THE GAP TOPOLOGY FOR LINEAR SYSTEMS

A GEOMETRIC APPROACH

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Contents

Acknowledgements **iii**
Preface **1**
1. Some Hilbert space geometry **3**
Chapter 1 Hilbert space geometry and robust stability **3**
2. Linear systems in Hilbert space **11**
3. Complementarity and robust 'internal' stability **14**
Chapter 2: Inner shift realizations **19**
1. The normalized shift realization in discrete time **19**
2. An attempt to characterize a class of first-order systems **23**
3. Continuous-time finite-dimensional systems **27**
4. Infinite-dimensional discrete-time systems **34**
Chapter 3: The gap topology **37**
1. Some alternative definitions of the gap topology **37**
2. Equivalence of graph topology and gap topology **38**
3. Equivalence of δ_{H_2} and the left halfline behaviour topology **40**
4. Relations with other topologies **41**
Chapter 4: Parameter variations **49**
1. An extension of l'Hôpital's rule to several variables **49**
2. Application of the rule to parameter variations **54**
Chapter 5: Computation of the gap **59**
1. Introduction **59**
2. State-space expression of parallel projections **59**
3. Computing the parallel projection norm **63**
4. Criterion for the gap to be equal to one **64**
5. Computations **67**
References **71**
Summary in Dutch **75**
List of symbolic abbreviations **77**
Index **79**

Preface

This thesis is devoted to robust control of linear systems.

A control algorithm is said to be *robust* when it also works if the system to be controlled is not known exactly or subject to disturbances. One way to quantify the robustness achieved by a controller is to introduce a metric on the class $\mathcal{P}$ of linear systems. A controller stabilizing a *nominal system* P_0 is then said to be robust with *robustness margin* $\geq \varepsilon$ if it achieves stabilization of all systems P with $d(P, P_0) < \varepsilon$.

We shall be concerned with a geometrical model of uncertainty where the set $\mathcal{P}$ of linear systems is considered as a set of subspaces of a Hilbert space, and the distance measure is the so-called *gap metric*, the sine of the *maximal angle* between subspaces.

Chapter 1 discusses some background on the geometry of the angles between closed subspaces of a Hilbert space. We give an interpretation of stabilization in terms of the complementarity of the solution sets to the equations that define the system and the controller. The main body of the chapter is devoted to the proof of some inequalities that are related to the continuity of elementary operations on the set of closed subspaces of a Hilbert space, like linear sum, intersection and orthogonal complementation. We also give a simple derivation of the relation between a system Σ and its adjoint system $\tilde{\Sigma}$, which describes the orthogonal complement of Σ.

Chapter 2 treats an aspect of the realization problem for linear systems. Using system-theoretic concepts only we prove a time-domain equivalent of the famous Beurling-Lax representation theorem for shift-invariant subspaces of H_2^+. Instead of the Beurling symbol we directly construct a realization of it.

A discussion of the properties of the topology and the various ways in which it can be obtained is to be found in chapter 3. We show the gap topology arises in various natural ways, and indicate the relationships with the L_2 induced operator norm topology on sets of transfer functions and the topology of minimal state space parameters. The topologies considered can all be interpreted as topologies of uniform convergence on the set of analytic functions from the extended complex plane to a Grassmannian manifold of subspaces of $\mathbf{C}^n$ (pointwise gap topologies). The connection with minimum driving variable state space parameters is made with the help of the realization procedure of chapter 2, which we can easily see to yield continuous parameters with respect to the topology of uniform convergence on the whole extended complex plane.

Chapter 4 investigates the continuity of a parametrized family of systems w.r.t. the gap topology. We derive a reasonably simple rule that answers this question.

The computation of the gap metric is the subject of chapter 5. We give a procedure that is based on an expression for the gap as a maximum of two projection norms, and a simple test for the occurrence of the important situation $\delta(V_1, V_2) = 1$. The method is based on a simple derivation of state space representations of various projection operators, which uses the interpretation of the adjoint system given in chapter 2. The efficiency of the implied method is hardly better than the conventional techniques. However the formulas are simple and there is no reason not to use them.

The general message of the booklet seems to be that it can be useful to analyze several phenomena in terms of the relative position of spaces that are solutions sets (in the space of square integrable functions) to the equations of the systems concerned. Most of our proofs are based on elementary Hilbert space geometry; a key role is played by the minimal angles and parallel projections of subspaces.

Chapter 1

Hilbert space geometry and robust stability

1. Some Hilbert space geometry

This section is devoted to the introduction of the gap topology and the proof and statement of some results about the continuity of various operations on the set of closed linear subspaces of a Hilbert space. Most of them can essentially be found in the classic [Kato], though we give some quantitative estimates not found there. The aim is to collect a few basic facts of elementary Hilbert space geometry and to put them into the right perspective from the point of view of control theory.

The main object of study in this thesis is the *gap topology* on the set of linear systems viewed as closed subspaces of a Hilbert space (cf. section 2). The gap topology on the set $\mathcal{SUB}(X)$ of all closed subspaces of a Hilbert space X can be introduced as follows: The *gap* between two closed subspaces X, Y of a Hilbert space is given by

$$\delta(X, Y) := \|\Pi_X - \Pi_Y\|,$$

where Π_X is the orthogonal projection on X. Equivalently

$$\delta(X, Y) = \max(\vec{\delta}(X, Y), \vec{\delta}(Y, X)),$$

where the directed gap

$$\vec{\delta}(X, Y) = \sup_{x \in X, \|x\|=1} d(x, Y).$$

The latter definition can also be used in Banach spaces. In Hilbert space it is clear that the gap is a metric because it inherits this property from the operator norm. The directed gaps need not be equal to each other; consider two spaces of which one is nontrivially contained in the other. However we do have the following:

Lemma 1.1. *Let V, V_1 be closed subspaces of a Hilbert space. Assume $\vec{\delta}(V, V_1) < 1$, and $\vec{\delta}(V, V_1) \neq \vec{\delta}(V_1, V)$. Then $V^\perp \cap V_1 \neq \{0\}$.*

Proof. This is a well-known fact, cf. [Georgiou,Smith], [Krasnosel'skiĭ et al.]. It is known that the two directed gaps are equal unless the restricted orthogonal projection $\Pi_{V_1}|V$ is not bijective. This means that either $V^\perp \cap V_1 \neq \{0\}$ or $V_1^\perp \cap V \neq \{0\}$. As we have assumed that $\vec{\delta}(V, V_1) < 1$, we must be in the first case. □

As a consequence we have also the following proposition, which is part of a collection of facts that is well-known to people working in the area but difficult to trace in the growing literature.

Proposition 1.2. *Consider two closed subspaces V, V_1 of a Hilbert space such that either* $\dim V = \dim V_1 < \infty$ *or* $\operatorname{codim} V = \operatorname{codim} V_1 < \infty$. *Then* $\vec{\delta}(V, V_1) = \vec{\delta}(V_1, V)$.

Proof. Suppose first $\dim V = \dim V_1$. If $\vec{\delta}(V, V_1) = \vec{\delta}(V_1, V) = 1$ there is nothing to prove, so assume $\vec{\delta}(V, V_1) < 1$. Again, the directed gaps are equal if the restricted projection $\Pi_{V_1}|V$ is bijective. But we know it must injective since otherwise $\ker \Pi_{V_1}|V \subset V$ is orthogonal to V_1, hence $\vec{\delta}(V, V_1) = 1$. So it is also surjective because of the equality of the dimensions. Hence the directed gaps are equal. If the codimensions are equal we can use the identity $\vec{\delta}(X, Y) := \|\Pi_{Y^\perp}|_X\| = \|(I - \Pi_Y)\Pi_X\| = \|((I - \Pi_Y)\Pi_X)^*\| = \|\Pi_X(I - \Pi_Y)\| = \|\Pi_X|_{Y^\perp}\| = \vec{\delta}(Y^\perp, X^\perp)$. So $\vec{\delta}(V_1, V) = \vec{\delta}(V^\perp, V_1^\perp) = \vec{\delta}(V_1^\perp, V) = \vec{\delta}(V, V_1)$. □

It is furthermore convenient to define the *maximal angle* $\vartheta(X, Y) \in [0, \frac{1}{2}\pi]$ by

$$\vartheta(X, Y) = \arcsin \delta(X, Y).$$

The maximal angle is a metric as well, and defines the same topology as the gap metric. The gap was introduced in the Russian literature as the 'opening' (also translated as 'aperture' in some texts) [Krein,Krasnosel'skiĭ] in order to study perturbations of unbounded closed operators.

The gap defines a metric on a set of operators between two Banach spaces X and Y by considering their graphs as subspaces of $X \times Y$. We shall denote the graph of an operator A by $\mathcal{G}(A)$. It is worth mentioning that the gap topology on the set $\mathcal{B}(X, Y)$ of bounded operators from X to Y is equivalent to the operator norm topology (cf. [Kato], Theorem IV.2.13).

Lemma 1.3. *Let X,Y be Banach spaces. Then the correspondence*

$$\mathcal{B}(X, Y) \to \mathcal{SUB}(X \times Y),$$
$$A \mapsto \mathcal{G}(A)$$

is a homeomorphic embedding of $\mathcal{B}(X, Y)$, equipped with the induced operator norm topology, in $\mathcal{SUB}(X \times Y)$.

Another notion we use is the *minimal angle* $\varphi(V, W) \in [0, \frac{1}{2}\pi]$ between two subspaces V and W of some Hilbert space X. It is defined by

$$\sin\varphi(V, W) = \inf\{d(x, W) \mid x \in V, \|x\| = 1\}.$$

An equivalent symmetrical definition is

$$\varphi(V, W) = \inf\{\vartheta(\operatorname{span} x, \operatorname{span} y) \mid x \in V, y \in W\}.$$

A generalization of the notion of minimal angle to subspaces with nonzero intersection is (cf. for instance [Kato], section IV.4.1):

Definition 1.4. Let U, V be subspaces of a Hilbert space X. Then the *minimal gap* $\gamma(U, V)$ is defined as

$$\gamma(U, V) := \inf_{v \in V \setminus U} \frac{d(v, U)}{d(v, U \cap V)}.$$

The corresponding angle will be denoted

$$\psi(U, V) := \arcsin\gamma(U, V) \in [0, \frac{1}{2}\pi]$$

Actually in Hilbert space $\psi(U, V)$ can easily be rewritten as a minimal angle in the following way:

Lemma 1.5. $\psi(U, V) = \varphi(V \ominus (U \cap V), U)$.

Proof. Choose $v \in V$ arbitrary and put $v = v_1 + v_2$, $v_1 \in U \cap V$, $v_2 \in V \ominus (U \cap V)$. This means $d(v, U \cap V) = \|v_2\|$ and $d(v, U) = d(v_2, U)$ so we can assume $v \in V \ominus (U \cap V)$ without loss of generality. Now obviously

$$\gamma(U, V) = \inf_{v \in V \ominus (U \cap V)} d(v, U)/\|v\| = \sin\varphi(V \ominus (U \cap V), U).$$

□

Also the following is used ([Kato], theorem IV.4.2):

Lemma 1.6. *For U, V closed subspaces of a Banach space we have: $U + V$ is closed $\Leftrightarrow \gamma(U, V) > 0$.*

The quantity γ occurs as a robustness margin for the so-called Fredholm indices of pairs of linear subspaces. Since these indices are related to quantities that have a meaning in control theory, like the number of poles of finite-dimensional systems, cf. chapter 3, proposition 4.6, we give the (Hilbert space) definitions here [Kato], IV.4.1.

Definition 1.7. Let U, V be two closed subspaces of a Hilbert space X with $U + V$ closed. Then the *nullity* $\mathrm{nul}(U, V)$ of U and V is $\dim U \cap V$. The *deficiency* $\mathrm{def}(U, V)$ is $\dim X \ominus (U + V)$.

The main robustness result is then (Corollary IV.4.25 in [Kato])

Proposition 1.8. *Let U, V be two closed subspaces of a Hilbert space X with $U + V$ closed. Then if $\delta(U', U) < \gamma(U, V)$ we have* $\mathrm{nul}(U', V) \leq \mathrm{nul}(U, V)$ *and* $\mathrm{def}(U', V) \leq \mathrm{def}(U, V)$.

We shall need the following lemma, a triangle inequality in terms of angles. It also appears in [Qiu, Davison 1992b]; we give it here with a different, less technical proof to emphasize the intuitive nature of the fact.

Lemma 1.9. *Let X,Y,Z be closed subspaces in a (real or complex) Hilbert space H. Then one has*

$$\varphi(Y, Z) \geq \varphi(X, Z) - \vartheta(X, Y).$$

Proof. We first prove that no loss of generality is entailed by the assumption that H is a real vector space. This is easy: we may replace a complex space H by the real Hilbert space structure H' on the same set that arises by restricting the scalar multiplication to $\mathbf{R}$ and replacing the complex-valued inner product by the real-valued Re $\langle x, y\rangle = \frac{1}{2}(\langle x, y\rangle + \langle y, x\rangle)$. Note that the transformation from H to H' is distance-preserving.

We first prove the inequality for one-dimensional (x, y, z), for which there is just one angle $\varphi = \vartheta$. Suppose (x, y, z) violate the inequality. Let V be the span of x and z over $\mathbf{R}$. Then take y' to be the orthogonal projection on V of y. Since the projection contracts distances, $\vartheta(x, y') \leq \vartheta(x, y)$ and $\vartheta(z, y') \leq \vartheta(z, y)$. So (x, y', z) would be a set of three lines in the plane such that $\vartheta(x, y') + \vartheta(y', z) < \vartheta(x, z)$, which is obviously impossible.

Now in the general case, let $\varepsilon > 0$ be arbitrary, let the lines $y \subset Y$ and $z \subset Z$ be such that $\varphi(y, z) \leq \varphi(Y, Z) + \varepsilon$, and let $x = \Pi_X y$. Since $\vec{\vartheta}(X, Y) = \sup_{y \subset Y} \vartheta(y, \Pi_X y)$ we then have $\vartheta(X, Y) \geq \vartheta(x, y)$, so $\varphi(Y, Z) + \vartheta(X, Y) + \varepsilon \geq \varphi(y, z) + \vartheta(x, y) \geq \varphi(x, z) \geq \varphi(X, Z)$. □

1.1 Continuity of skew projections

Definition 1.10. Two closed subspaces U, V of a Hilbert space X are said to be complementary if $U + V = X$ and $U \cap V = \{0\}$.

As a special case of proposition 1.8 we have (cf. [Berkson]): if $V \cap W = \{0\}$ then $\delta(V, V') < \sin\varphi(V, W) \Rightarrow V' \cap W = \{0\}$. Similarly also the situation $V + W = X$ is stable with respect to small perturbations in the gap topology. This fact has the control-theoretic meaning that closed loop stability is a robust property in the gap topology, cf. section 3.

If two subspaces $V, W \subset X$ are complementary one defines, as usual, the skew or parallel projection Π_W^V by $\Pi_W^V x = v$ such that $\exists w : v + w = x$. A useful fact is the following. Assume $V + W$ is closed and $V \cap W = \{0\}$. Let Π_W^V be the skew projection of $V + W$ along V on W. Then (cf. for instance [Gohberg, Krein, p. 339]):

Lemma 1.11. $\sin\varphi(V, W) = \frac{1}{\|\Pi_W^V\|}$.

We establish the fact that skew projections depend continuously on their kernels.

Lemma 1.12. *Let (U_1, W) and (U_2, W) be pairs of closed subspaces of a Hilbert space X with (U_1, W) complementary.*

Suppose $\vartheta(U_1, U_2) < \varphi(U_1, W)$. Then

$$\|\Pi_W^{U_1} - \Pi_W^{U_2}\| \leq \frac{1}{\sin\varphi(U_1, W)\sin(\varphi(U_1, W) - \vartheta(U_1, U_2))}\delta(U_1, U_2).$$

Proof. Let $\Psi_1 = \Pi_W^{U_1}$, $\Psi_2 = \Pi_W^{U_2}$, $\delta = \delta(U_1, U_2)$. Let $u_1 \in U_1$. Choose $u_2 \in U_2$ such that $\|u_1 - u_2\| \leq \delta\|u_1\|$. Then $\|(\Psi_1 - \Psi_2)u_1\| = \|\Psi_2 u_1\| = \|\Psi_2(u_1 - u_2)\| \leq \delta\|\Psi_2\|\|u_1\|$. For arbitrary x we have $x = u_1 + w$, $u_1 \in U_1$, $\|u_1\| \leq \|\Psi_1\|\|x\|$, and $(\Psi_1 - \Psi_2)x = (\Psi_1 - \Psi_2)u_1$, so $\|(\Psi_1 - \Psi_2)x\| \leq \|\Psi_1\|\|\Psi_2\|\,\delta\,\|x\|$. The minimal angle γ between U_2 and W is larger than or equal to $\varphi(U_1, W) - \vartheta(U_1, U_2)$, so because $\|\Psi_2\| = \frac{1}{\sin\gamma}$ we obtain the desired formula. □

Lemma 1.13. *Let U_1, V_1 be a pair of complementary subspaces. Then U_2, V_2 in a sufficiently small neighbourhood of U_1 resp. V_1 are also complementary, and one has*

$$\|\Pi_{V_1}^{U_1} - \Pi_{V_2}^{U_2}\| \leq \frac{\delta(U_1, U_2)}{\sin\varphi(U_1, V_1)\ \sin(\varphi(U_1, V_1) - \vartheta(U_1, U_2))} + \frac{\delta(V_1, V_2)}{\sin(\varphi(U_1, V_1) - \vartheta(U_1, U_2))\ \sin(\varphi(U_1, V_1) - \vartheta(U_1, U_2) - \vartheta(V_1, V_2))}.$$

Proof. Using lemma 1.12 and the inequality $\|\Pi_{V_1}^{U_1} - \Pi_{V_2}^{U_2}\| \leq \|\Pi_{V_1}^{U_1} - \Pi_{V_1}^{U_2}\| + \|\Pi_{V_1}^{U_2} - \Pi_{V_2}^{U_2}\|$ one easily obtains the desired inequality. □

The next proposition relates convergence in the gap topology of subspaces and convergence of their skew projections in operator norm.

Proposition 1.14. *Let (U, V) be complementary. We have:*

$$(U_n \to U) \wedge (V_n \to V) \iff \Pi_{U_n}^{V_n} \to_{\|\cdot\|} \Pi_U^V.$$

Proof. $\Rightarrow$: Obvious from lemma 1.13. $\Leftarrow$: We show $\delta(U_1, U_2)$, $\delta(V_1, V_2) \leq \|\Psi_1 - \Psi_2\|$, where $\Psi_i = \Pi_{U_i}^{V_i}$, $i = 1, 2$. Choose $x \in U_1$, $\|x\| = 1$. Now $d(x, U_2) \leq \|x - \Pi_{U_2}^{V_2} x\| = \|\Pi_{U_1}^{V_1} x - \Pi_{U_2}^{V_2} x\| \leq \|\Pi_{U_1}^{V_1} - \Pi_{U_2}^{V_2}\|$. Analogously (for the other directed gap) for $x \in U_2$, $\|x\| = 1$ we have $d(x, U_2) \leq \|\Psi_1 - \Psi_2\|$. And since $\|\Pi_{U_1}^{V_1} - \Pi_{U_2}^{V_2}\| = \|(I - \Pi_{U_1}^{V_1}) - (I - \Pi_{U_2}^{V_2})\| = \|\Pi_{V_2}^{U_2} - \Pi_{V_1}^{U_1}\|$, the same argument also gives $\delta(V_1, V_2) \leq \|\Pi_{U_1}^{V_1} - \Pi_{U_2}^{V_2}\|$. □

1.2 Continuity of linear sum and intersection

Continuity of feedback interconnection of linear systems was the main reason to introduce the gap topology in control theory. Interconnection of two systems is most naturally viewed simply as intersection of solution sets. So it is essential in this context to study the continuity of the lattice operations $\cap$, $+$, $\perp$ on subspaces of a Hilbert space Z with respect to the gap topology. The defining formula $\delta(X, Y) = \|\Pi_X - \Pi_Y\|$ implies that $\perp$ is actually isometric. As we already recalled in proposition 1.8, the behaviour of the *dimensions* of $X \cap Y$ and $Z \ominus (X + Y)$ under small perturbations was already studied in [Kato]. It is not difficult to extend his analysis to the continuity of the operations themselves.

Definition 1.15. Two closed subspaces U, V of a Hilbert space Z are in *general position* if $U + V$ is closed and either $U \cap V = \{0\}$ or $U + V = Z$.

This definition generalizes the generic (in the sense of the gap topology) finite-dimensional situation $\dim Z = n$, $\dim U = k$, $\dim V = l$, $\dim U \cap V = \max(k + l - n, 0)$.

Proposition 1.16. *Intersection is continuous for spaces in general position. For U, V in general position one has (for U' in a sufficiently small neighbourhood of U) $\delta(U \cap V, U' \cap V) \leq \frac{1}{\gamma(U,V)} \delta(U, U')$.*

In general we have for closed subspaces M, N that $M + N$ is closed iff $M^\perp + N^\perp$ is closed ([Kato], theorem IV.4.8). This means we can equivalently prove the dual form of this proposition.

Proposition 1.17. *Linear summation is continuous for spaces in general position. For U, V in general position one has (for V_1 in a sufficiently small neighbourhood of V) $\delta(U + V, U + V_1) \leq \frac{1}{\gamma(U,V)} \delta(V, V_1)$.*

Proof. We first prove separate continuity. For $x \in U + V$ with $\|x\| = 1$ we have $x = u + v$, $u \in U$, $v \in V$ with $\|v\| \leq 1/\gamma(U, V)$. It follows that in general

$$\vec{\delta}(U + V, U + V_1) \leq \delta(V, V_1)/\gamma(U, V).$$

We prove that the assumptions $\delta(V, V_1) < \gamma(U, V)$ and $(U+V = Z) \vee (U \cap V = \{0\})$ imply that this estimate also holds for the undirected gap $\delta(U + V, U + V_1)$. If $U + V = Z$, this also holds for V_1 with $\delta(V, V_1) < \gamma(U, V)$, cf. proposition 1.8, and the estimate trivially holds. So we may assume that $U \cap V = \{0\}$. Suppose $\vec{\delta}(U + V_1, U + V) > a := \delta(V, V_1)/\gamma(U, V)$. So there is $x = u + v_1 \in U + V_1$ with $\|x\| = 1$ such that $d(x, U + V) > a$. Let $v = \Pi_V v_1$. Then the triple $U' = \mathrm{span}(u)$, $V' = \mathrm{span}(v)$, $V_1' = \mathrm{span}(v_1)$ is such that $\dim(U' + V') = \dim(U' + V_1')$ and $\vec{\delta}(U' + V', U' + V_1') \neq \vec{\delta}(U' + V_1', U' + V')$. For $\vec{\delta}(U' + V_1', U' + V') \geq d(u + v_1, U + V) > a$, whereas $\vec{\delta}(U' + V', U' + V_1') \leq (1/\gamma(U', V'))\delta(V', V_1') \leq a$ since $\gamma(U', V') \geq \gamma(U, V)$ and $\delta(V, V_1) \geq (1/\|v_1\|)d(v_1, P_V v_1) = \delta(V', V_1')$; and v_1 and u are linearly independent by assumption. Of course u and v are independent because of the assumption $U \cap V = \{0\}$. So we have a contradiction by proposition 1.2.

