$

In order to apply Corollary 2.3.1, we will validate that φ is Lipschitz continuous and that $I'(x,y) \triangleq \inf_{\{\xi \in \mathbb{D}[0,1]:\ \varphi(\xi)=(x,y)\}} I(\xi)$ is a good rate function.

Lemma 2.3.2. *The function* $\varphi : \mathbb{D}[0,1] \to \mathbb{R}^2$ *is Lipschitz continuous w.r.t. the* J_1 *topology.*

Proof. For the Lipschitz continuity of φ, we claim that each component of φ is Lipschitz continuous. We first examine φ_1. Let $\xi, \zeta \in \mathbb{D}[0,1]$ and suppose w.l.o.g. that $\sup_{t\in[0,1]} \xi(t) > \sup_{t\in[0,1]} \zeta(t)$. For an arbitrary non-decreasing homeomorphism $\lambda : [0,1] \to [0,1]$,

$$\begin{aligned} |\varphi_1(\xi) - \varphi_1(\zeta)| &= \Big|\sup_{t\in[0,1]} \xi(t) - \sup_{t\in[0,1]} \zeta(t)\Big| = \Big|\sup_{t\in[0,1]} \xi(t) - \sup_{t\in[0,1]} \zeta\circ\lambda(t)\Big| \qquad (2.23)\\ &\le \sup_{t\in[0,1]} |\xi(t) - \zeta\circ\lambda(t)| \le \sup_{t\in[0,1]} |\xi(t) - \zeta\circ\lambda(t)| \vee |\lambda(t) - t|. \end{aligned}$$

Taking the infimum over λ, we conclude that

$$|\varphi_1(\xi) - \varphi_1(\zeta)| \le \inf_{\lambda\in\Lambda} \sup_{t\in[0,1]} \{|\xi(t) - \zeta(\lambda(t))| \vee |\lambda(t) - t|\} = d_{J_1}(\xi, \zeta).$$

Therefore, φ_1 is Lipschitz with the Lipschitz constant 1.

Now, in order to prove that $_2(\ )$ is Lipschitz, x two distinct paths $;\ 2$ D[0;1] and assume w.l.o.g. that $_2(\ ) > {}_2(\ )$. Let c , $_2(\ )\quad {}_2(\ ) > 0$, and let $t\ 2\ [0;1]$ be the maximum jump time of , i.e., $_2(\ ) = j\ (t\ )\quad (t\ )j$. For any > 0 there exists so that

$$d_{J_1}(\ ;\ )\ ,\ \inf_{2} fk \qquad k_1 \_ k \quad ek_1 g \quad k \qquad k_1 \_ k \quad ek_1 \quad :$$
$$j\ (t\ ) \qquad (t\ )j \_ j\ (t\ \ ) \qquad (t\ \ )j \quad : \tag{2.24}$$

From the general inequality $ja\quad bj \_ jc\quad dj \quad \frac{1}{2}(ja\quad cj\quad jb\quad dj)$,

$$j\ (t\ ) \qquad (t\ )j \_ j\ (t_1\ ) \qquad (t\ \ )j \tag{2.25}$$
$$\frac{1}{2}\ j\ (t\ ) \quad (t\ \ )j \quad j \quad (t\ ) \qquad (t\ \ )j$$
$$= \frac{1}{2}\quad {}_2(\ ) \quad {}_2(\ ) \ = c{=}2: \tag{2.26}$$

In view of (2.24) and (2.25), $d_{J_1}(\ ;\ )\quad \frac{1}{2}j\ (\ )\quad (\ )j\quad :$ Since is arbitrary, we get the desired Lipschitz bound with Lipschitz constant 2.

□

Now, we examine that the function I^0 satis es the necessary conditions of Corollary 2.3.1.

Lemma 2.3.3. *The rate function I^0 is a good rate function, and it is equal to*

$$I^0(c;b) = \begin{cases} \frac{c}{b}\ b\ +\ c \quad \frac{c}{b}\ b & \text{if } 0 < b \quad c; \\ 0 & \text{if } b = c = 0; \\ 1 & \text{otherwise.} \end{cases} \tag{2.27}$$

Proof. Note rst that (2.27) is obvious except for the rst case, and hence, we assume that $0 < b \quad c$. Note $I^0(c;b) = \inf f I(\ ) :\ \ 2\ \mathbb{D}_{6\,1}[0;1];\ \ (\ ) = (c;b)g$ since $I(\ ) = 1$ for $\ 2\ \mathrm{D}_{6\,1}[0;1]$. Now, de ne $C\ ,\ f\ \ 2\ \mathbb{D}_{6\,1}[0;1];(c;b) =\ (\ )g$ and remember that any $2\ \mathrm{D}_{6\,1}[0;1]$ admits the following representation:

$$= \sum_{i=1} x_i \mathbb{1}_{[u_i;1]}; \tag{2.28}$$

where u_i's are distinct in (0;1) and x_i's are non-negative and sorted in a decreasing order. Consider a step function $_0\ 2\ C$, with $\frac{c}{b}$ jumps of size b and one jump of size $c - \lfloor \frac{c}{b} \rfloor b$, so that $\xi_0 = \sum_{i=1}^{\lfloor \frac{c}{b} \rfloor} b \mathbb{1}_{[u_i,1]} + (c - \lfloor \frac{c}{b} \rfloor)\mathbb{1}_{[u_{\lfloor \frac{c}{b} \rfloor + 1},1]}$. It is clear that $\psi(\xi_0) = (c,b)$ and $I(\xi_0) = \lfloor \frac{c}{b} \rfloor b^\alpha + \left(c - \lfloor \frac{c}{b} \rfloor b\right)^\alpha$. Since $\xi_0 \in C$, the infimum of I over C should be at most $I(\xi_0)$ i.e., $I(\xi_0) \geq I'(c,b)$. To prove that ξ_0 is the minimizer of I over C, we will show that $I(\xi) \geq I(\xi_0)$ for any $\xi = \sum_{i=1}^{\infty} x_i \mathbb{1}_{[u_i,1]} \in C$ by constructing ξ' such that $I(\xi) \geq I(\xi')$ while $I(\xi') = I(\xi_0)$. There has to be an integer k such that $x_k' \triangleq \sum_{i=k}^{\infty} x_i \leq b$. Let $\xi^1 \triangleq \sum_{i=1}^{k} x_i^1 \mathbb{1}_{[u_i,1]}$ where x_i^1 is the i^{th} largest element of $\{x_1, \ldots, x_{k-1}, x_k'\}$. Then, $\xi^1 \in C$ and $I(\xi^1) \leq I(\xi)$ due to the sub-additivity of $x \mapsto x^\alpha$. Now, given $\xi^j = \sum_{i=1}^{k} x_i^j \mathbb{1}_{[u_i,1]}$, we construct ξ^{j+1} as follows. Find the first l such that $x_l^j < b$. If $x_l^j = 0$ or $x_{l+1}^j = 0$, set $\xi^{j+1} \triangleq \xi^j$. Otherwise, find the first m such that $x_{m+1}^j = 0$ and merge the l^{th} jump and the m^{th} jump. More specifically, set $x_l^{j+1} \triangleq x_l^j + x_m^j \wedge (b - x_l^j)$, $x_m^{j+1} \triangleq x_m^j - x_m^j \wedge (b - x_l^j)$, $x_i^{j+1} \triangleq x_i^j$ for $i \notin \{l, m\}$, and $\xi^{j+1} \triangleq \sum_{i=1}^{k} x_i^{j+1} \mathbb{1}_{[u_i,1]}$. Note that $x_l^{j+1} + x_m^{j+1} = x_l^j + x_m^j$ while either $x_l^{j+1} = b$ or $x_m^{j+1} = 0$. That is, compared to ξ^j, ξ^{j+1} has either one less jump or one more jump with size b, while the total sum of the jump sizes and the maximum jump size remain the same. From this observation and the concavity of $x \mapsto x^\alpha$, it is straightforward to check that $I(\xi^{j+1}) \leq I(\xi^j)$. By iterating this procedure k times, we arrive at $\xi' \triangleq \xi^k$ such that all the jump sizes of ξ' are b, or there is only one jump whose size is not b. From this, we see that ξ^k has to coincide with ξ_0. We conclude that $I(\xi) \geq I(\xi^1) \geq \cdots \geq I(\xi^k) = I(\xi') = I(\xi_0)$, proving that ξ_0 is indeed a minimizer.

Now we check that $\Psi_{I'}(\gamma) \triangleq \{(c,b) : I'(c,b) \leq \gamma\}$ is compact for each $\gamma \in [0, \infty)$ so that I' is a good rate function. It is clear that $\Psi_{I'}(\gamma)$ is bounded. To see that $\Psi_{I'}(\gamma)$ is closed, suppose that $(c_1,b_1) \notin \Psi_{I'}(\gamma)$. In case $0 < b_1 < c_1$, note that I' can be written as $I'(c,b) = b^\alpha \lfloor c/b \rfloor + \left(c - b\lfloor c/b \rfloor\right)^\alpha$, from which it is easy to see that I' is continuous at such (c_1,b_1)'s. Therefore, one can find an open ball around (c_1,b_1) in such a way that it doesn't intersect with $\Psi_{I'}(\gamma)$. By considering the cases $c_1 < b_1$, $b_1 = 0$, $b_1 = c_1$ separately, the rest of the cases are straightforward to check. We thus conclude that I' is a good rate function. □

Now we can apply Corollary 2.3.1. Note that

$$\inf_{(x,y) \in [c,\infty) \times [0,b]} I'(x,y) = \inf_{(x,y) \in (c,\infty) \times [0,b)} I'(x,y) = I'(c,b).$$

That is the large deviation lower and upper bound coincide and hence,

$$\lim_{n\to\infty} \frac{\log \mathbf{P}\left(\sup_{t\in[0,1]} X_n(t) \geq c; \sup_{t\in[0,1]} \left(X_n(t) - X_n(t-)\right) \leq b\right)}{L(n)n^{\alpha}}$$

$$= \begin{cases} -\left(\lfloor \frac{c}{b} \rfloor b^{\alpha} + \left(c - \lfloor \frac{c}{b} \rfloor b\right)^{\alpha}\right) & \text{if } 0 < b \leq c; \\ 0 & \text{if } b = c = 0; \\ -\infty & \text{otherwise.} \end{cases}$$

From the expression of the rate function, it can be inferred that the most likely way level c is reached is due to $\lfloor \frac{c}{b} \rfloor$ jumps of size b and one jump of size $\left(c - \lfloor \frac{c}{b} \rfloor b\right)$. If we compare this with the insights obtained from the case of truncated regularly-varying tails in [84], we see that the total number of jumps is identical, but the size of the jumps are deterministic and non-identical, while in the regularly-varying case, they are random and identically distributed.

2.4 LDP with respect to the M_1' topology

2.4.1 Nonexistence of LDP in the J_1 topology

Consider a compound Poisson process with arrival rate equal to 1 whose jump distribution is Weibull with shape parameter $1/2$. To elaborate more, let $X_n(t) \triangleq \frac{1}{n}\sum_{i=1}^{N(nt)} Z_i - t$ with $\mathbf{P}(Z_i \geq x) = \exp(-x^{\alpha})$, $\mathbf{E}Z_i = 1$, and $\alpha = 1/2$. If X_n satisfies a full LDP w.r.t. the J_1 topology, the rate function that controls the LDP (with speed n^{α}) associated with X_n should be of the same form as the one that controls the extended LDP:

$$I(\xi) = \begin{cases} \sum_{t\in[0,1]} \left(\xi(t) - \xi(t-)\right)^{\alpha} & \text{if } \xi \in \mathbb{D}_{\uparrow}^{\infty}[0,1]; \\ \infty; & \text{otherwise.} \end{cases}$$

To show that such an LDP is fundamentally impossible, we construct a closed set A for which

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{n^{\alpha}} > -\inf_{\xi\in A} I(\xi). \tag{2.29}$$

Let

$$\varphi_{s,t}(\xi) \triangleq \lim_{\epsilon\to 0} \sup_{(s-\epsilon)\vee 0 \leq u \leq v \leq 1\wedge(t+\epsilon)} \left(\xi(v) - \xi(u)\right).$$

Let $A_{c;s;t} \triangleq \{\xi : \varphi_{s,t}(\xi) \geq c\}$ be, in rough terms, the set of paths which increase at least by c between time s and t. Then $A_{c;s;t}$ is a closed set for each c, s, and

t. Next, define

$$A_m \triangleq A_{1;\frac{m+1}{m+2};\frac{m+1}{m+2}+mh_m} \cap A_{1;\frac{m}{m+2};\frac{m}{m+2}+mh_m} \cap \bigcap_{j=0}^{m-1} A_{\frac{1}{m^2};\frac{j}{m+2};\frac{j}{m+2}+mh_m}$$

where $h_m = \frac{1}{(m+1)(m+2)}$, and let

$$A \triangleq \bigcup_{m=1}^{\infty} A_m.$$

To see that A is closed, we first claim that $\in D[0,1]\setminus A$ implies the existence of > 0 and $N \geq 0$ such that $B(;) \cap A_m = \emptyset$ for all $m \geq N$. To prove this claim, we argue by contradiction. It is straightforward to check that for each n, there has to be $s_n, t_n \in [1-1/n, 1)$ such that $s_n \leq t_n$ and $(t_n) - (s_n) \geq 1/2$, which in turn implies that must possess an infinite number of increases of size at least 1/2 in $[1 - , 1)$ for any > 0. This implies that cannot possess a left limit, which is contradictory to the assumption that $\in D[0,1]\setminus A$. On the other hand, since each A_m is closed, $\bigcup_{i=1}^{N} A_i$ is also closed, and hence, there exists $^0 > 0$ such that $B(; ^0) \cap A_m = \emptyset$ for $m = 1, \dots, N$. Now, from the construction of and ^{0}, $B(; \wedge ^0) \cap A = \emptyset$, proving that A is closed.

Next, we show that A satisfies (2.29). First note that if is a pure jump function that belongs to A_m, has to possess m upward jumps of size $1/m^2$ and 2 upward jumps of size 1, and hence,

$$\inf_{\in A} I( ) \geq \inf_m \left(1^{1/2} + 1^{1/2} + m(1/m^2)^{1/2} \right) = 3. \tag{2.30}$$

On the other hand, letting $(t) \triangleq (t) - (t -)$,

$$\begin{aligned}
&\mathbf{P}(X_{(n+1)(n+2)} \in A_n)\\
&\geq \prod_{j=0}^{n-1} \mathbf{P}\left\{ \sup_{t\in[0,1]} \left| X_{(n+1)(n+2)}\left(\tfrac{(n+1)j+nt}{(n+1)(n+2)}\right) - X_{(n+1)(n+2)}\left(\tfrac{(n+1)j}{(n+1)(n+2)}\right) \right| \geq \frac{1}{n^2} \right\}\\
&\quad \cdot \mathbf{P}\left\{ \sup_{t\in(0,1]} X_{(n+1)(n+2)}\left(\tfrac{(n+1)n+nt}{(n+1)(n+2)}\right) \geq 1 \right\}\\
&\quad \cdot \mathbf{P}\left\{ \sup_{t\in(0,1]} X_{(n+1)(n+2)}\left(\tfrac{(n+1)(n+1)+nt}{(n+1)(n+2)}\right) \geq 1 \right\}\\
&= \mathbf{P}\left\{ \sup_{t\in[0,1]} X_{(n+1)(n+2)}\left(\tfrac{nt}{(n+1)(n+2)}\right) \geq \frac{1}{n^2} \right\}^n
\end{aligned}$$

$$
\begin{aligned}
&\mathbf{P}\Big(\sup_{t\in[0,1]}\Big\{X_{(n+1)(n+2)}\big(\tfrac{nt}{(n+1)(n+2)}\big)\Big\} \geq 1\Big)^2\\
=\ &\mathbf{P}\Big(\sup_{t\in[0,1]}\Big\{X(nt)\Big\} \geq \frac{(n+1)(n+2)}{n^2}\Big)^n\\
\geq\ &\mathbf{P}\Big(\sup_{t\in[0,1]}\Big\{X(nt)\Big\} \geq (n+1)(n+2)\Big)^2\\
\geq\ &\mathbf{P}\Big(\sup_{t\in[0,1]}\Big\{X(nt)\Big\} \geq 6\Big)^n\,\mathbf{P}\Big(\sup_{t\in[0,1]}\Big\{X(nt)\Big\} \geq (n+1)(n+2)\Big)^2;
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\limsup_{n\to\infty}\frac{\log\mathbf{P}(X_n\in A)}{n} &\geq \limsup_{n\to\infty}\frac{\log\mathbf{P}(X_{(n+1)(n+2)}\in A_n)}{(n+1)(n+2)}\\
&\geq \limsup_{n\to\infty}\frac{\log\mathbf{P}\Big(\sup_{t\in[0,1]}\Big\{X(nt)\Big\}\geq 6\Big)^n}{(n+1)(n+2)} \qquad (2.31)\\
&\quad + 2\limsup_{n\to\infty}\frac{\log\mathbf{P}\Big(\sup_{t\in[0,1]}\Big\{X(nt)\Big\}\geq (n+1)(n+2)\Big)}{(n+1)(n+2)}\\
&= (I) + (II).
\end{aligned}
$$

Letting $p_n \triangleq \mathbf{P}\Big(\sup_{t\in[0,n]}\Big\{X(t)\Big\} < 6\Big)$,

$$
\begin{aligned}
(I) &= \limsup_{n\to\infty}\frac{\log(1-p_n)^n}{(n+1)(n+2)} = \limsup_{n\to\infty}\frac{np_n\log(1-p_n)^{1/p_n}}{(n+1)(n+2)} \qquad (2.32)\\
&= \limsup_{n\to\infty}\frac{-np_n}{(n+1)(n+2)} = 0;
\end{aligned}
$$

since $p_n \to 0$ as $n\to\infty$. Next, observe that

$$
\{Z_1 \geq (n+1)(n+2)\} \Rightarrow \left(\sup_{t\in[0,1]}\Big\{X(nt)\Big\} \geq (n+1)(n+2)\right)
$$

and conclude

$$(II) = 2 \limsup_{n\to\infty} \frac{\log \mathbf{P}\left\{\sup_{t\in[0,1]} |X(nt)| \ge (n+1)(n+2)\right\}}{(n+1)(n+2)}$$

$$\ge 2 \limsup_{n\to\infty} \frac{\log \mathbf{P}\left(Z_1 \ge (n+1)(n+2)\right)}{(n+1)(n+2)} = 2 \limsup_{n\to\infty} \frac{\log e^{-(n+1)(n+2)}}{(n+1)(n+2)}$$
$$= -2. \tag{2.33}$$

From (2.31), (2.32), (2.33),

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{n} \ge -2. \tag{2.34}$$

This along with (2.30),

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(X_n \in A)}{n} \ge -2 > -3 \ge -\inf_{\xi\in A} I(\xi);$$

which means that A indeed is a counterexample for the desired LDP.

2.4.2 The LDP with the M_1' topology

Recall that $X_n(t) \triangleq \frac{1}{n}X(nt) - t\mathbf{E}X(1)$ and $S_n(t) = \frac{1}{n}\sum_{i=1}^{[nt]}(Z_i - \mathbf{E}Z).$ In this section, we establish the full LDP for X_n and S_n w.r.t. the M_1' topology. Let

$$I_{M_1'}(\xi) \triangleq \begin{cases} \sum_{t\in[0,1]} \xi(t) - \xi(t-) & \text{if } \xi \text{ is a non-decreasing pure jump function with } \xi(0) \ge 0; \\ \infty & \text{otherwise}. \end{cases} \tag{2.35}$$

The following lemma ensures that $I_{M_1'}$ is indeed a good rate function. Note that I and $I_{M_1'}$ differ only if the path has a jump at either 0 or 1.

Proposition 2.4.1. *$I_{M_1'}$ is a good rate function w.r.t. the M_1' topology.*

Corollary 2.4.2. *The stochastic processes X_n and S_n satisfy the LDP in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$ with speed $L(n)n$ and good rate function $I_{M_1'}$.*

Proof. Since the proof for S_n is essentially identical, we only provide the proof for X_n. From Proposition 2.4.1 we know that $I_{M_1'}$ is a good rate function. For the LDP upper bound, suppose that F is a closed set w.r.t. the M_1' topology. Then, it is also closed w.r.t. the J_1 topology. From the upper bound of Theorem 2.2.7 and the fact that $I_{M_1'}(\xi) \le I(\xi)$ for any $\xi \in \mathbb{D}[0,1]$,

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(X_n \in F)}{L(n)n} \le -\lim_{\epsilon\to 0} \inf_{\xi\in F^\epsilon} I(\xi) \le -\lim_{\epsilon\to 0} \inf_{\xi\in F^\epsilon} I_{M_1'}(\xi) = -\inf_{\xi\in F} I_{M_1'}(\xi).$$

Turning to the lower bound, suppose that G is an open set w.r.t. the M_1' topology. We claim that

$$\inf_{\xi\in G} I_{M_1'}(\xi) = \inf_{\xi\in G} I(\xi).$$

To show this, we only have to show that the RHS is not strictly larger than the LHS. Suppose that $I_{M_1'}(\xi) < I(\xi)$ for some $\xi \in G$. Since I and $I_{M_1'}$ differ only if the path has a jump at either 0 or 1, this means that ξ is a non-negative pure jump function of the following form:

$$\xi = \sum_{i=1}^{\infty} z_i \mathbb{1}_{[u_i,1]},$$

where $u_1 = 0$, $u_2 = 1$, u_i's are all distinct in $(0,1)$ for $i \ge 3$ and $z_i \ge 0$ for all i's. Note that one can pick an arbitrarily small δ so that $\sum_{i\in\{n:u_n<\delta\}} z_i < \epsilon$, $\sum_{i\in\{n:u_n>1-\delta\}} z_i < \epsilon$, $\delta \ne u_i$ for all $i \ge 2$, and $1-\delta \ne u_i$ for all $i \ge 2$. For such δ's, if we set

$$\xi_\delta \triangleq z_1 \mathbb{1}_{[\delta,1]} + z_2 \mathbb{1}_{[1-\delta,1]} + \sum_{i=3}^{\infty} z_i \mathbb{1}_{[u_i,1]},$$

then $d_{M_1'}(\xi,\xi_\delta) \le \epsilon$ while $I(\xi_\delta) = I_{M_1'}(\xi)$. That is, we can find an arbitrarily close element ξ_δ from ξ w.r.t. the M_1' metric by pushing the jump times at 0 and 1 slightly to the inside of $(0,1)$; at such an element, I assumes the same value as $I_{M_1'}(\xi)$. Since G is open w.r.t. M_1', one can choose δ small enough so that $\xi_\delta \in G$. This proves the claim. Now, the desired LDP lower bound is immediate from the LDP lower bound in Theorem 2.2.7 since G is also an open set in the J_1 topology. □

2.5 Technical proofs

2.5.1 Proofs of Lemma 2.2.3, 2.2.5

Proof of Lemma 2.2.3. We start with I. To show that the sub-level sets $_I(\ )$ are closed for each < 1, let be any given path that does not belong to $_I(\ )$. We will show that there exists an > 0 such that $d_{J_1}(\ ;\ _I(\ ))$. Note that $_I(\ )^c = (A \setminus B \setminus C \setminus D)^c = (A^c)\ [\ (A \setminus B^c)\ [\ (A \setminus B \setminus C^c)\ [\ (A \setminus B \setminus C \setminus D^c)$ where

$$A = f\ 2\ \mathbb{D}[0;1] : (0) = 0 \text{ and } (1) = (1\ )g;$$
$$B = f\ 2\ \mathbb{D}[0;1] : \text{ is non-decreasing}g;$$
$$C = f\ 2\ \mathbb{D}[0;1] : \text{ is a pure jump function}g;$$
$$D = f\ 2\ \mathbb{D}[0;1] : \textstyle\sum_{t\,2[0;1]}(\ (t)\quad (t\ ))\quad g:$$

For $2\ A^c$, we show that $d_{J_1}(\ ;\ _I(\ ))$ where $= \frac{1}{2}\max fj\ (0)j;j\ (1)$ $(1\ )jg$. We argue by contradiction. Suppose that there exists $2\ _I(\ )$ such that $d_{J_1}(\ ;\ ) <$. Then $j\ (0)j > j\ (0)j\ \ 2$ and $j\ (1)\quad (1\ )j > j\ (1)\quad (1\ )j$ 2 . That is, $\max fj\ (0)j;j\ (1)\quad (1\ )jg > \max fj\ (0)j\ \ 2\ ;j\ (1)\quad (1\ )j\ \ 2\ g = 0$. Therefore, $2\ A^c$, and hence, $I(\ ) = 1$, which contradicts the assumption that $2\ _I(\ )$.

If $2\ A \setminus B^c$, there are $T_s < T_t$ such that $c\ ,\ (T_s)\quad (T_t) > 0$. We claim that $d_{J_1}(\ ;\ )\quad c{=}2$ if $2\ _I(\ )$. Suppose that this is not the case and there exists $2\ _I(\ )$ such that $d_{J_1}(\ ;\ ) < c{=}2$. Let be a non-decreasing homeomorphism $: [0;1]\ !\ [0;1]$ such that $k\qquad k_1 < c{=}2$, in particular, $(T_s) > (T_s)\quad c{=}2$ and $(T_t) < (T_t) + c{=}2$. Subtracting the latter inequality from the former, we get $(T_s)\qquad (T_t) > (T_s)\quad (T_t)\quad c = 0$. That is, is not non-decreasing, which is contradictory to the assumption $2\ _I(\ )$. Therefore, the claim has to be the case.

If $2\ A \setminus B \setminus C^c$, there exists an interval $[T_s;T_t]$ so that is continuous and $c\ ,\ (T_t)\quad (T_s) > 0$. Pick small enough so that $(c\quad 2\ )(2\ )^{\ 1} >$. We will show that $d_{J_1}(\ ;\ _I(\ ))$. Suppose that $2\ _I(\ )$ and $d_{J_1}(\ ;\ ) <$, and let be a non-decreasing homeomorphism such that $k\qquad k_1 <$. Note that this implies that each of the jump sizes of in $[T_s;T_t]$ has to be less than 2 . On the other hand, $(T_t)\quad (T_t)$ and $(T_s)\quad (T_s) +$, which in turn implies that $(T_t)\qquad (T_s)\quad c\quad 2$. Since is a non-decreasing pure jump function,

$$c\quad 2\qquad (T_t)\qquad (T_s) = \sum_{t\,2(T_s;T_t]} (t)\qquad (t\ )$$

$$= \sum_{t \in (T_s; T_t]} \quad (t) \quad (t \quad) \quad (t) \quad (t \quad)^{1}$$
$$\sum_{t \in (T_s; T_t]} \quad (t) \quad (t \quad) \quad (2 \quad)^{1} \quad :$$

That is, $\sum_{t \in (T_s;T_t]} \quad (t) \quad (t \quad) \quad (2 \quad)^{1}(c \quad 2 \quad) >$, which is contradictory to our assumption that $\in {}_{l}(\ )$. Therefore, $d_{J_1}(\ ;\ {}_{l}(\ ))$.

Finally, let $\in A \cap B \cap C \cap D^c$. This implies that admits the following representation: $= \sum_{i=1}^{\infty} x_i \mathbb{1}_{[u_i;1]}$ where u_i's are all distinct in $(0;1)$ and $\sum_{i=1}^{\infty} x_i >$. Choose k and appropriately so that $\sum_{i=1}^{k}(x_i \quad 2 \quad) >$. We will show that $d_{J_1}(\ ;\ {}_{l}(\ ))$. Suppose that this is not the case. That is, there exists $\in {}_{l}(\ )$ so that $d_{J_1}(\ ;\ ) <$. Let be a non-decreasing homeomorphism such that $k \quad k_{\infty} <$. Thus for each $i \in \{1;\dots;k\}$, $(u_i) \quad (u_i \quad) \quad (u_i) \quad (u_i \quad) \quad 2 \quad = x_i \quad 2$, and hence,

$$I(\ ) = \sum_{t \in [0;1]} (\quad (t_i) \quad (t_i \quad)) \quad \sum_{i=1}^{k} (\quad (u_i) \quad (u_i \quad)) \quad \sum_{i=1}^{k} (x_i \quad 2 \quad) > \quad ;$$

which contradicts the assumption that $\in {}_{l}(\ )$. For I_k, notice that the effective domain $D_{\leqslant k}[0;1]$ is a closed subspace of $D_{\leqslant \infty}[0;1]$. Since I_k is the function I restricted in $D_{\leqslant k}[0;1]$, we have that I_k is also a lower semi-continuous function. □

Proof of Lemma 2.2.5. The inequality is obvious for $\inf_{\in G} I(\ ) = \infty$ therefore, we assume that $\inf_{\in G} I(\ ) < \infty$. Consequently, there exists a ${}_0 \in G$ such that $I({}_0) \quad \inf_{\in G} I(\ ) +$. Since G is open, we can pick > 0 such that $B_{J_1}({}_0; 2\ ) \quad G$ so that $B_{J_1}({}_0;\ ) \quad G$. Note that since $I({}_0) < \infty$, ${}_0$ has the representation ${}_0 = \sum_{i=1}^{\infty} x_i \mathbb{1}_{[u_i;1]}$ where $x_i \quad 0$ for all $i = 1; 2; \dots$, and the u_i's all distinct in $(0;1)$. Note also that since $I({}_0) = \sum_{i=1}^{\infty} x_i < \infty$ with < 1, $\sum_{i=1}^{\infty} x_i$ has to be finite as well. Thus, there exists K such that $k \quad K$ implies $\sum_{i=k+1}^{\infty} x_i <$. For these and K, we claim that $\inf_{\in G} \quad I_k(\ ) \quad \inf_{\in G} I(\ ) +$ holds. For any given $k \quad K$, let ${}_1 \quad \sum_{i=1}^{k} x_i \mathbb{1}_{[u_i;1]}$, then $I_k({}_1) \quad I({}_0)$ while $d_{J_1}({}_0; {}_1) \quad k \ {}_0 \quad {}_1 k_{\infty} \quad \sum_{i=k+1}^{\infty} x_i <$. That is, ${}_1 \in B_{J_1}({}_0;\ ) \quad G$. Therefore,

$$\inf_{\in G} \ I_k(\ ) \quad I({}_1) \quad I({}_0) \quad \inf_{\in G} I(\ ) + \quad :$$

□

2.5.2 Proof of Lemma 2.2.4

We prove Lemma 2.2.4 in several steps. Before we proceed, we introduce some notation and a distributional representation of the compound Poisson processes Y_n. The following representation for the time-scaled compound Poisson process is a straightforward modification of the distributional representation on page 305 of [55]; see also exercise 5.4 on page 163 of [83]:

$$\int_{x\geq 1} x N([0,n\cdot]\times dx) \overset{\mathcal{D}}{=} \sum_{l=1}^{\tilde N_n} Q_n^{\leftarrow}(\Gamma_l)\mathbb{1}_{[U_l,1]}(\cdot);$$

where $\Gamma_l = E_1 + E_2 + \dots + E_l$; E_i's are i.i.d. and standard exponential random variables; U_l's are i.i.d. and uniform variables in $[0,1]$; $\tilde N_n = N_n\big([0,1]\times[1,\infty)\big)$; $N_n = \sum_{l=1}^{\infty}\delta_{(U_l, Q_n^{\leftarrow}(\Gamma_l))}$; where $\delta_{(x,y)}$ is the Dirac measure concentrated on (x,y); $Q_n(x) \triangleq n\nu[x,\infty)$, and $Q_n^{\leftarrow}(y) \triangleq \inf\{s>0 : n\nu[s,\infty) < y\}$. It should be noted that $\tilde N_n$ is the largest l such that $\Gamma_l \leq n\nu_1$, where $\nu_1 \triangleq \nu[1,\infty)$, and hence, $\tilde N_n \sim \text{Poisson}(n\nu_1)$. Recall the definition of J_n^k — the process which keeps (up to) the k biggest Z_i's among $Z_1,\dots,Z_{N(n)}$. From this and the observation that $Q_n^{\leftarrow}(\Gamma_l)$ is decreasing in l, we conclude that J_n^k has the following distributional representation:

$$J_n^k(\cdot) \overset{\mathcal{D}}{=} \underbrace{\frac{1}{n}\sum_{i=1}^{k} Q_n^{\leftarrow}(\Gamma_i)\mathbb{1}_{[U_i,1]}(\cdot)}_{\triangleq \hat J_n^{\leqslant k}(\cdot)} - \underbrace{\frac{1}{n}\mathbb{1}\{\tilde N_n < k\}\sum_{i=\tilde N_n+1}^{k} Q_n^{\leftarrow}(\Gamma_i)\mathbb{1}_{[U_i,1]}(\cdot)}_{\triangleq J_n^{\leqslant k}(\cdot)}.$$

Roughly speaking, $(Q_n^{\leftarrow}(\Gamma_1)/n,\dots,Q_n^{\leftarrow}(\Gamma_k)/n)$ represents the k largest jump sizes of Y_n, and $\hat J_n^{\leqslant k}$ can be regarded as the process obtained by keeping only the k largest jumps of Y_n while disregarding the rest. Lemma 2.5.1 and Corollary 2.5.2 prove an LDP for $(Q_n^{\leftarrow}(\Gamma_1)/n,\dots,Q_n^{\leftarrow}(\Gamma_k)/n, U_1,\dots,U_k)$. Subsequently, Lemma 2.5.3 yields a sample path LDP for $\hat J_n^{\leqslant k}$. Finally, Lemma 2.2.4 is proved by showing that J_n^k satisfies the same LDP as the one satisfied by $\hat J_n^{\leqslant k}$.

Lemma 2.5.1. *$\big(Q_n^{\leftarrow}(\Gamma_1)/n, Q_n^{\leftarrow}(\Gamma_2)/n,\dots,Q_n^{\leftarrow}(\Gamma_k)/n\big)$ satisfies a large deviation principle in $\mathbb{R}_+^k$ with normalization $L(n)n^{\alpha}$; and with good rate function*

$$I_k(x_1,\dots,x_k) = \begin{cases} \sum_{i=1}^{k} x_i^{\alpha} & \text{if } x_1 \geq x_2 \geq \cdots \geq x_k \geq 0; \\ \infty & \text{otherwise.} \end{cases} \tag{2.36}$$

Proof. It is straightforward to check that I_k is a good rate function. For each $f \in C_b(\mathbb{R}_+^k)$, let

$$\Lambda_f \triangleq \lim_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{E} e^{L(n)n\, f\left(Q_n(\tau_1)/n,\, Q_n(\tau_2)/n, \dots, Q_n(\tau_k)/n\right)}. \tag{2.37}$$

Applying Bryc's inverse Varadhan lemma (see e.g. Theorem 4.4.13 of [22]), we can show that $(Q_n(\tau_1)/n, \dots, Q_n(\tau_k)/n)$ satisfies a large deviation principle with speed $L(n)n$ and good rate function $I_k(x)$ if

$$\Lambda_f = \sup_{x\in\mathbb{R}_+^k} \{f(x) - I_k(x)\} \tag{2.38}$$

for every $f \in C_b(\mathbb{R}_+^k)$.

To prove (2.38), fix $f \in C_b(\mathbb{R}_+^k)$ and let M be a constant such that $|f(x)| \leq M$ for all $x \in \mathbb{R}_+^k$. We first claim that the supremum of $\Psi_f \triangleq f - I_k$ is finite and attained. Pick a constant R so that $\theta R > 2M$. Since Ψ_f is upper semi-continuous on $[0,R]^k$, which is compact, there exists a maximizer $\hat{x} \triangleq (\hat{x}_1, \dots, \hat{x}_k)$ of Ψ_f on $[0,R]^k$. Since

$$\sup_{x\in[0,R]^k} \{f(x) - I_k(x)\} \geq \sup_{x\in[0,R]^k} f(x) \geq -M$$

and

$$\sup_{x\in\mathbb{R}_+^k\setminus[0,R]^k} \{f(x) - I_k(x)\} < \sup_{x\in\mathbb{R}_+^k\setminus[0,R]^k} \{f(x) - 2M\} \leq -M,$$

$\hat{x}$ is, in fact, a global maximizer. Now, it is enough to prove that

$$\Psi_f(\hat{x}) \leq \liminf_{n\to\infty} \frac{1}{L(n)n} \log \Gamma_f(n) \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{L(n)n} \log \Gamma_f(n) \leq \Psi_f(\hat{x}); \tag{2.39}$$

where

$$\Gamma_f(n) \triangleq \int_{\mathbb{R}_+^k} e^{L(n)n\, f\left(Q_n(y_1)/n, \dots, Q_n(y_1+\cdots+y_k)/n\right)} e^{-\sum_{i=1}^k y_i}\, dy_1 \dots dy_k.$$

We start with the lower bound — i.e., the first inequality of (2.39). Fix an arbitrary $\epsilon > 0$. Since Ψ_f is continuous on $A_\infty \triangleq \{(x_1, \dots, x_k) \in \mathbb{R}_+^k : x_1 \leq \cdots \leq x_k\}$, there exists $\delta > 0$ such that $x \in B(\hat{x}; 2\delta\sqrt{k}) \cap A_\infty$ implies $\Psi_f(x) \geq \Psi_f(\hat{x}) - \epsilon$. Since $\prod_{j=1}^k [\hat{x}_j + \delta, \hat{x}_j + 2\delta] \subset B(\hat{x}; 2\delta\sqrt{k})$ and $Q_n(\cdot)$ is non-increasing, $Q_n\left(\sum_{i=1}^{j} y_i/n\right) \in [\hat{x}_j + \epsilon, \hat{x}_j + 2\epsilon]$ for all $j = 1, \ldots, k$ implies $(Q_n(y_1)/n, \ldots, Q_n(y_1 + \cdots + y_k)/n) \in B(\hat{x}, 2\sqrt{k}\epsilon)$, and hence,

$$\ell_f(Q_n(y_1)/n, \ldots, Q_n(y_1 + \cdots + y_k)/n) \ge \ell_f(\hat{x}) - \delta. \tag{2.40}$$

That is, if we define $D_n^j (= D_n^{y_1,\ldots,y_{j-1}})$ as

$$D_n^j \triangleq \{y_j \in \mathbb{R}_+ : Q_n\left(\sum_{i=1}^{j} y_i/n\right) \in [\hat{x}_j + \epsilon, \hat{x}_j + 2\epsilon]\};$$

then (2.40) holds for $(y_1, \ldots, y_k)$'s such that $y_j \in D_n^j$ for $j = 1, \ldots, k$. Therefore,

$$\begin{aligned}
Z_f(n) \\
&= \int_{\mathbb{R}_+^k} \Big[e^{L(n)n\,\ell_f(Q_n(y_1)/n,\ldots,Q_n(y_1+\cdots+y_k)/n)} \\
&\qquad \cdot e^{-L(n)\sum_{i=1}^{k} Q_n(\sum_{j=1}^{i} y_j) - \sum_{i=1}^{k} y_i} \Big] dy_1 \cdots dy_k \\
&\ge \int_{D_n^1} \cdots \int_{D_n^k} \Big[e^{L(n)n\,\ell_f(Q_n(y_1)/n,\ldots,Q_n(y_1+\cdots+y_k)/n)} \\
&\qquad \cdot e^{-L(n)\sum_{i=1}^{k} Q_n(\sum_{j=1}^{i} y_j) - \sum_{i=1}^{k} y_i} \Big] dy_k \cdots dy_1 \\
&\ge \int_{D_n^1} \cdots \int_{D_n^k} e^{L(n)n\,\ell_f(\hat{x}_1,\ldots,\hat{x}_k)} e^{-L(n)\sum_{i=1}^{k} Q_n(\sum_{j=1}^{i} y_j) - \sum_{i=1}^{k} y_i} dy_k \cdots dy_1 \\
&\ge \int_{D_n^1} \cdots \int_{D_n^k} e^{L(n)n\,\ell_f(\hat{x}_1,\ldots,\hat{x}_k)} e^{-L(n)\sum_{i=1}^{k} n(\hat{x}_i+\epsilon) - \sum_{i=1}^{k} y_i} dy_k \cdots dy_1 \\
&= \underbrace{e^{L(n)n\,\ell_f(\hat{x}_1,\ldots,\hat{x}_k)}}_{\triangleq (I)_n} \underbrace{e^{-L(n)\sum_{i=1}^{k} n(\hat{x}_i+\epsilon)}}_{\triangleq (II)_n} \underbrace{\int_{D_n^1} \cdots \int_{D_n^k} e^{-\sum_{i=1}^{k} y_i} dy_k \cdots dy_1}_{\triangleq (III)_n};
\end{aligned} \tag{2.41, 2.42}$$

where the first equality is obtained by adding and subtracting the quantity $L(n)\sum_{i=1}^{k} Q_n(\sum_{j=1}^{i} y_j)$ to the exponent of the integrand. Note that by the construction of the D_n^j's,

$$Q_n(n(\hat{x}_j + 2\epsilon)) \ge y_1 + \cdots + y_j \ge Q_n(n(\hat{x}_j + \epsilon))$$

on the domain of the integral in $(III)_n$, and hence,

$$(III)_n \ge e^{-Q_n(n(\hat{x}_k+\delta))} \prod_{i=1} \left(Q_n\left(n(\hat{x}_i+\delta)\right) - Q_n\left(n(\hat{x}_i+2\delta)\right)\right) . \tag{2.43}$$

Since $Q_n\left(n(\hat{x}_k+\delta)\right) \to 0$ and

$$L\left(n(\hat{x}_i+\delta)\right) n\left(\hat{x}_i+\delta\right) - L\left(n(\hat{x}_i+2\delta)\right) n\left(\hat{x}_i+2\delta\right) \to -\infty$$

for each i,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{L(n)n} \log (III)_n \\
&\quad \ge \liminf_{n\to\infty} \frac{1}{L(n)n} \left(-Q_n\left(n(\hat{x}_k+\delta)\right)\right) \qquad (2.44)\\
&\qquad + \sum_{i=1} \liminf_{n\to\infty} \frac{1}{L(n)n} \log \left(Q_n\left(n(\hat{x}_i+\delta)\right) - Q_n\left(n(\hat{x}_i+2\delta)\right)\right) \\
&= \sum_{i=1} \liminf_{n\to\infty} \frac{1}{L(n)n} \log Q_n\left(n(\hat{x}_i+\delta)\right) \left(1 - \frac{Q_n\left(n(\hat{x}_i+2\delta)\right)}{Q_n\left(n(\hat{x}_i+\delta)\right)}\right) \\
&= \sum_{i=1} \liminf_{n\to\infty} \frac{-L\left(n(\hat{x}_i+\delta)\right) n\left(\hat{x}_i+\delta\right)}{L(n)n} \qquad (2.45)\\
&\qquad + \sum_{i=1} \liminf_{n\to\infty} \frac{\log\left(1 - e^{L(n(\hat{x}_i+\delta))n(\hat{x}_i+\delta) - L(n(\hat{x}_i+2\delta))n(\hat{x}_i+2\delta)}\right)}{L(n)n} \\
&= -\sum_{i=1} \left(\hat{x}_i+\delta\right) . \qquad (2.46)
\end{aligned}$$

From this, along with

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n\, L(n)} \log (I)_n &= \liminf_{n\to\infty} \frac{1}{n\, L(n)} \log \left(e^{n L(n) f(\hat{x}_1,\dots,\hat{x}_k)}\right) \\
&= f(\hat{x}_1, \dots, \hat{x}_k)
\end{aligned}$$

and

$$\liminf_{n\to\infty} \frac{1}{n\, L(n)} \log (II)_n = \liminf_{n\to\infty} \frac{1}{n\, L(n)} \log \left(e^{L(n) \sum_{i=1}^{k} (n(\hat{x}_i+\delta))}\right)$$

$$= \sum_{i=1}^{k} (\hat{x}_i + \delta) ;$$

we arrive at

$$\Lambda_f(\hat{x}) \geq \liminf_{n \to \infty} \frac{1}{L(n)n} \log \Gamma_f(n). \tag{2.47}$$

Letting $\delta \to 0$, we obtain the lower bound of (2.39).

Turning to the upper bound, consider

$$D_{R;n} \triangleq \{(y_1, y_2, \ldots, y_k) : Q_n^{\leftarrow}(y_1)/n \leq R\},$$

and decompose $\Gamma_f(n)$ into two parts:

$$\Gamma_f(n) = \int_{D_{R;n}} e^{L(n)n \cdot f\left(Q_n^{\leftarrow}(x_1)/n, \ldots, Q_n^{\leftarrow}(x_1+\cdots+x_k)/n\right)} e^{-\sum_{i=1}^{k} x_i} dx_1 \ldots dx_k$$
$$+ \int_{D^c_{R;n}} e^{L(n)n \cdot f\left(Q_n^{\leftarrow}(x_1)/n, \ldots, Q_n^{\leftarrow}(x_1+\cdots+x_k)/n\right)} e^{-\sum_{i=1}^{k} x_i} dx_1 \ldots dx_k.$$

We first evaluate the integral over $D^c_{R;n}$. Since $|f| \leq M$,

$$\int_{D^c_{R;n}} e^{L(n)n \cdot f\left(Q_n^{\leftarrow}(x_1)/n, \ldots, Q_n^{\leftarrow}(x_1+\cdots+x_k)/n\right)} e^{-\sum_{i=1}^{k} x_i} dx_1 \ldots dx_k$$
$$= \int e^{L(n)n \cdot f\left(Q_n^{\leftarrow}(x_1)/n, \ldots, Q_n^{\leftarrow}(x_1+\cdots+x_k)/n\right)} e^{-\sum_{i=1}^{k} x_i} \mathbb{1}_{\{Q_n^{\leftarrow}(x_1)/n > R\}} dx_1 \ldots dx_k$$
$$= \int e^{L(n)n \cdot f\left(Q_n^{\leftarrow}(x_1)/n, \ldots, Q_n^{\leftarrow}(x_1+\cdots+x_k)/n\right)} e^{-\sum_{i=1}^{k} x_i} \mathbb{1}_{\{x_1 \geq Q_n(nR)\}} dx_1 \ldots dx_k$$
$$\leq \int e^{L(n)n \cdot M} e^{-\sum_{i=1}^{k} x_i} \mathbb{1}_{\{x_1 \geq Q_n(nR)\}} dx_1 \ldots dx_k \leq e^{L(n)n \cdot M} \cdot 1 \cdot e^{-Q_n(nR)}$$
$$\leq e^{L(n)n \cdot M - Q_n(nR)}. \tag{2.48}$$

Turning to the integral over $D_{R;n}$, fix $\epsilon > 0$ and pick $\{x^{(1)}, \ldots, x^{(m)}\} \subseteq \mathbb{R}^k_+$ in such a way that $\left\{\prod_{j=1}^{k} [x_j^{(l)} - \epsilon, x_j^{(l)} + \epsilon]\right\}_{l=1,\ldots,m}$ covers A_R. Set

$$H_{R;n;l} \triangleq \left\{(y_1, \ldots, y_k) \in \mathbb{R}^k_+ : Q_n^{\leftarrow}\Big(\sum_{j=1}^{i} y_j\Big)/n \in [x_i^{(l)} - \epsilon, x_i^{(l)} + \epsilon] \;\; \forall i \in \{1, \ldots, k\}\right\}.$$

We see that $D_{R;n} \subseteq \bigcup_{l=1}^{m} H_{R;n;l}$, and hence,

$$
\begin{aligned}
&\int_{D_{R;n}} e^{L(n)n\, f\left(Q_n(y_1)/n,\dots,Q_n(y_1+\cdots+y_k)/n\right)}\, e^{-\sum_{i=1}^k y_i}\, dy_1\cdots dy_k \\
&\le \sum_{l=1}^{m} \int_{H_{R;n;l}} e^{L(n)n\, f\left(Q_n(y_1)/n,\dots,Q_n(y_1+\cdots+y_k)/n\right)}\, e^{-\sum_{i=1}^k y_i}\, dy_1\cdots dy_k \\
&= \sum_{l=1}^{m} \int_{H_{R;n;l}} \Big[e^{L(n)n\, f\left(Q_n(y_1)/n,\dots,Q_n(y_1+\cdots+y_k)/n\right) - L(n)\sum_{i=1}^k Q_n\left(\sum_{j=1}^i y_j\right)} \\
&\qquad\qquad e^{L(n)\sum_{i=1}^k Q_n\left(\sum_{j=1}^i y_j\right) - \sum_{i=1}^k y_i}\, dy_1\cdots dy_k \Big] \\
&\le \sum_{l=1}^{m} \int_{H_{R;n;l}} e^{L(n)n\, f(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_k)}\, e^{L(n)\sum_{i=1}^k Q_n\left(\sum_{j=1}^i y_j\right) - \sum_{i=1}^k y_i}\, dy_1\cdots dy_k \\
&= \sum_{l=1}^{m} e^{L(n)n\, f(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_k)} \underbrace{\int_{H_{R;n;l}} e^{L(n)\sum_{i=1}^k Q_n\left(\sum_{j=1}^i y_j\right) - \sum_{i=1}^k y_i}\, dy_1\cdots dy_k}_{=:\, \mathrm{H}(R;n;l)} .
\end{aligned}
\tag{2.49}
$$

Note that the first equality is obtained by adding and subtracting the term $L(n)\sum_{i=1}^k Q_n\left(\sum_{j=1}^i y_j\right)$ to the exponent of the integrand. Since

$$
\begin{aligned}
&Q_n\left(\textstyle\sum_{j=1}^i y_j\right)/n \in \left[x_i^{(l)} - \delta;\, x_i^{(l)} + \delta\right] \\
&\implies Q_n\left(n(x_i^{(l)} + \delta)\right) \le \textstyle\sum_{j=1}^i y_j \le Q_n\left(n(x_i^{(l)} - \delta)\right);
\end{aligned}
$$

we can bound the integral in (2.49) as follows:

$$
\begin{aligned}
&\int_{H_{R;n;l}} e^{L(n)\sum_{i=1}^k Q_n\left(\sum_{j=1}^i y_j\right) - \sum_{i=1}^k y_i}\, dy_1 \cdots dy_k \\
&\le \int_{H_{R;n;l}} e^{L(n)\sum_{i=1}^k n(x_i^{(l)} - \delta) - \sum_{i=1}^k y_i}\, dy_1 \dots dy_k \\
&\le \int_{H_{R;n;l}} e^{L(n)\sum_{i=1}^k n(x_i^{(l)} - \delta) - Q_n\left(n(x_k^{(l)} + \delta)\right)}\, dy_1 \dots dy_k \\
&= e^{L(n)\sum_{i=1}^k n(x_i^{(l)} - \delta) - Q_n\left(n(x_k^{(l)} + \delta)\right)} \int_{H_{R;n;l}} dy_1 \dots dy_k
\end{aligned}
$$

$$= e^{L(n)n \sum_{i=1}^{k}(x_i^{(l)} - \delta) - Q_n(n(x_k^{(l)}+\delta))} \prod_{i=1}^{k} \left(Q_n(n(x_i^{(l)} - \delta)) - Q_n(n(x_i^{(l)}+\delta)) \right) . \tag{2.50}$$

With (2.49) and (2.50), a straightforward calculation as in the lower bound leads to

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathrm{H}(R;n;l) \\
&\quad \le \limsup_{n\to\infty} \frac{1}{L(n)n} \log e^{L(n)n \Lambda_f(\hat{x}_1;\hat{x}_2;\dots;\hat{x}_k)} \\
&\qquad + \limsup_{n\to\infty} \frac{1}{L(n)n} \log e^{L(n)n \sum_{i=1}^{k}(\hat{x}_i^{(l)} - \delta) - Q_n(n(\hat{x}_k^{(l)}+\delta))} \\
&\qquad + \sum_{i=1}^{k} \limsup_{n\to\infty} \frac{1}{L(n)n} \log \left(Q_n(n(\hat{x}_i^{(l)} - \delta)) - Q_n(n(\hat{x}_i^{(l)}+\delta)) \right) \\
&\quad = \Lambda_f(\hat{x}_1;\dots;\hat{x}_k).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{L(n)n} \log \Lambda_f(n) \\
&= \limsup_{n\to\infty} \frac{1}{L(n)n} \log \left(e^{L(n)n M} Q_n(nR) \right) \vee \limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathrm{H}(R;n;l) \\
&\le (M - R) \vee \Lambda_f(\hat{x}_1;\dots;\hat{x}_k) = (M - R) \vee \sup_{x\in\mathbb{R}_+^k} \{f(x) - I_k(x)\}.
\end{aligned}$$

Since R was arbitrary, we can send $R \to \infty$ to arrive at the desired upper bound of (2.39). □

The following corollary is immediate from Lemma 2.5.1 and Theorem 4.14 of [39].

Corollary 2.5.2. *$(Q_n(\tau_1)/n;\dots;Q_n(\tau_k)/n;U_1;\dots;U_k)$ satisfies a large deviation principle in $\mathbb{R}_+^k \times [0;1]^k$ with speed $L(n)n$ and good rate function*

$$\hat{I}_k(x_1;\dots;x_k;u_1;\dots;u_k) := \begin{cases} \sum_{i=1}^{k} x_i & \text{if } x_1 \le x_2 \le \dots \le x_k \text{ and } u_1;\dots;u_k \in [0;1]; \\ \infty & \text{otherwise:} \end{cases} \tag{2.51}$$

Recall that $\hat{J}_n^{\leqslant k} = \frac{1}{n}\sum_{i=1}^{k} Q_n(\Gamma_i)\mathbb{1}_{[U_i,1]}$ and rate function I_k defined in (2.9). We next prove a sample path LDP for $\hat{J}_n^{\leqslant k}$.

Lemma 2.5.3. *$\hat{J}_n^{\leqslant k}$ satisfies the LDP in $(\mathbb{D}[0,1], \mathcal{T}_{J_1})$ with speed $L(n)n$ and rate function I_k.*

Proof. First, we note that I_k is indeed a rate function since the sublevel sets of I_k equal the intersection between the sublevel sets of I and a closed set $\mathbb{D}_{\leqslant k}[0,1]$, and I is a rate function (Lemma 2.2.3).

Next, we prove the LDP in $\mathbb{D}_{\leqslant k}[0,1]$ w.r.t. the relative topology induced by $\mathcal{T}_{J_1}$. (Note that I_k is a rate function in $\mathbb{D}_{\leqslant k}[0,1]$ as well.) Consider the map $T_k(x,u) \triangleq \sum_{i=1}^{k} x_i \mathbb{1}_{[u_i,1]}$. Since

$$\inf_{(x,u)\in T_k^{-1}(\xi)} \hat{I}_k(x,u) = I_k(\xi)$$

for $\xi \in \mathbb{D}_{\leqslant k}[0,1]$, the LDP in $\mathbb{D}_{\leqslant k}[0,1]$ is established once we show that for any closed set $F \subseteq \mathbb{D}_{\leqslant k}[0,1]$,

$$\limsup_{n\to\infty} \frac{1}{L(n)n}\log \mathbf{P}\left(\hat{J}_n^{\leqslant k} \in F\right) \leq -\inf_{(x,u)\in T_k^{-1}(F)} \hat{I}_k(x,u), \tag{2.52}$$

and for any open set $G \subseteq \mathbb{D}_{\leqslant k}[0,1]$,

$$-\inf_{(x,u)\in T_k^{-1}(G)} \hat{I}_k(x,u) \leq \liminf_{n\to\infty} \frac{1}{L(n)n}\log \mathbf{P}\left(\hat{J}_n^{\leqslant k} \in G\right). \tag{2.53}$$

We start with the upper bound. Note that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{L(n)n}\log \mathbf{P}\left(\hat{J}_n^{\leqslant k} \in F\right)\\
&= \limsup_{n\to\infty} \frac{1}{L(n)n}\log \mathbf{P}\left(\left(Q_n(\Gamma_1),\ldots,Q_n(\Gamma_k), U_1,\ldots,U_k\right) \in T_k^{-1}(F)\right)\\
&\leq \limsup_{n\to\infty} \frac{1}{L(n)n}\log \mathbf{P}\left(\left(Q_n(\Gamma_1),\ldots,Q_n(\Gamma_k), U_1,\ldots,U_k\right) \in \overline{T_k^{-1}(F)}\right)\\
&\leq -\inf_{(x_1,\ldots,x_k,u_1,\ldots,u_k)\in \overline{T_k^{-1}(F)}} \hat{I}_k(x_1,\ldots,x_k,u_1,\ldots,u_k).
\end{aligned}$$

In view of (2.52), it is therefore enough for the upper bound to show that

$$\inf_{(x,u)\in \overline{T_k^{-1}(F)}} \hat{I}_k(x,u) \leq \inf_{(x,u)\in T_k^{-1}(F)} \hat{I}_k(x,u).$$

To prove this, we proceed with proof by contradiction. Suppose that

$$c \triangleq \inf_{(x,u)\in T_k^{-1}(F)} \hat{I}_k(x,u) > \inf_{(x,u)\in T_k^{-1}(F)} \hat{I}_k(x,u). \qquad (2.54)$$

Pick an $\epsilon > 0$ in such a way that $\inf_{(x,u)\in T_k^{-1}(F)} \hat{I}_k(x,u) < c - 2\epsilon$. Then there exists $(x^*,u^*) \in T_k^{-1}(F)$ such that $\hat{I}_k(x^*,u^*) < c - 2\epsilon$. In addition, let $I_k(x_1,\dots,x_k,u_1,\dots,u_k) \triangleq \sum_{i=1}^{k} x_i$. Since I_k is continuous, one can find $(x',u') = (x'_1,\dots,x'_k,u'_1,\dots,u'_k) \in T_k^{-1}(F)$ sufficiently close to (x^*,u^*) so that $I_k(x',u') < c - \epsilon$. Note that for any permutation $p : \{1,\dots,k\} \to \{1,\dots,k\}$, $(x'',u'') \triangleq (x'_{p(1)},\dots,x'_{p(k)},u'_{p(1)},\dots,u'_{p(k)})$ also belongs to $T_k^{-1}(F)$ and $I_k(x'',u'') = I_k(x',u')$ due to the symmetric structure of T_k and I_k. If we pick p so that $x'_{p(1)} \ge \cdots \ge x'_{p(k)}$, then

$$\hat{I}_k(x'',u'') = I_k(x',u') < c - \epsilon \le \inf_{(x,u)\in T_k^{-1}(F)} \hat{I}_k(x,u),$$

which contradicts $(x'',u'') \in T_k^{-1}(F)$. Therefore, (2.54) cannot be the case, which proves the upper bound.

Turning to the lower bound, consider an open set $G \subseteq \mathbb{D}_{\leqslant k}[0,1]$.

