Performance Evaluation 148 (2021) 102200

Contents lists available at ScienceDirect

Performance Evaluation

journal homepage: www.elsevier.com/locate/peva

Commuter behavior under travel time uncertainty

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ARTICLE INFO

Article history:
Available online 12 March 2021

Keywords:
Traffic congestion
Travel time uncertainty
Vickrey model
Pure equilibrium
Mixed equilibrium
Dynamical system

ABSTRACT

We investigate the impact of random deviations in planned travel times using an extension of Vickrey's celebrated bottleneck model. The model is motivated by the fact that in real life, users can neither exactly plan the time at which they depart from home, nor the delay they experience before they join a particular congestion bottleneck under investigation. We show that the strategy advocated by the Nash equilibrium in Vickrey's model is not a user equilibrium in the model with travel time uncertainty. We then investigate the existence of a user equilibrium for the latter and show that in general such an equilibrium can neither be a pure Nash equilibrium, nor a mixed equilibrium with a continuous density. Our results imply that when random distortions influence user decisions, the dynamics of deterministic bottleneck models are inadequate to describe this more complex situation. We illustrate with numerical analysis how the mechanics of a bottleneck with delayed arrivals are unstable for any continuous strategy, thus shedding more light on the non-existence result. Alternatively, we formulate a logit model for the day-to-day dynamic adjustment in response to previous travel experiences. We further investigate the conditions for convergence of the dynamic choice model and numerically illustrate the resulting equilibrium strategy.

1. Introduction

In the standard bottleneck traffic models, travelers choose their departure times from their origins based on their desired arrival times at their destinations, taking into account possible delays on congested bottlenecks on their paths. Users are commonly assumed to choose their departure times with the goal of minimizing a cost function that takes into account waiting (due to congestion), earliness and tardiness penalties. A further common assumption is that given a departure time the time it takes to reach the bottleneck is known and deterministic, in which case it is convenient to disregard travel times (by a constant shift in time). However, departure and travel times may not be sufficiently predictable to justify this approach, as there may be randomness in the actual time of departure from home/work or in the travel time to the bottleneck. This paper explores the implications of taking this uncertainty into account. More specifically, people traveling through the examined bottleneck have an intended time at which they want to be at the bottleneck, but their actual arrival time will deviate day to day. We investigate the effects of these deviations on the dynamics of congestion at

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https://doi.org/10.1016/j.peva.2021.102200
0166-5316/

the bottleneck. Travelers can take the effect of deviations on the expected cost into account, posing an additional challenge to study the existence of a user equilibrium in which no individual traveler can reduce the individual cost.

In transportation literature, the $\alpha-\beta-\gamma$ bottleneck model is a popular approach to model congestion and user response in a tractable and isolated manner, we therefore use this model in our study. In brief summary, this bottleneck model was first introduced by Vickrey [1] and later adjusted by Arnott et al. [2]. It considers a single bottleneck through which a large number of travelers must pass to arrive at their destinations at a common time denoted by t^* (e.g., representing starting office hours). It is standard in this literature to model traffic flow as a fluid, without specific representation of individual travelers, the rationale for this assumption being that each individual traveler has a negligibly small impact on the whole system. Implications for individual travelers can nevertheless be captured by studying infinitesimally small particles in the traffic flow. That way, the model is equipped with costs of waiting (delay while traveling), and earliness and lateness (at the final destination). It is standard to further assume that all three components are properly represented by linear functions of time (with constants α, β and γ). Travelers can exactly choose their individual arrival times to the bottleneck if we assume that there is no variance in the travel time to the bottleneck from the origin (e.g. home to the highway). The outcome of this class of models is that travelers arrive at the bottleneck according to a Nash equilibrium: no traveler can improve its costs by unilaterally altering her/his departure time from the origin. The unique Nash equilibrium of the bottleneck model (and most of the subsequent extensions) is given by a continuous mixed strategy; there is a fluid arrival flow of travelers over some period of time around the peak t^*.

In this paper, we extend the bottleneck model by assuming that arrival times to the bottleneck do not perfectly match the planned time instants. Alternatively, we consider a system where people choose a time of arrival to the bottleneck, but the actual time of arrival deviates by some predefined probability distribution. We will show that this uncertainty results in having a non-convex cost function. In [3] a similar setting as in the current paper is used to investigate pricing schemes. Our focus is on the (non-)existence of a user equilibrium in absence of pricing.

The goal of our analysis is to gain insight into the effects of uncertainty on the responses of travelers and the resulting queueing behavior at the bottleneck. We analyze the impact of various scenarios with respect to the cost function and the travel time uncertainty. We then continue to investigate whether an equilibrium exists in our model, i.e., whether a common arrival strategy exists that ensures a constant expected cost throughout the arrival period, so that no (infinitesimal) user can reduce the individual cost by deviating from the common strategy. Our main result is that, as opposed to most bottleneck models, a continuous mixed user equilibrium neither exists, nor does a pure strategy equilibrium. This implies that if there exists an equilibrium, it must be a mixed strategy with one or more atoms. Under artificial parameter assumptions we indeed succeed in finding equilibrium strategies with multiple atoms, falsifying a general non-existence result.

This finding is not only of mathematical relevance, but its artificiality also leads us to consider a different dynamic adjustment logit-choice model (see [4]) that includes learning from past experience together with a smoothing function that takes into account possible random elements in the utility functions of users. Travelers adjust their arrival rates day by day and, under suitable circumstances, ultimately converge to an invariant arrival function. The convergence of the dynamic learning process depends on the learning parameters and the variance of the random arrival time distortion. We numerically characterize the ranges of parameters for which the dynamics converge and illustrate the resulting continuous steady-state arrival distribution.

Background and related literature

The Vickrey model has been extended in many directions, examples include the capacity drop observed at highways by Arnott [5], heterogeneity among travelers by [6,7], and many more. A recent overview of these extensions was written by Small [8]. A variety of extensions that include the effects of randomness, have also been considered. However, this research mostly considered randomness at the bottleneck only. Another related approach to model user departure patterns, which will be employed as well in the last section of this paper, is the logit dynamic choice model. The pioneering work for this model appeared in Ben-Akiva et al. [4] and extended in [9] they describe their dynamical model by the simulation and analyze the effects of related variables on the stationary state. In [10], the evolution of traffic flow over days is done by comparing the difference between the private cost and the average total cost.

Beyond the transportation literature, the response of travelers based on common preferences has been studied for a wide variety of applications that are closely related to the Vickrey model. These models use queueing theory in combination with game theory. The first model which uses a queueing approach was developed by Glazer and Hassin [11]. They consider a game where a population with a Poisson distributed size choose an arrival time with the aim of minimizing waiting time in the queue. Service times are assumed to be exponentially distributed. Many extensions have been studied with a broad range of application, such as a concert arrival game of Juneja et al. [12] and [13], at which the tardiness was added to the model, causing the order of arrivals to become relevant. In [14] the discrete stochastic queueing model is compared with the fluid approximation, which is commonly used in the transportation literature, and also in this paper a fluid model is assumed. A review of this line of research can be found in Chapter 4.1 of [15].

Another model that is related to the problem we consider here is the meeting game of Fosgerau et al. [16] and also synchronization under uncertainty by Ostrovsky [17] who studies the optimal strategy of individuals that incur a cost for waiting until the last arrival occurrences. Both study the problem in a cooperative manner while taking into account possible random deviations between the intended time chosen by the players and the realized time of the event.

Main contributions

We now summarize the main contributions of our work.

- The main finding of this paper is that, contrary to the standard bottleneck model, when there is random deviation between the intended and actual arrival times then a continuous arrival rate equilibrium is not possible. The special dynamics of the congestion process with distorted arrival times that lead to this negative result is studied in detail. Therefore, the very realistic assumption that arrival times have some inherent uncertainty in them calls for new solution methods because the standard results no longer apply.
- A pure strategy equilibrium that arises in some bottleneck models (e.g. [18]) is only possible if the capacity of the system is high enough to ensure that there is never any congestion. This implies that the only possible equilibrium is a mixed strategy with masses of travelers arriving at a discrete set of time instants. This type of solution seems to be very unnatural for a system with a large population of homogeneous travelers. Moreover, there is also an algorithmic issue to compute such an equilibrium and it makes it even less plausible predictive model for traveler behavior.
- A dynamic day-to-day model in which travelers update their departure time choice based on past experience is introduced and studied. For a logit based decision update model, these dynamics can converge to a continuous arrival flow of travelers. This solution is of course not a Nash equilibrium, but can be a useful predictive tool for systems where a continuous arrival flow is observed but no such equilibrium exists. The parameters for which convergence is achieved are characterized along with numerical analysis of the resulting arrival distribution. A noteworthy observation is that the presence of random distortions in travel times may increase social welfare if the variance of the distortion is low.

Paper organization

The remainder of the paper is organized as follows. Section 2 reviews the classical Vickrey model and its solution, and then introduces the current model with the additional assumption of randomly distorted arrival times. Section 3 illustrates how the introduction of distorted arrival times changes the dynamics of the congestion process and the corresponding cost structure when still imposing the standard Vickrey equilibrium solution. Section 4 studies the optimal behavior of travelers in such a system and establishes the main theoretical results of the paper regarding the existence, or non-existence, of a Nash equilibrium solution. In Section 5 the dynamic model is introduced and analyzed. Section 6 provides discussion and closing remarks.

2. Model description

We first describe the classical Vickrey bottleneck model and then proceed to the extension with uncertainty in the individual arrival times of travelers at the bottleneck.

2.1. Standard bottleneck model

A fluid population with a volume of N passes through a single bottleneck of capacity μ (flow per unit of time). For the discussion it will be convenient to identify N with a *large number* of travelers, although we will not be considering discrete individual travelers; in fact N need not be an integer value. Individual travelers are represented as infinitesimal particles in the flow, with no individual volume. Each traveler (infinitesimal particle) strategically decides when to arrive at the bottleneck in order to minimize its cost. It is assumed that each traveler wants to exit the bottleneck at time t^*, and incurs a penalty for deviations from this preference time and for delay. This penalty is captured by a linear cost function with coefficients α, β, γ, for waiting, early and late departure from the bottleneck respectively. In particular, a traveler who arrives at the bottleneck at time t and has to wait w units of time due to congestion incurs a cost of $\alpha w + \beta(t^* - (t+w))^+ + \gamma(t+w-t^*)^+$.

