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ON THE EDGE-COLOURING PROBLEM FOR UNIONS OF COMPLETE UNIFORM HYPERGRAPHS

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For given $n \in \mathbb{N}$ and $H \subset \{1, 2, \ldots, n\}$ we investigate whether the collection of subsets $A \subset \{1, 2, \ldots, n\}$ with $|A| \in H$ possesses a parallelism (1-factorization). A complete solution for the case $H = \{1, 2, \ldots, h\}$ is given.

0. Introduction

Let x be some fixed set of n elements. For $H \subset \{1, 2, \ldots, n\}$ let K_n^H denote the hypergraph (x, E) with vertex set x and collection of edges

$$E = \{y \subset x \mid |y| \in H\}.$$

(Since we never need symbols for the vertices of a hypergraph, but do use collections of collections of edges, we denote sets of vertices by lower case symbols, sets of edges by capitals and collections of sets of edges by upper case script letters.)

When H is non empty its largest element is denoted by h. When $H = \{h\}$ we write K_n^h instead of $K_n^{\{h\}}$, the complete h-uniform hypergraph on n vertices. When $H = \{1, 2, \ldots, h\}$ then, following Berge and Johnson, we write $\hat{K}_n^h$ for K_n^H, the hereditary closure of the complete h-uniform hypergraph on n vertices.

Baranyai [1] proved that K_n^h has a 1-factorization if and only if $h \mid n$. Bermond, Berge & Johnson [2, 5] then considered the case of $\hat{K}_n^h$ which they solved for $h \leqslant 4$ and in several other special cases. Our main result is

Theorem 1. *$\hat{K}_n^h$ possesses a 1-factorization exactly in the following cases:*

(i) *$n \leqslant 2h$ and $\hat{K}_n^{n-h-1}$ is 1-factorizable (or $n-h-1 \leqslant 0$).*

(ii) *$n = kh + l$, $k \geqslant 2$, $-1 \leqslant l \leqslant h-2$ and*

(iia) *$l = 0$ and $k \geqslant h-2$,*

or (iib) *$l = -1$ and $k \geqslant \frac{1}{2}h - 1$.*

[Note that in case $n = 2h$ or $n = 2h-1$ the conditions given under (i) and (ii) agree.]

In the course of proving this theorem we derive many results for general H. (Theorems 2 and 3 reduce the problem to the cases $n = kh$ and $n = kh-1$ with $k \geq 3$. Now we have:

(A) Let $n = kh$ with $k \geq 3$ and $h-1 \in H$. Then K_n^H is 1-factorizable if and only if either $k \geq h-1$ or ($k = h-2$ and $h-2 \in H$).

(B) Let $n = kh-1$ with $k \geq 3$ and $h-2 \in H$. Then K_n^H is 1-factorizable if and only if $h-1 \in H$ and $k \geq \frac{1}{2}h-1$ and $m(H)$ is odd (where the notation $m(H)$ is explained below).

All the existence results follow from Baranyai's theorem, which roughly says that a 1-factorization exists if (and only if) the numbers fit. For precise statements of this theorem see [1], [3] or [4]. It implies that a 1-factorization exists if and only if there exist non-negative numbers c_{ij} with $i = 1, \ldots, t$ and $j \in H$ such that $\sum_{i=1}^{t} c_{ij} = \binom{n}{j}$ for $j \in H$ and $\sum_{j \in H} jc_{ij} = n$ for $i = 1, \ldots, t$. (For this application of the theorem see [2], [3] or [5].)

For example $\hat{K}_8^3$ is 1-factorizable, since $\binom{8}{1} = 8$, $\binom{8}{2} = 28$, $\binom{8}{3} = 56$ and one can realize a 1-factorization consisting of one 1-factor containing the eight singletons, and 28 1-factors each containing two triples and a pair.

On the other hand, as was remarked by R.M. Wilson, $\hat{K}_7^3$ is not 1-factorizable since $\binom{7}{1} = 7$, $\binom{7}{2} = 21$, $\binom{7}{3} = 35$ and any 1-factor not containing a singleton must contain two pairs and a triple, so that there are at most 10 triples in a 1-factor without singleton and at most 14 triples in a 1-factor with singleton, which leaves 11 triples not in any 1-factor.