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\left(\hat{J}_n^{\leqslant k} \in G\right) \\
&= \liminf_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\left(\left(Q_n(\Gamma_1),\dots,Q_n(\Gamma_k),U_1,\dots,U_k\right) \in T_k^{-1}(G)\right) \\
&\ge \liminf_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\left(\left(Q_n(\Gamma_1),\dots,Q_n(\Gamma_k),U_1,\dots,U_k\right) \in T_k^{-1}(G)\right) \\
&\ge -\inf_{(x_1,\dots,x_k,u_1,\dots,u_k)\in T_k^{-1}(G)} \hat{I}_k(x_1,\dots,x_k,u_1,\dots,u_k).
\end{aligned}$$

In view of (2.53), we are done if we prove that

$$\inf_{(x,u)\in T_k^{-1}(G)} \hat{I}_k(x,u) \le \inf_{(x,u)\in T_k^{-1}(G)} \hat{I}_k(x,u). \qquad (2.55)$$

Let (x,u) be an arbitrary point in $T_k^{-1}(G)$ so that $T_k(x,u) \in G$. We will show that there exists $(x^*,u^*) \in T_k^{-1}(G)$ such that $I_k(x^*,u^*) \le I_k(x,u)$. Note first that if $u_i \in \{0,1\}$ for some i, then x_i has to be 0 since $G \subseteq \mathbb{D}_{\leqslant k}[0,1]$. This means that we can replace u_i with an arbitrary number in $(0,1)$ without changing the value of I_k and T_k. Therefore, we assume w.l.o.g. that $u_i > 0$ for each

$i = 1, \dots, k$. Now, suppose that $u_i = u_j$ for some $i \neq j$. Then one can find (x', u') such that $T_k(x', u') = T_k(x, u)$ by setting

$$(x', u') \triangleq (x_1, \dots, \underbrace{x_i + x_j}_{i^{th}\text{ coordinate}}, \dots, \underbrace{0}_{j^{th}\text{ coordinate}}, \dots, x_k, u_1, \dots, \underbrace{u_i}_{k+i^{th}\text{ coordinate}}, \dots, \underbrace{u'_j}_{k+j^{th}\text{coordinate}}, \dots, u_k),$$

where u'_j is an arbitrary number in $(0, 1)$; in particular, we can choose u'_j so that $u'_j \neq u_l$ for $l = 1, \dots, k$. It is easy to see that $I_k(x', u') \leq \hat{I}_k(x, u)$. Now one can permute the coordinates of (x', u') as in the upper bound to find (x'', u'') such that $T_k(x'', u'') = T_k(x, u)$ and $\hat{I}_k(x'', u'') \leq \hat{I}_k(x, u)$. Iterating this procedure until there is no $i \neq j$ for which $u_i = u_j$, we can find (x^*, u^*) such that $T_k(x^*, u^*) = T_k(x, u)$, u^*_i's are all distinct in $(0, 1)$, and $I_k(x^*, u^*) \leq I_k(x, u)$. Note that since T_k is continuous at (x^*, u^*), $T_k(x^*, u^*) \in G$, and G is open, we conclude that $(x^*, u^*) \in T_k^{-1}(G)^\circ$. Therefore,

$$\inf_{(x,u)\in T_k^{-1}(G)^\circ} I_k(x, u) \leq I_k(x, u).$$

Since (x, u) was arbitrarily chosen in $T_k^{-1}(G)$, (2.55) is proved. Along with the upper bound, this proves the LDP in $\mathbb{D}_{\leqslant k}[0, 1]$. Finally, since $\mathbb{D}_{\leqslant k}[0, 1]$ is a closed subset of $\mathbb{D}[0, 1]$, $\mathbf{P}(\hat{J}_n^{\leqslant k} \notin \mathbb{D}_{\leqslant k}[0, 1]) = 0$, and $I_k = \infty$ on $\mathbb{D}[0, 1] \setminus \mathbb{D}_{\leqslant k}[0, 1]$, Lemma 4.1.5 of [22] applies, proving the desired LDP in $\mathbb{D}[0, 1]$. □

Now we are ready to prove Lemma 2.2.4.

Proof of Lemma 2.2.4. Recall that

$$J_n^k \stackrel{\mathcal{D}}{=} \underbrace{\frac{1}{n}\sum_{i=1}^{k} Q_n^{\leftarrow}(\Gamma_i)\mathbb{1}_{[U_i,1]}}_{=\hat{J}_n^{\leqslant k}} - \underbrace{\frac{1}{n}\mathbb{1}\{\tilde{N}_n < k\}\sum_{i=\tilde{N}_n+1}^{k} Q_n^{\leftarrow}(\Gamma_i)\mathbb{1}_{[U_i,1]}}_{=\check{J}_n^{\leqslant k}}.$$

Let F be a closed set and note that

$$\begin{aligned}
\mathbf{P}(J_n^k \in F) &= \mathbf{P}(\hat{J}_n^{\leqslant k} - \check{J}_n^{\leqslant k} \in F) \\
&\leq \mathbf{P}\left(\hat{J}_n^{\leqslant k} - \check{J}_n^{\leqslant k} \in F, \mathbb{1}\{\tilde{N}(n) < k\} = 0\right) + \mathbf{P}\left(\mathbb{1}\{\tilde{N}(n) < k\} \neq 0\right) \\
&\leq \mathbf{P}(\hat{J}_n^{\leqslant k} \in F) + \mathbf{P}(\tilde{N}(n) < k).
\end{aligned}$$

From Lemma 2.5.3,

$$\limsup_{n\to\infty}\frac{\log\mathbf{P}(J_n^k \in F)}{L(n)n} \leq \limsup_{n\to\infty}\frac{\log\mathbf{P}(\hat{J}_n^{\leqslant k} \in F)}{L(n)n} \vee \limsup_{n\to\infty}\frac{\log\mathbf{P}(\tilde{N}(n) < k)}{L(n)n}$$

$$\inf_{\theta \in F} I_k(\theta);$$

since $\limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}(\bar{N}(n) < k) = -\infty$.

Turning to the lower bound, let G be an open set. Since the lower bound is trivial in case $\inf_{x\in G} I_k(x) = \infty$, we focus on the case $\inf_{x\in G} I_k(x) < \infty$. In this case,

$$
\begin{aligned}
&\liminf_{n\to\infty} \frac{\log \mathbf{P}(J_n^k \in G)}{L(n)n} \\
&\quad \geq \liminf_{n\to\infty} \frac{\log \mathbf{P}(J_n^k \in G;\ \bar{N}(n) \geq k)}{L(n)n} = \liminf_{n\to\infty} \frac{\log \mathbf{P}(\hat{J}_n^{\geqslant k} \in G;\ \bar{N}(n) \geq k)}{L(n)n} \\
&\quad \geq \liminf_{n\to\infty} \frac{1}{L(n)n} \log \left(\mathbf{P}(\hat{J}_n^{\geqslant k} \in G) - \mathbf{P}(\bar{N}(n) < k) \right) \\
&= \liminf_{n\to\infty} \frac{1}{L(n)n} \log \left(\mathbf{P}(\hat{J}_n^{\geqslant k} \in G) \left(1 - \frac{\mathbf{P}(\bar{N}(n) < k)}{\mathbf{P}(\hat{J}_n^{\geqslant k} \in G)} \right) \right) \\
&= \liminf_{n\to\infty} \frac{1}{L(n)n} \left\{ \log \left(\mathbf{P}(\hat{J}_n^{\geqslant k} \in G) \right) + \log \left(1 - \frac{\mathbf{P}(\bar{N}(n) < k)}{\mathbf{P}(\hat{J}_n^{\geqslant k} \in G)} \right) \right\} \\
&= \liminf_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}(\hat{J}_n^{\geqslant k} \in G) \geq -\inf_{\theta \in G} I_k(\theta).
\end{aligned}
$$

The last equality holds since

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{\mathbf{P}(\bar{N}(n) < k)}{\mathbf{P}(\hat{J}_n^{\geqslant k} \in G)} \\
&= \lim_{n\to\infty} \left\{ \exp\left(\frac{\log \mathbf{P}(\bar{N}(n) < k)}{L(n)n} - \frac{\log \mathbf{P}(\hat{J}_n^{\geqslant k} \in G)}{L(n)n} \right) \right\}^{L(n)n} = 0; \qquad (2.56)
\end{aligned}
$$

which in turn follows from

$$\limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}(\bar{N}(n) < k) = -\infty$$

and

$$\limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}(\hat{J}_n^{\geqslant k} \in G) \geq -\inf_{x\in G} I_k(x) < \infty.$$

$\square$

2.5.3 Proof of Lemma 2.2.6

In our proof of Lemma 2.2.6, the following lemmas (Lemma 2.5.4 and Lemma 2.5.5) play key roles.

Lemma 2.5.4. *For each* $\ > \ > 0$,

$$\limsup_{n \, ! \, 1} \frac{\log \mathbf{P} \left(\max_{1 \ j \ 2n} \sum_{i=1}^{j} Z_i \mathbb{1}_{\{Z_i \ n \}} \ \mathbf{E}Z \ > n \right)}{L(n)n} \quad (\ =3) \ (\ = \ )^{1} \ : \tag{2.57}$$

Proof. We re ne an argument developed in [47]. Note that for any $s > 0$ such that $1=s \ n$,

$$\mathbf{E}e^{sZ \mathbb{1}_{\{Z \ n \}}} = \mathbf{E}e^{sZ \mathbb{1}_{\{Z \ n \}}} \mathbb{1}_{\{Z \ \frac{1}{s}\}} + \mathbf{E}e^{sZ \mathbb{1}_{\{Z \ n \}}} \mathbb{1}_{\{Z < \frac{1}{s}\}} = (I) + (II); \tag{2.58}$$

and

$$\begin{aligned}
(I) &= \int_{[1=s;n \ ]} e^{sy} \mathrm{d}\mathbf{P}(Z \ y) + \int_{(n \ ;1)} \mathrm{d}\mathbf{P}(Z \ y) \\
&= \Big[e^{sy}\mathbf{P}(Z \ y) \Big]_{(1=s)}^{(n \ )} \ s \int_{[1=s;n \ ]} e^{sy}\mathbf{P}(Z \ y)\mathrm{d}y + \mathbf{P}(Z > n \ ) \\
&= e^{sn} \ \mathbf{P}(Z \ n \ ) \ e\mathbf{P}(Z < 1=s) \\
&\quad \ s \int_{[1=s;n \ ]} e^{sy}\mathrm{d}y + s \int_{[1=s;n \ ]} e^{sy}\mathbf{P}(Z > y)\mathrm{d}y + \mathbf{P}(Z > n \ ) \\
&= e^{sn} \ \mathbf{P}(Z \ n \ ) \ e\mathbf{P}(Z < 1=s) \ e^{sn} \ + e \\
&\quad + s \int_{[1=s;n \ ]} e^{sy}\mathbf{P}(Z > y)\mathrm{d}y + \mathbf{P}(Z > n \ ) \\
&= \ e^{sn} \ \mathbf{P}(Z > n \ ) + e\mathbf{P}(Z \ 1=s) + s \int_{[1=s;n \ ]} e^{sy}\mathbf{P}(Z > y)\mathrm{d}y + \mathbf{P}(Z > n \ ) \\
&\quad s \int_{[1=s;n \ ]} e^{sy}\mathbf{P}(Z > y)\mathrm{d}y + e\mathbf{P}(Z \ 1=s) + \mathbf{P}(Z > n \ ) \\
&\quad s \int_{[1=s;n \ ]} e^{sy}\mathbf{P}(Z > y)\mathrm{d}y + s^2(e+1)\mathbf{E}Z^2;
\end{aligned} \tag{2.59}$$

where the last inequality is from $\mathbf{P}(Z \ n \ ) \ \mathbf{P}(Z \ 1=s) \ s^2\,\mathbf{E}Z^2$; while

$$(II) \ \int_0^{1=s} e^{sy}\mathrm{d}\mathbf{P}(Z \ y) \ \int_0^{1=s} (1+sy+(sy)^2)\mathrm{d}\mathbf{P}(Z \ y) \ 1+s\,\mathbf{E}Z+s^2\,\mathbf{E}Z^2: \tag{2.60}$$

Therefore, from (2.58), (2.59) and (2.60), if $1/s \quad n$ and s is sufficiently small,

$$\mathbf{E}e^{sZ\mathbb{1}_{\{Z \quad n\}}} \quad s\int_{\frac{1}{s}}^{n} e^{sy}\mathbf{P}(Z>y)\,dy + 1 + s\mathbf{E}Z + s^2(e+2)\mathbf{E}Z^2$$
$$= s\int_{\frac{1}{s}}^{n} e^{sy \quad q(y)}dy + 1 + s\mathbf{E}Z + s^2(e+2)\mathbf{E}Z^2$$
$$\quad sn \; e^{sn \quad q(n \;)} + 1 + s\mathbf{E}Z + s^2(e+2)\mathbf{E}Z^2; \qquad (2.61)$$

where $q(x) \;, \quad \log \mathbf{P}(Z>x) = L(x)x$, and the last inequality is from the fact that $e^{sy \quad q(y)}$ is increasing over $[1/s; n \;]$ due to the assumption that $L(y)y \quad {}^1$ is non-increasing for sufficiently large y's. Now, from the Markov inequality,

$$\mathbf{P}\left(\sum_{i=1}^{j} Z_i\mathbb{1}_{\{Z_i \quad n\}} \quad \mathbf{E}Z > n\right)$$
$$\mathbf{P}\left(\exp\left(s\sum_{i=1}^{j} Z_i\mathbb{1}_{\{Z_i \quad n\}}\right) > \exp\; s(n \; + j\mathbf{E}Z)\right)$$
$$\exp\; s\; n \; + j\mathbf{E}Z \; + j\log\; \mathbf{E}e^{sZ\mathbb{1}_{\{Z \quad n\}}}$$
$$\exp\left\{ s(n \; + j\mathbf{E}Z) + j\; sn\; e^{sn \quad q(n \;)} + s\mathbf{E}Z + s^2(e+2)\mathbf{E}Z^2 \right\}$$
$$= \exp\left\{ sn \; + jsn\; e^{sn \quad q(n \;)} + js^2(e+2)\mathbf{E}Z^2 \right\}$$
$$\exp\left\{ sn \; + 2n^2 s\; e^{sn \quad q(n \;)} + 2ns^2(e+2)\mathbf{E}Z^2 \right\} \qquad (2.62)$$

for $j \quad 2n$, where the third inequality is from (2.61) and the generic inequality $\log(x+1) \quad x$. Fix $\;\in (0; (\;=\;)^{1} \quad)$ and set $s = \frac{q(n \;)}{n}$. Note that $1/s \to \infty$ as $n \to \infty$, while $1/s \quad n$ for sufficiently large n. From now on, we only consider sufficiently large n's such that $1/s < n$ and s is sufficiently small so that (2.61) and (2.62) are valid. To establish an upper bound for (2.62), we next examine $e^{sn \quad q(n \;)}$. Note that $q(n \;) = q(n \;)\frac{L(n \;)}{L(n \;)}(\;=\;)$, and hence,

$$sn \quad q(n \;) = \frac{q(n \;)}{n}n \quad q(n \;) = \quad q(n \;) \quad 1 \quad \frac{L(n \;)}{L(n \;)}(\;=\;)^{1} \quad ;$$

and

$$e^{sn \quad q(n \;)} \quad e^{\quad q(n \;)\left(1 \quad \frac{L(n \;)}{L(n \;)}(\;=\;)^{1} \quad\right)}. \qquad (2.63)$$

Plugging this $s = \frac{q(n)}{n}$ into (2.62) along with (2.63),

$$\max_{0 \quad j \quad 2n} \mathbf{P}\left(\sum_{i=1} Z_i \mathbb{1}_{\{Z_i \quad n\}} \quad \mathbf{E}Z > n \right)$$

$$\exp \quad q(n) + \frac{2 \quad nq(n)}{} e^{\quad q(n)(1 \quad \frac{L(n)}{L(n)}(\;=\;)^{1} \quad)} + \frac{10 \;^{2}\mathbf{E}Z^2}{2} \frac{q(n)^2}{n} \; :$$

Since

$$\limsup_{n \to \infty} \frac{1}{L(n)n} \frac{2 \quad nq(n)}{} e^{\quad q(n)(1 \quad \frac{L(n)}{L(n)}(\;=\;)^{1} \quad)} = 0;$$

and

$$\limsup_{n \to \infty} \frac{1}{L(n)n} \frac{10 \;^{2}\mathbf{E}Z^2}{2} \frac{q(n)^2}{n} = 0;$$

we conclude that

$$\limsup_{n \to \infty} \frac{1}{L(n)n} \log \max_{0 \quad j \quad 2n} \mathbf{P}\left(\sum_{i=1} Z_i \mathbb{1}_{\{Z_i \quad n\}} \quad \mathbf{E}Z > n \right)$$

$$= \limsup_{n \to \infty} \frac{q(n)}{L(n)n} = \quad :$$

From Etemadi's inequality,

$$\limsup_{n \to \infty} \frac{1}{L(n)n} \log \mathbf{P}\left(\max_{0 \quad j \quad 2n} \sum_{i=1} Z_i \mathbb{1}_{\{Z_i \quad n\}} \quad \mathbf{E}Z > 3n \right)$$

$$\limsup_{n \to \infty} \frac{1}{L(n)n} \log \left\{ 3 \max_{0 \quad j \quad 2n} \mathbf{P}\left(\sum_{i=1} Z_i \mathbb{1}_{\{Z_i \quad n\}} \quad \mathbf{E}Z > n \right) \right\} = \quad :$$

Since this is true for any such that $\in (0; (\;=\;)^{1} \quad)$, we arrive at the conclusion of the lemma. □

Lemma 2.5.5. *For every* $\;;\; > 0$,

$$\limsup_{n \to \infty} \frac{1}{L(n)n} \log \mathbf{P}\left(\sup_{1 \quad j \quad 2n} \sum_{i=1} \mathbf{E}Z \quad Z_i \mathbb{1}_{\{Z_i \quad n\}} > n \right) = \quad \infty :$$

Proof. Note first that there is n_0 such that $\mathbf{E}\left[Z_i \mathbb{1}_{\{Z_i>n\}}\right] \le \frac{\epsilon}{3}$ for $n \ge n_0$. For $n \ge n_0$ and $j \ge 2n$,

$$
\begin{aligned}
&\mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \le n\}}\right) > \epsilon n\right) \\
&= \mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z\mathbb{1}_{\{Z \le n\}} - Z_i \mathbb{1}_{\{Z_i \le n\}}\right) > \epsilon n - j\mathbf{E}Z\mathbb{1}_{\{Z>n\}}\right) \\
&\le \mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z\mathbb{1}_{\{Z \le n\}} - Z_i \mathbb{1}_{\{Z_i \le n\}}\right) > \epsilon n - j\epsilon/3\right) \\
&\le \mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z\mathbb{1}_{\{Z \le n\}} - Z_i \mathbb{1}_{\{Z_i \le n\}}\right) > \frac{\epsilon n}{3}\right).
\end{aligned}
$$

Let $Y_i^{(n)} \triangleq \mathbf{E}(Z_i \mathbb{1}_{\{Z_i \le n\}}) - Z_i \mathbb{1}_{\{Z_i \le n\}}$. Recall the definition of Z in Section 2.2.2 and note that it is bounded from below. Furthermore, $\mathbf{E}Y_i^{(n)} = 0$, $\mathbf{var}Y_i^{(n)} \le \mathbf{E}Z^2$, and $Y_i^{(n)} \le \mathbf{E}Z$ almost surely. From Bennet's inequality,

$$
\begin{aligned}
&\mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z\mathbb{1}_{\{Z_i \le n\}} - Z_i \mathbb{1}_{\{Z_i \le n\}}\right) > \frac{\epsilon n}{3}\right) \\
&\le \exp\left(-\frac{j\,\mathbf{var}Y^{(n)}}{(\mathbf{E}Z)^2}\left(\left(\frac{\epsilon n\,\mathbf{E}Z}{3j\,\mathbf{var}Y^{(n)}}\right)\log\left(1+\frac{\epsilon n\,\mathbf{E}Z}{3j\,\mathbf{var}Y^{(n)}}\right) - \frac{\epsilon n\,\mathbf{E}Z}{3j\,\mathbf{var}Y^{(n)}}\right)\right) \\
&\le \exp\left(-\left(\frac{\epsilon n}{3\,\mathbf{E}Z}\log\left(1+\frac{\epsilon n\,\mathbf{E}Z}{3j\,\mathbf{var}Y^{(n)}}\right) - \frac{\epsilon n}{3\,\mathbf{E}Z}\right)\right) \\
&\le \exp\left(-n\left(\frac{\epsilon}{3\,\mathbf{E}Z}\log\left(1+\frac{\epsilon\,\mathbf{E}Z}{6\,\mathbf{E}Z^2}\right) - \frac{\epsilon}{3\,\mathbf{E}Z}\right)\right)
\end{aligned}
\tag{2.64}
$$

for $j \ge 2n$. Therefore, for $n \ge n_0$ and $j \ge 2n$,

$$
\begin{aligned}
&\mathbf{P}\left(\sum_{i=1}^{j} \left(\mathbf{E}Z - Z_i \mathbb{1}_{\{Z_i \le n\}}\right) > \epsilon n\right) \\
&\le \exp\left(-n\left(\frac{\epsilon}{3\,\mathbf{E}Z}\log\left(1+\frac{\epsilon\,\mathbf{E}Z}{6\,\mathbf{E}Z^2}\right) - \frac{\epsilon}{3\,\mathbf{E}Z}\right)\right).
\end{aligned}
$$

Now, from Etemadi's inequality,

$$\limsup_{n\,!\,1} \frac{1}{L(n)n} \log \mathbf{P} \left(\sup_{1 \ j \ 2n} \sum_{i=1} \mathbf{E}Z \quad Z_i \mathbb{1}_{fZ_i \ \ n \ g} > 3n \right)$$

$$\limsup_{n\,!\,1} \frac{1}{L(n)n} \log \left(3 \max_{1 \ j \ 2n} \mathbf{P} \left(\sum_{i=1} \mathbf{E}Z \quad Z_i \mathbb{1}_{fZ_i \ \ n \ g} > n \right) \right)$$

$$\limsup_{n\,!\,1} \frac{1}{L(n)n} \log \quad 3 \exp \quad n \quad \frac{}{3\,\mathbf{E}Z} \quad \log \quad 1 + \frac{\mathbf{E}Z}{6\,\mathbf{E}Z^2} \qquad \frac{}{3\,\mathbf{E}Z}$$

$$= \quad 1:$$

Replacing with =3, we arrive at the conclusion of the lemma.

□

Now we are ready to prove Lemma 2.2.6.

Proof of Lemma 2.2.6.

$$\mathbf{P} \quad kH_n^k k_1 \ >$$

$$\mathbf{P} \quad kH_n^k k_1 \ > \ ; N(nt) \quad k \quad + \mathbf{P} \quad kH_n^k k_1 \ > \ ; N(nt) < k$$

$$\mathbf{P} \quad kH_n^k k_1 \ > \ ; N(nt) \quad k; Z_{R_n{}^1(k)} \quad n \tag{2.65}$$

$$+ \mathbf{P} \quad kH_n^k k_1 \ > \ ; N(nt) \quad k; Z_{R_n{}^1(k)} > n \quad + \mathbf{P}(N(nt) < k)$$

$$\mathbf{P} \quad kH_n^k k_1 \ > \ ; N(nt) \quad k; Z_{R_n{}^1(k)} \quad n$$

$$+ \mathbf{P} \quad N(nt) \quad k; Z_{R_n{}^1(k)} > n \quad + \mathbf{P}(N(nt) < k): \tag{2.66}$$

An explicit upper bound for the second term of (2.66) can be obtained:

$$\mathbf{P} \quad N(nt) \quad k; Z_{R_n{}^1(k)} > n \qquad \mathbf{P}(Q_n(\ _k) > n\ ) \quad \mathbf{P}(Q_n(\ _k) \quad n\ )$$

$$= \mathbf{P}(\ _k \quad Q_n(n\ )) = \int_0^{Q_n(n\ )} \frac{1}{k!} t^{k\ \ 1} e^{\ \ t} dt$$

$$= \int_0^{nv[n\ ;\,1\,)} \frac{1}{k!} t^{k\ \ 1} e^{\ \ t} dt \qquad \int_0^{nv(n\ ;\,1\,)} t^{k\ \ 1} dt$$

$$= \frac{1}{k} n^k e^{\ \ kL(n\ \ )n} \quad :$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\left(N(nt) \le k;\ Z_{R_n^{-1}(k)} > \delta n \right) \le -\delta k\ . \tag{2.67}$$

Turning to the first term of (2.65), we consider the following decomposition:

$$\begin{aligned}
&\mathbf{P}\left(\|H_n^k\|_\infty > \epsilon;\ N(nt) \le k;\ Z_{R_n^{-1}(k)} \le \delta n \right)\\
&= \underbrace{\mathbf{P}\left(N(nt) \le k;\ Z_{R_n^{-1}(k)} \le \delta n;\ \sup_{t\in[0,1]} H_n^k(t) > \epsilon \right)}_{(i)}\\
&\quad + \underbrace{\mathbf{P}\left(N(nt) \le k;\ Z_{R_n^{-1}(k)} \le \delta n;\ \sup_{t\in[0,1]} -H_n^k(t) > \epsilon \right)}_{(ii)}.
\end{aligned}$$

Since $Z_{R_n^{-1}(k)} \le \delta n$ implies $\mathbb{1}_{\{R_n(i)>k\}} \le \mathbb{1}_{\{Z_i \le \delta n\}}$,

$$\begin{aligned}
\text{(i)} \quad &\le \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \left(Z_i \mathbb{1}_{\{R_n(i)>k\}} - \mathbf{E}Z \right) > \epsilon n;\ N(nt) \le k;\ Z_{R_n^{-1}(k)} \le \delta n \right)\\
&\le \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \left(Z_i \mathbb{1}_{\{Z_i \le \delta n\}} - \mathbf{E}Z \right) > \epsilon n \right)\\
&= \mathbf{P}\left(\sup_{0\le j\le N(n)} \sum_{i=1}^{j} \left(Z_i \mathbb{1}_{\{Z_i \le \delta n\}} - \mathbf{E}Z \right) > \epsilon n \right)\\
&\le \mathbf{P}\left(\sup_{0\le j\le 2n} \sum_{i=1}^{j} \left(Z_i \mathbb{1}_{\{Z_i \le \delta n\}} - \mathbf{E}Z \right) > \epsilon n;\ N(n) < 2n \right) + \mathbf{P}\left(N(n) \ge 2n \right)\\
&\le \mathbf{P}\left(\sup_{0\le j\le 2n} \sum_{i=1}^{j} \left(Z_i \mathbb{1}_{\{Z_i \le \delta n\}} - \mathbf{E}Z \right) > \epsilon n \right) + \mathbf{P}\left(N(n) \ge 2n \right).
\end{aligned}$$

From Lemma 2.5.4 and the fact that the second term decays at an exponential rate,

$$\limsup_{n\to\infty} \frac{1}{L(n)n} \mathbf{P}\left(Z_{R_n^{-1}(k)} \le \delta n;\ \sup_{t\in[0,1]} H_n^k(t) > \epsilon \right) \le -(\epsilon/3)(\epsilon/\delta)^{1-\alpha}\ . \tag{2.68}$$

Turning to (ii),

$$\begin{aligned}
&\mathbf{P}\left(N(nt) \quad k; Z_{R_n^{-1}(k)} \quad n ; \sup_{t\in[0,1]} H_n^k(t) > \right)\\
&\quad \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \mathbf{E}Z \quad Z_i \mathbb{1}_{\{R_n(i)>k\}} > n \right)\\
&= \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \mathbf{E}Z \quad Z_i \mathbb{1}_{\{Z_i \quad n\}} + Z_i \quad \mathbb{1}_{\{Z_i \quad n\}} \quad \mathbb{1}_{\{R_n(i)>k\}} > n \right)\\
&\quad \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \mathbf{E}Z \quad Z_i \mathbb{1}_{\{Z_i \quad n\}} + Z_i \mathbb{1}_{\{Z_i \quad n\} \cap \{R_n(i) \quad k\}} > n \right)\\
&\quad \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \mathbf{E}Z \quad Z_i \mathbb{1}_{\{Z_i \quad n\}} + kn > n \right)\\
&= \mathbf{P}\left(\sup_{t\in[0,1]} \sum_{i=1}^{N(nt)} \mathbf{E}Z \quad Z_i \mathbb{1}_{\{Z_i \quad n\}} > n(\quad k) \right)\\
&\quad \mathbf{P}\left(\sup_{0 \quad j \quad 2n} \sum_{i=1}^{j} \mathbf{E}Z \quad Z_i \mathbb{1}_{\{Z_i \quad n\}} > n(\quad k); N(n) < 2n \right)\\
&\qquad + \mathbf{P}(N(n) \quad 2n)\\
&\quad \mathbf{P}\left(\sup_{0 \quad j \quad 2n} \sum_{i=1}^{j} \mathbf{E}Z \quad Z_i \mathbb{1}_{\{Z_i \quad n\}} > n(\quad k) \right) + \mathbf{P}(N(n) \quad 2n).
\end{aligned}$$

Applying Lemma 2.5.5 to the rst term and noticing that the second term vanishes at an exponential rate, we conclude that for and k such that $k \quad <$

$$\limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\left(Z_{R_n^{-1}(k)} \quad n ; \sup_{t\in[0,1]} H_n^k(t) > \right) = \quad \infty. \tag{2.69}$$

From (2.68) and (2.69),

$$\limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\left(Z_{R_n^{-1}(k)} \quad n ; \|H_n^k\|_\infty > \right) \quad (=3) \; (=)^{\infty} \quad . \tag{2.70}$$

This, together with (2.65) and (2.67),

$$\limsup_{n\to\infty} \frac{1}{L(n)n} \mathbf{P}\ \ \|H_n^k\|_\infty > \qquad \max\{\ \ (\ =3)\ \ (\ =\ )^1\ \ ;\ \ k\ \ \}$$

for any and k such that $k < $. Choosing, for example, $= \frac{}{2k}$ and letting $k \to \infty$, we arrive at the conclusion of the lemma.

□

2.5.4 Proofs of Lemma 2.2.8, 2.2.9, and 2.2.10

Proof of Lemma 2.2.8. We follow a similar program as in Lemma 2.2.4. First, we prove the finite-dimensional LDP for the k biggest jumps along with their jump times. Then, we transport the LDP to $\mathbb{D}_{6k}[0;1]$ by using an appropriate map. Recall that $Q\ (x) = \inf\{s > 0 : \mathbf{P}(Z\ \ s) < y\}$ and $V_{(1)}; : : : ; V_{(n\ 1)}$ are the order statistics of $n\ \ 1$ i.i.d. Uniform(0,1) random variables $V_1; : : : ; V_{n\ 1}$. We first claim that $(Q\ \ (V_{(1)})/n; : : : ; Q\ \ (V_{(k)})/n)$ satisfies the LDP with speed $L(n)n$ and good rate function I_k defined in (2.36). Let f be a bounded continuous function such that $|f(x)| < M$, $x \in \mathbb{R}_+^k$ for some $M \in \mathbb{R}$. We want to prove that

$$\lim_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{E} \exp\ \ L(n)n\ \ f\ \ Q\ \ (V_{(1)})/n; : : : ; Q\ \ (V_{(k)})/n\ \ \ = \sup_x\ \ _f(x);$$

where $_f = f\ \ \ I_k$; to invoke inverse Varadhan lemma and establish the LDP for $Q\ \ (V_{(1)})/n; : : : ; Q\ \ (V_{(k)})/n$. Recall that in the proof of Lemma 2.5.1, we have shown that the supremum of $f(x)\ \ \ I_k(x)$ over $\mathbb{R}_+^k$ is attained. Let $\hat{x}$ denote one of the optimizers that attain the supremum. Then, due to the form of I_k, for any given > 0, we can find > 0 and $x = (x_1; : : : ; x_k)$ such that $x_i\ \ \ x_{i+1} +$ for $i = 1; : : : ; k\ \ \ 1$ and $x \in \prod_{i=1}^k [x_i; x_i + \ ]$ implies

$$I_k(x)\ \ \ I_k(\hat{x}) \qquad\qquad \text{and} \qquad\qquad f(x)\ \ \ I_k(x)\ \ \ f(\hat{x})\ \ \ I_k(\hat{x})\ \ \ \ :$$

Therefore, if we set

$$A_n(\ ) \triangleq \{(y_1; : : : ; y_k) : Q\ \ (y_i)/n \in [x_i; x_i +\ \ ];\ i \in \{1; : : : ; k + 1\};$$

then $y \in A_n(\ )$ implies

$$I_k(Q\ \ (y_1)/n; : : : ; Q\ \ (y_k)/n)\ \ \ \ I_k(\hat{x})$$

and

$$f(Q\ \ (y_1)/n; : : : ; Q\ \ (y_k)/n)\ \ \ I_k(Q\ \ (y_1)/n; : : : ; Q\ \ (y_k)/n)\ \ \ \ f(\hat{x})\ \ \ I_k(\hat{x})\ \ \ \ ;$$

and hence,

$$f(Q\ \ (y_1)/n; \ldots; Q\ \ (y_k)/n) \quad f(\hat{x}) \quad 2\ .$$

Note also that $Q\ \ (y) < x$ if and only if $\mathbf{P}(Z \quad x) < y$, and hence,

$$\mathbf{P}(Z \quad n(x_i + \ \ )) < y_i \quad \mathbf{P}(Z \quad nx_i)$$

implies $Q\ \ (y_i)/n \in [x_i; x_i + \ \ ]$. We have that

$$\mathbb{1}_{\{Q\ \ (y_i)/n \in [x_i; x_i + \ ];\ i=1,\ldots,k\}} \quad \mathbb{1}_{\{\mathbf{P}(Z\ \ n(x_i+\ ))<y_i\ \ \mathbf{P}(Z\ \ nx_i);\ i=1,\ldots,k\}};$$

and hence, for $y_{k+1} \quad \mathbf{P}(Z \quad nx_i)$,

$$\int_0^{y_{k+1}} \cdots \int_0^{y_2} \mathbb{1}_{\{Q\ \ (y_i)/n \in [x_i; x_i + \ ];\ i=1,\ldots,k\}}\, dy_1 \quad dy_k$$

$$\int_0^{y_{k+1}} \cdots \int_0^{y_2} \mathbb{1}_{\{\mathbf{P}(Z\ \ n(x_i+\ ))<y_i\ \ \mathbf{P}(Z\ \ nx_i);\ i=1,\ldots,k\}}\, dy_1 \quad dy_k$$

$$= \int_0^1 \cdots \int_0^1 \mathbb{1}_{\{\mathbf{P}(Z\ \ n(x_i+\ ))<y_i\ \ \mathbf{P}(Z\ \ nx_i);\ i=1,\ldots,k\}}\, dy_1 \quad dy_k$$

$$= \prod_{i=1} \mathbf{P}(Z \quad nx_i) \quad \mathbf{P}(Z \quad nx_i + n\ \ )\ .$$

Therefore,

$$\mathbf{E}\exp\ \ L(n)n\ \ f\ \ Q\ \ (V_{(1)})/n; \ldots; Q\ \ (V_{(k)})/n$$

$$\mathbf{E}\exp\ \ L(n)n\ \ f\ \ Q\ \ (V_{(1)})/n; \ldots; Q\ \ (V_{(k)})/n \quad \mathbb{1}_{\{(V_{(1)},\ldots,V_{(k)}) \in A_n(\ )\}}$$

$$= \int_0^1 \int_0^{y_{n\ 1}} \cdots \int_0^{y_2} \Big[e^{L(n)n\ \ f(Q\ \ (y_1)/n,\ldots,Q\ \ (y_k)/n)} \Big] (n \quad 1)!\, \mathbb{1}_{\{(y_1,\ldots,y_k) \in A_n(\ )\}}\, dy_1 \ldots dy_{n\ 1}$$

$$(n \quad 1)!\, e^{L(n)n\ \ (f(\hat{x})\ \ 2\ )} \int_0^1 \int_0^{y_{n\ 1}} \cdots \int_0^{y_2} \mathbb{1}_{\{Q\ \ (y_i)/n \in [x_i; x_i + \ ];\ i=1,\ldots,k\}}\, dy_1 \quad dy_{n\ 2}\, dy_{n\ 1}$$

$$(n \quad 1)!\, e^{L(n)n\ \ (f(\hat{x})\ \ 2\ )} \prod_{i=1} \mathbf{P}(Z \quad nx_i) \quad \mathbf{P}(Z \quad nx_i + n\ \ )$$

$$\int_0^{\infty}\int_0^{y_{n-1}}\cdots\int_0^{y_{k+2}} \frac{\cdots}{\mathbf{P}(Z \ge nx_i)}\, dy_{k+1}\cdots dy_{n-2}\,dy_{n-1}$$
$$= (n-1)!\,e^{L(n)n\,(f(\hat{x})-2\delta)} \prod_{i=1}^{k} \left(\mathbf{P}(Z \ge nx_i) - \mathbf{P}(Z \ge nx_i + n\delta)\right)$$
$$\frac{1}{(n-k-1)!}\left(1-\mathbf{P}(Z \ge nx_i)\right)^{n-k-1}.$$

Since

$$\liminf_{n\to\infty}\frac{1}{L(n)n}\log\prod_{i=1}^{k}\left(\mathbf{P}(Z \ge nx_i) - \mathbf{P}(Z \ge nx_i+n\delta)\right) = -\sum_{i=1}^{k}\hat{x}_i = -I_k(\hat{x})$$

and

$$\liminf_{n\to\infty}\frac{1}{L(n)n}\log\left(1-\mathbf{P}(Z \ge nx_i)\right)^{n-k-1} = 0,$$

we get

$$\liminf_{n\to\infty}\frac{1}{L(n)n}\log \mathbf{E}e^{L(n)n\, f\left(Q_{(V_{(1)})/n},\ldots,Q_{(V_{(k)})/n}\right)} \ge f(\hat{x}) - 2\delta - I_k(\hat{x})$$
$$= \sup_{x\in\mathbb{R}_+^k}\{f(x) - I_k(x)\} - 2\delta.$$

Letting $\delta \to 0$, we arrive at the lower bound. Turning to the upper bound,

$$\mathbf{E}\exp\left(L(n)n\, f\left(Q_{(V_{(1)})/n},\ldots,Q_{(V_{(k)})/n}\right)\right)$$
$$= \mathbf{E}\exp\left(L(n)n\, f\left(Q_{(V_{(1)})/n},\ldots,Q_{(V_{(k)})/n}\right)\right)\mathbb{1}_{\{Q_{(V_{(1)})/n}>R\}}$$
$$+ \mathbf{E}\exp\left(L(n)n\, f\left(Q_{(V_{(1)})/n},\ldots,Q_{(V_{(k)})/n}\right)\right)\mathbb{1}_{\{Q_{(V_{(1)})/n}\le R\}}.$$

For the first term, note that

$$\mathbf{E}\exp\left(L(n)n\, f\left(Q_{(V_{(1)})/n},\ldots,Q_{(V_{(k)})/n}\right)\right)\mathbb{1}_{\{Q_{(V_{(1)})/n}>R\}}$$
$$\le \mathbf{E}\exp\{L(n)nM\}\mathbb{1}\{Q_{(V_{(1)})/n}>R\} \tag{2.71}$$
$$\le \exp\{L(n)nM\}\mathbf{P}(V_{(1)} \ge \mathbf{P}(Z \ge nR))$$
$$= \exp\{L(n)nM\}\left(1-\left(1-\mathbf{P}(Z\ge nR)\right)^{n-1}\right)$$
$$= \exp\{L(n)nM\}\left(1-\left(1-\exp\{-L(nR)(nR)^{\beta}\}\right)^{n-1}\right). \tag{2.72}$$

Also, from the generic inequality $1 - \exp(-z) \le z$,

$$1 - (1 - 1/x)^y = 1 - \{(1 - 1/x)^x\}^{y/x} = 1 - \exp\log\{(1 - 1/x)^x\}^{y/x}$$
$$= 1 - \exp\{(y/x)\log(1 - 1/x)^x\} \le -(y/x)\log(1 - 1/x)^x$$

for any $x, y > 0$. Setting $x = \exp(L(nR)(nR)^\gamma)$ and $y = n - 1$, we get

$$1 - \left(1 - \exp\{-L(nR)(nR)^\gamma\}\right)^{n-1}$$
$$\le -(n-1)\exp\{-L(nR)(nR)^\gamma\}\log\left(1 - 1/\exp\{L(nR)(nR)^\gamma\}\right)^{\exp\{L(nR)(nR)^\gamma\}}.$$

Substituting this into (2.72), we arrive at the upper bound for the first term:

$$\limsup_{n \to \infty} \frac{1}{L(n)n^\gamma} \log \mathbf{E} e^{L(n)n^\gamma f(Q^\leftarrow(V_{(1)})/n, \ldots, Q^\leftarrow(V_{(k)})/n)} \mathbb{1}_{\{Q^\leftarrow(V_{(1)})/n > R\}}$$
$$\le M - R^\gamma.$$

For the second term, fix $\epsilon > 0$ and pick $\{x^{(1)}, \ldots, x^{(m)}\} \subseteq \mathbb{R}_+^k$ in such a way that

$$\left\{\prod_{j=1}^{k} [x_j^{(l)} - \epsilon, x_j^{(l)} + \epsilon]\right\}_{l=1,\ldots,m}$$

covers $\{(x_1, \ldots, x_k) : R \ge x_1 \ge x_2 \ge \cdots \ge x_k \ge 0\}$ and $x_1^{(l)} \ge \cdots \ge x_k^{(l)} \ge 0$ for $l = 1, \ldots, m$. Set

$$A_{n,l}(R) \triangleq \left\{(y_1, \ldots, y_k) \in \mathbb{R}_+^k : y_1 \ge \cdots \ge y_k, \frac{Q^\leftarrow(y_j)}{n} \in [x_j^{(l)} - \epsilon, x_j^{(l)} + \epsilon], 1 \le j \le k\right\}.$$

Note that $y_1 \ge \cdots \ge y_k$ and $Q^\leftarrow(y_1)/n \le R$ implies

$$R \ge Q^\leftarrow(y_1)/n \ge Q^\leftarrow(y_2)/n \ge \cdots \ge Q^\leftarrow(y_k)/n,$$

which, in turn, implies $Q^\leftarrow(y_j)/n \in [x_j^{(l)} - \epsilon, x_j^{(l)} + \epsilon]$, $j = 1, \ldots, k$ for some $l \in \{1, \ldots, m\}$. Therefore,

$$\left\{(y_1, \ldots, y_k) \in \mathbb{R}_+^k : y_1 \ge \cdots \ge y_k, Q^\leftarrow(y_1)/n \le R\right\} \subseteq \bigcup_{l=1}^{m} A_{n,l}(R);$$

and hence,

$$\mathbf{E}\exp\left(L(n)n\, f\left(Q(V_{(1)})/n;\dots;Q(V_{(k)})/n\right)\right)\mathbb{1}_{\{Q(V_{(1)})\le R\}}$$
$$\le\sum_{i=1}^{M}\mathbf{E}\exp\left(L(n)n\, f\left(Q(V_{(1)})/n;\dots;Q(V_{(k)})/n\right)\right)\mathbb{1}_{\{(V_{(1)};\dots;V_{(k)})\in A_{n;l}(R)\}}.$$

Note that

$$\mathbf{E}\exp\left(L(n)n\, f\left(Q(V_{(1)})/n;\dots;Q(V_{(k)})/n\right)\right)\mathbb{1}_{\{(V_{(1)};\dots;V_{(k)})\in A_{n;l}(R)\}}$$
$$=\mathbf{E}e^{L(n)n\left(f\left(Q(V_{(1)})/n;\dots;Q(V_{(k)})/n\right)-l_k\left(Q(V_{(1)})/n;\dots;Q(V_{(k)})/n\right)\right)}e^{L(n)n\, l_k\left(Q(V_{(1)})/n;\dots;Q(V_{(k)})/n\right)}$$
$$\mathbb{1}_{\{(V_{(1)};\dots;V_{(k)})\in A_{n;l}(R)\}}$$
$$\le e^{L(n)n\,\delta_f(\hat{x}_1;\dots;\hat{x}_k)}\mathbf{E}e^{L(n)n\, l_k\left(Q(V_{(1)})/n;\dots;Q(V_{(k)})/n\right)}\mathbb{1}_{\{(V_{(1)};\dots;V_{(k)})\in A_{n;l}(R)\}}$$
$$\le e^{L(n)n\,\delta_f(\hat{x}_1;\dots;\hat{x}_k)}e^{L(n)n\, l_k\left(x_1^{(l)}+\delta;\dots;x_k^{(l)}+\delta\right)}\mathbf{E}\mathbb{1}_{\{(V_{(1)};\dots;V_{(k)})\in A_{n;l}(R)\}}$$

and

$$\mathbf{E}\mathbb{1}_{\{(V_{(1)};\dots;V_{(k)})\in A_{n;l}(R)\}}$$
$$=\int_0^1\int_0^{y_{n-1}}\cdots\int_0^{y_2}(n-1)!\,\mathbb{1}_{\{(y_1;\dots;y_k)\in A_{n;l}(R)\}}\,dy_1\cdots dy_{n-2}\,dy_{n-1}$$
$$\le(n-1)!\int_0^1\cdots\int_0^{y_2}\mathbb{1}_{\{Q(y_i)/n\in[x_i^{(l)}-\delta;\,x_i^{(l)}+\delta];\ i=1;\dots;k\}}\,dy_1\cdots dy_{n-2}\,dy_{n-1}$$
$$\le(n-1)!\prod_{i=1}^{k}\left(\mathbf{P}(Z\ge nx_i^{(l)}-n\delta)-\mathbf{P}(Z\ge nx_i^{(l)}+n\delta)\right)$$
$$\int_0^1\int_0^{y_{n-1}}\cdots\int_0^{y_{k+2}}dy_{k+1}\cdots dy_{n-2}\,dy_{n-1}$$
$$=(n-1)!\prod_{i=1}^{k}\left(\mathbf{P}(Z\ge nx_i^{(l)}-n\delta)-\mathbf{P}(Z\ge nx_i^{(l)}+n\delta)\right)\frac{1}{(n-k-1)!}$$
$$\le n^k\prod_{i=1}^{k}\left(\mathbf{P}(Z\ge nx_i^{(l)}-n\delta)-\mathbf{P}(Z\ge nx_i^{(l)}+n\delta)\right).$$

Therefore,

$$\mathbf{E}\exp\left(L(n)n\, f\left(Q(V_{(1)})/n;\dots;Q(V_{(k)})/n\right)\right)\mathbb{1}_{\{Q(V_{(1)})\le R\}}$$

$$n^k e^{L(n)n \quad f(\hat{x}_1,\dots,\hat{x}_k)} \sum_{i=1} e^{L(n)n \quad I_k \quad x_1^{(l)}+ \quad ,\dots,x_k^{(l)}+}$$

$$\prod_{i=1} \mathbf{P}(Z \quad nx_i^{(l)} \quad n\ ) \quad \mathbf{P}(Z \quad nx_i^{(l)} + n\ ) \ .$$

Note that

$$\limsup_{n\to\infty} \frac{1}{L(n)n} \log \prod_{i=1} \mathbf{P}(Z \quad nx_i^{(l)} \quad n\ ) \quad \mathbf{P}(Z \quad nx_i^{(l)} + n\ )$$

$$= \sum_{i=1} (x_i^{(l)} \quad)_+ = \quad I_k\ (x_1^{(l)} \quad ,\dots,x_k^{(l)} \quad)_+\ ,$$

where $(y)_+$ denotes $\max\{y,0\}$ and $(y_1,\dots,y_k)_+$ denotes $((y_1)_+,\dots,(y_k)_+)$. This, along with the principle of the largest term,

$$\limsup_{n\to\infty} \frac{\log \mathbf{E}\exp \quad L(n)n \quad f \quad Q \quad (V_{(1)})/n,\dots,Q \quad (V_{(k)})/n \quad \mathbb{1}_{\{Q \quad (V_{(1)}) \quad R\}}}{L(n)n}$$

$$\max_{l=1,\dots,m} \quad f(\hat{x}_1,\dots,\hat{x}_k) + \sum_{i=1} (x_i^{(l)} + \quad) \quad \sum_{i=1} (x_i^{(l)} \quad)_+$$

$$\max_{l=1,\dots,m} \quad f(\hat{x}_1,\dots,\hat{x}_k) + k \quad .$$

Sending $\to 0$, we get

$$\limsup_{n\to\infty} \frac{\log \mathbf{E}\exp \quad L(n)n \quad f \quad Q \quad (V_{(1)})/n,\dots,Q \quad (V_{(k)})/n \quad \mathbb{1}_{\{Q \quad (V_{(1)}) \quad R\}}}{L(n)n}$$

$$f(\hat{x}_1,\dots,\hat{x}_k).$$

Now, combining with the bound for the first term, and sending $R\to\infty$, we get the upper bound:

$$\limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{E}\exp \quad L(n)n \quad f \quad Q \quad (V_{(1)})/n,\dots,Q \quad (V_{(k)})/n$$

$$\max\{ \quad f(\hat{x}_1,\dots,\hat{x}_k), M \quad R \ \} \to \quad f(\hat{x}_1,\dots,\hat{x}_k).$$

Together with the lower bound, we get

$$\lim_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{E}\exp \quad L(n)n \quad f \quad Q \quad (V_{(1)})/n,\dots,Q \quad (V_{(k)})/n$$

$$= \Lambda_f(\hat{x}_1, \ldots, \hat{x}_k);$$

which in turn allows us to apply Bryc's inverse Varadhan Lemma to prove that $\left(Q^{\lambda}(V_{(1)})/n, \ldots, Q^{\lambda}(V_{(k)})/n\right)$ satisfies the LDP with rate function I_k. From Theorem 4.14 of [39], we see that $\left(Q^{\lambda}(V_{(1)})/n, \ldots, Q^{\lambda}(V_{(k)})/n, Z/n\right)$ satisfies the LDP with rate function I_k' given by

$$I_k'(x_1, \ldots, x_{k+1}) = \begin{cases} \sum_{i=1}^{k+1} x_i & \text{if } x_1 \geq x_2 \geq \cdots \geq x_k \geq 0 \text{ and } x_{k+1} \geq 0; \\ \infty & \text{otherwise.} \end{cases} \tag{2.73}$$

Proceeding similarly as in Corollary 2.5.2 and Lemma 2.5.3 (except for considering a mapping $\tilde{T}_k : (x_1, \ldots, x_{k+1}, u_1, \ldots, u_k) \mapsto \sum_{i=1}^{k} x_i \mathbb{1}_{[u_i,1]} + x_{k+1}\mathbb{1}_{\{1\}}$ instead of the mapping $T_k : (x_1, \ldots, x_k, u_1, \ldots, u_k) \mapsto \sum_{i=1}^{k} x_i \mathbb{1}_{[u_i,1]}$) and $\mathbb{D}_{\leqslant k}[0,1]$ instead of $\mathbb{D}_{\leqslant k}[0,1]$) we conclude that

$$\tilde{J}_n^k(t) = \frac{1}{n}\sum_{i=1}^{k} Q^{\lambda}(V_{(i)})\mathbb{1}_{[U_i,1]}(t) + \frac{1}{n} Z \mathbb{1}_{\{1\}}(t)$$

satisfies the LDP with speed $L(n)n^{\alpha}$ and rate function $\tilde{I}_k$ in (2.15).

□

Proof of Lemma 2.2.9. The proof is essentially identical to Lemma 2.2.5, and hence, omitted. □

Proof of Lemma 2.2.10. Let

$$H_n^k(t) \triangleq \bar{H}_n^k(t) + \frac{1}{n}\mathbf{E}Z\mathbb{1}_{\{1\}}(t) = \frac{1}{n}\sum_{i=k+1}^{N^{\lambda}_n} Q^{\lambda}(V_{(i)})\mathbb{1}_{[U_i,1]}(t) - \frac{1}{n}\sum_{i=1}^{N^{\lambda}_n}\mathbf{E}Z\mathbb{1}_{[U_i,1]}(t).$$

Since

$$\mathbf{P}(\|\bar{H}_n^k\|_{\infty} \geq \epsilon) \leq \mathbf{P}(\|\bar{H}_n^k\|_{\infty} \geq \epsilon/2) + \mathbf{P}(\|\frac{1}{n}\mathbf{E}Z\sum_{i=1}^{N^{\lambda}_n}\mathbb{1}_{[U_i,1]} + \frac{1}{n}\mathbf{E}Z\mathbb{1}_{\{1\}}\|_{\infty} \geq \epsilon/2)$$

and $\mathbf{P}(\|\frac{1}{n}\mathbf{E}Z\sum_{i=1}^{N^{\lambda}_n}\mathbb{1}_{[U_i,1]}(t) + \frac{1}{n}\mathbf{E}Z\mathbb{1}_{\{1\}}\|_{\infty} > \epsilon/2) = 0$ for large enough n, we only need to prove that

$$\lim_{k\to\infty}\limsup_{n\to\infty}\frac{1}{L(n)n^{\alpha}}\log\mathbf{P}(\|H_n^k\|_{\infty} > \delta) = -\infty.$$

To show this, we x an arbitrary $2\ (0; \ =k)$ and consider the following decomposition:

$$\mathbf{P}(kH_n^k k_1 > \ ) \quad \mathbf{P}(kH_n^k k_1 > \ ; Q \ (V_{(k)}) < n \ ) + \mathbf{P} \ Q \ (V_{(k)}) \quad n \quad :$$

We rst bound the second term. Since the density of the k-th order statistic of the uniform distribution on $[0;1]$ is $n \, {}^{n}_{k} \, {}^{1}_{1} \, x^{k} \ {}^{1}(1 \quad x)^{n \ k}$,

$$\mathbf{P} \ Q \ (V_{(k)}) \quad n \quad = \mathbf{P} \ V_{(k)} \quad \mathbf{P}(Z \quad n \ ) \qquad \int_0^{\mathbf{P}(Z \ n \ )} n \begin{matrix} n & 1 \\ k & 1 \end{matrix} \ x^{k \ 1} dx$$

$$= \begin{matrix} n \\ k \end{matrix} \ \mathbf{P}(Z \quad n \ ) \ ^{k} = \begin{matrix} n \\ k \end{matrix} \ \exp(\ kL(n \ )(n \ ) \ )$$

and hence, $\limsup_{n \, ! \, 1} \frac{1}{L(n)n} \log \mathbf{P}(Q_n \ (V_{(k)}) > n \ ) \qquad k \quad$. For the rst term,

$$\mathbf{P}(kH_n^k k_1 > \ ; Q \ (V_{(k)}) < n \ )$$

$$= \mathbf{P} \ \sup_{t2[0;1]} H_n^k(t) > \ ; Q \ (V_{(k)}) \quad n \quad + \mathbf{P} \ \sup_{t2[0;1]} \ H_n^k > \ ; Q \ (V_{(k)}) \quad n$$

$$\mathbf{P} \ \max_{1 \ j \ n \ 1} \sum_{i=1} (Z_i \mathbb{1}_{fZ_i \ n \ g} \quad \mathbf{E}Z) > n$$

$$+ \mathbf{P} \ \max_{1 \ j \ n \ 1} \sum_{i=1} (\mathbf{E}Z \quad Z_i \mathbb{1}_{fZ_i \ n \ g}) + kn \ > n \quad :$$

Note that from Lemma 2.5.4,

$$\limsup_{n \, ! \, 1} \frac{1}{L(n)n} \mathbf{P} \ \max_{1 \ j \ n \ 1} \sum_{i=1} (Z_i \mathbb{1}_{fZ_i \ n \ g} \quad \mathbf{E}Z) > n \qquad (\ =3) \ (\ = \ )^{1}$$

and from Lemma 2.5.5, since $< \ =k$,

$$\limsup_{n \, ! \, 1} \frac{1}{L(n)n} \mathbf{P} \ \max_{1 \ j \ n \ 1} \sum_{i=1} (\mathbf{E}Z \quad Z_i \mathbb{1}_{fZ_i \ n \ g}) + kn \ > n \quad = \quad 1 :$$

Therefore,

$$\limsup_{n \, ! \, 1} \frac{1}{L(n)n} \mathbf{P}(kH_n^k k_1 > \ ; Q_n \ (V_{(k)}) \quad n \ ) \quad \max f \ (\ =3) \ (\ = \ )^{1} \ ; \ 1 g$$

$$= \quad (\;=3) \;\; (\;= \;)^{1} \quad :$$

Applying the principle of the maximum term once again,

$$\lim_{k!1} \limsup_{n!1} \frac{1}{L(n)n} \mathbf{P}(k H_n^k k_1 > \;) \quad \lim_{k!1} \max f \;\; (\;=3) \;\; (\;= \;)^{1} \quad ; \quad k \quad g$$
$$= \quad (\;=3) \;\; (\;= \;)^{1} \quad :$$

Since can be chosen arbitrarily small,

$$\lim_{k!1} \limsup_{n!1} \frac{1}{L(n)n} \mathbf{P}(k H_n^k k_1 > \;) = \quad 1 :$$

□

2.5.5 Proof of Theorem 2.2.12

We follow a similar program as in Section 2.2.2 and the earlier subsections of this section. Let $Q_n^{(i)}(j)$, Q_n ($_j^{(i)}$)=n where Q_n (t) = inf$fs > 0 : n$ [$s; 1$) < tg and $_l^{(i)} = E_1^{(i)} + \quad + E_l^{(i)}$ where $E_j^{(i)}$'s are independent standard exponential random variables. Let $U_j^{(i)}$ be independent uniform random variables in [0,1] and $Z_n^{(i)}$, $Q_n^{(i)}(1); : : : ; Q_n^{(i)}(k); U_1^{(i)}; : : : ; U_k^{(i)}$. The following corollary is an immediate consequence of Corollary 2.5.2 and Theorem 4.14 of [39].

Corollary 2.5.6. *$Z_n^{(1)}; : : : ; Z_n^{(d)}$ satis es the LDP in $\prod_{i=1}^{d} \mathbb{R}_+^k \quad [0;1]^k$ with rate function $\sum_{j=1}^{d} I_k(z^{(j)})$ where $z^{(j)} = (x_1^{(j)}; : : : ; x_k^{(j)}; u_1^{(j)}; : : : ; u_k^{(j)})$ for each $j \;2\; f1; : : : ; dg$.*

Let $\hat{J}_n^{6k\,(i)}$, $\sum_{j=1}^{k} Q_n^{(i)}(j) \mathbb{1}_{[U_j^{(i)};1]}$.