Of course, the waiting time $w(t)$ of a traveler arriving to the queue at time t is a function of the queue length at time t. Hence $w(t)$ depends on the decisions of travelers who have already arrived at the bottleneck before time t. This interaction calls for strategic analysis as an optimal time must be chosen with respect to the choices of all other travelers.

In order to derive the waiting time function $w(t)$, we first assume that travelers arrive according to a fluid arrival rate function $a(t)$ on an interval $[t_a, t_b]$. Thus, $a(t)$ is the arrival rate corresponding to a continuous mixed arrival strategy, with total volume $N = \int_{t=t_a}^{t_b} a(t)dt$. Let $A(t)$ denote the cumulative arrival process, then the former assumption that it is differentiable with an arrival rate $a(t)$ implies that

$$A(t) = \int_{-\infty}^{t} dA(u) = \int_{-\infty}^{t} a(u)du.$$

We will later discuss the possibility of atoms (instantaneous masses) in the arrival process, but for now we focus on the continuous case which is the typical equilibrium solution in such models. Indeed, in the standard Vickrey model the unique equilibrium solution has this continuous form. Congestion occurs at any time t if the incoming rate of travelers

$a(t)$ is higher than the capacity μ, and thus a queue of waiting travelers is formed. Let $Q(t)$ denote the volume of travelers waiting at the bottleneck at time t and set $q(t) = \frac{d}{dt}Q(t)$, then for any $t \in [t_a, t_b]$,

$$q(t) = \begin{cases} a(t) - \mu & Q(t) > 0, \\ (a(t) - \mu)^+ & Q(t) = 0. \end{cases}$$

The waiting time of an arrival to the bottleneck at time t is then given by $w(t) = Q(t)/\mu$.

The time dependent cost function can be represented as follows:

$$C(t) := C(t, w(t)) = \alpha w(t) + \beta(t^* - (t + w(t)))^+ + \gamma(t + w(t) - t^*)^+. \tag{1}$$

Throughout this paper we make the standard assumption that $\gamma > \alpha > \beta$, (see, e.g., [19]). It is further assumed that travelers have full knowledge of each others behavior. The solution of the Vickrey model is given by a Nash equilibrium, meaning that no traveler can reduce its cost by altering its arrival time. Formally, a symmetric mixed strategy equilibrium is given by a rate function $a(t)$ and an interval $[t_a, t_b]$ such that for some $C \geq 0$,

$$C(t) = C, t \in [t_a, t_b],$$
$$C(t) \geq C, t \notin [t_a, t_b],$$

where the cost $C(t)$ is given by (1). This condition ensures that at any time in the support of the arrival strategy the cost is equal, and further that no traveler can choose an arrival time (or strategy) that achieves a lower cost. Recall that the integral of the resulting arrival rate function $a(t) > 0$ over a finite interval $t \in [t_a, t_b]$ should be equal to the total volume of travelers:

$$\int_{t_a}^{t_b} a(t)dt = N.$$

Given a total amount of fluid N, we want to find an inflow rate $a(t)$ such that no traveler can decrease its costs by altering its arrival time at the bottleneck. It has been shown (see, e.g., [20]) that such a Nash equilibrium is unique for $\gamma > \alpha > \beta$, and is given by

$$a(t) = \begin{cases} \mu + \frac{\beta\mu}{\alpha-\beta} & t \in [t_a, t_M), \\ \mu - \frac{\gamma\mu}{\alpha+\gamma} & t \in [t_M, t_b], \end{cases} \tag{2}$$

where

$$t_a = t^* - \frac{\eta N/\mu}{1+\eta}, \quad t_b = t^* + \frac{N/\mu}{1+\eta}, \quad t_M = t^* - \frac{\delta N/\mu}{\alpha},$$

with $\eta = \frac{\gamma}{\beta}$ and $\delta = \frac{\beta\gamma}{\beta+\gamma}$. This arrival curve gives all travelers equal cost of

$$C = \delta\frac{N}{\mu}. \tag{3}$$

For $\alpha > \beta$ the inflow rate function given in (2) generates a single busy period, i.e., a single interval (t_a, t_b) so that $w(t) > 0$ for all $t \in (t_a, t_b)$ and $w(t) = 0$ outside this interval [20]. The queueing dynamics resulting from the user equilibrium schedule of arrivals into the queue (2) – and the resulting departures – is illustrated in Fig. 1. The upper curve reflects the cumulative arrivals of travelers. The straight line emanating from the origin and connecting with the arrival curve represents the cumulative flow out of the queue. The vertical distance between the curves of cumulative arrivals and cumulative departures thus corresponds with the queue length at any time. Similarly, the horizontal distance between the curves of cumulative arrivals and cumulative departures corresponds to the waiting time of travelers. The waiting time is also depicted in the figure (the lower triangular shape). In this equilibrium, the first and last travelers will only incur costs for early, respectively late arrival (in both extremes there is no delay since the queue is empty). The traveler entering the queue at time t_M, subsequently leaves the queue (instantly arriving at the destination) exactly at the preferred time t^*. This traveler only incurs waiting cost and experiences the longest traveling time.

2.2. Bottleneck model with arrival time uncertainty

In reality, travelers do not necessarily arrive at their intended time. We therefore extend the above bottleneck model and assume that there is uncertainty about the actual time of arrival to the bottleneck. The deviation from the intended arrival time of each traveler is modeled by a continuous random variable $X \in (-\infty, +\infty)$ with density f, assuming the deviations of different traveler to be independent. Hence, if traveler i chooses time t_i then her actual arrival time is $t_i + X_i$. If $a(t)$ is continuous then f acts as a smoothing kernel over the arrival function $a(t)$. The resulting arrival rate function is given by

$$\tilde{a}(t) = \int_{u=-\infty}^{\infty} f(u)a(t-u)du. \tag{4}$$

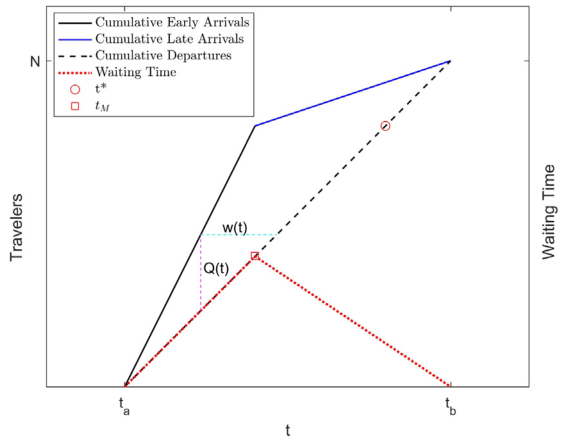


Fig. 1. Cumulative arrivals and departures and waiting time in Vickrey's single bottleneck model.

The time-dependent queue length at the bottleneck can be computed by accounting for the difference between the actual arrival rate of (4) and the departure rate μ

$$q(t) = \begin{cases} \tilde{a}(t) - \mu & Q(t) > 0, \\ (\tilde{a}(t) - \mu)^+ & Q(t) = 0, \end{cases} \tag{5}$$

where $Q(t)$ is the queue length at time t and $q(t)$ is its derivative. The waiting time of an arrival at time t can then be computed by

$$w(t) = Q(t)/\mu. \tag{6}$$

By plugging (6) into (1) we compute the expected cost of an arrival at time t,

$$\tilde{C}(t) = \alpha w(t) + \beta\big(t^* - t - w(t)\big)^+ + \gamma\big(t + w(t) - t^*\big)^+. \tag{7}$$

Given a time-dependent arrival rate $\tilde{a}(t)$ we can express the expected cost for a traveler who has an intended arrival time t by

$$\mathbb{E}[\tilde{C}(t)] = \int_{u=-\infty}^{\infty} \tilde{C}(t+u)f(u)du. \tag{8}$$

3. Preliminary analysis

To gain insight on the impact of uncertainty about the exact arrival time, we demonstrate the cost over time given that travelers are unaware of this uncertainty. We compute the impact of uncertainty for a number of delay functions. For each, we show the impact for an increasing level of uncertainty.

To obtain the arrival rate over time at which travelers are unaware of the uncertainty function f, we compute the Nash equilibrium $a(t)$ of the standard bottleneck model given by (2). Thus, the actual arrival rate $\tilde{a}(t)$ is computed by the convolution of the Nash equilibrium arrival rate of Eq. (2), and the arrival uncertainty distribution of X. The actual arrival rate can be obtained by taking the convolution as defined in Eq. (4). We then plug this rate into (7) to obtain the expected costs over time. Finally, the expected cost for a traveler who chooses time t is calculated by (8).