All the non-existence results are proved with a similar argument: Assume that there exists a 1-factorization $\mathcal{F}$ of K_n^H. For each 1-factor $P \in \mathcal{F}$ let

$$n(g) := n_P(g) := |\{a \in P \mid |a| = g\}|.$$

Suppose that $G \subset H$ and that for each $P \in \mathcal{F}$ we have

$$\sum_{i \in H} \alpha_i n(i) \geq \sum_{j \in G} n(j).$$

Then it follows that

$$\sum_{i \in H} \alpha_i \binom{n}{i} \geq \sum_{j \in G} \binom{n}{j}.$$

Using inequalities on binomial coefficients we then derive a contradiction. [E.g. in the above example $n = 7$, $h = 3$ we have $n(3) \leq 2n(1) + \frac{1}{2}n(2)$, hence $35 = \binom{7}{3} \leq 2\binom{7}{1} + \frac{1}{2}\binom{7}{2} = 24\frac{1}{2}$, a contradiction.]

1. Auxiliary lemma's

Let g, h, k, n be positive integers with $k \geq 2$.

Lemma 1. *Let $n \geq kh-1$ and $g \leq h$. Then*

$$\binom{n}{g} \leq (k-1)^{g-h}\binom{n}{h}.$$

Proof. Use induction on $h-g$. If $h=g$ the lemma is true. Next

$$\binom{n}{g-1} = \frac{g}{n-g+1}\binom{n}{g} \leq \frac{h}{n-h+1}\binom{n}{g} \leq \frac{1}{k-1}\binom{n}{g}$$

provides the induction step. □

Lemma 2. *Let $n \geq 2h$. Then*

$$\sum_{i=0}^{h-1}\binom{n}{i} < \frac{h}{n-2h+1}\binom{n}{h}.$$

Proof. As before we find

$$\binom{n}{h-j} \leq \left(\frac{h}{n-h+1}\right)^j \binom{n}{h},$$

so that

$$\sum_{i=0}^{h-1}\binom{n}{i} < \sum_{j=1}^{\infty}\left(\frac{h}{n-h+1}\right)^j\binom{n}{h} = \frac{h}{n-2h+1}\binom{n}{h}. \quad \square$$

We shall use these lemma's throughout the sequel without explicit reference. In Section 5 we shall use the following observations:

Lemma 3. *Let p, q, m be positive integers with $(p, q) \mid m$. If*

$$\frac{m}{(p,q)} \geq \left(\frac{p}{(p,q)}-1\right)\left(\frac{q}{(p,q)}-1\right)$$

then there exist nonnegative integers a, b such that $m = ap + bq$.

Proof. Easy exercise. □

Lemma 3a. *Let p, q, m be integers with $p<0<q$ and $(p, q) \mid m$. There exist nonnegative integers a, b such that $m = ap + bq$, and one may choose them in such a way that when $m \geq 0$, then*

$$a \leq \frac{q}{(p,q)} - 1 \quad \text{and} \quad b \leq \frac{m+p}{q} - \frac{p}{(p,q)},$$

and when $m \leq 0$, then

$$a \leq \frac{m+q}{q} + \frac{q}{(p,q)} \quad \text{and} \quad b \leq \frac{-p}{(p,q)} - 1.$$

Consequently we can always obtain

$$a+b\leq\max\left(\frac{m+p}{q},\frac{m+q}{p}\right)+q-p-1.$$

Proof. Trivial. □

2. The reflection principle

Theorem 2. *Let $n\leq 2h$. Then K_n^H has a 1-factorization if and only if ($h=n$ or $n-h\in H$) and $K_n^{H\setminus\{h,n-h\}}$ is 1-factorizable.*