Lemma 2.5.7. *$\hat{J}_n^{6k\,(1)}; : : : ; \hat{J}_n^{6k\,(d)}$ satis es the LDP in $\prod_{i=1}^{d} \mathbb{D}([0;1];\mathbb{R})$ with speed $L(n)n$ and rate function*

$$\sum_{i=1}^{d} I_k(\;_i) = \begin{cases} \sum_{i=1}^{d} \sum_{t: \;_i(t) \neq \;_i(t\;)} (\;_i(t) \quad \;_i(t \;)) & \text{if } \;_i \;2\; \mathbb{D}_{6k}[0;1] \\ & \text{for } i = 1; : : : ; d; \\ 1 ; & \text{otherwise:} \end{cases}$$

Proof. Since I_{k_i} is lower semi-continuous in $\prod_{i=1}^{d} \mathbb{D}([0;1];\mathbb{R})$ for each i, $\sum_{i=1}^{d} I_k$ is a sum of lower semi-continuous functions, and hence, is lower semi-continuous itself. The rest of the proof for the LDP upper bound and the lower bounds mirrors that of the one dimensional case (Lemma 2.5.3) closely, and hence, omitted. □

Proof of Lemma 2.2.13. Again, we consider the same distributional relation for each coordinate as in the 1-dimensional case:

$$J_n^{k\,(i)} \overset{D}{=} \underbrace{\frac{1}{n}\sum_{j=1}^{k} Q_n^{(i)}(j)\mathbb{1}_{[U_j,1]}}_{=\hat{J}_n^{\leqslant k\,(i)}} + \underbrace{\frac{1}{n}\mathbb{1}\{\tilde{N}_n^{(i)} < k\}\sum_{j=\tilde{N}_n^{(i)}+1}^{k} Q_n^{(i)}(j)\mathbb{1}_{[U_j^{(i)},1]}}_{=\check{J}_n^{\leqslant k\,(i)}}.$$

Note that this distributional equality holds jointly w.r.t. $i = 1,\ldots,d$ due to the assumed independence. Let F be a closed set and write

$$\begin{aligned}
&\mathbf{P}((J_n^{k\,(1)},\ldots,J_n^{k\,(d)}) \in F)\\
&\quad\leq \mathbf{P}\Big((\hat{J}_n^{\leqslant k\,(1)},\ldots,\hat{J}_n^{\leqslant k\,(d)}) \in F,\ \textstyle\sum_{i=1}^{d}\mathbb{1}\{\tilde{N}_n^{(i)} < k\} = 0\Big)\\
&\qquad + \sum_{i=1}^{d}\mathbf{P}\Big(\mathbb{1}\{\tilde{N}_n^{(i)} < k\} \neq 0\Big)\\
&\quad\leq \mathbf{P}\Big((\hat{J}_n^{\leqslant k\,(1)},\ldots,\hat{J}_n^{\leqslant k\,(d)}) \in F\Big) + \sum_{i=1}^{d}\mathbf{P}\Big(\mathbb{1}\{\tilde{N}_n^{(i)} < k\} \neq 0\Big).
\end{aligned}$$

From Lemma 2.5.7 and the principle of the largest term,

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{\log\mathbf{P}\big((J_n^{k\,(1)},\ldots,J_n^{k\,(d)}) \in F\big)}{L(n)n}\\
&\quad\leq \limsup_{n\to\infty}\frac{\log\mathbf{P}\big((\hat{J}_n^{\leqslant k\,(1)},\ldots,\hat{J}_n^{\leqslant k\,(d)}) \in F\big)}{L(n)n} \vee \max_{i=1,\ldots,d}\limsup_{n\to\infty}\frac{\log\mathbf{P}\big(\tilde{N}_n^{(i)} < k\big)}{L(n)n}\\
&\quad\leq -\inf_{(\xi_1,\ldots,\xi_d)\in F} I_k^d(\xi_1,\ldots,\xi_d).
\end{aligned}$$

Turning to the lower bound, let G be an open set. Since the lower bound is trivial in case $\inf_{x\in G} I_k(x) = \infty$, we focus on the case $\inf_{x\in G} I_k(x) < \infty$. In this case, using a reasoning similar to the one leading to (2.56),

$$\liminf_{n\to\infty}\frac{\log\mathbf{P}((J_n^{k\,(1)},\ldots,J_n^{k\,(d)}) \in G)}{L(n)n}$$

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{\log \mathbf{P}\left((J_n^{k(1)},\dots,J_n^{k(d)}) \in G,\ \sum_{i=1}^d \mathbb{1}\{N_n^{(i)} \geq k\} = 0\right)}{L(n)n}\\
&= \liminf_{n\to\infty} \frac{\log \mathbf{P}\left((\hat J_n^{\leqslant k(1)},\dots,\hat J_n^{\leqslant k(d)}) \in G,\ \sum_{i=1}^d \mathbb{1}\{N_n^{(i)} \geq k\} = 0\right)}{L(n)n}\\
&\geq \liminf_{n\to\infty} \frac{1}{L(n)n} \log\left(\mathbf{P}\left((\hat J_n^{\leqslant k(1)},\dots,\hat J_n^{\leqslant k(d)}) \in G\right) \cdot \mathbf{P}(N_n^{(1)} < k)^d\right)\\
&= \liminf_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\left((\hat J_n^{\leqslant k(1)},\dots,\hat J_n^{\leqslant k(d)}) \in G\right)\\
&\geq -\inf_{(\xi_1,\dots,\xi_d)\in G} I_k^d(\xi_1,\dots,\xi_d).
\end{aligned}$$

□

The proof of Lemma 2.2.14 is completely analogous to the one-dimensional case, and therefore omitted.

2.5.6 Proof of Proposition 2.4.1

In this section, we prove that $I_{M_1'}$ has compact level sets. To do so, we develop a criterion for relative compactness in the M_1' topology (Proposition 2.5.9). Based on Proposition 2.5.9, we verify that the sublevel sets of $I_{M_1'}$ are closed (proof of Proposition 2.4.1).

Let $\mathbb{D}[0,1]$ be the space of functions ξ from $[0,1]$ to $\mathbb{R}$ such that the left limit exists at each $t \in (0,1]$, the right limit exists at each $t \in [0,1)$, and

$$\xi(t) \in [\xi(t-) \wedge \xi(t+),\ \xi(t-) \vee \xi(t+)] \tag{2.74}$$

for each $t \in [0,1]$ where we interpret $\xi(0-)$ as 0 and $\xi(1+)$ as $\xi(1)$. Let $\mathbb{D}^{\uparrow}[0,1] \triangleq \{\xi \in \mathbb{D}[0,1] : \xi \text{ is nondecreasing and } \xi(0) \geq 0\}$.

Proposition 2.5.8. *Suppose that $\hat\xi_0 \in \mathbb{D}[0,1]$ with $\hat\xi_0(0) \geq 0$ and $\xi_n \in \mathbb{D}^{\uparrow}[0,1]$ for each $n \geq 1$. If $T \triangleq \{t \in [0,1] : \xi_n(t) \to \hat\xi_0(t)\}$ is dense on $[0,1]$ and $1 \in T$, then $\xi_n \overset{M_1'}{\to} \xi_0 \in \mathbb{D}^{\uparrow}$ where $\xi_0(t) \triangleq \lim_{s\downarrow t} \hat\xi_0(s)$ for $t \in [0,1)$ and $\xi_0(1) \triangleq \hat\xi_0(1)$.*

Proof. It is easy to check that $\hat\xi_0$ has to be non-negative and non-decreasing, and for such $\hat\xi_0$, ξ_0 should be in $\mathbb{D}^{\uparrow}[0,1]$. Let (x,t) be a parametrization of $\Gamma'(\hat\xi_0)$, and let $\epsilon > 0$ be given. Note that $\Gamma'(\xi_0)$ and $\Gamma'(\hat\xi_0)$ coincide. Therefore, the proposition is proved if we show that there exists an integer N_0 such that for each $n \geq N_0$, $'(\ _n)$ can be parametrized by some (y, r) such that

$$\|x - y\|_\infty + \|t - r\|_\infty \leq \epsilon . \tag{2.75}$$

We start with making an observation that one can always construct a finite number of points $S = \{s_i\}_{i=0,1,\dots,m} \subset [0, 1]$ such that

(S1) $0 = s_0 < s_1 < \cdots < s_m = 1$;

(S2) $t(s_i) - t(s_{i-1}) < \epsilon/4$ for $i = 1, \dots, m$;

(S3) $x(s_i) - x(s_{i-1}) < \epsilon/8$ for $i = 1, \dots, m$;

(S4) if $t(s_{k-1}) < t(s_k) < t(s_{k+1})$ then $t(s_k) \in T$;

(S5) if $t(s_{k-1}) < t(s_k) = t(s_{k+1})$, then $t(s_{k-1}) \in T$; if, in addition, $k - 1 > 0$, then $t(s_{k-2}) < t(s_{k-1})$;

(S6) if $t(s_{k-1}) = t(s_k) < t(s_{k+1})$, then $t(s_{k+1}) \in T$; if, in addition, $k + 1 < m$, then $t(s_{k+1}) < t(s_{k+2})$.

One way to construct such a set is to start with S such that (S1), (S2), and (S3) are satisfied. This is always possible because x and t are continuous and non-decreasing. Suppose that (S4) is violated for some three consecutive points in S, say s_{k-1}, s_k, s_{k+1}. We argue that it is always possible to eliminate this violation by either adding an additional point $\hat{s}_k$ or moving s_k slightly. More specifically, if there exists $\hat{s}_k \in (s_{k-1}, s_{k+1}) \setminus \{s_k\}$ such that $t(\hat{s}_k) = t(s_k)$, add $\hat{s}_k$ to S. If there is no such $\hat{s}_k$, $t(\cdot)$ has to be strictly increasing at s_k, and hence, from the continuity of x and t along with the fact that T is dense, we can deduce that there has to be $\bar{s}_k \in (s_{k-1}, s_{k+1})$ such that $t(\bar{s}_k) \in T$ and $|t(\bar{s}_k) - t(s_k)|$ and $|x(\bar{s}_k) - x(s_k)|$ are small enough so that (S2) and (S3) are still satisfied when we replace s_k with $\bar{s}_k$ in S. Iterating this procedure, we can construct S so that (S1)-(S4) are satisfied. Now turning to (S5), suppose that it is violated for three consecutive points s_{k-1}, s_k, s_{k+1} in S. Since T is dense and t is continuous, one can find $\hat{s}_k$ between s_{k-1} and s_k such that $t(s_{k-1}) < t(\hat{s}_k) < t(s_k)$ and $t(\hat{s}_k) \in T$. Note that after adding $\hat{s}_k$ to S, (S2), (S3), and (S4) should still hold while the number of triplets that violate (S5) is reduced by one. Repeating this procedure for each triplet that violates (S5), one can construct a new S which satisfies (S1)-(S5). One can also check that the same procedure for the triplets that violate (S6) can reduce the number of triplets that violate (S6) while not introducing any new violation for (S2), (S3), (S4), and (S5). Therefore, S can be augmented so that the resulting finite set satisfies (S6) as well. Set $\hat{S} \triangleq \{s_i \in S : t(s_i) \in T;\ t(s_{i-1}) < t(s_i)$ in case $i > 0;\ t(s_i) < t(s_{i+1})$ in case $i < m\}$ and let N_0 be such that $n \geq N_0$ implies $|\phi_n(t(s_i)) - \hat{\phi}_0(t(s_i))| < \epsilon/8$ for all $s_i \in \hat{S}$. Now we will fix $n \geq N_0$ and proceed to showing that we can re-parametrize an arbitrary parametrization (y', r') of $\Gamma(\phi_n)$ to obtain a new parametrization (y, r) such that (2.75) is satisfied. Let (y', r') be an arbitrary parametrization of $\Gamma(\phi_n)$. For each i such that $s_i \in \hat{S}$, let $s'_i \triangleq \max\{s \geq 0 : r'(s) = t(s_i)\}$ so that $r'(s'_i) = t(s_i)$ and $\phi_n(r'(s'_i)) = y'(s'_i)$. For i's such that $s_i \in S \setminus \hat{S}$, note that there are three possible cases: $t(s_i) \in (0, 1)$, $t(s_i) = 0$, and $t(s_i) = 1$. Since the other cases can be handled in similar (but simpler) manners, we focus on the case $t(s_i) \in (0, 1)$. In this case, one can check that there exist k and j such that $k \leq i \leq k + j$, $t(s_{k-1}) < t(s_k) = t(s_{k+j}) < t(s_{k+j+1})$, and $s_{k-1}, s_{k+j+1} \in \hat{S}$. Here we assume that $k > 1$; the case $k = 1$ is essentially identical but simpler — hence omitted. Note that from the monotonicity of $\hat{\phi}_0$ and (2.74),

$$
\begin{aligned}
x(s_{k-2}) \leq \hat{\phi}_0(t(s_{k-2})+) \leq \hat{\phi}_0(t(s_{k-1})-) \leq \hat{\phi}_0(t(s_{k-1})) \leq \hat{\phi}_0(t(s_{k-1})+) \\
\leq \hat{\phi}_0(t(s_k)-) \leq x(s_k);
\end{aligned}
$$

i.e., $\hat{\phi}_0(t(s_{k-1})) \in [x(s_{k-2}), x(s_k)]$, which along with (S3) implies $|\hat{\phi}_0(t(s_{k-1})) - x(s_{k-1})| < \epsilon/8$. From this, (S5), and the constructions of s'_{k-1} and N_0,

$$
\begin{aligned}
|y'(s'_{k-1}) - x(s_{k-1})| &= |\phi_n(r'(s'_{k-1})) - x(s_{k-1})| \\
&= |\phi_n(r'(s'_{k-1})) - \hat{\phi}_0(t(s_{k-1}))| + |\hat{\phi}_0(t(s_{k-1})) - x(s_{k-1})| \\
&= |\phi_n(t(s_{k-1})) - \hat{\phi}_0(t(s_{k-1}))| + |\hat{\phi}_0(t(s_{k-1})) - x(s_{k-1})| \\
&< \epsilon/4.
\end{aligned}
$$

Following the same line of reasoning, we can show that $|y'(s'_{k+j+1}) - x(s_{k+j+1})| < \epsilon/4$. Noting that both x and y' are nondecreasing, there have to exist $s'_k, \ldots, s'_{k+j}$ such that $s'_{k-1} < s'_k < \cdots < s'_{k+j} < s'_{k+j+1}$ and $|y'(s'_l) - x(s_l)| < \epsilon/4$ for $l = k, k+1, \ldots, k+j$. Note also that from (S2),

$$
\begin{aligned}
t(s_l) - \epsilon/4 = t(s_k) - \epsilon/4 < t(s_{k-1}) = r'(s'_{k-1}) \leq r'(s'_l) \leq r'(s'_{k+j+1}) \\
= t(s_{k+j+1}) < t(s_{k+j}) + \epsilon/4 = t(s_l) + \epsilon/4;
\end{aligned}
$$

and hence, $|r'(s'_l) - t(s_l)| < \epsilon/4$ for $l = k, \ldots, k+j$ as well. Repeating this procedure for the i's for which s'_i is not designated until there is no such i's left, we can construct $s'_1, \ldots, s'_m$ in such a way that

$$
|y'(s'_i) - x(s_i)| < \epsilon/4 \qquad \text{and} \qquad |r'(s'_i) - t(s_i)| < \epsilon/4
$$

for all i's. Now, define a (piecewise linear) map $\lambda : [0,1] \to [0,1]$ by setting $\lambda(s_i) = s_i'$ at each s_i's and interpolating (s_i, s_i')'s in between. Then, $y \equiv y' \circ \lambda$ and $r \equiv r' \circ \lambda$ consist a parameterization (y, r) of $\Gamma(\psi_n)$ such that $|x(s_i) - y(s_i)| < \epsilon/4$ and $|t(s_i) - r(s_i)| < \epsilon/4$ for each $i = 1, \dots, m$. Due to the monotonicity of x, y, t, and r along with (S2) and (S3), we conclude that $\|y - x\|_\infty < \epsilon/2$ and $\|t - r\|_\infty < \epsilon/2$, proving (2.75). □

Proposition 2.5.9. *Let K be a subset of $D^{\uparrow}[0,1]$. If $M \equiv \sup_{\psi \in K} \|\psi\|_\infty < \infty$ then K is relatively compact w.r.t. the M_1' topology.*

Proof. Let $\{\psi_n\}_{n=1,2,\dots}$ be a sequence in K. We prove that there exists a subsequence $\{\psi_{n_k}\}_{k=1,2,\dots}$ and $\psi_0 \in D[0,1]$ such that $\psi_{n_k} \xrightarrow{M_1'} \psi_0$ as $k \to \infty$. Let $T \equiv \{t_n\}_{n=1,2,\dots}$ be a dense subset of $[0,1]$ such that $1 \in T$. By the assumption, $\sup_{n=1,2,\dots} |\psi_n(t_1)| < M$, and hence there is a subsequence $\{n_k^{(1)}\}_{k=1,2,\dots}$ of $\{1, 2, \dots\}$ such that $\psi_{n_k^{(1)}}(t_1)$ converges to a real number $x_1 \in [-M, M]$. For each $i \geq 1$, given $\{n_k^{(i)}\}$, one can find a further subsequence $\{n_k^{(i+1)}\}_{k=1,2,\dots}$ of $\{n_k^{(i)}\}_{k=1,2,\dots}$ in such a way that $\psi_{n_k^{(i+1)}}(t_{i+1})$ converges to a real number x_{i+1}. Let $n_k \equiv n_k^{(k)}$ for each $k = 1, 2, \dots$. Then, $\psi_{n_k}(t_i) \to x_i$ as $k \to \infty$ for each $i = 1, 2, \dots$. Define a function $\hat{\psi}_0 : T \to \mathbb{R}$ on T so that $\hat{\psi}_0(t_i) = x_i$. We claim that $\hat{\psi}_0$ has left limit everywhere; more precisely, we claim that for each $s \in (0,1]$, if a sequence $\{s_n\} \subset T \cap [0, s)$ is such that $s_n \to s$ as $n \to \infty$, then $\hat{\psi}_0(s_n)$ converges as $n \to \infty$. (With a similar argument, one can show that $\hat{\psi}_0$ has right limit everywhere — i.e., for each $s \in [0,1)$, if a sequence $\{s_n\} \subset T \cap (s, 1]$ is such that $s_n \to s$ as $n \to \infty$, then $\hat{\psi}_0(s_n)$ converges as $n \to \infty$.) To prove this claim, we proceed with proof by contradiction; suppose that the conclusion of the claim is not true — i.e., $\hat{\psi}_0(s_n)$ is not convergent. Then, there exist a $\delta > 0$ and a subsequence r_n of s_n such that

$$|\hat{\psi}_0(r_{n+1}) - \hat{\psi}_0(r_n)| > \delta. \tag{2.76}$$

Note that since $\hat{\psi}_0$ is a pointwise limit of nondecreasing functions $\{\psi_{n_k}\}$ (restricted on T),

- $\hat{\psi}_0$ is also nondecreasing on T, (monotonicity)
- $\sup_{t \in T} |\hat{\psi}_0(t)| < M$. (boundedness)

However, these two are contradictory to each other since the monotonicity together with (2.76) implies $\hat{\psi}_0(r_{N_0+j}) > \hat{\psi}_0(r_{N_0}) + j\delta$, which leads to the contradiction to the boundedness for a large enough j. This proves the claim.

Note that the above claim means that $\hat{\xi}_0$ has both left and right limit at each point of $T \cap (0,1)$, and due to the monotonicity, the function value has to be between the left limit and the right limit. Since T is dense in $[0,1]$, we can extend $\hat{\xi}$ from T to $[0,1]$ by setting $\hat{\xi}_0(t) \triangleq \lim_{t_i \to t,\, t_i > t} \hat{\xi}_0(t_i)$ for $t \in [0,1] \setminus T$. Note that such $\hat{\xi}_0$ is an element of $\mathbb{D}[0,1]$ and satisfies the conditions of Proposition 2.5.8. We therefore conclude that $\xi_{n_k} \to \xi_0 \in \mathbb{D}^{\uparrow}[0,1]$ in M_1' as $k \to \infty$, where $\xi_0(t) \triangleq \lim_{s \downarrow t} \hat{\xi}_0(s)$ for $t \in [0,1)$ and $\xi_0(1) \triangleq \hat{\xi}_0(1)$. This proves that K is indeed relatively compact. □

Recall that our rate function with respect to the M_1' topology is as follows:

$$I_{M_1'}(\xi) = \begin{cases} \sum_{t\in[0,1]} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \text{ is a non-decreasing pure jump function with } \xi(0) \ge 0; \\ \infty & \text{otherwise.} \end{cases}$$

Now, we show that $I_{M_1'}$ has compact sublevel sets.

Proof of Proposition 2.4.1. In view of Proposition 2.5.9, it is enough to show that the sublevel sets of $I_{M_1'}$ are closed. Let a be an arbitrary finite constant, and consider the sublevel set $\Psi_{I_{M_1'}}(a) \triangleq \{\xi \in \mathbb{D}[0,1] : I_{M_1'}(\xi) \le a\}$. Let $\xi^c \in \mathbb{D}[0,1]$ be any given path that does not belong to $\Psi_{I_{M_1'}}(a)$. We will show that there exists $\epsilon > 0$ such that $d_{M_1'}\big(\xi^c, \Psi_{I_{M_1'}}(a)\big) \ge \epsilon$. Note that $\Psi_{I_{M_1'}}(a)^{\mathsf{c}} = A \cup B \cup C \cup D$ where

$A = \{\xi \in \mathbb{D}[0,1] : \xi(0) < 0\};$

$B = \{\xi \in \mathbb{D}[0,1] : \xi$ is not a non-decreasing function$\};$

$C = \{\xi \in \mathbb{D}[0,1] : \xi$ is non-decreasing but not a pure jump function$\};$

$D = \{\xi \in \mathbb{D}^{\uparrow}[0,1] : \xi$ is a pure jump function with $I_{M_1'}(\xi) > a\}.$

In each case, we will show that ξ^c is bounded away from $\Psi_{I_{M_1'}}(a)$. In case $\xi^c \in A$, note that for any parametrization (x,t) of ξ^c, there has to be $s^* \in [0,1]$ such that $x(s^*) = \xi^c(0) < 0$. On the other hand, $y(s) \ge 0$ for all $s \in [0,1]$ for any parametrization (y,r) of ξ such that $\xi \in \Psi_I(a)$, and hence, $\|x - y\|_\infty \ge |\xi^c(0)|$. Therefore,

$$d_{M_1'}(\xi^c, \xi) \ge \inf_{\substack{(x,t)\in\Gamma(\xi^c)\\(y,r)\in\Gamma(\xi)}} \|x - y\|_\infty \ge |\xi^c(0)|.$$

Since was an arbitrary element of $_{I_{M_1^0}}(a)$, we conclude that $d_{M_1^0}(\ ^c;\ \ _I(a))$ with $= j\ ^c(0)j$.

Using a similar argument, it is straightforward to show that any $^c\ 2\ B$ is bounded away from $_I(a)^c$.

If $^c\ 2\ C$, there has to be T_s and T_t such that $0\ \ T_s < T_t\ \ 1$, ^{c} is continuous on $[T_s; T_t]$, and $c\ ,\ \ ^c(T_t)\ \ ^c(T_s) > 0$. Pick a small enough $2\ (0; 1)$ so that

$$(4\ )\ \ ^1(c\ \ 5\ ) > a: \tag{2.77}$$

Note that since ^{c} is uniformly continuous on $[T_s; T_t]$, there exists > 0 such that $j\ ^c(t)\ \ ^c(s)j <$ if $jt\ \ sj$. In particular, we pick so that $<$ and $T_s + \ < T_t$. We claim that

$$d_{M_1^0}(\ \ _{I_{M_1^0}}(a);\ ^c)\ \ :$$

Suppose not. That is, there exists $2\ \ _{I_{M_1^0}}(a)$ such that $d_{M_1^0}(\ ;\ ^c) <$. Let $(x; t)\ 2\ \ (\ ^c)$ and $(y; r)\ 2\ \ (\ )$ be the parametrizations of ^{c} and , respectively, such that $kx\ \ yk_1 + kt\ \ rk_1 <$. Since $I_{M_1^0}(\ )\ \ a < 1$, one can nd a nite set $K\ \ ft\ 2\ [0; 1] :\ (t)\ \ (t\ \ ) > 0g$ of jump times of in such a way that $\sum_{t2K}\ (t)\ \ (t\ \ ) <$. Note that since $2\ (0; 1)$, this implies that $\sum_{t2K}\ (t)\ \ (t\ \ ) <$. Let $T_1; : : : ; T_k$ denote (the totality of) the jump times of in $K\ \backslash\ (T_s + \ ; T_t\ \ ]$, and let $T_0\ ,\ T_s +$ and $T_{k+1}\ ,\ T_t$. That is, $fT_1; : : : ; T_kg = K\ \backslash\ (T_s + \ ; T_t\ \ ] = K\ \backslash\ (T_0; T_{k+1}]$. Note that

- there exist s_0 and s_{k+1} in $[0; 1]$ such that

$$y(s_0) = \ (T_0); \qquad r(s_0) = T_0; \qquad y(s_{k+1}) = \ (T_{k+1}); \qquad r(s_{k+1}) = T_{k+1};$$

- for each $i = 1; : : : ; k$, there exists s_i^+ and s_i such that

$$r(s_i^+) = r(s_i\ ) = T_i; \qquad y(s_i^+) = \ (T_i); \qquad y(s_i\ ) = \ (T_i\ \ ):$$

Since $t(s_{k+1})\ 2\ [r(s_{k+1})\ \ ; r(s_{k+1}) + \ ]\ \ [T_s; T_t]$, and ^{c} is continuous on $[T_s; T_t]$ and non-decreasing,

$$\begin{aligned} y(s_{k+1})\ \ x(s_{k+1})\ \ &= \ ^c(t(s_{k+1})) \\ ^c(r(s_{k+1})\ \ \ )\ \ &= \ ^c(T_{k+1}\ \ \ ) \\ ^c(T_{k+1})\ \ & \ \ ^c(T_{k+1})\ \ 2\ : \end{aligned}$$

Similarly,

$$
\begin{aligned}
y(s_0) \quad x(s_0) + \quad &= {}^c(t(s_0)) + \quad {}^c(r(s_0) + \ ) + \\
&= {}^c(T_0 + \ ) + \quad {}^c(T_0) + \quad + \quad {}^c(T_0) + 2\ .
\end{aligned}
$$

Subtracting the two equations,

$$
y(s_{k+1}) \quad y(s_0) \quad {}^c(T_{k+1}) \quad {}^c(T_0) \quad 4 \ = c \quad 4\ .
$$

Note that

$$
\sum_{i=1}^{k} (T_i) \quad (T_i \ ) = (T_{k+1}) \quad (T_0) \quad \sum_{t \in (T_0; T_{k+1}] \setminus K^c} (\ (t) \quad (t \ )) \tag{2.78}
$$

$$
\begin{aligned}
&(T_{k+1}) \quad (T_0) \\
&= y(s_{k+1}) \quad y(s_0) \qquad c \quad 5\ .
\end{aligned} \tag{2.79}
$$

On the other hand,

$$
\begin{aligned}
y(s_i^+) \quad y(s_i\ ) \quad (x(s_i^+) + \ ) \quad (x(s_i\ ) \quad \ ) &= x(s_i^+) \quad x(s_i\ ) + 2 \\
&{}^c(t(s_i^+)) \quad {}^c(t(s_i\ )) + 2 \\
&{}^c(r(s_i^+) + \ ) \quad {}^c(r(s_i\ ) \quad \ ) + 2 \\
&{}^c(T_i + \ ) \quad {}^c(T_i \quad \ ) + 2 \quad 2 \ + 2 \quad 4\ .
\end{aligned}
$$

That is, $(\ (T_i) \quad (T_i \ ))^{\ 1} = (y(s_i^+) \quad y(s_i\ ))^{\ 1} \quad (4\ )^{\ 1}$. Combining this with (2.78),

$$
\begin{aligned}
I_{M_1^0}(\ ) \quad \sum_{i=1}^{k} (T_i) \quad (T_i \ ) &= \sum_{i=1}^{k} (T_i) \quad (T_i \ ) \quad (T_i) \quad (T_i \ )^{\ 1} \\
&(c \quad 5\ )(4\ )^{\ 1} > a;
\end{aligned}
$$

which is contradictory to the assumption that $\ 2 \ _{I_{M_1^0}}(a)$. Therefore, the claim that ^{c} is bounded away from $\ _{I_{M_1^0}}(a)$ by $\ $ is proved.

Finally, suppose that ${}^c \ 2 \ \mathbb{D}[0;1]$. That is, there exists $f(z_i; u_i) \ 2 \ \mathbb{R}_+ [0;1] g_{i=1;:::}$ such that ${}^c = \sum_{i=1}^{1} z_i \mathbb{1}_{[u_i;1]}$ where u_i's are all distinct and $\sum_{i=1}^{1} z_i > a$. Pick k and $\ > 0$ such that $\sum_{i=1}^{k}(z_i \quad 2\ ) > a$ and $u_{i+1} \quad u_i > 2$ for $i = 1; :::; k \quad 1$. We claim that $d_{M_1^0}(\ ;\ {}^c)$ for any $\ 2 \ _{I_{M_1^0}}(a)$. Suppose not and there is $\ 2 \ _{I_{M_1^0}}(a)$ such that $kx \quad yk_1 + kt \quad rk_1 <$

for some parametrizations $(x;t) \in (\ ^c)$ and $(y;r) \in (\ )$. Note rst that there are s_i^+'s and s_i 's for each $i = 1;:::;k$ such that $t(s_i\ ) = t(s_i^+) = u_i$, $x(s_i\ ) = \ ^c(u_i\ \ )$, and $x(s_i^+) = \ ^c(u_i)$. Since $y(s_i^+)\ \ x(s_i^+)\ \ \ = \ ^c(u_i)$ and $y(s_i\ )\ \ x(s_i\ ) + \ \ = \ ^c(u_i\ \ ) + \ $,

$$(r(s_i^+))\quad (r(s_i\ ))\quad y(s_i^+)\quad y(s_i\ )\quad \ ^c(u_i)\quad \ ^c(u_i\ )\quad 2\ \ = z_i\quad 2\ :$$

Note that by construction,

$$r(s_i^+) < t(s_i^+) + \ \ = u_i + \ \ < u_{i+1}\ \ \ = t(s_{i+1})\ \ \ < r(s_{i+1})$$

for each $i = 1;:::;k\ \ 1$, and hence, along with the subadditivity of $x \mapsto\ x\ $,

$$I_{M_1^0}(\ ) = \sum_{t\in[0;1]} (\ (t)\quad (t\ \ ) \quad \sum_{i=1}^{k} [\ (r(s_i^+))\quad (r(s_i\ ))] \quad \sum_{i=1}^{k} (z_i\quad 2\ ) \quad > a;$$

which is contradictory to the assumption $\ \in\ _{I_{M_1^0}}(a)$. □

Chapter 3

Large deviations for Markov random walks

3.1 Introduction

In this chapter we develop sample-path large deviation principles (LDP) for additive functionals of a Markov chain which is important in Operations Research (OR), namely, Lindley's recursion. This Markov chain describes the waiting time sequence in a single-server queue under a FIFO discipline and under independent and identically distributed (i.i.d.) inter-arrival times and service times. We focus on the case in which the input is light-tailed, i.e. the service times and inter-arrival times have a nite moment generating function in a neighborhood of the origin.

While the model that we consider is vital to many OR applications, and therefore important in its own right, the main contributions are also fundamental from a methodological standpoint. We contribute, as we shall explain, to the development of key tools in the study of sample-path large deviations for additive functionals of light-tailed geometrically and ergodic Markov chains.

A rich body of theory, pioneered by Donsker and Varadhan in classical work which goes back over forty years (see, for example, [24]) provides powerful tools designed to study large deviations for additive functionals of light-tailed and geometrically ergodic Markov chains. Roughly speaking, these are chains which converge exponentially fast to stationarity and whose stationary distribution is light-tailed.

Unfortunately, despite remarkable developments in the area, including the more recent contributions in [54], the prevailing assumptions in the literature are often not applicable to natural functionals of well-behaved geometrically ergodic models, such as Lindley's recursion with light-tailed input.

In particular, every existing general result describing sample-path large deviations of functionals of a process such as Lindley's recursion, must assume the function of interest to be bounded. Hence, the current state-of-the-art rules out very important cases, such as the sample path behavior of the empirical average of the waiting time sequence in single-server queue over large time scales. Our development allows one to study sample-path large deviations for the cumulative waiting time sequence of a single-server queue. In particular, we provide methodological ideas which, we believe, will be useful in further development of the general theory of sample-path large deviations for additive functionals of geometrically ergodic Markov processes. More precisely, our contributions are summarized as follows,

A) Let $X_k = \max\{X_{k-1} + U_k\}$ where $U_i; i \geq 1;$ is a sequence of i.i.d. random variables. Assume that the associated increments (U_i) have a finite moment generating function in a neighborhood of the origin and the traffic intensity is less than one, and let $f(x) = x^p$ for any $p > 0$. We establish a sample-path large deviations principle for $Y_n(\cdot) = \sum_{k=1}^{\lfloor n\cdot \rfloor} f(X_k)/n$ as $n \to \infty$ under the M_1' topology on $\mathbb{D}[0,1]$ with a good rate function and a sublinear speed function which is fully characterized in Theorem 3.3.1.

B) We believe that the overall strategy for establishing Theorem 3.3.1 can be applied generally to the sample-path large deviations analysis of additive functionals of geometrically ergodic Markov chains. The strategy involves splitting the sample path in cycles, roughly corresponding to returns to a compact set (in the case of the Lindley recursion, the origin). Then, we show that the additive functional in a cycle has a Weibullian tail. Finally, we use ideas similar to those developed in Chapter 2, involving sample-path large deviations for random walks with Weibullian increments for the analysis. The result in Chapter 2, however, cannot be applied directly to our setting here because of two reasons. First, the cycle in progress at the end of the time interval is different from the rest. Second, the number of cycles (and thus the number of terms in the decomposition) is random.

The sublinear speed of convergence highlighted in A) underscores the main qualitative difference between our result and those traditionally obtained in the Donsker-Varadhan setting. In our setting, as hinted in B), the large deviations behavior of Y_n is characterized by heavy-tailed phenomena (in the form of

Weibullian tails) which arise when studying the tails of the additive functional over a given busy period. Our choice of $f(\cdot)$ (growing slowly if $p > 0$) underscores the frailty of the assumptions required to apply the Donsker-Varadhan type theory (i.e. just a small amount of growth derails the application of the standard theory).

The choice of topology is an important aspect of our result. In Chapter 2 it is argued that M_1' is a natural topology to consider for developing a full sample path large deviation principle for random walks with Weibullian increments. It is explained that such a result is impossible in the context of the J_1 topology in $D[0,1]$. To be precise, the topology that we consider is a slightly stronger variation of the one considered by [80] and [81], who introduced the M_1' topology in $D[0,\infty)$, but in such a way that its direct projection onto $D[0,1]$ loses important continuous functions (such as the maximum of the path in the interval). The key aspect in this variation is the evaluation of the metric at the right endpoint. The version that we consider merges the jumps, in the same way in which it is done at the left endpoint in the standard M_1' description. This variation results in a stronger topology when restricted to functions on compact intervals and it includes the maximum as a continuous function. An important reason for why to use the M_1' topology is that it allows to merge jumps. This seems to be particularly relevant given that in our setting the large deviations behavior will eventually merge the increments within the busy periods.

In addition to the two elements mentioned in B), which make the result in Chapter 2 not directly applicable, our choice of a strong topology also makes the approach in Chapter 2 difficult to use. In fact, in contrast to Chapter 2, we use a projective limit strategy to directly obtain our large deviations principle. A direct approach, using the result in Chapter 2, which we explored, consisted in replacing the random number of busy periods by its fluid limits (for which there is a large deviations companion with a linear speed rate). However, this replacement does not constitute an exponentially good approximation. This would have been a successful strategy if the M_1' topology considered by [80] was used.

The development of Theorem 3.3.1 highlights interesting and somewhat surprising qualitative insights. For example, consider the case $f(x) = x$, corresponding to the area drawn under the waiting time as a curve. As we show, deviations of order $O(1)$ upwards from the typical behavior of the process $Y_n(\cdot)$ occur due to extreme behavior in a single busy period of duration $O(n^{1/2})$. A somewhat surprising insight involves the busy period in progress at time n, which is split into two parts of size $O(n^{1/2})$ involving the age and forward life time of the cycle (the former contributes to the area calculations, while the latter does not). This asymmetry, relative to the other busy periods during the time horizon $[0, n]$, which are completely accounted for inside the area calculation, raises the question of whether a correction in the LDP is needed, due to this effect, at the end of the time horizon. The answer is, no, the contribution to the current busy period and the ones inside the time horizon are symmetric. This result is highlighted in Theorems 3.4.3 and 3.4.4, which characterize the variational problem governing extreme busy periods.

There are several related works that deal with large deviations for the area under the waiting time sequence in a busy period. But they focus on queue length as in [5], or assume that the moment generating function of the increment is finite everywhere, as in [25]. None of these works obtain sample path results. Instead, we do not assume that the moment generating function of the service times or inter-arrival times is finite everywhere. To handle this level of generality, we employ recently developed sampled-path LDPs [10, 11, 94]. This level of generality requires us to put in a substantial amount of work to rule out discontinuous solutions of the functional optimization problems that appear in the large deviations analysis.

Another hurdle in developing tail asymptotics for the additive functional in a busy period is that the functional describing the area under the busy period is not continuous. To deal with this, we exploit path properties of the most probable — in asymptotic sense—trajectories of the busy period along with the continuity of the area functional over a fixed time horizon. In particular, we rigorously show how to approximate the area over the busy period (which has a random endpoint) with the area over a large, *fixed* horizon. This is counter-intuitive at first, because the former approach allows one to remove the reflection operator. However, the latter approach does not have a first passage time (which is a discontinuous function) as horizon, and this turns out to carry more weight. The most likely path leading to a large area is concave, as the area functional is continuous at such paths.

This chapter is organized as follows. We give a detailed model description and preliminary results in Section 3.2. Section 3.3 focuses on the sample-path LDP of the Lindley process. In Section 3.4 we present tail asymptotics for busy periods of the Lindley process and Section 3.5 contains technical proofs.

3.2 Model description, and preliminary results

3.2.1 The model

We consider the time-homogeneous Markov chain $\{X_n\}_{n \geq 0}$ that is induced by the Lindley recursion, i.e. $X_{n+1} \triangleq [X_n + U_{n+1}]^+; n \geq 0;$ such that $X_0 = 0$. Note that the r.v.'s $\{U_i\}_{i \geq 1}$ are i.i.d. such that $\mathbf{E}(U_1) = \quad < 0$. The state space of the Markov chain X_n is the half-line of non-negative real numbers. We make the following technical but necessary assumptions:

Assumption 3.2.1. Let $_+;$ be respectively, the supremum and infimum of the set $\{ : \mathbf{E}(e^{U}) < \infty\}$. We assume that $\infty \quad < 0 < \ _+ \quad \infty$.

Assumption 3.2.2. For $_+$ and in Assumption 3.2.1,

$$\lim_{n \to \infty} \frac{\log \mathbf{P}(U \quad n)}{n} = \quad _+; \lim_{n \to \infty} \frac{\log \mathbf{P}(\ U \quad n)}{n} = \quad :$$

Assumption 3.2.3. $\mathbf{P}(U > 0) > 0$.

Define the stochastic process Y_n, where

$$Y_n(\ ) \triangleq \frac{1}{n} \sum_{i=1}^{\lfloor n \rfloor} f(X_i); \quad f(x) = x^p$$

and $p > 0$ is a fixed constant. We develop a sample-path LDP for $Y_n(\ )$ therefore, we describe the topological space in which we derive the large deviation principle. Recall that $\mathbb{D}[0;1]$ denotes the Skorokhod space — the space of cadlag functions from $[0;1]$ to $\mathbb{R}$. We also consider the space $\mathbb{D}[0;\infty)$ of cadlag functions from $[0;\infty)$ to $\mathbb{R}$. Let $T_{M_1^0}$ denote the M_1^0 Skorokhod topology. Unless specified otherwise, we assume that $\mathbb{D}[0;1]$ is equipped with $T_{M_1^0}$ throughout the rest of this chapter.

Definition 3.2.1. For $\ \in \mathbb{D}[0;1]$, define the extended completed graph $^0(\ )$ of as

$$^0(\ ) \triangleq \{(u;t) \in \mathbb{R} \times [0;1] : u \in [\ (t \ ) \wedge \ (t); \ (t \ ) \vee \ (t)]\}$$

where $(0 \ ) \triangleq 0$. Define an order on the graph $^0(\ )$ by setting $(u_1;t_1) < (u_2;t_2)$ if either $t_1 < t_2$; or $t_1 = t_2$ and $| \ (t_1 \ ) \quad u_1| < | \ (t_2 \ ) \quad u_2|$. We call a continuous nondecreasing function $(u;t) = (u(s);t(s)); s \in [0;1]$ from $[0;1]$ to $\mathbb{R} \times [0;1]$ a parametrization of $^0(\ )$ if $^0(\ ) = \{(u(s);t(s)) : s \in [0;1]\}$. We also call such $(u;t)$ a parametrization of .

Definition 3.2.2. Define the M_1' metric on $\mathbb{D}[0,1]$ as follows

$$d_{M_1'}(\xi,\zeta) \triangleq \inf_{\substack{(u,t)\in\Gamma'(\xi)\\(v,r)\in\Gamma'(\zeta)}} \{\|u-v\|_\infty + \|t-r\|_\infty\}.$$

We say that $\xi \in \mathbb{D}[0,1]$ is a pure jump function if $\xi = \sum_{i=1}^{\infty} x_i \mathbb{1}_{[u_i,1]}$ for some x_i's and u_i's such that $x_i \in \mathbb{R}$ and $u_i \in [0,1]$ for each i and u_i's are all distinct. Let $\mathbb{D}_p^{\uparrow}[0,1]$ be the subspace of $\mathbb{D}[0,1]$ consisting of non-decreasing pure jump functions that assume non-negative values at the origin. Let $\mathbb{BV}[0,1]$ be the subspace of $\mathbb{D}[0,1]$ consisting of cadlag paths with finite variation. Every $\xi \in \mathbb{BV}[0,1]$ has a Lebesgue decomposition with respect to the Lebesgue measure. That is, $\xi = \xi^{(a)} + \xi^{(s)}$ where $\xi^{(a)}$ denotes the absolutely continuous part of ξ, and $\xi^{(s)}$ denotes the singular part of ξ. Subsequently, using Hahn's decomposition theorem we can decompose $\xi^{(s)}$ into its non-decreasing singular part $\xi^{(u)}$ and non-increasing singular part $\xi^{(d)}$ so that $\xi^{(s)} = \xi^{(u)} + \xi^{(d)}$. Without loss of generality (w.l.o.g.), we assume that $\xi^{(s)}(0) = \xi^{(u)}(0) = \xi^{(d)}(0) = 0$. We also consider the space $\mathbb{BV}[0,\infty)$ of cadlag paths that are of bounded variation on any compact interval.

3.2.2 Preliminary results

LDP results

We review some LDP results that have appeared in the literature. A straightforward adaptation of Corollary 2.4.1 in Chapter 2 to our context is the following

Result 3.2.1. *Let K_n be a random walk such that $K_0 = 0$ and $\mathbf{P}(K_1 \geq x) = e^{-L(x)x^\alpha}$ for $\alpha \in (0,1)$; suppose that L is a slowly-varying function, and $L(x)x^{\alpha-1}$ is eventually decreasing. Then, K_n satisfies the LDP in $(\mathbb{D}[0,T], \mathcal{T}_{M_1'})$ with speed $L(n)n^\alpha$ and rate function $I_{M_1'} : \mathbb{D}[0,T] \to [0,\infty]$,*

$$I_{M_1'}(\xi) \triangleq \begin{cases} \sum_{t\in[0,1]} (\xi(t) - \xi(t-))^\alpha & \text{if } \xi \in \mathbb{D}^{(\mathbf{E}S_1)}[0,T] \text{ with } \xi(0) \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

The following result, by [72], provides the logarithmic asymptotics for the steady state distribution of the reflected random walk. To this end, define $\theta^* = \sup\{\theta \geq 0 : \mathbf{E}(e^{\theta U}) \leq 1\}$.

Result 3.2.2 ([72])**.** *For the steady state distribution $\pi(\cdot)$ of the reflected random walk, it holds that,*

$$\lim_{n\to\infty} \frac{\log \pi([n,\infty))}{n} = -\theta^*.$$

Finally, we mention a recent sample path LDP for random walks, developed in [94] with light-tailed increments. Now, let $\{U_i\}_{i\geq 1}$ be i.i.d. random variables and define $K_n = \frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor} U_i;\ t \in [0,1]$. In the following result we consider the M_1 topology ([94]) instead of the M_1' topology.

Result 3.2.3. *Let U_1 satisfy* ***Assumptions*** *3.2.1 and 3.2.2. Define*

$$I_K(\xi) \triangleq \begin{cases} \int_0^1 \Lambda^*(\xi'_{(a)}(s))ds + \theta_+(\xi^{(u)}(1)) + \theta_-|\xi^{(l)}(1)| & \text{if } \xi \in \mathrm{BV}[0,1] \text{ and } \xi(0) = 0; \\ \infty & \text{otherwise:} \end{cases} \quad (3.1)$$

(i) ([10, 11]) K_n satisfies a large deviations lower bound in the M_1 topology with rate function I_K.

(ii) ([94]) Let ϕ be a real-valued function on $\mathrm{D}[0,1]$ which is uniformly continuous in the M_1 topology on the level sets $\{\xi : I_K(\xi) \leq \alpha\}$. Then $\phi(K_n)$ satisfies an LDP with rate function $J(u) = \inf_{\xi:\phi(\xi)=u} I_K(\xi)$.

Results on the theory of Markov chains

Let X_n be a geometrically ergodic Markov chain on the state space S, which includes an element 0, and invariant distribution π, such that $\pi(\{0\}) = \pi(0) \neq 0$. Let $\overleftarrow{X}_n$ be the time-reversed stationary version of the Markov chain X_n. Recall that for a two-sided stationary version of the chain $(X_n : -\infty < n < \infty)$, we have that $(\overleftarrow{X}_n : -\infty < n < \infty)$ satisfies the equality in distribution $(X_n, \dots, X_{n+m}) = (\overleftarrow{X}_{n+m}, \dots, \overleftarrow{X}_n)$ for any $-\infty < n < \infty$ and $m \geq 0$. Since $\pi(0) \neq 0$, the following lemma follows directly applying this distributional identity. In fact, the identity can be seen to hold path-wise since we can define $\overleftarrow{X}_n = X_{-n}$, assuming that X_0 follows π.

Lemma 3.2.1. *Let $\overleftarrow{X}_n$ be the time reversed chain of X_n. It holds that*

$$\mathbf{P}_0(X_i \in A_i : 1 \leq i \leq n-1) = \frac{1}{\pi(0)}\mathbf{P}_\pi(\overleftarrow{X}_i \in A_{n-i} : 1 \leq i \leq n; \overleftarrow{X}_n = 0) \quad (3.2)$$

$$\mathbf{E}_0[f(0, X_1, \dots, X_n)] = \frac{1}{\pi(0)}\mathbf{E}_\pi\left[f(0, \overleftarrow{X}_{n-1}, \dots, \overleftarrow{X}_0) I(\overleftarrow{X}_n = 0)\right]. \quad (3.3)$$

Building upon the previous result, we can establish the following lemma whose proof is deferred to a later section:

Lemma 3.2.2. *Define*

$$T = \inf\{n \in \{1, \ldots\} : X_n = 0\}; \text{ and } T^- = \inf\{n \in \{1, \ldots\} : X_n = 0\};$$

and suppose that $\mathbf{P}(T > n) = O(e^{-cn})$ *for some* $c > 0$. *In addition, let* n_0 *be such that* $\inf_{k \ge n_0} \mathbf{P}_0(X_k = 0) \ge \pi(0)/2$. *Then,*

$$\mathbf{P}\left(\sum_{k=0}^{T} (X_k)^p \ge x; X_n = 0\right) \le (n+1)\mathbf{P}\left(\sum_{k=0}^{T} X_k^p \ge x\right). \tag{3.4}$$

Moreover,

$$\mathbf{P}\left(\sum_{k=0}^{T} (X_k)^p \ge x; X_n = 0\right) \ge (\pi(0)^2/2)\mathbf{P}_0\left(\sum_{k=0}^{T} X_k^p \ge x\right) - O(e^{-cn}). \tag{3.5}$$

3.3 LDP for functionals of Markov chains

3.3.1 The main result

We present the sample-path large deviation principle for Y_n and the main ideas of its proof. We start with a few definitions. Let R be the reflection map i.e; $R(\phi)(t) = \phi(t) - \inf_{0 \le s \le t}\{\phi(s) \wedge 0\}$; $\forall t \ge 0$. Define

- $T(\phi) = \inf\{t > 0 : R(\phi)(t) \le 0\}$,
- $B_y \triangleq \left\{\phi \in BV[0, \infty) : \phi(0) = y; \int_0^{T(\phi)} R(\phi)(s)^p ds \ge 1\right\}$, and
- $\Lambda^*(y) \triangleq \sup_{\theta \in \mathbb{R}}\{\theta y - \log \mathbf{E}(e^{\theta U})\}$.

Set

$$I_y(\phi) \triangleq \begin{cases} \int_0^{T(\phi)} \Lambda^*(\dot{\phi}^{(a)}(s))ds + \theta_+ \psi^{(u)}(T(\phi)) + \theta_- \phi^{(d)}(T(\phi)) & \text{if } \phi(0) = y \text{ and } \phi \in BV[0, \infty); \\ \infty & \text{otherwise} \end{cases}$$

and denote with B_y the optimal value of the variational problem:

$$B_y \triangleq \inf_{\phi \in B_y} I_y(\phi). \tag{B_y}$$

Let $T_0 \triangleq 0$ and $T_i \triangleq \inf\{k > T_{i-1} : X_k = 0\}$ for $i \geq 1$, and subsequently, define $\mu = \mathbf{E}(\sum_{i=1}^{T_1} X_i^p)/\mathbf{E}(T_1)$. Define

$$\mathbb{D}^{(\mu)}[0,1] = \{\psi \in \mathbb{D}[0,1] : \psi(t) = \mu t + \xi(t),\ \forall t \in [0,1];\ \xi \in \mathbb{D}_p^{\uparrow}[0,1]\}$$

i.e., the subspace of increasing functions with slope μ and countable upward jumps. Lastly, let $\alpha = 1/(1+p)$.

Theorem 3.3.1. *The stochastic process Y_n satisfies a large deviation principle in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$ with speed n^α and rate function $I_{\mathbf{Y}} : \mathbb{D} \to \mathbb{R}_+$,*

$$I_{\mathbf{Y}}(\psi) \triangleq \begin{cases} B_0 \sum_{t:\, \psi(t) \neq \psi(t-)} (\psi(t) - \psi(t-))^{\alpha} & \text{if } \psi \in \mathbb{D}^{(\mu)}[0,1]; \\ \infty & \text{otherwise.} \end{cases} \tag{3.6}$$

That is, for any measurable set A,

$$-\inf_{A^\circ} I_{\mathbf{Y}}(\psi) \leq \liminf_{n\to\infty} \frac{\log \mathbf{P}(Y_n \in A)}{n^\alpha} \leq \limsup_{n\to\infty} \frac{\log \mathbf{P}(Y_n \in A)}{n^\alpha} \leq -\inf_{A^-} I_{\mathbf{Y}}(\psi). \tag{3.7}$$

3.3.2 Methodology

The strategy relies on a suitable representation for Y_n using renewal theory: the Markov chain $\{X_n\}_{n\geq 0}$ is regenerative with respect to the sequence $\{T_j : j \geq 0\}$ i.e;

- $\{X_{T_n}, \ldots, X_{T_{n+1}-1}\}_{n\geq 1}$ are i.i.d.; and
- $\{X_{T_n}, \ldots, X_{T_{n+1}-1}\}_{n\geq 1}$ are independent of $X_0, \ldots, X_{T_n - 1}$.

The sequence $\{T_j ; j \geq 1\}$ induces a renewal process $N(t) = \max\{k \geq 0 : T_k \leq t\}$. We decompose the process Y_n as follows:

$$Y_n(t) \triangleq \frac{1}{n} \sum_{j=1}^{N(nt)} \sum_{i=T_{j-1}+1}^{T_j} f(X_i) + \frac{1}{n} \sum_{i=T_{N(n)}+1}^{\lfloor nt \rfloor} f(X_i), \tag{3.8}$$

with the convention that $\sum_{i=T_{N(n)}}^{\lfloor nt \rfloor} f(X_i)$ is zero in case the superscript $\lfloor nt \rfloor$ is strictly smaller than the subscript $T_{N(n)}$.

We introduce some notation for the analysis of Y_n. Define

- $\tau_j = T_j - T_{j-1}$, the inter-arrival times of the renewal process N,

- $W_j = \sum_{i=T_{j-1}+1}^{T_j} f(X_i)$, the area under $f(X_i)$ during a busy period of X_n,
- $Z_n(\cdot) = \frac{1}{n}\sum_{j=1}^{N(n\cdot)} W_j$, the process up to the last regeneration point,
- $R_n(t) = \frac{1}{n}\sum_{i=T_{N(n)}+1}^{\lfloor nt \rfloor} f(X_i)$, the area under $f(X_i)$ starting from the previous regeneration,
- $V_n = \frac{1}{n}\sum_{i=T_{N(n)}+1}^{n} f(X_i)$, the area starting from the last regeneration point,
- $S_n(t) = V_n \mathbb{1}_{\{1\}}(t)$, the stochastic process with one jump of size V_n at the end of the time horizon.

The main result (Theorem 3.3.1) is derived by proving that;

1) the tail behavior of W_1 and V_n is asymptotically Weibull-like;
3) Z_n and S_n satisfy an LDP in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$;
3) $Z_n + S_n$ satisfies an LDP in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$; and
4) $Z_n + S_n$ and Y_n are exponentially equivalent in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$.

Regarding step 1), the logarithmic asymptotics of V_n and W_1 are presented in Theorems 3.4.3 and 3.4.4 in the next section. For the sample-path LDP of Y_n, we prove the exponential equivalence of $Z_n + S_n$ and Y_n in Lemma 3.3.2 by pushing the last cycle R_n to the end of the time horizon. Consequently, the LDP of R_n is deduced because of the LDP of S_n in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$. We derive an LDP for Z_n in $\mathbb{D}[0,1]$ with respect to the M_1' topology by obtaining an LDP with the point-wise convergence topology which is strengthened to the M_1' topology using the continuity of the identity map in the subspace of increasing cadlag paths. The LDP for $Z_n + S_n$ is deduced through the use of a continuous mapping approach, and hence, we obtain the LDP for Y_n.

The next technical lemmas are the buildling blocks for the sample-path LDP of Y_n. Let $\mathbb{D}^{\leqslant 1}[0,1] \triangleq \{\xi \in \mathbb{D}[0,1] : \xi = x\mathbb{1}_{\{1\}} \text{ for some } x \geq 0\}$.

Lemma 3.3.2. *S_n satisfies the LDP in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$ with speed n^α and the rate function $I_S : \mathbb{D}[0,1] \to \mathbb{R}_+$ where*

$$I_S(\xi) \triangleq \begin{cases} B_0(\xi(1) - \xi(1-))^\alpha; & \text{if } \xi \in \mathbb{D}^{\leqslant 1}[0,1]; \\ \infty; & \text{otherwise:} \end{cases} \tag{3.9}$$

Lemma 3.3.3. *The stochastic process Z_n satisfies a large deviation principle in $\mathbb{D}[0,1]$ w.r.t. the M_1' topology with speed n and the good rate function $I_Z : \mathbb{D}[0,1] \to \mathbb{R}_+$ where*

$$I_Z(\zeta) = \begin{cases} B_0 \sum_{t:\, \zeta(t) \neq \zeta(t-)} \big(\zeta(t) - \zeta(t-)\big) & \text{if } \zeta \in \mathbb{D}^{(\uparrow)}[0,1]; \\ \infty & \text{otherwise.} \end{cases} \tag{3.10}$$

Lemma 3.3.4. *Y_n and $Z_n + S_n$ are exponentially equivalent in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$; that is, for any $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big(d_{M_1'}(Y_n, Z_n + S_n) \geq \delta\big) = -\infty.$$

Now we are ready to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. The preceding sequence of lemmas has resulted in LDPs of Z_n (Lemma 3.3.3) and S_n (Lemma 3.3.2). Since Z_n and S_n are independent, (Z_n, S_n) satisfies an LDP in $\prod_{i=1}^{2} \mathbb{D}[0,1]$ with the rate function $I_{Z,S}(\xi,\zeta) = I_Z(\xi) + I_S(\zeta)$; see, for example, Theorem 4.14 of [39].

Let $\varphi : \prod_{i=1}^{2} \mathbb{D}[0,1] \to \mathbb{D}[0,1]$ denote the addition function $\varphi(\xi,\zeta) = \xi + \zeta$. Since φ is continuous on (ξ,ζ) as far as ξ and ζ do not share a jump time with opposite directions, φ is continuous on the effective domain of $I_{Z,S}$. Let $I_{\mathbf{W}}(\xi) \triangleq \inf\{I_{Z,S}(\xi_1,\xi_2) : \xi = \xi_1 + \xi_2,\ \xi_1 \in \mathbb{D}^{(\uparrow)}[0,1],\ \xi_2 \in \mathbb{D}^{\leqslant 1}[0,1]\}$, and note that it is straightforward to check that $I_W = I_{\mathbf{Y}}$. By the extended contraction principle — see p. 367 of [80] — we conclude that $Z_n + S_n$ satisfies the sample path LDP with the rate function $I_{\mathbf{Y}}$.