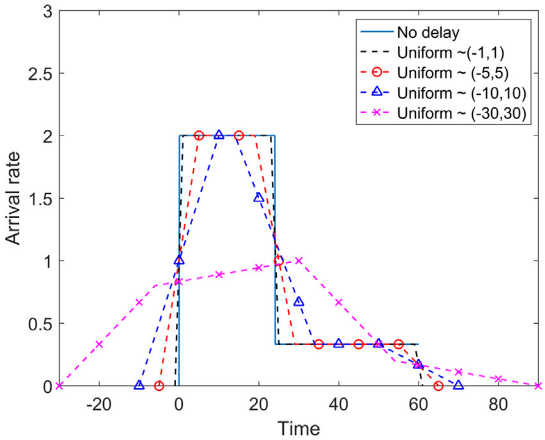
The numerical evaluation of the above described rates and cost function is carried out using the following approximating discretization scheme. The interval of the bottleneck period is split into n small segments of length Δ where:

$$n = \left\lfloor \frac{t_{end} - t_{start}}{\Delta} \right\rfloor. \tag{9}$$

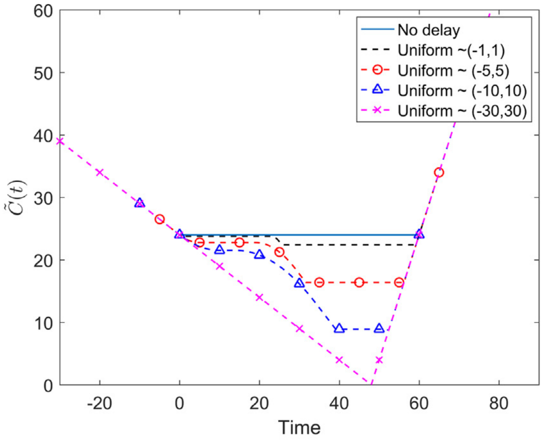
The probability volume of the kth segment is obtained by

$$p_k = \mathbb{P}[X \leq (k+1)\Delta + t] - \mathbb{P}[X \leq k\Delta + t]. \tag{10}$$

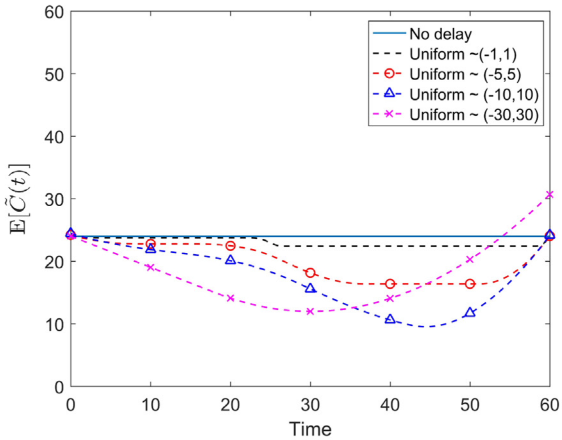
First, we consider a uniform uncertainty distribution X. The cost function is taken equal to the standard values from [19] where $\beta/\alpha = 0.5$ and $\gamma/\alpha = 2$ ($\alpha = 1$, $\beta = 0.5$, $\gamma = 2$, $N = 60$, $\mu = 1$). In Fig. 2 the results for $X \sim unif(-\tau, \tau)$ when $\tau \in \{0, 1, 5, 10, 30\}$ are visualized. In these examples, the bottleneck period is extended to $t_{start} = t_a - \tau$ and $t_{end} = t_b + \tau$, since deviations from the intended arrival times will cause travelers to arrive prior to t_a and later than t_b as well. The results of Fig. 2(c) show that the expected cost is below that of the cost without any delay, as long as a queue exists. In



(a) Arrival rate with Uniform delay



(b) Cost per time unit ($\tilde{C}(t)$)



(c) Expected cost for a traveler with intended time t ($\mathbb{E}[\tilde{C}(t)]$)

Fig. 2. Impact of the arrival pattern and costs over time for a uniform delay.

the boundaries of the arrival interval, for which the delay is equal to the period of the bottleneck, there will be no queue at all: travelers will only incur earliness or lateness cost, depending on their arrival time.

Similar results appear for $X \sim exp(\mu)$, where the delay distribution is on an infinite support. For both functions, the same observations are made: decrease in the average cost function for travelers while increasing the delay component. An important observation is that the expected cost is not constant throughout the arrival interval. Therefore, travelers will deviate from the Vickrey equilibrium arrival rate because they can reduce their cost. With the same approach, this phenomenon can be displayed for other delay distributions.

4. Optimal responses

We continue our analysis by considering the optimal response of travelers when they take into account the uncertainty function. We assume full information in the sense that all travelers are aware that arrival times deviate from the intended times according to f for themselves and all others. This will allow us to examine whether an equilibrium exists, and under which conditions. We explore both the options of pure and continuous mixed strategies, assuming a uniform delay function.

To explicitly study whether a pure or mixed equilibrium can exist, we specifically choose the arrival delay (that is the delay before arriving to the bottleneck) to be a uniformly distributed random variable $X \sim unif(0, 1)$

$$f(t) = \begin{cases} 1 & t \in [0, 1], \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The results can be extended to general $X \sim unif(0, \tau)$ by re-scaling time.

We first outline the simplifications in the model description that are due to the uniform delay assumption. Plugging Eq. (11) into Eq. (8) we obtain

$$\mathbb{E}[\tilde{C}(t)] = \int_{u=0}^{1} \tilde{C}(t+u)du.$$

Note that while the cost function is piecewise linear due to (1), and hence not smooth, the expected cost has a continuous derivative because the cost is continuous. Furthermore, as $\mathbb{E}[\tilde{C}(t)] \to \infty$ when $t \to \infty$ or $t \to -\infty$, a best response of a single customer is a global minimum that satisfies the necessary first order condition,

$$\frac{d\mathbb{E}[\tilde{C}(t)]}{dt} = \tilde{C}(t+1) - \tilde{C}(t) = 0 \;\Leftrightarrow\; \tilde{C}(t+1) = \tilde{C}(t). \tag{12}$$

Note that $\mathbb{E}[\tilde{C}(t)]$ is not convex and there may be multiple local minima or saddle points, that satisfy the necessary condition.

4.1. Pure strategy equilibrium

We first investigate the conditions for a pure strategy equilibrium to exist, namely a time instant such that if all travelers arrive together then no single traveler can reduce the cost by choosing any other arrival time. We then show that such an equilibrium is only possible in the degenerate case where the capacity is large enough for no queue to form, regardless of the strategies: $\mu > N$. Recall that N is the total fluid volume. Because of the random delay, having a uniform(0,1) distribution, the intended mass arrival of the volume N leads to a constant flow with rate N. Since μ is the outflow from the bottleneck per unit of time, $\mu > N$ implies that all the volume of travelers can flow instantaneously. For the general case, we further illustrate numerically how the non-convex shape of the expected cost function makes a pure strategy equilibrium impossible.

We first compute the cost for a tagged traveler arriving at time $s \in \mathbb{R}$, given that a total volume of N decides to arrive at time $t \in \mathbb{R}$. We distinguish between two cases. When $\mu \geq N$, the tagged traveler encounters no waiting time and the moment of arrival of the tagged traveler does not depend on the volume N at time t. The case where $\mu < N$, does lead to waiting times when the actual arrival time overlaps with the interval of arrival of the volume N. Formally, a pure strategy Nash equilibrium is given by the fixed point $t \in \text{argmin}_{s\in\mathbb{R}} \mathbb{E}[\tilde{C}_t(s)]$, where $\mathbb{E}[\tilde{C}_t(s)]$ is the expected cost for a traveler who intends to arrive at time s while the volume N intends to arrive at time t.

Case $\mu \geq N$

We determine the cost for a traveler arriving at s by

$$\begin{aligned}\mathbb{E}[\tilde{C}(s)] &= \int_{s}^{t^*} \beta(t^*-u)du + \int_{t^*}^{s+1} \gamma(u-t^*)du \\ &= \beta[t^*(t^*-s) - \frac{1}{2}((t^*)^2 - s^2)] + \gamma[\frac{(s+1)^2}{2} - \frac{(t^*)^2}{2} - t^*(s+1-t^*)].\end{aligned}$$

To find the time instant that gives the best response we compute the solution of $\frac{d\mathbb{E}[\tilde{C}(s)]}{ds} = 0$, yielding the unique solution

$$\begin{aligned}&\frac{d\mathbb{E}[\tilde{C}(s)]}{ds} = -\beta t^* + s\beta + \gamma(s+1-t^*) = 0, \\ &s = t^* - \frac{\gamma}{\beta+\gamma}.\end{aligned}$$

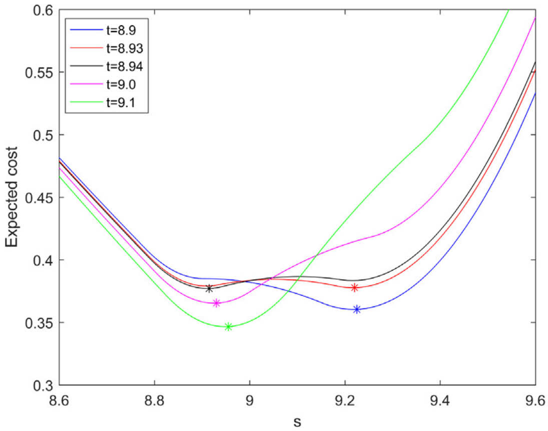
Thus, the best response does not depend on the arrival of the fluid volume N. This is the same solution as in the model with no waiting costs by [18].

Case $\mu < N$

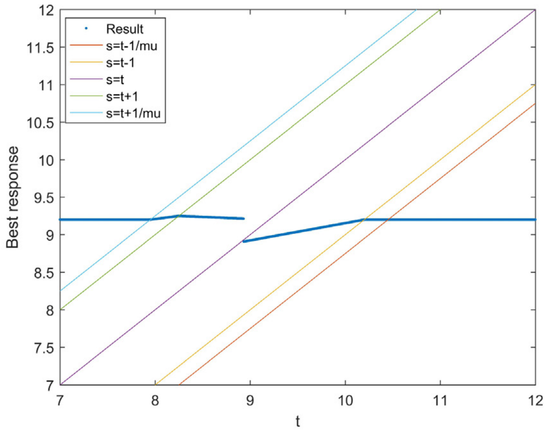
In case that $\mu < N$, a queue builds during the arrival interval of the volume N. For the sake of brevity, without loss of generality, we assume that the total volume of travelers is $N = 1$. Therefore, we need to consider the time of arrival of the volume, which is given by $\tilde{a}_t(u) = 1 \in [t, t+1]$. Including this in (5), we obtain the waiting time by

$$w_t(u) = \begin{cases} (\frac{1}{\mu} - 1)(u-t) & u \in [t, t+1], \\ \frac{1}{\mu} - (u-t) & u \in (t+1, t+\frac{1}{\mu}], \\ 0 & \text{otherwise,} \end{cases}$$

where u represents the intended arrival time.