Proof. First assume $n-h\in H$. Given a 1-factorization of $K_n^{H\setminus\{h,n-h\}}$ we obtain a 1-factorization of K_n^H by adding the 1-factors $\{a,x\setminus a\}$ for all $a\subset x$, $|a|=h$. Conversely, if there exists a 1-factorization of K_n^H then suppose that $\mathcal{F}$ is one such with the maximum number of 1-factors of the form $\{a,x\setminus a\}$, $|a|=h$. Let $a\subset x$, $|a|=h$. If the 1-factor P_1 containing a is not $\{a,x\setminus a\}$ but, say, $\{a,a_1,\ldots,a_r\}$ and the 1-factor P_2 containing $x\setminus a$ is $\{x\setminus a,b_1,\ldots,b_s\}$, then

$$(\mathcal{F}\setminus\{P_1,P_2\})\cup\{\{a,x\setminus a\},\{a_1,\ldots,a_r,b_1,\ldots,b_s\}\}$$

is a 1-factorization containing one more pair of complementary sets, a contradiction. Hence $\mathcal{F}$ contains all complementary pairs $\{a,x\setminus a\}$ for $a\subset x$, $|a|=h$, and removing these yields a 1-factorization of $K_n^{H\setminus\{h,n-h\}}$.

If $h=n$ one passes back and forth between 1-factorizations of K_n^H and $K_n^{H\setminus\{n\}}$ by adding or removing the 1-factor $\{x\}$.

Finally, suppose $0\neq n-h\notin H$, and assume that K_n^H has a 1-factorization $\mathcal{F}$. Each 1-factor in $\mathcal{F}$ containing an h-set contains at least two small sets, hence at least one set of cardinality at most $\frac{1}{2}(n-h)$. Therefore

$$\sum_{i\leq(n-h)/2}\binom{n}{i}\geq\binom{n}{h},$$

but, writing $g=n-h$, we have for $g\geq 2$

$$\sum_{i\leq g/2}\binom{n}{i}<\frac{\frac{1}{2}g+1}{n-g-1}\binom{n}{\lfloor\frac{1}{2}g+1\rfloor}\leq\binom{n}{\lfloor\frac{1}{2}g+1\rfloor}\leq\binom{n}{g}=\binom{n}{h}$$

(where $\lfloor\alpha\rfloor$ denotes the integral part of α; note that $g<n-g$), while of course

$$\sum_{i\leq g/2}\binom{n}{i}<\binom{n}{h}$$

is true also for $g=1$. Contradiction. □

Corollary. *Let $n\leq 2h$. Then $\hat{K}_n^h$ is 1-factorizable if and only if $\hat{K}_n^{n-h-1}$ is.*

This proves case (i) of Theorem 1. From now on we shall often tacitly assume $n>2h$.

3. Nonexistence for $l \geq 1$

Theorem 3. *Let $n = kh + l$, $k \geq 2$, $1 \leq l \leq h-2$. Then K_n^H is not 1-factorizable.*

Proof. Suppose K_n^H has a 1-factorization. Each partition (1-factor) contains at least one l-set or at least two other small sets (with size $<h$), and at most k h-sets, i.e.,

$$\binom{n}{h} \leq \tfrac{1}{2}k \sum_{i=1}^{h-1} \binom{n}{i} + \tfrac{1}{2}k \binom{n}{l}.$$

Using Lemmas 1 and 2 we find

$$1 < \tfrac{1}{2}k \left(\frac{1}{k-2} + \frac{1}{(k-1)^2} \right)$$

so that $k \leq 4$. In fact we found

$$1 < \tfrac{1}{2}k \left(\frac{1}{k-2+(l+1)/h} + \frac{1}{(k-1)^{h-l}} \right)$$

and since $h \geq l+2 \geq 3$ this is a contradiction also for $k=4$. (One gets $3^{h-l} < 2 + 4h/(l+1) < 4 + 2(h-l)$.)

Now let $n = 3h + l$. Note that each partition containing an h-set also contains a set of cardinality at most $\frac{1}{3}(2h+l)$, while if it contains two h-sets it must contain a set of cardinality at most $\frac{1}{2}(h+l)$, and finally if it contains three h-sets and not two smaller sets then it is $n = 3 * h + 1 * l$. Hence

$$\binom{n}{h} \leq \binom{n}{l} + \sum_{i \leq (h+l)/2} \binom{n}{i} + \sum_{i \leq (2h+l)/3} \binom{n}{i}.$$

For $l \leq h-4$ we find (estimating both sums by $\sum_{i \leq h-2} \binom{n}{i}$):

$$1 \leq \frac{1}{2^{h-l}} + 2 \cdot \frac{h-1}{h+l+3} \cdot \frac{1}{2}.$$

which is a contradiction.