To prove joint continuity, first notice that our assumptions imply that either $\gamma(U, V) = \sin\varphi(U, V)$ or $\gamma(U, V) = \sin\varphi(U^\perp, V^\perp)$. We may use lemma 1.9 to see that

$$\psi(U, V_1) \geq \psi(U, V) - \vartheta(V, V_1).$$

Hence for any U_1 close enough to U, we finally have

$$\begin{aligned}
\delta(U_1 + V_1, U + V) &\leq \delta(U + V_1, U + V) + \delta(U + V_1, U_1 + V_1) \\
&\leq \frac{1}{\gamma(U, V)}\delta(V, V_1) + \frac{1}{\gamma(U, V_1)}\delta(U, U_1) \\
&\leq \frac{1}{\gamma(U, V)}\delta(V, V_1) + \frac{1}{\sin(\psi(U, V) - \vartheta(V, V_1))}\delta(U, U_1).
\end{aligned}$$

□

As a consequence of the proof we can state another lemma.

Lemma 1.18. *Let the spaces $U_0, V_0 \subset X$ be in general position. Then the correspondence*

$$\mathcal{SUB}(X) \times \mathcal{SUB}(X) \to \mathbf{R}$$
$$(U, V) \mapsto \gamma(U, V)$$

is continuous at (U_0, V_0) w.r.t. the gap topology.

1.3 Continuity results involving bounded operators

We shall occasionally need the fact that, with some assumptions, images and kernels of linear operators are continuous under bounded perturbations.

Definition 1.19. The *reduced minimum modulus* $\tau(A)$ of an operator is defined by

$$\tau(A) := \inf_{x\in(\ker A)^\perp} \frac{\|Ax\|}{\|x\|} = \inf_{x\notin(\ker A)} \frac{\|Ax\|}{d(x, \ker A)}.$$

The reduced minimum modulus can be thought of as a kind of condition number with respect to the null space of an operator. It is easily shown that one directed gap can be estimated in terms of it.

Lemma 1.20. *Let A, B be bounded operators Then*

$$\vec{\delta}(\ker B, \ker A) \leq \frac{\|A - B\|}{\tau(A)}.$$

Proof. Suppose $\vec{\delta}(\ker B, \ker A) > M$. Choose $x \in \ker B$, $\|x\| = 1$ with $d(x, \ker A) > M$. Then $\|Ax\| \geq \tau(A)M \Rightarrow \|(A - B)x\| \geq \tau(A)M \Rightarrow M \leq \frac{\|A-B\|}{\tau(A)}$. □

For images of operators we have the same thing. First it is useful to recall theorem IV.5.2 of [Kato].

Theorem 1.21. *A closed operator T has closed range iff* $\tau(T) > 0$.

Lemma 1.22. *Let A, B be bounded operators with closed range. Then*

$$\vec{\delta}(\operatorname{im} B, \operatorname{im} A) \leq \frac{\|A - B\|}{\tau(A)}.$$

Proof. To prove this, choose $y = Ax \in \operatorname{im} A$ with $\|y\| = 1$, $x \in (\ker A)^\perp$. Then obviously $\|x\| \leq 1/\tau(A)$. So $d(y, \operatorname{im} B) \leq \|Ax - Bx\| \leq (1/\tau(A))\|A - B\|$. □

We can use the previous section to determine the conditions that are sufficient to take care also of the other directed gap. First, there is a simple relation between the reduced minimum modulus and the 'minimal gap' γ.

Lemma 1.23. *Let A be a closed operator from X to Y. Then* $\gamma(\mathcal{G}(A), X \times \{0\}) = \sqrt{\frac{\tau(A)^2}{1+\tau(A)^2}}$.

Proof. Recall the formula $\gamma(U, V) = \sin\varphi(U \ominus (U \cap V), V)$, and note that $\mathcal{G}(A) \cap (X \times \{0\}) = \ker A \times \{0\}$. Now let $\|x\| = 1, x \perp \ker A$, $v = (x, Ax)$. Then $\frac{d(v, X\times\{0\})^2}{\|v\|^2} = \frac{\|Ax\|^2}{1+\|Ax\|^2}$. Obviously the right hand side is increasing in $\|Ax\|$. So it is clearly minimized for x with $\|Ax\|$ minimal. Whence the desired relation. □

We note that $\mathcal{G}(A)$ and $X \times \{0\}$ are in general position if A is either surjective or injective with $\tau(A) > 0$, whereas $\mathcal{G}(A)$ and $\{0\} \times Y$ are always in general position if A is bounded.

Lemma 1.24. *Let $B \in \mathcal{B}(X, Y)$ be surjective or injective. Then the correspondence $A \mapsto \tau(A)$ is continuous at B if $\tau(A) > 0$.*

Proof. This follows from the previous lemma and the continuity of $\gamma(.,.)$ for subspaces in general position.

Lemma 1.25. *Let $A \in \mathcal{B}(X, Y)$ be surjective. Then the correspondence $B \mapsto \ker B$ is continuous at A.*

Proof. First note that in general for $V_i \subset X \times Y$ we have $\delta(\Pi_X V_1, \Pi_X V_2) = \delta(\Pi_X V_1 + Y, \Pi_X V_2 + Y) = \delta(V_1 + Y, V_2 + Y)$ by orthogonality of X and Y. Then the result follows from $\ker A = \Pi_X(\mathcal{G}(A) \cap (X \times \{0\}))$ and the continuity of the correspondence $A \mapsto \mathcal{G}(A)$. □

Lemma 1.26. *Let A be a bounded operator from X to Y. Then the mapping*

$$\mathcal{B}(X, Y) \times \mathcal{SUB}(X) \to \mathcal{SUB}(Y)$$
$$(A, V) \mapsto AV$$

is continuous at (A, V) if either $AV = Y$ or A is injective on V with $\tau(A|_V) > 0$.

Proof. We have $\delta(AV_1, AV_2) = \delta(\Pi_Y(\mathcal{G}(A) \cap (V_1 \times Y)), \Pi_Y(\mathcal{G}(A) \cap (V_2 \times Y)) = \delta(\mathcal{G}(A) \cap (V_1 \times Y) + X, \mathcal{G}(A) \cap (V_2 \times Y) + X)$. Here the intersection is always in general position if A is everywhere defined; the requirement that the sum be in general position is exactly the hypothesis. □

2. Linear systems in Hilbert space

We make a few introductory remarks about the various ways we can view a linear system as a subspace of a Hilbert space and the different system representations we use. We shall identify a system in various ways with a subset of a space of quadratically integrable functions over a fixed finite dimensional vector space W of *external variables*. In the time domain, we consider solutions in $L_2(W)$, $L_2^+(W) = L_2(0, \infty; W)$, $L_2^-(W)$. In the frequency domain, we consider $L_2(i\mathbf{R}; W)$ and the Hardy spaces $H_2^+(W)$, $H_2^-(W)$. In the sequel we shall often omit the external variable space when no confusion can arise, so we refer to $L_2(T; W)$ as $L_2(T)$. Using the standard embedding of $L_2^+ = L_2(0, \infty)$ in $L_2(-\infty, \infty)$, $f \mapsto f'$ with $f'(t) = 0$ for $t < 0$ we can view L_2^+ as a subspace of $L_2(-\infty, \infty)$. Analogously for $L_2(-\infty, 0)$. Let $\mathcal{L}$ be the Fourier-Laplace transform isomorphism from $L_2(-\infty, \infty)$ to $L_2(i\mathbf{R}) = H_2^- \oplus H_2$. We consider only time-invariant systems, which means that we consider subspaces invariant for the shift operator. So on the whole line, we stay within the class of subspaces of $L_2(W)$ that are invariant for the shifts σ_d defined by $\sigma_d(f)(t) = f(t-d)$ for any $d \in \mathbf{R}$. When $d \geq 0$, we call σ_d a *forward shift*. For $d \leq 0$, σ_d is a *backward shift* which we shall also denote as τ_{-d}. On the halflines, so for instance in L_2^+, we can consider those subspaces that are invariant for all forward shifts or those that are invariant for all backward shifts. Abusing terminology a bit, we also call, for instance, the multiplication operators M_λ with symbol $e^{-\lambda s}$ on H_2 forward shifts. So a forwards invariant subspace of H_2 is meant to be a subspace invariant for all M_λ for $\lambda > 0$.

The point of view that is adopted in most of H_∞ optimization theory is to identify a system with its transfer function. So one presupposes a division $W = Y \times U$ of the external variable space W in an *input space* U and an *output space* Y and one starts with some shift-invariant closed operator G from $H_2^+(U)$ to $H_2^+(Y)$. For a transfer function G from U to Y, the *graph* is defined as a subspace of $H_2 := H_2(\mathbf{C}^+, Y \times U)$ by

$$\mathcal{G}(G) = \{(y, u) \in H_2(W) \mid y = Gu\}.$$

So one studies a class of closed forwards invariant subspaces of $H_2^+(W)$. At this point we must mention the following famous result, the Beurling-Lax representation theorem.

Theorem 2.1. *For each closed forwards invariant subspace V of $H_2(W)$ there exists a D such that* $\dim D \leq \dim V$ *and a* $\Theta \in H_\infty(\mathcal{L}(D, W))$ *such that*

$$V = \Theta H_2(D)$$

and $\Theta^*(s)\Theta(s) = I_D$ *for almost all* $s \in i\mathbf{R}$.

One calls Θ a *Beurling symbol* of V. One can also identify a transfer function G with an input-output operator in L_2 of the whole time axis. We write $\mathcal{G}_{L_2}(G)$ for the graph of this operator.

The external *behaviour* $\mathcal{B}_W(\Sigma)$ of a linear system in a certain space of W-valued functions is the set of its external solution trajectories in that space (cf. [Willems

1991]). By

$$V = \operatorname{im} \Sigma \subset L_2(\mathbf{R}, W)$$

it is understood that Σ is an input-output system with output space W, and V consists of the W-parts of the trajectories of Σ:

$$V = \Pi_{L_2(W)} \mathcal{B}_{L_2}(\Sigma).$$

In other words, Σ is a *driving-variable representation* of V [Willems 1983]. Taking behaviours as the basic objects, one starts from shift-invariant subspaces on the whole time axis; on the halfline, one also considers those solutions to the system that arise with nonzero initial state. Thus, on the halfline we are induced to study backwards invariant subspaces instead of the forwards invariant ones.

We proceed to sketch the relation between graphs of transfer functions and the different notions of behaviour. As a starting point we choose the set $\mathcal{S}(W)$ of closed shift-invariant subspaces of some $L_2(\mathbf{R}, W)$. From $V \in \mathcal{S}(W)$ we can obtain a backwards invariant subspace of L_2^+ by projecting on L_2^+:

$$V_+ = \Pi_+ V,$$

where Π_+ is the orthogonal projection on L_2^+ embedded in L_2. The space V_+ is not necessarily closed in general, but it is for finite-dimensional systems V. One obtains a forwards invariant subspace of L_2^+ by taking the intersection

$$V_+^0 = V \cap L_2^+.$$

If the space W is written as $U \times Y$, in such a way that U is an input space for the system V, then there exists a transfer function G from U to Y such that the Beurling symbol $\Theta \in H_\infty$ of V_+^0 is of the form $[{N \atop M}]$, with M invertible and $G = NM^{-1}$.

This means that we have

$$\mathcal{L}V = \mathcal{G}_{L_2}(G).$$

If this last equality holds we take V, V_+ to be the 'behaviours of G' in $L_2(\mathbf{R})$ resp. L_2^+, which relations we symbolize by

$$V = \mathcal{B}(G),$$

and

$$V_+ = \mathcal{B}_+(G).$$

It is a consequence of the previous discussion that

$$\mathcal{G}(G) = \mathcal{L}(\mathcal{B}_+^0(G)) = \mathcal{L}(\mathcal{B}(G) \cap L_2^+).$$

We can also consider $V_- = \mathcal{B}_-(G)$ and $\mathcal{G}_-(G) = \mathcal{L}(\mathcal{B}^0_-(G)) = \mathcal{L}(\mathcal{B}(G) \cap L_2(-\infty, 0))$. It is possible for any V in the class we consider (finite-dimensional systems) to find a decomposition of the external variable space such that $V = \mathcal{B}(G)$ for some G. So considering graphs of transfer functions entails no loss of generality.

It is often useful to have a description of the orthogonal complement of a graph or behaviour. In $L_2(-\infty, \infty)$, the orthogonal complements of shift-invariant spaces are shift-invariant themselves, and the adjoint of a multiplication operator is also a multiplication operator, so the orthogonal complement of a behaviour in $L_2(-\infty, \infty)$ is easy to describe. First assume we deal with a rational transfer function G. Let $\tilde{G}(s) = G^T(-s)$ as usual.

Lemma 2.2. $\mathcal{B}(G)^\perp = \mathcal{B}(-\tilde{G})$.

Proof. Write

$$\mathcal{L}\mathcal{B}(G) = \mathcal{M}_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]} L_2(i\mathbf{R}),$$

where $G = NM^{-1}$ is a coprime factorization over L_∞, and $\mathcal{M}_\Theta$ is the multiplication operator with symbol Θ. The adjoint of the multiplication operator by $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is of course the operator

$$\mathcal{M}_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]}^{*} = \mathcal{M}_{[\tilde{N} \quad \tilde{M}]},$$

so $\mathcal{L}\mathcal{B}(G)^\perp = \ker \mathcal{M}_{[\tilde{N} \quad \tilde{M}]}$, and the latter space is equal to $\mathcal{L}\mathcal{B}(-\tilde{G})$. □

On the halfline, the orthogonal complement of a graph (shift-invariant) is a behaviour (backwards invariant).

Proposition 2.3. $\mathcal{G}(G)^\perp = \mathcal{L}\mathcal{B}_+(-\tilde{G})$.

Proof. It is easy to verify that for V, W closed subspaces of a Hilbert space, one has in general $V \ominus \Pi_V W = V \cap W^\perp$. We can apply this to $V = H_2$, $W = \mathcal{L}\mathcal{B}(G)$, $W^\perp = \mathcal{L}\mathcal{B}(-\tilde{G})$, $V \cap W = \mathcal{G}(G)$, $\Pi_V(W^\perp) = \mathcal{L}\mathcal{B}_+(-\tilde{G})$. □

3. Complementarity and robust 'internal' stability

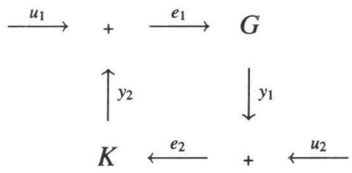
Assume we have input space U and output space Y. We consider a system G as a closed operator from $L_2(U)$ to $L_2(Y)$, and a controller C as a closed operator from $L_2(Y)$ to $L_2(U)$. Put $P = \mathcal{G}(G) = \ker [-G \quad I]$, $C = \mathcal{G}(K) = \ker [I \quad -K]$.

The feedback configuration that usually is studied in robustness analysis is obtained in the following way. One considers signals $(e_1, e_2, u_1, u_2, y_1, y_2) \in Z :=$

$L_2((0,\infty), E_1 \times E_2 \times U_1 \times U_2 \times Y_1 \times Y_2)$, and associates to the controller the subset $\mathcal{C} = \{z \mid y_2 = Ke_2\}$ of Z and to the system G the subset $\mathcal{P} = \{z \mid y_1 = Ge_1\}$. Then one considers the interconnection $\{e_1 = u_1 + y_2, e_2 = u_2 + y_1\}$ of the two systems. So let $\mathcal{I} := \{z \in Z \mid e_1 = u_1 + y_2, e_2 = u_2 + y_1\} \subset Z$ and let $\mathcal{H}(\mathcal{P}, \mathcal{C}) := \mathcal{I} \cap \mathcal{P} \cap \mathcal{C} \subset Z$. Now the *closed loop transfer function* $H(P, C)$ corresponds to the operator from $L_2(U_1 \times U_2)$ to $L_2(E_1 \times E_2)$ the graph of which is the projection of $\mathcal{H}(\mathcal{P}, \mathcal{C})$ on $L_2(E_1 \times E_2 \times U_1 \times U_2)$, or

$$H(P, C) = \begin{bmatrix} I & -G \\ -K & I \end{bmatrix}^{-1}.$$

The diagram corresponding to the configuration is of course well-known.



For the following theorem cf. [Foias et al. 1993].

Theorem 3.1. *The following are equivalent:*
(i) The configuration (G, K) is stable.
(ii) The subspaces P and C are complementary, so Π_P^C is a well defined bounded operator.
(iii) The closed loop transfer function $H(P, C)$ is a bounded operator.

3.1 Continuity of the closed loop behaviour

One of the main motivations for the use of the gap topology in the analysis of robustness is that it can be characterized as the weakest topology on the set of linear systems for which the closed loop transfer function is in continuous correspondence with the interconnected systems.

Definition 3.2. The topology $\mathcal{O}_{\text{robust}}$ on the set of closed operators from $L_2(U)$ to $L_2(Y)$ is defined by its subbasis elements

$$B_{P_0, C_0, \varepsilon} = \{P : \|H(P, C_0) - H(P_0, C_0)\| < \varepsilon\}$$

for (P_0, C_0) stable. Similarly $\mathcal{O}_\Pi$ is defined by the subbasis elements

$$B_{P_0, C_0, \varepsilon} = \{P : \|\Pi_P^{C_0} - \Pi_{P_0}^{C_0}\| < \varepsilon\}.$$

There is a simple relation between $H(P, C)$ and Π_P^C:

$$H(P, C) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \Pi_P^C \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

which is a consequence of the expression

$$\Pi_P^C = \begin{bmatrix} I \\ G \end{bmatrix} (I - KG)^{-1} \begin{bmatrix} I & -K \end{bmatrix},$$

as is readily verified. So it is clear that for issues of convergence we may look at Π_P^C instead of the closed loop transfer function $H(P, C)$. Consequently, we have:

Proposition 3.3. $\mathcal{O}_{\text{robust}} = \mathcal{O}_{\Pi}$.

We can apply this proposition and proposition 1.14 on the convergence of skew projections to obtain an easy proof of the fact that $\mathcal{O}_{\text{robust}}$ is just the gap topology.

Proposition 3.4. *The gap topology on the set of closed operators from $L_2(U)$ to $L_2(Y)$ is equivalent to the topology $\mathcal{O}_{\text{robust}}$.*

We can base a second easy proof of proposition 3.4 on the continuity of intersection for subspaces in general position. We also do this, because we believe this is a natural way to go. The gaps between the graphs P, P' of G and G', and corresponding spaces $\mathcal{P}, \mathcal{P}' \subset Z$ are of course equal. Similarly, the topology defined by the operator norm on stable transfer functions $H(P, C)$ coincides with the topology defined by the gaps between the spaces $\mathcal{H}(\mathcal{P}, \mathcal{C})$, since gap topology and norm topology are equivalent for bounded operators.

Second ***Proof*** of 3.4: We have $\mathcal{H}(\mathcal{P}, \mathcal{C}) = \mathcal{I} \cap \mathcal{P} \cap \mathcal{C} \subset Z$. We prove that the subspaces we intersect are in general position. The fact that $(\mathcal{P} \cap \mathcal{I}) + \mathcal{C} = Z$ is a consequence of the interpretation of stability as complementarity. Indeed every (e_2, y_2) can be written as $(e_{21}, Ke_{21}) + (Gy_{21}, y_{21})$, so $(e_1, e_2, u_1, u_2, y_1, y_2)$ is equal to $(e_1 - y_{22}, e_{21}, u_1, u_2, y_1 - Gy_{22}, Ke_{21}) + (y_{22}, Gy_{22}, 0, 0, Gy_{22}, y_{22}) \in \mathcal{C} + (\mathcal{P} \cap \mathcal{I})$. The same reasoning also gives $(\mathcal{C} \cap \mathcal{I}) + \mathcal{P} = Z$, so certainly $\mathcal{P} + \mathcal{I} = Z$. Thus we can conclude that the closed loop behaviour $\mathcal{H}(\mathcal{P}, \mathcal{C})$ is continuous in $\mathcal{P}$ and $\mathcal{C}$.

Put $V = \{z \mid u_1 = 0, u_2 = 0\}$. Then from $\mathcal{H}(\mathcal{P}, \mathcal{C})$ we can reconstruct $\mathcal{P} \cap V$ continuously as $\mathcal{P} \cap V = \mathcal{H}(\mathcal{P}, \mathcal{C}) \oplus A$, where $A = \{z \mid y_1 = 0, e_1 = 0, u_1 = 0, u_2 = 0\}$, since $A \cap \mathcal{H}(\mathcal{P}, \mathcal{C}) = \{0\}$. This implies that two systems P and P' are close to each other in the gap topology if the closed loop transfer functions $H(P, C)$ and $H(P', C)$ are close in operator norm. □

3.2 The stability margin

We now turn to the investigation of the robust stability margin.

Definition 3.5. Given a stable configuration (P, C) and a set V of subspaces of $U \times Y$, the *robust stability margin* $\rho_V(P, C)$ is given by

$$\rho_V(P, C) := \inf\{\delta(P, P') | P' \in V \text{ and } (P', C) \text{ is not stable}\}.$$

The robust stability margin turns out to be the sine of the minimal angle $\varphi(P, C)$ in Hilbert spaces. In disguise, this has been known since [Vidyasagar,Kimura] for finite-dimensional systems. The geometry behind it was uncovered only later. For the most general class of perturbations, the proof is easy.