(a) Costs s for specific t



(b) Best response of s for range $t \in T$

Fig. 3. Best response of s for $t \in T$ for $\mu = 0.8$, $t^* = 10$, $N = 1$, $\alpha = 1$, $\beta = 0.5$, $\gamma = 2$.

We insert the $w_t(u)$ in (7), and compute the cost for a traveler who intends to arrive at time u given that the volume N intends to arrive at time t by

$$\tilde{C}_t(u) = \alpha w_t(u) + \beta\big(t^* - (t + w_t(u))\big)^+ + \gamma\big(t + w_t(u) - t^*\big)^+.$$

Finally, we compute the expected cost for a traveler who intends to arrive at time s by

$$\mathbb{E}[\tilde{C}_t(s)] = \int_s^{s+1} \tilde{C}_t(u)du. \tag{13}$$

Proposition 1. *A pure Nash equilibrium does not exist for $N > \mu$ when delay is uniformly distributed.*

Before proving Proposition 1, we wish to provide intuition to the non-existence result by numerically illustrating the shape of the cost function. In Fig. 3(a), we plot the expected cost of the tagged traveler based on different s values for a fixed t and observe that there are either two local minima or a single local minimum and a saddle point. The best time instant for the tagged traveler to arrive at the bottleneck should minimize its expected cost function and are shown by star marks in Fig. 3(a). Moving t affects the best response, as is illustrated in Fig. 3(b). The jump corresponds to the point where the global minimum changes from the earlier local minimum (on the left in Fig. 3(a)) to the one at a later time (on the right in Fig. 3(a)). This jump in the best response function is the reason why no fixed point exists because none of the best responses lie on the purple line where $s = t$.

Proof of Proposition 1. We firstly compute the expected cost for a traveler choosing intended arrival time s, given that the volume N intends to arrive at time t. Therefore, we split the integral of Eq. (13) into several cases. We make a division between the case where $s \le t$ and $s \ge t$, and we separate between the point where the earliness cost changes to lateness cost denoted by $x^* = t + \mu(t^* - t)$.

For $s \le t < s + 1 \le x^*$,

$$\mathbb{E}[\tilde{C}_t(s)] = \int_s^t \beta(t^* - u)du + \int_t^{s+1} \alpha w_t(u) + \beta(t^* - u - w_t(u))du,$$

for $s \le t < x^* < s + 1 \le t + 1$,

$$\mathbb{E}[\tilde{C}_t(s)] = \int_s^t \beta(t^* - u)du + \int_t^{s+1} \alpha w_t(u)du + \int_t^{x^*} \beta(t^* - u - w_t(u))du + \int_{x^*}^{s+1} \gamma(u + w_t(u) - t^*)du,$$

and for $t \le s < x^* < t + 1 \le s + 1$,

$$\mathbb{E}[\tilde{C}_t(s)] = \int_s^{t+1} \alpha w_t(u)du + \int_s^{x^*} \beta(t^* - u - w_t(u))du + \int_{x^*}^{t+1} \gamma(u + w_t(u) - t^*)du + \int_{t+1}^{s+1} \alpha w_t(u) + \gamma(u + w_t(u) - t^*)du.$$

We excluded the cases where $s, t > x^*$, and also $s+1, t+1 < t^*$ which cannot be an equilibrium because the expected cost of a traveler arriving to the queue at t can be trivially improved. To find the optimal time s when all others arrive at time t we solve the necessary condition (12) for each of the above cases.

Case 1: For $s \le t < s+1 \le x^*$,

$$s = \frac{\alpha(\mu - 1 + t - \mu t) + \beta(1 - \mu - t + \mu t)}{\alpha(1-\mu) - \beta(2\mu - 1)}$$

then, for $s = t = t_e$ we obtain

$$t_e = \frac{\alpha}{\beta}(1 - \frac{1}{\mu}) + \frac{1}{\mu} - 1.$$

This will give a negative value for any $\mu < 1$ and $\alpha > \beta$.

Case 2: For $s \le t < x^* < s+1 \le t+1$,

$$s = \frac{\alpha(\mu + t - \mu t - 1) + \beta\mu t^* + \gamma(\mu t^* + t - \mu t - 1)}{\alpha + \gamma + \mu(\beta - \alpha)}$$

then, for $s = t = t_e$ we obtain

$$t_e = t^* - \frac{\gamma}{\beta+\gamma}(\frac{1}{\mu}) - \frac{\alpha}{\beta+\gamma}(\frac{1}{\mu} - 1).$$

As t_e has to meet the criteria of $t_e \ge t^* - \frac{1}{\mu}$, we can only find a pure equilibrium when $\mu > \frac{\alpha-\beta}{\alpha}$.

Case 3: For $t \le s < x^* < t+1 \le s+1$, we obtain the same solution as in case 2 because the cost functions coincide when taking $s \uparrow t$ or $s \downarrow t$.

The above suggests that a pure equilibrium solution can only be at t_e. However, it can be shown that the cost of the tagged traveler arriving at time t_e when all others arrive at t_e, is not the global minimum. This is done by considering the derivative of the expected cost in the range $t \le s < x^* < t+1 \le s+1$ (i.e., case 3 above),

$$\frac{\mathbb{E}[\tilde{C}_t(s)]}{ds} = \frac{-\alpha(\mu - 1 + s - t) + \beta(s - \mu t^* + (\mu-1)t) + \gamma(1 - t^*\mu + \mu t)}{\mu},$$

and observing that it is a decreasing function with s, as $\alpha > \beta$. This implies that t_e is a saddle point and the global minimum is at $t > t_e$. The global minimum is obtained in the range $t \le s \le x^* \le t+1 \le t + \frac{1}{\mu} \le s+1$. The cost of s for this case is

$$\begin{aligned}\mathbb{E}[\tilde{C}_t(s)] &= \int_s^{t+1} \alpha w(u)du + \int_s^{x^*} \beta(t^* - u - w(u))du \\ &\quad + \int_{x^*}^{t+1} \gamma(u + w(u) - t^*)du + \int_{t+1}^{t+\frac{1}{\mu}} \alpha w(u) \\ &\quad + \gamma(u + w(u) - t^*)du + \int_{t+\frac{1}{\mu}}^{s+1} \gamma(u - t^*)du,\end{aligned}$$

with a first derivative $\frac{d\mathbb{E}[\tilde{C}_t(s)]}{ds}$ that equals

$$\frac{-t^*(\beta+\gamma)\mu - \alpha s + \beta s - \alpha\mu s + \gamma\mu(1+s) + \alpha t - \beta t - \alpha\mu t + \beta\mu t}{\mu}.$$

Solving the first order condition $\frac{d\mathbb{E}[\tilde{C}_t(s)]}{ds} = 0$ yields the best response

$$s^* = \frac{t^*\mu(\beta+\gamma) - \gamma\mu + t(\alpha-\beta)(\mu-1)}{\alpha(\mu-1) + \beta + \gamma\mu}. \tag{14}$$

In Fig. 4, we illustrate for a numerical example the saddle point at t_e with the global minimum $s^* > t_e$ corresponding to the best response. Finally, we conclude that a pure equilibrium does not exist.

4.2. Continuous mixed strategy equilibrium

We continue our analysis by considering a continuous mixed arrival strategy. We formulate the conditions for such an equilibrium and show that there exists no solution for the uniform delay function. We further highlight why a continuous mixed strategy equilibrium fails by a numerical approximation. We focus here only on the non-degenerate case of $\mu < N = 1$.

A symmetric mixed arrival strategy is given by an arrival density $g(t)$ and cumulative arrival function $G(t)$ such that all travelers select their intended arrival time according to this distribution. Let $[t_a, t_b]$ be the support of the distribution

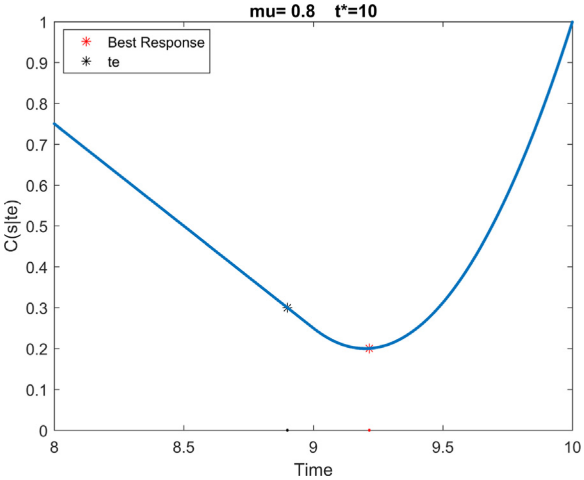


Fig. 4. Best response time s of Eq. (14) for a volume N intending to arrive at time t, where $t = t_e$.

g, and note that it is finite in equilibrium. For a traveler who intends to arrive at $t \in [t_a, t_b]$ the actual arrival time is $t+X$, where $X \sim unif(0, 1)$, as before. The arrival rate function of Eq. (4) then simplifies to

$$\tilde{a}(t) = \int_{t_a \vee (t-1)}^{t_b \wedge t} g(t-u)du = G(t_b \wedge t) - G(t_a \vee (t-1)), \tag{15}$$

where $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$.

This allows us to determine the waiting time when $q(s) \geq 0$ for $t_a \leq s \leq t$,

$$w(t) = \frac{1}{\mu} \int_{t_a}^{t} \Big(\tilde{a}(u) - \mu\Big)^{+} du. \tag{16}$$

Let $\mathbb{E}_g C(t)$ denote the expected cost for a traveler with intended arrival time t when all others arrive according to g. The realized cost of a traveler arriving at t is then given by $\tilde{C}$ as defined in (7). A distribution g is a Nash equilibrium if both of the following conditions are satisfied for every t such that $g(t) > 0$

1. $\mathbb{E}_g C(t) \leq \mathbb{E}_g C(s)$ for every $s \in \mathbb{R}$,
2. the necessary condition (12): $\tilde{C}(t) = \tilde{C}(t+1)$ is satisfied.