We now prove the large deviation upper bound. Let F be a closed set w.r.t. the M_1' topology, and let $F_\delta \triangleq \{\xi \in \mathbb{D}[0,1] : d_{M_1'}(\xi, F) \leq \delta\}$. Then,

$$\begin{aligned}
&\mathbf{P}\big(Y_n \in F\big) \\
&\quad\leq \mathbf{P}\big(Y_n \in F,\ d_{M_1'}(Y_n, Z_n + S_n) \leq \delta\big) + \mathbf{P}\big(Y_n \in F,\ d_{M_1'}(Y_n, Z_n + S_n) > \delta\big) \\
&\quad\leq \mathbf{P}\big(Y_n \in F,\ d_{M_1'}(Y_n, Z_n + S_n) \leq \delta\big) + \mathbf{P}\big(d_{M_1'}(Y_n, Z_n + S_n) > \delta\big) \\
&\quad\leq \mathbf{P}\big(Z_n + S_n \in F_\delta,\ d_{M_1'}(Y_n, Z_n + S_n) \leq \delta\big) + \mathbf{P}\big(d_{M_1'}(Y_n, Z_n + S_n) > \delta\big) \\
&\quad\leq \mathbf{P}\big(Z_n + S_n \in F_\delta\big) + \mathbf{P}\big(d_{M_1'}(Y_n, Z_n + S_n) > \delta\big).
\end{aligned}$$

In view of the principle of the largest term and Lemma 3.3.4,

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\log\mathbf{P}\;\; Y_n\in F\\
&\quad \limsup_{n\to\infty}\frac{1}{n}\log\;\;\;\mathbf{P}\;\; Z_n+S_n\in F\;\;\;+\mathbf{P}\;\; d_{M_1^0}(Y_n;Z_n+S_n)>\\
&\quad \limsup_{n\to\infty}\frac{1}{n}\log\mathbf{P}\;\; Z_n+S_n\in F\;\;\;\vee\;\limsup_{n\to\infty}\frac{1}{n}\log\mathbf{P}\;\; d_{M_1^0}(Y_n;Z_n+S_n)>\\
&= \limsup_{n\to\infty}\frac{1}{n}\log\mathbf{P}\;\; Z_n+S_n\in F\\
&\quad\;\; \inf_{\in F} I_{\mathbf{Y}}(\;):
\end{aligned}
$$

Since I_W is good w.r.t. $T_{M_1^0}$, $\lim_{\to 0}\inf_{\in F} I_{\mathbf{Y}}(\;)=\inf_{\in F} I_{\mathbf{Y}}(\;)$. The desired large deviation upper bound follows by taking $\;\to 0$.

For the lower bound, let G be an open set in $T_{M_1^0}$. We assume that $\inf_{\in G} I_{\mathbf{Y}}(\;)<\infty$ since the lower bound is trivial otherwise. For any given $\;>0$, pick $\;\in G$ such that $I(\;)\;\;\inf_{\in G} I_{\mathbf{Y}}(\;)+\;$. Let $\;>0$ be such that $B_{M_1^0}(\;;2\;)\in G$. Then, from Lemma 3.3.4,

$$
\frac{\mathbf{P}\;\; d(Y_n;Z_n+S_n)<}{\mathbf{P}\;\; Z_n+S_n\in B_{M_1^0}(\;;\;)}\;\to 0;
$$

and hence,

$$
\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n}\log\mathbf{P}\;\; Y_n\in G\\
&\quad \liminf_{n\to\infty}\frac{1}{n}\log\mathbf{P}\;\; Z_n+S_n\in B_{M_1^0}(\;;\;);\; d(Y_n;Z_n+S_n)<\\
&\quad \liminf_{n\to\infty}\frac{1}{n}\log\mathbf{P}\;\; Z_n+S_n\in B_{M_1^0}(\;;\;)\quad 1\;\;\left(\frac{\mathbf{P}\;\; d(Y_n;Z_n+S_n)<}{\mathbf{P}\;\; Z_n+S_n\in B_{M_1^0}(\;;\;)}\right)\\
&= \liminf_{n\to\infty}\frac{1}{n}\log\mathbf{P}\;\; Z_n+S_n\in B_{M_1^0}(\;;\;)\\
&\quad\;\; \inf_{\in B_{M_1^0}(\;;\;)} I_{\mathbf{Y}}(\;)\quad I_{\mathbf{Y}}(\;)\quad \inf_{\in G} I_{\mathbf{Y}}(\;)\quad :
\end{aligned}
$$

Taking $\;\to 0$, we arrive at the desired lower bound.

□

Discussion of the main result

It is worth commenting on the role of the R_n, since this element allows us to expose the importance of a careful analysis involving the area during a busy period. As mentioned in the introduction, one may wonder if the contribution of $R_n(t)$ may end up counting different in the form of the LDP. The typical path for $Y_n(\cdot)$ is a straight line with drift equal to the steady-state waiting time. Our result indicates that most likely large deviations behavior away from the most likely path occur due to isolated busy periods which exhibit extreme behavior. For example, in the case $f(x) = x$, substantially extreme busy periods (leading to large deviations of order $O(n)$) have a duration of order $O\left(n^{1/2}\right)$ and exhibit excursions of order $O\left(n^{1/2}\right)$ therefore, accumulating an area of order $O(n)$.

The results in the next section characterize the variational problem which governs such extreme busy periods. But the fact that each busy period, including the one in progress at the end of the time horizon, contributes the same way in the rate function may be somewhat remarkable. The reason is that when the cycle in progress at the end of the time horizon is extreme, as indicated in the introduction, its duration is of order $O\left(n^{1/2}\right)$. This suggests that the remainder of the cycle is also of order $O\left(n^{1/2}\right)$, and hence, one may wonder if this long time duration may have a significant contribution to the total area. It turns out that this does not happen and the reason is the following. While the remaining part of the cycle in progress may be large, the position of the chain is actually $o\left(n^{1/2}\right)$ from the end of the time horizon, so the total contribution to the area of the remaining portion of the cycle is negligible.

3.4 Busy period asymptotics

As it has been discussed, a large deviations analysis of the area under a busy period is an indispensable component for deriving the sample path LDP of Y_n in Theorem 3.3.1. The next two theorems focus on the tails of W_1 and V_n, showing that they exhibit Weibull behavior. Recall B_y and denote with B the optimal value of the following variational problem B:

$$B \triangleq \inf_{y \in [0,\infty);\ \in B_y} \{ y + I_y(\ ) \}; \qquad (B\ )$$

where $\triangleq \sup\{ \geq 0 : \mathbf{E}(e^{\ U}) \leq 1\}$. Note that $_+$ and is strictly positive in view of Assumption 3.2.1 and the assumption that < 0. Note also that

$$B = \inf_{y \in [0,\infty)} \{ y + B_y \}:$$

3.4.1 Methodology

The tail asymptotics for W_1 and V_n are derived using a recently developed LDP for random walks with light-tailed increments due to [10, 11, 94], cf. Result 3.2.3 below. Specifically, W_1 is the image of the unrestricted random walk $K_n = \frac{1}{n}\sum_{i=1}^{n} U_i$ to which the functional $\phi(\cdot) \triangleq \int_0^{T(\cdot)} (R(\cdot)(s))^p ds$ is applied. Note that $\phi : D[0,\infty) \to \mathbb{R}_+$ is not continuous, and hence, the proof for the tail asymptotics of W_1 gets more involved than simply applying the contraction principle. We derive large deviations upper and lower bounds and show that they coincide.

For the upper bound, we replace the hitting time T_1 with a sufficiently large value T. This enables us to study the area of X_n over the finite-time horizon $[0;T]$. For T large enough, we show that the area of the reflected random walk over the whole time horizon $[0;T]$ serves as an asymptotic upper bound for W_1, and it is expressed as a functional of K_n. This functional is shown to be uniformly continuous in the (standard) M_1 topology on level sets of the rate function associated with the LDP for K_n. Invoking Result 3.2.3, recently established in [94], we get a large deviation upper bound.

For the lower bound, we confine the functional of the area under the busy period, over a fixed time horizon by imposing an extra condition. Subsequently, we derive a variational problem associated with the lower bound. Lastly, we show that B_0 has the same value as the variational problem associated with the large deviation upper and lower bound.

For V_n we follow the same approach with some slight modifications. In order to carry out our analysis for V_n, we associate the tail of W_1 with the tail of V_n through Lemmas 3.2.1, and 3.2.2. We prove that V_n has similar tail asymptotics to that of W_1, initialized from the steady state of X_n i.e;

$$\lim_{n\to\infty} \frac{\log \mathbf{P}_0(V_n > x)}{n^{1/(1+p)}} = \lim_{n\to\infty} \frac{\log \mathbf{P}_\pi(W_1 > nx)}{n^{(1/1+p)}}.$$

For this reason, it is necessary to invoke tail asymptotics for the steady state distribution π of X_n. To this end, we use a result in [72] (see Result 3.2.2) regarding the asymptotic behavior of the invariant measure of homogeneous Markov chains. Lastly, we repeat the same steps as in the analysis of W_1. Namely, we derive large deviation upper and lower bounds and we show that they coincide.

3.4.2 Tail asymtotics

The main results (Theorem 3.4.3, and 3.4.4) are a consequence of the next two technical propositions.

Proposition 3.4.1. *(i) Recall that $T_1 = \inf\{k > 0 : X_k = 0\}$; then,*

$$\limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^{T_1/x} (X(\lfloor ux\rfloor)/x)^p du \ge 1\right) \le -B_y.$$

(ii) Recall that $W_1 = \sum_{i=1}^{T_1} X_i^p$; then,

$$\liminf_{u\to\infty} \frac{1}{u^{1/(1+p)}} \log \mathbf{P}_0 (W_1 > u) \ge -B_0. \tag{3.11}$$

Proposition 3.4.2. *(i) $\sum_{k=0}^{m-1} X_k^p > x^{1+p} \iff \int_0^{m/x} \left(\frac{X_{\lfloor xs\rfloor}}{x}\right)^p ds > 1$*

(ii) Let $y^* = (|\mu|(p+1))^{1/1+p}$. For any $y \ge y^*$,

$$B_y = 0.$$

(iii) It holds that

$$B_0 = B^*.$$

(iv) Finally,

$$\lim_{k\to\infty} \min_{i\ge 1} \left\{ \frac{i-1}{k} y^* + B_{\frac{i}{k}y^*} \right\} = \inf_{y\in[0,\infty)} \left\{ y + B_y \right\} = B^*.$$

Theorem 3.4.3. *Let $W_1 \triangleq \sum_{k=1}^{T_1} X_k^p$. Then*

$$\lim_{t\to\infty} \frac{1}{t^{1/1+p}} \log \mathbf{P}(W_1 \ge t) = -B_0. \tag{3.12}$$

Proof. For the upper bound, setting $t = x^{p+1}$,

$$\limsup_{t\to\infty} \frac{1}{t^{1/(1+p)}} \log \mathbf{P}(W_1 \ge t) = \limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_0 \left(\sum_{k=0}^{T_1-1} X_k^p \ge x^{1+p} \right)$$

$$= \limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_0 \left(\int_0^{T_1/x} \left(\frac{X(\lfloor ux\rfloor)}{x} \right)^p du \ge 1 \right) \le -B_0$$

where we applied part *(i)* of Proposition 3.4.2 for the second equality and part *(i)* of Proposition 3.4.1 for the inequality. This together with the matching lower bound in part *(ii)* of Proposition 3.4.1, we arrive at the desired asymptotics (3.12). □

For V_n, we notice again a Weibull-like asymptotic behavior similar to W_1 except that the prefactor associated with V_n is B_π (instead of B_0). It turns out that the prefactor B_π is equal to B_0. This leads to the conclusion that every busy period, including the one in progress at the end of the time horizon, has the same tail asymptotics.

Theorem 3.4.4. *The area of the busy period starting from the steady state (π) satisfies*

$$\lim_{n\to\infty} \frac{1}{n^{1/(p+1)}} \log \mathbf{P}_\pi(V_n \geq b) = -B_0\, b^{1/(1+p)}. \tag{3.13}$$

Proof. We start with proving the large deviation upper bound for V_n. Denote the time-reversed Markov process of $\{X_n\}_{n\geq 0}$ with $\{\tilde{X}_n\}_{n\geq 0}$, and let $\tilde{T}_1 = \inf\{i > 0 : \tilde{X}_i \leq 0\}$. Let $y^* \triangleq (\gamma\, j(p+1))^{1/(1+p)}$ and $x \triangleq b > 0$. Setting $x^{p+1} = nb$,

$$\begin{aligned}
\mathbf{P}_0(V_n \geq b) &= \mathbf{P}_0\left(\frac{1}{n}\sum_{i=T_{N(n)}}^{n} X_i^p \geq b\right) = \frac{1}{\pi(0)}\mathbf{P}\left(\frac{1}{n}\sum_{i=0}^{\tilde{T}_1} (\tilde{X}_i)^p \geq b;\ \tilde{X}_n = 0\right)\\
&\leq \frac{n+1}{\pi(0)}\mathbf{P}\left(\frac{1}{nb}\sum_{i=0}^{T_1-1} X_i^p \geq 1\right)\\
&= \frac{n}{\pi(0)}\mathbf{P}\left(\int_0^{T_1/x}\left(\frac{X(\lfloor ux\rfloor)}{x}\right)^p du \geq 1\right),
\end{aligned} \tag{3.14}$$

where the second equality follows from Lemma 3.2.1 applying the function $g(y_0,\ldots,y_n) = I(\sum_{\max\{i\leq n: y_i=0\}}^{n} y_i^p > nb)$, the inequality follows from the upper bound in Lemma 3.2.2, and the last equality follows from part *(i)* of Proposition 3.4.2. From the tower property, we have that

$$\begin{aligned}
&\mathbf{P}\left(\int_0^{T_1/x} (X(\lfloor ux\rfloor)/x)^p du \geq 1\right) \\
&= \mathbf{E}\left[\mathbb{1}\{X(0) \leq xy^*\}\mathbf{P}\left(\int_0^{T_1/x} (X(\lfloor ux\rfloor)/x)^p du \geq 1 \,\Big|\, X(0)\right)\right.
\end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&+\mathbf{E}\left[\sum_{i=1}^{k} \mathbb{1}\left\{X(0) \in \left(\tfrac{i-1}{k}xy, \tfrac{i}{k}xy\right]\right\} \mathbf{P}\left(\int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \geq 1 \,\middle|\, X(0)\right)\right] \\
&\leq \mathbf{E}\,\mathbb{1}\{X(0) \geq xy\} \\
&\quad + \sum_{i=1}^{k} \mathbf{E}\left[\mathbb{1}\left\{X(0) \in \left(\frac{i-1}{k}xy, \infty\right)\right\} \mathbf{P}_{\frac{i}{k}xy}\left(\int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \geq 1\right)\right] \\
&\leq \pi[xy, \infty) + \sum_{i=1}^{k} \pi\left(\frac{i-1}{k}xy, \infty\right) \underbrace{\mathbf{P}_{\frac{i}{k}xy}\left(\int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \geq 1\right)}_{\triangleq\, \mathbf{b}(i;k;x;y)}
\end{aligned}
$$

where in the first inequality we used that the Markov chain X_n is stochastically monotone. Therefore, by the principle of the maximum term and part *(i)* of Proposition 3.4.1,

$$
\begin{aligned}
&\limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}\left(\int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \geq 1\right) \\
&\quad \leq \left\{\limsup_{x\to\infty} \frac{1}{x} \log \pi[xy, \infty)\right\} \vee \max_{i=1,\dots,k} \left\{\limsup_{x\to\infty} \frac{1}{x} \log \pi\left(\frac{i-1}{k}xy, \infty\right) \mathbf{b}(i;k;x;y)\right\} \\
&= (-\gamma y) \vee \max_{i=1,\dots,k} \left\{-\gamma\frac{i-1}{k} y + \limsup_{x\to\infty} \frac{1}{x} \log \mathbf{b}(i;k;x;y)\right\}. \\
&\leq (-\gamma y) \vee \max_{i=1,\dots,k} \left\{-\gamma\frac{i-1}{k} y - B_{\frac{i}{k}y}\right\}.
\end{aligned}
$$

Note that since $B_y = 0$ for $y \geq y$ due to part *(ii)* of Proposition 3.4.2,

$$
\begin{aligned}
(-\gamma y) \vee \max_{i=1,\dots,k} \left\{-\gamma\frac{i-1}{k} y - B_{\frac{i}{k}y}\right\} &= \max_{i \geq 1} \left\{-\gamma\frac{i-1}{k} y - B_{\frac{i}{k}y}\right\} \\
&= -\min_{i \geq 1} \left\{\gamma\frac{i-1}{k} y + B_{\frac{i}{k}y}\right\}.
\end{aligned}
$$

Taking $k \to \infty$ and applying part *(iii)* and *(iv)* of Proposition 3.4.2,

$$
\limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}\left(\int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \geq 1\right) \leq -B = -B_0.
$$

From this along with (3.14), we arrive at the desired upper bound:

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{1}{n^{1/(1+p)}} \log \mathbf{P}_0(V_n \geq b) \\
&\leq \limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}\left(\int_0^{T_1/x} (X(\lfloor ux \rfloor)/x)^p du \geq 1 \right) b^{1/(1+p)} \\
&\leq -B_0 \, b^{1/(1+p)}.
\end{aligned}$$

Next, for n sufficiently large, using the lower bound of Lemma 3.2.2 for $n \geq n_0$:

$$\mathbf{P}_0(V_n \geq b) = \mathbf{P}_0\left(\frac{1}{n} \sum_{i=T_{N(n)}}^{n} X_i^p \geq b \right) = \frac{1}{\pi(0)} \mathbf{P}\left(\frac{1}{n} \sum_{i=0}^{T_1} (X_i)^p \geq b; X_n = 0 \right)$$

$$\geq \frac{\pi(0)}{2} \mathbf{P}_0\left(\frac{1}{nb} \sum_{i=0}^{T_1} X_i^p > 1 \right) = \frac{\pi(0)}{2} \mathbf{P}_0(W_1 > nb). \tag{3.16}$$

From here, we can directly apply part *(ii)* of Proposition 3.4.1 to (3.16) and obtain the matching lower bound:

$$\liminf_{n\to\infty} \frac{1}{n^{1/(1+p)}} \log \mathbf{P}_0(V_n > b) \geq -B_0 \, b^{1/(1+p)}.$$

□

Discussion on the computation of B_0

We conclude with a discussion on how to compute B_0. Note that it is not straightforward that the infimum in the representation (B_y) of B_0 is attained since the associated objective function does not have compact level sets unless the moment generating function of U_1 is finite everywhere, (cf. [57]). The following proposition, of which proof is deferred to a technical section, facilitates the characterization of the optimal solution of B_0:

Proposition 3.4.5. *Let*

- $B_y^{AC} \triangleq B_y \cap AC[0,\infty)$,
- $B_y^{CNCV} \triangleq B_y^{AC} \cap \{\xi \in AC[0,\infty) : \xi$ *is concave*$\}$,

and recall that $B_y = \inf_{\xi \in B_y} I_y(\xi)$. *Then,*

$$B_y = \inf_{\xi \in B_y^{AC}} I_y(\xi) = \inf_{\xi \in B_y^{CNCV}} I_y(\xi).$$

Now, the feasible region can be reduced by writing

$$B_0 = \inf_{2 B_0^{\mathrm{CNCV}}} I_0(\ ) = \inf_{z} \inf_{T \ \ 0} \inf_{2 F_{z;T}} I_0(\ ) \tag{3.17}$$

where $F_{z;T} = f \ : \ 2\ B_0^{\mathrm{CNCV}}; \ (0) = z; \ (T) = 0g$. Every element in the set $F_{z;T}$ can be written as $(t) = \int_0^t \dot{}(s)ds$ with $\dot{}(s)\ 2\ [\ ;z]$. Using this, it can be shown that $F_{z;T}$ is compact. Since $I_0(\ )$ is lower semi-continuous, the inner in mum in (3.17) is attained by some function . To characterize , it is convenient to remove the re ection operator. Given that we require $(T) = 0$, the concavity requirement implies that we can restrict our search to functions for which $\dot{}(s) =$ for $s > T$. Thus, the inner in mum of (3.17) is equivalent to minimizing $\int_0^T (\dot{}(s))ds$ subject to the constraints $(t) = \int_0^t \dot{}(s)ds$ with $\dot{}$ decreasing, $(0) = 0; \dot{}(0) = z$, $(T) = 0$ and $\int_0^T R(\ )(s)^p ds \quad 1$. In turn, this is equivalent to requiring that $\dot{}$ is decreasing, $(0) = 0; \dot{}(0) = z$, $(T) = 0$ and $\int_0^T (s)^p ds \quad 1$. Applying standard variational methods (see, for example, [56]), there exist constants c and $\ell \quad 0$ such that satis es the di erential equation $r \quad (\dot{}(s)) = c \quad \ell p \int_0^s (t)^{p \ 1} dt$: Since $\dot{}(0) = z$, $c = r \quad (z)$. Since $r \quad (z) = (r \quad)^{\ 1}(z)$, we can write

$$\dot{}(s) = r \qquad r \quad (z) \quad \ell p \int_0^s (t)^{p \ 1} dt \quad : \tag{3.18}$$

To summarize this discussion, we conclude that we can compute B_0 by minimizing $\int_0^T (\dot{}(s))ds$ with $\dot{}(s)$ satisfying (3.18), over $z \quad$, $T \quad 0$, $\ell \quad 0$.

3.5 Proofs

3.5.1 Proof of Proposition 3.4.5

The next two lemmas facilitate the proof of Proposition 3.4.5. Recall that R is the one-dimensional re ection map.

Lemma 3.5.1. *Suppose that* $\ ;\ ;\ \ 2\ \mathrm{D}[0;T]$, $(s) = \ (s) + \ (s)$, *and* (s) *is non-negative and non-decreasing. Then,* $R(\ )(t) \quad R(\ )(t)$ *for all* $t\ 2\ [0;T]$.

Proof. Recall rst that if $z \quad 0$ then $x \wedge (y + z) \quad (x \wedge y) + z$ for any $x; y\ 2\ \mathrm{R}$. From the non-negativity and monotonicity assumptions on , we have that

$$0 \wedge \ (s) \quad 0 \wedge \ (s) + \ (s) \quad 0 \wedge \ (s) + \ (t); \quad 0 \quad s \quad t$$

and hence,

$$0 \wedge \inf_{0 \le s \le t} \ (s) \le 0 \wedge \ (s) \le 0 \wedge \ (s) + \ (t); \quad 0 \le s \le t.$$

Taking infimum over $s \in [0,t]$, we get $0 \wedge \inf_{s \in [0,t]} \ (s) \le 0 \wedge \inf_{s \in [0,t]} \ (s) + \ (t).$ Therefore,

$$R(\ )(t) = \ (t) - 0 \wedge \inf_{s \in [0,t]} \ (s) \ge \ (t) - 0 \wedge \inf_{s \in [0,t]} \ (s) - \ (t)$$
$$= \ (t) - 0 \wedge \inf_{s \in [0,t]} \ (s) = R(\ )(t).$$

□

Fix $T > 0$ and consider the functional $\ _T : D[0,T] \to \mathbb{R}_+$, where $\ _T(\ ) = \int_0^T (R(\ )(s))^p ds$. Now, let V_y^T denote the optimal value of the following optimization problem V_y^T:

$$V_y^T \triangleq \inf_{\in V_y^T} I_y^{BV[0,T]}(\ ); \qquad (V_y^T)$$

where

$$V_y^T \triangleq \{ \in D[0,T] : \ (0) = y; \ \ _T(\ ) \ge 1\};$$

and

$$I_y^{BV[0,T]}(\ ) \triangleq \begin{cases} \int_0^T \ (\dot{}(s))ds + \ _+ \ ^{(u)}(T) + \ \ ^{(d)}(T) & \text{if } \ (0) = y \text{ and } \in BV[0,T]; \\ \infty & \text{otherwise.} \end{cases}$$

Lemma 3.5.2. *Suppose that* $\in BV[0,T]$ *and set* $y \triangleq \ (0)$*. Then*

(i) there exists a path $\ _1 \in BV[0,T]$ *such that*

- *i-1)* $\ _1(0) = y;$
- *i-2)* $\ _T(\ _1) \ge \ _T(\ );$
- *i-3)* $I_y^{BV[0,T]}(\ _1) \le I_y^{BV[0,T]}(\ );$
- *i-4) For some* $t \in [0,T]$*,* $\ _1$ *is nonnegative over* $[0,t]$ *and* $\ _1$ *is linear with slope over* $[t,T]$.

(ii) there exists a path $\ _2 \in AC[0,T]$ *such that*

- *ii-1)* $\ _2(0) = y + z$ *for some* $z \in [0, \ ^{(u)}(T)];$

ii-2) $_T(\ _2)\quad\ _T(\ )$;

ii-3) $_+\ z + I^{\mathrm{BV}[0;T]}_{y+z}(\ _2)\quad I^{\mathrm{BV}[0;T]}_{y}(\ )$;

ii-4) *For some* $t\ 2\ [0;T]$, $_2$ *is nonnegative over* $[0;t]$ *and* $_2$ *is linear with slope* *over* $[t;T]$.

Suppose further that $\ 2\ \mathrm{AC}[0;T]$. *Then*

(iii) *there exists a path* $_3\ 2\ \mathrm{AC}[0;T]$ *such that*

iii-1) $_3(0) = y$;

iii-2) $_T(\ _3)\quad\ _T(\ )$;

iii-3) $I^{\mathrm{BV}[0;T]}_{y}(\ _3)\quad I^{\mathrm{BV}[0;T]}_{y}(\ )$;

iii-4) $_3$ *is concave over* $[0;T]$ *and its derivative is bounded by* *from below.*

Proof. For part *(i)*, we rst construct a new trajectory $_1$ from by discarding the downward jumps, i.e., $_1 = \ ^{(a)} + \ ^{(u)}$. Obviously, $I^{\mathrm{BV}[0;T]}_{y}(\ _1)\quad I^{\mathrm{BV}[0;T]}_{y}(\ )$. Note that $_1 = \ + (\ ^{(d)})$ where $^{(d)}$ is non-negative and non-decreasing. From Lemma 3.5.1 we have that $R(\ _1)(t)\quad R(\ )(t)$ for all $t\ 2\ [0;T]$, and hence, $_T(\ _1)\quad\ _T(\ )$. For each $t\ 2\ [0;T]$, let

- $l(t)\ ,\ \inf fs\ 2\ [0;T] : R(\ )(u) > 0$ for all $u\ 2\ [s;t]g$
- $r(t)\ ,\ \sup fs\ 2\ [0;T] : R(\ )(u) > 0$ for all $u\ 2\ [t;s]g$,

and $(t)\ ,\ [l(t);r(t))$. Set $C_1^+\ ,\ f\ (t)\quad [0;T] : t\ 2\ [0;T]g$. Note that, by construction, the elements of C_1^+ cannot overlap, and hence, there can be at most countable number of elements in C_1^+. In view of this, we write $C_1^+ = f[l_i;r_i) : i\ 2\ \mathrm{N}g$ and let $_i\ ,\ [l_i;r_i)$. The following observations are immediate from the construction of C_1^+, the right continuity of , and the fact that $_1$ does not have any downward jumps.

O1. If $t\ 2\ [0;T)$ does not belong to any of the elements of C_1^+, then $R(\ _1)(t) = 0$.

O2. $R(\ _1)$ is continuous on the right end of the intervals $_i$ except for the case $r_i = T$.

Note that O1 also implies that $_1(t) = \ _1(t\ \ )$ for such t's. Let $s_n\ ,\ \mathrm{P}_{i=1}^{n\ 1}(r_i\quad l_i)$ for $n\ 2\ \mathrm{N}$. Note that $s_n\ !\ s_1\ 2\ [0;T]$ as $n\ !\ 1$. Let $\dot{}^{(a)}(t)$ denote the time derivative $\frac{d}{dt}\ ^{(a)}(t)$ of $^{(a)}$ at t, and set

$$_1(t)\ ,\ y + \int_0^t \dot{}_1(s)ds + \ ^{(u)}_1(t);$$

where
$$\_1(t) \;, \sum_{i \in \mathbb{N}} \_^{(a)}(t \quad s_i + l_i)\mathbb{1}_{[s_i;s_{i+1})}(t) + \quad \mathbb{1}_{[s_1;T]}(t);$$
and
$$_1^{(u)}(t) \;, \sum_{i \in \mathbb{N}} {}^{(u)}(t \wedge s_{i+1} \quad s_i + l_i) \quad {}^{(u)}(l_i \quad) \; \mathbb{1}_{[s_i;T]}(t):$$

That is, on the interval $[s_i; s_{i+1})$, $_1$ behaves the same way as $_1$ does on the interval $[l_i; r_i)$; whereas $_1$ decreases linearly at the rate $j \; j$ outside of those intervals. Given this, it can be checked that

O3. $\int_{s_i}^{s_{i+1}} R(\;_1)(s)\;^p ds \quad \int_{l_i}^{r_i} R(\;_1)(s)\;^p ds$

O4. $\int_{l_i}^{r_i} (\_^{(a)}(s))ds = \int_{s_i}^{s_{i+1}} (\_1^{(a)}(s))ds$

O5. $_1^{(u)}(s_{i+1} \quad) \quad _1^{(u)}(s_i \quad) = \;_1^{(u)}(r_i \quad) \quad _1^{(u)}(l_i \quad)$

Now, we verify the conditions *i-1)*, *i-2)*, *i-3)*, *i-4)*. Note rst that the conditions *i-1)* and *i-4)* are obvious from the construction of $_1$. We can verify *i-2)* as follows:
$$
\begin{aligned}
{}_T(\;_2) = \int_0^T R(\;_1)(s)\;^p ds \quad \int_0^{s_1} R(\;_1)(s)\;^p ds &= \sum_{i=1}^{1} \int_{s_i}^{s_{i+1}} R(\;_1)(s)\;^p ds \\
\sum_{i=1}^{1} \int_{l_i}^{r_i} R(\;_1)(s)\;^p ds = \int_0^T R(\;_1)(s)\;^p ds &= \;_T(\;_1);
\end{aligned}
$$

where the second inequality is from O3, and the second last equality is from O1. Moving onto *i-3)*, note that due to the left continuity of $_1$, $s_n \; ! \; s_1$ implies that $(s_n \;) \; ! \; (s_1 \;)$. Also, $_1^{(u)}(s_1) \quad _1^{(u)}(s_1 \quad) = 0$ and $_1^{(u)}$ is constant on $[s_1; T]$. Therefore, $\sum_{i=1}^{1} \; _1^{(u)}(s_{i+1} \quad) \quad _1^{(u)}(s_i \quad) = \lim_{n \,!\, 1} \; _1^{(u)}(s_{n+1} \quad) = \;_1^{(u)}(s_1 \quad) = \;_1^{(u)}(T)$ where we adopted the convention that $_1^{(u)}(0 \quad) = 0$. From O4, O5, and this observation,
$$
\begin{aligned}
I_y^{\mathrm{BV}[0;T]}(\;_1) &= \int_0^T (\_1(t))ds + \;_+ \;\; _1^{(u)}(T) \\
&= \sum_{i=1}^{1} \int_{s_i}^{s_{i+1}} (\_1(t))ds + \;_+ \sum_{i=1}^{1} \; _1^{(u)}(s_{i+1} \quad) \quad _1^{(u)}(s_i \quad)
\end{aligned}
$$

$$= \sum_{i=1} \int_{l_i}^{r_i} (\dot{}^{(a)}(t))ds + {}_{+} \sum_{i=1} {}^{(u)}(r_i \quad) \quad {}^{(u)}(l_i \quad)$$

$$\int_0^T (\dot{}^{(a)}(t))ds + {}_{+} \quad {}^{(u)}(T) = I_y^{\mathrm{BV}[0;T]}(\ _1):$$

For part *(ii)*, we construct $_2$ from $_1$ by moving all the jumps of $^{(u)}$ to time 0. This neither increases $I_y^{\mathrm{BV}[0;T]}$ nor decreases $_T$. That is, if we set

$$_2(t) \ , \ y + \int_0^t \dot{}_1(s)ds + \ {}_1^{(u)}(T);$$

then $_T(\ _2) \quad _T(\ _1)$ obviously, and $_{+} \quad {}_1^{(u)}(T) + I^T_{y+\ {}_1^{(u)}(T)}(\ _2) \quad I_y^T(\ _1)$. Noting that $_1^{(u)}(T) \quad {}^{(u)}(T)$, we see that satis es all the claims of the lemma.

For part *(iii)*, let $\ 2$ AC$[0;T]$ be a concave majorant of . Then there exists a non-increasing $\dot{}\ 2$ D$[0;T]$ such that $(t) = (0) + \int_0^t \dot{}(s)ds$. (Due to the continuity of , (0) and (0) should coincide.) Let $_3(t)$, (0) + $\int_0^t$ _ $\dot{}(s)ds$. Note that *iii-1)*, *iii-2)*, and *iii-4)* are straightforward to check from the construction. To show that *iii-3)* is also satis ed, we construct $C_2^+ = f(l_i^0; r_i^0)$ $[0;T]$: $i\ 2$ Ng in a similar way to C_1^+ so that the elements of C_2^+ are non-overlapping, and $(s) <\ _3(s)$ if and only if $s\ 2\ (l_i^0; r_i^0)$ for some $i\ 2$ N. Note that due to the continuity of and , $(l_i^0) = (l_i^0)$ and $(r_i^0) = (r_i^0)$, and has to be a straight line on $(l_i^0; r_i^0)$ for each $i\ 2$ N. Set s_0 , 0 _ sup$ft\ 2\ [0;T] : \dot{}(t) \quad g$. Then, no interval in C_2^+ contains s_0, because otherwise, has to be a straight line in a neighborhood of s_0, and hence, $\dot{}$ has to be constant there, but this is contradictory to the de nition of s_0. Now, let $\dot{}$ denote a derivative of . Then $\int_{l_i^0}^{r_i^0} (\ \wedge \dot{}(s))ds = \int_{l_i^0}^{r_i^0} (\ )ds = 0$ for i's such that $r_i^0 > s_0$, and hence,

$$I_y^{\mathrm{BV}[0;T]}(\ ) \quad I_y^{\mathrm{BV}[0;T]}(\ _3) = \int_0^T (\dot{}(s))ds \quad \int_0^T (\ \_\ \dot{}(s))ds$$

$$\sum_{i 2 \mathrm{N}:\ r_i^0 \quad s_0} \int_{l_i^0}^{r_i^0} (\dot{}(s)) \quad (\dot{}(s)) \ ds:$$

Note that from the construction of C_2^+, if $s\ 2\ [l_i^0; r_i^0]$ for some i such that $r_i^0 \quad s_0$, we have that $\dot{}(s) = (\ _3(r_i^0) \quad _3(l_i^0))=(r_i^0 \quad l_i^0) = (\ (r_i^0) \quad (l_i^0))=(r_i^0 \quad l_i^0)$, and hence, from Jensen's inequality,

$$\int_{l_i^0}^{r_i^0} (\dot{}(s)) \quad (\dot{}(s)) \ ds$$

$$= \int_{l_i^0}^{r_i^0} (\dot{\ }(s))\mathrm{d}s \quad \int_{l_i^0}^{r_i^0} (\ (r_i^0) \quad (l_i^0))/(r_i^0 \quad l_i^0)\ \mathrm{d}s$$
$$= \int_{l_i^0}^{r_i^0} (\dot{\ }(s))\mathrm{d}s \quad (r_i^0 \quad l_i^0) \quad \int_{l_i^0}^{r_i^0} \dot{\ }(s)\mathrm{d}s/(r_i^0 \quad l_i^0)$$
$$0.$$

Therefore, $_3$ satisfies *iii-3)* as well. □

Now we are ready to prove Proposition 3.4.5.

Proof of Proposition 3.4.5. Since $B_y^{\mathrm{CNCV}} \quad B_y^{\mathrm{AC}} \quad B_y$, we only have to prove that $B_y \quad \inf_{\ 2 B_y^{\mathrm{CNCV}}} I_y(\ )$. For this, we show that for any given $\ 2\ B_y$ and any given $\ > 0$, there is $\ 2\ B_y^{\mathrm{CNCV}}$ such that $I_y(\ ) \quad I_y(\ ) +\ $. To construct such , we first note that we can find $_1\ 2\ B_y$ such that $T(\ _1) < 1$ and $I_y(\ _1) \quad I_y(\ )$ thanks to Lemma 3.5.3. Now set $T = T(\ _1)$ and denote the restriction of $_1$ on $[0; T]$ with $_1$ | i.e., $_1\ 2\ D[0; T]$ and $_1(t) = \ _1(t)$ for $t\ 2\ [0; T]$. We appeal to Lemma 3.5.2 to pick a path $_2\ 2\ \mathrm{AC}[0; T]$ such that $_2(0) = y + z$, $0 \quad z \quad _1^{(u)}(T) = \ _1^{(u)}(T)$, $_T(\ _2) \quad _T(\ _1) \quad 1$, and $_+ \ z + I_{y+z}^{\mathrm{BV}[0;T]}(\ _2) \quad I_y^{\mathrm{BV}[0;T]}(\ _1) = I_y(\ _1) \quad I_y(\ )$. Due to Equation (5.5) in [67], $\lim_{x\,!\,1} \frac{(x)}{x} = \ _+$. As a consequence, we can choose a $u > 0$ large enough so that

$$(u)/u \qquad _+ + \ /z. \tag{3.19}$$

Set

$$_3(s)\ ,\ (y + us)\mathbb{1}_{[0;z/u]}(s) + \ _2(s \quad z/u)\,\mathbb{1}_{(z/u;z/u+T]}(s).$$

Then, $_3\ 2\ \mathrm{AC}[0; z/u + T]$, $_3(0) = y$, $_3(z/u) = y + z$, and that $_{z/u+T}(\ _3) \quad _T(\ _2) \quad 1$. Moreover,

$$I_y^{\mathrm{BV}[0;z/u+T]}(\ _3) = (z/u) \quad (u) + \int_0^T (\dot{\ }_2(s))ds \qquad _+z + \quad + I_{y+z}^{\mathrm{BV}[0;T]}(\ _2)$$
$$I_y(\ ) + \ .$$

Next, we appeal to the part *(iii)* of Lemma 3.5.2 to find a $\ 2\ \mathrm{AC}[0; z/u + T]$ such that $(0) = y$, $_{z/u+T}(\ ) \quad 1$, $I_y^{\mathrm{BV}[0;z/u+T]}(\ ) \quad I_y^{\mathrm{BV}[0;z/u+T]}(\ _3) \quad I_y(\ ) +\ $, and is concave on $[0; z/u + T]$ with the derivative bounded by from below. Now, if we set

$$(t) = \ (t \wedge (z/u + T)) + \ ([t \quad (z/u + T)]^+); \quad t \quad 0;$$

then $\ 2\ B_y^{\mathrm{CNCV}}$ and $I_y(\ ) = I_y^{\mathrm{BV}[0;z/u+T]}(\ ) \quad I_y(\ ) +\ $. □

3.5.2 Proof of Proposition 3.4.1

The proof of Proposition 3.4.1 hinges upon the following technical lemmas. Recall that $B_y^{AC} = B_y \cap AC[0;\infty)$. For a fixed $M > 0$, let

i) $B_y^{AC;M} \triangleq B_y^{AC} \cap \{ \in D[0;\infty) : T( ) \quad M\}$, and let

ii) $B_y^{M} \triangleq B_y \cap \{ \in D[0;\infty) : T( ) \quad M\}$.

Lemma 3.5.3. *For any given y 0, there exists a constant $M = M(y) > 0$ such that*

- *for each $\in B_y$, there exists a path $\in B_y^M$ such that $I_y( ) \quad I_y( )$;*
- *therefore,*

$$\inf_{\in B_y} I_y( ) = \inf_{\in B_y^M} I_y( ); \tag{3.20}$$

- *moreover, $M(y) \quad cy + d$ for some $c > 0$ and $d > 0$.*

Proof. Let $y \triangleq (j \; j(p+1))^{1=1+p}$. In case $y \quad y$, the equality in (3.20) holds with the optimal values of the LHS and RHS both being zero: to see this, set $M \triangleq \quad y=$ and $(t) \triangleq y + \quad t$, and note that $\int_0^{T( )} R( )(s)^p ds \quad 1$ and $T( ) = M$, and hence, $\in B_y^M$ while $I_y( ) = 0$. Therefore, we assume for the rest of the proof that $y < y$. It is enough to show that there exists $M > 0$ such that

$$\text{For any given } \in B_y \setminus B_y^M, \text{ one can find } \in B_y^M \text{ such that } I_y( ) \quad I_y( ). \tag{3.21}$$

To construct such M, consider w and z such that $< w < 0 < z$, $(w) < \infty$ and $(z) < \infty$. We consider a piece-wise linear path

$$(t) \triangleq (y + zt)\mathbb{1}_{[0;(y \quad y)=z]}(t) + (y + \quad t \quad (y \quad y)=z)\,\mathbb{1}_{[(y \quad y)=z;\,\infty)}(t)$$

and

$$M \triangleq \max\left\{\frac{(y \quad y) \quad (z)}{z \quad (w)};\ (y \quad y)=z \quad y= ;\ \frac{y + \frac{y \quad y}{z} + \quad (z)}{w}\right\}:$$

Then, $\in B_y^M$ and $I_y( ) = \quad (z)\frac{y \quad y}{z}$. Suppose that $\in B_y \setminus B_y^M$ so that $T( ) > M$. If $\notin BV[0;\infty)$, $I( ) = \infty$, from which (3.21) is immediate. Suppose that $\in BV[0;\infty)$ so that $= {}^{(a)} + {}^{(u)} + {}^{(d)}$. Note that if we set ${}^{0} \triangleq {}^{(a)} + {}^{(u)}$, then $T(\ ^0)\quad T(\ )$, $I_y(\ ^0)\quad I_y(\ )$, and $^0\ 2\ B_y$. Therefore, we assume w.l.o.g. $^{(d)} = 0$. Note that if $^{(u)}(T(\ ))\quad (z)\frac{y\ \ y}{z\ \ +}$, then

$$I_y(\ )\quad {}_{+}\ {}^{(u)}(T(\ )) > \quad (z)\frac{y\quad y}{z} = I_y(\ ):$$

On the other hand, if $^{(u)}(T(\ )) < \quad (z)\frac{y\ \ y}{z\ \ +}$, then $^{(a)}(T(\ ))\qquad (z)\frac{y\ \ y}{z\ \ +}$, and hence, by the construction of M,

$$< w < \quad y + \frac{y\quad y}{z\ \ +}\quad (z)\ =T(\ ) < \quad {}^{(a)}(T(\ ))\quad y\ =T(\ ):$$

Therefore,

$$\begin{aligned} I_y(\ ) &\quad \int_0^{T(\ )} (\ \_(s))ds \quad T(\ ) \qquad {}^{(a)}(T(\ ))\quad y\ =T(\ ) \\ &T(\ ) \qquad y + \frac{y\quad y}{z\ \ +}\quad (z)\ =T(\ ) \qquad T(\ ) \qquad (w) \\ &M \qquad (w) \quad \frac{(y\quad y)}{z}\quad (z) = I_y(\ ); \end{aligned}$$

where the second inequality is from Jensen's inequality, the third and fourth inequalities are from the monotonicity of on $[\ ;1\ )$, and the fth and the sixth inequalities are from the construction of M and , respectively. This concludes the proof of (3.21) and (3.5.3).

To see the existence of $c > 0$ and $d > 0$, note that for the case $y\quad y$, our construction of $M(y)$ is linear in y, whereas $M(y)$ is bounded for the case $y < y$. □

Lemma 3.5.4. *Let $M > 0$ be the constant in Lemma 3.5.3. Then,*

$$B_y = V_y^T$$

for any $T\quad M$.

Proof. The conclusion of the lemma follows immediately from the following claims.

Claim 1: V_y^T is nonincreasing in T.

Proof of Claim 1. Let $t_1 < t_2$. For each $_1\ 2\ V_y^{t_1}$, consider $_2(s)\ ,\ \ _1(s \wedge t_1) + (s\quad t_1)\mathbb{1}_{(t_1;t_2]}(t)$. Then, $_2\ 2\ V_y^{t_2}$ and $I_y^{\mathrm{BV}[0;t_1]}(\ _1) = I_y^{\mathrm{BV}[0;t_2]}(\ _2)$. Therefore, $V_y^{t_2}$ is at least as small as $V_y^{t_1}$.

Claim 2: If $M > 0$ is such that $\inf_{\varphi \in B_y^M} I_y(\varphi) = \inf_{\varphi \in B_y} I_y(\varphi)$ as in Lemma 3.5.3, then

$$\inf_{\varphi \in B_y^M} I_y(\varphi) \ge V_y^M .$$

Proof of Claim 2. Given an $\epsilon > 0$, consider $\varphi \in B_y^M$ such that $I_y(\varphi) \le \inf_{\varphi \in B_y^M} I_y(\varphi) + \epsilon$. Set $\psi(t) := \varphi(t \wedge T(\varphi)) + (t - T(\varphi))\mathbb{1}_{(T(\varphi),M]}(t)$. Then, $\psi \in V_y^M$ and hence,

$$V_y^M = \inf_{\psi \in V_y^M} I_y^{\mathrm{BV}[0,M]}(\psi) \le I_y^{\mathrm{BV}[0,M]}(\psi) \le I_y(\varphi) \le \inf_{\varphi \in B_y^M} I_y(\varphi) + \epsilon .$$

Taking $\epsilon \to 0$, we arrive at Claim 2.

Claim 3: For any $T \ge M$,

$$V_y^T \ge \inf_{\varphi \in B_y} I_y(\varphi).$$

Proof of Claim 3. By (i), and (iii) of Lemma 3.5.2, given an $\epsilon > 0$, consider $\varphi \in V_y^T$ so that $I_y^{\mathrm{BV}[0,T]}(\varphi) \le \inf_{\varphi \in V_y^T} I_y^{\mathrm{BV}[0,T]}(\varphi) + \epsilon$, φ is concave over $[0,T]$, φ is non-negative over $[0,t]$, φ is linear with slope μ over $[t,T]$, and $\tau_T(\varphi) \le 1$. Set $\psi(t) := \varphi(t \wedge T) + (t - T)\mathbb{1}_{(T,\infty)}(t)$. Then, $\psi \in B_y$ and hence,

$$V_y^T = \inf_{\varphi \in V_y^T} I_y^{\mathrm{BV}[0,T]}(\varphi) + \epsilon \ge I_y^{\mathrm{BV}[0,T]}(\varphi) \ge I_y(\psi) \ge \inf_{\varphi \in B_y^T} I_y(\varphi).$$

Taking $\epsilon \to 0$, we arrive at Claim 3. □

Set

$$K_t := \left\{ \varphi \in D[0,t] : \varphi(0) = 0, \int_0^t (R(\varphi)(s))^p ds \le 1, \ \varphi(s) \ge 0 \text{ for } s \in [0,t] \right\}.$$

The following corollary is immediate from the two previous lemmas:

Corollary 3.5.5. *Let $M > 0$ be the constant in Lemma 3.5.3. For any $y \ge 0$,*

$$\inf_{t \in [0,M]} \inf_{\varphi \in K_t} I_0^{\mathrm{BV}[0,t]}(\varphi) = V_0^M = B_0.$$

Proposition 3.5.6. *The optimal value B_y, associated with $\mathcal{B}_y$, satisfies*

(i) $y \mapsto B_y$ is non-increasing in y;

(ii) $y \mapsto B_y$ *is Lipschitz continuous.*

Proof. For part *(i)*, let x, y be such that $0 \quad x < y$. We will show that for any $\quad > 0$, there exists $\quad \in B_y$ such that $I_y(\quad) < B_x + \quad$. Due to Lemma 3.5.3, we can pick $\quad \in B_x$ such that $I_x(\quad) < B_x + \quad$ and $T(\quad) < \infty$. Set

$$(t) \, , \, (y \quad x) + \quad (t \wedge T(\quad)) + \quad [t \quad T(\quad)]^+ .$$

Then, since $(0) = y$, $R(\quad)(t) \quad R(\quad)(t)$ on $t \in [0, T(\quad)]$, we see that $\quad \in B_y$. On the other hand, since $\quad$ has no jump on $[T(\quad), \infty)$, and $(\_^{(a)}(s)) = \quad (\quad) = 0$ on $s \in [T(\quad), \infty)$, as well as $T(\quad) \quad T(\quad)$,

$$\begin{aligned} I_y(\quad) &= \int_0^{T(\quad)} (\_^{(a)}(s))ds + \quad_+ \ ^{(u)}(T(\quad)) + \quad ^{(d)}(T(\quad)) \\ &= \int_0^{T(\quad)} (\_^{(a)}(s))ds + \quad_+ \ ^{(u)}(T(\quad)) + \quad ^{(d)}(T(\quad)) \\ &= \int_0^{T(\quad)} (\_^{(a)}(s))ds + \quad_+ \ ^{(u)}(T(\quad)) + \quad ^{(d)}(T(\quad)) \\ &= I_x(\quad) < B_x + \quad . \end{aligned}$$

For part *ii)*, note that we only need to prove one side of the inequality thanks to part *i)*. That is, it is enough to show that if $0 \quad x < y$, then $B_x \quad B_y + (y \quad x) \quad (1)$. Fix an $\quad > 0$ and pick $\quad \in B_y$ such that $I_y(\quad) \quad B_y + \quad$. Set

$$(t) \, , \, (x + t)\mathbb{1}_{[0, y \quad x]}(t) + \quad (t \quad (y \quad x))\mathbb{1}_{[y \quad x, \infty)}(t).$$

Then $^{(u)}(s) = \ ^{(u)}(s \quad (y \quad x))$ and $^{(d)}(s) = \ ^{(d)}(s \quad (y \quad x))$ on $s \in [y \quad x, \infty]$, and $^{(u)}(s) = \ ^{(d)}(s) = 0$ on $s \in [0, y \quad x]$, and $T(\quad) = T(\quad) + y \quad x$. Hence,

$$\begin{aligned} I_x(\quad) &= \int_0^{y \quad x} (1)ds + \int_{y \quad x}^{T(\quad)} (\_^{(a)}(s))ds + \quad_+ \ ^{(u)}(T(\quad)) + \quad ^{(d)}(T(\quad)) \\ &= (y \quad x) \quad (1) + \int_{y \quad x}^{T(\quad)+y \quad x} (\_^{(a)}(s))ds + \quad_+ \ ^{(u)}(T(\quad) + y \quad x) \\ &\quad + \quad ^{(d)}(T(\quad) + y \quad x) \\ &= (y \quad x) \quad (1) + \int_0^{T(\quad)} (\_^{(a)}(s + (y \quad x)))ds + \quad_+ \ ^{(u)}(T(\quad)) \\ &\quad + \quad ^{(d)}(T(\quad)) \end{aligned}$$

$$= (y \quad x) \quad (1) + \int_0^{T(\ )} (\dot{\ }^{(a)}(s))ds + \ _+ \ ^{(u)}(T(\ )) + \ ^{(d)}(T(\ ))$$

$$= (y \quad x) \quad (1) + I_y(\ ) \quad (y \quad x) \quad (1) + B_y + \ :$$

Since $\ 2\ B_x$, this implies that $B_x \quad (y \quad x) \quad (1) + B_y + \$. Taking $\ !\ 0$, we arrive at the desired inequality.

□

The main preparatory result for the asymptotic upper bound relies on a result of [94]. The goal of the next two lemmas is to verify a uniform continuity result. Let TV( ) be the total variation of .

Lemma 3.5.7. *The function $H : D[0;T]\ !\ [0;1)$ given by $H(\ ) = \int_0^T (s)ds$ is Lipschitz continuous on the set of $f\ : TV(\ ) \quad Mg$ for every $M < 1$.*

Proof. Let be such that TV( ) M and let be such that $d_{M_1}(\ ;\ )$. Set $(t)\ ,\ \inf f x : d((t;x);\ (\ )) \quad g$ where ( ) is the completed graph of and d is the L_1 distance in R^2, i.e., $d((t;x);(s;y)) = jt \quad sj + jx \quad yj$. Then, $d_{M_1}(\ ;\ )$ implies that $(t) \quad (t)$ for all $t\ 2\ [0;T]$. Due to the construction of and the fact that L_1 balls are contained in L_2 balls of the same radius, the di erence between the area below and the area below is bounded by len(( )) , where the lenght len(( )) of ( ) is bounded by T + TV( ). Putting everything together, we conclude that

$$\int_0^T (s)ds \quad \int_0^T (s) \quad \int_0^T (s)ds \quad \int_0^T (s) \quad (T + M)\ : \tag{3.22}$$

The upper bound can be established in the same way. □

Recall the function $\ _T : D[0;T]\ !\ [0;1)$ de ned as $\ _T(\ ) = \int_0^T R(\ )(s)^p ds$.

Lemma 3.5.8. *$\ _T$ is Hölder continuous with index $\min f p;1g$ on the set $f\ : I_K(\ ) \quad g$.*

Proof. Let be such that $I_K(\ )$. Let $\ 2\ (0;\min f\ _+;j \quad jg)$. Observe that $(\dot{\ }(s)) \quad j\dot{\ }(s)j \quad (\ )$. Hence,

$$\int_0^1 j\dot{\ }(s)jds + \ ^u(1) + j\ ^d(1)j \quad (\ + \ (\ ))= \ ,\ M\ :$$

Consequently, if $I_K(\ )$, then TV( ) M . The re ection map R is a Lipschitz continuous map from $D[0;T]$ to $D[0;T]$ w.r.t. the M_1 topology with

Lipschitz constant 2 (cf. [96], Theorem 13.5.1), and if the total variation of ϕ is bounded by M_ϕ, the total variation of $R(\phi)$ is bounded by $2M_\phi$. Consequently, the total variation of $R(\phi)^p$ is bounded by $2^p(2M_\phi)^p \equiv \tilde{M}_\phi$. Moreover, the map $\phi \to R(\phi)^p$ is Hölder continuous on $\{\phi : I_K(\phi) \le \infty\}$ with index $\min\{p, 1\}$. Since the composition of a Lipschitz and Hölder continuous map is again Hölder continuous (in this case, with exponent $\min\{p, 1\}$), the proof follows from Lemma 3.5.7.

□

Lemma 3.5.9. *(i) For any $t, y \ge 0$ and $T > 0$,*

$$\limsup_{x \to \infty} \frac{1}{x} \log \mathbf{P}_{xy}(T_1/x > T) \le -ty + T \log \mathbf{E} e^{tU}. \tag{3.23}$$

(ii) For any $y \ge 0$ and $T > 0$,

$$\limsup_{x \to \infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^T (X(\lfloor sx \rfloor)/x)^p ds \ge 1 \right) \le -V_y^T. \tag{3.24}$$

Proof. For part *(i)*, note that

$$\mathbf{P}_{xy}(T_1 > xT) \le \mathbf{P}_{xy}\left(X_{\lfloor xT \rfloor} > 0\right) = \mathbf{P}\left(\sum_{i=1}^{\lfloor xT \rfloor} U_i > xy \right) \le e^{-txy} \left(\mathbf{E}\, e^{tU}\right)^{\lfloor xT \rfloor};$$

where the last inequality is from the Markov inequality. Taking logarithms, dividing both sides by x, and taking lim sup, we get (3.23).

For part *(ii)*, as a Hölder continuous map is uniformly continuous, Lemma 3.5.7 allows us to apply Result 3.2.3 (ii) to obtain

$$\lim_{x \to \infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^T (X(\lfloor sx \rfloor)/x)^p ds \ge 1 \right) = -\inf_{a \in [1, \infty)} J_y(a);$$

where $J_y(a) = \inf\{I_K(\phi) : \phi \in \mathbb{D}[0, T];\ \phi(0) = y;\ \Phi_T(\phi) = a\}$. It is easy to see that $\inf_{a \in [1, \infty)} J_y(a) = V_y^T$ hence, (3.24) follows.

□

Now we are ready to prove Proposition 3.4.1.

Proof of Proposition 3.4.1. For part *(i)*, consider a small enough $t_0 > 0$ so that $\mathbf{E}e^{t_0 U} < 1$. Then, by to Lemma 3.5.4, we can pick a sufficiently large $T > 0$ so that $B_y = V_y^T - \epsilon$ and $t_0 y + T \log \mathbf{E}e^{t_0 U} < -B_y$. Considering the case $T_1/x \le T$ and $T_1/x > T$ separately and then applying the principle of the maximum term,

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^{T_1/x} \phi(X(\lfloor ux \rfloor)/x)^p du \ge 1 \right) \\
&\le \limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\left\{ \int_0^{T_1/x} \phi(X(\lfloor ux \rfloor)/x)^p du \ge 1; T_1 \le xT \right\} \cup \{T_1 > xT\} \right) \\
&\le \limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}\left(\int_0^{T} \phi(X(\lfloor ux \rfloor)/x)^p du \ge 1 \right) \vee \limsup_{x\to\infty} \frac{1}{x} \log \mathbf{P}_{xy}(T_1/x > T) \\
&\le -V_y^T \vee t_0 y + T \log \mathbf{E}e^{t_0 U} = -B_y - \epsilon \vee t_0 y + T \log \mathbf{E}e^{t_0 U} = -B_y;
\end{aligned} \tag{3.25}$$

where we used Lemma 3.5.9 for the third inequality.

Next, we move on to part *(ii)*. For any given $t > 0$, let

- $A_{t;\epsilon} = \{\phi \in \mathbb{D}[0;t] : \phi(0) = \epsilon; \int_0^t R(\phi)(s)^p ds > 1; \phi(s) > 0; \forall s \in [0;t]\}$ and
- $\bar{A}_{t;\epsilon} = \{\phi \in \mathbb{D}[0;t] : \phi(0) = \epsilon; \int_0^t R(\phi)(s)^p ds > 1; \phi(s) > \epsilon/2; \forall s \in [0;t]\}$.