Lemma 3.6. *Let a subspace V of a Hilbert space X be given. Assume L is a line not on V. Then a subspace V' of X exists such that $L \subset V'$, the codimension $[V + L : V'] = 1$ and $\delta(V', V) = \vec{\delta}(L, V)$.*

Proof. Let $N = \Pi_V L$, $V' = (V \ominus N) + L$. Now for any $x \in V'$ with $\|x\| = 1$ we have $x = x_1 + x_2$, $x_1 \in L$, $x_2 \in V \ominus N$, $x_1 \perp x_2$. So $d(x, V) = \|x_1\| \leq \vec{\delta}(L, V)$. Taking x on L leads to $\vec{\delta}(V', V) = \vec{\delta}(L, V)$. Because of the equality of the codimensions $[V + L : V']$ and $[V + L : V]$ we have by 1.2 also $\delta(V', V) = \vec{\delta}(L, V)$. □

Theorem 3.7. *With V the set of all subspaces of the space X, we have $\rho_V(P, C) = \sin \varphi(P, C)$.*

Proof. For arbitrary ε we can take a line L in C with $\vec{\delta}(L, P) < \sin \varphi(P, C) + \varepsilon$. Then let P' be the perturbation from the previous lemma. □

Various other versions of this result exist, where the perturbation P' is restricted to some smaller class. Important classes are the graphs of operators [Foias et al. 1993], the graphs of causal operators [Foias et al. 1991], graphs of shift-invariant operators [Vidyasagar,Kimura].

Chapter 2
Inner shift realizations

We devise a canonical realization for systems in continuous time that is intended to be a counterpart to the discrete time shift realization given in [Kuijper,Schumacher]. Let W be a finite-dimensional Hilbert space, let $\mathcal{D}$ be the differentiation operator on $L_2(W)$, and let $\mathcal{I}_{Y\to Z}$ be the canonical embedding for $Y \subset Z$. Let the space $\mathcal{H}_1$ be the domain of $\mathcal{D}$. The main objective of this chapter is to prove the following:

Theorem. *Let V be a closed shift-invariant subspace of $L_2(W)$ with $V_+^0 = V \cap L_2^+$ and $V_+ = \Pi_+ V$ such that the codimension $[V_+ : V_+^0]$ is finite. Then V is given as the the image of a system $\Sigma(V)$ whose trajectories are determined from the elements w of V by the following construction:*

$$\begin{aligned} X &= V_+ \ominus V_+^0, \\ x_t(w) &= \Pi_X \tau_t w, \\ u_t(w) &= w(t) - x_t(w)(0), \\ U \subset W &= \operatorname{span}\{u_t(w) \mid w \in V \cap \mathcal{H}_1\}. \end{aligned}$$

A driving variable representation is obtained by parametrizing

$$\begin{aligned} A : X \to X &:= \mathcal{D}|_X, \\ C : X \to W &:= x \mapsto x(0), \\ B : U \to X &:= u \mapsto -C^* u, \\ D : U \to W &:= \mathcal{I}_{U\to W}. \end{aligned}$$

Moreover, the system $\Sigma(V)$ is inner.

This theorem is a finite-dimensional time-domain version of the famous Beurling-Lax representation theorem for closed shift-invariant subspaces of H_2^+. In fact the easiest proof of something like this is obtained by realizing the Beurling symbol of V_0. However we think it may be not without interest to have an alternative proof of it that is based only on the properties of V viewed directly as a dynamical system. The idea of the proof is to use the fact that under certain conditions a set of functions that has the state property is a solution set to a set of first order differential equations. A lemma to this effect is given in section 2; the details of the realization are worked out in section 3. Section 1 reviews the discrete time realization and section 4 is devoted to an infinite-dimensional version of the main theorem in discrete time.

1. The normalized shift realization in discrete time

By way of comparison we also give the discrete time equivalent of our procedure. The proofs are indicated a bit more sparsely than for the continuous time case.

Theorem 1.1. *Let V be a closed shift-invariant subspace V of $l_2(W)$ with $V_+^0 = V \cap l_2^+(W)$ such that the codimension $[V_+ : V_+^0]$ is finite. Then V is given as the the image of a system $\Sigma(V)$ whose trajectories are determined from the elements w of V by the the following construction:*

$$\begin{aligned} X &= V_+ \ominus V_+^0, \\ U &= V_+^0 \ominus \sigma V_+^0, \\ x_t(w) &= \Pi_X \tau_t w, \\ u_t(w) &= \Pi_U \tau_t w. \end{aligned}$$

A driving variable representation is obtained by parametrizing

$$\begin{aligned} A &: X \to X := \tau\,|_X, \\ B &: U \to X := \tau\,|_U, \\ C &: X \to W := x \mapsto x(0). \\ D &: U \to W := u \mapsto u(0). \end{aligned}$$

Moreover, the system obtained in this way is inner.

We need to prove first that this description indeed fits the system. We first recall the definition of state with the help of the switching property ([Willems 1991], definition VII.1). We denote the concatenation of (or switching between) two trajectories as $a \wedge b$. Let W and X be finite-dimensional vector spaces.

Definition 1.2. Let V be some subset of a function space $(W \times X)^T$ over a time axis T. We say that X *is a state space for* V if for each $t \in T$ and all $(w(.), x(.)), (w_1(.), (x_1(.)) \in V$ we have

$$x(t) = x_1(t) \Rightarrow (w(.)|_{(-\infty,t)}, x(.)|_{(-\infty,t)}) \wedge (w_1(.)|_{[t,\infty]}, x_1(.)|_{[t,\infty]}) \in V.$$

The state property is also referred to as the *switching property*. The main ingredient in the proof of 1.1 is the following observation by Jan Willems:

Proposition 1.3. *Let $T = \mathbf{Z}$. Suppose V is a time-invariant subset of $(W \times X)^T$ that has the properties*

(i) X is a state space for V.
(ii) $V_W := \Pi_{W^T} V$ is closed in the topology of pointwise convergence.

Then there exists a subset $E \subset W \times X \times X$ and a first order system

$$V_E = \{(w, x) \in (W \times X)^T \mid \forall t\, (w(t), x(t+1), x(t)) \in E\}$$

such that $\Pi_{W^T} V = \Pi_{W^T} V_E$. Furthermore if V is such that the external part of the trajectories uniquely defines the internal part in a continuous way with respect to the topology of pointwise convergence, actually $V = V_E$.

Proof. Let $E = \{(w(0), x(0), x(1)) \mid (w, x) \in V\}$. We show that that $\Pi_{W^T} V = \Pi_{W^T} V_E$. So let $(w(.), f(.)) \in V_E$. Define (ω_n, φ_n) by the following inductive construction. Let $(\omega_0, \varphi_0) \in V$ be such that $(\omega(0), \varphi(0), \varphi(1)) = (w(0), f(0), f(1))$. Now assuming ω_n and φ_n defined, choose (α_n, χ_n) and $(\beta_n, \psi_n) \in V$ such that we have $(\alpha_n(0), \chi_n(0), \chi_n(1)) = (w(n+1), f(n+1), f(n+2))$ and $(\beta_n(0), \psi_n(0), \psi_n(1)) = (w(-n-1), f(-n-1), f(-n))$. Let

$$(\omega_{n+1}, \varphi_{n+1}) = \tau_{n+1}(\beta_n, \psi_n)|_{(-\infty,-n-1]} \wedge (\omega_n, \varphi_n)|_{[-n,-n]} \wedge \sigma_{n+1}(\alpha_n, \chi_n)|_{[n+1,\infty)}.$$

Then it is clear that the ω_n converge to $w(.)$ in the pointwise topology, so $w(.) \in V$. If $w(.)$ uniquely and continuously defines a state trajectory, the set V is closed is the topology of pointwise convergence, so $(w, f) \in V$.

The reverse inclusion is trivial. □

This lemma yields first order descriptions in *pencil form*. A driving variable representation of V_E can be obtained from such a description if $\Pi_X E = X$ (which is our case as the state space is constructed in such a way that this automatically holds) and we parametrize the freedom in the choice of $x(t+1)$ and $w(t)$ given $x(t)$ by a suitable input.

Lemma 1.4. *A closed shift-invariant subspace of l_2^+ that has finite state dimension is closed in the topology of pointwise convergence.*

Proof. Like the continuous-time equivalent 3.7. □

In fact this is if and only if: completeness (closure in the topology of pointwise convergence) also entails finite state dimension. So using this property of the set of trajectories really is a restriction to finite state dimension. We try to remedy this with a slightly different approach (using the properties of input trajectories instead of those of state trajectories) in section 4.

Proposition 1.5. *With the definitions of 1.1, X is indeed a state for W.*

Proof. Like the continuous time equivalent 3.4. □

Proposition 1.6. *The set V_e of combined state space and external trajectories arising from an l_2 external behaviour V according to the definitions of theorem 1.1 is a subset of l_2 closed in the topology of pointwise convergence.*

Proof. The fact that the state space trajectories are l_2 follows like in the continuous time case (lemma 3.9). So by 1.3 it is a solution set to a first order linear difference equation, hence closed in l_2. Pointwise closure is then again a consequence of the finite dimension. □

Proof of 1.1: It follows from the previous that X is a state, and that the (w, x) trajectories are indeed a first order system. It is easily checked that the suggested (A, B, C, D) indeed describe the (W, X, U) trajectories. So we check: $x(1) = \Pi_X\tau w = \Pi_X\tau(\Pi_X w + \Pi_U w + \Pi_{\sigma V_0} w) = \tau\Pi_X w + \Pi_X\tau\Pi_U w = Ax(0) + Bu(0)$ since $\tau\Pi_{\sigma V_0} w \in V_0 \perp X$ and $\tau\Pi_U w \in X$. Also $w(0) = (\Pi_X w + \Pi_U w + \Pi_{\sigma V_0} w)(0) = (\Pi_X w)(0)+(\Pi_U w)(0) = Cx+Du$ since $w(0) = 0$ for $w \in \sigma V_0$. Next we need to show that we get indeed all the l_2 solutions to $x(t+1) = Ax(t)+Bu(t)$, $w(t) = Cx+Du$. So let (x_0, x_1) be such that there exists u_0 such that $x_1 = \tau x_0+\Pi_X\tau u_0$. We need to show that (x_0, x_1) is an element of the of the set E from the proof of (1.2). This is easy: Let $w \in V = x_0 + u_0$. Then $x_0(w) = x_0$ and $x_1(w) = \Pi_X\tau(x_0 + u_0) = \tau x_0 + \Pi_X\tau u_0$. □

Lemma 1.7. *The discrete time shift realization is normalized.*

Proof. We first show that

$$\|x(t)\|^2 - \|x(t+1)\|^2 = \|w(t)\|^2 - \|u(t)\|^2.$$

At time $t = 0$ this equality can be written

$$\|\Pi_X w\|^2 - \|\Pi_X\tau w\|^2 = \|w\|^2 - \|\tau w\|^2 - \|\Pi_U w\|^2 \Leftrightarrow$$
$$0 = \|\Pi_{\sigma V_0} w\|^2 - \|\Pi_{V_0}\tau w\|^2$$

The equivalence follows by writing out $\|w\|^2 = \|\Pi_X w\|^2 + \|\Pi_U w\|^2 + \|\Pi_{\sigma V_0} w\|^2$ and similarly for $\|\tau w\|^2$, and then cancelling some terms. Finally if $w = x+u+w_0$,

$x \in X, u \in U, w_0 \in \sigma V_0$, then $\Pi_{\sigma V_0} w = w_0$, and we have $\tau w = \tau x + \tau u + \tau w_0$. Now $\tau u \perp V_0$ since $u \perp \sigma V_0$, so $\Pi_{V_0} \tau w = \tau w_0$. Since $w_0(0) = 0$, we have $\|w_0\| = \|\tau w_0\|$, so indeed

$$0 = \|\Pi_{\sigma V_0} w\|^2 - \|\Pi_{V_0} \tau w\|^2$$

Now the isometric nature of the input-output relation follows:

$$\|w\|_{l_2}^2 - \|u\|_{l_2}^2 = \Sigma_{i=-\infty}^{+\infty} \|x(t)\|^2 - \|x(t+1)\|^2 = 0.$$

The sum converges as $\Sigma_{i=-\infty}^{+\infty} \|x(t)\|^2 - \|x(t+1)\|^2 = \lim_{n\to\infty} \|x(-n)\|^2 - \|x(n)\|^2$, which tends to zero because $\forall w \in l_2^+ \lim_{n\to\infty} \|\tau_n w\| = 0$. □

This proof depends in no way on the finite dimension of the state space.

2. An attempt to characterize a class of first-order systems

First-order differential or difference equations have the 'state' or 'switching' property that the elements of their solution sets can be concatenated freely to yield another solution trajectory as long as the resulting function is continuous. As we have seen, it is almost trivial to show that a closed set of functions from the integers to some vector space that has the state property actually is a solution set to a suitably defined difference equation. In continuous time things are not so obvious. We are able to give only a very tentative characterization of a certain class of first-order (possibly nonlinear) systems on the time axis $T = \mathbf{R}^+$. The systems are of the following type: we study differential inclusions with output of the form

$$(w, \dot{x}) \in F(x).$$

The solutions to such an equation we consider here are those with continuous state trajectory. So we shall say a trajectory $(w(.), x(.))$ is a solution if $x(.)$ is continuous and there exists a function $z(.) \in L_1^{\text{loc}}(T, X)$ such that for all $t \in T$

$$x(t) = \int_0^t z(s)ds,$$
$$(w(t), z(t)) \in F(x(t)).$$

Our first limitation is that we consider only differential inclusions of the form $\dot{x} \in F(x)$ where the set-valued mapping F is continuous; we need a form of continuity for set-valued mappings that also applies to unbounded sets $F(x)$.

Definition 2.1. A set-valued mapping $x \mapsto F(x)$ is said to be *uniformly Lipschitz continuous* if

$$\exists A > 0 \, \forall x, x' \in X \, \forall y \in F(x) \, \exists y' \in F(x') \, \|y - y'\| \leq A \max(1, \|y\|) \, \|x - x'\|.$$

We call A the *Lipschitz constant.*

In order to obtain a characterization we have to assume the presence of a sufficiently large subset of the behaviour that behaves not too wildly with respect to linearization.

Definition 2.2. A family $\mathcal{F}$ of differentiable functions $(w(.), g(.)) \in (W \times X)^T$ is said to be *uniformly differentiable in X at* 0 if for all $c \geq 0$ a time interval $[0, T]$ exists such that for all $(w(.), g(.)) \in \mathcal{F}$ and $t \in [0, T]$ the following two requirements are satisfied:

$$(i) \quad \|g(t) - (g(0) + g'(0)t)\| < c(\|g(0)\| + \|g'(0)\| + \|w(0)\|)t, \text{ and}$$
$$(ii) \quad \|w(t) - (w(0) + w'(0)t)\| < c(\|g(0)\| + \|g'(0)\| + \|w(0)\|)t.$$

The asymmetry in the treatment of the X part and the W part of the trajectory is motivated by the desire not to refer to the derivatives w', on which we do not wish to impose a Lipschitz condition.

Definition 2.3. A set $\mathcal{F}$ of functions is said to be *generating* for a set-valued mapping $F : X \to 2^{W \times X}$ if for each tuple $(w, x, \dot{x}) \in W \times X \times X$ satisfying $(w, \dot{x}) \in F(x)$ there is a $(p(.), g(.)) \in \mathcal{F}$ such that $p(0) = w$, $g(0) = x$ and $g'(0) = \dot{x}$.

Now the next lemma gives some sufficient conditions for a subset $\mathcal{B} \subset (W \times X)^T$ to be the projection on W^T of the set of all solutions to $(w, \dot{x}) \in F(x)$ in the sense discussed above.

Lemma 2.4. *Let $\mathcal{B}$ be a subset of $L_2^{\text{loc}}(\mathbf{R}^+, W \times X)$ with the following properties:*

(i) $\mathcal{B}$ is invariant for the backward shifts τ_d.
(ii) $\mathcal{B}_W := \Pi_W \mathcal{B}$ is closed in L_2^{loc}.
(iii) X is a state for $\mathcal{B}$ and the elements of $\Pi_X \mathcal{B}$ are continuous.
(iv) The differentiable functions in $\mathcal{B}$ are dense in $\mathcal{B}$ in the L_2^{loc} sense.
Let

$$F(x) = \{(w, \dot{x}) \in W \times X \mid \exists f \in \mathcal{B} \cap C^1(\mathbf{R}^+, W \times X) \ : f(0) = (w, x) \wedge \Pi_X f'(0) = \dot{x}\}.$$

(v) F is uniformly Lipschitz continuous.
(vi) A uniformly differentiable family $\mathcal{F} \subset \mathcal{B}$ exists that is generating for F.

(vii) The C^1 functions are dense in the set of solutions to the differential inclusion $(w, \dot{x}) \in F(x)$.

Then $\mathcal{B}_W$ is equal to the set of external trajectories corresponding to solutions with continuous state trajectory to the differential inclusion $(w, \dot{x}) \in F(x)$. If $\mathcal{B}$ is such that the W trajectories determine the X trajectories in a unique way continuous with respect to the L_2^{loc} topology, then B actually is the solution set of $(w, \dot{x}) \in F(x)$.

Proof. Let an arbitrary C^1 differentiable solution $(w(.), f(.))$ to $(w, \dot{x}) \in F(x)$ be given. We must prove that $f \in \mathcal{B}$ by approximating the external trajectory w in the L_2^{loc} sense. It is sufficient for this that we can do the approximation on any fixed interval I. On the interval I the function f satisfies

$$(*)\ \forall c > 0 \exists T > 0 \forall t_0 \in I \forall t \in [t_0, t_0 + T]\ \|f(t) - (f(t_0) + (t - t_0)f'(t_0))\| < c(t - t_0).$$

This follows from the uniform continuity of f' on an interval. Define the approximations $\{\psi_n = (\chi_n(t), \varphi_n(t))\}$ by the following construction. For each tuple $(w, x, \dot{x})$ we choose a function $(w(.), f(.))$ from the family $\mathcal{F}$ having $w(0) = w, f(0) = x$ and $f'(0) = \dot{x}$. Let a be a Lipschitz constant for F. Divide an interval $[0, C]$ into n equal parts by by $t_k = kC/n$, $0 \leq k \leq n$, and define ψ_n by an inductive construction as follows

$$\varphi_n(0) = f(0),$$
$$\chi_n(0) = w(0).$$

Now suppose ψ_n has been defined for $t \leq t_k$. Now if $k = n$, continue ψ_n after t_k in any arbitrary way such that $\psi \in \mathcal{B}$. Otherwise choose $\gamma = (\alpha, \beta) \in F(\chi_n(t_k), \varphi_n(t_k))$ to be such that $\|\gamma - (w(t_k), f'(t_k))\| \leq a \max(1, \|(w(t_k, f'(t_k))\|)$ $\|\psi_n(t_k) - (w(t_k), f(t_k))\|$. Now let

$$\varphi_n(t) = g_{\alpha,\beta}(t - t_k),\ t_k < t \leq t_{k+1},$$
$$\chi_n(t) = p_{\alpha,\beta}(t - t_k),\ t_k < t \leq t_{k+1}.$$

It is clear by the switching property that the functions ψ_n are all in $\mathcal{B}$. We show that $\varphi_n \rightarrow f$ in the topology of uniform convergence on I. To this end, define:

$$d_{k,n} = \|\varphi_n(t_k) - f(t_k)\|$$
$$e_{k,n} = \|\varphi_n'(t_k) - f'(t_k)\|$$
$$h_{k_n} = \|\varphi_n(t_k) + \frac{1}{n}\varphi_n'(t_k) - (f(t_k) + \frac{1}{n}f'(t_k))\|$$
$$j_{k,n} = \|\chi_n(t_k) - w(t_k)\|.$$

Abbreviate $w_{k,n} = \chi_n(t_k)$, $x_{k,n} = \varphi_n(t_k)$, $x'_{k,n} = \varphi_n'(t_k)$. Let

$$b := \max(1, \sup_{t \in [0,C]} \|f'(t)\| + \|f(t)\| + \|w(t)\|)$$

Now it is possible to see that for any $c > 0$ the following inequalities are satisfied for large enough n:

$$h_{k,n} \le d_{k,n} + \frac{1}{n} e_{k,n}$$
$$\|x_{k,n}\| + \|x'_{k,n}\| + \|w_{k,n}\| \le d_{k,n} + e_{k,n} + j_{k,n} + b$$
$$e_{k,n} \le abd_{k,n}$$
$$j_{k,n} \le abd_{k,n}$$
$$d_{k+1,n} \le h_{k,n} + \frac{2bc}{n} + \frac{c}{n}(d_{k,n} + e_{k,n} + j_{k,n}).$$

The first and second inequalities are obvious. The third and fourth inequalities reflect the continuity of $F(x)$. The last uses the condition (vi) and the differentiability of f. It is derived as follows: $\|\varphi_n(t_{k+1}) - f(t_{k+1})\| \le \|f(t_k + 1) - (f(t_k) + \frac{1}{n} f'(t_k))\| + \|\varphi_n(t_{k+1}) - (\varphi_n(t_k) + \frac{1}{n}\varphi'_n(t_k))\| + \|\varphi_n(t_k) + \frac{1}{n}\varphi'_n(t_k) - (f(t_k) + \frac{1}{n} f'(t_k))\|$. Now $(*)$ and the ugly condition (vi) imply that for sufficiently large n the first term on the right hand side is bounded by $\frac{cb}{n}$, and the second is bounded by $\frac{c}{n}(d_{k,n} + e_{k,n} + j_{k,n} + b)$. The third term is just $h_{k,n}$. Some substitutions lead to

$$d_0 = 0$$
$$d_1 \le \frac{2cb}{n}$$
$$d_{k+1} \le (1 + \frac{1}{n}(ab + 2abc + c))d_k + \frac{2cb}{n}.$$

Let

$$z_1 = 2cb,$$
$$z_{k+1} = (1 + \frac{1}{n}(1 + ab + 2abc + c))z_k.$$

Then $\frac{z_k}{n} \ge \frac{2cb}{n}$ for all $k > 1$, as the z_k are an increasing sequence. So, for $k > 1$ we have $d_{k,n} \le z_{k,n}$. Since it is well-known that $\lim_{n\to\infty}(1 + x/n)^n = e^x$, it follows that $\limsup_{n\to\infty} z_{k,n} = e^{(1+ab+2abc+c)} z_1$. As the constant c can be made arbitrarily small, it follows that the $z_{k,n}$, and also $d_{k,n}$, tend uniformly to zero. Hence the state space parts of the trajectories converge. The w-trajectories must then also converge in the sense of L_2^{loc} by virtue of the fourth inequality. If the state trajectory is determined continuously by the external trajectory, it also follows that $(w, f) \in \mathcal{B}$. □

It would be interesting to try to arrive at nicer conditions than the ones given here. Certainly some kind of continuity of $F(x)$ seems to be needed, as can be seen from the example of the set $\mathcal{B} = \{f \mid f(t) = (t + C)^3\}$ for $C \in \mathbf{R}$, which is closed and shift-invariant and also trivially has the switching property since no two different trajectories intersect, but is *not* the solution set to any first-order differential inclusion. The kind of continuity that we require here is motivated by the fact that we have to deal with unbounded sets $F(x)$ in order to be able to apply the lemma to linear systems. The simple precise form of the continuity condition is also motivated by the application to the linear situation. The condition that really spoils the lemma is, of course, (vi), which we imposed just to make the proof we had in mind work out. It could also have been replaced by something of a more local nature.