Proposition 2. *A continuous mixed equilibrium does not exist for non-degenerate case of $\mu < N = 1$ when delay is uniformly distributed.*

Proof. Let $k(t) := \tilde{C}(t+1) - \tilde{C}(t)$, then by Eq. (7) we have

$$k(t) = \begin{cases} (\alpha-\beta)(w(t+1) - w(t)) - \beta & t+1+w(t+1) < t^*, \\ (\alpha+\gamma)w(t+1) + (\beta-\alpha)w(t) + (\gamma+\beta)(t-t^*) + \gamma & t+w(t) < t^* \leq t+1+w(t+1), \\ (\alpha+\gamma)(w(t+1) - w(t)) + \gamma & t+w(t) \geq t^*. \end{cases} \tag{17}$$

For any continuous distribution G there is some $\hat{t} > t_a$ such that $G(\hat{t}) = \int_{t_a}^{\hat{t}} g(u)du = \mu$, hence $\tilde{a}(t) < \mu$ and $w(t) = 0$ by (16) for every $t \in [t_a, \hat{t}]$. We will show that the equilibrium condition $k(t_a) = 0$ leads to a contradiction for the three possible cases in (17).

1. If $t_a + 1 + w(t_a+1) < t^*$ then $k(t) = 0$ yields for some $\delta > 0$,

$$w(t+1) = \frac{\beta}{\alpha - \beta} \quad \forall t \in [t_a, t_a + \delta].$$

Therefore, in particular $\frac{d}{dt} w(t+1)|_{t=t_a} = 0$. Next, observe that $t_a + 1 > \hat{t}$, otherwise $w(t_a+1) = 0$, and we thus have that

$$\frac{d}{dt} w(t+1)|_{t=t_a} = 0 = \frac{1}{\mu} G(t_b \wedge (t_a+1)) - 1 > \frac{1}{\mu} G(\hat{t}) - 1 = 0, \tag{18}$$

yielding a contradiction.

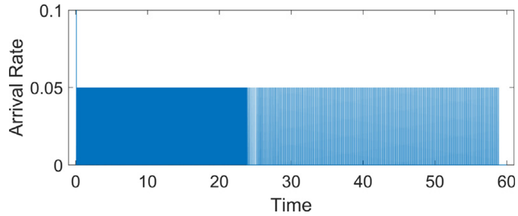


Fig. 5. Arrival rate intensities over time.

2. If $t_a + w(t_a) < t^* \leq t_a + 1 + W(t_a + 1)$ then $k(t) = 0$ for some $\delta > 0$,
$$w(t+1) = \frac{(\gamma+\beta)(t^* - t) - \gamma}{\alpha + \gamma} \quad \forall t \in [t_a, t_a + \delta],$$
and in particular $\frac{d}{dt} w(t+1)|_{t=t_a} = -\frac{\gamma+\beta}{\alpha+\gamma} < 0$. Again, this contradicts the above conclusion in (18) that $\frac{d}{dt} w(t_a+1) > \frac{1}{\mu} G(\hat{t}) - 1 = 0$.
3. If $t_a + 1 + w(t_a + 1) < t^*$ then $k(t_a) = 0$ yields
$$w(t_a + 1) = -\frac{\gamma}{\alpha + \gamma} < 0,$$
which is a contradiction as $w(t) \geq 0$ for all t by definition.

4.3. Approximation of continuous equilibrium

We propose a numerical procedure to obtain an arrival rate function for which the expected cost per traveler remains constant on most of the support. This method can be applied to non-uniform delays distributions and exhibits similar properties.

We want to obtain an arrival rate $\tilde{a}(t)$, for which the $\mathbb{E}[\tilde{C}(t)] \approx C$. To obtain a numerical solution, we discretize the functions of Section 4.2 by Eq. (9) where $[t_{start}, t_{end}]$ denotes the interval including the support of the delay function and Δ is the stepsize. We can also compute the probability distribution of arrival as computed in Eq. (10).

C_{target} is equal to Eq. (3), the arrival rate $\tilde{a}(t)$ is captured in the vector $\bar{r} = (r_1, \ldots, r_n)$, and ϵ is a small value with which we increase the rate at the indicated location.

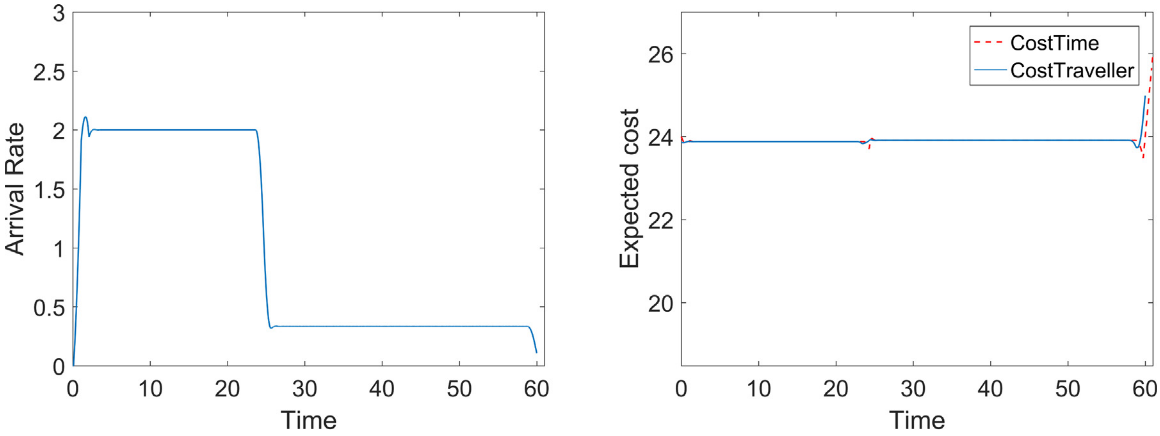
Algorithm 1 shows the numerical procedure that results in an arrival rate function for which the costs over time remain relatively constant. In summary, the procedure consists of the following steps. We first set a target cost denoted by C_{target}, which we want to keep constant. We search for the earliest moment of arrival such that this cost constraint is met. At this specific time instant we add a small arrival volume of rate ϵ. Given the updated arrival vector, we compute the new cost function over time. Again, we compute the earliest moment of arrival t such that $\mathbb{E}[\tilde{C}(t)] \leq C_{target}$. We continue this procedure until this condition cannot be met anymore.

Algorithm 1 Procedure to approximate a mixed equilibrium.

Inputs:
  $n, t^*, \beta, C_{target}, v, w, p_j$
Initialize:
  $i_{\text{Loc}} = \frac{t^* - \beta C_{target} - t_{start}}{\Delta}$;
  $r_i = 0$ for $i = 0, \ldots, n$
while $i_{\text{Loc}} \neq \emptyset$ **do**
  $r_{i_{\text{Loc}}} = r_{i_{\text{Loc}}} + \epsilon$
  for $i = 0, \ldots M$ **do**
    $C_i = \sum_{j=0}^{i+w-v} p_j \tilde{C}_{j+v}$
  end for
  $i_{\text{Loc}} = \arg\min_i \{i : C_i < C_{target}\}$
end while

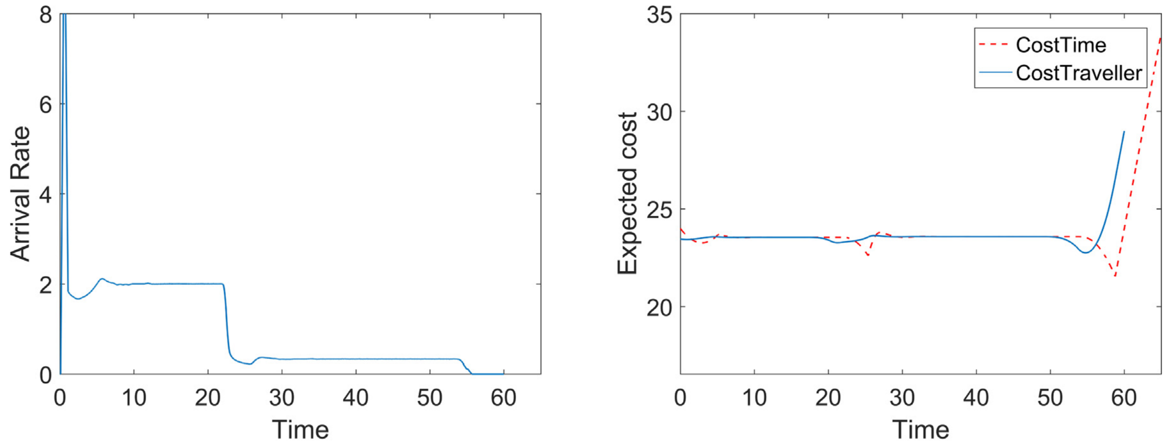
In Fig. 5, a representation of the outcome of the approximation procedure of Algorithm 1 is visualized. The line density indicates the arrival rate intensity over time. We observe a large density in the beginning, followed by a reduced density at the peak moment x^* ($t = 24$), which increases again shortly after. The arrival rate over a specified period of time is given by the sum of the lines. We apply a moving average filter to obtain the arrival rate function over time.

In Figs. 6 and 7, the results of these rates and the costs over time are visualized for uniform delay function. These figures show that for a larger uncertainty, obtaining a constant cost function becomes more difficult.



(a) Arrival rate (b) Cost function

Fig. 6. Approximated equilibrium function for $X \sim \text{unif}(0, 1)$.



(a) Arrival rate (b) Cost function

Fig. 7. Approximated equilibrium function for $X \sim \text{unif}(0, 5)$.

Table 1
Total amount of travelers (N) for fixed costs with varying delay function $f(t)$ and mean delay τ.

	Uniform	Exponential
$\tau = 0$	60	60
$\tau = 1$	59.6	59.2
$\tau = 5$	58.7	56.5
$\tau = 10$	57.5	54.1

In Table 1, we observe that the total amount of travelers passing the bottleneck decreases with respect to delay, while fixing the expected cost to the value of Eq. (3). Conclusions on the impact of arrival time uncertainty with respect to disutility cannot be made, as the current results are not in equilibrium. However, this does suggest that uncertainty increases the disutility of individual travelers.