For $n = 4h-3$ we find (in the same way, this time dividing by $\binom{n}{h-1}$):

$$\frac{3h-2}{h} \leq \frac{1}{4} + 2 \cdot \frac{h-1}{2h} + 1,$$

also a contradiction.

For $n = 4h-2$ we obtain (examining the possible partitions of n)

$$\binom{n}{h} \leq 2 \sum_{i \leq h-2} \binom{n}{i} + \binom{n}{h-2} + \binom{n}{h-1}$$

from which it follows that

$$\frac{3h-1}{h} \leq 2 \cdot \frac{h-1}{2h+1} + \frac{h-1}{3h} + 1,$$

again a contradiction.

The only remaining case is $k=2$. But from now familiar considerations it follows that

$$\binom{n}{h} \leq \sum_{i<(h+l)/2} \binom{n}{i} + \binom{n}{l} + \frac{1}{2}\binom{n}{\frac{1}{2}(h+l)}$$

where the last term is absent when $h+l$ is odd. Subsequently we obtain the following inequalities:

$$\binom{n}{h} < \frac{\frac{1}{2}(h+l+1)}{n-(h+l+1)+1}\binom{n}{\lfloor\frac{1}{2}(h+l+1)\rfloor} + \binom{n}{l} + \frac{1}{2}\binom{n}{\lfloor\frac{1}{2}(h+l)\rfloor},$$

$$\frac{n-h+1}{h}\binom{n}{h-1} < \frac{h+l+1}{2h}\binom{n}{h-1} + \binom{n}{l} + \frac{1}{2}\binom{n}{h-1},$$

$$\frac{l+1}{2h}\binom{n}{h-1} < \binom{n}{l} = \frac{l+1}{2h}\binom{n}{l+1},$$

$$h-1 < l+1,$$

which gives a contradiction. □

By the remark at the end of the previous section, and this theorem, we may and shall assume henceforth that $n=kh+l$ where $k \geq 3$ and $l=0$ or $l=-1$.

4. A few other nonexistence results

Theorem 4. *Let $n=kh$, $3 \leq k \leq h-3$. Then if K_n^H is 1-factorizable, $(h-1) \notin H$.*

Proof. Assume K_n^H is 1-factorizable. A partition containing $(h-1)$-sets contains either a g-set with $g \leq k$ or at least two g-sets with $k+1 \leq g \leq h-2$ (for: $n = kh = k(h-1)+k$). Hence

$$\binom{n}{h-1} \leq \frac{k}{2}\sum_{i \leq k}\binom{n}{i} + \frac{k}{2}\sum_{i \leq h-2}\binom{n}{i},$$

so

$$1 \leq \frac{k}{2} \cdot \frac{k+1}{n-2k-1} \cdot (k-1)^{-(h-k-2)} + \frac{k}{2} \cdot \frac{h-1}{n-2h+3}, \tag{1}$$

and, since $h-2 \geq k+1$:

$$1 \leq \frac{k}{2} \cdot \frac{k+1}{(k+1)k-1} \cdot \frac{1}{k-1} + \frac{k}{2} \cdot \frac{1}{k-2},$$

a contradiction for $k \geq 5$. Returning to (1) we find for $k=4$:

$$1 \leq \frac{10}{n-9} \cdot \frac{1}{3} + 2 \cdot \frac{h-1}{2h+3},$$

$$\frac{1}{2h+3} \leq \frac{2}{3(n-9)} = \frac{2}{12h-27},$$

$$h \leq 4,$$

a contradiction. Hence $k=3$. Again examining the possible partitions of n we find

$$\binom{n}{h-1} \leq \sum_{i \leq h-2} \binom{n}{i} + \binom{n}{2} + 2\binom{n}{3} \leq \frac{h-1}{n-2h+3}\binom{n}{h-1} + 3\binom{n}{3},$$

hence, since $h \geq 6$,

$$\frac{4}{h+3}\binom{n}{5} \leq \frac{4}{h+3}\binom{n}{h-1} \leq 3\binom{n}{3} = \frac{3 \cdot 4 \cdot 5}{(n-3)(n-4)}\binom{n}{5},$$

$$(h-1)(3h-4) \leq 5(h+3),$$

$$h < 5,$$

a contradiction. □

Just the opposite conclusion holds when $n = kh-1$:

Theorem 5. *Let $n = kh-1$, $k \geq 3$. If K_n^H is 1-factorizable, then $(h-1) \in H$.*

Proof. Suppose $h-1 \notin H$. Each partition containing an h-set must also contain smaller sets, hence

$$\frac{1}{k-1}\binom{n}{h} \leq \sum_{i \leq h-2} \binom{n}{i} < \frac{h-1}{n-2h+3}\binom{n}{h-1} < \binom{n}{h-1} \leq \frac{1}{k-1}\binom{n}{h},$$

a contradiction. □

More generally we have (for $k \geq 3$):

Lemma 4. *Let $\mathcal{F}$ be any 1-factorization of K_n^H. For each $g \in H$ there are partitions in $\mathcal{F}$ not containing any f-set for $f < g-1$.*

Proof. Assume the contrary, then there is some element $h \in H$ (where for the duration of this proof we drop the convention that $h = \max H$) such that any partition containing an h-set also contains sets of size at most $h-2$. Write $n = kh + l$ with $-1 \leq l \leq h-2$ and $k \geq 3$. If $l = -1$, then the proof of the previous theorem produces a contradiction. Hence $0 \leq l \leq h-2$. By assumption each partition containing an h-set also contains a 'small' set, and if it contains k h-sets and only one 'small' set, the latter must have size l. Hence

$$\binom{n}{h} \leqslant (k-1) \sum_{i \leqslant h-2} \binom{n}{i} + \binom{n}{l} < \frac{h-1}{n-2h+3} \binom{n}{h} + \binom{n}{l},$$

$$\frac{n-3h+4}{n-2h+3} \binom{n}{h} < \binom{n}{l} = \frac{l+1}{kh} \binom{n}{l+1} < \frac{l+1}{kh} \binom{n}{h},$$

$$\frac{l+4}{n-2h+3} < \frac{l+1}{kh} < \frac{l+4}{kh},$$

a contradiction. □

For $k = h-2$ the conclusion of Theorem 4 no longer holds. But we can say the following:

Theorem 6. *Let $n = (h-2)h$, and suppose that K_n^H is 1-factorizable. Then if $(h-1) \in H$, also $(h-2) \in H$.*

Proof. For any partition $n = a * h + b * (h-1)$ we have $b \equiv 0 \pmod h$, and, since $n < h(h-1)$, $b = 0$. Therefore any partition containing $(h-1)$-sets also contains smaller sets. Now the conclusion follows from Lemma 4. □

Theorem 7. *Let $n = kh-1$, $3 \leqslant k \leqslant \frac{1}{2}(h-3)$. Then if K_n^H is 1-factorizable $(h-2) \notin H$.*

Proof. Suppose K_n^H is 1-factorizable, and $(h-2) \in H$. Consider the partitions containing $(h-2)$-sets. Since

$$n = k(h-2) + 2k - 1 \leqslant (k+1)(h-2) - 2 \tag{2}$$

the number of $(h-2)$-sets in such a partition is at most k. Moreover, such a partition cannot contain only h-, $(h-1)$- or $(h-2)$-sets since

$$n = ah + b(h-1) + c(h-2)$$

implies

$$b + 2c \equiv 1 \pmod h;$$

but by (2) $a+b+c \leqslant k$, so that $b+2c \leqslant 2k < h$, and it follows that $b+2c = 1$ and so $c = 0$, i.e., the partition did not contain any $(h-2)$-sets.