Another weak point of the lemma is that it does not do much in infinite-dimensional situations, for which the presence of a generating uniformly differentiable set is more problematic.

Lemma 2.5. *Let $\mathcal{B}$ be a subset of $L_2^{\text{loc}}(\mathbf{R}, W \times X)$ that satisfies (i)-(vii) of lemma 2.4 and in addition:*

(i)′ $\mathcal{B}$ is invariant for the forward shifts σ_d.
(ii)′ The uniformly differentiable family $\mathcal{F} \subset \mathcal{B}$ is such that also the reversed family $\{f(-t) \mid f \in \mathcal{F}\}$ is uniformly differentiable.

Then $\mathcal{B}_W$ is equal to the set of external trajectories corresponding to solutions with continuous state trajectory to the differential inclusion $(w, \dot{x}) \in F(x)$.

Proof. Apply the construction in the proof of 2.4 both in the forward and in the backward direction. □

3. Continuous-time finite-dimensional systems

This section is devoted to a realization procedure for finite-dimensional systems in continuous time. The idea is to obtain a state-space representation of a shift-invariant subspace of L_2 by viewing it directly as a dynamical system. We introduce a canonical state space and obtain a description of the combined set of external and state-space trajectories.

As a starting point we choose the set $\mathcal{S}(W)$ of closed shift-invariant subspaces of $L_2(\mathbf{R}, W)$. Let a closed shift-invariant subspace V of $L_2(-\infty, \infty)$ be given. Recall the definitions

$$V_+ := \Pi_+(V) := \Pi_{L_2^+}(V)$$

and

$$V_+^0 := V \cap L_2^+.$$

We now state the fact we want to prove.

Proposition 3.1. *Suppose* $V \in \mathcal{S}(W)$ *is such that the quotient space* V_+ / V_+^0 *has dimension* $n < \infty$.
Then:

(a) An inner finite-dimensional input-state-output system $\Sigma = \Sigma(A, B, C, D)$ *exists such that* $V = \operatorname{im} \Sigma$ *and* $\deg \Sigma = n$.

(b) if $V = \operatorname{im} \Sigma$, *then the dimension of the state space of* Σ *is at least equal to the codimension* $[V_+ : V_+^0] = \dim V_+ / V_+^0$.

We refer to the set of closed shift-invariant spaces with finite state dimension as $\mathcal{S}_{\mathrm{fd}}$. Working on the whole time axis, the autonomous stable finite-dimensional systems are excluded. For completeness' sake we also state a version of the previous on the right halfline.

Proposition 3.2. *Suppose* $V \subset L_2^+$ *is closed and invariant for the backward shifts* τ_d. *Let* V^0 *be the largest closed subspace invariant for the forward shifts* σ_d. *Suppose the dimension of* V / V^0 *is finite. Then a finite-dimensional input-state-output system* $\Sigma = \Sigma(A, B, C, D)$ *exists such that* $V = \operatorname{im} \Sigma$ *and* $\deg \Sigma = \dim V / V^0$.

One possible easy proof of these propositions proceeds by means of (i) the Beurling-Lax theorem to obtain a representation in the first place and (ii) some version of Kronecker's theorem to characterize the degree of the Beurling symbol. We shall however give a different proof by constructing a realization "from the L_2 behaviour" in a way analogous to the discrete time realization procedure discussed in [Kuijper,Schumacher] and [Fuhrmann]'s shift realization. The idea is to obtain a realization in which the A-operator is just differentiation on the canonical state space. For our purposes this has the advantage that it allows us to look at the continuity of the realization parameters in the original data, and this helps us to compare the gap topology and topology of state space parameters. The whole procedure is a little less tidy than the discrete one, which can be done in a completely algebraic way. The difference with the usual Hilbert space theory [Fuhrmann] here is that we start from shift-invariant subspaces and not from input-output operators.

Now let some shift-invariant space V be given, and let $V_+^0 = V \cap L_2^+$ be the intersection with the embedded L_2^+; assume that we have a finite-dimensional candidate state space

$$X := V_+ \ominus V_+^0.$$

With $w \in V_+$ one defines the associated *state trajectory* $x_t(w)$ and the *state derivative trajectory* $\dot{x}_t(w)$ by

$$x_t(w) = \Pi_X \tau_t(w)$$
$$\dot{x}_t(w) = \lim_{h \downarrow 0} \frac{x_{t+h}(w) - x_t(w)}{h}$$

At this point we need not elaborate on the exact conditions for the correctness of the second definition. We return to it below. However for the time being note that it is correct at least on the space $\mathcal{H}_1$ of L_2 functions with derivative in L_2, which space is dense in V^+.

Lemma 3.3. *For any n, and any closed backwards invariant subspace V of L_2^+, the subspace of n times differentiable elements of V is dense in V.*

Proof. The derivation operator generates a C_0 semigroup of backwards shifts. So we can use the fact that the domain of any power of the infinitesimal generator of such a semigroup is always dense. □

First one verifies that X is a state space in the sense that the state has the (deterministic) Markovian property that it contains all the information about the past of a trajectory that determines its possible futures.

Lemma 3.4. *X is a (minimal) state space for V and the state space trajectories are continuous.*

Proof. Suppose two trajectories on the whole time axis $w_1, w_2 \in V$ have the same state $x = \Pi_X w$ at time 0. Then $w_2^+ - w_1^+$ is in V_+^0, and this means that the concatenation $w_{12} := w_1^- \wedge w_2^+ := t \longrightarrow w_1(t)$ for $t \leq 0$ and $t \longrightarrow w_2(t)$ for $t > 0$ is equal to $w_1 + (0^- \wedge (w_2^+ - w_1^+))$. The continuity of the state space trajectories follows from the strong continuity of the backward shift semigroup and the continuity of the projection on X. Note that it is immediate that the state defined in this way is minimal in the following sense: one can switch to the zero trajectory if and only if the state at time 0 is zero. From this, minimality in terms of the dimension of the state space readily follows. □

One can do the same thing on the left halfline, accordingly defining the backward state $x_0^-(w)$ at time 0. Using the minimality of the state space we can deduce easily that the forward and backward state contain the same information about a trajectory.

Lemma 3.5. *For linear systems on the whole time axis, the forward and the backward state mappings are equivalent in the sense that*

$$x_0^+(w) = 0 \Leftrightarrow x_0^-(w) = 0.$$

If the state dimension is finite, one in fact has

$$\exists\, C_1, C_2\, \forall w \in V\;\; C_1\|x_0^+(w)\| \leq \|x_0^-(w)\| \leq C_2\|x_0^+(w)\|.$$

Proof. Suppose $w \in V$ is such that $x_0^+(w) = 0$. Then $w^- \wedge 0 \in V$, so also $0 \wedge w^+ = (w^- \wedge w^+) - (w^- \wedge 0)$ is an element of V, so $x_0^-(w) = 0$. The uniformity of the equivalence for finite-dimensional systems follows from the finite dimension of the state space and the continuity of the projections. □

It should be noted that among all trajectories with the same initial condition x_0, the trajectory x_0 itself as an element of V_+ is the one of minimal L_2 norm. This is easily seen to be a consequence of the fact that x_0 is orthogonal to all trajectories with initial state zero, so x_0 is the projection of the origin on the set of trajectories with the same initial state. It follows that the state space norm we obtain is just the *future Gramian* of [Weiland].

The idea of the proof of (1.1) is to apply lemma (1.5) to the combined set of external and state space trajectories of V. We first prove some more properties of the sets of state space trajectories and state derivative trajectories that we shall need later on. In the proof of the discrete-time version of the main theorem we gave, closure in the topology of pointwise convergence played an important role. In continuous time, we need L_2^{loc} closure of the behaviours with finite-dimensional state. In order to prove this, we first obtain a lemma about the memory of finite-dimensional systems. This lemma is rather weak, as we know that one actually has the property of *finite memory*: the state is determined from an arbitrarily short part of the past of trajectory. At this point, when we do not have the realization yet, it is however easier to prove the following weaker property that will also do.

Lemma 3.6. *A system with finite-dimensional state has fading memory in the sense that*

$$\exists T\, \exists C\, \forall w \in V\ \ \|x_0(w)\| \leq C\|w|_{[0,T]}\|.$$

Proof. Since the norm of the initial state is dominated by a finite sum of inner products with L_2 functions (cf. 3.9) , it follows that for any λ_2 we have for large enough T that $\|x_0(w)\| \leq \lambda_1\|w|_{[0,T]}\| + \lambda_2\|w|_{[T,\infty]}\|$. By the equivalence of the forward and the backward state it follows that $\exists w^- : w^- \wedge w^+ \in V$ and $\|w^-\| \leq C_2\|x_0(w)\|$. Hence also $\|x_T(w)\| \leq C_1(C_2\|x_0(w)\| + \|w|_{[0,T]}\|)$. But then we could have chosen the continuation of w after T to be such that $\|w|_{[T,\infty]}\| \leq C_1(C_2\|x_0(w)\| + \|w|_{[0,T]}\|)$. Substituting, we get

$$\|x_0(w)\| \leq (\lambda_1 + C_1C_2\lambda_2)\|w|_{[0,T]}\| + \lambda_2C_1C_2\|x_0(w)\|.$$

So the constant we are looking for is

$$C = \frac{\lambda_1 + C_1C_2\lambda_2}{1 - \lambda_2C_1C_2}.$$

□

Proposition 3.7. *The elements of $\mathcal{S}_{\mathrm{fd}}$ are closed in the stronger sense of the L_2^{loc}-topology.*

Proof. Assume $w_n \longrightarrow w$ in L_2^{loc}, $w_n \in V$. Then we may assume that $\|(w_n - w)|_{[-2n,2n]}\| \longrightarrow 0$. So let $w_n' = x_{-2n}^-(w_n) \wedge w_n|_{[-2n,2n]} \wedge x_{2n}^+(w_n)$. Then we have $w_n' \longrightarrow w$ in the strong (L_2-) sense. This can be seen as follows. We have $\|w|_{[n,2n]}\| \longrightarrow 0$ so also $\|w_n|_{[n,2n]}\| \longrightarrow 0$, so applying the previous lemma one has $\|x_{2n}^+(w_n)\| \longrightarrow 0$. Similarly $\|x_{-2n}^-(w_n)\| \longrightarrow 0$. This implies the convergence of $w_n' \longrightarrow w$ in L_2. □

The same reasoning applies as well on the halfline, so also also the spaces V_+ are closed in the weaker topology.

Let $\{x_i\}$ be an orthonormal basis of the state space. The x_i are analytic functions, in fact exponential functions as follows from the fact that a finite-dimensional shift invariant space is always the solution set to an autonomous system of equations (we shall prove later on that it is a solution set; because of the finite dimension, the equations must be autonomous).

Lemma 3.8. *For any $w \in L_2^+$ and for any stable exponential function x, the L_2 norm of the convolution product*

$$t \longmapsto \langle \tau_t w, x \rangle$$

is finite and not larger than $\|\mathcal{L}x\|_\infty \|w\|$.

Proof. As is well known we have $\mathcal{L}(w * x) = \mathcal{L}w\mathcal{L}x$, from which the inequality obviously follows. □

This implies that the state trajectory of a finite dimensional system is L_2, since it is dominated by a finite sum of such convolution products. It follows that we can conveniently define, for any $w \in V$, the state derivative trajectory up to L_2 equivalence by means of integration by parts:

$$\dot{x}_t(w) = \Sigma_{i=1}^n - \langle w(t), x_i(0) \rangle \, x_i - \langle \tau_t w, x_i' \rangle \, x_i.$$

Hence also the state derivative trajectories are elements of L_2. We summarize a few observations about the relation between state and external trajectories in the next lemma.

Lemma 3.9. *For all $w \in V$ the state space trajectories are in L_2. The state part of a trajectory is determined continuously from the external part in the sense of L_2^{loc}. So the combined set V_e of state and external trajectories is closed in the L_2^{loc}*

sense. Furthermore if $\{x_i\}$ is an orthonormal basis of the canonical state space X we have:

$$x_t(w) = \Sigma_{i=1}^{n} \langle x_i, \tau_t w\rangle x_i$$

$$\dot{x}_t(w) = \Sigma_{i=1}^{n} - \langle w(t), x_i(0)\rangle x_i - \langle \tau_t w, x_i'\rangle x_i.$$

Proof. The continuity of the state trajectory as a function of the external one in L_2^{loc} sense follows from the fading memory property 3.6. The other observations are even more elementary. □

One more lemma is needed before we can move on to the proof of the main theorem.

Lemma 3.10. *The combined set of external and state trajectories of a finite-dimensional linear system satisfies condition (vi) of lemma 2.5, i.e. a generating uniformly differentiable subset exists.*

Proof. The fact that the ugly condition is satisfied is a consequence of linearity. Choose for the family $\mathcal{F}$ for instance the span of a finite number of linearly independent functions $(w_i(.), g_i(.))$ such that span$\{(w_i(0), g_i(0), g_i'(0)\} = V$ and the vectors $(w_i(0), g_i(0), g_i'(0))$ are orthonormal. Then an interval $[0, T]$ that works for all of the g_i works for the whole family. □

Before we move on to the proof of 3.1 and 3.2 we say a few words about the precise form of the representation we want to obtain of our systems. Application of lemma 1.4 yields a description of the form $(w, \dot{x}) \in F(x)$, which is not the driving-variable state space form we promise in the statement of 3.1. Of course in general an (A, B, C, D) representation is easily obtained from any first order description. In this case, however, a very simple explicit description of the quadruple of linear operators is available. First we define appropriate inputs.

Definition 3.11. The *shift realization* of an element V of $\mathcal{S}_{\text{fd}}$ is the system $\Sigma \subset L_2(W \times X \times U)$ whose trajectories are defined by

$$\begin{aligned} X &= V_+ \ominus V_+^0, \\ x_t(w) &= \Pi_X \tau_t(w), \\ u_t(w) &= w(t) - x_t(w)(0), \\ U \subset W &= \text{span}\{u_t(w) \mid w \in V \cap \mathcal{H}_1\}. \end{aligned}$$

We want describe the above system as the set of L_2 solutions to

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ w &= Cx + Du. \end{aligned}$$

The main difference with the discrete time procedure is that we do not have the convenient canonical input space $V_+^0 \ominus \sigma V_+^0$ here. Let $\mathcal{D}$ be the differentiation operator on $L_2(W)$ restricted to X, let $\mathcal{I}$ be the embedding of U in W. Now we define

$$\begin{aligned} A : X \to X &:= \mathcal{D}\,|_X, \\ C : X \to W &:= x \mapsto x(0), \\ B : U \to X &:= u \mapsto -C^* u, \\ D : U \to W &:= \mathcal{I} u \end{aligned}$$

.

Note that the operators A and C are well defined and bounded for systems with finite-dimensional state, as a consequence of the analyticity of the elements of the state space.

Proof of 3.1 and 3.2: Define

$$V_e := \{(w, x) \in L_2(\mathbf{R}, W \times X) \,|\, w \in V \wedge \forall t \geq 0\, x(t) = x_t(w)\}$$

We first need to verify that the set V_e satisfies the assumptions for the application of lemma (2.5) (respectively 2.4 on the halfline), so we may conclude it is indeed a first order system.

So: (*i, ii*) Hold by assumption. The switching property for state space trajectories (lemma 3.4) is (*iii*). Lemma 3.3 is (*iv*). The Lipschitz continuity condition (*v*) is satisfied by any linear differential inclusion given in the form $G(w, x, \dot{x}) = 0$ for some constant matrix G. Lemma 3.10 gives the presence of a generating uniformly differentiable family (*iv*). Finally (*vii*) is well-known in the linear case.

This finishes the proof inasfar as we are interested only in proving that a first-order description of V_e exists at all. It remains to be shown that the suggested description fits. By time-invariance, we need only check at $t = 0$. So first we must see that indeed $Bu_0(w) = \dot{x} - Ax = \Pi_X \mathcal{D} w - \mathcal{D} \Pi_X w$. Let z be any element of X and let $w \in V$ be in $\mathcal{H}_1$, let $x = \Pi_X w$ and $u_0 = w(0) - x(0)$. Then $\langle z, \Pi_X \mathcal{D} w - \mathcal{D} \Pi_X w \rangle = \langle z, \mathcal{D} w \rangle - \langle z, \mathcal{D} \Pi_X w \rangle$. Integrating by parts, this becomes $-\langle z(0), w(0) \rangle - \langle \mathcal{D} z, w \rangle - (-\langle z(0), x(0) \rangle - \langle \mathcal{D} z, w \rangle) = \langle z(0), x(0) - w(0) \rangle = \langle z, -C^* u_0 \rangle$. Of course we also have $w(0) = u_0 + x(0) = Cx + Du_0$. To see that we obtain all solutions to the suggested system one has to show that all pairs (x, u), $x \in X, u \in U$ indeed occur as state and input at time zero for some $w \in V$. So, if $u = w_1(0) - x_0(w_1)(0)$ let w_2 be such that $w_2^+ = x - x_0(w_1)$. Then $w = w_1 + w_2$ has state x and input u at time $t = 0$. □

It is interesting to note that our realization scheme leads to a normalized representation.

Lemma 3.12. *The shift realization is normalized.*

Proof. We first show that along trajectories of the system we have

$$-\frac{d}{dt}\|x(t)\|^2 = \|w(t)\|^2 - \|u(t)\|^2.$$

So (take $t = 0$),

$$\frac{d}{dt}\|x(t)\|^2|_{t=0} = \langle x, \dot{x}\rangle + \langle \dot{x}, x\rangle .$$

Now

$$\langle x, \dot{x}\rangle = \langle x, \Pi_X \mathcal{D}w\rangle = \langle x, \mathcal{D}w\rangle$$

Integrating by parts,

$$\langle x, \mathcal{D}w\rangle = -\langle x(0), w(0)\rangle - \langle \mathcal{D}x, w\rangle .$$

So (using $\langle x, \mathcal{D}x\rangle + \langle \mathcal{D}x, x\rangle = -\|x(0)\|^2$ and $\langle \mathcal{D}x, w\rangle = \langle \mathcal{D}x, \Pi_X w\rangle = \langle \mathcal{D}x, x\rangle$) we get

$$\frac{d}{dt}\|x(t)\|^2|_{t=0} = -\langle x(0), w(0)\rangle - \langle w(0, x(0))\rangle + \|x(0)\|^2.$$

Also

$$\|w(0)\|^2 - \|u(0)\|^2 = \langle w(0), x(0)\rangle + \langle x(0), w(0)\rangle - \|x(0)\|^2.$$

By time-invariance the equality follows for general t.

Now the isometric nature of the input-output realization follows:

$$\|w\|_{L_2}^2 - \|u\|_{L_2}^2 = \int_{t=-\infty}^{\infty} -\frac{d}{dt}\|x(t)\|^2 = 0.$$

□

We make a final comment on the relation between 3.1 and 3.2: the suggested state spaces are in fact the same if the system on the halfline is the projection of a system on the whole line.

Lemma 3.13. *For finite-dimensional systems V on the whole time axis, we have that V_+^0 is the largest forwards invariant subspace of V_+.*

Proof. This follows from the L_2^{loc} closure of V. If w^+ is in the largest forwards invariant subspace of V^+, it follows that for any n, there exists a $w_n \in V$ such that $w_n^+ = w^+$ and $w_n|_{[-n,0]} \equiv 0$. Then obviously $w_n \to 0 \wedge w^+$ in L_2^{loc}-sense. So $0 \wedge w^+ \in V$. □

4. Infinite-dimensional discrete-time systems

We only deal with the realization of a closed shift-invariant subspace V of l_2 on the whole line.

Whereas state trajectories are characterized by the property of free continuous concatenation, the set of input trajectories of an input-state-output system has the property of arbitrary switching.

Lemma 4.1. *The set of driving input functions arising from definitions of section 1 has the property of free concatenation.*

Proof. Let u, u_1 be the driving inputs for external trajectories w, w_1. Let $x = x_0(w)$, $x_1 = x_0(w_1)$. Then let $w' \in L_2^+ = w^+ + x_1 - x$. Since $x_0(w') = x_1 = x_0(w_1)$ it follows that $w_2 = w_1^- \wedge w'^+ \in V$. Obviously for $t > 0$ we have $u_t(w_2) = u(t)$. We show that for $t < 0$ one has $u_t(w_2) = u_1(t)$. First, it is clear by the definition of the backward state that for $t < 0$ we have $x_t^-(w_2) = x_t^-(w_1)$. So, as a consequence of the equivalence of forward and backward state it follows that also $x_t^+(w_2) = x_t^+(w_1)$. It is a consequence of the definitions made in the statement of theorem 1.1 that $x_t(w) = 0 \wedge w(t) = 0 \Rightarrow u_t(w) = 0$. Hence, since state and external value of w_1 and w_2 coincide for $t < 0$, we get $u_t(w_1) = u_t(w_1) = u_1(t)$ for $t < 0$. □

Lemma 4.2. *The set of occurring input functions is closed in l_2.*

Proof. As shown in section 1, the relation from external trajectory to driving input is isometric. Hence it has closed range. □

Lemma 4.3. *A time-invariant closed subspace $\mathcal{U}$ of $l_2^+(U)$ that has the property of arbitrary switching and additionally satisfies $\forall u \in U\, \exists f \in \mathcal{U} : f(0) = u$ is in fact equal to $l_2^+(U)$.*

Proof. One can proceed like in the proof of proposition 1.3, or just notice that this lemma is the case with state dimension zero of theorem 1.1. □

Theorem 4.4. *For systems on the whole time axis, the discrete time shift realization is also valid if the state dimension is not finite.*

Proof. In section 1, The finite state dimension was only used to establish that indeed the set of system trajectories contains the image of the system (A, B, C, D). As is easily seen, it is sufficient for this to see that

$$\mathcal{U} := \{u(.) \in l_2(U) \mid \exists w(.) \in V : \forall t \in T : u(t) = u_t(w(t))\} = l_2(U).$$

But this follows from the previous lemmata. □

Chapter 3

The gap topology on linear time-invariant systems

We survey several possible descriptions of the gap topology on the set of linear time-invariant systems, and we compare it to other topologies that have been discussed in the literature. Simple proofs are given of the equivalence of three different ways of introducing the gap topology, — the gap topology as defined by the gaps between the graphs of the transfer functions, the topology of uniform convergence of the associated Hermann-Martin mappings from $\mathbf{C}^+$ to a Grassmannian manifold $Grass(m, m+p)$, where m is the number of inputs to the system, p the number of outputs ("point-wise gap"), and lastly the gap topology $\mathcal{O}_{L_2^-}$ defined by the gaps between the $L_2(-\infty, 0)$-behaviours of the systems involved. In the last section, we compare some gap topologies that are really different. We use the shift realization from the previous chapter to draw conclusions about the relations between the gap topologies and the state space parameter topology. This chapter is based on the earlier publications [de Does et al.] and [de Does, Schumacher 1994b].