4.4. Equilibrium by discontinuous arrivals

We showed that there is neither pure-strategy Nash equilibrium nor a continuous density g for which the equilibrium condition holds. Therefore, an equilibrium distribution necessarily has one or more atoms. Below we outline how such a discrete distribution can be constructed.

If the delay function is defined as $unif(0, \tau)$, we set a mass with a volume of $r_1 > \mu$ arriving at the time of $x^* - n_1$. The size of the mass should be defined such that it mimics the Vickrey solution after it is convolved with the delay function. The time distance between two consecutive intended arrivals is exactly equal to τ. After x^* the arrival mass changes into the lower rate of $r_2 < \mu$ to allow the built up queue dissipates gradually till the time of $x^* + n_2$ that the queue totally diminishes. This setting is mainly a result of a forced mass arrival in the very beginning of the congestion period as it has been shown in Figs. 7(a) and 6(a). The existence of the equilibrium depends on the proper spacing of the mass arrivals such that the condition of $r_1 n_1 + r_2 n_2 = N$ is met.

The so constructed equilibrium is theoretically interesting, since it illustrates that a Nash equilibrium can exist, but only under very specific artificial parameter choices. Because of that, from a practical perspective, this Nash equilibrium is of little use. We therefore set out in the next section to develop a different *dynamic* model in which users adapt their choices sequentially (day by day). In that model we will no longer assume that users are driven purely by cost minimization, but rather conform to a choice that is likely to have low cost (this will be made precise next).

5. A dynamic choice model

We define a dynamic logit model as an alternative for the user equilibrium model. The underlying assumption is that the utility of a traveler is related to the cost, but travelers are not necessarily seeking the minimum cost. In the model, the choice is given by a probability distribution that has travel cost as its input, such that times with high (expected) costs are chosen with a lower probability. In the dynamic setup, the expected cost relies on the past experience of the travelers.

We will first formally define the model and then examine the long-term behavior of the dynamic model by simulation. In particular, we will see that for certain parameters the departure rate from home/work that is updated dynamically converges to steady-state distribution.

We define $a(t, \omega)$ to be the intended arrival time at time t on day ω; after observing the cost on a given day, the travelers will update their choice for the next day. If the utility (cost) on day ω is observed, the likeliness of choosing time t on day $\omega + 1$ is given by a continuous logit model as

$$p(t) = \frac{e^{-C(t)/\theta}}{\int_{t \in \mathcal{T}} e^{-C(u)/\theta} \mathrm{d}u} ,$$

where the parameter $\theta > 0$ scales the cost function $C(t)$.

The scale parameter reflects the variance of the unobserved portion of utility. In [21], θ is defined as a parameter related to changes of departure times and can take a range of $(0, 14)$. In [4], θ is a parameter that measures the randomness of the traffic behavior and it can be changed by applying different policies. In this paper we examine different values of θ to see its effect on convergence to an equilibrium. Note that if θ is large, then the utility is rather insensitive to the cost function and the distribution $p(t)$ will be quite flat. On the contrary, if θ is near 0, $p(t)$ drives users close to minimizing their (expected) cost.

As is common in dynamic models, we assume that possibly not the entire traveler population adapts its choice every day. Instead we assume that the arrival rate at time t on day ω is determined by

$$a(t, \omega) = RNp(t) + (1 - R)a(t, \omega - 1), \tag{19}$$

where R is the fraction of drivers who (can) modify their arrival time choices.

This recursion lends itself well for a numerical simulation procedure: First, the chosen interval of arrivals is divided into n segment of equal length. The initial arrival rate to the bottleneck is uniformly distributed such that there is no congestion period. For all t, we compute the actual arrival rate, cost function and the probability of time t being chosen. Based on the logit model of Eq. (19), the arrival rate is updated; which is the intended arrival rate for the next day. Subsequently we calculate $\tilde{a}$, $\tilde{C}$ and $p(t)$. The procedure stops when the maximum relative difference between the arrival rates at each time of t for two consecutive days is smaller than a given tolerance.

5.1. Existence of an equilibrium

We will use the above procedure to investigate the convergence of the dynamic model to an equilibrium. If the delay function is $unif(0, 1)$, we will see that, depending on the fraction of travelers who review their arrival choices, the process converges for large enough values of θ (recall that large values of θ reflects that travelers are quite insensitive to cost). The gray area in Fig. 8 shows the possible combinations of R and θ in which the process converges and for the combinations that lie on the white area, we observe oscillation of arrival rates. Note that for lower values of the scale parameter θ (corresponding to more cost-sensitive travelers), the process can be stabilized by reducing the daily fraction R of travelers who revise their choices.

For illustration, the equilibrium arrival rate function at the bottleneck is depicted in Fig. 9, for the parameter choices $\alpha = 1$, $\beta = 0.5$, $\gamma = 2$, $N = 60$, $\mu = 1$, $\theta = 12$ and with a delay of $unif(0, 1)$. The congestion period begins at time t_a when the arrival rate to the bottleneck exceeds the capacity of the bottleneck and it ends at time t_b when the queue dissipates. In this model, unlike the Vickrey model, there are arrivals into the system outside the congestion period.

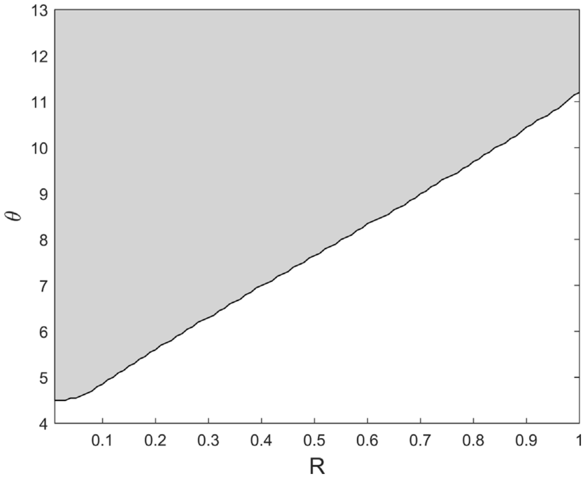


Fig. 8. A user equilibrium exists for combinations of θ and R in the gray area.

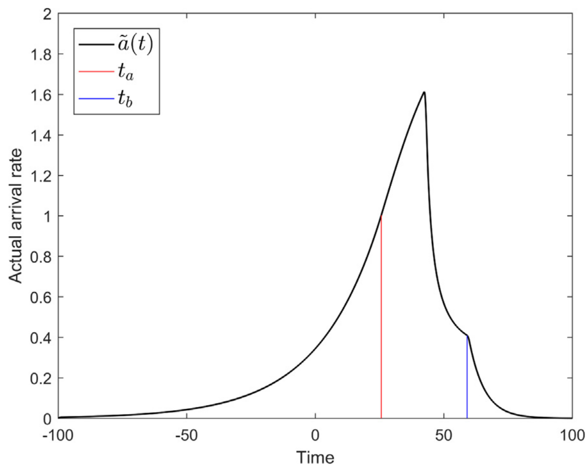


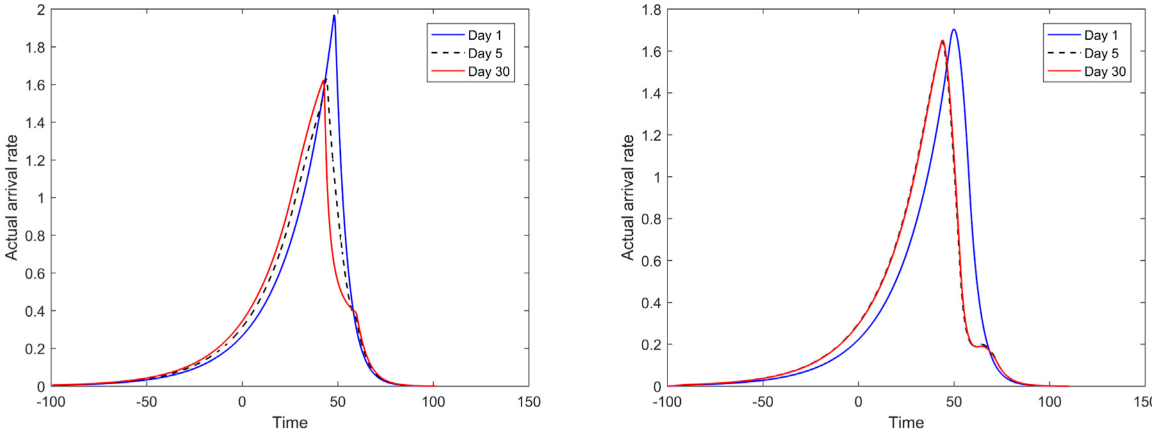
Fig. 9. The stationary arrival rate for the *unif*(0, 1) delay function.

The dynamical evolution of the arrival distribution and the congestion period toward the stationary state for delay functions of *unif*(0, 1) and *unif*(0, 10) are demonstrated in Figs. 10(a) and 10(b) respectively. We can see that for the same value of the scale parameter θ, the larger the support of the delay function, the lesser days it takes the process to converge.

We further conduct simulation experiments to analyze the effect of the scale parameter θ on τ for the delay function of *unif*$(0, \tau)$ when $R = 1$. The results of possible combinations of τ and θ to reach an equilibrium are depicted in the gray area in Fig. 11. Note that the figure corroborates with our earlier findings. For small values of θ (i.e., with close to cost-minimizing travelers) an equilibrium only exists when the degree of uncertainty τ in arrival time is large enough (which spreads the travelers to such an extent that no significant congestion occurs).

To see the effect of different values of θ on the congestion period, the queue length and average expected cost, we run the simulation keeping $\tau = 22$ while θ is changed; the results are demonstrated in the Figs. 12. Fig. 12(a) shows that t_b is a decreasing function of θ and t_a is a decreasing function up to a critical value after which it increases until the case when there is no congestion in the system. The beginning and the end of congestion period curves show that for larger values of θ the queue becomes shorter (Fig. 12(b)) and it is more centered toward t^* and because of that the average expected cost is a decreasing function as it is illustrated in Fig. 12(c).