Set $u = x^{1+p}$. Let δ be small enough such that $\mathbf{P}(U_1 > \sqrt[p]{\delta}) > 0$. Define the event $B_{x;\delta} = \{U_i > \sqrt[p]{\delta}; i = 1;\dots;\lceil x\sqrt[p]{\delta}\rceil\}$. Setting $k_\delta = \lceil x\sqrt[p]{\delta}\rceil + 1$, we obtain

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{1}{u^{1/(1+p)}} \log \mathbf{P}_0(W_1 > u) \\
&= \liminf_{x\to\infty} \frac{1}{x} \log \mathbf{P}_0\left(\sum_{k=0}^{T_1-1} X_k^p > u \right) \\
&\ge \liminf_{x\to\infty} \frac{1}{x} \log \mathbf{P}_0\left(\sum_{k=k_\delta}^{T_1-1} X_k^p > x^{1+p}; B_{x;\delta} \right) \\
&= \liminf_{x\to\infty} \frac{1}{x} \log \left[\mathbf{P}_0\left(\sum_{k=k_\delta}^{T_1-1} X_k^p > x^{1+p} \,\middle|\, B_{x;\delta} \right) \mathbf{P}_0(B_{x;\delta}) \right] \\
&\ge \liminf_{x\to\infty} \frac{1}{x} \log \left[\mathbf{P}_{\epsilon x}\left(\sum_{k=0}^{T_1-1} X_k^p > x^{1+p} \right) \mathbf{P}_0(B_{x;\delta}) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \liminf_{x\to\infty} \frac{1}{x}\log \left[\mathbf{P}_x\left(\int_0^{T_1=x} X_{\lfloor xs\rfloor}/x^{\,p}\, ds > 1 \right) \mathbf{P}_0(B_{x;\ }) \right] \\
&\quad \liminf_{x\to\infty} \frac{1}{x}\log \mathbf{P}_x\left(\int_0^{t} X_{\lfloor xs\rfloor}/x^{\,p}\, ds > 1;\ T_1 > xt \right) \mathbf{P}_0(B_{x;\ }) \\
&= \liminf_{x\to\infty} \frac{1}{x}\log \mathbf{P}_x\left(\int_0^{t} X_{\lfloor xs\rfloor}/x^{\,p}\, ds > 1;\ \frac{X_{\lfloor xs\rfloor}}{x} > 0;\ \forall s \in [0;t] \right) \mathbf{P}_0(B_{x;\ }) \\
&\quad \liminf_{x\to\infty} \frac{1}{x}\log \mathbf{P}\left(K_x \in A_{t;\ } \right) \mathbf{P}_0(B_{x;\ }) \\
&\quad \inf_{\in (A_{t;\ })} I^{BV[0;t]}(\ ) + {}^{p}\!\!- \log \mathbf{P}(U_1 > {}^{p}\!\!-) \\
&\quad \inf_{\in \bar{A}_{t;\ }} I^{BV[0;t]}(\ ) + {}^{p}\!\!- \log \mathbf{P}(U_1 > {}^{p}\!\!-).
\end{aligned}
$$

where the third equality is from part *(i)* of Proposition 3.4.2. The second to last inequality follows from part *(i)* of Result 3.2.3 since the integral and the infimum are both continuous in the M_1 topology (see, respectively Theorem 11.5.1 and Theorem 13.4.1 of [96]). Recall that

$$
K_t = \left\{ \ \in D[0;t] : \ (0) = 0;\ \int_0^t (R(\ )(s))^p ds \le 1;\ \ (s) \ge 0 \text{ for } s \in [0;t] \right\}.
$$

Note that for all $\ > 0$,

$$
\inf_{\in \bar{A}_{t;\ }} I^{BV[0;t]}(\ ) \le \inf_{\in K_t} I_0^{BV[0;t]}(\ ). \tag{3.26}
$$

To see this, suppose that $\ \in K_t$. Then, $\tilde{\ } = \ + \ $ belongs to $\bar{A}_{t;\ }$ and $I^{BV[0;t]}(\tilde{\ }) = I_0^{BV[0;t]}(\ )$. Since the construction holds for every $\ \in K_t$, we have that $\inf_{\in K_t} I_0^{BV[0;t]}(\ ) \ge \inf_{\in \bar{A}_{t;\ }} I^{BV[0;t]}(\ )$. Therefore,

$$
\liminf_{u\to\infty} \frac{1}{u^{1/(1+p)}} \log \mathbf{P}_0(W_1 > u) \ge \inf_{\in K_t} I_0^{BV[0;t]}(\ ) + {}^{p}\!\!- \log \mathbf{P}(U_1 > {}^{p}\!\!-).
$$

Since $\ $ and t are arbitrary, taking $\ \to 0$ and taking the infimum over $t \in [0;M]$, Corollary 3.5.5 gives

$$
\liminf_{u\to\infty} \frac{1}{u^{1/(1+p)}} \log \mathbf{P}_0(W_1 > u) \ge \inf_{t\in[0;M]} \inf_{\in K_t} I_0^{BV[0;t]}(\ ) = \ B_0.
$$

□

3.5.3 Proof of Proposition 3.4.2, and Lemma 3.2.2

We start with the proof of Lemma 3.2.2.

Proof of Lemma 3.2.2. We first derive the upper bound, by noting that

$$
\begin{aligned}
&\mathbf{P}\left(\sum_{k=0}^{n} (X_k)^p \ge x; X_n = 0\right) \\
&= \sum_{m=0}^{n} \mathbf{P}\left(\sum_{k=0}^{m-1} (X_k)^p \ge x; T > m-1; X_m = 0; X_n = 0\right) \\
&\le \sum_{m=0}^{n} \mathbf{P}\left(\sum_{k=0}^{m-1} (X_{k-(m-1)})^p \ge x; X_{-(m-1)} > 0; \dots; X_0 > 0\right) \\
&= \sum_{m=0}^{n} \mathbf{P}\left(\sum_{k=0}^{m-1} X_{m-1-k}^p \ge x; X_{m-1} > 0; \dots; X_0 > 0\right) \\
&= \sum_{m=0}^{n} \mathbf{P}\left(\sum_{k=0}^{m-1} X_k^p \ge x; T > m-1\right) \le \sum_{m=0}^{n} \mathbf{P}\left(\sum_{k=0}^{T} X_k^p \ge x; T > m-1\right) \\
&\le (n+1)\mathbf{P}\left(\sum_{k=0}^{T} X_k^p \ge x\right).
\end{aligned}
$$

For the lower bound, first write

$$
\begin{aligned}
&\mathbf{P}\left(\sum_{k=0}^{n} (X_k)^p \ge x; X_n = 0\right) \\
&= \sum_{m=1}^{n} \mathbf{P}\left(\sum_{k=0}^{m-1} (X_k)^p \ge x; T = m; X_n = 0\right) \\
&= \sum_{m=1}^{n} \mathbf{P}\left(\sum_{k=0}^{m-1} (X_k)^p \ge x; T = m\right) \mathbf{P}_0(X_{n-m} = 0)
\end{aligned}
$$

Apply Lemma 3.2.1 [by using $g(y_1; \dots; y_n) = I(\sum_i y_i^p > x; y_i > 0 : i < n)$] to observe that

$$
\mathbf{P}\left(\sum_{k=0}^{m-1} (X_k)^p \ge x; T = m\right)
$$

$$
\begin{aligned}
&= \mathbf{P}\left(\sum_{k=0}^{m-1}(X_k)^p \quad x; X_i > 0; i = 1;\ldots;m \quad 1; X_m = 0\right)\\
&= \quad (0)\mathbf{P}_0\left(\sum_{k=1}^{m} X_k^p \quad x; X_i > 0; i = 1;\ldots;m \quad 1;\right)\\
&= \quad (0)\mathbf{P}_0\left(\sum_{k=1}^{m} X_k^p \quad x; T \quad m\right)\\
&\quad\;\; (0)\mathbf{P}_0\left(\sum_{k=1}^{m} X_k^p \quad x; T = m\right)\\
&= \quad (0)\mathbf{P}_0\left(\sum_{k=0}^{T} X_k^p \quad x; T = m\right)
\end{aligned}
$$

Consequently, for every xed n_0 such that $\inf_{k \quad n_0} \mathbf{P}_0(X_k = 0) \qquad (0)=2$,

$$
\begin{aligned}
\mathbf{P}&\left(\sum_{k=0}^{T}(X_k)^p \quad x; X_n = 0\right)\\
&\quad (0)\sum_{m=0}^{n \quad n_0}\mathbf{P}_0\left(\sum_{k=0}^{T} X_k^p \quad x; T = m\right)\mathbf{P}_0(X_{n \quad m} = 0)\\
&\quad (0)\mathbf{P}_0\left(\sum_{k=0}^{T} X_k^p \quad x; T \quad n \quad n_0\right)\inf_{k \quad n_0}\mathbf{P}_0(X_k = 0):\\
&\quad (\quad (0)^2=2)\mathbf{P}_0\left(\sum_{k=0}^{T} X_k^p \quad x\right) \quad O(e^{\quad cn}):
\end{aligned}
$$

□

Now, we move on to the proof of Proposition 3.4.2.

Proof. Proof of Proposition 3.4.2. For part *(i)*, note that

$$
\frac{1}{x^{1+p}}\sum_{k=0}^{m-1} X_k^p = \frac{1}{x^{1+p}}\int_0^m X_{\lfloor u \rfloor}^p du = \frac{1}{x^{1+p}}\int_0^{m=x} x X_{\lfloor xs \rfloor}^p ds
$$

$$
= \int_0^{m=x}\left(\frac{X_{\lfloor xs \rfloor}}{x}\right)^p ds
$$

where the second equality is from the change of variable with $u = xs$. The claimed equivalence is immediate from this.

For part *(ii)*, note that if we set $(t)\ ,\ y\quad t$, then $I_y(\ ) = 0$ while $\ \in B_y$, and hence, $B_y = 0$.

For part *(iii)*, note that

$$\lim_{k \to \infty} \min_{i\ 1} \left(\frac{i\ \ 1}{k}\ y + B_{\frac{i}{k}y} \right) = \lim_{k \to \infty} \left(\min_{i\ 1} \left(\frac{i}{k}y + B_{\frac{i}{k}y} \right) \quad \frac{1}{k}\ y \right)$$

$$= \lim_{k \to \infty} \min_{i\ 1} \left(\frac{i}{k}y + B_{\frac{i}{k}y} \right) :$$

Moreover, from part *(ii)* of Proposition 3.5.6,

$$\lim_{k \to \infty} \min_{i\ 1} \left(\frac{i}{k}\ y + B_{\frac{i}{k}y} \right) = \inf_{y \in [0;\infty)} \quad y + B_y \ :$$

For part *(iv)*, note that by definition, $B_0 \quad B$. Therefore, we only have to prove that $B_0 \quad B$. Recall that $\ = \sup\{\ > 0 : \mathbf{E}(e^{\ U}) \quad 1\}$ and $_+ = \sup\{\ \in \mathbb{R} : \mathbf{E}(e^{\ U}) < \infty\}$. For the rest of this proof, let be the log-moment generating function and let D denote the effective domain of i.e; $D = \{x : (x) < \infty\}$. We start with a claim: for any $\ > 0$ there exists a $u > 0$ such that

$$(u)/u \qquad + \ : \tag{3.27}$$

To prove (3.27) we distinguish between the cases $\ < \ _+$ and $\ = \ _+$. For the first case note that $\ \in D$. In view of the convexity and continuity of $\mathbf{E}(e^{\ U})$, $\mathbf{E}(e^{\ U}) = 1$. Due to Lemma 2.2.5 (c) of [22], is a differentiable function in D with $^0(\ ) = \frac{\mathbf{E}(Ue^{\ U})}{\mathbf{E}(e^{\ U})}$. Since $\ \in D$ we have that $^0(\ ) = \mathbf{E}(Ue^{\ U}) < \infty$. In addition, $^0(0) = \mathbf{E}(U) < 0$ implies that $(\ )$ is decreasing for small values of . Now, the convexity and differentiability of over its effective domain implies that ^{0} should be increasing at and thus $\mathbf{E}(Ue^{\ U}) > 0$. It can be checked that for $u = \mathbf{E}(Ue^{\ U})$,

$$\frac{(u)}{u} = \frac{\mathbf{E}(Ue^{\ U}) \quad \log \mathbf{E}(e^{\ U})}{\mathbf{E}(Ue^{\ U})} = \ ;$$

and hence our claim is proved. Consider now the case $\ = \ _+$. In view of Equation (5.5) in [67], $\lim_{x \to \infty} \frac{(x)}{x} = \ _+$. That is, for any $\ > 0$ we can choose a u so that $(u)/u \quad _+ + \ = \ + \ $. We proved the claim (3.27).

Back to the inequality $B_0 \le B$, we will show that for any given $\epsilon > 0$ and any given path $\phi \in B_y$, we can construct a path $\psi \in B_0$ so that $I_0(\psi) \le I_y(\phi) + \mu y + \epsilon$. To this end, let $u > 0$ be such that $\Lambda(u)/u \le \mu + \epsilon/y$ and set

$$\psi(s) \triangleq us\mathbb{1}_{\{s \le y/u\}} + \phi(s - y/u)\mathbb{1}_{\{s > y/u\}}.$$

Then $\psi(0) = 0$, $\psi(y/u) = y$, and $\psi \in B_0$. Furthermore, one can see that

$$I_y(\psi) = (y/u)\Lambda(u) + \int_0^{T(\phi)} \Lambda(\dot\phi(s))ds + \Lambda_+^{(u)}(T(\phi)) = (y/u)\Lambda(u) + I_y(\phi).$$

From the construction of u,

$$I_0(\psi) \le \mu y + \epsilon + I_y(\phi)$$

as desired. This concludes the proof of part *(iv)*.

□

3.5.4 Proof of Proposition 3.3.3

In this section we prove a sample-path LDP for Z_n. We employ a well-known technique, based on the projective limit theorem by Dawson and Gärtner; see Theorem 4.6.1 in [22]. The following three lemmas lead to the first key step in this approach, which consists of obtaining the finite-dimensional LDP for Z_n.

Lemma 3.5.10. *For any given $0 = t_0 < t_1 < t_2 < \dots < t_k$, let $\Delta t_i = t_i - t_{i-1}$ for $i = 1, \dots, k$. Then,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(\sum_{j=1}^{N(nt_1)} W_j \ge na_1, \dots, \sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j \ge na_k \right)$$
$$\le -B_0 \left(\sum_{i=1}^{k} (a_i - \mu\Delta t_i)_+ \right); \tag{3.28}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(\sum_{j=1}^{N(nt_1)} W_j \ge na_1, \dots, \sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j \ge na_k \right)$$
$$\ge -B_0 \left(\sum_{i=1}^{k} (a_i - \mu\Delta t_i)_+ \right); \tag{3.29}$$

where $(x)_+ \triangleq x \vee 0$.

Proof. Firstly, for notational convenience, let $E_i^{(n)}(\ ) := n[t_i/\mathbf{E}\ \ ; t_i/\mathbf{E}\ \ + \ ]$. We will use this notation throughout the proof of this lemma. For the upper bound in equation (3.28), notice that

$$\mathbf{P}\left(\sum_{j=1}^{N(nt_1)} W_j \ge na_1; \dots; \sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j \ge na_k \right)$$

$$\le \underbrace{\sum_{i=1}^{k} \mathbf{P}\left(N(nt_i) \notin E_i^{(n)}(\ ) \right)}_{=(\mathrm{I})}$$

$$+ \mathbf{P}\left(\sum_{j=1}^{N(nt_1)} W_j \ge na_1; \dots; \sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j \ge na_k; \right.$$

$$\left. N(nt_i) \in E_i^{(n)}(\ ) \text{ for } i = 1; \dots; k \right) = (\mathrm{II}).$$

For (I), by Theorem 6.1 in ([80]),

$$\limsup_{n \to \infty} \frac{\log(\mathrm{I})}{n} = -\infty. \qquad (3.30)$$

Shifting our attention to (II),

$$\mathbf{P}\left(\sum_{j=1}^{N(nt_1)} W_j \ge na_1; \dots; \sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j \ge na_k; \right.$$

$$\left. N(nt_i) \in E_i^{(n)}(\ ) \text{ for } i = 1; \dots; k \right)$$

$$\le \sum_{i_1=\lceil n(t_1/\mathbf{E}\ -\ )\rceil}^{\lfloor n(t_1/\mathbf{E}\ +\ )\rfloor} \cdots \sum_{i_k=\lceil n(t_k/\mathbf{E}\ -\ )\rceil}^{\lfloor n(t_k/\mathbf{E}\ +\ )\rfloor} \mathbf{P}\left(\sum_{j=1}^{i_1} W_j \ge na_1; \dots; \sum_{j=i_{k-1}+1}^{i_k} W_j \ge na_k; \right.$$

$$\left. N(nt_l) = i_l \text{ for } l = 1; \dots; k \right)$$

$$\le \sum_{i_1=\lceil n(t_1/\mathbf{E}\ -\ )\rceil}^{\lfloor n(t_1/\mathbf{E}\ +\ )\rfloor} \cdots \sum_{i_k=\lceil n(t_k/\mathbf{E}\ -\ )\rceil}^{\lfloor n(t_k/\mathbf{E}\ +\ )\rfloor}$$

$$\mathbf{P}\left(\sum_{j=1}^{i_1} W_j \geq na_1, \ldots, \sum_{j=i_{k-1}+1}^{i_k} W_j \geq na_k\right) I(i_1 \leq \cdots \leq i_k)$$

$$= \sum_{i_1=\lceil n(t_1/\mathbf{E}(\cdot)-\delta)\rceil}^{\lfloor n(t_1/\mathbf{E}(\cdot)+\delta)\rfloor} \cdots \sum_{i_k=\lceil n(t_k/\mathbf{E}(\cdot)-\delta)\rceil}^{\lfloor n(t_k/\mathbf{E}(\cdot)+\delta)\rfloor} \mathbf{P}\left(\sum_{j=1}^{i_1} W_j \geq na_1\right) \cdots \mathbf{P}\left(\sum_{j=i_{k-1}+1}^{i_k} W_j \geq na_k\right)$$

$$\leq (2\delta n)^k\, \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/\mathbf{E}(\cdot)+\delta)\rfloor} W_j \geq na_1\right) \cdots \mathbf{P}\left(\sum_{j=\lceil n(t_{k-1}/\mathbf{E}(\cdot)-\delta)\rceil}^{\lfloor n(t_k/\mathbf{E}(\cdot)+\delta)\rfloor} W_j \geq na_k\right).$$

Now, we have that from Result 3.2.1,

$$\limsup_{n\to\infty} \frac{1}{n}\log(II) \leq \sum_{i=1}^{k} \limsup_{n\to\infty}\frac{1}{n}\log \mathbf{P}\left(\sum_{j=\lceil n(t_{i-1}/\mathbf{E}(\cdot)-\delta)\rceil}^{\lfloor n(t_i/\mathbf{E}(\cdot)+\delta)\rfloor} W_j \geq na_i\right) + \limsup_{n\to\infty}\frac{\log(2\delta n)^k}{n}$$

$$\leq -B_0 \sum_{i=1}^{k} (a_i - (t_i + 2\delta\,\mathbf{E}(\cdot)))_+.$$

Taking $\delta \to 0$, we arrive at

$$\limsup_{n\to\infty}\frac{1}{n}\log(II) \leq -B_0 \sum_{i=1}^{k}(a_i - t_i)_+. \tag{3.31}$$

In view of (3.30) and (3.31),

$$\limsup_{n\to\infty}\frac{1}{n}\log \mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j \geq na_1, \ldots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j \geq na_k\right)$$

$$\leq \max\left(\limsup_{n\to\infty}\frac{\log(I)}{n}; \limsup_{n\to\infty}\frac{\log(II)}{n}\right)$$

$$\leq -B_0\sum_{i=1}^{k}(a_i - t_i)_+.$$

For the lower bound in Equation (3.29), notice that

$$
\begin{aligned}
&\mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j > na_1, \ldots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j > na_k \right) \\
\geq\ & \mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j > na_1, \ldots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j > na_k, \right. \\
&\qquad\qquad\qquad \left. N(nt_i) \in E_i^{(n)}(\delta) \text{ for } i = 1, \ldots, k \right) \\
\geq\ & \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E - \delta)\rfloor - 1} W_j > na_1, \ldots, \sum_{j=\lceil n(t_{k-1}/E + \delta)\rceil}^{\lfloor n(t_k/E - \delta)\rfloor - 1} W_j > na_k, \right. \\
&\qquad\qquad\qquad \left. N(nt_i) \in E_i^{(n)}(\delta) \text{ for } i = 1, \ldots, k \right) \\
\geq\ & \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E - \delta)\rfloor - 1} W_j > na_1, \ldots, \sum_{j=\lceil n(t_{k-1}/E + \delta)\rceil}^{\lfloor n(t_k/E - \delta)\rfloor - 1} W_j > na_i \right) - (I) \\
=\ & \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E - \delta)\rfloor - 1} W_j > na_1 \right) \prod_{i=2}^{k} \mathbf{P}\left(\sum_{j=\lceil n(t_{i-1}/E + \delta)\rceil}^{\lfloor n(t_i/E - \delta)\rfloor - 1} W_j > na_i \right) - (I) \\
=\ & \underbrace{\mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_1/E - \delta)\rfloor - 1} W_j > na_1 \right) \prod_{i=2}^{k} \mathbf{P}\left(\sum_{j=1}^{\lfloor n(t_i/E - \delta)\rfloor - \lceil n(t_{i-1}/E + \delta)\rceil} W_j > na_i \right)}_{=(III)} \\
&- (I). \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.32)
\end{aligned}
$$

From Theorem 3.4.3, Result 3.2.1 and (3.30), we get $\frac{(I)}{(III)} \to 0$ as $n \to \infty$. Therefore, (3.32) leads to

$$
\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(\sum_{j=1}^{N(nt_1)-1} W_j > na_1, \ldots, \sum_{j=N(nt_{k-1})}^{N(nt_k)-1} W_j > na_k \right) \\
\geq\ & \liminf_{n\to\infty} \frac{1}{n} \log \left((III) \left(1 - \frac{(I)}{(III)} \right) \right) = \liminf_{n\to\infty} \frac{1}{n} \log (III)
\end{aligned}
$$

$$= B_0 \sum_{i=1}^{k} (a_i - (\mu t_i - 2\epsilon\mathbf{E}))_+.$$

Taking $\epsilon \to 0$, we arrive at (3.29) concluding the proof.

□

Lemma 3.5.11. *For any given $\mathbf{t} = (t_1, \dots, t_k)$ such that $0 = t_0 \le t_1 < \dots < t_k \le 1$, the probability measures μ_n of $\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)} W_j, \dots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right)$ satisfy the LDP in $\mathbb{R}_+^k$ w.r.t. Euclidean topology with speed n and the good rate function $I_{\mathbf{t}} : \mathbb{R}_+^k \to \mathbb{R}_+$:*

$$I_{\mathbf{t}}(x_1, \dots, x_k) = \begin{cases} B_0 \sum_{i=1}^{k} (x_i - \mu t_i) & \text{if } x_i \ge \mu t_i, \ \forall i = 1, \dots, k; \\ \infty; & \text{otherwise.} \end{cases} \tag{3.33}$$

Proof. We claim that $\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \dots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right)$ satisfies a weak LDP. Once our claim is established, since $I_{\mathbf{t}}$ is a good rate function, and $\mathbb{R}_+^k$ is Polish, $\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \dots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right)$ is exponentially tight, and consequently, Lemma 1.2.18 of [22] applies, showing that the full LDP is satisfied. Now, to prove the claimed weak LDP, we start by showing that

$$\underbrace{\limsup_{n\to\infty} \frac{\log \mu_n(A)}{n}}_{\triangleq \overline{L}_A} = \underbrace{\liminf_{n\to\infty} \frac{\log \mu_n(A)}{n}}_{\triangleq \underline{L}_A} \tag{3.34}$$

for every $A \in \mathcal{A} \triangleq \{\prod_{i=1}^{k} (a_i, b_i) \cap \mathbb{R}_+^k : a_i < b_i\}$. Let

$$L_A \triangleq \begin{cases} B_0 \sum_{i=1}^{k} (a_i - \mu t_i)_+ & \text{if } b_i \ge \mu t_i \text{ for } i = 1, \dots, k; \\ \infty & \text{otherwise.} \end{cases}$$

We will prove (3.34) by showing that $\overline{L}_A \le -L_A \le \underline{L}_A$. We consider the two cases separately:

case 1. $b_i \ge \mu t_i$ for $i = 1, \dots, k$;

case 2. $b_i < \mu t_i$ for some $i \in \{1, \dots, k\}$.

Let $A = \prod_{i=1}^k (a_i, b_i) \cap \mathbb{R}_+$ and $a_i < b_i$ for $i = 1, \ldots, k$. We start with case 1. Since $A \subset \prod_{i=1}^k [a_i, b_i)$,

$$\overline{L}_A \le \limsup_{n \to \infty} \frac{1}{n} \log \mathbf{P}\left(\sum_{j=1}^{N(nt_1)} W_j \ge na_1, \ldots, \sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j \ge na_k \right)$$

$$\le -B_0 \sum_{i=1}^k (a_i - \ t_i) = -L_A, \tag{3.35}$$

where the second inequality is from (3.28). Since $\prod_{i=1}^k [a_i + \epsilon, b_i) \subset A$ for small enough $\epsilon > 0$,

$$\begin{aligned} \underline{L}_A & \\ &\ge \liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right) \in \prod_{i=1}^k [a_i + \epsilon, b_i)\right) \\ &\ge \liminf_{n\to\infty} \frac{1}{n} \log \left\{ \mathbf{P}\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j > a_1 + \epsilon, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j > a_k + \epsilon\right) \right. \\ &\quad \left. - \sum_{l=1}^k \mathbf{P}\left(\sum_{j=N(nt_{i-1})+1}^{N(nt_i)} W_j \ge na_i \ \forall i \ne l, \sum_{j=N(nt_{l-1})+1}^{N(nt_l)} W_j \ge nb_l\right) \right\} \\ &\ge \liminf_{n\to\infty} \frac{1}{n} \log \left(\mathbf{P}\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j > a_1 + \epsilon, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j > a_k + \epsilon\right) \right. \\ &\quad \left. \left(1 - \frac{\sum_{l=1}^k \mathbf{P}\left(\frac{1}{n}\sum_{j=N(nt_{i-1})+1}^{N(nt_i)} W_j \ge a_i + \epsilon \ \forall i \ne l, \frac{1}{n}\sum_{j=N(nt_{l-1})+1}^{N(nt_l)} W_j \ge b_l\right)}{\mathbf{P}\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)} W_j > a_1 + \epsilon, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j > a_k + \epsilon\right)}\right)\right). \end{aligned} \tag{3.36}$$

Note that due to the logarithmic asymptotics of Lemma 3.5.10, for every $l \in \{1, \ldots, k\}$,

$$\frac{\mathbf{P}\left(\frac{1}{n}\sum_{j=N(nt_{i-1})+1}^{N(nt_i)} W_j \ge a_i + \epsilon \text{ for } i \ne l, \ \frac{1}{n}\sum_{j=N(nt_{j-1})+1}^{N(nt_l)} W_j \ge b_l\right)}{\mathbf{P}\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)} W_j > a_1 + \epsilon, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j > a_k + \epsilon\right)} \to 0,$$

and hence, the second term of (3.36) disappears. Therefore,

$\underline{L}_A$

$$\liminf_{n\to\infty} \frac{\log \mathbf{P}\left(\frac{1}{n}\sum_{j=1}^{N(nt_1)} W_j > a_1 + \delta, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j > a_k + \delta\right)}{n}$$
$$\geq -B_0 \sum_{i=1}^{k} (a_i + \delta - \mu t_i) .$$

Taking $\delta \to 0$, we arrive at $\underline{L}_A \geq L_A$, which, together with (3.35), proves (3.34) for case 1.

For case 2, note that by Result 3.2.1,

$$\overline{L}_A \leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(\sum_{j=N(nt_{i-1})+1}^{N(nt_i)} W_j < nb_i\right) = -\infty;$$

and hence, $\overline{L}_A = \underline{L}_A = L_A = -\infty$.

Now note also that

$$I_{\mathbf{k}}(x_1, \ldots, x_k) = -\inf\{L_A : A \ni (x_1, \ldots, x_k)\}. \tag{3.37}$$

Since $\mathcal{A}$ is a base of the Euclidean topology, the desired weak LDP follows from (3.34), (3.37), and Theorem 4.1.11 of [22]. □

The following is an immediate Corollary of Lemma 3.5.11.

Lemma 3.5.12. *For any given* $\mathbf{t} = (t_1, \ldots, t_k)$ *such that* $0 = t_0 \leq t_1 < \cdots < t_k \leq 1$*, the probability measures* (μ_n) *of* $\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=0}^{N(nt_k)} W_j\right)$ *satisfy an LDP in* $\mathbb{R}_+^k$ *with speed* n *and with good rate function,* $\tilde{I}_{\mathbf{t}} : \mathbb{R}_+^k \to \mathbb{R}_+$,

$$\tilde{I}_{\mathbf{t}}(x_1, \ldots, x_k) = \begin{cases} B_0 \sum_{i=1}^{k} (x_i - x_{i-1} - \mu t_i) & \text{if } x_i - x_{i-1} \geq \mu t_i \\ & \text{for } i = 1, \ldots, k; \\ \infty; & \text{otherwise.} \end{cases} \tag{3.38}$$

Proof. The proof is an application of the contraction principle. To this end, consider the function $f : \mathbb{R}_+^k \to \mathbb{R}_+^k$, $f(x_1, x_2, \ldots, x_k) = (x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_k)$. Notice that

$$\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=0}^{N(nt_k)} W_j\right) = f\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \ldots, \frac{1}{n}\sum_{j=N(nt_{k-1})+1}^{N(nt_k)} W_j\right);$$

where f is a continuous function. That is $\left(\frac{1}{n}\sum_{j=0}^{N(nt_1)} W_j, \dots, \frac{1}{n}\sum_{j=0}^{N(nt_k)} W_j\right)$ satisfies a large deviation principle with the rate function

$$\tilde{I}_{\mathbf{t}}(y_1,\dots,y_k) = \inf\{I_{\mathbf{t}}(x) : y = f(x_1,\dots,x_k)\}.$$

Since $(y_1,\dots,y_k) = f(x_1,\dots,x_k)$, it is immediate that $y_1 \le y_2 \le \dots \le y_k$. Therefore,

$$\tilde{I}_{\mathbf{t}}(y_1,\dots,y_k) = \begin{cases} B_0 \sum_{i=1}^{k}(y_i - y_{i-1} - \mu \Delta t_i), & \text{if } y_{i+1} - y_i \ge \mu \Delta t_i \\ & \text{for } i = 1,\dots,k; \\ \infty, & \text{otherwise.} \end{cases}$$

□

Now, for a path $\phi \in D[0,1]$ let

$$I(\phi) = \begin{cases} B_0 \sum_{t:\, \phi(t) \ne \phi(t-)} (\phi(t) - \phi(t-)), & \text{for } \phi \in D^{(\mu)}[0,1]; \\ \infty & \text{otherwise.} \end{cases}$$

Since Z_n satisfies a finite-dimensional LDP, the Dawson and Gärtner projective limit theorem implies that Z_n obeys a sample path LDP in $D[0,1]$ endowed with the pointwise convergence topology. The next lemma verifies that the rate function associated with the LDP of Z_n, is indeed I.

Lemma 3.5.13. *Let $\mathbf{T} = \{(t_1,\dots,t_k) \in [0,1]^k : k \ge 1\}$ be the collection of all ordered finite subsets of $[0,1]$. Then*

$$\sup_{\mathbf{t} \in \mathbf{T}} \tilde{I}_{\mathbf{t}}(\phi) = I(\phi).$$

Proof. This proof is essentially identical to the proof of Lemma 4 of [40] and hence omitted. □

We derive the sample path LDP for the stochastic process Z_n w.r.t. the pointwise convergence topology, which we denote with W. Recall that $D^{(\mu)}[0,1]$ denotes the subspace of increasing piecewise linear jump functions with slope μ.

Lemma 3.5.14. *The stochastic process Z_n satisfies a large deviation principle in $(D[0,1], W)$, with speed n and good rate function $I_Z : D \to \mathbb{R}_+$ where*

$$I_Z(\phi) = \begin{cases} B_0 \sum_{t:\, \phi(t) \ne \phi(t-)} (\phi(t) - \phi(t-)) & \text{if } \phi \in D^{(\mu)}[0,1]; \\ \infty & \textit{otherwise.} \end{cases} \tag{3.39}$$

Proof. The proof is an immediate consequence of the Dawson and Gärtner's projective limit theorem, (Theorem 4.6.1 of [22]), and Lemma 3.5.13. □

Next, we establish the sample-path LDP for the stochastic process Z_n in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$.

3.5.5 Proof of Lemma 3.3.3

Proof of Lemma 3.3.3. For the upper bound, consider the following set $K_M \triangleq \{\xi \in \mathbb{D}[0,1] : \xi \text{ is nondecreasing}, \ \xi(0) \ge 0, \ \|\xi\|_\infty \le M\}$. Let F be a closed set in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$. Then,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(Z_n \in F\right) \\
&\quad \le \limsup_{n\to\infty} \frac{1}{n} \log \left(\mathbf{P}\left(Z_n \in F \cap K_M\right) + \mathbf{P}\left(Z_n \in K_M^c\right)\right) \\
&\quad \le \limsup_{n\to\infty} \frac{1}{n} \log \left(\mathbf{P}\left(Z_n \in F \cap K_M\right) + \mathbf{P}\left(\sum_{j=1}^{N(nt)} W_j \ge M\right)\right).
\end{aligned}$$

From Proposition 2.5.8 of Chapter 2, one can check that point-wise convergence in K_M implies the convergence w.r.t. the M_1' topology, and K_M (and hence $F \cap K_M$ as well) is closed w.r.t. $\mathcal{T}_{M_1'}$. Suppose that ξ is in the closure of $F \cap K_M$ w.r.t. $\mathcal{W}$. Then, because of the above mentioned properties of K_M, there exists a sequence of paths $\{\xi_n\}$ in $F \cap K_M$ such that $\xi_n \to \xi$ w.r.t. $\mathcal{T}_{M_1'}$, which, in turn, implies that $\xi \in F \cap K_M$. That is, $F \cap K_M$ is closed in $\mathcal{W}$ as well. Now, applying the sample-path LDP w.r.t. $\mathcal{W}$ we have proved in the above lemma, and then picking M large enough,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(Z_n \in F\right) &\le \max\left\{-\inf_{\xi \in F \cap K_M} I_Z(\xi), \ -B_0 M\right\} \\
&= -\inf_{\xi \in F \cap K_M} I_Z(\xi) \\
&\le -\inf_{\xi \in F} I_Z(\xi).
\end{aligned}$$

Moving on to the lower bound, let G be an open set in $(\mathbb{D}[0,1], \mathcal{T}_{M_1'})$. We assume that $I(G) < \infty$ since we have nothing to show otherwise. Fix an arbitrary $\xi \in G \cap \mathbb{D}^{(\epsilon)}[0,1]$, and let k be such that an open ball of radius $\frac{1+\epsilon}{k}$ around ξ is inside of G; that is,

$$B_{M_1'}\left(\xi; \frac{1+\epsilon}{k}\right) \triangleq \left\{\zeta \in \mathbb{D}[0,1] : d_{M_1'}(\zeta, \xi) < \frac{1+\epsilon}{k}\right\} \subseteq G.$$

Note that since $\xi \in \mathbb{D}^{(\ )}[0,1]$ and Z_n is non-decreasing,

$$\{|Z_n(i/k) - \xi(i/k)| < 1/k; \text{ for } i = 0,\dots,k\} \subseteq \{Z_n \in B_{M_1'}(\xi; \tfrac{1+\ }{k})\}.$$

Therefore, in view of Lemma 3.5.12,

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(Z_n \in G \right) \\
&\quad \ge \liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(Z_n \in B_{M_1'}\left(\xi; \frac{1+\ }{k}\right)\right) \\
&\quad \ge \liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(|Z_n(i/k) - \xi(i/k)| < 1/k; \text{ for } i = 0,\dots,k \right) \\
&= - \inf_{(y_1,\dots,y_k) \in \prod_{i=1}^{k} (\xi(i/k) - 1/k,\ \xi(i/k)+1/k)} \Gamma_k(y_1,\dots,y_k) \\
&\ge - B_0^{(p)} \sum_{i=1}^{k} \left(\xi(i/k) - \xi((i-1)/k) \right)^{\ } / k \\
&\ge - B_0^{(p)} \sum_{t:\ \xi(t) \ne \xi(t-)} \left(\xi(t) - \xi(t-) \right)^{\ } = - I_Z(\xi).
\end{aligned}$$

Since ξ was an arbitrary element of $G \cap \mathbb{D}^{(\ )}[0,1]$, we arrive at the desired large deviation lower bound:

$$-\inf_{\xi\in G} I_Z(\xi) = -\inf_{\xi \in G \cap \mathbb{D}^{(\ )}[0,1]} I_Z(\xi) \le \liminf_{n\to\infty} \frac{1}{n} \log \mathbf{P}\left(Z_n \in G \right).$$

□

3.5.6 Proof of Lemma 3.3.2, and 3.3.4

We start with the proof of Lemma 3.3.2. To this end, define $\mathbb{D}^{\le 1}[0,1] \triangleq \{\xi \in \mathbb{D}[0,1] : \xi = x\mathbb{1}_{\{1\}}$ for some $x \ge 0\}$ and recall the definition of $S_n = V_n \mathbb{1}_{\{1\}}(t)$ and $V_n = \frac{1}{n}\sum_{i=T_{N(n)}+1}^{n} f(X_i)$.

Proof of Lemma 3.3.2. Define a function $T : \mathbb{R}_+ \to \mathbb{D}^{\le 1}[0,1]$ as $T(x) \triangleq x\mathbb{1}_{\{1\}}$. Then, $S_n = T(V_n)$ and T is a continuous function w.r.t. the M_1' topology. Therefore, the desired LDP follows from the contraction principle if we prove that V_n satisfies an LDP in $\mathbb{R}_+$ with sub-linear speed $n^{\ }$ and the rate function $I_V : \mathbb{R}_+ \to \mathbb{R}_+$ where $I_V(x) = B_0\, x^{\ }$. To prove the LDP for V_n, note first that since $\mathbf{P}(V_n \in \cdot)$ is exponentially tight (w.r.t. the speed n) from Theorem 3.4.4, it is enough to establish the weak LDP. For the weak LDP, we start with showing that for any $a, b \in \mathbb{R}$, $B \triangleq (a, b) \cap \mathbb{R}_+$ satisfies

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(V_n \in B)}{n} = \liminf_{n\to\infty} \frac{\log \mathbf{P}(V_n \in B)}{n}.$$

Since this holds trivially if $b \le 0$ or $a \ge b$, we assume that $0 \vee a < b$. Note that from Theorem 3.4.4, $\mathbf{P}(V_n \ge b)/\mathbf{P}(V_n \ge 0 \vee a) \to 0$. Therefore,

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(V_n \in B)}{n} \le \limsup_{n\to\infty} \frac{\log \mathbf{P}(V_n \ge 0 \vee a)}{n} \le -B_0(0 \vee a).$$

Similarly, for small enough $\epsilon > 0$,

$$\begin{aligned} \liminf_{n\to\infty} \frac{\log \mathbf{P}(V_n \in B)}{n} &\ge \liminf_{n\to\infty} \frac{\log\left(\mathbf{P}(V_n \ge 0 \vee a + \epsilon)\left(1 - \frac{\mathbf{P}(V_n \ge b)}{\mathbf{P}(V_n \ge 0 \vee a + \epsilon)}\right)\right)}{n} \\ &= \liminf_{n\to\infty} \frac{\log \mathbf{P}(V_n \ge 0 \vee a + \epsilon)}{n} = -B_0(0 \vee a + \epsilon). \end{aligned}$$

Taking $\epsilon \to 0$, we see that the limit supremum and the limit infimum coincide. Since $\mathcal{C} = \{(a, b) \cap \mathbb{R}_+ : a, b \in \mathbb{R}, a < b\}$ forms a base of the Euclidean topology on $\mathbb{R}_+$, Theorem 4.1.11 of [22] applies, and hence, proves the desired weak LDP. This concludes the proof. □

Now, we focus on the exponential equivalence of Y_n and $Z + S_n$.

Proof of Lemma 3.3.4. Fix an $\epsilon > 0$, and define $D_n(\epsilon) = \{N(n)/n \ge 1/\mathbf{E}\tau - \epsilon\}$. Due to the construction of Y_n, Z_n, and S_n, we have that for any $\delta > 0$,

$$\{d_{M_1'}(Y_n, Z_n + S_n) \ge \delta\} \subseteq \{(n - T_{N(n)})/n \ge \delta\} \cup \{\exists j \le N(n) : \zeta_j \ge n\delta\}. \tag{3.40}$$

To bound the probability of the first set, note that

$$\begin{aligned} \mathbf{P}\big((n - T_{N(n)})/n > \delta\big) &= \mathbf{P}\big(T_{N(n)} < n(1-\delta)\big) \\ &\le \mathbf{P}\big(T_{N(n)} < n(1-\delta); D_n(\epsilon)\big) + \mathbf{P}\big(T_{N(n)} < n(1-\delta); D_n(\epsilon)^c\big). \end{aligned}$$

Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbf{P}\big((n - T_{N(n)})/n$$

$$\limsup_{n \to \infty} \frac{1}{n} \log \left(\mathbf{P} \left(T_{N(n)} \quad n(1 \quad); D_n(\ ) \right) + \mathbf{P}\left(D_n(\ )^c\right) \right)$$
$$= \limsup_{n \to \infty} \frac{1}{n} \log \mathbf{P} \left(T_{N(n)} \quad n(1 \quad); D_n(\ ) \right) \vee \limsup_{n \to \infty} \frac{1}{n} \log \mathbf{P}\left(D_n(\ )^c\right). \qquad (3.41)$$

Let $\ < \ =(2\mathbf{E}\ )$, then,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbf{P} \left(T_{N(n)} \quad n(1 \quad); D_n(\ ) \right)$$
$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbf{P} \left(T_{bn(\frac{1}{\mathbf{E}} \quad)c} \quad n(1 \quad); D_n(\ ) \right)$$
$$= \limsup_{n \to \infty} \frac{1}{n} \log \mathbf{P} \left(N(n(1 \quad)) \quad n \left(\frac{1}{\mathbf{E}} \quad \right); D_n(\ ) \right)$$
$$= -\infty.$$

Using the definition of a renewal process and Cramer's theorem we obtain

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbf{P}\left(D_n(\ )^c\right) = -\infty. \qquad (3.42)$$

Therefore,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbf{P} \left((n \quad T_{N(n)})/n > \ \right) = -\infty. \qquad (3.43)$$

Moving on to the bound for the probability of the second term in (3.40), for any $\ > 0$,

$$\mathbf{P}\left(\{\exists j \quad N(n) : \ _j \quad n\ \}\right)$$
$$= \mathbf{P}\left(\exists j \quad N(n) : \ _j \quad n\ ; N(n)/n \quad 1/\mathbf{E}\ +\ \right) + \mathbf{P}\left(N(n)/n > 1/\mathbf{E}\ +\ \right)$$
$$\mathbf{P}\left(\exists j \quad dn/\mathbf{E}(\ ) + n\ e : \ _j \quad n\ \right) + \mathbf{P}\left(N(n)/n > 1/\mathbf{E}\ +\ \right)$$
$$dn/\mathbf{E}(\ ) + n\ e\,\mathbf{P}\left(\ _1 \quad n\ \right) + \mathbf{P}\left(N(n)/n > 1/\mathbf{E}\ +\ \right).$$

Since $\mathbf{P}(\ _1 \quad n\ )$ and $\mathbf{P}(N(n)/n > 1/\mathbf{E}\ +\ )$ decay at exponential rate,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbf{P}\left(\{\exists j \quad N(n) : \ _j \quad n\ \}\right) = -\infty.$$

This along with (3.43) and (3.40) proves the desired exponential equivalence. □

Chapter 4

Asymptotics for the multiple server queue

4.1 Introduction

The queue with multiple servers, known as the $GI/GI/d$ queue, is a fundamental model in queueing theory. Its use in everyday applications such as call centers and supermarkets is well documented and, despite being significantly studied over decades, it continues to pose interesting research challenges. Early work [78, 51] focused on exact analysis of the invariant waiting-time distribution but finding tractable solutions has turned out to be challenging. This has led to lines of research that focus on approximations, either considering heavily-loaded systems [46, 73] or investigating the frequency of rare events, e.g. the probability of a long waiting time or large queue length. For light-tailed service times, such problems have been considered in [86, 79].

In this chapter we focus on rare event analysis of the queue length in the case of heavy-tailed service times, a topic that is more recent. For a single server, the literature on this topic is extensive, as there is an explicit connection between waiting times and first passage times of random walks, a textbook treatment can be found in [33]. Tail asymptotics for the steady-state queue length has been treated in [30].

One of the earliest works on heavy tails in the setting of a queue with multiple servers is [95], in which there is a conjecture regarding the form of the tail of the waiting time distribution in steady state, assuming that the service time distribution is sub-exponential. This has led to follow-up work on necessary and sufficient conditions for finite moments of the waiting time distribution in steady state [89], and on tail asymptotics [31, 32]. Most of the results in the latter two papers focus on the case of regularly varying service times. An insight is that, if the system load ρ is not an integer, a large waiting time occurs due to the arrival of $d - \lceil \rho \rceil$ big jobs. The case of other heavy-tailed service times is poorly understood.

We assume that the service time distribution has a tail of the form $e^{-L(x)x^{\alpha}};$ where $\alpha \in (0;1);$ and L is a slowly varying function (a more comprehensive definition is given later on). Tail distributions of this form are also known as semi-exponential. Their analysis poses challenges as this category of tails falls in between the Pareto (very heavy tailed) case, and the classical light-tailed case. In particular, in the case of $d = 2$ and $\rho < 1$, the results in [31] imply that two big jobs are necessary to cause a large waiting time when service times have a Weibull distribution. The arguments in [31] cannot be extended to the case $\rho > 1$. In the 2009 Erlang centennial conference, Sergey Foss posed the question "how many big service times are needed to cause a large waiting time to occur, if the system is in steady state?". He noted that even a physical or heuristic treatment has been absent.

In this chapter we investigate a strongly related question, namely we analyze the event that the queue length $Q(\gamma n)$ at a large time γn exceeds a value n. A key result that we utilize in our analysis is a powerful upper bound of Gamarnik and Goldberg, see [38], for $\mathbf{P}(Q(t) > x)$. This upper bound can be combined with the large deviations principle for random walks with heavy-tailed Weibull-type increments (see Chapter 2), which is another key result that we use. Consequently, we can estimate the probability of a large queue length of the $GI/GI/d$ queue with heavy-tailed Weibull-type service times and obtain physical insights about "the most likely way" in which a large queue length builds up.

The main result of this chapter, given in Theorem 4.3.1, states the following. If $Q(t)$ is the queue length at time t (assuming an empty system at time zero) and $\gamma \in (0; \infty)$, then

$$\lim_{n \to \infty} \frac{\log \mathbf{P}(Q(\gamma n) > n)}{L(n)n^{\alpha}} = -c^{*}; \tag{4.1}$$

with c^{*} the value of the optimization problem

$$\min_{x_1;\ldots;x_d} \sum_{i=1}^{d} x_i^{\alpha} \quad \text{subject to} \tag{4.2}$$

$$\sup_{s \in [0, \;]} \left\{ s - \sum_{i=1}^{d} (s - x_i)^+ \right\} \geq 1;$$

$$x_1, \ldots, x_d \geq 0;$$

where is the arrival rate, and service times are normalized to have unit mean. Note that this problem is equivalent to an L -norm minimization problem with $\in (0, 1)$. Such problems also appear in applications such as compressed sensing, and are strongly NP-hard in general, see [41] and references therein. In our particular case, we can analyze this problem exactly, and if $1/(\; b \; c)$, the solution takes the simple form

$$c = \min_{l \in \{0, \ldots, \lfloor \; \rfloor\}} (d \; l) \; \frac{1}{l} \; . \tag{4.3}$$

This simple minimization problem has at most two optimal solutions, which represent the most likely number of big jumps that are responsible for a large queue length to occur, and the most likely buildup of the queue length is through a linear path. For smaller values of , asymmetric solutions can occur, leading to a piecewise linear buildup of the queue length; this phenomenon is discussed further in the chapter.

Note that the intuition that the solution to (4.2) yields is qualitatively different from the case in which service times have a power law. In the latter case, the optimal number of big jobs equals the minimum number of servers that need to be removed to make the system unstable. In the Weibull-type case, there is a nontrivial trade-off between the *number* of big jobs and their *size*, and this trade-off is captured by (4.2) and (4.3).

Although we do not make these claims rigorous for $= \infty$ (which requires an interchange of limits argument), it makes a clear suggestion of what the tail behavior of the steady-state queue length should be. This can then be related to the steady-state waiting time distribution, and the original question posed by Foss, using the distributional Little's law.

As mentioned before, we obtain (4.1) by utilizing a tail bound for $Q(t)$, which is derived in [38]. This tail bound is given in terms of functionals of superpositions of renewal processes. We show that these functionals are (almost) continuous in the M_1' topology (in the sense of being amenable to the use of the extended contraction principle). The M_1' topology is precisely the topology used in the development of the large deviations principle for random walks with Weibull-type increments; see Chapter 2. So, our approach here makes the new large deviations principle directly applicable.

The chapter is organized as follows. Section 4.2 provides a model description and some useful tools used in our proofs. Section 4.3 provides our main result and some mathematical insights associated with it. Section 4.4 contains the lemmas needed to construct the main result of this paper, Theorem 4.3.1, along with its proof. In Section 4.5, we present an explicit computation of the decay rate associated with large queue length build ups. Finally, Section 4.6 contains technical proofs.

4.2 Model description and preliminary results

We consider the FCFS $GI/GI/d$ queuing model with d servers in which inter-arrival times are independent and identically distributed (i.i.d.) random variables (r.v.'s) and service times are i.i.d. r.v.'s independent of the arrival process. Let $A \geq 0$ and $S \geq 0$ be a pair of generic inter-arrival and service time, respectively. We introduce the following assumptions:

Assumption 4.2.1. There exists $\theta_+ > 0$ such that $\mathbf{E}(e^{\theta A}) < \infty$ for every $\theta \leq \theta_+$.

Assumption 4.2.2. $\mathbf{P}(S \geq x) = e^{-L(x)x^{\alpha}}$, $\alpha \in (0,1)$ where $L(\cdot)$ is a slowly varying function at infinity and $L(x)x^{\alpha-1}$ is eventually non-increasing.

Let $Q(t)$ denote the queue-length process at time t in the FCFS $GI/GI/d$ queuing system with inter-arrival times being i.i.d. copies of A and service times being i.i.d. copies of S. We assume that $Q(0) = 0$. The goal is to identify the limiting behavior of $\mathbf{P}(Q(\gamma n) > n)$ as $n \to \infty$ in terms of the distributions of A and S.

To simplify the notation, let $\lambda = 1/\mathbf{E}[A]$ and assume without loss of generality that $\mathbf{E}[S] = 1$. To ensure stability, let $\lambda < d$. Let M be the renewal process associated with A. That is,

$$M(t) = \inf\{s : A(s) > t\},$$

and $A(t) \triangleq A_1 + A_2 + \cdots + A_{\lfloor t \rfloor}$ where $A_1, A_2, \ldots$ are i.i.d. copies of A, and $A(0) = 0$. Similarly, for each $i = 1, \ldots, d$, let $S^{(i)}(t) \triangleq S_1^{(i)} + S_2^{(i)} + \cdots + S_{\lfloor t \rfloor}^{(i)}$ where $S_1^{(i)}, S_1^{(i)}, \ldots$ are i.i.d. copies of S, and $N^{(i)}$ be the renewal process associated with S. Let M_n and $N_n^{(i)}$ be scaled processes of M and $N^{(i)}$. More precisely, $M_n(t) = M(nt)/n$ and $N_n^{(i)}(t) = N^{(i)}(nt)/n$ for $t \geq 0$. Our analysis hinges on Corollary 1 of [38], which for the $GI/GI/d$ queue states the following result:

Result 4.2.1. *For all $x > 0$ and $t \geq 0$,*

$$\mathbf{P}(Q(t) > x) \leq \mathbf{P}\left(\sup_{0 \leq s \leq t} \left\{ (M(t) - M(t-s)) - \sum_{i=1}^{d} (N^{(i)}(t) - N^{(i)}(t-s)) \right\} > x \right). \quad (4.4)$$

Now, from (4.4), we conclude that for each $\tau \in (0, \infty)$

$$\mathbf{P}\left(Q(\tau n) > n\right) \leq \mathbf{P}\left(\sup_{0 \leq s \leq \tau} \left\{ M_n(\tau) - M_n(s) - \sum_{i=1}^{d} \left(N_n^{(i)}(\tau) - N_n^{(i)}(s)\right) \right\} \geq 1 \right). \quad (4.5)$$

Though this is only an upper bound, our main result implies that (4.5) is an asymptotically tight upper bound as $n \to \infty$. We establish this later on by deriving a lower bound with the same asymptotic behavior.

In view of the above, a natural way to proceed is to establish large-deviations principles for M_n and $N_n^{(i)}$, $i = 1, \ldots, d$. By deriving an LDP one can have an estimate of the magnitude of probabilities of rare events on an exponential scale: if the upper and lower bounds of the LDP match, then $\mathbf{P}(X_n \in G) \approx e^{-a_n \inf_{x \in G} I(x)}$. The optimizers of the infimum typically provide insight in the most likely way a rare event occurs (i.e. the conditional distribution given the rare event of interest). For more background we refer to [39] and [28]. An important factor in establishing an LDP on function spaces is the topology of the space under consideration. Let $\mathbb{D}[0, T]$ denote the Skorokhod space (i.e. the space of cadlag functions from $[0, T]$ to $\mathbb{R}$). We shall use $\mathcal{T}_{M_1'}$ to denote the M_1' Skorokhod topology on $\mathbb{D}[0, T]$, which is generated by a metric $d_{M_1'}$ defined in terms of the graphs induced by the elements of $\mathbb{D}[0, T]$. The precise definitions of the graph and the metric are the same as in the introduction of the thesis.

The continuity of certain maps w.r.t. the M_1' topology is a key component in our whole argument. Therefore, we note some important related properties used in our proofs. We refer to the following lemmas for these results.

Lemma 4.2.1. *For any $T > 0$,*

i) The functional $E : \mathbb{D}[0, T] \to \mathbb{R}$, where $E(\xi) = \xi(T)$ is continuous w.r.t. the M_1' topology on $\mathbb{D}[0, T]$.

ii) The functional $S : \mathbb{D}[0, T] \to \mathbb{R}$, where $S(\xi) = \sup_{t \in [0, T]} \xi(t)$ is continuous w.r.t. the M_1' topology on $\xi \in \mathbb{D}[0, T]$ such that $\xi(0) \geq 0$.

Lemma 4.2.2. *The map* $: \mathbb{D}[0; =] \; ! \; \mathbb{D}[0; =]$ *where* $( ) \, , \quad + \quad$ *is continuous w.r.t. the* M_1^0 *topology on* $\mathbb{D}[0; =]$.

Note, the addition map $(\, ; \,) \; 7! \quad + \quad$ is a continuous map w.r.t. the M_1^0 topology if the functions and do not have jumps of the opposite sign at the same jump times.

Now, we present a straightforward adaptation of Corollary 2.4.2 derived in Chapter 2 on sample path large deviations for random walks with heavy-tailed semi-exponential increments which constitutes an important cornerstone of our whole argument. We say that 2 D$[0; T]$ is a pure jump function if $= \sum_{i=1}^{1} x_i \mathbb{1}_{[u_i;T]}$ for some x_i's and u_i's such that $x_i \; 2$ R and $u_i \; 2 \; [0; T]$ for each i and u_i's are all distinct. Let $\mathbb{D}_p^{"}[0; T]$ be the subspace of $\mathbb{D}[0; T]$ consisting of non-decreasing pure jump functions that assume non-negative values at the origin.

Result 4.2.2. *Let* $S_n; n \quad 1$ *be a mean-zero random walk such that* $\mathbf{E}(e^{\quad S_1}) < 1$ *for some* > 0, $\mathbf{P}(S_1 \quad x) = e^{\quad L(x)x}$ *for some* $2 \; (0; 1)$, *and assume that* $L(x)x^{\quad 1}$ *is eventually non-increasing. Then,* S_n *satis es the LDP in* $(\mathbb{D}[0; T]; T_{M_1^0})$ *with speed* $L(n)n$ *and good rate function* $I_{M_1^0} : \mathbb{D}[0; T] \; ! \; [0; 1]$,

$$I_{M_1^0}(\;) \, , \quad \begin{cases} \sum_{t2[0;1]} (\; (t) \quad (t \quad)) & \text{if} \quad 2 \; \mathbb{D}_p^{"}[0; T]; \\ 1 & \text{otherwise:} \end{cases} \tag{4.6}$$

Note that M_n and $N_n^{(i)}$'s depend on a random number of A_j's and $S_j^{(i)}$'s, and hence may depend on an arbitrarily large number of A_j's and $S_j^{(i)}$'s. This does not exactly correspond to the large deviations framework presented in Result 3.2.1. To accommodate such a context, we introduce the following maps. Fix > 0. For any path , let $(\;)(t)$ denote the running supremum of up to time t:

$$(\;)(t) \, , \sup_{s2[0;t]} (s):$$

For each , de ne a map $:$ D$[0; =] \; ! \;$ D$[0; \;]$ as

$$(\;)(t) \, , \; ' \; (\;)(t) \wedge \quad (\;)(t);$$

where

$$' \; (\;)(t) \, , \; \inf fs \; 2 \; [0; =] : \; (s) > tg; \quad \text{and} \tag{4.7}$$

$$(\ )(t) \ , \ \frac{1}{\ } \quad + \ t \qquad (\ )(\ =\ )_{+} \ : \tag{4.8}$$

Here we denoted max$fx;0g$ with $[x]_+$. In words, between the origin and the supremum of , ( )(s) is the rst passage time of crossing the level s; from there to the nal point , ( ) increases linearly from = at rate 1= (instead of jumping to 1 and staying there). De ne A_n 2 D$[0; \ =\mathbf{E}A]$ as $A_n(t)$, $A(nt)=n$ for t 2 $[0; \ =\mathbf{E}A]$ and $S_n^{(i)}$ 2 D$[0; \ ]$ as $S_n^{(i)}(t)$, $S^{(i)}(nt)=n = \frac{1}{n} \sum_{j=1}^{bntc} S_j^{(i)}$ for t 2 $[0; \ ]$. In deriving LDPs for M_n and $N_n^{(i)}$, we use the fact that $_{\mathbf{E}A}(A_n)$ is a function of $fA_n(t) : t$ 2 $[0; \ =\mathbf{E}A]g$ (and hence, the LDP associated with it can be derived from the LDP we have for A_n) as well as the fact that $_{\mathbf{E}A}(A_n)$ is close enough to M_n so that they satisfy the same LDP. Similarly, we derive the LDP for $N_n^{(i)}$ from the LDP for $S_n^{(i)}$ using the fact that $_1(S_n^{(i)})$ is close enough to $N_n^{(i)}$ for our purpose.