Likewise for $2k \leqslant g < h-2$ a partition $n = a * h + b * (h-1) + c * (h-2) + g$ is impossible. (Again we find $b + 2c \equiv g+1 \pmod h$ and $b + 2c < h$, so $b + 2c = g+$

$1 \geq 2k+1$; but by (2) $b+c \leq k$, which is impossible.) Hence

$$\binom{n}{h-2} \leq \frac{k}{2} \sum_{i \leq 2k-1} \binom{n}{i} + \frac{k}{2} \sum_{i \leq h-4} \binom{n}{i} + \frac{k-1}{2} \binom{n}{h-3}.$$

For $k \geq 5$ we find

$$\binom{n}{h-2} \leq k \sum_{i \leq h-4} \binom{n}{i} + \frac{k-1}{2} \binom{n}{n-3} < \left(\frac{k}{k-2} + \frac{k-1}{2}\right) \cdot \frac{1}{k-1} \cdot \binom{n}{h-2},$$

a contradiction.

For $k=4$ the above inequality implies

$$\frac{3h+2}{h-2} \binom{n}{h-3} = \binom{n}{h-2} \leq \frac{4}{h-4} \binom{n}{8} + \left(\frac{h-3}{h+3} + \frac{3}{2}\right) \binom{n}{h-3}$$

$$\leq \left(\frac{5}{2} + \frac{2}{h+3}\right) \binom{n}{h-3}$$

(since $h \geq 2k+3 = 11$), and so produces the required contradiction.

Finally, let $k=3$, $n=3h-1$, $h \geq 9$. By the usual arguments we find

$$\binom{n}{h-2} \leq \binom{n}{1} + \binom{n}{2} + \binom{n}{5} + \sum_{i \leq (h+2)/2} \binom{n}{i} + \sum_{i \leq 2h/3} \binom{n}{i} \frac{1}{3} \binom{n}{\frac{1}{3}(2h+1)}$$

where terms $\binom{n}{\alpha}$ with α not an integer are zero. [As follows: If a partition contains three $(h-2)$-sets then by $n = 3(h-2)+5$ it also contains a g-set for $g = 1, 2$, or 5. If a partition contains two $(h-2)$-sets then by $n = 2(h-2)+h+3$ it also contains a g-set for $g \leq \frac{1}{2}(h+2)$ or two $\frac{1}{2}(h+3)$-sets. Finally if it contains only one $(h-2)$-set and it does not contain a g-set for $g \leq \frac{2}{3}h$, then it was $n = (h-2)+3 * \frac{1}{3}(2h+1)$.] Estimating roughly we find for $g = \lfloor\frac{1}{3}(2h+2)\rfloor$:

$$\binom{n}{h-2} \leq 3 \sum_{i \leq g-1} \binom{n}{i} + \binom{n}{g} < \left(\frac{3g}{n-2g+1} + 1\right) \binom{n}{g} < \binom{n}{g+1}$$

and it follows that

$$h-2 < \lfloor\tfrac{1}{3}(2h+2)\rfloor + 1,$$

a contradiction. □

At this point we have shown that the conditions of Theorem 1 are necessary. The next section is devoted to the proof of the sufficiency.

5. Positive results

Theorem 8. *Let $n = hk$ and $k \geq h-1$. Then K_n^H is 1-factorizable.*

Proof. Use 1-factors of the form

$$n=\frac{h}{(h,g)}*g+\left(k-\frac{g}{(h,g)}\right)*h$$

for $g\in H\backslash\{h\}$. In order to accommodate all g-sets we need

$$N_g:=\frac{(h,g)}{h}\binom{n}{g}$$

such 1-factors for each g, and

$$N_h:=\frac{1}{k}\left(\binom{n}{h}-\sum_{g\in H,\,g\neq h}N_g\left(k-\frac{g}{(h,g)}\right)\right)$$

1-factors of the type

$$n=k*h$$

are needed for the remaining h-sets.

By Baranyai's theorem this setup will produce a solution if

(i) N_g is integral for each $g\in H$,

(ii) $N_g\geq 0$ for each $g\in H$,

(iii) $k\geq g/(h,g)$ for each $g\in H\backslash\{h\}$.

Ad(i): if a and b are integers with $(h,g)=ah+bg$, then

$$N_g=\frac{(h,g)}{h}\binom{n}{g}=a\binom{n}{g}+bk\binom{n-1}{g-1}$$

is integral for $g\in H$, $g\neq h$. Also, since $\sum_{g\in H}N_g=\sum_{g\in H}\binom{n-1}{g-1}$, it follows that N_h is integral.