Variation of the support of the delay function can be also investigated in the stable dynamic model. We assume that $\theta = 12$ and is constant for different support values of *unif*$(0, \tau)$ delay functions. From Fig. 13(a), it can be observed that the duration of congestion is the same and both curves of t_a and t_b are increasing functions of τ up to a critical point after which t_b starts decreasing. Although average expected cost is increasing during the whole process as it has been shown in Fig. 13(c), as the congestion period becomes shorter and the maximum queue length decreases after a certain value of



(a) Delay function of $unif(0,1)$　　(b) Delay function of $unif(0,10)$

Fig. 10. Transient distribution of arrival rates.

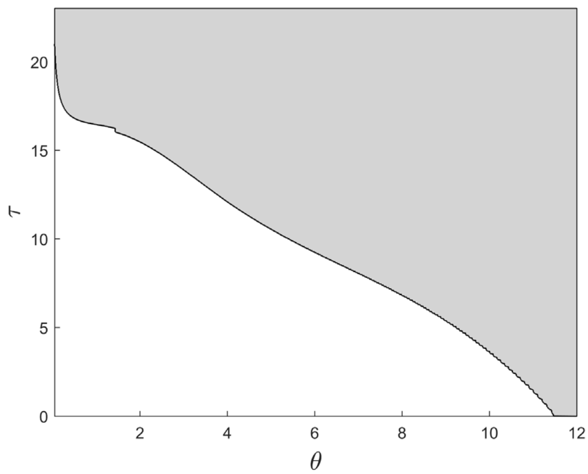


Fig. 11. A user equilibrium exists for combinations of θ and τ in the gray area.

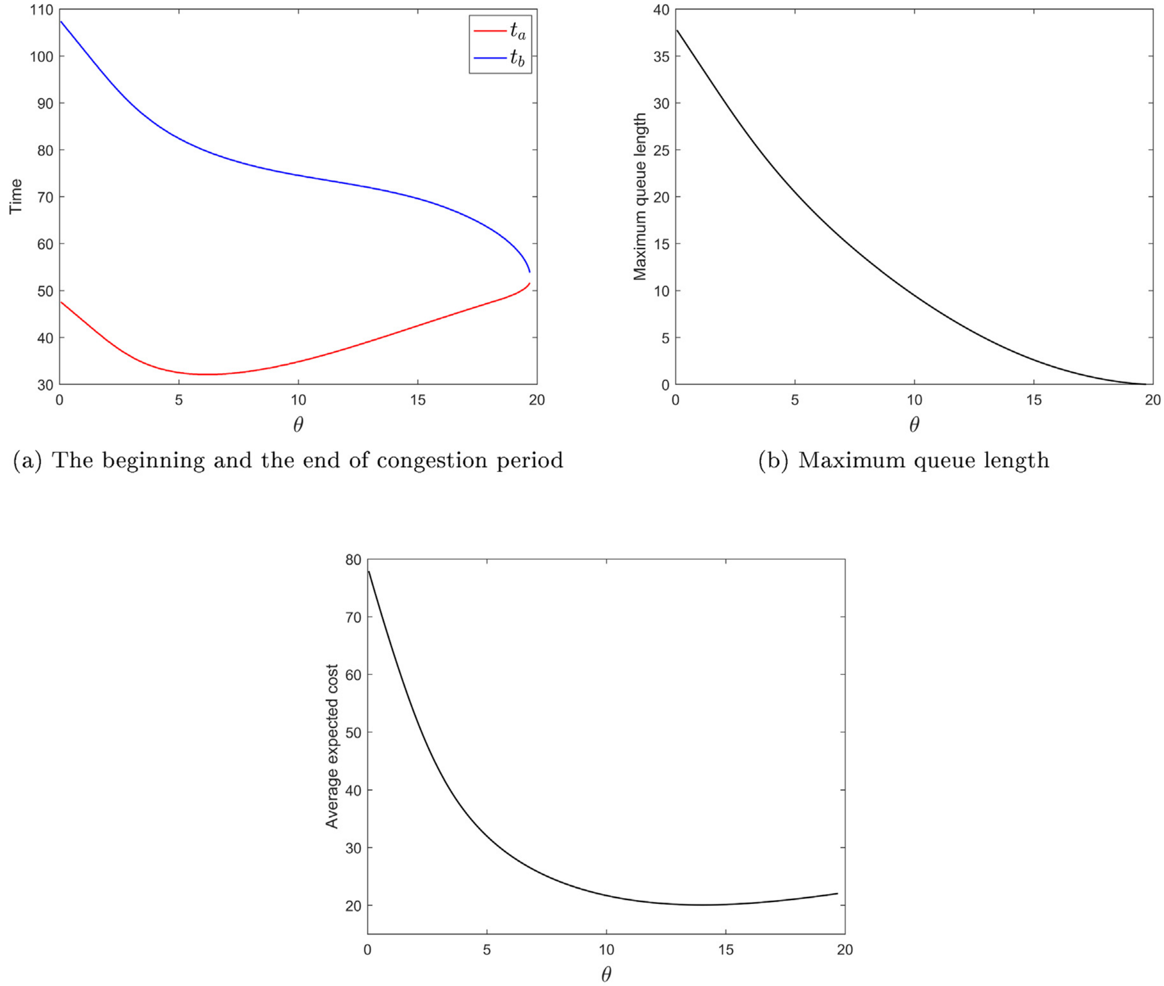
τ (Fig. 13(b)), there is still some social welfare gain by increasing the flexibility of travelers' choices on their arrival times into the bottleneck.

The results from effects of different related variables in the dynamic model on the stationary state are summarized as below.

- The larger the fraction R of travelers that revise their arrival time choices, the larger the scale parameter θ must be to ensure an equilibrium (Fig. 8).
- The larger the support τ of the delay function becomes, the lesser the number of days it takes for the system to stabilize (Fig. 10).
- The larger the scale parameter θ, the lesser the flexibility travelers have in arrival time choices (Fig. 11).
- Increasing the scale parameter θ makes the system less congested and decreases average expected cost (Fig. 12).
- Increasing τ makes the system less congested (after a certain amount of randomness) but increases average expected cost (Fig. 13).

5.2. Social cost

The proposed framework for the logit model is focused on a dynamical adaption of arrival time choices by travelers in response to a bottleneck with travel time uncertainties. In some cases the result of the dynamic adjustment is a stable situation where the arrivals have an equilibrium distribution in terms of the recursion governing the adaptations. Note however that the cost of travelers may not be the same. The social effects of uncertainties on the average expected cost and



(a) The beginning and the end of congestion period

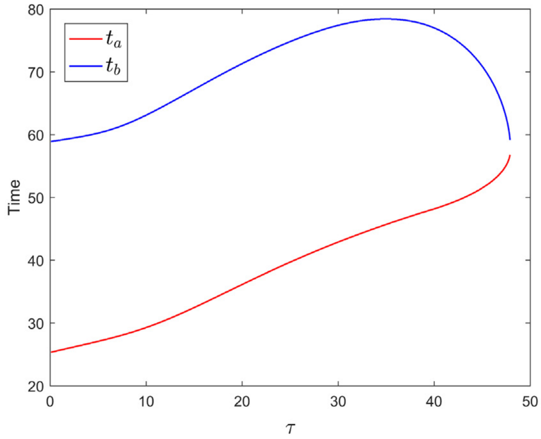
(b) Maximum queue length

(c) Average expected cost

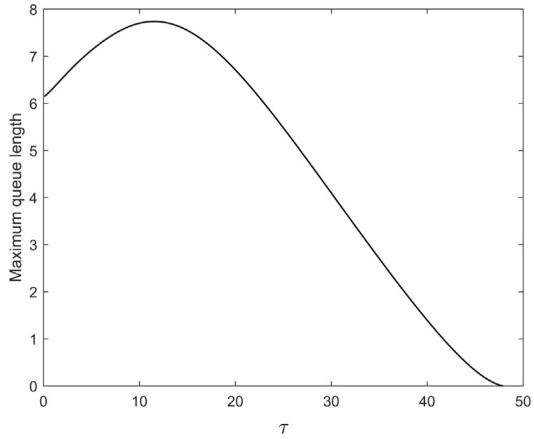
Fig. 12. Impact of scale parameter on congestion and cost.

the duration of the congestion period can be further studied by comparing the difference between the derived results of the logit model and the standard Vickrey model. We numerically analyze the effects of different uniform delay functions on the expected cost and the congestion period while θ and R are kept unchanged. The analysis is carried out by considering two values of the scale parameter θ to see the direction of changes. Fig. 14 demonstrates the effects of increased travel time uncertainties on the average expected cost and the duration of the congestion period. It shows that if travelers react on their past experiences, the social cost and congestion period can decrease by giving them more flexibility in their actual time to arrive at the bottleneck. We conclude with a summary of the observations from this numerical experiment.

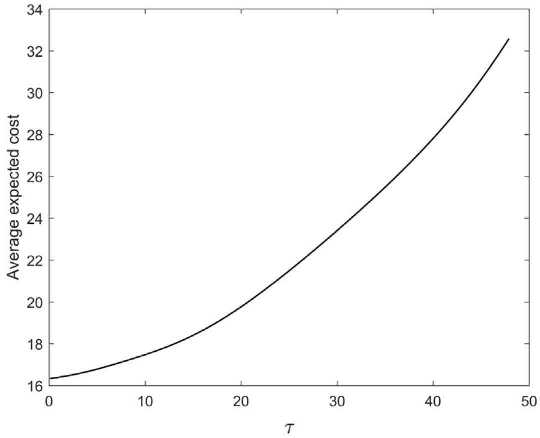
- In Fig. 14(a) we observe that the congestion period is shorter in all of the computed examples. The reason for this phenomenon is that the random delays in travel time makes the effective arrival flow more spread out and thus reduces congestion.
- Fig. 14(b) illustrates that for low values of τ, i.e., distortions with a small support and variance, social welfare is lower than in the Vickrey outcome. This relates to the previous point that congestion can be reduced by the random distortions. It is well known that a toll that eliminates congestion is the socially optimal policy for the Vickrey model [2] and so here we observe that small random travel time delay may improve the social welfare without any control mechanism. For high values of τ we observe that the high variance in arrivals can decrease social welfare significantly.
- The effect on social welfare remains consistent for high and low levels of the scale parameter θ. However, the length of the congestion period in Fig. 14(a) appears to be very sensitive to θ.