4.3 Main result

Recall that $Q(t)$ denotes the queue length of the $GI/GI/d$ queue at time t.

Theorem 4.3.1. *For each* 2 $(0; 1\ )$, *it holds that*

$$\lim_{n\ !\ 1} \frac{1}{L(n)n} \log \mathbf{P}\,(Q(\ n) > n) = \quad c \ ;$$

where c *is de ned as follows: for* $\quad$ 1= , c *is equal to*

$$\min \left(\inf_{0<k \quad b \ c; \quad <1=(\quad k)} \Big\{ (d \quad k) \qquad + (1 \qquad + \ k) \ (k \quad b \qquad 1= \ c)^1 \Big\} \ ; \right. \tag{4.9}$$

$$\left. \min_{l=0}^{b \quad 1= \ c} (d \quad l) \ \frac{1}{\qquad l} \qquad \right) ;$$

while for $< 1=$, $c \ = \ 1$.

Theorem 4.3.1 is stated under the assumption that $\mathbf{E}S = 1$ for the sake of simplicity. Following a completely analogous argument with slightly more involved notations, one can obtain the following expression for c for the general case where $= 1/\mathbf{E}S \neq 1$:

$$\min_{\substack{0<k\ \ b\,=\,c \\ <1=(\quad k\ )}} \left((d \quad k) \quad + (1 \quad (\quad k\ )) \qquad k \right) = \frac{1}{\quad}^{1}$$

$$\wedge \min_{l=0}^{b\,=\,\frac{1}{\ }c} (d \quad l) \frac{1}{l} \quad :$$

Proof methodology

The proof of Theorem 4.3.1 is provided in Section 4.6 by implementing the following strategy:

1) We rst prove that A_n and $S_n^{(i)}; i = 1; : : : ; d$, satisfy certain LDPs in Proposition 4.4.1. The LDPs for the $S_n^{(i)}$'s are a consequence of Result 4.2.2, while the LDP of A_n is deduced by the sample-path LDP in [79].

2) We prove that $_{\mathbf{E}A}(\ )$ and $_1(\ )$ are essentially continuous maps | see Proposition 4.4.3 for the precise statement | and hence, $_{\mathbf{E}A}(A_n)$ and $_1(S_n^{(i)})$ satisfy the LDPs deduced by the extended contraction principle (cf.[79]).

3) We show that M_n and $N_n^{(i)}$ are equivalent to $_{\mathbf{E}A}(A_n)$ and $_1(S_n^{(i)})$, respectively, in terms of their large deviations (Proposition 4.4.2); so M_n and $N_n^{(i)}$ satisfy the same LDPs (Proposition 4.4.4).

4) By applying the contraction principle to the $N^{(i)}$'s with the continuous maps in Proposition 4.4.3, we infer the (logarithmic) asymptotic upper bound of $\mathbf{P}(Q(\ n) > n)$, which can be characterized by the solution of a (non-standard) variational problem. On the other hand, the lower bound is derived by keeping track of the optimal solution associated with the LDP upper bound. The complete argument is presented in Proposition 4.4.5.

5) We solve the variational problem in Proposition 4.5.1 to explicitly compute its optimal solution. The optimal solution of the variational problem provides the limiting exponent and information on the trajectory leading to a large queue length.

Discussion of the main result

In the remainder of this section, we further investigate properties of the solution of the optimization problem that defines c . In large-deviations theory, solutions of such problems are known to provide insights into the most likely way a specific rare event occurs. Such insights are physical, and more technical work is typically needed to make such insights rigorous; we refer to Lemma 4.2 of [39] for more background. The latter lemma can be applied in a relatively straightforward manner to derive a rigorous statement for the most likely way that the functional in the Gamarnik and Goldberg upper bound (cf. Result 4.4) becomes large. The computations below are mainly intended to provide physical insight, and highlight differences from the well-studied case where the job sizes follow a regularly varying distribution.

We consider two different cases based on the value of . If $< 1=$, no finite number of large jobs suffices, and we conjecture that the large deviations behavior is driven by a combination of light-tailed and heavy-tailed phenomena in which the light-tailed dynamics involve pushing the arrival rate by exponential tilting to the critical value 1= , followed by the heavy-tailed contribution evaluated as we explain in the following development. If $> 1=$, we observe the following features that come in contrast with the case of regularly varying service-time tails:

1. The large-deviations behavior may not be driven by the smallest number of jumps which drives the queueing system to instability (i.e. $dd \quad e$). In other words, in the Weibull setting, it might be more efficient to block more servers.

2. It is not necessary that the servers are blocked by the same amount i.e; the asymmetry in job sizes may be the most probable scenario in certain cases.

To illustrate the first point, assume that $> 1=(\quad b\ c)$, in which case $b\ c \quad b \quad 1= c:$ In that particular case, the first infimum in (4.9) is over an empty set and we interpret it as 1. So the optimal solution of c reduces to

$$\min_{l=0}^{b\ c} (d \quad l) \quad \frac{1}{\quad l} \quad :$$

Let l denote the index associated with the optimal value of the expression above. Intuitively, $d \quad l$ represents the optimal number of blocked servers so that the queue gets congested. Observe that $d \quad b \;\; c = dd \qquad e$ corresponds to the number of servers blocked in the regularly varying case. Note that if we examine

$$f\,(t) = (d \quad t)\,(\qquad t) \quad ;$$

for $t\ 2\ [0; b \;\; c]$, then the derivative $f\_(\ )$ is equal to

$$f\_(t) = \quad (d \quad t)\,(\qquad t) \qquad ^{1} \quad (\qquad t) \quad :$$

Hence,

$$f\_(t) < 0 \;\; (\;) \;\; t < \frac{d}{1};$$

and

$$f\_(t) > 0 \;\; (\;) \;\; t > \frac{d}{1};$$

with $f\_(t) = 0$ if and only if $t = (\qquad d) = (1 \qquad)$. This observation allows us to conclude that whenever $\quad > 1 = (\qquad b \;\; c)$ we can distinguish two cases. The rst one occurs if

$$b \;\; c \qquad \frac{d}{1};$$

in which case $l \quad = b \;\; c$. This case is qualitatively consistent with the way in which large deviations occur in the regularly varying case. On the other hand, if $b \;\; c > \frac{d}{1};$ then we must have that

$$l \quad = \quad \frac{d}{1} \quad \text{or } l \quad = \quad \frac{d}{1} \quad :$$

This case is the one highlighted in Feature 1 in which we may obtain $d \quad l \quad > dd \qquad e$ and thus more servers are blocked contrary to the large-deviations behavior observed in the regularly varying case. However, the blocked servers are symmetric in the sense that they are treated in exactly the same way.

In contrast, the second feature indicates that the typical trajectory leading to congestion may be obtained by blocking not only a speci c amount to drive the system to instability, but also by blocking the corresponding servers by di erent loads in the large deviations scaling. To appreciate this we must assume that

$$1 = \quad < \qquad 1 = (\qquad b \;\; c) :$$

In this case, the contribution of the in mum in (4.9) becomes relevant. To illustrate that we can obtain solutions satisfying the second feature; consider the case $d = 2$, $1 < \quad < 2$, and

$$1 = \quad < \quad < 1 = (\qquad 1) :$$

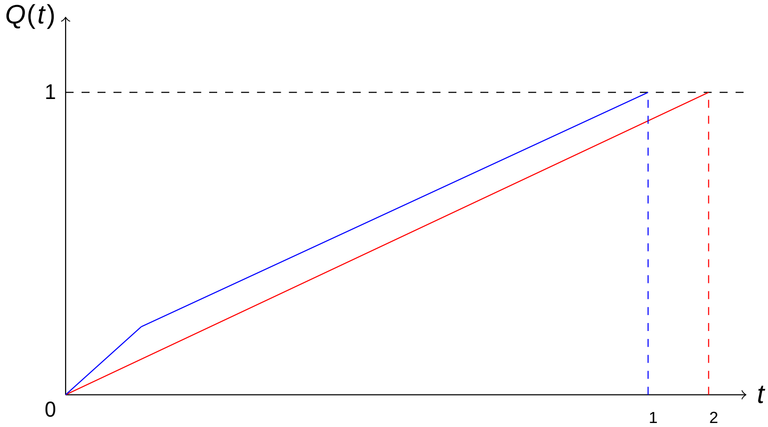

Figure 4.1: Most likely path for the queue build-up up to times $_1 = \frac{1}{\ \ 1}$ $0{:}1$ and $_2 = \frac{1}{\ \ 1}$ where the number of servers is $d = 2$, the arrival rate is $= 1{:}49$, and the Weibull shape parameter of the service time is $= 0{:}1$.

Choose $= 1{=}(\quad 1)$ and $= 2 \quad ^3$ for > 0 su ciently small, we derive

$$+ (1 \quad (\quad 1)) = 1 \quad + \quad + o \quad ^2 \quad 2^1 \quad ;$$

concluding that

$$+ (1 \quad (\quad 1)) < 2 \ \frac{1}{\ } \quad :$$

More explicitly consider the case $d = 2$, $= 1{:}49$, $= 0{:}1$ and $= \frac{1}{\ \ 1}$ $0{:}1$. For these values, $_1 + (1 \quad _1 (\quad 1)) < 2 \ \frac{1}{\ }$, and the most likely scenario leading to a large queue length is two big jobs arriving at the beginning and blocking both servers with di erent loads. On the other hand, if $= \frac{1}{\ \ 1}$, the most likely scenario is a single big job blocking one server. These two scenarios are illustrated in Figure 4.1.

We conclude by presenting a future research direction. We provide asymptotics only for the transient model of the queue length process Q. For a queue in steady state, more work is needed to overcome the technicalities arising with the large-deviations framework. Speci cally, one has to prove that the interchange of limits as and n tend to in nity,

$$\lim_{\ ! \ 1} \lim_{n ! \ 1} \frac{1}{L(n)n} \log \mathbf{P}(Q(\ n) > n) = \lim_{n ! \ 1} \frac{1}{L(n)n} \lim_{\ ! \ 1} \log \mathbf{P}(Q(\ n) > n);$$

is valid. We conjecture that the optimal value, similar to (4.9), of the variational problem associated with the steady state model will consist solely of the term, $\min_{l=0}^{\lfloor c \rfloor} \left\{ (d \quad l) \quad \frac{1}{l} \right\}$, obtained by taking $\;= 1$ in (4.2).

4.4 Proof of Theorem 4.3.1

We follow the general strategy outlined in the previous section. The rst step consists of deriving the LDP's for $A_n; S_n^{(i)}$ which subsequently provides us with the LDPs of M_n and $N_n^{(i)}$. Let $D_p^{"}[0; \;=\;]$ be the subspace of $D[0; \;=\;]$ consisting of non-decreasing pure jump functions that assume non-negative values at the origin, and de ne $\; 2\, D[0; \;=\;]$ as $\;(t) \;,\; t$. Let $D\;[0; \;=\;] \;,\; + D_p^{"}[0; \;=\;]$ the subspace of non-decreasing piecewise linear functions that have slope and assume non-negative values at the origin.

4.4.1 Intermediate propositions

Sample path LDPs for fundamental components of the queue length upper bound. Recall that $A_n(t) = \frac{1}{n}\sum_{j=1}^{\lfloor nt \rfloor} A_j$ and $S_n^{(i)}(t) = \frac{1}{n}\sum_{j=1}^{\lfloor nt \rfloor} S_j^{(i)}$.

Proposition 4.4.1. *A_n satis es the LDP on $D[0; \;=\mathbf{E}A]; d_{M_1^0}$ with speed $L(n)n$ and good rate function*

$$I_0(\;) = \begin{cases} 0 & \text{if } \;= \mathbf{E}A; \\ 1 & \text{otherwise;} \end{cases} \tag{4.10}$$

and $S_n^{(i)}$ satis es the LDP on $D[0; \;]; d_{M_1^0}$ with speed $L(n)n$ and the good rate function

$$I_i(\;) = \begin{cases} \sum_{t 2[0; \;]} (\;(t) \quad (t\;)) & \text{if } \; 2\, D^1[0; \;]; \\ 1 & \text{otherwise:} \end{cases}$$

Exponential equivalence of useful processes. To carry out the second step of our approach, we next prove that $_{\mathbf{E}A}(A_n)$ and $_1(S_n^{(i)})$ satisfy the same LDP's as M_n and $N_n^{(i)}$ for each $i = 1; : : : ; d$, respectively. To show this, we next prove that $_{\mathbf{E}A}(A_n)$ and $_1(S_n^{(i)})$ are exponentially equivalent to M_n and $N_n^{(i)}$ for each $i = 1; : : : ; d$, respectively.

Proposition 4.4.2. *M_n and $_{\mathbf{E}A}(A_n)$ are exponentially equivalent in* D[0;] *equipped with the $T_{M_1^0}$ topology and $N_n^{(i)}$ and $_1(S_n^{(i)})$ are exponentially equivalent in* D[0;]; $T_{M_1^0}$ *for each $i = 1;:::;d$.*

Due to the continuity of over the e ective domain of the rate functions $I_i; i = 1;:::;d$ — see step 2) of the methodology — we can appeal to the extended contraction principle — to establish LDP's for $_{\mathbf{E}A}(A_n)$ and $_1(S_n^{(i)})$ for each $i = 1;:::;d$. With the next proposition, we prove that the map is su ciently continuous for the application of the extended contraction principle. De ne

$$D \;, f \;2\; \mathrm{D}[0; \;=\;] : \quad (\;)(\;) \qquad (\;)(\quad) > 0 \text{ and } (0) \quad 0g:$$

Proposition 4.4.3. *For each 2* R, : D[0; =] ! D[0;] *is continuous on D^c w.r.t. the M_1^0 topology.*

Our next proposition, which constitutes the third step of our strategy, characterizes the LDPs satis ed by $_{\mathbf{E}A}(A_n)$ and $_1(S_n^{(i)})$ — and hence, by M_n and $N_n^{(i)}$ as well. De ne C [0;] , f 2 C[0;] : = ' ( ) for some 2 D [0; =]g where C[0;] is the subspace of D[0;] consisting of continuous paths, and $_s(\;) = \max\{0; \sup\{t \in [0; \;] : (t) = s\} \quad \inf\{t \in [0; \;] : (t) = s\}\}$.

Proposition 4.4.4. *$_{\mathbf{E}A}(A_n)$ and M_n satisfy the LDP with speed $L(n)n$ and the good rate function*

$$I_0^0(\;) \,, \begin{cases} 0 & \text{if } \;=\; _{1=\mathbf{E}A}; \\ 1 & \text{otherwise;} \end{cases}$$

and for $i = 1;::::;d$, $_1(S_n^{(i)})$ and $N_n^{(i)}$ satisfy the LDP with speed $L(n)n$ and the good rate function

$$I_i^0(\;) \,, \begin{cases} \sum_{s \in [0; \;]} \;_s(\;) & \text{if } \;\in \mathrm{C}^1[0; \;]; \\ 1 & \text{otherwise.} \end{cases}$$

4.4.2 Large deviations for the queue length

Now we are ready to follow step 4) of our outlined strategy and characterize the log asymptotics of $\mathbf{P}\; Q(\;n) > n$. Recall that

$$_s(\;) \,, \max\left\{0; \sup\{t \in [0; \;] : (t) = s\} \quad \inf\{t \in [0; \;] : (t) = s\}\right\}:$$

Proposition 4.4.5.

$$\lim_{n\to\infty}\frac{1}{L(n)n}\log\mathbf{P}\left(Q(\ n) > n\right) = \ c$$

where c is the solution of the following variational problem:

$$\inf_{1,\dots,d}\ \sum_{i=1}^{d}\ \sum_{s\in[0,\ ]}\ {}_s(\ _i) \qquad (4.11)$$

$$\text{subject to}\quad \sup_{0\ s}\ \frac{s}{\mathbf{E}A}\ \sum_{i=1}^{d}\ _i(s)\ \ 1;$$

$$_i \in C^1[0,\ ] \quad \text{for} \quad i = 1,\dots,d.$$

Proof. From Corollary 1 of [38], for any $\ > 0$,

$$\mathbf{P}\left(Q(\ n) > n\right)$$

$$\mathbf{P}\left(\sup_{0\ s}\left\{M_n(\ )\ \ M_n(s)\ \ \sum_{i=1}^{d}\ N_n^{(i)}(\ )\ \ N_n^{(i)}(s)\right\}\ 1\right)$$

$$\mathbf{P}\left(\sup_{0\ s}\left\{M_n(\ )\ \ M_n(s)\ \ \frac{s}{\mathbf{E}A}\right\}\right.$$

$$\left.+\sup_{0\ s}\left\{\frac{s}{\mathbf{E}A}\ \sum_{i=1}^{d}\ N_n^{(i)}(\ )\ \ N_n^{(i)}(s)\right\}\ 1\right)$$

$$\underbrace{\mathbf{P}\left(M_n(\ )\ \ \frac{}{\mathbf{E}A}\right)}_{(I)} + \underbrace{\mathbf{P}\left(\inf_{0\ s}\ M_n(s)\ \ \frac{s}{\mathbf{E}A}\right)}_{(II)}$$

$$+\underbrace{\mathbf{P}\left(\sup_{0\ s}\left\{\frac{s}{\mathbf{E}A}\ \sum_{i=1}^{d}\ N_n^{(i)}(\ )\ \ N_n^{(i)}(s)\right\}\ 1\ \ 2\ \right)}_{(III)}.$$

By the LDP for M_n (Proposition 4.4.4), it is straightforward to deduce that

$$\limsup_{n\to\infty}\frac{1}{L(n)n}\log\mathbf{P}\left(M_n(\ )\ \ \frac{}{\mathbf{E}A}\right) = \ \infty$$

and

$$\limsup_{n\to\infty}\frac{1}{L(n)n}\log\mathbf{P}\left(\inf_{0\ s}\ M_n(s)\ \ \frac{s}{\mathbf{E}A}\right) = \ \infty.$$

Therefore, by the principle of the maximum term

$$\limsup_{n\to\infty}\frac{\log \mathbf{P}\left(Q(\ \ n) > \ \ n\right)}{L(n)n}$$
$$\max\left\{\limsup_{n\to\infty}\frac{\log(\mathrm{I})}{L(n)n};\ \limsup_{n\to\infty}\frac{\log(\mathrm{II})}{L(n)n};\ \limsup_{n\to\infty}\frac{\log(\mathrm{III})}{L(n)n}\right\}$$
$$=\limsup_{n\to\infty}\frac{\log \mathbf{P}\left(\sup_{0\ \ s\ \ }\left\{\frac{s}{\mathbf{E}A}\ \ \sum_{i=1}^{d} N_n^{(i)}(\ \ )\ \ N_n^{(i)}(s)\right\}\ \ 1\ \ 2\right)}{L(n)n}.$$

To bound the limit supremum in the equality above, we derive an LDP for

$$\frac{\ }{\mathbf{E}A}\sum_{i=1}^{d} N_n^{(i)}(\ \ ) + \sup_{0\ \ s\ \ }\left(\sum_{i=1}^{d} N_n^{(i)}(s)\ \ \frac{s}{\mathbf{E}A}\right).$$

Due to Proposition 4.4.4 and Theorem 4.14 of [39], $(N_n^{(1)},\dots,N_n^{(d)})$ satisfy the LDP in $\prod_{i=1}^{d}\mathrm{D}[0,\ \ ]$ (w.r.t. the d-fold product topology of $T_{M_1'}$) with speed $L(n)n$ and rate function

$$I'(\ _1,\dots,\ _d) \triangleq \sum_{i=1}^{d} I_i'(\ _i).$$

Let $\mathrm{D}^{\uparrow}[0,\ \ ]$ denote the subspace of $\mathrm{D}[0,\ \ ]$ consisting of non-decreasing functions. Since $N_n^{(i)}\in\mathrm{D}^{\uparrow}[0,\ \ ]$ with probability 1 for each $i=1,\dots,d$, we can apply Lemma 4.1.5 (b) of [22] to deduce the same LDP for $(N_n^{(1)},\dots,N_n^{(d)})$ in $\prod_{i=1}^{d}\mathrm{D}^{\uparrow}[0,\ \ ]$. We define $f_1:\prod_{i=1}^{d}\mathrm{D}^{\uparrow}[0,\ \ ]\to\mathrm{D}[0,\ \ ]$ as

$$f_1(\ _1,\dots,\ _d) \triangleq \sum_{i=1}^{d}\ \ _i\ \ \ 1/\mathbf{E}A.$$

Note that f_1 is continuous since all the jumps are in one direction in its domain. Since the supremum functional $f_2:\ \ \mapsto\sup_{0\ \ s\ \ }\ \ (s)$ is continuous in the range of f_1 — see Lemma (4.2.1) — $f_2\circ f_1$ is a continuous map as well. The functional $f_3:\ \ \mapsto\ \ (\ \ )$ is also continuous w.r.t. the M_1' topology on $\mathrm{D}[0,\ \ ]$ due to Lemma (4.2.1). Therefore, the continuous map $f:\prod_{i=1}^{d}\mathrm{D}^{\uparrow}[0,\ \ ]\to\mathbb{R}$ where

$$f(\ _1,\dots,\ _d) \triangleq \frac{\ }{\mathbf{E}A}\sum_{i=1}^{d} f_3(\ _i) + f_2\circ f_1(\ _1,\dots,\ _d)$$

is continuous, and hence, we can apply the contraction principle with f to establish the LDP for

$$f(N_n^{(1)};\dots;N_n^{(d)}) = \frac{}{\mathbf{E}A}\sum_{i=1}N_n^{(i)}(\ ) + \sup_{0\ s}\sum_{i=1}N_n^{(i)}(s)\quad\frac{s}{\mathbf{E}A}\ :$$

The LDP is controlled by the good rate function

$$I''(x) \triangleq \inf\left\{ I'(\ _1;\dots;\ _d) : \frac{}{\mathbf{E}A}\sum_{i=1}\ _i(\ ) + \sup_{0\ s}\sum_{i=1}\ _i(s)\quad\frac{s}{\mathbf{E}A} = x\right\}:$$

Note that since $I'(\ ) = \infty$ for $\ \notin C^1[0;\ ]$, and $(\ )\in C^1[0;\ ]$ if and only if $(\ )\quad(\quad)\in C^1[0;\ ]$,

$$\begin{aligned}
I''(x)&\\
&= \inf\Bigg\{ I'(\ _1;\dots;\ _d) : \frac{}{\mathbf{E}A}\sum_{i=1}\ _i(\ )\\
&\qquad + \sup_{0\ s}\sum_{i=1}\ _i(s)\quad\frac{s}{\mathbf{E}A} = x;\ \ _i\in C^1[0;\ ]\Bigg\}\\
&= \inf\left\{ I'(\ _1;\dots;\ _d) : \sup_{0\ s}\left\{\frac{s}{\mathbf{E}A}\sum_{i=1}\ _i(\ )\quad\ _i(s)\right\} = x;\ \ _i\in C^1[0;\ ]\right\}\\
&= \inf\Bigg\{ I'(\ _1;\dots;\ _d) : \sup_{0\ s}\left\{\frac{s}{\mathbf{E}A}\sum_{i=1}\ _i(\ )\quad\ _i(\quad s)\right\} = x;\\
&\qquad \ _i\in C^1[0;\ ]\Bigg\}\\
&= \inf\left\{ I'(\ _1;\dots;\ _d) : \sup_{0\ s}\left\{\frac{s}{\mathbf{E}A}\sum_{i=1}\ _i(s)\right\} = x;\ \ _i\in C^1[0;\ ]\right\}:
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{\log\mathbf{P}(Q(\ n) > n)}{L(n)n}\\
&\quad\limsup_{n\to\infty}\frac{\log\mathbf{P}\ f(N_n^{(1)};\dots N_n^{(d)})\quad 1\quad 2}{L(n)n}\qquad\inf_{x\in[1\ \ 2\ ;\infty)} I''(x)
\end{aligned}$$

$$= \inf \left\{ \sum_{i=1}^{d} \sum_{s \in [0,\infty)} \ \cdot_s(\phi_i) \ : \ \sup_{0 \le s} \left(\frac{s}{\mathbf{E}A} - \sum_{i=1}^{d} \phi_i(s) \right) \ge 1 - 2\delta; \ \phi_i \in C^1[0,\infty) \right\} .$$

Taking $\delta \to 0$, we see that c is the upper bound for the left-hand-side.

We move on to the matching lower bound in case $\alpha > 1/\beta$. Considering the obvious coupling between Q and $(M; N^{(1)}; \cdots ; N^{(d)})$, one can see that $M(s) - \sum_{i=1}^{d} N^{(i)}(s)$ can be interpreted as (a lower bound of) the length of an imaginary queue at time s where the servers can start working on the jobs that have not arrived yet. Therefore, $\mathbf{P}(Q((a+s)n) > n) \ge \mathbf{P}(Q((a+s)n) > n | Q(a) = 0) \ge \mathbf{P}(M_n(s) - \sum_{i=1}^{d} N_n^{(i)}(s) > 1)$ for any $a \ge 0$. Let s^* be the level crossing time of the optimal solution of (4.11). Then, for any $\epsilon > 0$,

$$\begin{aligned} &\mathbf{P}(Q(\cdot n) > n) && (4.12)\\ &\ge \mathbf{P}\left(M_n(s^*) - \sum_{i=1}^{d} N_n^{(i)}(s^*) > 1 \right) \\ &\ge \mathbf{P}\left(M_n(s^*) - s^*/\mathbf{E}A > -\epsilon \text{ and } s^*/\mathbf{E}A - \sum_{i=1}^{d} N_n^{(i)}(s^*) > 1 + \epsilon \right) \\ &\ge \mathbf{P}\left(s^*/\mathbf{E}A - \sum_{i=1}^{d} N_n^{(i)}(s^*) > 1 + \epsilon \right) - \mathbf{P}\left(M_n(s^*) - s^*/\mathbf{E}A \le -\epsilon \right) . \end{aligned}$$

Due to Proposition 4.4.4,

$$\limsup_{n \to \infty} \frac{1}{L(n)n} \log \mathbf{P}(M_n(s^*) - s^*/\mathbf{E}A \le -\epsilon) = -\infty ;$$

and hence, due to (4.12), it is straightforward to deduce that,

$$\begin{aligned} &\liminf_{n \to \infty} \frac{\log \mathbf{P}(Q(\cdot n) > n)}{L(n)n} \\ &\ge \liminf_{n \to \infty} \frac{\log \mathbf{P}(s^*/\mathbf{E}A - \sum_{i=1}^{d} N_n^{(i)}(s^*) > 1 + \epsilon)}{L(n)n} \\ &\ge -\inf_{(\xi_1, \ldots, \xi_d) \in A} I'(\xi_1, \ldots, \xi_d) \end{aligned}$$

where $A = \{(\xi_1, \ldots, \xi_d) : s^*/\mathbf{E}A - \sum_{i=1}^{d} \xi_i(s^*) > 1 + \epsilon\}$. Note that the optimizer $(\xi_1, \ldots, \xi_d)$ of (4.11) satisfies $s^*/\mathbf{E}A - \sum_{i=1}^{d} \xi_i(s^*) \ge 1$. Consider $(\xi'_1, \ldots, \xi'_d)$

obtained by increasing one of the job sizes of ($_1; \ldots ;$ $_d$) by > 0. One can always nd a small enough such since > 1= . Note that there exists > 0 such that s^0=**E**A $\sum_{i=1}^{d}$ $^0(s^0) > 1 +$. Therefore,

$$\liminf_{n ! 1} \frac{\log \mathbf{P}(Q(n) > n)}{L(n)n} \quad I^0({}^0_1; \ldots ; {}^0_d) \quad c$$

where the second inequality is from the subadditivity of x 7! x . Since can be chosen arbitrarily small, letting ! 0, we arrive at the matching lower bound.
□

4.5 Solving the associated variational problem

We now simplify the expression of c given in Proposition 4.4.5.

Proposition 4.5.1. *If < 1= , c = 1 . If 1= , c can be computed via*

$$\min_{x_1; \ldots ; x_d} \sum_{i=1} x_i \tag{4.13}$$

$$\text{subject to} \quad \sup_{s 2 [0;]} \left\{ s \quad \sum_{i=1} (s \quad x_i)^+ \right\} \quad 1$$

$$x_1; \ldots ; x_d \quad 0;$$

which in turn equals

$$\min \left(\inf_{0<k \; b \; c: \; <1=(\; k)} \left\{ (d \quad k) \quad + (1 \quad + \; k) \; (k \quad b \quad 1= \; c)^1 \right\} ; \right. \tag{4.14}$$

$$\left. \min_{l=0}^{b \;\; 1= \; c} (d \quad l) \quad \frac{1}{l} \right) :$$

Proof. Recall that $\mathbb{D}^1[0; \,]$ is the subspace of the Skorokhod space and consists of non-decreasing piecewise linear functions with slope 1 almost everywhere over the time horizon [0;] and non-negative values at the origin. Recall $'_1( )$ de ned in (4.7) as well. From these de nitions, it is easy to see that Proposition 4.4.5 implies that the constant c is equal to

$$\inf_{1; \ldots ; \, d} \sum_{i=1} \sum_{s 2 [0;]} {}_s(\, _i) \tag{4.15}$$

$$\text{subject to} \quad \sup_{0 \quad s} \quad s \quad \sum_{i=1}^{d} \quad _i(s) \quad 1$$

$$_i = \varphi_1(\ _i); \quad _i \in D^1[0; \ ] \quad \text{for } i = 1; \ldots; d.$$

Note that this is an infinite-dimensional (functional) optimization problem. We reduce this optimization problem to a more standard problem in two main steps:

1. We first show that it suffices to optimize over $_i$'s of the form $_i(t) = t + x_0$ for some $x_0 \quad 0$.
2. Next, we reduce the infinite-dimensional problem over the previously mentioned set into a finite-dimensional optimization problem where the aim is to minimize a concave function over a compact polyhedral set. This allows us to invoke Corollary 32.3.1 of [85], which enables us to calculate the optimal solution by finding the extreme points of the feasible region.

Step 1.

Suppose that $(\ _1; \ldots; \ _d)$ is an optimal solution associated with (4.15) and recall that $_i = \varphi_1(\ _i)$. We now claim that the corresponding functions $_1; \ldots; \ _d$ have at most one jump. We prove this by contradiction. Assume that at least one of the $_i$'s exhibits two jumps at times u_0 and u_1 of size x_0 and x_1, respectively, with $0 \quad u_0 < u_1 \quad$. Let

$$_i(\ ) = \ _i(\ ) \quad x_1 I_{[u_1; \ ]}(\ ) + x_1 I_{[u_0; \ ]}(\ ).$$

Intuitively we constructed a new path, $_i(\ )$ by merging the two jumps into a big jump at time u_0. Since $x_0; x_1$ are non-negative then, we have that

$$_i(t) \quad _i(t); \ \forall t > 0.$$

Figure 4.2 illustrates this. Now, let $_i = \varphi_1 \quad _i$. From the definition of φ_1, we obviously have that

$$_i(s) \quad _i(s) \quad \text{for } s \in [0; \ ]. \tag{4.16}$$

Therefore, due to (4.16), $(\ _1; \ldots; \ _{i-1}; \ _i; \ _{i+1}; \ldots; \ _d)$ is also a feasible solution for (4.15). Moreover, by the following observation,

$$\sum_{s \in [0; \ ]} \ _s \ _i = \sum_{s \in [0; \ ]} \ _s(\ _i) + (x_0 + x_1) \quad x_0 \quad x_1;$$

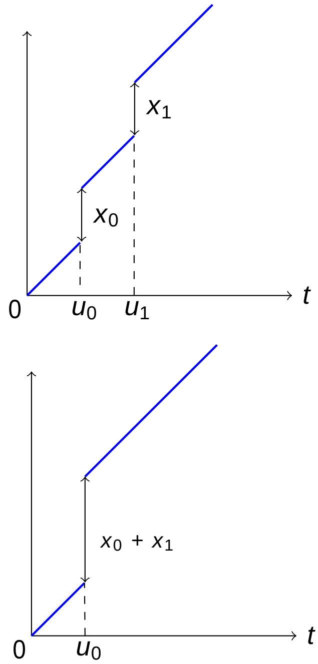

Figure 4.2: The 2 gures above depict the graphs of two jump functions, , and . By merging the two jumps of into one big jump, at time u_0, the resulting step function is bigger than or equal to .

along with the fact that $(x_0 + x_1) \quad < x_0 + x_1;$ we deduce that the candidate solution $(\ _1; :::; \ _{i-1}; \ _i; \ _{i+1}; :::; \ _d)$ strictly improves the value of the objective function in (4.15). That is, $(\ _1; :::; \ _d)$ cannot be an optimal solution. The argument can be iterated when $\ _i$ exhibits more than two jumps.

In conclusion, we proceed assuming that every $\ _i(\ )$ has a single jump of size $x_i > 0$ at some time $u_i \ 2 \ [0; \ ]$, and hence, we can use the following representation:

$$\ _i(s) = \min(s; u_i) + (s \quad x_i \quad u_i)^+ ; \text{ for } i = 1; : : : ; d: \tag{4.17}$$

To complete the rst step of our construction, we show that, without loss of generality, jumps can be assumed to occur at time 0. Suppose that $u_i > 0$ for some $i \ 2 \ f1; : : : ; dg$. De ne

$$\ ^0_i(s) = \ _i(s) \quad x_i \mathrm{I}_{[u_i; \ ]}(s) + x_i \mathrm{I}_{[0; \ ]}(s):$$

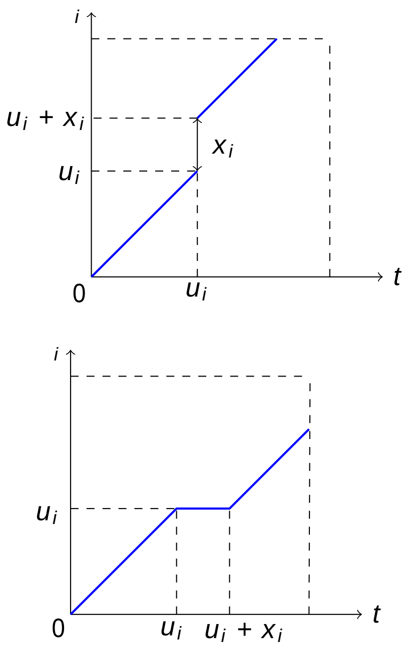

Figure 4.3: The pictures above depict the graph of a function $_i$ in $D^1[0; \ ]$ and the graph of the function $_i = '_1(_i)$. The function $_i$ has one jump of size x_i and this translates to a at line under the transformation $'_1$. In conclusion, we infer that $_i$ has the representation: $_i(s) = \min(s; u_i) + (s \quad x_i \quad u_i)^+$.

We constructed a new path ^{0} by moving the jump time to 0. Again, it is easy to verify that $^0(s) \quad (s)$ for all s 2 $[0; \ ]$, and if we let $^0_i = '_1(^0_i)$, then $^0_i(s) \quad _i(s)$ for all s 2 $[0; \ ]$. Consequently, we preserve feasibility without increasing the value of the objective function in (4.15). Therefore, w.l.o.g. we can assume that the $_i$'s that correspond to the optimal solution of (4.15) are those paths that have at most one discontinuity at time zero and then they linearly increase with slope 1. That is, the solution $(_1; : : : ; _d)$ takes the following form: for each $i = 1; : : : ; d$,

$$_i(s) = (s \quad x_i)^+ \quad \text{for some } x_i \quad 0: \tag{4.18}$$

Step 2. Thanks to the reduction in (4.18), we see that for each $i = 1; : : : ; d;$ we have that $_0(_i) = x_i$, while $_s(_i) = 0$ for every $s > 0$. Thus, we see that

(4.11) takes the form

$$
\begin{aligned}
&\min_{x_1,\ldots,x_d} && \sum_{i=1}^{d} x_i \qquad (4.19)\\
&\text{subject to} && \sup_{s\in[0,\infty)} \left\{ \lambda s - \sum_{i=1}^{d} (s - x_i)^+ \right\} \le 1;\\
& && x_1,\ldots,x_d \in [0,\infty].
\end{aligned}
$$

We continue simplifying the optimization problem in (4.19), reducing it to a polyhedral optimization problem. Let $x = (x_1,\ldots,x_d)$ be an optimal solution so that its coordinates are sorted in increasing order: $0 \le x_1 \le \ldots \le x_d \le \infty$. Note that the supremum of $l(s;x) \triangleq \lambda s - \sum_{i=1}^{d}(s - x_i)^+$ over $s \in [0,\infty]$ cannot be obtained strictly before x_d, since in such a case, a sufficiently small perturbation of x_d to its left leads to a strictly smaller value of the objective function without changing the supremum of $l(s;x)$, which is a contradiction to the assumption that x is an optimal solution. On the other hand, from the stability assumption $\lambda < d$, the slope of $l(s;x)$ is negative after x_d, and hence, its supremum cannot be obtained strictly after x_d. Therefore, the supremum of $l(s;x)$ has to be attained at $s = x_d$. Now, set $a_1 = x_1$ and $a_i = x_i - x_{i-1}$ for $i = 2,\ldots,d$. Then, $x_i = a_1 + \ldots + a_i$ for $i = 1,\ldots,d$, and

$$
l(x_d;x) = \lambda(a_1 + \ldots + a_d) - \sum_{i=1}^{d} \left(a_1 + \ldots + a_d - \sum_{j=1}^{i} a_j\right);
$$

and hence, (4.19) is equivalent to

$$
\begin{aligned}
&\min_{a_1,\ldots,a_d} && \sum_{i=1}^{d} \left(\sum_{j=1}^{i} a_j \right)\\
&\text{subject to} && \lambda(a_1 + \ldots + a_d) - \sum_{i=1}^{d} \left(a_1 + \ldots + a_d - \sum_{j=1}^{i} a_j\right) \le 1;\\
& && a_1 + \ldots + a_d \le \infty;\ a_1,\ldots,a_d \ge 0;
\end{aligned}
$$

and by simplifying the constraints we arrive at,

$$
\min_{a_1,\ldots,a_d} \sum_{i=1}^{d} \left(\sum_{j=1}^{i} a_j \right)
$$

$$\text{subject to} \quad a_1 + (\rho - 1)a_2 + \dots + (\rho - d + 1)a_d \geq 1;$$
$$a_1 + \dots + a_d \leq \rho;\; a_1, \dots, a_d \geq 0.$$

Recall $0 < \rho < d$, and let m be any of the integers in the set $\{1, \dots, d-1\}$. If $(\rho - m) < 0$, we deduce that $a_{m+1} = 0$. If this was not the case, we could construct a feasible solution which reduces the value of the objective function and also satisfies the previously mentioned conditions. That is, the variational problem has an even simpler representation than the one above:

$$\min_{a_1, \dots, a_d} \sum_{i=1}^{\lfloor \rho \rfloor} \left(\sum_{j=1}^{i} a_j \right) + (d - \lfloor \rho \rfloor) \left(\sum_{j=1}^{\lfloor \rho \rfloor + 1} a_j \right) \tag{4.20}$$

$$\text{subject to} \quad a_1 + (\rho - 1)a_2 + \dots + (\rho - \lfloor \rho \rfloor)a_{\lfloor \rho \rfloor + 1} = 1; \tag{4.21}$$
$$a_1 + \dots + a_{\lfloor \rho \rfloor + 1} \leq \rho; \tag{4.22}$$
$$a_1, \dots, a_{\lfloor \rho \rfloor + 1} \geq 0. \tag{4.23}$$

Recall that $c = 1$ if $\rho < 1/$. Assuming $\rho > 1/$, we recover the optimal solution by evaluating the extreme points associated with the polyhedron described by the constraints (4.21), (4.22), and (4.23). The objective function in (4.20) is concave and lower bounded inside the feasible region. In addition, the feasible region is a compact polyhedron. Therefore, the optimizer is achieved at some extreme point in the feasible region (see Corollary 32.3.1 in [85]).

Depending on the value of ρ we indicate how to compute the basic feasible solutions related to (4.20). Firstly, we treat the case $\rho > 1/(\rho - \lfloor \rho \rfloor)$ where ρ is not an integer. After that, we treat the general case $\rho > 1/$. Given that $\rho > \lfloor \rho \rfloor$, observe that if $\rho \geq 1/(\rho - \lfloor \rho \rfloor)$ then any solution satisfying (4.21) and (4.23) automatically satisfies (4.22). That is, we can ignore the constraint (4.22) by assuming that $\rho \geq 1/(\rho - \lfloor \rho \rfloor)$. Consequently, we only need to characterize the extreme points of (4.21), (4.23). Let $a_i = 1/(\rho - i + 1)$ for $i = 1, \dots, \lfloor \rho \rfloor + 1$. Let x_i denote the vector of the i'th extreme point. This is, $x_i = (0, \dots, a_i, \dots, 0)$. Calculating the value of the objective function over all extreme points, assuming that $\rho \geq 1/(\rho - \lfloor \rho \rfloor)$, we get

$$\min \left\{ d a_1, (d-1)a_2, \dots, (d - \lfloor \rho \rfloor)a_{\lfloor \rho \rfloor + 1} \right\}$$
$$= \min_{i=1}^{\lfloor \rho \rfloor + 1} (d - i + 1) \left(\frac{1}{\rho - i + 1} \right). \tag{4.24}$$

Next, we consider the general case $> 1=$. We show that additional extreme points arise by considering the inclusion of (4.22) and this might potentially give rise to solutions in which large service requirements are not equal across all the servers. Note that, if $= b\ \ c$ we must have that $a_{b\ \ c+1} = 0$. To see this, suppose that is not the case. Then, a feasible solution would be of the form $v = (a_1; : : : ; a_i; : : : ; a_{b\ \ c+1})$. By setting $a_{b\ \ c+1} = 0$, we construct another solution, $v^0 = (a_1; : : : ; a_i; : : : ; a_{b\ \ c}; 0)$. Observe that v^0 is a feasible solution and it reduces the value of the objective function (4.20) in comparison to v. Our subsequent analysis also includes the case $= b\ \ c$.

We identify the extreme points of (4.21), (4.22), (4.23). For that we introduce the slack variable $a_0 \quad 0$.

$$a_1 + (\quad 1)\, a_2 + ::: + (\quad b\ \ c)\, a_{b\ \ c+1} = 1\,; \tag{4.25}$$

$$a_0 + a_1 + ::: + a_{b\ \ c+1} = \quad ; \tag{4.26}$$

$$a_0; a_1; :::; a_{b\ \ c+1} \quad 0\,: \tag{4.27}$$

From elementary results in polyhedral combinatorics, we know that extreme points correspond to basic feasible solutions. By choosing $a_{i+1} = 1=(\quad i)$ and $a_0 = \quad a_{i+1}$ we recover basic solutions which correspond to the extreme points identi ed by the equations above. Recall, if $= b\ \ c$ we must have that $a_{b\ \ c+1} = 0$. That is, we can assume that $\quad i > 0$. We observe that $\quad 1=(\quad i)$ implies that $a_{i+1} = 1=(\quad i)$ and $a_j = 0$ for $j \neq i + 1$ which is a basic feasible solution for (4.25). Additional basic solutions are obtained by solving

$$\begin{aligned} 1 &= (\quad k)\, a_{k+1} + (\quad l)\, a_{l+1}; \\ &= \quad a_{k+1} + a_{l+1}: \end{aligned}$$

Suppose that $0 \quad l < k <$. This system of equations always has a unique solution because the equations are linearly independent, and hence,

$$1 = k a_{k+1} + l a_{l+1}:$$

Therefore, the solution $(a_{k+1}; a_{l+1})$ is given by

$$\begin{aligned} (k \quad l)\, a_{k+1} &= (\quad l) \quad 1; \\ (k \quad l)\, a_{l+1} &= 1 \quad (\quad k)\,: \end{aligned}$$

If we want $(a_{k+1}; a_{l+1})$ to be both basic and feasible we must have that $1/(\ \ l) \quad 1/(\ \ k)$. Now, we calculate the value of the objective function for $a_{k+1} = a_{k+1}$, $a_{l+1} = a_{l+1}$, and $a_{i+1} = 0$ for $i \notin \{k; l\}$. That is,

$$\begin{aligned} &\sum_{i=1}^{\lfloor c\rfloor}\left(\sum_{j=1}^{i} a_j\right) + (d \ \ \lfloor c\rfloor)\left(\sum_{j=1}^{\lfloor c\rfloor+1} a_j\right) \\ = \;& a_{l+1}(k \ \ l) + (\lfloor c\rfloor \ \ k)(a_{k+1} + a_{l+1}) \ + (d \ \ \lfloor c\rfloor)(a_{k+1} + a_{l+1}) \\ = \;& a_{l+1}(k \ \ l) + (d \ \ k)(a_{k+1} + a_{l+1}) \ . \end{aligned} \tag{4.28}$$

Recall, $1/(\ \ l) \quad 1/(\ \ k)$. As we mentioned before, if $\ = 1/(\ \ k)$, then we have that $a_{k+1} = 1/(\ \ k)$ and $a_i = 0$ for $i \neq k+1$ which is a feasible extreme point. Furthermore, we see that under this particular solution the objective function has a smaller value than the solution involving a_{k+1} and a_{l+1}. To illustrate this, observe that,

$$a_{l+1}(k \ \ l) + (d \ \ k)(a_{k+1} + a_{l+1}) \ > (d \ \ k)\,a_{k+1} .$$

Therefore, (a_{k+1} and a_{l+1}) would be an optimal solution under the condition $1/(\ \ l) \quad < 1/(\ \ k)$. Due to (4.24) and (4.28) we conclude that the optimal value of the variational problem (4.15) is given by

$$\min_{0<k \ \lfloor c\rfloor \ <1/(\ \ k)} \left((d \ \ k) \ + (1 \ \ (\ \ k)) \min_{0 \ \ l<\lfloor c\rfloor;\ 1/(\ \ l)} \frac{1}{k \ \ l} (k \ \ l) \right) \wedge \min_{l=0}^{\lfloor c\rfloor \wedge \lfloor 1/\ c\rfloor} (d \ \ l) \frac{1}{\ \ l} \ .$$

By simplifying the expression above, we arrive at (4.14).

□

4.6 Technical proofs

4.6.1 Proofs of Proposition 4.4.3, Lemma 4.2.2, and 4.2.1

We start with the continuity of the functional .

Proof of Lemma 4.2.2. Suppose that, $_n \to$ in $\mathbb{D}[0, =]$ w.r.t. the M_1^0 topology. As a result, there exist parametrizations $(u_n(s), t_n(s))$ of $_n$ and $(u(s), t(s))$ of so that,

$$\sup_{s =} \{|u_n(s) \quad u(s)| + |t_n(s) \quad t(s)|\} \to 0 \qquad \text{as } n \to \infty.$$

This implies that

$$\max\{\sup_{s =} |u_n(s) \quad u(s)|, \sup_{s =} |t_n(s) \quad t(s)|\} \to 0 \text{ as } n \to \infty.$$

Observe that, if $(u(s), t(s))$ is a parametrization for , then $(u(s) + \quad t(s), t(s))$ is a parametrization for $(\ )$. Consequently,

$$\sup_{s =} \{|u_n(s) + \quad t_n(s) \quad u(s) \qquad t(s)| + |t_n(s) \quad t(s)|\}$$
$$\sup_{s =} \{|u_n(s) \quad u(s)|\} + \sup_{s =} \{(\ + 1)|t_n(s) \quad t(s)|\} \to 0.$$

Thus, $(\ _n) \to (\ )$ in the M_1^0 topology, proving that the map is continuous. □ □

The next lemma provides the continuity of two functionals used in our large deviation analysis.

Proof of Lemma 4.2.1. Consider a sequence $_n$ such that $d_{M_1^0}(\ _n, \ ) \to 0$. From the definition of the M_1^0 topology, there exists a parametrization $(u(s), t(s))$ of the completed graph of and a parametrization $(u_n(s), t_n(s))$ of the completed graph of $_n$ such that

$$\sup_{s\in[0,T]} \{|u_n(s) \quad u(s)|\} + \sup_{s\in[0,T]} \{|t_n(s) \quad t(s)|\} \to 0, \quad \text{as } n \to \infty. \tag{4.29}$$

For i), note that $|u_n(T) \quad u(T)| \quad \sup_{s\in[0,T]} |u_n(s) \quad u(s)| \to 0$, while $_n(T) = u_n(T)$ and $(T) = u(T)$. Therefore, $|E(\ ) \quad E(\ _n)| = |\ _n(T) \quad (T)| \to 0$ as $n \to \infty$. Therefore, E is a continuous functional. For ii), suppose that $(0) \quad 0$. For any > 0, there exists N such that $_n(0)$ for $n > N$. Now, from the definition of parametrization and the nonnegativity of (0), we see that $\sup_{s\in[0,T]} u(s) = \sup_{s\in[0,T]} (s)$. Similarly, we can show that $|\sup_{s\in[0,T]} u_n(s) \quad \sup_{s\in[0,T]} \ _n(s)| <$. Therefore,

$$\limsup_{n\to\infty} \sup_{s\in[0,T]} \ _n(s) \qquad \sup_{s\in[0,T]} (s)$$

$$\limsup_{n\to\infty} \sup_{s\in[0;T]} u_n(s) \quad \sup_{s\in[0;T]} u(s) +$$

$$\limsup_{n\to\infty} \sup_{s\in[0;T]} u_n(s) \quad u(s) + \quad = :$$

Since was arbitrary, this proves the continuity of S at .

□

In the next proof, we show continuity properties of the map . Recall,

$$D \quad , f \in \mathrm{D}[0; =] : \quad ( )( ) \quad ( )(\quad) > 0 \text{ and } (0) \quad 0g:$$

Proof of Proposition 4.4.3. Note that = and is continuous, so we only need to check the continuity of over the range of , in particular, non-decreasing functions. Let be a non-decreasing function in D[0; =]. We consider two cases separately: ( )( ) > = and ( )( ) = .

We start with the case ( )( ) > = . Pick > 0 such that ( )( ) > = + 2 and (=) + 2 < . For such an , it is straightforward to check that $d_{M_1^0}(;) <$ implies ( )( ) > = and never exceeds on [0; =]. Therefore, the parametrizations of ( ) and ( ) consist of the parametrizations — with the roles of space and time interchanged — of the original and concatenated with the linear part coming from . More specifically, suppose that $(x; t) \in ( )$ and $(y; r) \in ( )$ are parametrizations of and . Since is non-decreasing, if we define on $s \in [0; T]$

$$x^0(s) \triangleq \begin{cases} t(2s) & \text{if } s \quad T=2 \\ \frac{1}{} t^0(s) \quad ( )(=) + & \text{if } s > T=2 \end{cases};$$

$$t^0(s) \triangleq \begin{cases} x(2s) & \text{if } s \quad T=2 \\ ( )(=) (2s=T \quad 1) + ( )(=) & \text{if } s > T=2 \end{cases};$$

$$y^0(s) \triangleq \begin{cases} r(2s) & \text{if } s \quad T=2 \\ \frac{1}{} r^0(s) \quad ( )(=) + & \text{if } s > T=2 \end{cases};$$

$$r^0(s) \triangleq \begin{cases} y(2s) & \text{if } s \quad 1=2 \\ ( )(=) (2s=T \quad 1) + ( )(=) & \text{if } s > 1=2 \end{cases};$$

then $(x^0; t^0) \in (\; ( ))$, $(y^0; r^0) \in (\; ( ))$. Noting that

$$\|x^0 \quad y^0\|_\infty + \|t^0 \quad r^0\|_\infty$$

$$= \sup_{s\in[0;1/2]} |t(2s) \quad r(2s)| \vee \sup_{s\in(1/2;1]} |x'(s) \quad y'(s)|$$

$$+ \sup_{s\in[0;1/2]} |x(2s) \quad y(2s)| \vee \sup_{s\in(1/2;1]} |t'(s) \quad r'(s)|$$

$$= \|t \quad r\|_1 \vee \quad ^1| \ (\ )(\ ) \qquad (\ )(\ )| + \|x \quad y\|_1 \vee | \ (\ )(\ ) \qquad (\ )(\ )|$$

$$^1\|t \quad r\|_1 \vee \|x \quad y\|_1 + \|x \quad y\|_1$$

$$(1 + \quad ^1)(\|x \quad y\|_1 + \|t \quad r\|_1);$$

and taking the in mum over all possible parametrizations, we conclude that $d_{M_1^0}(\quad (\ );\quad (\ ))\quad (1+\quad ^1)d_{M_1^0}(\ ;\ )\quad (1+\quad ^1)$, and hence, is continuous at .

Turning to the case $(\ )(\ ) \qquad = $, let > 0 be given. Due to the assumption that $(\ )$ is continuous at , there has to be a > 0 such that $'\ (\ )(\ ) + \quad < \ '\ (\ )(\qquad) \quad '\ (\ )(\ + \ ) \quad '\ (\ )(\ ) +$. We prove that if $d_{M_1^0}(\ ;\ ) < \quad \wedge$, then $d_{M_1^0}(\quad (\ );\quad (\ ))\quad 8$. Since the case where $(\ )(\ )$ $=$ is similar to the above argument, we focus on the case $(\ )(\ ) < \ = $; that is, also crosses level before $=$. Let $(x; t) \in \ (\ )$ and $(y; r) \in \ (\ )$ be such that $\|x \quad y\|_1 + \|t \quad r\|_1 <$. Let $s_x \triangleq \inf\{s \quad 0 : x(s) > \quad \}$ and $s_y \triangleq \inf\{s \quad 0 : y(s) > \quad \}$. Then it is straightforward to check $t(s_x) = \ '\ (\ )(\ )$ and $r(s_y) = \ '\ (\ )(\ )$. Of course, $x(s_x) = $ and $y(s_y) = $. If we set $x'(s) \triangleq t(s \wedge s_x)$, $t'(s) \triangleq x(s \wedge s_x)$, and $y'(s) \triangleq r(s \wedge s_y)$, $r'(s) \triangleq y(s \wedge s_y)$, then

$$\|x' \quad y'\|_1$$

$$\|t \quad r\|_1 + \sup_{s\in[s_x \wedge s_y; s_x \vee s_y]} |t(s_x) \quad r(s)| \vee |t(s) \quad r(s_y)|$$

$$\|t \quad r\|_1$$

$$+ \sup_{s\in[s_x \wedge s_y; s_x \vee s_y]} |t(s_x) \quad t(s)| + |t(s) \quad r(s)|$$

$$\vee \ |t(s) \quad t(s_y)| + |t(s_y) \quad r(s_y)|$$

$$\|t \quad r\|_1 + \ |t(s_x) \quad t(s_y)| + \|t \quad r\|_1 \ \vee \ |t(s_y) \quad t(s_x)| + \|t \quad r\|_1$$

$$2\|t \quad r\|_1 + 2|t(s_x) \quad t(s_y)|.$$

Now we argue that $t(s_x) \qquad t(s_y) \quad t(s_x) + $. To see this, note rst that $x(s_y) < x(s_x) + \quad = \quad +$, and hence,

$$t(s_y) \quad '\ (\ )(x(s_y)) \quad '\ (\ )(\ + \ ) \quad '\ (\ )(\ ) + \quad = t(s_x) + \ .$$

On the other hand,

$$t(s_x) \qquad = \ '\ (\ )(\ ) \qquad '\ (\qquad) \quad t(s_y);$$

where the last inequality is from $(t(s_y)) \quad x(s_y) > x(s_x) \quad =$ and the definition of $'$. Therefore, $\|x' \quad y'\|_\infty \quad 2 \; + 2 \; < 4$. Now we are left with showing that $\|t' \quad r'\|_\infty$ can be bounded in terms of .

$$
\begin{aligned}
&\|t' \quad r'\|_\infty \\
&\quad \|x \quad y\|_\infty + \sup_{s \in [s_x \wedge s_y, s_x \vee s_y]} \{|x(s_x) \quad y(s)| \vee |x(s) \quad y(s_y)|\} \\
&\quad \|x \quad y\|_\infty \\
&\qquad + \sup_{s \in [s_x \wedge s_y, s_x \vee s_y]} \quad |x(s_x) \quad x(s)| + |x(s) \quad y(s)| \\
&\qquad\qquad \vee \; |x(s) \quad x(s_y)| + |x(s_y) \quad y(s_y)| \\
&\quad \|x \quad y\|_\infty + \; |t(s_x) \quad t(s_y)| + \|x \quad y\|_\infty \vee \; |x(s_x) \quad x(s_y)| + \|x \quad y\|_\infty \\
&\quad 2\|x \quad y\|_\infty + 2|x(s_x) \quad x(s_y)| \\
&= 2\|x \quad y\|_\infty + 2|y(s_y) \quad x(s_y)| \\
&\quad 4\|x \quad y\|_\infty < 4 \; .
\end{aligned}
$$

Therefore, $d_{M_1^0}(\;); \; (\;) \quad \|x' \quad y'\|_\infty + \|t' \quad r'\|_\infty < 8$.

□

4.6.2 Proof of Proposition 4.4.1

Proof of Proposition 4.4.1. In view of Lemma 3.2 of [79] it is easy to deduce that $\frac{1}{n}\sum_{j=1}^{\lfloor n t \rfloor}(A_j \quad \mathbf{E}A)$ satisfies the LDP on $(D[0; \; = \mathbf{E}A]; d_{M_1^0})$ with speed $L(n)n$ and with good rate function

$$
I_A(\;) = \begin{cases} 0 & \text{if} \quad = 0; \\ \infty & \text{otherwise.} \end{cases}
$$

On the other hand, due to Result 3.2.1, $\frac{1}{n}\sum_{j=1}^{\lfloor n t \rfloor} S_j^{(i)} \quad 1$ satisfies the LDP on $(D[0; \;]; d_{M_1^0})$ with the good rate function

$$
I_{S^{(i)}}(\;) = \begin{cases} \sum_{t \in [0; \;]} (\; (t) \quad (t \;)) & \text{if} \quad \in D_p^{\uparrow}[0; \;]; \\ \infty & \text{otherwise}. \end{cases} \tag{4.30}
$$

Clearly, $\frac{1}{n}\sum_{j=1}^{\lfloor n t \rfloor} A_j \quad t \; \mathbf{E}A$ and $\frac{1}{n}\sum_{j=1}^{\lfloor n t \rfloor} S_j^{(i)} \quad t$ are exponentially equivalent to $\frac{1}{n}\sum_{j=1}^{\lfloor n t \rfloor}(A_j \quad \mathbf{E}A)$ and $\frac{1}{n}\sum_{j=1}^{\lfloor n t \rfloor} S_j^{(i)} \quad 1$, respectively. Therefore, $\frac{1}{n}\sum_{j=1}^{\lfloor n t \rfloor} A_j$

t $\mathbf{E}A$ and $\frac{1}{n}\sum_{j=1}^{\lfloor nt \rfloor} S_j^{(i)}$ t satisfy the LDPs with the good rate functions I_A and $I_{S^{(i)}}$, respectively.