1. Some alternative definitions of the gap topology

The *gap topology* on linear systems is usually defined as the topology induced by the gaps between the graphs of the transfer functions. This means that we obtain the standard gap topology considered in robustness analysis as the topology induced by the gap

$$\delta_{H_2}(U, V) = \delta(\mathcal{L}U_+^0, \mathcal{L}V_+^0).$$

It is natural also to consider the gaps between solution sets. Here one must be careful: the "right" topology for robust stabilization is induced by the gap between the behaviours on the *left* half-line. So we introduce topology $\mathcal{O}_{L_2^-}$ as the topology induced by the gap

$$\delta_-(U, V) = \delta(U_-, V_-).$$

The topology of behaviours on the whole time axis is strictly weaker, and the behaviour topology on the right halfline is not comparable to the gap topology.

Instead of describing a system in the frequency domain by its transfer function, one can also identify it with the associated *Hermann-Martin mapping* to a Grassmannian manifold. Let us briefly explain this. Let Ω be a subset of the extended complex plane without isolated points. For $F(s)$ any $m \times q$ rational matrix, we can extend the associated mapping

$$\begin{aligned} f : \Omega &\mapsto Grass(m, q) \\ s &\mapsto \operatorname{im} F(s) \end{aligned}$$

across the poles and zeros of $F(s)$ in Ω in a unique way to a mapping that is continuous with respect to the spherical metric of $\mathbf{C}_\infty$, and the gap metric on $Grass(m, n)$. Any f arising in this way is a *rational mapping* from Ω to $Grass(m, q)$. Denote the set of all such f by $\mathcal{R}(\Omega, m, q)$. Now the *Hermann-Martin mapping* of the system V is the mapping f associated in this way to the Beurling symbol Θ of V_+^0. This particular choice of representation is rather arbitrary of course; it is more elegant to construct from V, for instance, the curve f defined on $\mathbf{C}^+$ by

$$f(s) = \{g(s) \mid g \in \mathcal{L}V_+^0\},$$

which is the way it is done in [Qiu, Davison 1992a]. Put

$$\delta_{\sup}(f, g) := \sup\{\delta(f(s), g(s)) : s \in \mathbf{C}^+\}.$$

The *pointwise gap topology* is the topology on the space of linear systems induced by $\delta_{\sup}$ on $\mathcal{R}(\mathbf{C}^+, m, q)$. We shall see that the pointwise definition makes it easier to obtain conclusions about the continuity of parameter variations with respect to the gap topology. Historically the first definition of the gap topology (originally called *graph topology*) was by closeness of stable coprime factor representations. We rephrase [Vidyasagar]'s original definition of the graph topology here in the setting of Hermann-Martin maps:

Definition 1.1. The *graph topology* on $\mathcal{R}(\mathbf{C}^+, m, q)$ is generated by the neighbourhood basis consisting of the sets $U_{\varepsilon,F}(f)$ for f in $\mathcal{R}(\mathbf{C}^+, m, q)$, $\varepsilon > 0$, and F a rational matrix in $H_\infty^{n \times q}(\mathbf{C}^+)$ that $f(s) = \operatorname{im} F(s)$, defined by

$$U_{\varepsilon,F}(f) = \{g \in \mathcal{R}(\mathbf{C}^+, m, q) \mid \exists G \in H_\infty^{m \times q}(\mathbf{C}^+) : g(s) = \operatorname{im} G(s) \wedge \|F - G\|_\infty < \varepsilon\}.$$

2. Equivalence of graph topology and gap topology

We give a simple proof due to Hans Schumacher of the equivalence of the graph topology and the pointwise gap topology. The fact by itself is known [Qiu, Davison 1992a], but there is no simple direct proof to be found in the literature.

Lemma 2.1. *Let $\Omega = \mathbf{C}^+$ or $\Omega =$ some disk D. If $F \in RH_\infty(\Omega)$ is such that $f(s) = \operatorname{im} F(s)$, then for all $\delta > 0$ we can find ε such that for all g:*

$$\delta_{\sup}(f, g) < \varepsilon \Rightarrow \exists G \in H_\infty(\Omega) : g(s) = \operatorname{im} G(s) \wedge \|F - G\|_\infty < \delta.$$

Proof. Let $f(s) = \operatorname{im} F(s)$. Choose $Y(s) \in H_\infty(\Omega)$ solving the Bezout equation $YF = I$. Then $\ker Y(s)$ and $f(s)$ are complementary for all $s \in \overline{\Omega}$. Now for $g(s) = \operatorname{im} G(s)$ in a sufficiently small neighbourhood of $f(s)$, $\ker Y(s)$ and $g(s)$ must also be complementary for all $s \in \overline{\Omega}$, which implies that YG is unimodular in $H_\infty(\Omega)$. It follows that a representation $g(s) = \operatorname{im} G'(s)$ can be chosen such that $YG' = I$ ($G' = G(YG)^{-1}$).

We have

$$\|F - G'\| = \|FYF - G'YF\| \leq \|FY - G'Y\|\|F\|.$$

$\|F(s)\|$ is of course bounded on $\overline{\Omega}$; furthermore, because $YF = I$ and $YG' = I$, it follows that FY equals the skew projection $\Pi_{\ker Y}^{\operatorname{im} F}$, and $G'Y$ equals $\Pi_{\ker Y}^{\operatorname{im} G'}$. By lemma 1.12 of chapter 1, we obtain

$$\|F(s) - G'(s)\| \leq \frac{\|F(s)\|}{\sin \varphi(s) \sin(\varphi(s) - \alpha(s))} \delta_{\sup}(f, g)$$

with $\alpha(s) \in [0, \frac{1}{2}\pi]$ such that $\sin \alpha(s) = \delta(f(s), g(s))$, and $\varphi(s)$ the minimal angle between $f(s)$ and $\ker Y(s)$. A compactness argument shows that $\sin \varphi$ and $\sin(\varphi - \alpha)$ are bounded away from zero on $\overline{\Omega}$ when α is sufficiently small (note that it follows from lemma 1.18 of chapter 1 that $\varphi(s)$ is continuous). □

Let $\lambda(f)$ be defined as $\inf\{\sigma_{\min}(F) : s \in \mathbf{C}^+\}$, where F is an inner matrix (i.e. $F(s)^* F(s) = I$ on $i\mathbf{R}$) of full column rank in $RH_\infty(\mathbf{C}^+)$ such that $f(s) = \operatorname{im} F(s)$. Note that λ is well defined as a consequence of the uniqueness of the Beurling symbol up to a constant unitary factor. Let $g(s) = \operatorname{im} G(s)$.

Proposition 2.2. $\delta_{\sup}(f, g) \leq \frac{1}{\lambda(f)} \|F - G\|_\infty$.

Proof. Follows easily from the fact that for constant matrices A, B of full row rank we have $\delta(\operatorname{im} A, \operatorname{im} B) \leq (1/\sigma_{\min}(A))\|A - B\|$. □

Proposition 2.3. *The graph topology is equivalent to the topology induced by* $\delta_{\sup}$.

Proof. Immediate from 2.1 and 2.2. □

Essentially in the same way we also obtain the equivalence of the graph topology and the gap topology.

Lemma 2.4. *If* $F \in RH_\infty(\mathbf{C}^+)$ *and* $V \subset H_2^+$ *are such that* $V = FH_2$, *then for all* $\delta > 0$ *we can find* ε *such that for all* V':

$$\delta(V, V') < \varepsilon \Rightarrow \exists G \in H_\infty(\Omega) : g(s) = G(s)H_2 \wedge \|F - G\|_\infty < \delta.$$

Proof. Completely analogous to 2.1, now using the H_∞ norm instead of reasoning pointwise. □

Proposition 2.5. *The graph topology is equivalent to the topology induced by* δ_{H_2}.

3. Equivalence of the gap and the left halfline behaviour topology

This section establishes the link between the time-domain angles between behaviours of linear time-invariant systems and the angles between graphs of transfer functions in H_2. By means of the isometric operator $Jf = \overline{f(-\overline{z})}$ we can flip H_2^- onto $H_2{}^2$ and vice versa.

Proposition 3.1. *Let* Σ_1, Σ_2 *be linear systems with transfer functions* G_1, G_2 *and* $L_2(-\infty, 0)$*-behaviours* B_1, B_2. *Then*

$$\delta(B_1, B_2) = \delta(\mathcal{G}(-G_1^T), \mathcal{G}(-G_2^T)).$$

Proof. Completely analogously to proposition 2.3 of chapter 1, we have $\mathcal{G}_-(G)^\perp = \mathcal{L}(\mathcal{B}_-(-\tilde{G}))$. Using the isometric character of J and the orthogonal complementation operator $\perp$, we have $\delta(B_1, B_2) = \delta(J\mathcal{L}B_1{}^\perp, J\mathcal{L}B_2{}^\perp)$. □

Proposition 3.2. *Let* Σ_G, Σ_K *be linear systems with transfer functions* G *from* U *to* Y *and* K *from* Y *to* U *respectively, and* $L_2(-\infty, 0)$*-behaviours* B_G, B_K. *Then if the feedback interconnection of* G *and* K *is stable, we have*

$$(i)\ \sin\varphi(\mathcal{G}(-G^T), \mathcal{G}(-K^T)) = \sin\varphi(\mathcal{G}(G), \mathcal{G}(K)),$$
$$(ii)\ \sin\varphi(B_G, B_K) = \sin\varphi(\mathcal{G}(G), \mathcal{G}(K)).$$

Proof. The second statement follows from the first (proof analogous to proposition 3.1). Let P be the graph of G, C the graph of K, and P^T, C^T the graphs of $-G^T$, $-K^T$. We know (identifying a multiplication operator with its symbol for typographical reasons) that

$$\Pi_P^C = \begin{bmatrix} I \\ G \end{bmatrix} (I - KG)^{-1} [I \quad -K]$$

and that $\|\Pi_P^C\|^{-1} = \sin \varphi(P, C)$. Now

$$\Pi_{C^T}^{P^T} = \begin{bmatrix} I \\ -K^T \end{bmatrix} (I - G^T K^T)^{-1} [I \quad G^T]$$

Furthermore, also $(\Pi_P^C)^T$ is equal to this last expression, and of course the L_∞ norms of a matrix and its transpose are equal. □

Proposition 3.3. *The topology induced on the set of finite-dimensional input-state-output systems by the gap between the $L_2(-\infty, 0)$-behaviours is the same as the gap topology.*

Proof. The gap topology is the weakest topology on systems such that Π_P^C is a continuous function of P. By proposition 3.1 and the proof of proposition 3.2, the $L_2(-\infty, 0)$-gap topology is the weakest topology such that $(\Pi_P^C)^T$ is continuous in P. This is obviously the same thing. □

Note that it follows that complementarity of the L_2^--behaviours is the same thing as complementarity of the H_2^+-graphs. Thus, we can also model stability robustness in terms of the $L_2(-\infty, 0)$-behaviours of systems. This is what we should expect, a feedback interconnection being stable iff the autonomous L_2^--behaviour is $\{0\}$.

4. Relations with other topologies

So far, we have been considering different interpretations of the same topology; it is perhaps enlightening to compare a few of the genuinely different topologies that have been studied in the literature. Worthy of consideration are:

(i) We can make a few changes in the definition of the gap topology. In the first place, we can change the set of trajectories we identify a system to. The gaps δ_{H^+}, δ_{H^-} are between the graphs in H_2^+ resp. H_2^-, and δ_L, δ_{L^+}, δ_{L^-} refer to the gaps between the behaviours in L_2, L_2^+ resp. L_2^-. In the definition of the pointwise gap, we can vary the subset of the extended complex plane on which the supremum of the pointwise gaps is taken. So for instance δ_∞ is the gap δ_{sup} on $\mathcal{R}(\mathbf{C}_\infty, m, q)$, and $\delta_{i\mathbf{R}}$ is the supremum of the pointwise gaps over the imaginary axis. The topologies $\delta_{L^+} \equiv \delta_{H^-}$ and $\delta_{L^-} \equiv \delta_{H^+}$ are not comparable.

(ii) The L_∞ norm topology on transfer functions. On RL_∞ this topology is equivalent to to the topology induced by $\delta_{i\mathbf{R}}$, which is in turn equivalent to δ_L (and thus it is weaker than the gap topology), So it makes sense to replace this topology by the L_2 gap topology that is its natural extension to the set of transfer functions not in L_∞.

(iii) Parameter topologies. The continuity of perturbations caused by varying parameters is the subject of the next chapter. Here we consider only the parameter topology of minimal driving variable realizations.

Proposition 4.1. *Let V_1, V_2 be behaviours in $L_2(\mathbf{R})$ and let Θ_i be the Beurling symbols of V^0_{i+}. For s on the imaginary axis let $V_i(s) = \operatorname{span}\Theta_i(s)$. Then*

$$\delta_L(V_1, V_2) = \sup\{\delta(V_1(s), V_2(s)) \mid s \in i\mathbf{R}\}.$$

Proof. Let $\Psi_2(s) = \Theta_2(s)\Theta_2(s)^*\Theta_1(s)$. The matrix $\Theta_2(s)\Theta_2(s)^*$ represents the orthogonal projection on $V_2(s)$, and the associated multiplication operator represents the projection on V_2. So we have

$$\begin{aligned}\vec{\delta}(V_1, V_2) &= \|(I - \Theta_2\Theta_2^*)\Theta_1\|_\infty = \sup\{\|\Pi_{V_2(s)^\perp}\Theta_1(s)\| \mid s \in i\mathbf{R}\}\\ &= \sup\{\|\Pi_{V_2(s)^\perp}|_{V_1(s)}\| \mid s \in i\mathbf{R}\} = \sup\{\vec{\delta}(V_1(s), V_2(s)) \mid s \in i\mathbf{R}\}.\end{aligned}$$

□

Proposition 4.2. *The different gap topologies are related according to the following diagram, where the arrows point from weaker to stronger topologies.*

$$\begin{array}{ccc} & \delta_L & \\ \swarrow & & \searrow \\ \delta_{L^+} \equiv \delta_{H^-} & & \delta_{L^-} \equiv \delta_{H^+} \\ \searrow & & \swarrow \\ & \delta_\infty & \end{array}$$

Proof. Follows from the interpretation of all the topologies in the diagram as pointwise gap topologies. To prove that the inclusions are strict, it is sufficient to observe that perturbations of the form $G_\varepsilon = G + \varepsilon/(s-1)$ for G stable are continuous in δ_{H_-}, but certainly not in the gap topology, in which stability is a robust property. □

It can be shown that $\mathcal{S}_{\rm fd}$ equipped with the topology induced by δ_∞ is not connected and falls apart into components according to the McMillan degrees of the transfer functions, on which components it is equivalent the parameter topology of minimal realizations modulo state space isomorphism [Byrnes,Duncan]. Part of this result can easily be obtained as a corollary to some of the observations in this chapter.

Proposition 4.3. *Equipped with the topology induced by δ_∞, $\mathcal{S}_{\rm fd}$ is not connected. In particular, systems with different McMillan degree are in different components.*

Proof. Following the reasoning of the previous section we know that δ_{L^+} is equivalent to δ_{H^-}, and from the equivalence of the gap topology and the pointwise gap topology we know δ_∞ is stronger than both δ_{H^-} and δ_{L^+}. So δ_∞ is stronger than δ_{H^+} and δ_{L^+}. Hence, for g in a sufficiently small neighbourhood of any curve f on the Grassmannian, by continuity of orthogonal complementation and intersection, the minimal state space $\mathcal{L}(\mathcal{B}_+(g))/\mathcal{G}(g) \cong \mathcal{B}_+(g) \ominus \mathcal{B}_+^0(g) = \mathcal{B}_+(g) \cap \mathcal{B}_+^0(g)^\perp$ will be close to the minimal state space of f. It follows that they must have the same dimension. □

Lemma 4.4. *Let V_1, V_2 be finite-dimensional systems on the whole time axis. Then in general $\delta_{L_2}(V_1, V_2) \leq \delta_{H_2}(V_1, V_2)$, and*

$$(i)\quad \sqrt{1 - \vec{\delta}_{H_2}(V_1, V_2)^2} = \|\Pi^{V_{1+}^0}_{(V_{2+}^0)^\perp}\|^{-1} = \|\Pi^{\mathcal{G}(G_1)}_{\mathcal{L}\mathcal{B}_+(-\tilde{G}_2)}\|^{-1},$$

$$(ii)\quad \sqrt{1 - \vec{\delta}_{L_2}(V_1, V_2)^2} = \|\Pi^{V_1}_{V_2^\perp}\|^{-1} = \|\Pi^{\mathcal{G}(G_1)}_{\mathcal{G}(-\tilde{G}_2)}\|^{-1},$$

where it is assumed that $\vec{\delta}_{H_2}(V_1, V_2) < 1$, respectively $\vec{\delta}_{L_2}(V_1, V_2) < 1$ for (i) and (ii).

Proof. To obtain the relation between the directed gaps and the projection norms one uses lemma 1.11 of chapter 1 in combination with the following simple geometric fact:

$$\sqrt{1 - \vec{\delta}(V_1, V_2)^2} = \sin\varphi(V_1, V_2^\perp).$$

For (*i*), one furthermore appeals to proposition 2.3 of chapter 1. For the final statement of (*ii*), we use the fact that both $\Pi^{\mathcal{L}V_1}_{\mathcal{L}V_2^\perp}$ and $\Pi^{\mathcal{G}(G_1)}_{\mathcal{G}(-\tilde{G}_2)}$ are multiplication operators with the symbol

$$\begin{bmatrix} G_1 \\ I \end{bmatrix} (I + \tilde{G}_2 G_1)^{-1} [\,\tilde{G}_2 \quad I\,],$$

respectively mapping from $L_2(i\mathbf{R})$ to $L_2(i\mathbf{R})$ and from H_2 to H_2. Since in general the induced operator norm of a rational L_∞ matrix G as a multiplication operator from $\operatorname{dom} G \subset H_2$ to H_2 is equal to its L_∞ norm, we can conclude the two induced norms are equal.

To see that the L_2 gap is the smaller of the two, it suffices to point out that $\mathcal{G}(-\tilde{G}_2)$ is in general a subspace of $\mathcal{LB}_+(-\tilde{G}_2)$, so the infimum that gives the minimal angle with $\mathcal{G}(G_1)$ is computed over a smaller set. □

It is interesting to note that, though the metrics δ_{L_2} and δ_{H_2} are not equivalent, we can estimate one of the directed gaps of the stronger topology in terms of the weaker metric.

Lemma 4.5. *Let V_1 and V_2 be finite-dimensional systems on the whole time axis. Then for V_2 such that δ_{L_2} is sufficiently small, we have*

$$(i)\quad \vec{\delta}(V_{1+}, V_{2+}) \leq \frac{1}{\gamma(V_1, L_2^-)}\delta_{L_2}(V_1, V_2)$$

$$(ii)\quad \vec{\delta}(V_{2+}^{\,0}, V_{1+}^{\,0}) \leq \frac{1}{\gamma(V_1, L_2^+)}\delta_{L_2}(V_1, V_2)$$

Proof. First we check that $\gamma(V, L_2^+) > 0$ for finite-dimensional systems. This is the case since we have by lemma 1.5 of chapter 1

$$\gamma(V, L_2^+) = \sin \varphi(V_+ \ominus V_+^0, L_2^+),$$

and the space $V_+ \ominus V_+^0$ is finite-dimensional. For the rest, use that in general the inequalities for the directed gaps $\vec{\delta}(U + V, U + V') \leq \frac{1}{\gamma(U,V)}\delta(V, V')$ and its dual $\vec{\delta}(U \cap V', U \cap V) \leq \frac{1}{\gamma(U,V)}\delta(V, V')$ do not need the general position of U and V, cf. the proof of proposition 1.17 of chapter 1. Furthermore for (*i*), one uses the general fact $\delta(\Pi_U V, \Pi_U V_1) = \delta(V + U^\perp, V_1 + U^\perp)$. □

If we write $V_n \uparrow V$ for $\vec{\delta}(V_n, V) \to 0$ and $V_n \downarrow V$ for $\vec{\delta}(V, V_n) \to 0$, this can be paraphrased as: $V_n \to V$ in L_2 implies $V_{n+} \downarrow V_+$ and $V_{n+}^{\,0} \uparrow V_+^0$ in L_2^+.

We shall examine the relations between the different topologies a bit closer. For several classes of systems, the topologies coincide. Consider the following sets:

Definition 4.6.
(a) The set of systems of McMillan degree n is denoted $\mathcal{S}_n$.
(b) The set of systems of McMillan degree $\leq n$ is denoted $\mathcal{S}_{\leq n}$.
(c) The set of systems with n right open halfplane poles and no poles on the imaginary axis is denoted $\mathcal{S}_n^+$.

(d) The set of systems with at most n right open halfplane poles and no poles on the imaginary axis is denoted $\mathcal{S}^+_{\le n}$.

For systems without imaginary axis poles the number of poles can be retrieved in the geometrical point of view as a Fredholm index; this helps to analyze their robustness properties. First we recall:

Proposition 4.7. *Let $G \in RL_\infty$. Then* $\dim H_2(U) \ominus \operatorname{dom} G$ *is equal to the number of right halfplane poles of G.*

Proposition 4.8. *The number $\#(P, C)$ of closed loop poles in the right halfplane of a well posed feedback interconnection is equal to* $\operatorname{def}(P, C)$. *In particular, the number of right halfplane poles of $G \in L_\infty$ is equal to* $\operatorname{def}(\mathcal{G}(G), \{0\} \times Y)$.

Proof. Apply the previous lemma to $G = H(P, C)$. This makes sense because of the connection between the closed loop transfer function and the parallel projection Π^C_P ensures that $\operatorname{dom} H(P, C) = P + C$. □

The robustness of the number of right halfplane poles was also shown in the book [Vidyasagar], but without the geometric interpretation given here.

Corollary 4.9. *The number of right halfplane poles is robust in the gap topology with margin* $\sin \varphi(\mathcal{G}(G), \{0\} \times Y)$.