(a) The beginning and the end of congestion period

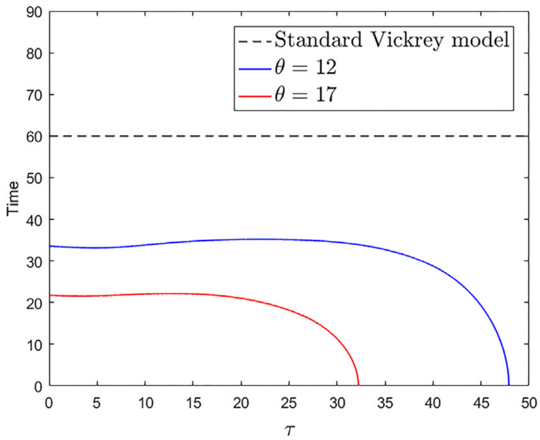


(b) Maximum queue length

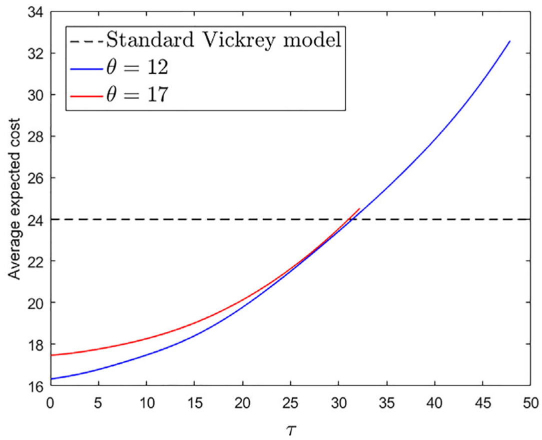


(c) Average expected cost

Fig. 13. Impact of support of uniform delay on congestion and cost.



(a) Duration of congestion period



(b) Average expected cost

Fig. 14. Social effects of scale parameter and support of uniform delay functions on cost and level of congestion.

6. Conclusion and discussion

In this paper we have investigated the impact of uncertainty when travelers plan their arrivals to a bottleneck (congestion point). In our model the actual arrival times at the bottleneck and the intended arrival times may deviate from each other according to some distribution. Such a random distortion models the fact that the actual arrival time of users at a specific congestion point cannot be completely controlled by the users. In reality it is common that the departure times from the points of origin, and the delays incurred before reaching the specific bottleneck can only be estimated up to a certain confidence range.

The equilibrium rate of the Vickrey model without distorted arrivals is, in general, not a good approximation for a possible equilibrium in the model with distortions. The numerical analysis further showed that costs are reduced for almost the entire bottleneck period, while for highly variable uncertainty distributions, the costs spike up at the end of the bottleneck period. This is due to the possibility of arrival after the end of the standard bottleneck period. As a consequence, users will deviate from this arrival profile, and the next natural question is whether a new profile exists such that users have no incentive to deviate from it.

We have shown that in general such an equilibrium does not exist for a model with uniformly distributed delay functions; neither as a pure strategy nor as a mixed-continuous strategy. We showed how the dynamics of delayed arrivals inhibit the existence of a constant expected cost function, thus ruling out a Nash equilibrium. We have seen that a Nash equilibrium solution with multiple masses of arrivals can be constructed for specific parameter choices. This special case is of mathematical interest, but the precise equilibrium has no practical importance as it depends on very particular parameter choices.

We have further presented an alternative dynamic model with arrival times being updated daily according to a logit choice function.

We numerically showed that the day-to-day arrival rate evolution will converge to a stationary profile when travelers have larger uncertainties regarding their travel times (τ is large) or when travelers are not very sensitive to the cost (the scale parameter θ in the logit model is large). In both cases, arrivals are widely spread out, and congestion (queue length and waiting times) is not that significant. To the contrary, for small values of the scale parameter θ, the discrete choice model closely approximates cost minimization. The lack of convergence of the dynamic model in that case, is thus well explained by the non-existence, in general, of a Nash equilibrium distribution.

Variations/extensions

The main contribution of this paper is that if travelers are uncertain as to their delays in arriving at the congestion point, a Nash equilibrium does not exist in general, and hence, the congestion period may show large fluctuations from day to day (in terms of time of congestion, severity of congestion and duration of congestion) even under similar exogenous conditions. It motivated us to look at a different paradigm (the logit model) in which users are not strictly minimizing cost. We have seen that in this setting the dynamic model guarantees convergence for suitable parameters. Note that in this model the cost is not the same for all travelers, which would be necessary for equilibrium if users were minimizing cost.

A natural question is to what extent the exact setting of the paper is due to the negative result regarding the existence of a Nash equilibrium. We discuss two variations that were suggested by one of the reviewers of our paper.

A public transport model

Suppose that we let travelers choose from a discrete set of departure times, as is natural in the context of public transport. This leads to an intriguing setting, both from the perspective of modeling and mathematical analysis. The discrete nature of this model makes it natural to investigate the existence of a Wardrop equilibrium, which in its simplest setting is guaranteed. It is not clear however whether this is true if we only allow for finite (volume) capacity of public transport units, which would be of practical relevance.

Heterogeneous travelers

Heterogeneity in travel time distortion is a natural assumption. It has been studied, for example, in [8] where the travel times from home to the bottleneck are independent and identically distributed random variables from a certain probability distribution. In our model, users do not a-priori know their own delay. In their decisions they do take into account the delay *distribution*. Our model can be extended with traveler-specific delay distributions, which are known to them. It is not hard to see that in general no Nash equilibrium can exist: In such an equilibrium, each traveler type would have a specific distribution for the *chosen* arrival time, and no traveler would profit by deviating from those distributions. Hence the *unconditional* distribution for the *chosen* arrival time (of an arbitrary traveler; weighing the type-specific distortion distributions with the respective proportions of the types) will satisfy the Nash equilibrium condition in our model, which we know does not have a solution in general.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We thank the editor and anonymous reviewers of our paper for their valuable input, and in particular the reviewer who suggested several variations and extensions, some of which we discussed in the last section. Funding from the project Dynafloat of the program in Sustainable Logistics of the Netherlands Organization of Scientific Research is gratefully acknowledged.

References

[1] W.S. Vickrey, Congestion theory and transport investment, Am. Econ. Rev. 59 (2) (1969) 251–260.
[2] R. Arnott, A. De Palma, R. Lindsey, Economics of a bottleneck, J. Urban Econ. 27 (1) (1990) 111–130.
[3] W. Xin, D. Levinson, Stochastic congestion and pricing model with endogenous departure time selection and heterogeneous travelers, Math. Popul. Stud. 22 (1) (2015) 37–52.
[4] M. Ben-Akiva, M. Cyna, A. de Palma, Dynamic model of peak period congestion, Transp. Res. B 18 (4) (1984) 339–355.
[5] R. Arnott, A bathtub model of downtown traffic congestion, J. Urban Econ. 76 (2013) 110–121.
[6] G.F. Newell, The morning commute for nonidentical travelers, Transp. Sci. 21 (2) (1987) 74–88.
[7] R. Arnott, A. De Palma, R. Lindsey, The welfare effects of congestion tolls with heterogeneous commuters, J. Transp. Econ. Policy (1994) 139–161.
[8] K.A. Small, The bottleneck model: An assessment and interpretation, Econ. Transp. 4 (1) (2015) 110–117.
[9] M. Ben-Akiva, A.D. Palma, P. Kanaroglou, Dynamic model of peak period traffic congestion with elastic arrival rates, Transp. Sci. 20 (3) (1986) 164–181.
[10] W. Wu, Day-to-day departure time adjustment behaviors in a single bottleneck model, in: Proceedings of the 2nd International Conference on Transportation Engineering, ICTE 2009, 2009, pp. 3375–3380.
[11] A. Glazer, R. Hassin, ?/M/1: On the equilibrium distribution of customer arrivals, European J. Oper. Res. 13 (2) (1983) 146–150.
[12] R. Jain, S. Juneja, N. Shimkin, The concert queueing game: to wait or to be late, Discrete Event Dyn. Syst. 21 (1) (2011) 103–138.
[13] S. Juneja, N. Shimkin, The concert queueing game: strategic arrivals with waiting and tardiness costs, Queueing Syst. 74 (4) (2013) 369–402.
[14] M. Haviv, When to arrive at a queue with tardiness costs? Perform. Eval. 70 (6) (2013) 387–399.
[15] R. Hassin, Rational Queueing, CRC Press, 2016.
[16] M. Fosgerau, L. Engelson, J.P. Franklin, Commuting for meetings, J. Urban Econ. 81 (2014) 104–113.
[17] M. Ostrovsky, M. Schwarz, Synchronization under uncertainty, Int. J. Econ. Theory 2 (1) (2006) 1–16.
[18] A. Glazer, R. Hassin, L. Ravner, A strategic model of job arrivals to a single machine with earliness and tardiness penalties, IISE Trans. 50 (4) (2018) 265–278.
[19] K.A. Small, The scheduling of consumer activities: Work trips, Am. Econ. Rev. 72 (3) (1982) 467–479.
[20] M.J. Smith, The existence of a time-dependent equilibrium distribution of arrivals at a single bottleneck, Transp. Sci. 18 (4) (1984) 385–394.
[21] C. Antoniou, M. Ben-Akiva, M. Bierlaire, R. Mishalani, Demand simulation for dynamic traffic assignment, IFAC Proc. Vol. 30 (8) (1997) 633–637, 8th IFAC/IFIP/IFORS Symposium on Transportation Systems 1997 (TS '97), Chania, Greece, 16–18 June.