Now, consider the map : D[0; =]; $T_{M_1^0}$! D[0; =]; $T_{M_1^0}$ where ( ) , + . Let $I_0(\ ) , \inf\{I_A(\ ) :\ 2\ \mathbb{D}[0;\ = \mathbf{E}A];\ = \ _{\mathbf{E}A}(\ )\}$. From the form of I_A, it is easy to see that I_0 coincides with the right-hand-side of (4.10). Since this map is continuous (Lemma 4.2.2), the contraction principle (Result 1.3.5) applies showing that $A_n = \ _{\mathbf{E}A}\ \frac{1}{n}\sum_{j=1}^{\lfloor nt \rfloor} A_j$ t $\mathbf{E}A$ satisfies the desired LDP with the good rate function I_0. We next consider $S_n^{(i)}$. Let $I_i(\ ) , \inf\{I_{S^{(i)}}(\ ) :\ 2\ \mathbb{D}[0;\ ];\ = \ _1(\ )\}$. Note that $I_{S^{(i)}}(\ ) = 1$ whenever $2\ \mathbb{D}_p^{"}$, and $2\ \mathbb{D}_p^{"}$ if and only if $= \ _1(\ )$ belongs to $\mathbb{D}^1[0;\ ]$. Again, it is easy to check that I_i coincides with the right-hand-side of (4.30). We apply the contraction principle once more to conclude that $S_n^{(i)} = \ _1\ \frac{1}{n}\sum_{j=1}^{\lfloor nt \rfloor} S_j^{(i)}$ t satisfies the desired LDP with the good rate function I_i.

□

4.6.3 Proof of Proposition 4.4.2

Proof of Proposition 4.4.2. We first claim that $d_{M_1^0}(N_n^{(i)};\ _1(S_n^{(i)}))$ implies either

$$(S_n^{(i)})(\ ) \quad \frac{1}{2} \qquad \text{or} \qquad N_n^{(i)}(\ ) \qquad =2.$$

To see this, suppose not. That is,

$$(S_n^{(i)})(\ ) < \frac{1}{2} \qquad \text{and} \qquad N_n^{(i)}(\ ) \quad < \ =2. \tag{4.31}$$

By the construction of $S_n^{(i)}$ and $N_n^{(i)}$, we see that $N_n^{(i)}(\ )$ is non-decreasing and $N_n^{(i)}(t)$ for t $(S_n^{(i)})(\ )$. Therefore, the second condition of (4.31) implies

$$\sup_{t \in [\ (S_n^{(i)})(\ );\ ]} |N_n^{(i)}(t) \quad | < \ =2.$$

On the other hand, since the slope of $_1(S_n^{(i)})$ is 1 on $[\ (S_n^{(i)})(\ );\ ]$, the first condition of (4.31) implies that

$$\sup_{t \in [\ (S_n^{(i)})(\ );\ ]} |\ _1(S_n^{(i)})(t) \quad | < \ =2;$$

and hence,

$$\sup_{t \in [\ (S_n^{(i)})(\ );\ ]} | \ _1(S_n^{(i)})(t) \quad N_n^{(i)}(t)| < \ : \tag{4.32}$$

Note also that by the construction of $\ _1$, $N_n^{(i)}(\ )$ and $\ _1(S_n^{(i)})(\ )$ coincide on $[0;\ (S_n^{(i)})(\ ))$. From this, along with (4.32), we see that

$$\sup_{t \in [0;\ ]} | \ _1(S_n^{(i)})(t) \quad N_n^{(i)}(t)| < \ ;$$

which implies that $d_{M_1^0}(\ _1(S_n^{(i)}); N_n^{(i)}) < \ $. The claim is proved. Therefore,

$$\limsup_{n \to \infty} \frac{\log \mathbf{P}\ \ d_{M_1^0}(N_n^{(i)};\ _1(S_n^{(i)}))}{L(n)n}$$

$$\limsup_{n \to \infty} \frac{\log\ \mathbf{P}\ \ (S_n^{(i)})(\ )\ \ =2\ + \mathbf{P}\ \ N_n^{(i)}(\ )\ \ =2}{L(n)n}$$

$$\limsup_{n \to \infty} \frac{\log \mathbf{P}\ \ (S_n^{(i)})(\ )\ \ =2}{L(n)n} \vee \limsup_{n \to \infty} \frac{\log \mathbf{P}\ \ N_n^{(i)}(\ )\ \ +\ =2}{L(n)n};$$

and we are done for the exponential equivalence between $N_n^{(i)}$ and $\ _1(S_n^{(i)})$ if we prove that

$$\limsup_{n \to \infty} \frac{\log \mathbf{P}\ \ (S_n^{(i)})(\ )\ \ =2}{L(n)n} = \ \infty \tag{4.33}$$

and

$$\limsup_{n \to \infty} \frac{\log \mathbf{P}\ \ N_n^{(i)}(\ )\ \ =2}{L(n)n} = \ \infty : \tag{4.34}$$

For (4.33), note that $\ (S_n^{(i)})(\ )\ \ =2$ implies that $S_n^{(i)}(\ )\ \ =2$, and hence,

$$\limsup_{n \to \infty} \frac{\log \mathbf{P}\ \ (S_n^{(i)})(\ )\ \ =2}{L(n)n} \quad \limsup_{n \to \infty} \frac{\log \mathbf{P}\ \ S_n^{(i)}(\ )\ \ =2}{L(n)n}$$

$$\inf_{(\ )\ \ =2} I_0(\ ) \quad \infty ;$$

where the second inequality is due to the LDP upper bound for $S_n^{(i)}$ in Proposition 4.4.1 and the continuity of the map $\mapsto$ ( ) as a functional from (D[0;]; $d_{M_1^0}$) to R. For (4.34), note that $N_n^{(i)}$( ) =2 implies $S_n^{(i)}$(+ =2) . Considering the LDP for $S_n^{(i)}$ on D[0; + =2], we arrive at the same conclusion. This concludes the proof for the exponential equivalence between M_n and $_1(S_n^{(i)})$. The exponential equivalence between M_n and $_{\mathbf{E}A}(A_n)$ is essentially identical, and hence, omitted.

□

Proof of Proposition 4.4.4. Let $\hat{I}_0^0$( ) , inf $f I_0$( ) : 2 $\mathbb{D}$[0; =**E**A]; = $_{\mathbf{E}A}$( )g and $\hat{I}_i^0$( ) , inf $f I_i$( ) : 2 $\mathbb{D}$[0;]; = $_1$( )g for $i = 1; : : : ; d$. Recall that in Proposition 4.4.1 we established the LDP for A_n and $S_n^{(i)}$ for each $i = 1; : : : ; d$. Note that if 2 D $_{\mathbf{E}A}$, f 2 D[0; =**E**A] : $_{\mathbf{E}A}$( )( ) $_{\mathbf{E}A}$( )( ) > 0g, then there has to be $s; t$ such that 0 $s < t <$ =**E**A and ( )(s) = . For such , I_0( ) = 1 . This along with Proposition 4.4.3, we see that $_{\mathbf{E}A}$ is continuous on the e ective domain of I_0. Therefore the extended contraction principle (see [79]) applies, establishing the LDP for $_{\mathbf{E}A}(A_n)$ with rate function $\hat{I}_0^0$. The LDP for $_1(S_n^{(i)})$ with rate function $\hat{I}_i^0$ follows from the same argument. Due to the exponential equivalence derived in Proposition 4.4.2, M_n and $N_n^{(i)}$ satisfy the same LDP as $_{\mathbf{E}A}(A_n)$ and $_1(S_n^{(i)})$. Therefore, we are done once we prove that the rate functions $\hat{I}_i^0$ deduced from the extended contraction principle satisfy, $I_i^0 = \hat{I}_i^0$ for $i = 0; : : : ; d$.

Starting with $i = 0$, note that I_0( ) = 1 if $\not\in$ $_{\mathbf{E}A}$, and hence,

$$\hat{I}_0^0(\ ) = \inf f I_0(\ ) : \ 2\ \mathbb{D}[0;\ =\mathbf{E}A];\ =\ _{\mathbf{E}A}(\ )g = \begin{cases} 0 & \text{if } =\ _{\mathbf{E}A}(\ _{\mathbf{E}A}); \\ 1 & \text{otherwise}; \end{cases} \tag{4.35}$$

where it is straightforward to check that $_{\mathbf{E}A}(\ _{\mathbf{E}A}) =\ _{1=\mathbf{E}A}$. Therefore, $I_0^0 = \hat{I}_0^0$.

Turning to $i = 1; : : : ; d$, note rst that since I_i( ) = 1 for any $\not2$ D^1[0;],

$$\hat{I}_i^0(\ ) = \inf f I_i(\ ) : \ 2\ \mathbb{D}^1[0;\ ];\ =\ _1(\ )g:$$

Note also that $_1$ can be simpli ed on D^1[0;]: it is easy to check that if 2 D^1[0;], $_1$( )(t) = and $'_1$( )(t) for t 2 [0;]. Therefore, $_1$( ) = $'_1$( ), and hence,

$$\hat{I}_i^0(\ ) = \inf f I_i(\ ) : \ 2\ \mathbb{D}^1[0;\ ];\ =\ '_1(\ )g:$$

Now if we define $\varrho_1 : D[0, \ ] \to D[0, \ ]$ as

$$\varrho_1(\ )(t) \triangleq \begin{cases} (t) & t \in [0, \varphi_1(\ )(\ )) \\ \ + (t - \varphi_1(\ )(\ )) & t \in [\varphi_1(\ )(\ ), \ ] \end{cases},$$

then it is straightforward to check that $I_i(\ ) \ \ I_i(\varrho_1(\ ))$ and $\varphi_1(\ ) = \varphi_1(\varrho_1(\ ))$ whenever $\ \in D^1[0, \ ]$. Moreover, $\varrho_1(D^1[0, \ ]) \ \ D^1[0, \ ]$. From these observations, we see that

$$\hat{I}_i^0(\ ) = \inf\{I_i(\ ) : \ \in \varrho_1(D^1[0, \ ]), \ = \varphi_1(\ )\}. \tag{4.36}$$

Note that $\ \in \varrho_1(D^1[0, \ ])$ and $\ = \varphi_1(\ )$ implies that $\ \in C^1[0, \ ]$. Therefore, in case $\ \notin C^1[0, \ ]$, no $\ \in D[0, \ ]$ satisfies the two conditions simultaneously, and hence,

$$\hat{I}_i^0(\ ) = \inf \emptyset = \infty = I_i^0(\ ). \tag{4.37}$$

Now we prove that $\hat{I}_i^0(\ ) = I_i^0(\ )$ for $\ \in C^1[0, \ ]$. We claim that if $\ \in \varrho_1(D^1[0, \ ])$,

$$\ _s(\varphi_1(\ )) = \ (s) \ \ (s \ \ )$$

for all $s \in [0, \ ]$. The proof of this claim is provided at the end of the proof of the current proposition. Using this claim,

$$\begin{aligned} \hat{I}_i(\ ) &= \inf\left\{\sum_{s\in[0, \ ]} (\ (s) \ \ (s \ \ )) : \ \in \varrho_1(D^1[0, \ ]), \ = \varphi_1(\ )\right\} \\ &= \inf\left\{\sum_{s\in[0, \ ]} \ _s(\varphi_1(\ )) : \ \in \varrho_1(D^1[0, \ ]), \ = \varphi_1(\ )\right\} \\ &= \inf\left\{\sum_{s\in[0, \ ]} \ _s(\ ) : \ \in \varrho_1(D^1[0, \ ]), \ = \varphi_1(\ )\right\}. \end{aligned}$$

Note also that $\ \in C^1[0, \ ]$ implies the existence of such that $\ = \varphi_1(\ )$ and $\ \in \varrho_1(D^1[0, \ ])$. To see why, note that there exists $\ ^{\prime} \in D^1[0, \ ]$ such that $\ = \varphi_1(\ ^{\prime})$ due to the definition of $C^1[0, \ ]$. Let $\ \triangleq \varrho_1(\ ^{\prime})$. Then, $\ = \varphi_1(\ )$ and $\ \in \varrho_1(D^1[0, \ ])$. From this observation, we see that

$$\left\{\sum_{s\in[0, \ ]} \ _s(\ ) : \ \in \varrho_1(D^1[0, \ ]), \ = \varphi_1(\ )\right\} = \left\{\sum_{s\in[0, \ ]} \ _s(\ )\right\},$$

and hence,

$$\hat{I}_i(\ ) = \sum_{s\in[0, \ ]} \ _s(\ ) = I_i^0(\ ) \tag{4.38}$$

for $\ 2\ C^1[0;\ ]$. From (4.37) and (4.38), we conclude that $I_i^0 = \hat{I}_i$ for $i = 1; : : : ; d$.

All that remains is to prove that $_s('_1(\ )) = \ (s)\ \ \ (s\ \ )$ for all $s\ 2\ [0;\ ]$. We consider the cases $s > \ '_1(\ )(\ )$ and $s\ \ \ '_1(\ )(\ )$ separately. First, suppose that $s > \ '_1(\ )(\ )$. Since $'_1(\ )$ is non-decreasing, this means that $'_1(\ )(t) < s$ for all $t\ 2\ [0;\ ]$, and hence, $ft\ 2\ [0;\ ] : '_1(t) = sg = ;$. Therefore,

$$
\begin{aligned}
&_s('_1(\ )) \\
&= 0\ \_\ \ \sup ft\ 2\ [0;\ ] : '_1(t) = sg\ \ \ \inf ft\ 2\ [0;\ ] : '_1(t) = sg \\
&= 0\ \_\ (\ \ 1\ \ \ \ 1\ ) = 0:
\end{aligned}
$$

On the other hand, since is continuous on $['_1(\ )(\ );\ ]$ by its construction,

$$
(s)\ \ \ \ (s\ \ ) = 0:
$$

Therefore,

$$
_s('_1(\ )) = 0 = \ (s)\ \ \ \ (s\ \ )
$$

for $s > \ '_1(\ )(\ )$.

Now we turn to the case $s\ \ \ \ '_1(\ )(\ )$. Since $'_1(\ )$ is continuous, this implies that there exists $u\ 2\ [0;\ ]$ such that $'_1(\ )(u) = s$. From the de nition of $'_1(\ )(u)$, it is straightforward to check that

$$
u\ 2\ [\ (s\ \ );\ (s)]\ \ \ \ \ (\ )\ \ \ \ \ s = \ '_1(\ )(u): \tag{4.39}
$$

Note that $[\ (s\ \ );\ (s)]\ \ \ \ [0;\ ]$ for $s\ \ \ \ '_1(\ )(\ )$ due to the construction of . Therefore, the above equivalence (4.39) implies that $[\ (s\ \ );\ (s)] = fu\ 2\ [0;\ ] : '_1(\ )(u) = sg$, which in turn implies that $(s\ \ ) = \inf fu\ 2\ [0;\ ] : '_1(\ )(u) = sg$ and $(s) = \sup fu\ 2\ [0;\ ] : '_1(\ )(u) = sg$. We conclude that

$$
_s('_1(\ )) = \ (s)\ \ \ \ (s\ \ )
$$

for $s\ \ \ \ '_1(\ )(\ )$.

□

Chapter 5

Asymptotics for stochastic networks

5.1 Introduction

In this chapter, we obtain logarithmic asymptotics for stochastic uid networks with heavy-tailed Weibull input. Our results comprise of upper and lower large deviation bounds for the bu er content process of the network in the vector valued Skorokhod space which is endowed with the product J_1 topology. We also provide asymptotic estimates for over ow probabilities of subsets of the system's nodes. Lastly, we apply our results to a special network: the so-called multiple on-o sources uid network with heavy-tailed Weibull inputs.

The stochastic network is a key model within applied probability and is connected to many applications. Some real-life examples include computer communication and manufacturing networks. Stochastic uid network models have been a subject of intense research activity. The stability of queueing networks is examined in [21] where, for a multiclass queueing network with any initial conditions, and i.i.d. interarrival and service times within each class, the network is positive Harris recurrent. Under di erent assumptions, the authors in [63] considered the single-class queueing network, whereas, in [4], assuming ergodicity and stationarity, the single-class Jackson-type queueing network is studied. In this chapter, we focus on stochastic uid networks, which are networks in which the content in each node is the di erence between a non-decreasing input process $\mathbf{J}$ and a deterministic linear output ow.

Stochastic fluid networks with respect to heavy-tailed input processes are not fully understood. The two-node case with feedback loops and heavy-tailed input has been studied in [34]. In the case of feedforward networks with heavy-tailed distributions we refer to the survey paper [17]. More specifically, feedforward networks with heavy-tailed input have been studied in [59], [34]. In [97], the multiple on-off sources model with regularly varying inputs has been studied. Although in [97] the authors establish exact asymptotics for the tail behavior of the workload process, they note that these methods do not hold for other subexponential distributions such as the lognormal, and the heavy-tailed Weibull distribution. Finally, we point out that all of the above results focus on stationary distributions; the behavior of time-dependent performance measures in this context has not been studied.

Here, we consider stochastic fluid networks comprised of d nodes and we do not pose any restrictions on the topology of the network. We restrict our analysis to the case of compound Poisson input processes with semi-exponential increments. Specifically, by large deviations we investigate in which way the presence of heavy tails affects the tail asymptotics in multidimensional complex stochastic networks.

Using an appropriate map (the multidimensional reflection map) we describe the movement of the fluid/customers in the network. In our model, a superposition of the fluid — emitted from the network's nodes — feeds into buffers which are emptied at constant rates. Usually one would be interested in the probability that a subset of the buffer contents exceeds some level. We use a representation of the stochastic network model which fits the large deviations framework of Chapter 2.

Within the large deviations framework, continuity properties of mappings between random processes are the basis upon which large deviation principles from the original process to the image process are induced. This approach has been formalized as the contraction principle. Thus far, it has generally not been possible to establish heavy-tailed large deviation principles for the behavior of single-class stochastic networks with feedback via a continuous-mapping approach, since one needs a large deviation principle for the input processes.

Let us introduce our results. We prove large deviation upper and lower bounds for trajectories of the network's buffer content process. To do so, we utilize the continuity of the reflection map with respect to the product J_1 topology on an appropriate subspace of the Skorokhod space. Using the product J_1 topology, we establish an extended sample path large deviation principle for sequences of linearly scaled compound Poisson input processes on appropriate subspaces. This result hinges on the sample path large deviations for Levy processes with heavy-tailed Weibull increments, (Theorem 2.2.12). Exploiting the large deviation upper and lower bounds for the buffer content process we prove logarithmic asymptotics for overflow probabilities associated with a subset of the system's nodes. Our asymptotic results depend on the average input flows and the deterministic output rates of each station. This is a consequence of the previously mentioned extended sample path LDP for the buffer content process coupled with a continuous mapping approach in order to deduce an LDP for the buffer content process at each node in the system. To employ the continuous mapping approach, since the buffer content process is a function of the unregulated content process through the multidimensional reflection map, we prove an independent result for large deviation bounds for Lipschitz continuous maps.

In the case of the specific variation of the multiple on-off sources model (which we study in this chapter) with heavy-tailed Weibull inputs, and heterogeneous sources (nodes) we explicitly compute the decay rate associated with the tail probabilities. As a result, we contribute to understanding the most likely way buffer overflow occurs: to overflow a buffer, there is a trade-off between the intensity of the deviant behavior, namely to what extent the sources transmit with rate bigger than the mean rate, and the duration of this extreme behavior. For the multiple on–off sources network, it is known that sources alternate between on and off to overflow the buffer. For sources with subexponential on periods, we have the following intuition based on [59]. During the path to overflow, a source either sends at peak rate for the entire period, or constantly alternates between on and off, and effectively contributes at mean rate. We perform explicit calculations for the case of inputs with semiexponential distributions to precisely describe this trade-off. In the power law case, the number of sources sending at peak rate is just enough so that the peak rates of the transmitting sources plus the mean rates of the other sources exceed the output rate of the sink. However, the semiexponential case displays a delicate solution. That is, the number of sources that transmit at peak rate depends on the shape parameter of the semi-exponential distributions. Moreover, the time each source transmits at peak rate is proportional to the overflow threshold. A similar phenomenon has been observed in Chapter 4 where the multiple server queue with semi-exponential service times has been studied.

The outline of this chapter is as follows: Section 5.2 contains a description of our model, the topological space in which the input processes are defined, a mathematical introduction to the reflection map, and preliminary results on large deviations. In Section 5.3 we present our main results: upper and lower large deviation bounds for the buffer content process, and logarithmic asymptotics for overflow probabilities of the buffer content process. In particular, we include an explicit computation of the decay rate for the special case of the multiple on-off sources model. Section 5.4 contains complementary proofs that support our main results.

5.2 Model description and preliminary results

5.2.1 The Model

In this section, we describe our model and we present some preliminary results that are used in our analysis. We consider a single-class open stochastic fluid network with d nodes. We allow the possibility to assign a dedicated exogenous input to a subset of the d nodes. For this reason, let $\mathcal{J}$ denote the subset of nodes that have an exogenous input. At each node $i \in \{1,\dots,d\}$, the fluid is processed and released at a deterministic rate r_i. Fractions of the processed fluid from each node is then routed to other nodes or out of the network. We characterize the stochastic fluid network by a four-tuple $(\mathbf{J}, \mathbf{r}, Q, \mathbf{X}(0))$, where $\mathbf{J}(\cdot) = \left(J^{(1)}(\cdot),\dots,J^{(d)}(\cdot)\right)^{\intercal}$ is the vector of the assigned exogenous input stochastic processes at each one of the d nodes, respectively. The random variable $J^{(i)}(t)$ represents the total amount of exogenous input to node i during the time interval $[0,t]$. The vector $\mathbf{r} \triangleq (r_1,\dots,r_d)^{\intercal}$ is the vector of deterministic output rates at the d nodes, $Q \triangleq [q_{i,j}]_{i,j\in\{1,\dots,d\}}$ is the $d \times d$ substochastic routing matrix, and $\mathbf{X}(0) \triangleq (X^{(1)}(0),\dots,X^{(d)}(0))$ is the nonnegative random vector of initial contents at the d nodes.

Now, we make our model more specific. Regarding the d-dimensional stochastic fluid model, we assume that the input flow streams to the i-th station/node are Poisson processes with unit rate. Let $\{N^{(i)}(t)\}_{t\geq 0}$ denote the Poisson process of unit rate that is associated with each station i, which is also independent from $\{N^{(j)}(t)\}_{t\geq 0}$ for every $j = 1,\dots,d$. At each node, the arrival of the kth job in station i generates a workload $J_k^{(i)}$. In addition, let $\mathbf{J}_k = (J_k^{(1)},\dots,J_k^{(i)},\dots,J_k^{(d)})^{\intercal}$ denote a sequence of i.i.d. positive random vectors with i.i.d. increments such that $\{\mathbf{J}_k\}_{k\geq 1}$ is independent of $\{N^{(i)}(t)\}_{t\geq 0}$, for each $i \in \{1,\dots,d\}$. The total amount of external workload that arrives at station i is equal to $J^{(i)}(t) \triangleq \sum_{j=1}^{N^{(i)}(t)} J_j^{(i)},\ t \in [0,T],$ which is a compound Poisson process with mean λ_i. If no exogenous input is assigned to node i, then set $J^{(i)}(\cdot) \equiv 0$, and $\lambda_i \triangleq 0$. We pose an assumption on the distribution of $J_1^{(i)}$, for $i \in J$, making it semi-exponential:

Assumption 5.2.1. For each $i \in J \equiv \{1,\dots,d\}$, $\mathbf{P}\left(J_1^{(i)} > x\right) = e^{-c_i L(x) x^{\alpha}}$ where $c_i \in (0,\infty)$, and L is a slowly varying function such that $L(x)/x^{1-\alpha}$ is non-increasing for sufficiently large x's.

Naturally, the stochastic process $\mathbf{J}$ is non-decreasing, non-negative, and its sample paths are allowed to be discontinuous. If the buffer at node i and at time t is nonempty, then there is fluid output from node i at a constant rate r_i. On the other hand, if the buffer of node i is empty at time t, the output rate equals the minimum of the combined external input plus internal input rate and the output rate r_i.

We provide more details about the stochastic dynamics of our network. A proportion $q_{i,j}$ of all output from node i is immediately routed to node j, while a proportion $q_i := 1-\sum_{j=1}^{k} q_{ij}$ is routed out of the network. We assume that $q_{ii} := 0$, and the routing matrix Q is substochastic, so that $q_{i,j} \geq 0$, and $q_i \geq 0$ for all i,j. We also assume that $Q^n \to 0$ as $n \to \infty$ which implies that all input eventually leaves the network. Let $Q^{\top}$ be the transpose matrix of Q. We ensure the stability of the network by posing the following assumption based on [50]:

Assumption 5.2.2. Let $\lambda = (\lambda_1,\dots,\lambda_d)^{\top}$, and assume that $(I - Q^{\top})\mathbf{r} - \lambda > 0$.

Due to our model specifics, the total workload at station i is processed at a constant rate r_i from the i-th server; and a proportion q_{ij} is routed from the i-th station to the j-th server. Let $Q = (I - Q^{\top})$. The potential content vector $\mathbf{X}(t)$, $\mathbf{X}(t) := \mathbf{J}(t) - Q\mathbf{r}t$, at time t, would be the initial value $\mathbf{X}(0)$ plus the exogenous input $\mathbf{J}(t)$ minus the output $\mathbf{r} \cdot t$ plus the internal input $Q\mathbf{r} \cdot t$. Let $\mathbf{Z}^{(i)}(t)$ denote the buffer content of the i-th station at time t. We are interested in the buffer content process whose dynamics are expressed formally by the so-called reflection map. Intuitively, the reflection map is defined in terms of a pair of processes $(\mathbf{Z},\mathbf{Y})$ that solve the differential equation

$$d\mathbf{Z}(t) = d\mathbf{X}(t) + Q\,d\mathbf{Y}(t). \tag{5.1}$$

where $\mathbf{Y}(\cdot)$ is the minimal amount required to keep $\mathbf{Z}(\cdot)$ non-negative. The component $X^{(i)}(t)$ represents what the content of buffer i would be at time t if the output occurred continuously at rate r_j from node j, for all j, whether station j had fluid to emit. Consequently, as we assume $Z(0) = 0$ the buffer content is

$$\mathbf{Z}(t) = \mathbf{X}(t) + Q \cdot \mathbf{Y}(t), \quad t \in [0,T]. \tag{5.2}$$

We call the map from $\mathbf{X} \mapsto (\mathbf{Y}; \mathbf{Z})$ the reflection map. We now provide a more rigorous definition of this map.

5.2.2 Preliminary results on the reflection map

We start with the definition of the reflection map. Let $\mathbb{D}[0; T]$ denote the Skorokhod space: the space of cadlag functions over $[0; T]$. Denote with $\mathbb{D}^{\uparrow}[0; T]$ the subspace of the Skorokhod space containing non-decreasing functions.

Definition 5.2.1. [96] For any $\in \prod_{i=1}^{k} \mathbb{D}[0; T]$ and any reflection matrix $Q = (I \quad Q^{|})$, let the feasible regulator set be

$$( ) , \left\{ \in \prod_{i=1} \mathbb{D}^{\uparrow}[0; T] : \quad + Q \qquad 0 \right\}$$

and let the reflection map be

$$\mathbf{R} , (;) : \prod_{i=1} \mathbb{D}[0; T] \mapsto \prod_{i=1} \mathbb{D}[0; T] \qquad \prod_{i=1} \mathbb{D}[0; T]$$

with regular component

$$( ) , \inf \{ \quad ( ) \} = \inf \left\{ w \in \prod_{i=1} \mathbb{D}[0; T] : w \in \quad ( ) \right\}$$

i.e;

$$^{(i)}(t) = \inf \{ w_i(t) \in \mathbb{R} : w \in \quad ( ) \} \text{ for all } i \in \{1; \ldots; k\} \text{ and } t \in [0; T];$$

and content component

$$( ) , \quad + Q \quad ( ):$$

The infimum in the definition of may not exist in general, however, in Theorem 14.2.1 of [96], it is proven that the reflection map is properly defined with the component-wise order. In addition, the regulator set is non-empty and its infimum is attained in $( )$. If $\mathbf{R} = (;)$ is a continuous map, then the reflection map solves the Skorokhod problem implied by (5.1). Now, we state some important results regarding the properties of $(;)$. The following result gives an explicit representation of the solution of the Skorokhod problem given by (5.1).

Result 5.2.1. *[96] If $Y( ) = (\mathbf{X})( )$ and $\mathbf{Z}( ) = (\mathbf{X})( )$ then, $(\mathbf{Y}( ); \mathbf{Z}( ))$ solves the Skorokhod problem implied by the equation (5.1). The mappings and are Lipschitz continuous maps w.r.t. to the uniform metric.*

The next result is a useful property of the Skorokhod map; it allows us to describe the discontinuities of the re ection map under some mild assumptions.

Result 5.2.2. *[96] Let $2 \; \mathrm{Q}_{i=1}^{k} D[0; T]$. For the set of discontinuity points of $( )$ $(Disc(( )))$ and $( )$ $(Disc(( )))$, it holds that $Disc(( )) [Disc(( )) = Disc( )$. In addition, if has only positive jumps then, $( )$ is continuous and*

$$( )(t) \quad ( )(t \quad) = (t) \quad (t \quad):$$

Result 5.2.3 (Theorem 14.2.6. of [96]). *If in $\mathrm{Q}_{i=1}^{d} D[0; a]$, $a > 0$, then $( ) \quad ( )$.*

5.2.3 Some useful tools on large deviations

We start with a result which deduces an extended LDP for closed subspaces of a metric space $(X; d)$ given that the original process satis es an LDP on the bigger space X. Let D_I , $fx \; 2 \; X : I(x) < 1g$.

Lemma 5.2.1. *Let E be a closed subset of X . Let X_n be a stochastic process such that $\mathbf{P}(X_n \; 2 \; E) = 1$ for all $n \quad 1$. Suppose that E is equipped with the topology induced by X. Then, if the probability measures of X_n satisfy the extended LDP in X with speed a_n, and with rate function I so that $D_I \quad E$, then the same extended LDP holds in E.*

Proof. In the topology induced on E by X, the open sets are the sets of the form $G \setminus E$ with $G \quad X$ open. Similarly, the closed sets in this topology are the sets of the form $F \setminus E$ with $F \quad X$ closed. Furthermore, $\mathbf{P}(X_n \; 2 \quad) = \mathbf{P}(X_n \; 2 \quad \setminus E)$ for any $\; 2 \; B$ where B is the Borel sigma-algebra. Suppose that an extended LDP holds in X. Now, for the upper bound, let F be a closed subset of E. Then, F is a closed subset of X. Hence,

$$\limsup_{n \; ! \; 1} \frac{1}{a_n} \log \mathbf{P}\,(X_n \; 2 \; F) \qquad \inf_{x 2 F} I(x) = \qquad \inf_{x 2 F \; \setminus E} I(x):$$

For the lower bound, let G be an open subset of E. That is, $G = G^0 \setminus E$ where G^0 is an open subset of X. Then,

$$\limsup_{n\to\infty}\frac{1}{a_n}\log\mathbf{P}\left(X_n\in G\right)=\liminf_{n\to\infty}\frac{1}{a_n}\log\mathbf{P}\left(X_n\in G'\cap E\right)$$
$$=\liminf_{n\to\infty}\frac{1}{a_n}\log\mathbf{P}\left(X_n\in G'\right)\geq-\inf_{x\in G'\cap E}I(x)=-\inf_{x\in G}I(x).$$

Now, since the level sets $\Gamma_I(\alpha)$ are closed subsets of X, the rate function I remains lower semicontinuous when restricted to E. □

We continue with a useful lemma on pre-images of Lipschitz continuous maps on metric spaces. For a closed subset of the metric space (X,d), recall, $A^{\delta}\triangleq\{\xi\in X:d(\xi,A)\leq\delta\}$ where $d(\xi,A)=\inf_{\zeta\in A}d(\xi,\zeta)$.

Lemma 5.2.2. *Let* $(\mathbb{S},\rho)$ *and* $(\mathbb{X},d)$ *be metric spaces. Suppose that* $\phi:(\mathbb{X},d)\to(\mathbb{S},\rho)$ *is a Lipschitz continuous mapping with Lipschitz constant* $\|\phi\|_{\mathrm{Lip}}$*. Then, for any set* $F\subseteq\mathbb{X}$ *it holds that*

$$\phi^{-1}(F)^{\delta}\subseteq\phi^{-1}\left(F^{\|\phi\|_{\mathrm{Lip}}\delta}\right).$$

Proof. Suppose that $\xi\in\phi^{-1}(F)^{\delta}$. For each $n\geq1$, since $\phi^{-1}(F)$ is a closed set, there exists a $\xi_n\in\phi^{-1}(F)$ so that $d(\xi_n,\xi)\leq\delta+1/n$. Since $\xi_n\in\phi^{-1}(F)$ we have that $\phi(\xi_n)\in F$. Furthermore, $\rho(\phi(\xi_n),\phi(\xi))\leq\|\phi\|_{\mathrm{Lip}}(\delta+1/n)$ and hence, $d\left(\phi(\xi),F\right)\leq\|\phi\|_{\mathrm{Lip}}(\delta+1/n)$. Letting $n\to\infty$, we get $d\left(\phi(\xi),F\right)\leq\|\phi\|_{\mathrm{Lip}}\delta$, that is, $\xi\in\phi^{-1}(F^{\|\phi\|\delta})$. Since this holds for any $\xi\in\phi^{-1}(F)^{\delta}$, the statement holds true.

□

Lemma 5.2.3. *Let* $(\mathbb{S},\rho)$ *and* $(\mathbb{X},d)$ *be metric spaces. Suppose that the sequence of probability measures of* $\mathbf{X}_n$ *satisfies the extended LDP in* $(\mathbb{X},d)$ *with speed* a_n *and rate function* I*. Moreover, let* $\phi:(\mathbb{X},d)\to(\mathbb{S},\rho)$ *be a Lipschitz continuous mapping and set*

$$I'(y)\triangleq\inf_{\phi(x)=y}I(x).$$

Then,

i) the stochastic process $\mathbf{S}_n=\phi(\mathbf{X}_n)$ *satisfies the following lower and upper bounds: for any open set* $G\subseteq\mathbb{S}$,

$$\liminf_{n\to\infty}\frac{1}{a_n}\log\mathbf{P}\left(\mathbf{S}_n\in G\right)\geq-\inf_{x\in G}I'(x);$$

and for any closed set $F \subset \mathbb{S}$,

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}(\mathbf{S}_n \in F) \le -\lim_{\delta\to 0} \inf_{x\in F^\delta} I'(x).$$

ii) In addition to Φ *being a Lipschitz map, suppose that* Φ *is a homeomorphism; that is,* Φ *is injective, surjective, and* Φ^{-1} *is continuous. Then,* $\mathbf{S}_n$ *satisfies the extended LDP in* $(\mathbb{S};\rho)$ *with speed* a_n *and rate function* I'*.*

iii) If I' *is a good rate function — i.e.,* $\Psi_{I'}(M) \triangleq \{y \in \mathbb{S} : I'(y) \le M\}$ *is compact for each* $a \in [0,\infty)$*, then* $\mathbf{S}_n$ *satisfies the large deviation principle in* $(\mathbb{S}; d)$ *with speed* a_n *and good rate function* I'*.*

Proof. i). For the upper bound let F be a closed subset of $(\mathbb{S};\rho)$. Thanks to Lemma 5.2.2, for any $\delta > 0$, we have that $\left(\Phi^{-1}(F)\right)^\delta \subseteq \Phi^{-1}\left(F^{\delta\|\Phi\|_{\mathrm{Lip}}}\right)$. and hence,

$$\inf_{x\in(\Phi^{-1}(F))^\delta} I(x) \ge \inf_{x\in\Phi^{-1}\left(F^{\delta\|\Phi\|_{\mathrm{Lip}}}\right)} I(x). \tag{5.3}$$

Furthermore, by the extended LDP of $\mathbf{X}_n$, for $\epsilon > 0$ there exists an $n(\epsilon)$ such that for any $n \ge n(\epsilon)$

$$\mathbf{P}(\mathbf{X}_n \in \Phi^{-1}(F)) \le \exp\left(a_n\left[-\inf_{x\in(\Phi^{-1}(F))^\delta} I(x) + \epsilon\right]\right) \quad \text{for any } \delta > 0. \tag{5.4}$$

Consequently, (5.3), and (5.4) lead to

$$\mathbf{P}(\mathbf{X}_n \in \Phi^{-1}(F)) \le \exp\left[a_n\left(-\inf_{x\in\Phi^{-1}\left(F^{\delta\|\Phi\|_{\mathrm{Lip}}}\right)} I(x) + \epsilon\right)\right], \tag{5.5}$$

for any $n \ge n(\epsilon)$ and $\delta > 0$. Next, for $n \ge n(\epsilon)$,

$$\begin{aligned}\mathbf{P}(\mathbf{S}_n \in F) &= \mathbf{P}(\Phi(\mathbf{X}_n) \in F)\\ &= \mathbf{P}\left(\mathbf{X}_n \in \Phi^{-1} F\right)\\ &\le \exp\left[a_n\left(-\inf_{x\in\Phi^{-1}\left(F^{\delta\|\Phi\|_{\mathrm{Lip}}}\right)} I(x) + \epsilon\right)\right]\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}(\mathbf{S}_n \in F) \leq -\inf_{x \in \Phi^{-1}(F^{\delta \|\cdot\|_{\mathrm{Lip}}})} I(x) + \epsilon$$
$$= -\inf_{y \in F^{\delta \|\cdot\|_{\mathrm{Lip}}}} I'(y) + \epsilon. \tag{5.6}$$

Letting $\epsilon \to 0$, and $\delta \to 0$ in (5.6), we arrive at the desired large deviation upper bound.

Turning to the lower bound, consider an open set G. Since $\Phi^{-1}(G)$ is open,

$$\liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}(\mathbf{S}_n) \in G) = \liminf_{n\to\infty} \frac{1}{a_n} \log \mathbf{P}\left(\mathbf{X}_n \in \Phi^{-1}(G)\right)$$
$$\geq -\inf_{y \in \Phi^{-1}(G)} I(\cdot) = -\inf_{x \in G} I'(x).$$

ii). The upper and lower bounds for the extended large deviation principle have been proved in *i*). Since $\mathbf{X}_n$ satisfies a large deviation principle, we have that the level sets of I are closed sets i.e., $\Psi_I(M) \triangleq \{x \in \mathbb{X} : I(x) \leq M\}$ is a closed set for every $M > 0$. Now, we verify that I' is lower-semicontinuous. The level sets of I' are $\Psi_{I'}(M) \triangleq \{y \in \mathbb{S} : I'(y) \leq M\}$, for every $M > 0$. Note that

$$\{y \in \mathbb{S} : I'(y) \leq M\} = \{\Phi(x) : I(x) \leq M\} = \Phi(\Psi_I(M)).$$

Since Φ is a homeomorphism the image set of a closed set is a closed set. Hence, $\mathbf{S}_n$ satisfies the extended LDP.

iii). The upper and lower bounds for the extended large deviation principle have been proved in *i*); due to our assumption, I' is a good rate function. Hence

$$\lim_{\delta\to 0} \inf_{y \in F^{\delta \|\cdot\|_{\mathrm{Lip}}}} I'(y) = \inf_{y \in F} I'(y).$$

Consequently,

$$\limsup_{n\to\infty} \frac{\log \mathbf{P}(\mathbf{S}_n \in F)}{a_n} \leq -\lim_{\delta\to 0} \inf_{y \in F^{\delta \|\cdot\|_{\mathrm{Lip}}}} I'(y) = -\inf_{y \in F} I'(y).$$

□

5.2.4 The topology of the function space

In this section, we introduce our preliminary results on sample-path large deviations for the input and the content process. For our results we will use the

J_1 topology. In order to study networks, we need a multidimensional functional setting. That is, we work on the functional space $(\prod_{i=1}^{k} D[0;T]; \prod_{i=1}^{k} T_{J_1})$ which is a product space equipped with the product J_1 topology. The product topology | which we denote with $\prod_{i=1}^{k} T_{J_1}$ | is induced by the product metric d_p which in turn is de ned in terms of the J_1 topology. More precisely, for ; 2 $\prod_{i=1}^{k} D[0;T]$ such that = ($^{(1)};:::;$ $^{(k)}$) and = ($^{(1)};:::;$ $^{(k)}$) we have that

$$d_p(\; ; \;) , \sum_{i=1}^{k} d_{J_1}({}^{(i)}; {}^{(i)}):$$

Note that we use the component-wise partial order on $D[0;T]$ and R^k. That is,

$$x_1 , (x_1^{(1)};:::;x_1^{(k)}) \quad x_2 , (x_2^{(1)};:::;x_2^{(k)}) \text{ in } R^k$$

$$\text{if } x_1^{(i)} \quad x_2^{(i)} \text{ in } R \text{ for all } i \; 2 \; f1;:::;kg:$$

Also, we write in $D[0;T]$ if (t) (t) in R^k for all t 2 $[0;T]$.

Some useful continuous functions

Lemma 5.2.4. *For k 2 R^d, let* : $\prod_{i=1}^{d} D[0;T]$ 7! $\prod_{i=1}^{d} D[0;T]$ *be such that* ( )(t) = (t) t. *Then,*

i) is a Lipschitz continuous map w.r.t. the product J_1 topology; and

ii) is a homeomorphism.

Proof. i). Suppose that $d_p(\; ; \;) <$ w.r.t. the product J_1 topology. Then, there exists a homeomorphism $^{(i)}$, i 2 $f1;:::;dg$, so that

$$k \qquad {}^{(i)}k_1 + k \; {}^{(i)} \quad e( )k_1 < 2 :$$

Therefore,

$$\begin{aligned} d_{J_1} \quad {}^{(i)}_k( ); \; {}^{(i)}_k( ) \quad & k \; {}^{(i)} \quad {}^{(i)} \quad {}^{(i)} + k^{(i)}(\; {}^{(i)} \quad e( ))k_1 + k \; {}^{(i)} \quad e( )k_1 \\ & k \; {}^{(i)} \quad {}^{(i)} \quad {}^{(i)}k_1 + (1 + jk^{(i)}j)k \; {}^{(i)} \quad e( )k_1 \\ & 2(1 + jk^{(i)}j) : \end{aligned} \tag{5.7}$$

Consequently,

$$
\begin{aligned}
d_p(\ \ (\ );\ \ \ (\ )) \quad & \sum_{i=1} d_p(\ {}_k^{(i)}(\ );\ {}_k^{(i)}(\ )) \\
& 2\sum_{i=1}(1 + |k^{(i)}|) \\
& = 2(d + \sum_{i=1} |k^{(i)}|) \ :
\end{aligned}
$$

ii). Note that $\ ^{1}(\ )(\ ) = \ (\ ) + k(\ ) = \ \ {}_k(\ )$. Hence, is injective and surjective. Furthermore, the continuity of $\ ^{1}$ is obtained by applying *i)* to $_k$. ☐

Lemma 5.2.5. *The function* $:\ \prod_{i=1}^{d} \mathrm{D}[0;T] \mapsto \mathrm{R}^d$, $(\ ) = \ (T)$ *is Lipschitz continuous in* $\prod_{i=1}^{k} \mathbb{D}[0;T]$ *w.r.t. the product* J_1 *topology.*

Proof. Suppose that $d_p(\ ;\ )\ \ $. Then, we have that $d_{J_1}(\ ^{(i)};\ ^{(i)})\ \ $ for every $i = 1;:::;k$. For any homeomorphism $\ : [0;T] \mapsto [0;T]$ that satisfies $(0) = 0$ and $(T) = T$, we have that

$$
\begin{aligned}
|\ ^{(i)}(\ )\ \ \ ^{(i)}(\ )| &= |\ ^{(i)}(T)\ \ \ ^{(i)}(T)| \\
&= |\ ^{(i)}(T)\ \ \ ^{(i)}(\ (T))| \\
& \|\ ^{(i)}\ \ \ ^{(i)}\ \ \|_1
\end{aligned}
$$

Since this is true for any $\ $, $|\ ^{(i)}(\ )\ \ \ ^{(i)}(\ )|\ \ \ d_{J_1}(\ ^{(i)};\ ^{(i)})\ \ \ :$ Since this holds for every $i = 1;:::;d;$ we have that $\|\ (\ )\ \ \ (\ )\|_1\ \ \ .$ ☐

We end this section with a key result regarding the Lipschitz continuity of the reflection map with the product J_1 topology. The proof of Theorem 5.2.6 is deferred to Section 5.4. Let $\mathrm{D}^{\uparrow}[0;T]$ be the subspace of non-decreasing paths. Subsequently, let $\mathrm{D}^{\ i}[0;T]$ be the subspace of non-decreasing paths with slope $_i$, namely

$$
\mathrm{D}^{\ i}[0;T] \triangleq \{\ \in \mathrm{D}[0;T] : \ (t) = \ (t) + \ _i\ t;\ \ \in \mathrm{D}^{\uparrow}[0;T]\}:
$$

Theorem 5.2.6. *The reflection map* $\mathbf{R} : \prod_{i=1}^{d} \mathrm{D}^{\ i}[0;T] \mapsto \prod_{i=1}^{2d} \mathrm{D}[0;T]$ *where* $\mathbf{R} = (\ ;\ \ )$ *is Lipschitz continuous with the product* J_1 *topology.*

Note that the restriction to paths without downwards jumps is essential for this result to hold. Since the order in which the jumps take place matters for the action of the reflection map, we cannot ensure the continuity of the reflection map without any extra regularity conditions. The main difficulty arises when at limit paths we have simultaneous jumps with different signs in multiple coordinates (K. Ramanan, personal communication).

The extended sample path LDP for the content process

Lastly, we state our preliminary results on large deviations regarding the multidimensional input process of the stochastic fluid network. Recall that $\mathbf{J}(\cdot)$ denotes the input process which is a vector of independent compound Poisson processes, and with mean vector . We consider the scaled version $\mathbf{J}_n \triangleq \frac{1}{n}\mathbf{J}(n\cdot)$. Consequently, $\mathbf{J}_n$ satisfies the following LDP which is a consequence of Theorem 2.2.12 and Lemma 5.2.1. For any $\in \prod_{i=1}^{d} \mathbb{D}[0,T]$, let

$$I(\ ^{(j)}) = \sum_{\{t:\ ^{(j)}(t) \neq\ ^{(j)}(t-)\}} \left(\ ^{(j)}(t) - \ ^{(j)}(t-)\right) :$$

Result 5.2.4. *The probability measures of $\mathbf{J}_n$ satisfy the extended LDP in the function space $\prod_{i=1}^{d} \mathbb{D}^{\uparrow}[0,T]$, $\prod_{i=1}^{d} T_{J_1}$ with speed $L(n)n$, and with rate function $I^{(d)} : \prod_{i=1}^{d} \mathbb{D}^{\uparrow}[0,T] \to [0,\infty]$ where*

$$I^{(d)}(\ ^{(1)},\ldots,\ ^{(d)}) = \begin{cases} \sum_{j \in J} c_j I(\ ^{(j)}); & \text{if } \ ^{(j)} \in \mathbb{D}\ ^{j}[0,T] \text{ for } j \in J \text{ and } \ ^{(j)} \equiv 0 \text{ for } j \notin J; \\ \infty; & \text{otherwise.} \end{cases} \tag{5.8}$$

Recall that the content vector is a function of the exogenous input plus the internal input; that is, $\mathbf{X}(t) = \mathbf{J}(t) - Q\mathbf{r}t$. We define the scaled version of the potential content vector $\mathbf{X}_n(\cdot) = \frac{1}{n}\mathbf{X}(n\cdot)$. Obviously, $\mathbf{X}_n$ is the image of $\mathbf{J}_n$ where the map $_{Q\mathbf{r}}$ is applied. Due to Lemma 5.2.4, $_{Q\mathbf{r}}$ is Lipschitz continuous and a homeomorphism with respect to the product J_1 topology. The following large deviation principle for $\mathbf{X}_n(\cdot)$ is a direct consequence of *ii)* in Lemma 5.2.3.

Result 5.2.5 (Consequence of Result 5.2.4 and Lemma 5.2.3). *The probability measures of $\mathbf{X}_n = \left(X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(d)}\right)$ satisfy the extended LDP in*

$\prod_{i=1}^{d} D^{(Qr)_i}[0;T]; \prod_{i=1}^{d} T_{J_1}$ with speed $L(n)n$ and the rate function

$$\Gamma^{(d)}(\phi^{(1)};\dots;\phi^{(d)}) = \begin{cases} \sum_{j\in J} c_j I(\phi^{(j)}) & \text{if } \phi^{(j)} \in D^{(Qr)_i}[0;T] \\ & \text{for } j \in J; \\ & \text{and } \phi^{(j)}(t) = (Qr)_j(t) \\ & \text{for } j \notin J; \\ \infty; & \text{otherwise:} \end{cases} \tag{5.9}$$

5.3 Main results

In this section, we state our main results (Theorem 5.3.1, 5.3.6, and 5.3.8) along with the lemmas which are used to prove the main results.

5.3.1 Large deviations for the buffer content process

In this subsection, we state sample path large deviation bounds for the scaled buffer content process $\mathbf{Z}_n(\cdot) \triangleq \frac{1}{n}\mathbf{Z}(n\cdot)$ with $\mathbf{Z}(\cdot)$ defined in (5.2). The reflection map enables us to represent the buffer content process in terms of the content process $\mathbf{X}_n$ and the map Ψ i.e; $\mathbf{Z}_n = \Psi(\mathbf{X}_n)$. In view of Theorem 5.2.6, the large deviation bounds for the buffer content process are a consequence of Lemma 5.2.3, and the continuity of the reflection map with respect to the product J_1 topology.

Theorem 5.3.1. *The buffer content process $\mathbf{Z}_n$ satisfies the following upper and lower bounds in the function space $\left(\prod_{i=1}^{d} D[0;T]; \prod_{i=1}^{d} T_{J_1}\right)$:*

i) For any closed set $F \subseteq \prod_{i=1}^{d} D[0;T]$,

$$\limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}(\mathbf{Z}_n \in F) \le -\lim_{\epsilon\to 0} \inf_{\xi\in F^{\epsilon}} I_S(\xi);$$

ii) and for any open set $G \subseteq \prod_{i=1}^{d} D[0;T]$,

$$\liminf_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}(\mathbf{Z}_n \in F) \ge -\inf_{\xi\in G} I_S(\xi);$$

where

$$I_S(\xi) = \inf\left\{ \Gamma^{(d)}(\zeta) : \xi = \Psi(\zeta);\ \zeta \in \prod_{i=1}^{d} D^{(Qr)_i}[0;T] \right\}.$$

Proof. The large deviation bounds for the buffer content process are a consequence of Lemma 5.2.3, and the continuity of the reflection map with respect to the product J_1 topology. Specifically, Theorem 5.2.6 ensures the continuity of the content component is Lipschitz continuous with the product J_1 topology. Then, *i)* of Lemma 5.2.3, and Result 5.2.5 imply the large deviation upper and lower bounds of $\mathbf{Z}_n$.

□

5.3.2 Asymptotics for overflow probabilities

In this subsection, we examine the probability that the workload, associated with a subset of nodes in the system, exceeds a high level. We estimate the probability of large exceedance for the solution of the stochastic differential equation (5.1). Recall, $\ (\ ) = \ (T)$. Let $\mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{R}^d_+$. Let $B(\ ) = \mathbf{b}^\intercal \ (\ )$ and consider $\mathbf{Z}_n(T) = B(\mathbf{X}_n)$. Let

$$I^0(a) \triangleq \inf \left\{ I^{(d)}(\ _1, \ldots, \ _d) \text{ s.t. } a = B(\ );\ \ \in \prod_{i=1}^{} \mathbb{D}^{(\ \ \mathbf{Qr})_i}[0,T] \right\}.$$

Moreover, let $\mathbb{D}_{\leq 1}[0,T]$ be the subspace of paths that have at most one discontinuity. Subsequently, define

$$V_{\geq}(c) \triangleq \left\{ \ \in \prod_{i=1}^{} \mathbb{D}^{(\ \ \mathbf{Qr})_i}[0,T] \cap \mathbb{D}_{\leq 1}[0,T] : B(\ ) \ \ c \right\},$$

and

$$V_{\geq}(c) = \inf_{\ \in V_{\geq}(c)} I^{(d)}(\ ).$$

Similarly, let

$$V_{>}(c) \triangleq \left\{ \ \in \prod_{i=1}^{} \mathbb{D}^{(\ \ \mathbf{Qr})_i}[0,T] \cap \mathbb{D}_{\leq 1}[0,T] : B(\ ) > c \right\}.$$

Subsequently, let

$$V_{>}(c) = \inf_{\ \in V_{>}(c)} I^{(d)}(\ ).$$

Note that $V_{\geq}(c)$, $V_{>}(c)$ may depend on T. The next lemma enables us to reduce the feasible region of the optimization problem $\inf_{x \in A} I^0(c)$ — where A is a half interval | from $\prod_{i=1}^{d} D^{(\quad Qr)_i}[0;T]$ to $\prod_{i=1}^{d} D^{(\quad Qr)_i}[0;T] \setminus D^{\leq 1}[0;T]$ i.e; to the subspace of feasible step functions that have at most one discontinuity.

Lemma 5.3.2. *Let $\ \in \prod_{i=1}^{d} D^{(\quad Qr)_i}[0;T]$. Then, there exists a path $\tilde{\ }$ in $\prod_{i=1}^{d} D^{(\quad Qr)_i}[0;T] \setminus D^{\leq 1}[0;T]$ such that*

i) $I^{(d)}(\tilde{\ }) \quad I^{(d)}(\ )$, and

ii) $(\tilde{\ })(T) \quad (\ )(T)$.

Proof. The proof of this lemma is deferred to Section 5.4. □

An immediate consequence of Lemma 5.3.2 is that

$$V_{>}(c) = \inf_{x \in [c;\infty)} I^{0}(x) \qquad \text{and} \qquad V_{>}(c) = \inf_{x \in (c;\infty)} I^{0}(x):$$

Let $D_{+}[0;T]$ be the subspace of $D[0;T]$ that contains paths with only positive discontinuities i.e; $D_{+}[0;T] = \{\ \in D[0;T] : \ (t) \quad (t\ ) \quad 0\}$.

Lemma 5.3.3. *For $a = (a_1;\dots;a_d) \in \mathbb{R}^d_+$ and $\ \in \prod_{i=1}^{d} D_{+}[0;T] \setminus D^{\leq 1}[0;T]$ let $\ = \ + a\mathbb{1}_{\{T\}}$. Then,*

i) $(\ ) = (\ )$;

ii) $(\ )(T) = (\ )(T) + a$; and

iii) $I^{(d)}(\ ) \quad I^{(d)}(\ ) + \sum_{i=1}^{d} c_i a_i$:

Proof. The proof of this lemma is deferred to Section 5.4. □

Recall, J is the index set of nodes with exogenous input. Next, let $I^{+} = \{j \in \{1;\dots;d\} : b_j > 0\}$.

Lemma 5.3.4. *Assume that $J \setminus I^{+} \neq \emptyset$. Then, the map $x \mapsto V_{>}(x)$ is -Hölder continuous i.e.,*

$$|V_{>}(y) \quad V_{>}(x)| \qquad \max_{\{1 \quad i \quad d : b_i > 0\}} \frac{c_i}{b_i} \quad |y \quad x| \ :$$

Proof. Let $y \quad x \quad 0$. It is obvious that $V_{>}(y) \quad V_{>}(x) \quad 0$: By the definition of the infimum, for any $\ > 0$ there exists a $\ \in V_{>}(x)$ so that $I^{(d)}(\ ) < V_{>}(x) + \ $.

Next, let $i \in I^+$ where $I^+ = \{j \in \{1, \ldots, d\} : b_j > 0\}$. Subsequently, let $\psi = \varphi + (\mathbf{y} - \mathbf{x})\mathbb{1}_{\{T\}}$ so that $\mathbf{y} - \mathbf{x} = (0, \ldots, \frac{y-x}{b_i}, \ldots, 0)$. Due to *ii)* of Lemma 5.3.3,

$$\mathbf{b}^\intercal \Gamma(\psi)(T) = \mathbf{b}^\intercal(\Gamma(\varphi)(T) + \mathbf{y} - \mathbf{x}) \geq \mathbf{b}^\intercal \Gamma(\varphi)(T) + b_i \frac{(y-x)}{b_i} \geq x + y - x;$$

hence, $\psi \in V_>(y)$. Moreover, due to *iii)* of Lemma 5.3.3,

$$I^{(d)}(\psi) \leq I^{(d)}(\varphi) + \max_{\{1 \leq i \leq d : b_i > 0\}} \frac{c_i}{b_i}(y - x) \, .$$

Hence,

$$\begin{aligned} V_>(y) \leq I^{(d)}(\psi) &\leq I^{(d)}(\varphi) + \max_{1 \leq i \leq d : b_i > 0} \frac{c_i}{b_i}(y - x) \\ &< V_>(x) + \max_{\{1 \leq i \leq d : b_i > 0\}} \frac{c_i}{b_i}(y - x) + \epsilon \, . \end{aligned}$$

This leads to $V_>(y) - V_>(x) \leq \max_{\{1 \leq i \leq d : b_i > 0\}} \frac{c_i}{b_i}(y - x) + \epsilon$. We obtain the desired result by letting ϵ tend to 0. Thus,

$$|V_>(y) - V_>(x)| \leq \max_{\{1 \leq i \leq d : b_i > 0\}} \frac{c_i}{b_i} |y - x| \, .$$

□

Lemma 5.3.5. *Assume that $J \cap I^+ \neq \emptyset$. Then, it holds that $V_>(c) = V_\geq(c)$.*

Proof. For any $\delta > 0$ we have that $V_>(c + \delta) \leq V_\geq(c)$. Hence, in view of Lemma 5.3.4,

$$\begin{aligned} |V_>(c) - V_\geq(c)| &= V_>(c) - V_\geq(c) \\ &\leq V_>(c + \delta) - V_\geq(c) \\ &\leq \max_{\{1 \leq i \leq d : b_i > 0\}} \frac{c_i}{b_i} |\delta| \, . \end{aligned}$$

Now, we let δ go to 0 and we obtain the stated result.