Ad(ii): Since $N_g\leq\frac{1}{2}\binom{n}{g}$ for $g<h$ it suffices to prove that

$$\binom{n}{h}\geq\sum_{g\leq h-1}\tfrac{1}{2}(k-1)\binom{n}{g}.$$

But

$$\sum_{g\leq h-1}\binom{n}{g}\leq\frac{h}{n-2h+1}\binom{n}{h},$$

and for $k\geq 3$ we indeed have

$$\tfrac{1}{2}(k-1)\cdot\frac{h}{n-2h+1}\leq\frac{1}{2}\frac{k-1}{k-2}\leq 1.$$

(For $k\leq 2$, $h\leq k+1$ one may verify directly that $N_h\geq 0$.)

Ad(iii): $g/(h,g)\leq g\leq h-1\leq k$. □

In fact we proved the more general

Theorem 8a. *Let $n=hk$ and let $k\geq g/(h,g)$ for each $g\in H$. Then K_n^H is 1-factorizable.* □

(Strictly speaking we proved this for $k \geq 3$. For $k \leq 2$, however, the condition $g/(h, g) \leq k$ is equivalent to $g \mid n$ and we find a 1-factorization with partitions of the form $n = (n/g) * g$ for each $g \in H$.)

Remark. Let $n = kh$, $k \geq 3$. The proof that $N_h \geq 0$ did not really use the structure of the 1-factors. Hence "*there are always enough h-sets*" or: "*if there is a 1-factorization of K_n^H with possibly repeated h-sets then there is a proper 1-factorization*". For: a 1-factor not containing only h-sets contains at most $k-1$ h-sets and at least two smaller sets. But we proved

$$\frac{1}{k-1}\binom{n}{h} > \frac{1}{2}\sum_{i \leq h-1}\binom{n}{i}.$$

Proposition 1. *Let $n = hk-1$ and $k \geq h-1$ or ($k \geq \frac{1}{2}h-1$ and h even). Then $\hat{K}_n^h$ is 1-factorizable.*

This proposition is an immediate consequence of Theorem 8 (or Theorem 8a) and the following proposition.

Proposition 2. *Let H contain no two consecutive integers, and let $H' = H \cup (H-1) \setminus \{0\}$. Then if K_{n+1}^H is 1-factorizable, $K_n^{H'}$ is 1-factorizable too.*

Proof. Let $x' = x \cup \{\infty\}$ be some set of $n+1$ elements. Given a 1-factorization of K_{n+1}^H (with vertex set x'), remove the point ∞ from each set containing it. This yields a 1-factorization of $K_n^{H'}$. □ □

Theorem 9. *If $(h-2) \in H$, then $K_{h(h-2)}^H$ is 1-factorizable.*

Proof. If $(h-1) \notin H$ then this follows immediately from Theorem 8a. Hence assume that $(h-1) \in H$ and let $H' = H \setminus \{h-1, h-2\}$. By Theorem 8a $K_n^{H'}$ is 1-factorizable and by Proposition 2, $K_n^{\{h-1, h-2\}}$ is 1-factorizable (note that $n = (h-1)^2 - 1$ so that K_{n+1}^{h-1} is 1-factorizable) hence $K_n^H = K_n^{H'} \cup K_n^{\{h-1, h-2\}}$ is 1-factorizable too. □

This finishes the proof of part (iia) of Theorem 1. In fact statement (A) of the introduction follows from the Theorems 4, 6, 8 and 9.

What remains to be proved in Theorem 1 is the 1-factorizability of $\hat{K}_n^h$ for $n = kh-1$ and h odd and $\frac{1}{2}(h-1) \leq k \leq h-2$. The general idea is that just as in the above remark for $n = kh$ also here we have plenty of h-sets: each partition differing from $n = (k-1) * h + 1 * (h-1)$ contains at most $(k-1)$ h-sets and at least two smaller sets. But if the g-sets for $g \leq h-2$ are used up, the only way to get rid of the remaining h-sets is to use the partition $n = (k-1) * h + 1 * (h-1)$. Therefore it is necessary that at this moment the number of remaining h-sets be exactly $k-1$ times the number of remaining $(h-1)$-sets. On the other hand, if we keep the numbers of $(h-1)$-sets and h-sets in proportion $1:k-1$, then we can never run short of $(h-1)$-sets, since the number of h-sets remains positive. (Note that initially $\binom{n}{h-1}=\binom{n}{h}/k-1$.)