Lemma 4.10. *Let $G_1 \in RL_\infty$. Then a continuous function f with $f(0) = 0$ exists such that for $G_2 \in RL_\infty$ with $\|G_1 - G_2\|$, $\vec{\delta}(\mathcal{G}(G_1), \mathcal{G}(G_2))$ and* $\delta(\operatorname{dom} G_1, \operatorname{dom} G_2)$ *sufficiently small, we have*

$$\delta(\mathcal{G}(G_1), \mathcal{G}(G_2)) \le f(\|G_1 - G_2\|, \vec{\delta}(\mathcal{G}(G_1), \mathcal{G}(G_2)), \delta(\operatorname{dom} G_1, \operatorname{dom} G_2)).$$

Proof. Assume $\vec{\delta}(P_1, P_2) < d_0$, $\delta(\operatorname{dom} G_1, \operatorname{dom} G_2) < d_1$. Let $x_2 = (u_2, G_2u_2) \in P_2$. Choose $x_1 = (u_1, G_1u_1) \in P_1$ such that $\|u_1 - u_2\| < d_1\|u_2\|$. Let $x'_2 = (u'_2, G_2u'_2) \in P_2$ be such that $\|x'_2 - x_1\| < d_0\|x_1\|$. Then

$$\begin{aligned}\|x_1 - x_2\| &\le \|x_1 - x'_2\| + \|x_2 - x'_2\| \\ &\le d_0\|x_1\| + \|u_2 - u'_2\| + \|G_2\|\|u_2 - u'_2\| \\ &\le d_0(\|u_1\| + \|G_1\|\|u_1\|) + \|u_1 - u_2\| + \|u_1 - u'_2\| + \\ &\quad \|G_2\|(\|u_1 - u_2\| + \|u_1 - u'_2\|)\end{aligned}$$

Now $\|u_1 - u'_2\| \le d_0\|x_1\|$, and $\|u_1\| \le \|x_1\|$, and also $\|x_1\|$ is bounded in terms of $\|x_2\|$ by $\|x_1\| \le (1 + \|G_1\|)\|u_1\| \le (1 + \|G_1\|)(1 + d_1)\|u_2\|$, so we can obtain an estimate as required. □

The following was shown in [Zhu 1991] using expansions by partial fractions:

Proposition 4.11. *On the set $\mathcal{S}_n^+$ the L_2 gap topology coincides with the gap topology.*

Proof. Fix some system $V \in \mathcal{S}_n^+$. If $V_n \to V$ in L_2, it follows by 4.5 that $V_{+n}^0 \uparrow V_+^0$ in L_2^+. Assume $P = \mathcal{L}V_+^0 = \mathcal{G}(G)$ where G is in RL_∞ with exactly k right halfplane poles, similarly $P_n = \mathcal{L}V_{n+}^0 = \mathcal{G}(G_n)$. Now the spaces V_+^0 and $\{0\} \times Y$ have zero intersection and $\gamma(V_+^0, \{0\} \times Y) > 0$, from which it follows that also $V_{+n}^0 + (\{0\} \times Y) \uparrow V_+^0(\{0\} \times Y)$.

Now if the the number of right halfplane poles of G and all the G_n are equal, so are the codimensions of the spaces $P + (\{0\} \times Y)$ and $P_n + (\{0\} \times Y)$ in L_2^+. It follows by proposition 1.2 of chapter 1 that $P_n + (\{0\} \times Y) \to P + (\{0\} \times Y)$. Then since $\delta(\operatorname{dom} G, \operatorname{dom} G') = \delta(\Pi_U P, \Pi_U P') = \delta(P + Y, P' + Y)$ we also get $\operatorname{dom} G \to \operatorname{dom} G_n$. Hence also $P \to P'$ by the previous lemma. □

On the set of systems with fixed order we have a stronger result, cf. [Meyer].

Proposition 4.12. *On the set $\mathcal{S}_n$ the L_2 gap topology coincides with the topology induced by δ_∞.*

Proof. Fix a system V on the whole time axis. Let $X = V_+ \ominus V_+^0$, $k = \dim X$. Assume $V_n \to V$ in L_2, and all the V_n have order k. We know from lemma 4.5 that $V_{n+} \downarrow V_+$ and $V_{n+}^0 \uparrow V_+^0$. Now we show that the equal codimensions $[V_{n+} : V_{n+}^0]$ imply that $V_{n+}^0 \to V_+^0$. Suppose not. Then V_+^0 contains, for n sufficiently large, a line l almost orthogonal to V_{n+}^0. On the other hand $V_{n+} \downarrow V_+$ implies that the orthogonal projection $\Pi_{V_{n+}}$ is injective on V and arbitrarily close to the identity on V. Hence $\Pi_{V_{n+}}(X + l)$ is a $k + 1$ dimensional subspace of V_{n+} that is almost orthogonal, hence complementary to V_{n+}^0. But this is a contradiction with the assumption that the order of V_n is k.

So we have shown the L_2 gap topology is equivalent to the H_2 gap topology on $\mathcal{S}_n$. Since by the same argument it must also be equivalent to the H_- topology, it follows that all the topologies in the diagram of proposition 4.2 coincide on $\mathcal{S}_n$. □

Our next topic is the continuity of the realization procedure of chapter 2 with respect to the δ_∞ topology. The proof is facilitated by the fact that the (A_n, B_n, C_n, D_n) state space realization operators of a converging sequence of systems $V_n \to V$ can be compared to the (A, B, C, D) of the limit system V as operators between the spaces $L_2(W)$ and W. Operators having different domains can be compared in the gap topology. Because of the fixed finite dimensions of the state and input spaces X_n, U_n this amounts to a notion of convergence that can also be phrased as follows: $F_n \to F$ if the domains of definition D_{F_n} converge to D_F in the gap topology and

$$\forall c \exists n \forall m > n, x \in D_F, y \in D_{F_m} : \|Fx - F_m y\| \leq c\|x - y\|.$$

It is readily seen that this also implies the possibility of choosing converging parameters for our (A_n, B_n, C_n, D_n): One chooses sets $\{\varphi_n\}$, $\{\psi_n\}$ of unitary automorphisms of L_2 respectively W mapping X_n to X respectively U_n to U in such a way that $\|\varphi_n - I\| \to 0$, $\|\psi_n - I\| \to 0$. Then a parametrization of $A'_n = \varphi_n A_n \varphi_n^{-1}$, $B'_n = \varphi_n B_n \psi_n^{-1}$, $C'_n = \psi_n C_n \varphi_n^{-1}$, $D'_n = \psi_n D_n \psi_n^{-1}$ will be converging.

First we need to see that convergence of semigroups implies convergence of the infinitesimal generators.

Lemma 4.13. *Let $T_i(t)$ be a sequence of contraction semigroups on a finite-dimensional space X. Then if $T_i(t) \to T(t)$ pointwise in t, the infinitesimal generators A_i of $T_i(t)$ converge to the infinitesimal generator A of $T(t)$.*

Proof. It is easy to see by Lebesgue's dominated convergence theorem that $e^{A_i t} \to e^{At}$ implies that $R(\lambda, A_i) = (sI - A_i)^{-1} = \int_0^\infty e^{-\lambda t} e^{A_i t} dt \to R(\lambda, A)$ (cf. [Pazy], theorem 4.2 of chapter 3). Hence also $A_i \to A$ as a consequence of the continuity of matrix inversion. □

The following lemma enables us to derive the convergence of the suggested C and B operators from the convergence of the A operators.

Lemma 4.14. *The evaluation operator at zero $x \mapsto x(0)$ is bounded on $\mathcal{H}_1$ equipped with the graph norm of the differentiation operator $\mathcal{D}$ defined by $\|x\|^2_{\mathcal{H}_1} = \|x\|^2 + \|\mathcal{D}x\|^2$.*

Proof. [Fuhrmann], proof of theorem III-6-2. □

Lemma 4.15. *If the systems V_1, V_2 have a different number of inputs, it follows that $\delta(V_1, V_2) = 1$.*

Proof. It is immediate that the pointwise gaps are 1 for all s. Hence also $\delta_{L_2}(V_1, V_2) = 1$. □

Theorem 4.16. *The realization procedure of section 1 yields continuous parameters with respect to the topology induced by δ_∞.*

Proof. Suppose we have a converging sequence $V_n \to V$. As shown in the proof of proposition 4.3, we also have convergence of the state spaces $X_n \to X$. Now choose a set $\{\varphi_n\}$ of unitary automorphisms of L_2 mapping X_n to X in such a way that $\|\varphi_n - I\| \to 0$. Now let $T_n(t) = \tau_t|_{X_n}$. Since the backward shift operator is continuous, it follows that we have convergence of the semigroups $\varphi_n T_n(t) \varphi_n^{-1}$ on X to $T(t)$. Hence by lemma 4.13, the infinitesimal generators A'_n converge to A.

Now in turn this implies that $A_n \longrightarrow A$ in the sense discussed above. By lemma 4.14 the evaluation mapping at 0 is A-bounded, from which it follows that we have $C_n \longrightarrow C$ in the sense discussed above.

The spaces U_n can be seen to converge to U as follows: We show that for all $u \in L_2(U)$ there exists a sequence $u_n \in L_2(U_n)$ converging to u in the sense of L_2^{loc}. This can only be the case if $U_n \downarrow U$. Then since the number of inputs to the systems V_i is equal by virtue of lemma 4.13, we must have $U_n \longrightarrow U$. So, fix $u \in L_2(U)$ as the input corresponding to external trajectory $w \in V$. Now if $w_n \in V_n$ are chosen such that $w_n \longrightarrow w$, it follows that $x(.)_n \longrightarrow x(.)$ in L_2^{loc}. Then since $C_n \longrightarrow C$ also $Cx(.)_n \longrightarrow Cx(.)$ in the space of continuous functions, and hence $u(.)_n = w_n - C_n x(.)_n \longrightarrow u = w - Cx(.)$ in L_2^{loc}.

Also $B_n = C_n^* \longrightarrow B, D_n \longrightarrow D$. □

Chapter 4

Continuity of parameter variations in the graph topology

Consider a family of linear time-invariant systems given in *AR* form as $\ker P(s, \lambda)$, where the parameters λ are taken from some closed subcone of $\mathbf{R}^k$. The question we pose is the following: under which conditions do we have convergence in the graph topology of the systems $\ker P(s, \lambda)$ towards $\ker P(s, 0)$ as $\lambda \to 0$? As the gap is not given by closed form formulas but has to be computed by an iterative procedure (see the next chapter), it is usually difficult to estimate the distance between the nominal system and the perturbed systems. Therefore we try to provide here only a partial answer in a qualitative sense. It turns out that the pointwise perspective is helpful here. This chapter is an attempt to find a simple underlying rule covering the various instances of continuity and discontinuity discussed in [Vidyasagar], [Cobb], and [de Does, Schumacher 1994a].

1. An extension of l'Hôpital's rule to several variables

In this section, we give a multivariable version of l'Hôpital's rule. Let M be a Grassmann manifold of m-dimensional subspaces of some n-dimensional vector space V. We are interested in the continuity of subspace-valued mappings f of the form

$$\begin{aligned} K &\to M \\ z &\mapsto \ker F(z), \end{aligned}$$

where K is a closed subset of $\mathbf{R}^k$, and $F(z)$ is an analytic $m \times n$ matrix-valued function. The following is an easy consequence of lemma 1.20 of chapter 1.

Proposition 1.1. *Suppose $F(z)$ is a continuous matrix-valued function on K. Then if $F(z)$ is of full row rank for all z in K the function $z \mapsto \ker F(z)$ is continuous.*

When F does lose rank the situation is a bit more delicate. If an analytic $F(z)$ depends on one real or complex parameter z it is always possible to extend f

continuously to z_0, and (in the case of a curve on a Grassmann manifold of 1-dimensional spaces) the value that must be assigned to z_0 can be determined by l'Hôpital's rule; one looks at the leading terms of the series expansion of F. In fact a discontinuity of a curve depending on one parameter is always due to a factor in the matrix $F(z)$ that can be eliminated; we can write $F(z) = G(z)H(z)$ in such a way that the generic ranks of $H(z)$ and $F(z)$ are the same and $H(z_0)$ is of full row rank on K, and assign continuously the value $f(z_0) = \ker H(z_0)$. In the case of several variables such an assignment need no longer be possible; consider for example the curve $(z_1, z_2) \mapsto \ker [z_1 \quad z_2]$, at $z_1 = z_2 = 0$. We must determine when it is possible to extend f continuously. In the applications we have in mind the interesting case is the one in which z_0 is on the boundary of the set K. In the sequel, we order the terms of multivariate polynomials according to the total degree ordering: the multi-index $N = n_1, ..., n_k$ precedes $M = m_1, ..., m_l$ iff the total degree $\Sigma_{i=1,k} n_i$ is smaller than $\Sigma_{i=1,l} m_i$. In the next definition, the phrase "minimal degree N_i of the i-th row" of an analytic matrix means the smallest total degree occurring in a term of the series expansion of one of the entries of that row.

Definition 1.2. The *leading term matrix* of a matrix-valued function with analytic entries at a point a is the polynomial matrix that arises by truncating, for each i, the series expansion at a of the entries in the i-th row to the terms having total degree equal to the minimal degree N_i of that row.

Lemma 1.3. *Consider a matrix-valued analytic function $A(z)$, defined on some closed cone K in $\mathbf{R}^n$. Suppose the leading term matrix $B(z)$ of $A(z)$ has full row rank on $K\backslash\{0\}$, and the subspace-valued function*

$$f : K\backslash\{0\} \to M$$
$$z \mapsto \ker B(z)$$

is constant. Then $z \mapsto \ker A(z)$ can be extended continuously to 0. If the other requirements are satisfied and $\ker B(z)$ is not constant, the mapping $z \mapsto \ker A(z)$ has an essential discontinuity at 0.

Proof. First suppose all requirements are satisfied. Let $B_i(z)$ be the i-th row of $B(z)$, let N_i be the total degree of the entries of $B_i(z)$. Then the $B_i(z)$ are homogeneous polynomial matrices. Let $A_i(z)$ be the corresponding row of $A(z)$.

Then $\ker A(z) = \bigcap_{i=1,m} \ker A_i(z)$, $\ker B(z) = \bigcap_{i=1,m} \ker B_i(z)$, and each B_i is the leading term matrix of A_i. Let the sequences of subspaces $\{V_i(z)\}$, $\{W_i(z)\}$ be defined by $V_0 = W_0 = \mathbf{R}^n$, $V_{i+1}(z) = \ker A_i(z) \cap V_i(z)$, $W_{i+1}(z) = \ker B_i(z) \cap W_i(z)$. It is part of the hypothesis that the matrix $B(z)$ is of full row rank, so for each $z \neq 0$ we have that $W_{i-1}(z)$ and $\ker B_i(z)$ intersect only in 0, so $\gamma(W_{i-1}(z), \ker B_i(z)) > 0$

is a continuous function of z on $K\backslash\{0\}$. We show there is $\gamma_i > 0$ such that $\forall z\ \gamma(W_{i-1}(z), \ker B_i(z)) > \gamma_i$. On a closed subset $\|z\| = R$ of $K\backslash\{0\}$ this follows from the continuity of γ in general position (1.18 of chapter1) and a standard compactness argument. This is sufficient, since

$$B_i(\lambda z_1, ..., \lambda z_m) = \lambda^{N_i} B_i(z_1, ..., z_m),$$

so multiplication with λ does not affect the position of the spaces $W_i(z)$.

Similarly, on a subset $\{z \mid \|z\| = R\}$ of K it follows from the full rank of $B(z)$ that a number τ_0 exists such that $\tau(B_i(z)) > \tau_0$ for all z. Since one has $\tau(B_i(\lambda z_1, ...\lambda z_m)) = \lambda^{N_i}\tau(B_i(z_1, ...z_m))$ it follows that a number $c > 0$ exists such that $\tau(B_i(z)) > c\|z\|^{N_i}$ on $K\backslash\{0\}$. From the fact that B_i is the leading term matrix of A_i it fo lows that for any ε we can find δ such that whenever $\|z\| < \delta$, the inequality

$$\|A_i(z) - B_i(z)\| < \varepsilon\|z\|^{N_i}$$

holds. By lemma 1.20 of chapter 1 this means that $\delta(\ker A_i(z), \ker B_i(z)) < \frac{1}{c}\varepsilon$ for $\|z\| < \delta$. Now use (cf. the proof of I.1.17, where a similar relation is given for the sum of subspaces, which by duality also applies to intersections)

$$\begin{aligned}\delta(V_i, W_i) = \delta(V_{i-1} \cap \ker A_i, W_{i-1} \cap \ker B_i) &\leq \frac{1}{\gamma_i}\delta(V_{i-1}, W_{i-1}) \\ &+ \frac{\delta(\ker A_i(z), \ker B_i(z))}{\sin(\psi(W_{i-1}(z), \ker B_i(z)) - \vartheta(\ker A_i(z), \ker B_i(z)))}.\end{aligned}$$

It then follows that the second tends to zero as $\|z\| \to 0$. Now by induction we have that $V_m(z)$ converges to $W_m(z) = \ker B$ for $\|z\| \to 0$ in K.

Now for the converse statement, if it is the case that z_0, z_1 with $\ker B(z_0) \neq \ker B(z_1)$ exist, then for sufficiently small λ, λz_0 and λz_1 are arbitrarily close to 0 and we have $\delta(\ker B(\lambda z_0), \ker B(\lambda z_1)) = \delta(\ker B(z_0), \ker B(z_1))$. Hence we do not have convergence of $\delta(\ker A(\lambda z_0), \ker A(\lambda z_1))$ to zero as $\lambda \to 0$. □

The rule 1.3 is appropriate for the study of AR systems; for the study of systems with auxiliary variables we need a slightly more general version.

Lemma 1.4. *Consider a matrix-valued analytic function $A(z)$, defined on closed some cone K in $\mathbf{R}^n$; let $H : \mathbf{R}^n \to \mathbf{R}^k$ be a constant linear mapping. Suppose the leading term matrix $B(z)$ of $A(z)$ has full row rank on $K\backslash\{0\}$, and the subspace-valued function*

$$\begin{aligned} f : K\backslash\{0\} &\to M \\ z &\mapsto H[\ker B(z)] \end{aligned}$$

is constant, and H is injective on $\ker B(z)$ *for all* z *in a neighbourhood of* 0 *in* K. *Then* $z \mapsto H[\ker A(z)]$ *can be extended continuously to* 0. *If the other requirements are satisfied and* $H[\ker B(z)]$ *is not constant, the mapping* $z \mapsto \ker A(z)$ *has an essential discontinuity at* 0.

Proof. Apply the same reasoning as in the proof of 1.3, to see that $W_m(z)$ converges to $V_m(z)$, but now insert a final application of lemma 1.26 of chapter 1 to see that also $H[V_m(z)] \to H[W_m(z)]$. The converse statement follows like in the proof of the previous lemma. □

The rule as given here is not without defects. It may well be the case that the leading term matrix of a certain $A(z)$ does not have full rank, but after some transformation of it by row operations the leading term matrix does have full rank. So it is not invariant with respect to coordinate transformation, whereas the question we would like to investigate (continuity) obviously is. Part of this can be remedied by looking at the image of the curve we are interested in under the Plücker embedding. We may do this because of the homeomorphic nature of the embedding. In the next section we shall see an example where this is useful. First we need to recall the following:

Definition 1.5. The *Plücker embedding* is the mapping $\mathcal{P}$ from $Grass(m, n)$ to the projective space $P^{\binom{m}{n}}$ such that the image $\mathcal{P}(W)$ of an m-dimensional subspace W spanned by vectors $w_1, ..., w_m$ is the equivalence class of the m-linear form $w_1 \wedge .... \wedge w_m$ in the projective space $P(\Lambda^m(V^*))$ over the space of alternating m-forms on V^*.

The Plücker embedding can be computed explicitly as follows:

Lemma 1.6. *The Plücker embedding can be calculated by*

$$\mathcal{P}(\ker A) = \operatorname{span}(\det A_i)_{i=1,\binom{m}{n}}$$

where (A_i) *is an enumeration of the square minors of* A *in some fixed order.*

Proposition 1.7. *The Plücker embedding is well-defined, and is a homeomorphism onto its image.*

Proof. For the definition of the Plücker embedding and the fact that its image is a subvariety of $P^{\binom{m}{n}}$ cf. for instance [Harris], page 64. The fact that it is a homeomorphism is somewhat more difficult to trace in the literature but easy to prove. It is obvious from 1.6 that the mapping is continuous. So we can appeal to the fact that a continuous bijection between compact metric spaces is automatically open. □

It follows that the metrics $\delta(.,.)$ and $\delta(\mathcal{P}(.), \mathcal{P}(.))$ on the Grassmannian are equivalent (for the metric on the projective space we can also use the gap). By compactness it follows that they are also uniformly equivalent in the sense that $\forall\delta\exists\varepsilon\forall x\, B'_\varepsilon(x) \subset B_\delta(x)$, where B, B' are balls in the two respective metrics. As a consequence the topologies of uniform convergence of mappings from K to the manifolds are also the same.

Definition 1.8. A *leading vector* of a vector-valued function $a(z)$ with analytic entries is the polynomial vector that arises by truncating the series expansion of the entries of $a(z)$ to the terms having total degree less than or equal to some fixed number N.

In fact this time we shall consider the possibility of a nonhomogeneous truncation, which is a slight improvement.

Lemma 1.9. *Consider a vector-valued analytic function $a(z)$, defined on some closed cone K in $\mathbf{R}^n$. Suppose a leading vector $b(z)$ of $a(z)$ does not vanish on a neighbourhood of 0 in $K\backslash\{0\}$, and the function*

$$f : K\backslash\{0\} \to M$$
$$z \mapsto \operatorname{span} b(z)$$

is constant. Then $z \mapsto \operatorname{span} a(z)$ can be extended continuously to 0.

Proof. Suppose we truncate to total degree N. We first prove that a constant α exists such that the inequality

$$\tau(b(z)) \geq \alpha\|z\|^N$$

holds. This was obvious in the case of homogeneous truncation. In the nonhomogeneous case we go about as follows: Choose ε such that $b(z)$ does not vanish on $K\backslash\{0\} \cap \{z \mid \|z\| \leq \varepsilon\}$. For each $z \in K$ with $\|z\| = \varepsilon$ let the polynomial $b_z(\lambda) = b(\lambda z)$, $\lambda \geq 0 \in \mathbf{R}$. Now for any nonzero univariate polynomial f of degree N, a constant α exists such that $\|f(\lambda)\| \geq \alpha\|\lambda\|^N$ on a positive neighbourhood of zero. Furthermore this constant can be chosen as depending continuously on the parameters of the polynomial, and also the radius of the neighbourhood can be chosen continuously. So again by compactness, a constant α and a neighbourhood $[0, \varepsilon)$ can be chosen that does the job uniformly for all the polynomials $b_z(\lambda)$. Then, again, since a constant β exists such that $\delta(\operatorname{span} a(z), \operatorname{span} b(z)) \leq \beta\|z\|^{N+1} / \tau(b(z))$ it follows that $\operatorname{span} a(z)$ is continuous at 0. □

Here, it should of course be noted that the condition that $\operatorname{span} b(z)$ is constant is easy to check: all its entries must be equal up to a constant factor. In the next section we shall see an example where the rule 1.9 decides the issue whereas 1.3 does not.