□

For a general stochastic network, with routing matrix Q and reflection matrix $Q = (I - Q^\intercal)$, the following large deviation principle holds.

Theorem 5.3.6. *For a fixed* $\mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{R}^d_+$ *assume that* $J \cap I^+ \neq \emptyset$. *The overflow probabilities* $\mathbf{P}(B(\mathbf{Z}_n) \geq c)$ *satisfy the following logarithmic asymptotics:*

$$\lim_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}(B(\mathbf{Z}_n) \geq c) = -V_>(c). \tag{5.10}$$

Proof. Note that the map B is a Lipschitz continuous map — being the composition of Lipschitz continuous continuous maps — w.r.t. the J_1 topology. More specifically, due to Lemma 5.2.5, is continuous w.r.t. to the product J_1 topology. Due to Theorem 5.2.6, the content component map is continuous with respect to the product J_1 topology. Thanks to *i)* of Lemma 5.2.3, we deduce upper and lower bounds for $B(\mathbf{Z}_n)$. For the upper bound, let $c > 0$ and recall that $I'(c) = \inf\{I^{(d)}(\xi) : B(\xi) = c\}$. Thanks to the upper bound in *i)* of Lemma 5.2.3, Lemma 5.3.2, and Result 5.2.5 we have that

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}(B(\mathbf{Z}_n) \geq c) &\leq -\lim_{\epsilon\to 0} \inf_{x \in [c-\epsilon\|B\|_{\mathrm{Lip}}, \infty)} I'(x) \\ &= -\lim_{\epsilon\to 0} V_>(c - \epsilon\|B\|_{\mathrm{Lip}}).\end{aligned}$$

Due to Lemma 5.3.4 we have that $V_>(\cdot)$ is continuous, and hence,

$$\lim_{\epsilon\to 0} V_>(c - \epsilon\|B\|_{\mathrm{Lip}}) \to V_>(c).$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}(B(\mathbf{Z}_n) \geq c) \leq -V_>(c),$$

and we obtain the desired upper bound.

For the lower bound, thanks to *i)* of Lemma 5.2.3, Lemma 5.3.2, and Result 5.2.5 we get that

$$\liminf_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}(B(\mathbf{Z}_n) \geq c) \geq -\inf_{x\in(c,\infty)} I'(x) = -V_>(c).$$

Invoking Lemma 5.3.5, we have that $V_>(c) = V_>(c)$, hence,

$$\liminf_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}(B(\mathbf{Z}_n) > c) \geq -V_>(c).$$

□

5.3.3 The multiple heavy-tailed on–off sources model

In this subsection, we study a model of stochastic fluid networks which is strongly related to the multiple on-off sources model. Let us give a description of this specific network. The network comprises of d nodes; each of the $d-1$ nodes has an exogenous input and their output is directed only to the node d. The node d does not have a dedicated exogenous input; moreover, it is the only node that allows the produced output of the $\{1,\dots,d-1\}$ nodes out of the network. The exogenous input at node $i \in \{1,\dots,d-1\}$ is generated by a compound Poisson process $J^{(i)}$ with mean λ_i and whose increments satisfy Assumption 5.2.1. The routing matrix Q is given by the following transition probabilities:

$$q_{ij} = \begin{cases} 1, & \text{for } i \in \{1,\dots,d\} \ \& \ j = d, \\ 0, & \textit{otherwise.} \end{cases}$$

The next figure is an example of this particular network.

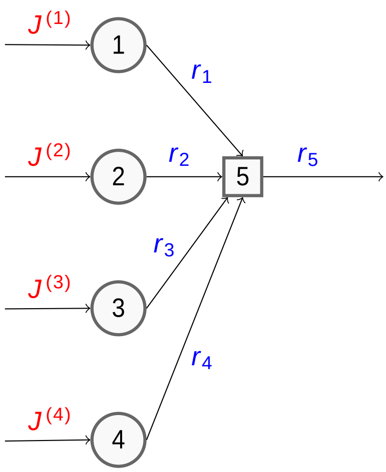

Figure 5.1: The figure depicts the dynamics of the network under consideration (with 5 nodes). Input at each one of the first 4 nodes is according to a compound Poisson process with mean λ_i. Each node is emptied at a constant rate and the processed fluid is routed to node 5 which is emptied at a constant rate r_5.

We derive an explicit computation of the decay rate for overflow probabilities for the buffer content process of the d node. Recall, $\mathbf{r} = (r_1,\dots,r_d)$, and $\lambda = (\lambda_1,\dots,\lambda_d)$. For a set $A \subseteq \{1,\dots,d-1\}$ let

$$J(\mathbf{r}, \lambda; A) \triangleq \frac{\sum_{i\in A} c_i(r_i - \lambda_i)}{\sum_{i\in A} r_i + \sum_{i\in\{1,\dots,d-1\}\setminus A} \lambda_i - r_d}.$$

Let the optimization problem $K(\mathbf{r}, \lambda; T)$ be as follows:

$$\min_{A\subseteq\{1,2,\dots,d-1\}} J(\mathbf{r}, \lambda; A) \tag{5.11}$$
$$\text{s.t.} \left(\sum_{i\in A} r_i + \sum_{i\in\{1,\dots,d-1\}\setminus A} \lambda_i - r_d\right) T \geq c;$$

Define the collection of K-dominant sets A^* with respect to $K(\mathbf{r}, \lambda; T)$. That is,

$$A^* = \arg\min_{A\subseteq\{1,\dots,d-1\}} \{K(\mathbf{r}, \lambda; T)\}. \tag{5.12}$$

Note that A^* may not be unique; A^* may contain more than one set. Let $K^*(\mathbf{r}, \lambda; T)$ be the optimal value of $K(\mathbf{r}, \lambda; T)$.

Lemma 5.3.7. *Consider the functional optimization problem*

$$F^*(c) \triangleq \inf_{\phi_1,\phi_2,\dots,\phi_d} \sum_{i=1}^{d-1} c_i \left[\sum_{\{t:\,\phi_i(t)\neq\phi_i(t-)\}} (\phi_i(t) - \phi_i(t-))\right]$$
$$s.t.\quad \psi^{(d)}(\phi)(T) \geq c;$$
$$\phi^{(d)}(t) = -(Q\mathbf{r})_d\, t;\ t\in[0,T];\ and$$
$$\phi^{(i)} \in \mathbb{D}_{(\lambda - Q\mathbf{r})_i}[0,T] \cap \mathbb{D}^{\leqslant 1}[0,T]\ for\ i = 1,\dots,d. \tag{5.13}$$

Then,

$$F^*(c) = \begin{cases} c\,K^*(\mathbf{r}, \lambda; T) & \text{if } T\left(\sum_{i=1}^{d-1} r_i - r_d\right) \geq c \\ \infty & \text{if } T\left(\sum_{i=1}^{d-1} r_i - r_d\right) < c. \end{cases} \tag{5.14}$$

Proof. The proof is deferred to Section 5.4. □

Theorem 5.3.8. *It holds that,*

$$\lim_{n\to\infty} \frac{1}{L(n)n} \log \mathbf{P}\left(\mathbf{Z}_n^{(d)}(T) \geq y\right) = -K^*(\mathbf{r}, \lambda; T)(y \vee 0);$$

Proof. Thanks to Lemma 5.3.7 we have an explicit computation of the decay rate. Applying *iii*) of Lemma 5.2.3 we derive the logarithmic asymptotics for overflow probabilities for the buffer content process of the d-labeled node. □

5.4 Complementary proofs

This section contains proofs for key results used in the main body of this chapter.

5.4.1 Proofs of Lemma 5.3.3, and Lemma 5.3.2

Proof of Lemma 5.3.3. We prove *i)* by using induction and the continuity of ( ), and ( ). Note that for a $\in \prod_{i=1}^{k} D[0;T]$ the regular component ( ) is the limit function of $y_n(\ )$ where $y_{n+1}(\ ) = \ (y_n(\ ))$ and $y_0(\ )(s) = \max\{\sup_{s \ t} \ (s)\}; 0\}$. Recall that $=$ over $[0;t]$; $t \quad T$. It can be easily checked that $\max\{\sup_{s \ t} \ (s)\}; 0\} = \max\{\sup_{s \ t} \ (s)\}; 0\}$ for $t \quad T$. Therefore, $y_0(\ ) = y_0(\ )$ over $[0;T \quad]$. Since $y_1(\ ) = \ (y_0(\ )$, and $y_1(\ ) = \ (y_0(\ )$ we see that for any $t \quad T$

$$
\begin{aligned}
(y_0(\ )(t) &= \sup_{s \ t} \{0; \ (s) + Qy_0(\ )(s)\} \\
&= \sup_{s \ t} \left\{ 0; \ (s) + Qy_0(\ )(s) \right\} = \ (y_0(\ )(t).
\end{aligned}
$$

For the induction step, let $y_k(\ ) = y_k(\ )$ over $[0;T \quad]$. Then, for any $t \quad T$ we have

$$
\begin{aligned}
y_{k+1}(\ )(t) &= \sup_{s \ t} \{0; \ (s) + Qy_k(\ )(s)\} \\
&= \sup_{s \ t} \left\{ 0; \ (s) + Qy_k(\ )(s) \right\} = y_{k+1}(\ )(t).
\end{aligned}
$$

Since the inequality holds for every $n \in \mathbb{N}$, we have that $(\ )(t) = \ (\ )(t)$, $t \in [0;T \quad]$. Lastly, in view of the continuity of ( ), and ( ) we get

$$
(\ )(T) = \lim_{\ \to 0} \ (\ )(T \quad) = \lim_{\ \to 0} \ (\ )(T \quad) = \ (\ )(T);
$$

hence ( ) = ( ). For *ii)*, observe that

$$
(\ )(T) = \ (T) + \ (\ )(T) = \ (T) + a + \ (\ )(T) = \ (\ )(T) + a.
$$

Lastly, we prove *iii)*. Let $= (^{(1)}, \dots, ^{(d)})$, and $= (^{(1)}, \dots, ^{(d)})$. Recall, the function $x \mapsto x$; $\in (0,1)$ is subadditive. Let

$$I_i(^{(i)}) = c_i \sum_{t \in [0,T]:\ ^{(i)}(t) \neq ^{(i)}(t-)} \left| ^{(i)}(t) - ^{(i)}(t-) \right| .$$

For any $i \in \{1, \dots, d\}$, we have

$$\begin{aligned} I_i(^{(i)}) &= c_i \sum_{t \in [0,T]:\ ^{(i)}(t) \neq ^{(i)}(t-)} \left| ^{(i)}(t) - ^{(i)}(t-) \right| \\ &= c_i \sum_{t \in [0,T):\ ^{(i)}(t) \neq ^{(i)}(t-)} \left| ^{(i)}(t) - ^{(i)}(t-) \right| \\ &\quad + \left(^{(i)}(T) - ^{(i)}(T-) + a_i \right) \\ &\leq c_i \sum_{t \in [0,T):\ ^{(i)}(t) \neq ^{(i)}(t-)} \left| ^{(i)}(t) - ^{(i)}(t-) \right| \\ &\quad + \left(^{(i)}(T) - ^{(i)}(T-) \right) + c_i a_i \\ &= I_i(^{(i)}) + c_i a_i . \end{aligned}$$

Therefore,

$$I^{(d)}(\ ) = \sum_{i=1}^{d} I_i(^{(i)}) \leq \sum_{i=1}^{d} I_i(^{(i)}) + \sum_{i=1}^{d} c_i a_i = I^{(d)}(\ ) + \sum_{i=1}^{d} c_i a_i .$$

□

We continue with Lemma 5.3.2.

Proof of Lemma 5.3.2. We start with some preliminary observations. Recall that Q is the reflection matrix, and it is invertible with

$$Q^{-1} = (I - Q^{\intercal})^{-1} = I + Q + Q^2 + \dots .$$

Consequently, Q^{-1} is a matrix with non-negative entries. Next, we continue with an observation (O1): if $u, v \in \mathbb{R}^k$, and A is such that $A \in \mathbb{R}_+^{k \times k}$, then $u \leq v$ — in every component — implies $Au \leq Av$.

Let $= (^{(1)}, \dots, ^{(d)}) \in \prod_{i=1}^{d} \mathbb{D}^{(\ \mathbf{Qr})_i}[0,T]$. Since $^{(i)} \in \mathbb{D}^{(\ \mathbf{Qr})_i}[0,T]$, we have the following representation:

$$^{(i)}(t) = (\quad \mathbf{Qr})_i \; t + \sum_{j=1} x_j^{(i)} \mathbb{1}\left\{\left[u_j^{(i)}; T \right]\right\}(t); \; t \in [0;T]:$$

The proof of our statement is a consequence of the following steps.

Step 1: There exists a path $\tilde{\ } \in \prod_{i=1}^{d} \mathbb{D}^{(\quad \mathbf{Qr})_i}[0;T] \cap \mathbb{D}^{\leq 1}[0;T]$ such that $\tilde{\ } \quad$, and $\tilde{\ }(T) = \ (T)$.

Proof of Step 1. Let $s^{(i)} = \sup\{t : \ ^{(i)}(t) \quad ^{(i)}(t \ ) > 0\}$ for each $i = 1; : : : ; d$, and let

$$\tilde{\ }^{(i)}(t) = (\quad Qr)_i \; t + \sum_{j=1} x_j^{(i)} \; \mathbb{1}\left\{\left[s^{(i)}; T \right]\right\}(t); \; t \in [0;T]:$$

Due to construction, $\tilde{\ } \quad$ component-wise, and $\tilde{\ }(T) = \ (T)$.

Step 2: It holds that $\ (\tilde{\ })(T) \quad (\ )(T)$.

Proof of Step 2. Due to Result 5.2.3, the statement is obvious.

Step 3: It holds that $Q \ (\tilde{\ })(T) \quad Q \ (\ )(T)$.

Proof of Step 3. We prove **Step 3** by contradiction. Suppose that $Q \ (\tilde{\ })(T) < Q \ (\ )(T)$. Since the matrix Q is invertible and $Q^{\ 1} \in \mathbb{R}_+^{k \ k}$, invoking observation (O1), we end up with $\ (\tilde{\ })(T) < \ (\ )(T)$ which is not possible due to **Step 2**.

Now, we conclude the proof of our statement. Recall that $\ (\ )(T) = \ (T) + Q \ (\ )(T)$. From **Step 3** and $\tilde{\ }(T) = \ (T)$, *ii)* of our lemma is obvious. For *i)*, due to the sub-additivity of the function $x \mapsto x \ ; \quad \in (0;1)$, we notice that

$$
\begin{aligned}
\Gamma^{(d)}(\tilde{\ }) &= \sum_{i=1} c_i \left[\sum_{\{t : \tilde{\ }^{(i)}(t) \ \tilde{\ }^{(i)}(t \ ) > 0\}} (\tilde{\ }^{(i)}(t) \quad \tilde{\ }^{(i)}(t \ )) \right] \\
&= \sum_{i=1} c_i \sum_{j=1} x_j^{(i)} \qquad \sum_{i=1} c_i \sum_{j=1} x_j^{(i)} \\
&= \sum_{i=1} c_i \left[\sum_{\{t : \ ^{(i)}(t) \ \ ^{(i)}(t \ ) > 0\}} (\ ^{(i)}(t) \quad ^{(i)}(t \ )) \right] \\
&= \Gamma^{(d)}(\ ):
\end{aligned}
$$

□

5.4.2 Proof of Lemma 5.3.7

In the next lemma, we compute the decay rate associated with a certain stochastic uid network, namely the so-called multiple on-o sources network with semiexponential input.

The exogenous input at node $i \in \{1, \ldots, d-1\}$ is generated by a compound Poisson process $J^{(i)}$ with mean $_i$ and whose increments satisfy Assumption 5.2.1. In this case, the probability measures of the content vector $\mathbf{X}_n = X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(d)}$ satisfy the extended LDP in $\prod_{i=1}^{d} \mathbb{D}^{(Q\mathbf{r})_i}[0,T]$ with the product J_1 topology, with speed $L(n)n$, and rate function

$$I^{(d)}(^{(1)}, \ldots, ^{(d)}) = \begin{cases} \sum_{j=1}^{d} c_j \sum_{t \in [0,1]} I(^{(j)}) & \text{if } ^{(j)} \in \mathbb{D}^{(Q\mathbf{r})_i}[0,T] \\ & \text{for } j = 1, \ldots, d-1, \\ & \text{and } ^{(d)}( ) = (Q\mathbf{r})_d( ), \\ \infty, & \text{otherwise,} \end{cases} \tag{5.15}$$

where $I(^{(j)}) = ^{(j)}(t) \quad ^{(j)}(t \quad) \quad .$

Proof of Lemma 5.3.7. Before we embark on solving the functional optimization problem, we explicitly present the main components needed for the computation of the regulator component and the content component map. To this end, we give the routing, and the re ection matrix of our network. Recall, the routing matrix Q is given by the following transition probabilities:

$$q_{ij} = \begin{cases} 1, & \text{for } i \in \{1, \ldots, d\} \ \& \ j = d, \\ 0, & \textit{otherwise.} \end{cases} \tag{5.16}$$

Subsequently, the re ection matrix Q is

$$Q_{ij} = \begin{cases} 1, & \text{for } i = j, \\ 1, & \text{for } j \in \{1, \ldots, d\} \ \& \ i = d, \\ 0, & \textit{otherwise.} \end{cases} \tag{5.17}$$

The existence and uniqueness of the regulator process (( )) has been established in the preliminaries. For that reason, the optimization problem in (5.13) is well-de ned and a solution exists. The optimization problem is meaningful only for one-step functions, and for this reason, we will focus on the interaction of one step-functions — with slope $(\quad Q\mathbf{r})_i$ for each $i = 1, \ldots, d-1$ — with the d coordinates of the re ection map $\mathbf{R}$ i.e, $(\mathbf{R}^{(1)}, \ldots, \mathbf{R}^{(d)})$. Recall our terminology:

$^{(i)}(\ )$ is the i-th coordinate of the regulator component map, and we apply similar terminology to the content component map .

Before we embark on calculations, we provide a sketch of our strategy for the computation of the optimal value of (5.13):

1) for paths in $\prod_{i=1}^{d} D^{(\quad Qr)_i}[0;T] \setminus D^{61}[0;T]$; we explicitly compute the regulator component $(\ )$ and the content component map $(\ )$;

2) we reduce the optimization problem to a nite dimensional optimization problem and subsequently compute the optimal value $F\ (c)$ for every $c > 0$.

Step 1. Recall that we are interested in the bu er content of the d-labeled node. Recall the de nition of the regulator component; for a $\ 2\ \prod_{i=1}^{d} D[0;T]$

$$(\ ) = \inf f\ (\ )g \qquad \text{where} \qquad (\ ),\ f\ 2\ \prod_{i=1}^{d} D^{"}[0;T]:\ \ + Q\ \ \ 0g:$$

Due to the form of the matrix Q, the derivation of the regulator process $(\ )$ is the in mum (coordinate-wise) of all the functionals $\mathbf{w} = (w_1(\ ); : : : ; w_d(\ ))\ 2\ \prod_{i=1}^{d} D^{"}[0;T]$ that satisfy the following set of inequalities:

$$\mathbf{w} = \begin{cases} w_i(t) & ^{(i)}(t); \text{ for } i\ 2\ f1; : : : ; d\quad 1g\ \&\ t\ 2\ [0;T]; \\ w_d(t) & ^{(d)}(t) + \sum_{i=1}^{d} w_i(t); \qquad t\ 2\ [0;T]: \end{cases} \tag{5.18}$$

For every $i \neq d$ the coordinate-wise regulator component process $^{(i)}(\ )$ is the smallest functional such that $w_i\qquad ^{(i)}$. By default, $^{(i)}$ is an one-step function with discontinuity size equal to x_i which takes place at time $u_i\ 2\ [0;T]$. De ne $t(x_i;u_i) = u_i + \frac{x_i}{r_i\quad i}$; then, the i-th coordinate of the regulator component is as follows

$$^{(i)}(\ )(t) = \begin{cases} (Q\mathbf{r}\qquad)_i\ \ t; & \text{for } t\ 2\ [0;u_i]; \\ (Q\mathbf{r}\qquad)_i\ \ u_i; & \text{for } t\ 2\ (u_i; t(x_i;u_i)]; \\ (Q\mathbf{r}\qquad)_i\ \ u_i + (Q\mathbf{r}\qquad)_i\ \ (t\quad t(x_i;u_i)); & \text{for } t\ 2\ [t(x_i;u_i);T]: \end{cases} \tag{5.19}$$

By the de nition of the in mum, it is obvious that no other con guration, except for the proposed functionals $f\ ^{(i)}(\ )g_{i=1}^{d\ \ 1}$, achieves the in mum coordinate-wise. Given the $d\quad 1$ regulator components which were displayed in (5.19), for each

$i \in \{1, \ldots, d-1\}$ we can compute the $d-1$ coordinates of the content component of the reflection map i.e.,

$$^{(i)}(\ ) = \begin{cases} 0; & \text{for } t \le u_i; \\ x_i + (\ \ Q\mathbf{r})_i\ (t - u_i); & \text{for } t \in (u_i, t(x_i, u_i)]; \\ 0; & \text{for } t \ge t(x_i, u_i). \end{cases} \tag{5.20}$$

Now, we focus on the calculation of $^{(d)}(\ )$, or, in words, the d-th coordinate of regulator component. Due to the definition of the regulator map, $^{(d)}(\ )$ must be the smallest non-decreasing functional so that

$$^{(d)}(\ )(\ ) = {}^{(d)}(\ ) + w_d(\ ) - \sum_{i=1}^{d} {}^{(i)}(\ )(\ ) \ge 0; \quad w_d(\ ) \in \mathbb{D}^{\uparrow}[0,T] \cap \mathbb{D}^{\le 1}[0,T].$$

From the inequality above, we can deduce that

$$^{(d)}(\ )(t) = \sup_{s<t} \left\{ \sum_{i=1}^{d} {}^{(i)}(\ )(s) - {}^{(d)}(s) \right\}. \tag{5.21}$$

Now, we can explicitly compute $^{(d)}(\ )(T)$. In view of (5.21), (5.17), (5.19), and (5.21),

$$\begin{aligned}
&^{(d)}(\ )(T) \\
&= {}^{(d)}(T) + {}^{(d)}(\ )(T) - \sum_{i=1}^{d-1} {}^{(i)}(\ )(T) \\
&= {}^{(d)}(T) - \sum_{i=1}^{d-1} {}^{(i)}(\ )(T) + \sup_{s<T} \left\{ \sum_{i=1}^{d-1} {}^{(i)}(\ )(s) - {}_d(s) \right\} \\
&= \sup_{s<T} \left\{ {}^{(d)}(T) - \sum_{i=1}^{d-1} {}^{(i)}(\ )(T) - {}_d(s) - \left(\sum_{i=1}^{d-1} {}^{(i)}(\ )(s) \right) \right\} \\
&= \sup_{s<T} \left\{ \left(r_d - \sum_{i=1}^{d-1} r_i \right)(T - s) - \sum_{i=1}^{d-1} [(Q\mathbf{r}\ \ )_i\ T - x_i] + \sum_{i=1}^{d-1} {}^{(i)}(\ )(s) \right\}.
\end{aligned}$$

We simplify the above expression,

$$
\begin{aligned}
&{}^{(d)}(\ )(T) \qquad (5.22)\\
&= \sup_{s<T}\left(r_d \left(\sum_{i=1}^{d-1} r_i \quad T \sum_{i=1}^{d-1}[(Q\mathbf{r} \quad)_i \;\; T \quad x_i] \right) \quad _d(s) \quad \sum_{i=1}^{d-1} {}^{(i)}(\ )(s) \right)\\
&= \sup_{s<T}\left(r_d \;\; T + T \sum_{i=1}^{d-1} r_i \sum_{i=1}^{d-1}[(Q\mathbf{r} \quad)_i \;\; T \quad x_i] \quad _d(s) \quad \sum_{i=1}^{d-1} {}^{(i)}(\ )(s) \right)\\
&= \sup_{s<T}\left(r_d \;\; T + T \sum_{i=1}^{d-1} r_i \sum_{i=1}^{d-1}[(Q\mathbf{r} \quad)_i \;\; T \quad x_i] \quad _d(s) \quad \sum_{i=1}^{d-1} {}^{(i)}(\ )(s) \right)\\
&= \sup_{s<T}\left(r_d \;\; T + T \sum_{i=1}^{d-1} \;_i + \sum_{i=1}^{d-1} x_i \quad r_d \sum_{i=1}^{d-1} r_i \quad s \sum_{i=1}^{d-1} {}^{(i)}(\ )(s) \right)\\
&= \sup_{s<T}\left\{ \underbrace{ r_d \;\; (T \quad s) + T \sum_{i=1}^{d-1} \;_i + \sum_{i=1}^{d-1} x_i + \sum_{i=1}^{d-1} [\ {}^{(i)}(\ )(s) \quad r_i \;\; s] }_{,\ '(s)} \right\} : \qquad (5.23)
\end{aligned}
$$

Since we have explicit computations for the regular, and content component maps (5.23), we make further computations to reduce the feasible set to the set that contains paths of the following form:

$$
{}^{(i)}(t) = x_i \;\; \mathbb{1} \quad T \quad \frac{x_i}{(Q\mathbf{r} \quad)_i}; T \quad (t) + (\quad Q\mathbf{r})_i \;\; t; \;\; t \in [0; T];
$$

where $x_i \in [0; T(r_i \quad _i)]$ and for each $i = 1; \ldots; d \quad 1:$

For a step function ${}^{(1)}; \ldots; {}^{(d)}$ in $\prod_{i=1}^{d} D^{(\quad Q\mathbf{r})_i}[0; T] \cap D^{\uparrow 1}[0; 1]$ we have the following representation:

$$
{}^{(i)}(t) = x_i \mathbb{1}\{[u_i; T]\}(t) + (\quad Q\mathbf{r})_i \;\; t; \qquad \text{for each } i = 1; \ldots; d \quad 1:
$$

Note, the ${}^{(i)}$'s are cadlag functions. Consequenlty, the right limit exists, $x_i < \infty$ for every $i = 1; \ldots; d \quad 1$. In view of (5.19), the i-th coordinate of the regulator component $\ (\ )$ is equal to $\sup_{s \le t}\{\ {}^{(i)}(s)\} \vee 0$, which, in turn, is equal to

$$^{(i)}(\ ) = \begin{cases} 0; & \text{for } t \quad u_i; \\ x_i + (\quad Qr)_i \ (t \quad u_i); & \text{for } t \ 2 \ (u_i; t(x_i; u_i)]; \\ 0; & \text{for } t \quad t(x_i; u_i): \end{cases} \tag{5.24}$$

where $t_i(x_i; u_i) = u_i + \frac{x_i}{(Qr \quad)_i} \quad T$. Moreover, due to, 5.22,

$$^{(d)}(\ )(T) = \sup_{s<T} \left({}^{(d)}(T \quad s) + \sum_{i=1}^{d \ 1} {}^{(i)}(\ )(s) \quad \sum_{i=1}^{d \ 1} {}^{(i)}(\ )(T) \right)$$

where $^{(i)}(\ )(T) = (Qr \quad)_i \ T \quad x_i$. Based on the $^{(i)}$'s de ned above, we construct paths $\tilde{\ } = (\tilde{\ }^{(1)}; \ldots; \tilde{\ }^{(d \ 1)})$ which have discontinuities at speci c times of the domain $[0; T]$ and are still feasible solutions to the optimization problem in (5.13) i.e;

$$^{(d)}(\tilde{\ })(T) \qquad ^{(d)}(\ )(T): \tag{5.25}$$

For every $^{(i)}$ we consider the step function $\tilde{\ }^{(i)}$ where

$$\tilde{\ }^{(i)}(t) = x_i \ \mathbb{1} \quad T \quad \frac{x_i}{(Qr \quad)_i}; T \quad (t) + (\quad Qr)_i \ t; \ t \ 2 \ [0; T] \tag{5.26}$$

which induces a regulator component $^{(i)}(\tilde{\ })$ where

$$^{(i)}(\tilde{\ })(t) = \begin{cases} (Qr \quad)_i \ t; & \text{if } t \ 2 \ \left[0; T \quad \frac{x_i}{(Qr \quad)_i}\right); \\ (Qr \quad)_i \quad T \quad \frac{x_i}{(Qr \quad)_i} \ ; & \text{if } t \ 2 \ \left[T \quad \frac{x_i}{(Qr \quad)_i}; T\right]: \end{cases} \tag{5.27}$$

It is easy to check that $^{(i)}(\tilde{\ })(\ ) \quad ^{(i)}(\ )(\ )$, and $^{(i)}(\tilde{\ })(T) = (Qr \quad)_i \ T \quad x_i = {}^{(i)}(\ )(T)$. In view of the previously mentioned inequalities, (5.25) holds, and hence, the $\tilde{\ }^{(i)}$'s are a feasible solution to (5.13). Furthermore,

$$r^{(d)}(\tilde{\ }) = \sum_{i=1}^{d} c_i \left[\sum_{\{t: \tilde{\ }^{(i)}(t) \neq \tilde{\ }^{(i)}(t)\}} \tilde{\ }^{(i)}(t) \quad \tilde{\ }^{(i)}(t) \right]$$

$$= \sum_{i=1}^{d} c_i \ x_i = \sum_{i=1}^{d} c_i \left[\sum_{\{t: \ ^{(i)}(t) \neq \ ^{(i)}(t)\}} {}^{(i)}(t) \quad ^{(i)}(t) \right] = r^{(d)}(\ ):$$

In conclusion, for every feasible solution — feasible for (5.13) — in the subspace $\prod_{i=1}^{d} D^{(\lambda - Q\mathbf{r})_i}[0,T] \cap D^{\uparrow 1}[0,1]$, we constructed a feasible configuration $\{\tilde\theta^{(i)}\}_{i=1}^{d}$, where $\tilde\theta^{(i)} \in \prod_{i=1}^{d} D^{(\lambda - Q\mathbf{r})_i}[0,T] \cap D^{\uparrow 1}[0,1]$, which induces the same cost w.r.t. the objection function in (5.13) and with specified jump times as seen in (5.26).

Step 2. Before we reduce the optimization problem to a discrete one, we study its associated constraints. For each $i = 1,\dots,d-1$, set $v_i = T - \frac{x_i}{(Q\mathbf{r}-\lambda)_i}$. We compute the supremum in (5.22). Towards this end, we claim that $\varphi(\tilde\theta)(\cdot)$ in (5.23) achieves maximum over one of the following points $\{0, v_1,\dots,v_{d-1}, T\}$. To see this, observe that $\varphi(\tilde\theta)(\cdot)$ is a piecewise linear function whose slope changes at times v_i, for $i = 1,\dots,d-1$. Consequently, the maximum is achieved over $\{0, v_1,\dots,v_{d-1}, T\}$. Note that $\varphi'(\tilde\theta)(\cdot)$ is positive over $[0, \min_{1\le i\le d-1} v_i)$, and hence, $\varphi(\tilde\theta)(\cdot)$ cannot attain maximum over $s = 0$; moreover, $\varphi(\tilde\theta)(T) = 0$ therefore, $\varphi(\tilde\theta)(\cdot)$ attains maximum over $\{v_1,\dots,v_{d-1}\}$.

We compute (5.23) for every $s = v_i$, $i \in \{1,\dots,d-1\}$, and we solve the associated variational problem. Now, choose v_{i^*} where $i^* \in \{1,\dots,d\}$. Naturally, there would exist some $v_j; j \neq i^*$; such that $v_j \le v_{i^*}$. Denote with S_{i^*} the set which contains such v_j's i.e., $S_{i^*} = \{j \in \{1,\dots,d-1\} : v_j \le v_{i^*}\}$. Obviously, S_{i^*} contains at least one maximal element which is equal to v_{i^*}. Since $v_i = T - \frac{x_i}{(Q\mathbf{r}-\lambda)_i}$, the maximal element v_{i^*} is achieved with $\frac{x_{i^*}}{(Q\mathbf{r}-\lambda)_{i^*}} = \min_{j\in S_{i^*}} \frac{x_j}{(Q\mathbf{r}-\lambda)_j}$. In this case, based on (5.26), and (5.27) the optimization problem (5.13) reduces to

$$
\begin{aligned}
&\min_{x_1,\dots,x_{d-1}\in\mathbb{R}_+} \sum_{i=1}^{d-1} c_i x_i \\
&s.t.\ \left(\sum_{j\in S_{i^*}} r_j + \sum_{j\notin S_{i^*}} \lambda_j - r_d\right) \min_{j\in S_{i^*}} \frac{x_j}{r_j-\lambda_j} + \sum_{j\notin S_{i^*}} x_j \ge c; \\
&T(r_j-\lambda_j) \ge x_j \text{ for } j \in \{1,\dots,d-1\}; \\
&\frac{x_j}{r_j-\lambda_j} \ge \frac{x_{i^*}}{r_{i^*}-\lambda_{i^*}} \text{ for } j \in S_{i^*}; \\
&\frac{x_j}{r_j-\lambda_j} \le \frac{x_{i^*}}{r_{i^*}-\lambda_{i^*}} \text{ for } j \notin S_{i^*}.
\end{aligned}
\tag{5.28}
$$

Obviously, this problem is equal to

$$\min_{x_1,\dots,x_{d-1}\in\mathbb{R}_+} \sum_{i=1}^{d-1} c_i x_i \tag{5.29}$$

$$
\begin{aligned}
s.t. \; & \left(\sum_{j\in S_i} r_j + \sum_{j\notin S_i} \delta_j - r_d\right) \min_{j\in S_i} \frac{x_j}{r_j-\delta_j} + \sum_{j\notin S_i} x_j = c; \\
& T(r_j-\delta_j) \ge x_j \text{ for } j\in\{1,\dots,d-1\}; \\
& \frac{x_j}{r_j-\delta_j} \ge \frac{x_i}{r_i-\delta_i} \text{ for } j\in S_i; \\
& \frac{x_j}{r_j-\delta_j} \le \frac{x_i}{r_i-\delta_i} \text{ for } j\notin S_i;
\end{aligned}
$$

otherwise, we would be able to construct a feasible solution that achieves a smaller value in the objective function. Observe that only the value of $\frac{x_i}{r_i-\delta_i}$ is relevant in the third constraint of the above optimization problem. That is, for the rest of x_j's in S_i we have that $x_j = \frac{x_i(r_j-\delta_j)}{r_i-\delta_i}$. The optimization problem (5.29) can be reduced to

$$\min_{x_i,\{x_j\in\mathbb{R}_+ : j\notin S_i\}} x_i \sum_{j\in S_i} c_j \left(\frac{r_j-\delta_j}{r_i-\delta_i}\right) + \sum_{j\notin S_i} c_j x_j \tag{5.30}$$

$$s.t. \left(\sum_{j\in S_i} r_j + \sum_{j\notin S_i} \delta_j - r_d\right) \min_{j\in S_i} \frac{x_j}{r_j-\delta_j} + \sum_{j\notin S_i} x_j = c; \tag{5.31}$$

$$
\begin{aligned}
& T(r_j-\delta_j) \ge x_j \text{ for } j=i \text{ ; and } j\notin S_i; \\
& x_j \le \frac{x_i(r_j-\delta_j)}{r_i-\delta_i} \text{ for } j\notin S_i :
\end{aligned}
$$

The feasible region of the optimization problem above is the intersection of the closed boxes $\mathrm{H}_i = \left\{0 \le x_i \le \frac{x_i(r_j-\delta_j)}{r_i-\delta_i}\right\}$, $\mathrm{B}_i = \{0 \le x_i \le T(r_i-\delta_i)\}$, and the hyperplane P induced by (5.31) i.e; $P = \bigcap_{j\in\{1,\dots,d-1\}\setminus S_i} \mathrm{H}_j \cap \mathrm{B}_j \cap \mathrm{B}_i \cap \mathrm{P}$. We can easily see that P is a closed and bounded convex set. From convex optimization theory, see Corollary 33.2.1 in [85], the minimum of the concave objective function at (5.30) is achieved over the extreme points of P. Consequently, the optimal solution is a vector comprised by the elements $x_j = x_i \frac{(r_j-\delta_j)}{r_i-\delta_i}$, or, $x_j = 0$ (we can easily see that these elements form the extreme points of P). In conclusion, our analysis above entails that the optimal solution can be achieved by the elements

$$x_i = \frac{x_i\,(r_i - \lambda_i)}{r_i - \lambda_i}; \text{ or } x_i = 0 \text{ for } i \in \{1, \ldots, d-1\} \setminus S_i\,. \tag{5.32}$$

Due to (5.32), (5.31) is equivalent to

$$\left(\sum_{j \in A} r_j + \sum_{j \notin A} \lambda_j - r_d\right) \frac{x_i}{r_i - \lambda_i} = c, \tag{5.33}$$

where $A \subseteq \{1, \ldots, d-1\}$ such that $S_i \subseteq A$. Obviously, x_i is such that

$$x_i = \frac{c\,(r_i - \lambda_i)}{\sum_{j \in A} r_j + \sum_{j \notin A} \lambda_j - r_d}. \tag{5.34}$$

Since $x_i \leq T\,(r_i - \lambda_i)$, we must have that $\sum_{j \in A} r_j + \sum_{j \notin A} \lambda_j - r_d \geq T^{-1} c$. Therefore, in view of (5.32), the objective function is equal to

$$\left(\frac{c\,(r_i - \lambda_i)}{\sum_{j \in A} r_j + \sum_{j \notin A} \lambda_j - r_d}\right) \sum_{j \in A} \frac{r_j - \lambda_j}{r_i - \lambda_i}\,. \tag{5.35}$$

In the case of $T\left(\sum_{j=1}^{d} r_j - r_d\right) < c$, the optimization problem is infeasible and we interpret its value as ∞. To conclude our analysis, we relax the constraint $S_i \subseteq A$ optimizing (5.35) over all subsets $A \subseteq \{1, \ldots, d\}$ which satisfy the inequality $T\left(\sum_{j \in A} r_j + \sum_{j \notin A} \lambda_j - r_d\right) \geq c$, and hence, we reach the conclusion of our lemma. □

5.4.3 Proof of Theorem 5.2.6

Recall, $\prod_{i=1}^{d} \mathbb{D}[0, T]$ is the Skorokhod space equipped with the product J_1 topology and $\mathbb{D}^{\uparrow}[0, T] \triangleq \{\xi \in \mathbb{D}[0, T] : \xi \text{ is non-decreasing on } [0, T]\}.$ Due to [96] (p.g. 486), $\mathbb{D}^{\uparrow}[0, T]$ is a closed subspace of $\mathbb{D}[0, T]$ with the J_1 topology, hence, $\prod_{i=1}^{d} \mathbb{D}^{\uparrow}[0, T]$ is a closed subspace of $\prod_{i=1}^{d} \mathbb{D}[0, T]$ with the product J_1 topology. Since $\mathbb{D}^{\lambda}[0, T]$ is the image of $\mathbb{D}^{\uparrow}[0, T]$ using the homeomorphism Φ we have that $\prod_{i=1}^{d} \mathbb{D}^{\lambda_i}[0, T]$ is a closed subset of $\prod_{i=1}^{d} \mathbb{D}[0, T]$. Let $\lambda = (\lambda_1, \ldots, \lambda_d)$, and let $\|\lambda\| \equiv \|\lambda\|_\infty = \max_{1 \leq i \leq d} |\lambda_i|$.

Some supporting lemmas

Lemma 5.4.1. *Suppose that ; are strictly increasing functions such that*

i) $(0) = (0) = 0$,

ii) $(T) = (T) = T$,

iii) $\| \; e\|_\infty <$, *and* $\| \; e\|_\infty <$.

Then, $\| \; e\|_\infty < 2$.

Proof. Since $\| \; e\|_\infty = \| \;^{1}\|_\infty \; \| \; e\|_\infty + \| \;^{1} \; e\|_\infty$ and $\max\{\| \; e\|_\infty ; \| \; e\|_\infty\} <$, the statement follows. □

Lemma 5.4.2. *If* $w^{(i)}$ *is increasing, continuous so that* $w^{(i)}(0) = 0$, *and* $w^{(i)}(T) = T$ *for each* $i = 1; \ldots ; d$, *then* $w(s) = \min\{w^{(1)}(s); \ldots ; w^{(d)}(s)\}$, *and* $\hat{w}(s) = \max\{w^{(1)}(s); \ldots ; w^{(d)}(s)\}$ *are increasing, continuous so that* $w(0) = \hat{w}(0) = 0$, *and* $w(T) = \hat{w}(T) = T$.

Proof. The min and max of continuous and increasing functions is increasing and continuous. The other properties are easily verified. □

Lemma 5.4.3. *Let* $\in \prod_{i=1}^{d} D^{\,i}[0;T]$ *and* w *be an increasing function such that* $w(0) = 0; \; w(T) = T$. *Then, it holds that*

$$\| \; (\;)(w) \quad (\;)\|_\infty < K \max_{1 \le i \le d} | \;_i| \|w \; e\|_\infty ;$$

where K *is the Lipschitz constant associated with the Lipschitz continuity of w.r.t. the uniform metric.*

Proof. The proof is a consequence of the following two claims:

i) for every $s \in [0;T]$ there exists a $U(s) \in (0; \infty)$ so that $| \; (\;)(w(s)) \quad (\;)(s)| \le U(s)$; and

ii) it holds that $\sup_{s \in [0;T]} U(s) \le K \max_{1 \le i \le d} | \;_i| \|w \quad \|_\infty$.

For claim *i*), pick an $s \in [0;T]$. Then, there are two cases:

1) $w(s) \le s$, or

2) $w(s) \ge s$.

In case 1), since ( ) is an increasing function, ( )($w(s)$) ( )(s): Hence, we only need to bound ( )($w(s)$) ( )(s). Moreover, since $2 \prod_{i=1}^{d} D^{i}[0;T]$ we have that

$$(w(s)) = ((w(s) \quad s) + s) \quad (s) + \quad (w(s) \quad s): \tag{5.36}$$

Next, consider the path $\tilde{}_1$ where

$$\tilde{}_1 = \begin{cases} (t); & t \,2\, [0;s] \\ (s) + \quad (t \quad s); & t \,2\, [s;w(s)]: \end{cases}$$

Since $\tilde{}_1$ over $[0;w(s)]$, due to Result 5.2.3, we have that $(\tilde{}_1)(w(s))$ ( )($w(s)$). On the other hand, let

$$\tilde{}_2 = \begin{cases} (t); & t \,2\, [0;s] \\ (s); & t \,2\, [s;w(s)]: \end{cases}$$

By the de nition of $\tilde{}_2$ we have that $(\tilde{}_2)(w(s)) =$ ( )(s). Moreover, due to the construction of $\tilde{}_1$ we have that

$$\begin{aligned} (\ )(w(s)) \quad (\ )(s) \quad & (\tilde{}_1)(w(s)) \quad (\tilde{}_2)(w(s)) \\ & K \sup_{t2[0;w(s)]} j\tilde{}_1(s) \quad \tilde{}_2(s)j \\ & Kk \quad k_1 jw(s) \quad sj: \end{aligned}$$

For case 2), since ( ) is an increasing function, ( )(s) ( )($w(s)$): Moreover, since $2 \prod_{i=1}^{d} D^{i}[0;T]$ we have that

$$(s) = ((s \quad w(s)) + w(s)) \quad (w(s)) + \quad (s \quad w(s)): \tag{5.37}$$

Next consider the path $\tilde{}_1$ where

$$\tilde{}_1 = \begin{cases} (t); & t \,2\, [0;w(s)] \\ (s) + \quad (t \quad w(s)); & t \,2\, [w(s);s]: \end{cases}$$

Since $\tilde{}_1$ over $[0;s]$, due to Result 5.2.3, we have that $(\tilde{}_1)(s)$ ( )(s). On the other hand, let

$$\tilde{}_2(t) = \begin{cases} (t); & t \,2\, [0;w(s)] \\ (s); & t \,2\, [w(s);s]: \end{cases}$$

Due to the construction of $\tilde{\psi}_2$ we have that $\Phi(\tilde{\psi}_2)(w(s)) = \Phi(\psi)(s)$. Moreover,

$$\begin{aligned}\Phi(\psi)(w(s)) - \Phi(\psi)(s) &\le \Phi(\tilde{\psi}_1)(w(s)) - \Phi(\tilde{\psi}_2)(w(s)) \\ &\le K \sup_{t\in[0,w(s)]} |\tilde{\psi}_1(s) - \tilde{\psi}_2(s)| \\ &\le K\|\psi\|_\infty |w(s) - s|.\end{aligned}$$

For *ii)* of our statement observe that for every $s \in [0,T]$,

$$K\|\psi\|_\infty |w(s) - s| \le K\|\psi\|_\infty \|w - e\|_\infty.$$

Hence,

$$\|\Phi(\psi)(w) - \Phi(\psi)\|_\infty \le K \max_{1\le i\le d} |\psi_i| \|w - e\|_\infty.$$

□

Next, we need two more lemmas which will solidify our proof for the Lipschitz continuity of the regulator map in $\prod_{i=1}^{d} D^{\psi_i}[0,T]$.

Lemma 5.4.4. *Consider a vector $w = (w^{(1)},\dots,w^{d})$ of time deformations, $\hat{w}(\cdot) = \max\{w^{(1)}(\cdot),\dots,w^{(d)}(\cdot)\}$, and $\check{w}(\cdot) = \min\{w^{(1)}(\cdot),\dots,w^{(d)}(\cdot)\}$. For any $\psi \in \prod_{i=1}^{d} D^{\psi_i}[0,T]$,*

i) $\Phi(\psi(w^{(1)}),\dots,\psi(w^{(d)})) \le \Phi(\psi)(\check{w}) + 2K\|\psi\|_\infty(\|w - e\|_\infty)$, *and*

ii) $\Phi(\psi(w^{(1)}),\dots,\psi(w^{(d)})) + 2K\|\psi\|_\infty(\|w - e\|_\infty) \ge \Phi(\psi)(\hat{w})$.

Proof. We start with *i)*. Since $\psi \in \prod_{i=1}^{d} D^{\psi_i}[0,T]$ and $\check{w} \le \min\{w^{(1)},\dots,w^{(d)}\}$ we have that for each $i = 1,\dots,d$,

$$\psi^{(i)}(w^{(i)}(s)) - \psi^{(i)}(\check{w}(s)) \le \|\psi\|_\infty (w^{(i)}(s) - \check{w}(s)), \; s \in [0,T].$$

Therefore, due to Result 5.2.3, and the Lipschitz continuity of Φ with the uniform metric

$$\begin{aligned}&\Phi(\psi^{(1)}(w^{(1)}),\dots,\psi^{(d)}(w^{(d)})) \\ &\le \Phi(\psi^{(1)}(\check{w}) - \|\psi\|_\infty(w^{(1)} - \check{w}),\dots,\psi^{(d)}(\check{w}) - \|\psi\|_\infty(w^{(d)} - \check{w})) \\ &\le \Phi(\psi)(\check{w}) + K\|\psi\|_\infty \max_{1\le i\le d}(\|\check{w} - w^{(i)}\|_\infty) \\ &\le \Phi(\psi)(\check{w}) + 2K\|\psi\|_\infty \max_{1\le i\le d}(\|w^{(i)} - e\|_\infty).\end{aligned}$$

For *ii)*, since $2 \prod_{i=1}^{d} D^{i}[0;T]$, and $\hat{w}$ $\max\{w^{(1)};\ldots;w^{(d)}\}$ we have that

$$^{(i)}(\hat{w}(s)) \quad ^{(i)}(w^{(i)}(s)) \quad k \quad k_1 (\hat{w}(s) \quad w^{(i)}(s));$$

for each $i = 1;\ldots;d;$ and $s\ 2\ [0;T]$. Therefore, due to Result 5.2.3, and the Lipschitz continuity of with the uniform metric

$$
\begin{aligned}
&(\ ^{(1)};\ldots;\ ^{(d)})(\hat{w}) \\
&\quad (\ ^{(1)}(\hat{w}) \quad k \quad k_1 (\hat{w} \quad w^{(1)});\ldots;\ ^{(d)}(\hat{w}) \quad k \quad k_1 (\hat{w} \quad w^{(d)})) \\
&\quad (\ )(\hat{w}) + Kk \quad k_1 \max_{1 \ i \ d}(k\hat{w} \quad w^{(i)}k_1) \\
&\quad (\ )(w) + 2Kk \quad k_1 \max_{1 \ i \ d}(kw^{(i)} \quad ek_1):
\end{aligned}
$$

□

Lemma 5.4.5. *For any* $2 \prod_{i=1}^{d} D^{i}[0;T]$,

$$k \ (\ (w^{(1)});\ldots;\ (w^{(d)})) \quad (\ ^{(1)};\ldots;\ ^{(d)})k_1 \quad 3Kk \quad k_1 kw \quad ek_1 :$$

Proof. Due to Lemma 5.4.3, and Lemma 5.4.4,

$$
\begin{aligned}
&(\ (w^{(1)});\ldots;\ (w^{(d)})) \quad (\ ^{(1)};\ldots;\ ^{(d)}) \\
&\quad (\ )(w) \quad (\ ^{(1)};\ldots;\ ^{(d)}) + 2Kk \quad k_1 (kw \quad ek_1) \\
&\quad (\ ) + Kk \quad k_1 (kw \quad ek_1) \quad (\ ) + 2Kk \quad k_1 (kw \quad ek_1) \\
&= 3Kk \quad k_1 (kw \quad ek_1):
\end{aligned}
$$

For the other inequality, notice that

$$
\begin{aligned}
&(\ ^{(1)};\ldots;\ ^{(d)}) \quad (\ (w^{(1)});\ldots;\ (w^{(d)})) \\
&\quad (\ ) \quad (\ )(\hat{w}) + 2Kk \quad k_1 kw \quad ek_1 \\
&\quad 3Kk \quad k_1 kw \quad ek_1 :
\end{aligned}
$$

□

Lipschitz continuity of the reflection map

Proposition 5.4.6. *The regulator map* $: \prod_{i=1}^{d} D^{i}[0;T] \mapsto \prod_{i=1}^{d} D[0;T]$ *is Lipschitz continuous with the product J_1 topology.*

Proof. Given and , let be such that $d_p(\; ; \;) <$. Then, there exists $^{(i)}$ s.t. $k^{(i)} \quad ^{(i)} \quad ^{(i)}k_1 \_ k^{(i)} \quad ek_1 <$ for each $i = 1; : : : ; d$. Notice that

$$d_p(\; (\;); \; (\;)) \quad \inf_{w^{(1)};:::;w^{(d)} 2} \left(\sum_{i=1} k^{(i)}(\;) \quad w^{(i)} \quad ^{(i)}(\;)k_1 \right)$$

$$\sum_{i=1} k^{(i)}(\;) \quad ^{(1)} \quad ^{(i)}(\;)k_1$$

$$\sum_{i=1} k^{(i)}(\;) \quad ^{(1)} \quad ^{(i)}(^1 \quad ^{(1)}; : : : ; \; ^{(d)} \quad ^{(d)})k_1$$

$$+ \sum_{i=1} k^{(i)}(^1 \quad ^{(1)}; : : : ; \; ^{(d)} \quad ^{(d)}) \quad ^{(i)}(\; ^{(1)}; : : : ; \; ^{(d)})k_1$$

$$\sum_{i=1} k^{(i)}(\;) \quad ^{(1)} \quad ^{(i)}(^1 \quad ^{(1)}; : : : ; \; ^{(d)} \quad ^{(d)})k_1$$

$$+ d \quad K \max_{1 \; i \; d} k^{(i)} \quad ^{(i)} \quad ^{(i)}k_1$$

$$\sum_{i=1} k^{(i)}(\;) \quad ^{(1)} \quad ^{(i)}(^1 \quad ^{(1)}; : : : ; \; ^{(d)} \quad ^{(d)})k_1 + d \quad K \quad =2$$

$$= \sum_{i=1} k^{(i)}(\;) \quad ^{(i)}(^1 \quad ^{(1)}; : : : ; \; ^{(d)} \quad ^{(d)}) \quad ^{(1)} \; ^1k_1 + d \quad K \quad =2$$

$$= \sum_{i=1} k^{(i)}(\;) \quad ^{(i)}(^1 \quad ^{(1)} \quad ^{(1)} \; ^1; : : : ; \; ^{(d)} \quad ^{(d)} \quad ^{(1)} \; ^1)k_1 + d \quad K \quad =2 \tag{5.38}$$

Since $k^{(i)} \quad ^{(1)} \; ^1 \quad ek < 2$ for each $i = 1; : : : ; d$, by Lemma 5.4.5 we have that

$$k \; (\;) \quad (^1 \quad ^{(1)} \quad ^{(1)} \; ^1; : : : ; \; ^{(d)} \quad ^{(d)} \quad ^{(1)} \; ^1)k_1 \quad d6Kk \quad k_1 : \tag{5.39}$$

Combing (5.38), and (5.39) we have that

$$d_p(\; (\;); \; (\;)) \quad Kd(6k \quad k_1 + 1) :$$

Since was arbitrary as far as $d_p(\; ; \;) <$, letting $\# d_p(\; ; \;)$ we obtain the Lipschitz continuity of .

Proof of Theorem 5.2.6. The Lipschitz continuity of the regulator map has been proven in Proposition 5.4.6. We only need to verify the Lipschitz continuity of the content component map . Let be such that $d_p(\; ; \;) <$. Then, $d_{J_1}(\; ^{(i)}; \; ^{(i)}) <$ for each $i = 1; \ldots ; d$. Note that $^{(i)}(\;) = \; ^{(i)} + \; ^{(i)}(\;) \; \sum_{j \in \{1;\ldots;d\} n \{i\}} p_{ji} \; ^{(j)}$. Hence,

$$d_{J_1}(\; ^{(i)}(\;); \; ^{(i)}(\;)) \quad d_{J_1}(\; ^{(i)}; \; ^{(i)}) + \sum_{i=1}^{d} d_{J_1}(\; ^{(i)}(\;); \; ^{(i)}(\;))$$

$$< \; + Kd(6k \; k_1 + 1) \; = \; (1 + Kd(6k \; k_1 + 1)):$$

Since $d_p(\; (\;); \; (\;)) \quad \sum_{i=1}^{d} d_{J_1}(\; ^{(i)}(\;); \; ^{(i)}(\;)) \quad d(1 + Kd(6k \; k_1 + 1))$, we have that is Lipschitz continuous in $\prod_{i=1}^{d} D^{\; i}[0; T]$ by letting $\# d_p(\; ; \;)$. □

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Summary

Large Deviations for Semi-exponential Distributions: Theory and Applications

The focus of my research has been mainly on large deviations theory with semi-exponential distributions along with subsequent applications to queueing theory, stochastic networks, and large deviations theory for Markov random walks.

The starting point of the dissertation is the sample-path large deviation principle developed in Chapter 2. Speci cally, in Chapter 2, we prove the large deviation principle for Levy processes and random walks with heavy-tailed Weibull (semi-exponential) increments. The large deviation principle holds in the Skorokhod space with respect to the M_1' topology. In addition, we prove the extended sample path LDP in the Skorokhod space with the ner J_1 topology. The above results have been extended to multidimensional settings in the case of independent Levy processes and random walks. To enhance the applicability of the extended LDP, we develop theoretical tools for the extended LDP and a form of the contraction principle. Moreover, we show that the extended LDP is the optimal result one can achieve with respect to the J_1 topology. This is demonstrated by constructing a counterexample; showing that the LDP in the J_1 topology is not possible. For the random processes treated in this chapter, our large deviation results demonstrate that associated rare events are caused by big discontinuities of their sample paths.

In Chapter 3, we prove the sample-path large deviation principle for unbounded additive functionals of processes with light-tailed increments that are induced by the Lindley recursion. The LDP holds in the Skorokhod space equipped with the M_1' topology and with sub-linear speed. Our technique hinges on a suitable decomposition of the Markov chain in terms of regeneration cycles.

At each regeneration cycle we study the accumulated area of the Lindley process and we show that the accumulated area displays an asymptotic semi-exponential behavior. By aggregating the trajectory of the process at each regeneration cycle, we attain a process with heavy-tailed i.i.d. jump distributions. This method allows us to use the main results of Chapter 2 and eventually acquire the sample-path large deviation principle for the aggregated process. In conclusion, the main results of Chapter 3, establish that the structure of light-tailed random processes can induce (asymptotically) a heavy-tailed behavior.

In the fourth chapter, we focus on the multiple server queue ($G=G=d$) with semi-exponential service times. The main results, in Chapter 4, provide asymptotic estimates for the probability of large queue lengths as well as the detailed answers on how large queue lengths occur. For the latter part, we determine the number of big jobs and their sizes that lead to congestion. Since the Weibull (semi-exponential) case is near the boundary of the light-tailed and heavy-tailed cases, our results show qualitative and quantitative di erences in comparison to both the power law case and the light-tailed cases.

In Chapter 5, we study a stochastic uid network model with heavy-tailed input (compound Poisson processes with semi-exponential increments). Our results include the continuity of the multidimensional re ection map w.r.t. the product J_1 topology on certain subspaces of the Skorokhod space, and asymptotic estimates of over ow probabilities for a subset of nodes of the stochastic uid network. Based on the continuity of the multidimensional re ection map, we prove large deviation bounds for the multidimensional bu er content process of the stochastic uid network. Then, we use the large deviation bounds of the bu er content process to estimate over ow probabilities for a subset of nodes of the stochastic uid network and we associate the over ow probabilities with a simpli ed optimization problem. Finally, we perform explicit computations and obtain detailed answers in the case of a certain network which relates to | w.r.t. its network topology | the multiple on-o sources model.

About the author

Mihail Bazhba was born in Athens (Greece) on September 26, 1992. He nished his secondary education in 2010 at the second general lyceum of Gerakas, Greece. From 2010 to 2014, Mihail proceeded to study Mathematics at the National and Kapodistrian University of Athens. Subsequently, he completed his master's programme Statistics and Operations Research at the same university in August 2016 and obtained his master's degree. In September 2016, he started his PhD project within the Stochastic group at Centrum Wiskunde & Informatica, in Amsterdam, the Netherlands under the supervision of Bert Zwart and Chang-Han Rhee. His research focused on sample-path large deviations with semi-exponential distributions. The results of his PhD project are presented in this dissertation. Since 2020 he is employed at the University of Amsterdam.