We state a parametrized version of the rule for future reference.

Lemma 1.10. *Consider a vector-valued analytic function $a(z, z_1)$, defined on some cone K in $\mathbf{R}^{n+m}$. Suppose a leading vector $b(z, z_1)$ of $a(z, z_1)$, with respect to the total degree term ordering of z (not counting z_1), does not vanish on a neighbourhood of 0 in $K\backslash\{0\}$, and for all z_1 the function*

$$f : K\backslash\{0\} \to M$$
$$z \mapsto \operatorname{span} b(z, z_1)$$

is constant. Then $z \mapsto (z_1 \mapsto \operatorname{span} a(z, z_1))$ can be extended continuously to 0 in the topology of pointwise convergence with the limit curve

$$z_1 \mapsto \operatorname{span} b(z, z_1).$$

Proof. Apply 1.9 pointwise for each z_1. □

Also, for clarity we state the fact that the presence of a factor more or less in the matrix $A(z)$ does not matter.

Lemma 1.11. *Let $A(z) = C(z)B(z)$, with $C(z)$ square, such that $A(z)$ and $B(z)$ both have generically full column rank on K. Then $z \mapsto \ker A(z)$ can continuously extended to the points at which $A(z)$ loses rank iff $z \mapsto \ker B(z)$ can be extended continuously to the points at which $B(z)$ loses rank.*

Proof. Obvious, since the mappings we want to extend coincide on a dense open subset of K. □

2. Application of the rule to parameter variations

We return to the question of the continuity in the gap topology of a set of systems with parameters taken from some subcone P of $\mathbf{R}^k$. We first recall two facts that enable us to reduce our question to the question of joint continuity of a mapping from the set $\mathbf{C}^+ \times P$ to a Grassmannian manifold.

Proposition 2.1. (proposition 2.3 of chapter 3) *The gap topology is equivalent to the topology of uniform convergence on the set of mappings from* $\mathbf{C}^+$ *to* $Grass(m, n)$.

Lemma 2.2. *Joint continuity of a mapping* f *from a product* $A \times B$ *of compacta to a metric space* X *implies the continuity of the correspondence*

$$\begin{aligned} A &\to C(B, X) \\ a &\mapsto f(a, .) \end{aligned}$$

in the topology of uniform convergence.

So we can conclude that a family of systems $\Sigma(\lambda)$ is continuous in the graph topology if the function $(s, \lambda) \to \Sigma(s, \lambda)$ is continuous at all points of $K = \mathbf{C}^+ \times P$. In the examples we give the point at which the matrices in the system description lose rank has $s = \infty$, which means we are discussing *singular perturbations.*

Proposition 2.3. (Singular perturbations, first model) *Suppose we have a first order system* $P(\lambda)$ *depending analytically on a real parameter* $\lambda \geq 0$

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u \\ \lambda\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2(\lambda)u \\ y &= C_1 x_1 + C_2 x_2 + Du \end{aligned}$$

satisfying $B_2(\lambda) = o(\lambda)$ *as* $\lambda \downarrow 0$, A_{22} *Hurwitz. Let*

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21},\ B_0 = B_1,\ C_0 = C_1 - C_2A_{22}^{-1}A_{21}.$$

Assume (A_0, B_0) *is stabilizable, and* (C_0, A_0) *is detectable. Then the family* $P(\lambda)$ *is continuous at* 0 *in the graph topology.*

Proof. First transform with $t = 1/s$. We can apply lemma 1.4 to $z = (\lambda, t)$, $K = \{z \mid s \in \mathbf{C}^+, \lambda \geq 0)\}$,

$$\begin{aligned} A(z) &= \begin{bmatrix} I - tA_{11} & -tA_{12} & -tB_1 \\ -tA_{21} & \lambda I - tA_{22} & -tB_2(\lambda) \end{bmatrix}, \\ H &= \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}. \end{aligned}$$

An expansion as required in section 1 has the truncation

$$B(z) = \begin{bmatrix} I & 0 & 0 \\ -tA_{21} & \lambda I - tA_{22} & 0 \end{bmatrix}.$$

Stabilizability implies that no loss of rank of $A(z)$ occurs outside the point 0. By detectability, H is injective on $\ker A(z)$ outside $z = 0$, thus ensuring continuity outside the point zero. The requirement that $B(z)$ does not lose rank is fulfilled by A_{22} being Hurwitz. Furthermore if A_{22} is Hurwitz it is easy to see that $\ker B(z)$ is the set $\{(x, u) | x = 0\}$. So also the requirement that H is injective on $\ker B(z)$ is obviously fulfilled. Hence the curve

$$\begin{aligned} K &\to Grass(U \times Y) \\ (t, \lambda) &\mapsto H[\ker A(t, \lambda)] \end{aligned}$$

can be extended continuously to 0. This implies continuity of the family of systems by 2.1 and 2.2. □

The previous example may seem in contradiction the rather restrictive result on the continuity of singular perturbations that was obtained in [Vidyasagar] (sufficiency of the condition given) and [Cobb] (necessity). However, the model used there is different. For completeness, we also state and prove Vidyasagar's result.

Proposition 2.4. (Singular perturbations, second model) *Suppose we have a first order system $P(\lambda)$ depending analytically on a real parameter $\lambda \geq 0$*

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u \\ \lambda \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2 u \\ y &= C_1 x_1 + C_2 x_2 + Du \end{aligned}$$

satisfying A_{22} Hurwitz. Let

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, B_0 = B_1, C_0 = C_1 - C_2 A_{22}^{-1} A_{21}.$$

Assume (A_0, B_0) stabilizable and (C_0, A_0) detectable. Then the family $P(\lambda)$ is continuous at 0 in the graph topology if and only if the transfer matrix $C_2(sI - A_{22})^{-1}B_2$ is identically zero.

Proof. This time we get

$$B(z) = \begin{bmatrix} I & 0 & 0 \\ -tA_{21} & \lambda I - tA_{22} & -tB_2 \end{bmatrix},$$

so $H[\ker B(z)]$ is the set $\{(y, u) | y = (C_2(\lambda I - tA_{22})^{-1}tB_2 + D)u\}$. This set clearly does not depend on z iff $C_2(sI - A_{22})^{-1}B_2 \equiv 0$. □

Example 2.5. Let us consider an example of a singular perturbation not in state space form. The equations are supposed to represent two masses interconnected by a spring, where the input u is a force on one of the masses and χ_1 and χ_2 are the displacements of the respective masses. We take the output of the system to be equal to (χ_1, χ_2).

$$\begin{aligned} s^2\chi_1 &= u - k(\chi_1 - \chi_2) - \alpha(s\chi_1 - s\chi_2), \\ s^2\chi_2 &= k(\chi_1 - \chi_2) + \alpha(s\chi_1 - s\chi_2). \end{aligned}$$

We are interested in what happens if α and k tend to infinity. So, rewriting the system with $s = 1/t$, $\alpha = 1/x$, $k = 1/y$ and multiplying out the fractions (we may do this because of lemma 1.11), we get

$$\begin{aligned} xy\chi_1 &= t^2(yxu - x(\chi_1 - \chi_2) - y(s\chi_1 - s\chi_2)), \\ xy\chi_2 &= t^2x((\chi_1 - \chi_2) + y(s\chi_1 - s\chi_2)). \end{aligned}$$

Rearranging things a bit one gets

$$\begin{bmatrix} xy + t^2x + ty & -t^2x - ty & -t^2yx \\ -t^2x - ty & xy + t^2x + ty & 0 \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ u \end{bmatrix} = 0.$$

The leading term matrix is

$$\begin{bmatrix} xy & -ty & 0 \\ -ty & xy & 0 \end{bmatrix},$$

which loses rank whenever $y = 0$.

So this does not lead to an answer to the question of continuity. On the other hand, if one computes the Plücker coordinates, one gets the vector

$$a(z) = \begin{bmatrix} x^2y^2 + 2xy(t^2x + ty) \\ -t^2xy(-t^2x - ty) \\ -t^2xy(xy + t^2x + ty) \end{bmatrix}.$$

Extracting a factor xy, the leading terms are of order 2, and a leading vector of order 3 is

$$b(z) = \begin{bmatrix} xy + 2(ty + t^2x) \\ 0 \\ 0 \end{bmatrix}.$$

This has constant image and does not vanish on $K\backslash\{0\}$ since x and y are positive, so we can conclude to continuity. We can compute the pointwise limit system (which must also be the uniform limit since we know there is a uniform limit) with the help of lemma 1.10 by taking lowest degree terms not counting t:

$$b(z) = \begin{bmatrix} 2(t^2x + ty) \\ -t^2(-t^2x - ty) \\ -t^2(t^2x + ty) \end{bmatrix}.$$

So (transforming back to s) the limiting system in terms of the Plücker embedding is given by

$$b(s) = \begin{bmatrix} 2s^2 \\ 1 \\ -1 \end{bmatrix}.$$

Without taking the Plücker coordinates one could have obtained the limit system by adding the first row to the second and then taking a leading term matrix not counting t:

$$\begin{bmatrix} xy & xy & -t^2yx \\ -t^2x - ty & t^2x + ty & 0 \end{bmatrix}.$$

In terms of s this amounts to the limit system

$$\ker \begin{bmatrix} s^2 & s^2 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Chapter 5

Computation of the gap

We consider the computation of the gap between the graphs of rational transfer functions. The gap is related to the norm of a skew projection, which can be computed by solving a regular indefinite linear quadratic optimal control problem with partially specified initial conditions. The use of normalized coprime factorizations is avoided in this way, and Riccati equations are derived directly in terms of the state-space realizations. This chapter was published as [de Does].

1. Introduction

As can be seen from the definition, the directed gap can be expressed as the norm of a projection:

$$\vec{\delta}(V_1, V_2) = \Pi_1 := \|\Pi_{V_2^\perp}|_{V_1}\|$$

It is on this expression that the computation of the gap in [Georgiou] is based: writing shift-invariant V_1 and V_2 as the images of isometric shift-invariant operators Θ_1 and Θ_2 respectively, we get

$$\|\Pi_1\| = \|\Pi_{(\mathrm{im}\,\Theta_2)^\perp}\Theta_1\|.$$

Now by the commutant lifting theorem

$$\|\Pi_1\| = \|\Pi_{(\mathrm{im}\,\Theta_2)^\perp}\Theta_1\| = \inf_{Q\in H_\infty} \|\Theta_1 + \Theta_2 Q\|_\infty,$$

and the rightmost term in this equality defines a model-matching problem that can be solved by standard techniques of H_∞-optimization. The initial idea behind this chapter was that the leftmost term looks easier. Indeed, it can be shown directly, without using any H_∞ optimization theory, to define a linear quadratic optimal control problem (cf. remark 2.3). However, there are some difficulties in the case of nonstrictly proper systems in this approach: the Riccati equations one obtains are of higher order than necessary.

These difficulties can be circumvented by using an other way of relating the gap to the norm of a projection. Let $\Pi_{V_2}^{V_1}$ denote the skew projection of $V_1 + V_2$ along V_1 on V_2.

We now appeal to lemma 4.4 of chapter 3:

$$\sqrt{1 - \vec{\delta}(V_1, V_2)^2} = \sin\varphi(V_1, V_2^\perp) = \|\Pi_2\|^{-1} := \|\Pi_{V_2^\perp}^{V_1}\|^{-1}.$$

The norm of Π_2 turns out to be more convenient to compute.

In the remainder of this chapter, let $V = \mathcal{G}(G)$, $V_1 = \mathcal{G}(G_1)$.

2. State-space expression of parallel projections

As a consequence of the description of the orthogonal complement of the graph of a transfer function we gave in chapter 1, proposition 2.3, it is possible to derive state-space equations for the skew projection along $\mathcal{G}(G)$ on $\mathcal{G}(G_1)^\perp = \mathcal{LB}_+(-\tilde{G}_1)$. Using the fact that the graph of a system corresponds in the time domain to solutions with initial state zero, and $(C(sI-A)^{-1}B+D)^\sim = -B^{\mathrm{T}}(sI-(-A^{\mathrm{T}}))^{-1}C^{\mathrm{T}}+D^{\mathrm{T}}$, we can render the situation

$$\begin{aligned} y_1 &= Gu_1, \\ (y_2, u_2) &\in \mathcal{LB}_+(-\tilde{G}_1), \\ (y_3, u_3) &= (y_1, u_1) + (y_2, u_2) \end{aligned}$$

in the time domain as follows:

$$\begin{aligned} \dot{x}_1 &= Ax_1 + Bu_1,\ x_1(0) = 0, \\ y_1 &= Cx_1 + Du_1, \\ \dot{x}_2 &= -A_1^{\mathrm{T}}x_2 + C_1^{\mathrm{T}}y_2, \\ u_2 &= B_1^{\mathrm{T}}x_2 - D_1^{\mathrm{T}}y_2, \\ \begin{bmatrix} y_3 \\ u_3 \end{bmatrix} &= \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} y_2 \\ u_2 \end{bmatrix}. \end{aligned}$$

Perhaps it should be emphasized that the main idea of this section is a slight enhancement of the expressive power of differential equations with constant coefficients, obtained by specifying the initial conditions $x(0)$ partially. One can give state-space descriptions of mappings commuting with the forward shift (transfer functions) from L_2^+ to L_2^+ by fixing initial conditions $x(0) = 0$ for the whole state x. Backwards invariant subspaces of L_2^+ ("behaviours") are described by differential equations without restrictions on the initial state. By imposing partial constraints on the initial state, the time-variant projections that play a role in the geometric theory of robust stabilization can be provided with a state-space description (cf. remark 2.3).

The we can rewrite the equations on the previous page in the form of state-space system that takes (y_3, u_3) as input and outputs the projection (y_2, u_2) on $\mathcal{B}_+(-\tilde{G}_1)$:

Lemma 2.1. *Let V be a forwards invariant subspace of $L_2^+(Y \times U)$, with finite-dimensional state-space representation $V = \mathcal{B}_+^0(\Sigma(A, B, C, D))$, where the system (A, B, C, D) is assumed to be minimal. Let V_1 be analogously represented by $\Sigma_1 := (A_1, B_1, C_1, D_1)$. Assume $I + DD_1^{\mathrm{T}}$ is nonsingular.*
Let

$$\mathcal{A} = \begin{bmatrix} A - BD_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1}C & -BB_1^{\mathrm{T}} + BD_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1}DB_1^{\mathrm{T}} \\ -C_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1}C & -A_1^{\mathrm{T}} + C_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1}DB_1^{\mathrm{T}} \end{bmatrix}$$

$$\mathcal{B} = \begin{bmatrix} BD_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1} & -BD_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1}D + B \\ C_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1} & -C_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1}D \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} -(I + DD_1^{\mathrm{T}})^{-1}C & (I + DD_1^{\mathrm{T}})^{-1}DB_1^{\mathrm{T}} \\ D_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1}C & B_1^{\mathrm{T}} - D_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1}DB_1^{\mathrm{T}} \end{bmatrix}$$

$$\mathcal{D} = \begin{bmatrix} (I + DD_1^{\mathrm{T}})^{-1} & -(I + DD_1^{\mathrm{T}})^{-1}D \\ -D_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1} & D_1^{\mathrm{T}}(I + DD_1^{\mathrm{T}})^{-1}D \end{bmatrix}$$

Furthermore assume that the stable subspace of $\mathcal{A}$ has intersection $\{0\}$ with the space $\{x \mid x_1 = 0\}$, where X is partitioned $X = X_1 \times X_2$ according to the division of the blocks of $\mathcal{A}$. Then

(i) The intersection $V \cap V_1^{\perp} = \{0\}$, so the skew projection $\Pi_{V_1^{\perp}}^{V}$ of $V + V_1^{\perp}$ along V on $V_1^{\perp}$ is well-defined.

(ii) $\Pi_{V_1^{\perp}}^{V}$ is given by the system of equations

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \mathcal{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathcal{B}w, \ x_1(0) = 0, \\ \Pi_{V_1^{\perp}}^{V} w &= \mathcal{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathcal{D}w, \\ w &= \begin{bmatrix} y(.) \\ u(.) \end{bmatrix} \in L_2^+, \\ x(.) &\in C([0, \infty), X). \end{aligned}$$

Furthermore, the system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is minimal.

Proof. The formulas are obtained from those on the previous page by straightforward manipulation. The equations given for the skew projection do not define a mapping if the system with input zero can give a nonzero output. This is the case iff there exists an initial condition x that has $x_1 = 0$ and is such that the autonomous evolution $x(.)$ of $\dot{x} = \mathcal{A}x$ with this initial condition gives an L_2 output trajectory $\mathcal{C}x(.)$. By minimality of $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ (to be shown below), this implies $x(.) \in L_2$, so the initial state x must be in the stable invariant subspace of $\mathcal{A}$.

So what remains to be shown is the minimality of the system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$. It is clear that the order of the system is the sum $n + n_1$ of the McMillan degrees of G and $-\tilde{G}_1$. We use a relation between skew projections and closed loop behaviours to see that the transfer function $\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$ also has degree $n + n_1$. For a feedback interconnection of a plant G and a controller K, put $P = \ker[I \quad -G], C = \ker[-K \quad I]$. The *closed loop transfer function* (cf. [Vidyasagar]) is defined by

$$H(P, C) = \begin{bmatrix} I & -G \\ -K & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} G \\ I \end{bmatrix} (I - KG)^{-1} [K \quad I].$$

It is easy to verify that

$$\Pi_P^C = \begin{bmatrix} G \\ I \end{bmatrix} (I - KG)^{-1} [-K \quad I], \; H(P, C) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \Pi_P^C \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix},$$

so the McMillan degrees of the matrix expressions for Π_P^C and $H(P, C)$ are the same. But the McMillan degree of $H(P, C)$ is equal to that of its inverse, which equals the degree of

$$\begin{bmatrix} 0 & -G \\ -K & 0 \end{bmatrix},$$

which is obviously equal to the sum of the degrees of G and K. Apply this argument to $K = -\tilde{G}_1$, which gives $\Pi_P^C = \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$. Note that the feedback connection of G and $-\tilde{G}_1$ is well-defined in Vidyasagar's sense by the assumption that $I + DD_1^T$ is nonsingular. □

Remark 2.2. For $V = V_1$, one gets the orthogonal projection on $V^\perp$. In this case the equations are of the type that can arise from a variational approach to optimal control, and this is of course not accidental. To solve an optimal control problem with cost criterion $\int \|u\|^2 + \|y\|^2$, we can go about as follows: among the external trajectories corresponding to the initial state x_0, the one with minimal norm is the orthogonal projection of the origin on the set of trajectories with initial condition x_0, so we must solve (for a strictly proper system)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A & -BB^T \\ -C^TC & -A^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; x_1(0) = x_0,$$

and the problem of finding the optimal state-space trajectory becomes the problem of finding x_2 such that (x_0, x_2) is in the stable invariant subspace $\mathcal{X}_-(\mathcal{A})$ of the Hamiltonian matrix $\mathcal{A}$. This is in turn the problem of writing

$$\mathcal{X}_-(\mathcal{A}) = \operatorname{im} \begin{bmatrix} I \\ X \end{bmatrix}.$$

For a similar derivation of a solution to a smoothing problem cf. [Weinert,Desai].

Remark 2.3. Along the same lines, one can obtain state-space formulas for operators of the form

$$\Pi_{\{\Theta H_2\}^\perp} \mathcal{M}_{\Theta_1}, \Theta \in H_\infty, \Theta_1 \in L_\infty.$$

Our first attempt to derive state-space formulas for the computation of the gap was based on the state-space expression of $\Pi_{\mathcal{G}(G)^\perp} \left[\begin{smallmatrix} I \\ G_1 \end{smallmatrix}\right]$. The formulas were not entirely satisfactory because the system obtained has higher McMillan degree than the one in (2.3) for non-strictly proper systems. However, it is suggested that the direct translation of this type of operator to state space equations may be useful in other contexts, as it plays an important role in the branch of operator theory that is connected with H_∞ optimization (cf. for instance [Nikol'skiĭ]).

3. Computing the parallel projection norm

Recall lemma 4.4 of chapter 3:

$$\sqrt{1 - \vec{\delta}(V, V_1)^2} = \sin \varphi(V, V_1^\perp) = \|\Pi^V_{V_1^\perp}\|^{-1}.$$

So we must compute the norm of Π_2 as defined in §1. Two special cases must be singled out:

(*i*): The input-output behaviour determined by the equations of lemma 2.1 is not a mapping. Then $V \cap V_1^\perp \neq \{0\}$, and the gap is 1. This occurs when the stable subspace of the $\mathcal{A}$-matrix of the system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of lemma 2.1 has nontrivial intersection with the space $\{x \mid x_1 = 0\}$.

(*ii*): The system is not minimal, or $I + DD_1^{\mathrm{T}}$ is singular. We have already shown that the system is minimal if $I + DD_1^{\mathrm{T}}$ is nonsingular. If it is singular the gap is 1:

Lemma 3.1. *Suppose $I + DD_1^{\mathrm{T}}$ is singular. Then $\delta(\mathcal{G}(G), \mathcal{G}(G_1)) = 1$.*

Proof. Using lemma 4.4 of chapter 3, it follows that if the mapping

$$\mathcal{P} := \Pi_{\mathcal{G}(G)}^{\mathcal{G}(-\tilde{G}_1)} = \begin{bmatrix} G \\ I \end{bmatrix} (I + \tilde{G}_1 G)^{-1} [\tilde{G}_1 \quad I]$$

is not bounded from L_2 to L_2, the L_2 gap between the behaviours of G and G_1 is 1. When $I + DD_1^{\mathrm{T}}$ is singular, $\mathcal{P}$ is not bounded, so the L_2-gap and the H_2-gap must both be 1 in this case. □

So we may assume $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is minimal. As is well-known (cf. for instance [Boyd et al.] for the computation of the H_∞ norm), computing the norm of an operator is related to solving optimal control problems with the cost criterion

$$\omega(y, u) = \int_0^\infty \gamma^2 \|u(t)\|^2 - \|y(t)\|^2 dt.$$

In this case, it is more convenient to state the solution directly in terms of the associated Hamiltonian matrix than to solve the problem in terms of Riccati equations. The Hamiltonian is

$$\mathcal{H}_\gamma = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & -\mathcal{A}^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \mathcal{B} & 0 \\ 0 & -\mathcal{C}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} -\mathcal{D} & \gamma I \\ \gamma I & -\mathcal{D}^{\mathrm{T}} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{C} & 0 \\ 0 & \mathcal{B}^{\mathrm{T}} \end{bmatrix}$$

Let W be the subspace $\{x \mid x_1 = 0\}$ of the state space X, and let $\mathcal{X}_\gamma$ be the stable subspace of $\mathcal{H}_\gamma$. We are now in a position to state the main result of this chapter. In the appendix we summarize its use for the computation of the gap by a γ-iteration procedure.

Proposition 3.2. $\|\Pi_2\| < \gamma \Leftrightarrow \mathcal{H}_\gamma$ *has no imaginary eigenvalues, there is a symmetric matrix K such that $\mathcal{X}_\gamma = \operatorname{im} \left[\begin{smallmatrix} I \\ K \end{smallmatrix}\right]$, and K is negative definite on W.*

Proof. Define the available storage $V_a(x_0)$ of a minimal state-space system Σ with respect to ω as

$$V_a(x_0) = \inf \left\{ \omega(u(.), y(.)) \mid \exists x(.) : (x(.), y(.), u(.)) \in \mathcal{B}_+(\Sigma) \wedge x(0) = x_0 \right\}.$$

Let the optimal cost $K^+(x) = -V_a(x)$. It is well-known how to compute the optimal cost from a Riccati equation (cf. for instance [Trentelman,Willems]): if K is such that $\mathcal{X}_\gamma = \operatorname{im} \left[\begin{smallmatrix} I \\ K \end{smallmatrix}\right]$, then $\langle Kx, x \rangle = K^+(x)$. First note that if such a K does not exist, then ([Boyd et al.]), the L_∞ norm of the system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is greater than γ, and we know by lemma III.4.4 that this also implies $\|\Pi_2\| > \gamma$. (In fact the infimal value of γ for which the Hamiltonian has no imaginary eigenvalues corresponds to the L_2-gap.)

By the definition of the system of lemma 2.1, it is clear that the optimal cost K is negative definite on the subspace $\{x \mid x_1 = 0\}$ iff the skew projection norm is less than γ. □

4. Criterion for the gap to be equal to one

After the computation of one directed gap, it must be checked whether the other directed gap is equal to it. It is known that the gap is 1 when the two directed gaps are not equal (chapter 1, lemma 1.1), so one needs to do just one step in the γ-iteration for the computation of $\|\Pi_{V^\perp}^{V_1}\|$ to verify this. We give yet another criterion.

Proposition 4.1. *Suppose V, V_1 are graphs of transfer functions with minimal state-space realizations (A, B, C, D) and (A_1, B_1, C_1, D_1) respectively. Then*

$$\vec{\delta}(V, V_1) < 1, \ \vec{\delta}(V, V_1) \neq \vec{\delta}(V_1, V)$$

implies that the stable subspace of the matrix

$$\mathcal{A}' = \begin{bmatrix} A_1 - B_1 D^{\mathrm{T}}(I + D_1 D^{\mathrm{T}})^{-1} C_1 & -B_1 B^{\mathrm{T}} + B_1 D^{\mathrm{T}}(I + D_1 D^{\mathrm{T}})^{-1} D_1 B^{\mathrm{T}} \\ -C^{\mathrm{T}}(I + D_1 D^{\mathrm{T}})^{-1} C_1 & -A^{\mathrm{T}} + C^{\mathrm{T}}(I + D_1 D^{\mathrm{T}})^{-1} D_1 B^{\mathrm{T}} \end{bmatrix}$$

has nontrivial intersection with the subspace $\{x \mid x_1 = 0\}$.

Proof. Combine (*i*) of lemma 2.1 with lemma 1.1 of chapter 1. □

Lemma 4.2. *Let $V = \mathcal{G}(G)$, $V_1 = \mathcal{G}(G_1)$. Suppose $\delta(\mathcal{B}(G), \mathcal{B}(G_1)) < 1$. Then $V + V_1^\perp = \mathcal{G}(G) + \mathcal{G}(G_1)^\perp$ is closed, and $\vec{\delta}(V, V_1) = 1 \Rightarrow V \cap V_1^\perp \neq \{0\}$.*

Proof. By lemma III.4.4 it follows from $\delta_{L_2}(\mathcal{B}(G), \mathcal{B}(G_1)) < 1$ that the mapping $\mathcal{P} = \Pi_{\mathcal{G}(G)}^{\mathcal{G}(-\tilde{G}_1)}$ is a multiplication operator with symbol in RL_∞. This implies that $\mathcal{G}(G) + \mathcal{G}(-\tilde{G}_1)$ is closed, as $\gamma(\mathcal{G}(G), \mathcal{G}(-\tilde{G}_1)) = \|\mathcal{P}\|^{-1}$ when the intersection is trivial (which it is in our case by the assumption $V \cap V_1^\perp = \{0\}$). But it follows from the relation $\mathcal{G}(F) = \mathcal{L}(B_+^0(F))$ that the codimension $[\mathcal{L}(\mathcal{B}_+(-\tilde{G}_1)) : \mathcal{G}(-\tilde{G}_1)] = [\mathcal{B}_+(-\tilde{G}_1)) : B_+^0(-\tilde{G}_1)]$ is finite (in fact it is equal to the McMillan degree of G_1). This implies that also $V_1^\perp + V = \mathcal{L}(\mathcal{B}_+(-\tilde{G}_1)) + \mathcal{G}(G)$ is closed. Hence $\gamma(V, V_1^\perp) > 0$ by chapter 1, lemma 1.6, so $V \cap V_1^\perp = \{0\}$ implies that $\vec{\delta}(V, V_1) < 1$. □

Lemma 4.3. *Let $V = \mathcal{G}(G)$, $V_1 = \mathcal{G}(G_1)$, let $\mathcal{A}$ and $\mathcal{A}'$ be as in lemmata 2.1 and 4.1 respectively. Then we have $\delta(V, V_1) = 1$ iff one of the following possibilities holds:*

(i) The stable space of $\mathcal{A}$ has nontrivial intersection with $\{x \mid x_1 = 0\}$.
(ii) The stable space of $\mathcal{A}'$ has nontrivial intersection with $\{x \mid x_1 = 0\}$.
(iii) $\mathcal{A}$ has eigenvalues on the imaginary axis.
(iv) $\mathcal{A}'$ has eigenvalues on the imaginary axis.
(v) $I + DD_1^{\mathrm{T}}$ is singular.

Proof. We proceed by elimination. It is clear that (*iii*) – (*v*) correspond to the L_2 gap being 1, and that (*i*), (*ii*) correspond respectively to $V \cap V_1^{\perp} \neq \{0\}$, $V_1 \cap V^{\perp} \neq \{0\}$. So all of (*i*) – (*v*) imply that the gap is 1. If none of (*iii*) – (*v*) holds, the mapping $\Pi^{\mathcal{G}(G)}_{\mathcal{G}(-\tilde{G}_1)}$ is bounded from L_2 to L_2, so by lemma 4.4 of chapter 3 we have $\delta(\mathcal{B}_{L_2}(G), \mathcal{B}_{L_2}(G_1)) < 1$. By the previous lemma this implies that if neither (*i*) nor (*ii*) holds, then $\delta(V, V_1) < 1$. □

Note that it suffices to calculate the eigenvalues of $\mathcal{A}$, as $-\lambda$ is an eigenvalue of $\mathcal{A}'$ for each $\lambda \in \sigma(\mathcal{A})$, and vice versa. In fact, we can simplify the calculations a bit further. Let $\mathcal{X}_-(A)$ be the maximal stable invariant subspace of a matrix, $\mathcal{X}_+(A)$ its antistable space.

Proposition 4.4. *We have $\delta(V, V_1) < 1$ iff $I + DD_1^{\mathrm{T}}$ is nonsingular, $\mathcal{A}$ has no eigenvalues on the imaginary axis, and $\mathcal{X}_-(\mathcal{A})$ is complementary to $\{x \mid x_1 = 0\}$.*

Proof. It is easily verified that $\mathcal{A}' = -\mathcal{A}^{\mathrm{T}}$. Now since for matrices without eigenvalues on the imaginary axis we have $\mathcal{X}_-(X^{\mathrm{T}}) = \mathcal{X}_+(X)^{\perp}$ it follows that

$$\mathcal{X}_-(\mathcal{A}') = \mathcal{X}_-(\mathcal{A})^{\perp}.$$

Hence $\mathcal{X}_-(\mathcal{A}') \cap \{x \mid x_2 = 0\} = \{0\} \Leftrightarrow \mathcal{X}_-(\mathcal{A})^{\perp} \cap \{x \mid x_2 = 0\} = \{0\} \Leftrightarrow \mathcal{X}_-(\mathcal{A}) + \{x \mid x_1 = 0\}$ is the whole space. □

An alternative to the gap that has the advantage of giving a less conservative estimate of robustness is the following, proposed by [Vinnicombe]. For completeness we sketch the relation with the formulas of this section.

Definition 4.5. The *Vinnicombe gap* δ_V between $V = \mathcal{G}(G)$ and $V_1 = \mathcal{G}(G_1)$ is given by the following definition: If $\dim V \cap V_1^{\perp} = \dim V_1 \cap V^{\perp}$ then $\delta_V(V, V_1) = \delta_{L_2}(V, V_1)$; else $\delta_V(V, V_1) = 1$.

It can be shown that this distance measure is a metric equivalent to the gap, and that the robustness margin for the two metrics is identical. The next proposition shows how it can be computed in terms of the formulas of this chapter.

Proposition 4.6. *We have* $\delta_V(V, V_1) = 1$ *iff*
(*i*) $\delta_{L_2}(V, V_1) = 1$ *or*
(*ii*) $\dim \mathcal{X}_-(\mathcal{A}) \cap \{x \mid x_1 = 0\} \neq \operatorname{codim} \mathcal{X}_-(\mathcal{A}) + \{x \mid x_1 = 0\}$

Proof. Using the proof of the previous proposition, we can see that it is sufficient to obtain

$$\dim V \cap V_1^{\perp} = \dim \mathcal{X}_-(\mathcal{A}) \cap \{x \mid x_1 = 0\}.$$

This however a consequence of lemma 2.1, as the space of stable outputs with input 0 of the system of equations given there has dimension equal to $\dim \mathcal{X}_-(\mathcal{A}) \cap \{x \mid x_1 = 0\}$. □

So the Vinnicombe gap is somewhat easier to compute than the standard one, as the calculation consists of one test for the position of $\mathcal{X}_-(\mathcal{A})$ and the computation of an L_∞ norm, for which fast optimized routines are available.

5. Computations

On the last page of this chapter we summarize the computation of the gap with accuracy ε in the form of a pseudo-Algol procedure. By $\mathcal{H}_\gamma \in \mathrm{Dom}(\mathit{Ric})$ it is meant that the stable invariant subspace of $\mathcal{H}_\gamma$ can be written as im $\left[\begin{smallmatrix} I \\ K \end{smallmatrix}\right]$, and in this case $X = \mathit{Ric}(\mathcal{H}_\gamma)$.

The test for definiteness is likely to be numerically unreliable when X is close to being singular, i.e. when we are close to the L_2 gap. We briefly compare this computation to the one that follows from [Georgiou]'s reduction to a model matching problem. To do a computation in Georgiou's approach one needs to solve a Riccati equation in order to obtain the normalized coprime factorization, and then the resulting model matching problem can be solved using state-space formulas. An implementation of such a procedure is available for instance in the matlab package "mu-tools". The amount of work that needs to be done to solve a model matching problem in such a way is comparable to the running time of our procedure.

We have three sources for differences in efficiency:

(1) We have a more efficient test for $\delta = 1$ that involves calculating the stable subspace of a matrix of dimension $n + n_1$ only (where n, n_1 are the degrees of the two original systems) and not of the full size $2(n + n_1)$ of the Hamiltonian used in the iteration, and we do not need any normalized or other factorizations.

(2) The normalized coprime factorization does not have to be performed. A normalized coprime factorization is less expensive than one step in the typical

bisection procedure that is commonly used to solve H_∞ optimization problems; so on the whole this makes for some difference in efficiency in cases when the gap turns out to be 1, or no high accuracy is required and not many iteration steps are performed.

(3) The problem $X = Ric(\mathcal{H})$ that has to be solved in both procedures is of the same dimension, but we only have to test the definiteness of one block of X.

We have compared a matlab procedure of our own to the one in mutools. For fairness, our procedure relies on the same mutools routine *ric_schr*. Sample systems $[A, B, C, D]$, $[A_1, B_1, C_1, D_1]$ were chosen according to the following procedure (not that it matters much): First, A, B, C, D were chosen randomly with parameters between 0 and 1. Then, a random weight $p \in [0, 1]$ was chosen, and a random perturbation $\Delta A, \Delta B, \Delta C, \Delta D$, again with parameters in $[0, 1]$. Our A_1, B_1, C_1, D_1 were then $A + p\Delta A, B + p\Delta B, C + p\Delta C, D + p\Delta D$. On a sample of a hundred randomly chosen gap problems between systems of order 3, with accuracy 1 / 100 we had the following results:

Mutools: Average number of floating point operations: 737350. The gap was computed as 1 twenty-five times; the number of operations needed to find out about this was on the average 197656. With gap not equal to one mutools needed on the average 917248 flops.

Our own routine: It needed on the average 590422 operations. It decided 16 times that the gap was equal to one, and needed on the average 67184 flops to find out. The average number of operations needed for gaps not equal to 1 was 690086.

So we may conclude there is some increase in efficiency, though it is not quite sensational.

real $\delta(A, B, C, D, A_1, B_1, C_1, D_1, \varepsilon)$ /* *Assumes minimality of both systems* */
begin
    if Singular($I + DD_1^{\mathrm{T}}$) **then return** 1 **fi**
    $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$:= SkewProjection($A, B, C, D, A_1, B_1, C_1, D_1$)
        /* *Form the system of lemma 2.1* */
    $n := \dim A$
    $m := \dim A_1$
    $N := n + m$

    /* *First check whether* $\delta = 1$ *as in section 4* */

    Perform an ordered (complex) Schur decomposition $\mathcal{A} = UTU^*$
    i := 1
    while $i \leq N \wedge \operatorname{Re} T_{ii} < 0$ **do** $i := i + 1$ **od**
    if $i \neq m + 1 \vee \operatorname{Re} T_{ii} = 0 \vee$ Singular($U_{1:m,1:m}$) **return** 1 **fi**

    /* *Now do the* γ*-iteration of section 3* */

    $\delta_- := \sqrt{1 - \frac{1}{\|\mathcal{D}\|^2}}$
    $\delta_+ := 1$
    Error := $\delta_+ - \delta_-$
    $\delta := 1$
    while Error > ε **do**
        $\delta := (\delta_- + \delta_+)/2$
        $\gamma := \frac{1}{\sqrt{1-\delta^2}}$
        $\mathcal{H}_\gamma := \begin{bmatrix} \mathcal{A} & 0 \\ 0 & -\mathcal{A}^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \mathcal{B} & 0 \\ 0 & -\mathcal{C}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} -\mathcal{D} & \gamma I \\ \gamma I & -\mathcal{D}^{\mathrm{T}} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{C} & 0 \\ 0 & \mathcal{B}^{\mathrm{T}} \end{bmatrix}$
        if $\mathcal{H}_\gamma \in \mathrm{Dom}(Ric)$ **then**
            SolutionExists := **true**
            $X := Ric(\mathcal{H}_\gamma)$
            SolutionDefinite := NegativeDefinite($X_{m+1:n,m+1:n}$)
        else
            SolutionDefinite := SolutionExists := **false**
        fi
        if SolutionExists $\wedge$ SolutionDefinite **then** $\delta_+ := \delta$ **else** $\delta_- := \delta$ **fi**
        Error := $\delta_+ - \delta_-$
    od
    return δ
end

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Samenvatting

Dit proefschrift is het resultaat van de afgelopen vier jaar in het kader van het project *Control computations for element models* op het Centrum voor Wiskunde en Informatica bij de afdeling Besliskunde, Statistiek en Systeemtheorie verricht onderzoek naar robuuste regeling van systemen beschreven door gewone lineaire differentiaal- en differentievergelijkingen met constante coëfficienten.

Onder robuuste regeling verstaat men het ontwerpen van een regelaar voor een systeem waarvan sommige parameters onbekend zijn, of dat onderhevig is aan storingen. Een hulpmiddel om de robuustheid van een regeling te beoordelen is het introduceren van een afstandsmaat (metriek) op de verzameling lineaire systemen. Men kan dan zeggen dat een regeling ontworpen voor een bepaald systeem $\mathcal{P}$ robuust met een marge $\geq \rho$ is wanneer hij alle systemen stabiliseert die afstand kleiner dan ρ tot $\mathcal{P}$ hebben.

Hier wordt een model van deze onzekerheid gehanteerd dat duidelijk meetkundig van aard is. De gebruikte afstandsmaat is de sinus van de hoek tussen de oplossingsverzamelingen van de twee systemen opgevat als deelruimtes van de ruimte van kwadratisch integreerbare functies. Het proefschrift bestudeert de door deze afstandsmaat, de zogenaamde *gap*, geïnduceerde topologie.

Hoofdstuk 1 geeft wat achtergrond over de meetkunde van de hoeken tussen deelruimtes van een Hilbertruimte. Er wordt een meetkundige interpretatie van stabilisatie gegeven in termen van complementariteit van de oplossingsruimten van de stelsels vergelijkingen die systeem en regelaar beschrijven. Voorts worden enige ongelijkheden gegeven die betrekking hebben op de continuiteit van elementaire operaties op de verzameling gesloten deelruimten van een Hilbertruimte, zoals lineaire som, doorsnede en orthogonale complementatie. Verder geven we een simpele afleiding van de meetkundige relatie tussen een lineair systeem Σ en zijn gadjungeerde systeem $\tilde{\Sigma}$, dat het orthogonale complement van de oplossingsruimte van Σ beschrijft.

Hoofdstuk 2 behandelt een aspect van het realisatieprobleem voor lineaire systemen, dat wil zeggen het verkrijgen van een beschrijving in toestandsruimtevorm van een systeem dat gedefinieerd is als een deelruimte van een functieruimte. Er wordt met systeemtheoretische technieken een tijdsdomein equivalent van de bekende representatiestelling van Beurling-Lax bewezen. In plaats van het Beurling-symbool van een schuifinvariante ruimte construeren we direct een zogenaamde shiftrealisatie hiervan in continue tijd.

Hoofdstuk 3 geeft verbanden aan met andere manieren om topologieën te definieren op de verzameling lineaire systemen. We laten zien dat de gaptopologie op diverse natuurlijke manieren naar voren komt, en geven relaties aan met de topologie gegeven door de L_2-geïnduceerde operatornorm op de verzameling transferfuncties en de topologie van toestandsruimteparameters. De bestudeerde topologieën kunnen alle worden geinterpreteerd als topologie van uniforme convergentie van analytische functies van de Riemannbol naar een Grassmannvariëteit van deelruimten van een euclidische ruimte $\mathbf{C}^n$. Om het verband met de toestandsruimteparameters te leggen gebruiken we de realisatieprocedure uit hoofdstuk 2, waarvan we eenvoudig kunnen inzien dat hij continue parameters oplevert ten opzichte van de topologie van uniforme convergentie op de hele Riemannbol $\mathbf{C}_\infty$.

Hoofdstuk 4 behandelt de vraag wanneer een "gestructureerde" perturbatie (een perturbatie die bepaald wordt door variaties van parameters in de vergelijkingen van een model) continu is in de gaptopologie. We zijn in staat een betrekkelijk eenvoudige regel te geven die deze vraag beantwoordt.

Hoofdstuk 5 tenslotte behandelt het berekenen van de gap. We geven een procedure die gebaseerd is op een expressie voor de gap als maximum van twee operatornormen, en een simpel criterium voor het optreden van de veel voorkomende situatie $\delta(V_1, V_2) = 1$ voor twee systemen V_1 en V_2. De berekeningsmethode is gebaseerd op het afleiden van toestandsruimterepresentaties voor diverse projectieoperatoren, waarbij we gebruik maken van de karakteristiek van het geadjungeerde systeem uit hoofdstuk 2.

List of symbolic abbreviations

$\mathcal{SUB}(X)$	**3**
Π_X	**3**
$\delta(X, Y)$	**3**
$\vec{\delta}(X, Y)$	**3**
$\vartheta(X, Y)$	**4**
$\mathcal{B}(X, Y)$	**4**
$\mathcal{G}(A)$	**4**
$\varphi(V, W)$	**4**
$\sin\varphi(V, W)$	**4**
$\gamma(U, V)$	**5**
$\psi(U, V)$	**5**
$\mathrm{nul}(U, V)$	**5**
$\mathrm{def}(U, V)$	**5**
Π_W^V	**6**
$\tau(A)$	**9**
$L_2(W), L_2^+(W), L_2^-(W)$	**11**
$H_2^+(W), H_2^-(W)$	**11**
$\mathcal{L}$	**11**
σ_d	**11**
τ_d	**11**
$\mathcal{B}_W(\Sigma)$	**12**
Π_+	**13**
V_+	**13**
V_+^0	**13**
$\mathcal{B}(G)$	**13**
$\mathcal{M}_\Theta$	**14**
$H(P, C)$	**14**
$\mathcal{O}_{\mathrm{robust}}$	**15**
$\mathcal{O}_\Pi$	**15**
$\rho(P, C)$	**16**
$\mathcal{H}_1$	**19**
$a \wedge b$	**20**
$\mathcal{S}_{\mathrm{fd}}$	**28**

$x_t(w)$, $\dot{x}_t(w)$ **28**
$x^-(w)$ **29**
$x^+(w)$ **29**
V_e **31**
$\delta_{H_2}(U, V)$ **37**
$\mathcal{O}_{L_2^-}$ **37**
$\delta_-(U, V)$ **37**
$Grass(m, q)$ **37**
$\mathcal{R}(\Omega, m, q)$ **37**
$\delta_{\sup}$ **38**
δ_L, δ_{L^+}, δ_{L^-}, δ_∞ **41**
$\mathcal{S}_n$, $\mathcal{S}_{\leq n}$, $\mathcal{S}_n^+$, $\mathcal{S}_{\leq n}^+$ **44**
$\mathcal{X}_-(A)$, $\mathcal{X}_+(A)$ **66**
δ_V **66**

Index

aperture, 4
AR system, 49
backward shift, 12
closed loop transfer function, 15
complementary subspaces, 6
directed gap, 3
forward shift, 12
Fredholm indices, 5
future Gramian, 30
gap topology, 3
gap, 3
general position, 8
graph topology, 38
Grassmannian manifold, 37
Hermann-Martin mapping, 38
internal stability, 14
l'Hôpital's rule, 49
leading term matrix, 50
maximal angle, 4
minimal angle, 5
minimal gap, 5
opening, 4
parallel projection, 7
Plücker embedding, 52
pointwise gap topology, 38
rational mapping, 38
reduced minimum modulus, 10
shift operator, 12
singular perturbation, 55
skew projection, 7
stability margin, 17
Vinnicombe gap, 66