Looking for partitions $n=a*h+b*(h-1)+c*(h-2)$ with $c\neq 0$ we find (since $\lfloor n/(h-2)\rfloor = k+1$) $b+c\leq k+1$ and $b+2c\equiv 1 \pmod h$, so $b+2c=h+1$. With $b=0$ we have a unique partition

$$(\beta)\qquad n=\left(k-\frac{h-1}{2}\right)*h+\frac{h+1}{2}*(h-2)$$

and partitions with $b\neq 0$ exist if and only if $k\geq\frac{1}{2}(h+1)$, e.g.,

$$(\gamma)\qquad n=\left(k-\frac{h+1}{2}\right)*h+2*(h-1)+\frac{h-1}{2}*(h-2).$$

Hence if $k\geq\frac{1}{2}(h+1)$ we can first get rid of all the small sets in an almost arbitrary way, next use 1-factors (β) and (γ) to cover the $(h-2)$-sets, where (β) and (γ) are taken in such a proportion as to make the proportion of the remaining $(h-1)$- and h-sets $1:k-1$, and finally cover the rest with 1-factors (α):

$$(\alpha)\qquad n=(k-1)*h+1*(h-1).$$

The case $k=\frac{1}{2}(h-1)$ will be dealt with separately.

So, let $n=kh-1$ and h odd, $h\geq 5$, $k\geq\frac{1}{2}(h+1)$, $\{h-2, h-1, h\}\subset H$. For $g\in H$, $g\leq h-4$ we use $\binom{n}{g}$ partitions of type

$$(\delta)\qquad n=(k-\tfrac{1}{2}g-1)*h+1*(h-1)+\tfrac{1}{2}g*(h-2)+1*g$$

if g is even, and of type

$$(\varepsilon)\qquad n=(k-\tfrac{1}{2}(g+1))*h+\tfrac{1}{2}(g+1)*(h-2)+1*g$$

if g is odd.

If $h-3\in H$ then we cannot use partitions (δ) only, since this would disturb the proportion of remaining h- and $(h-1)$-sets too much. Therefore, besides

$$(\delta_0)\qquad n=(k-\tfrac{1}{2}(h-1))*h+1*(h-1)+\tfrac{1}{2}(h-3)*(h-2)+1*(h-3)$$

we also use

$$(\rho)\qquad n=(k-\tfrac{1}{2}(h-3))*h+\tfrac{1}{2}(h-5)*(h-2)+2*(h-3).$$

If we take

$$N_\rho:=\left\lfloor\frac{h-3}{2k+h-3}\binom{n}{h-3}\right\rfloor$$

partitions of type (ρ) and $N_{\delta_0}:=\binom{n}{h-3}-2N_\rho$ partitions of type (δ_0), then we cover all $(h-3)$-sets, and we have

$$-\left(k+\frac{h-3}{2}\right)<\sum_{P,n_P(h-3)\neq 0}(n_P(h)-(k-1)n_P(h-1))\leq 0$$

(recall that for a partition P, we defined $n_P(g)$ as the number of g-sets it contains). Let $d_P := n_P(h) - (k-1)n_P(h-1)$ and $D := \sum\{d_P \mid n_P(g) \neq 0 \text{ for some } g \leq h-3\}$. Now

$$-\left(k+\frac{h-3}{2}\right) - \sum_{\substack{g \leq h-5 \\ g \text{ even}}} \tfrac{1}{2}g\binom{n}{g} < D \leq \sum_{\substack{g \leq h-4 \\ g \text{ odd}}} \left(k - \frac{g+1}{2}\right)\binom{n}{g},$$

so

$$-\left(k+\frac{h-3}{2}\right) - \frac{h-5}{2}\cdot\frac{1}{k}\binom{n}{h-4} < D < \binom{n}{h-3}.$$

Also

$$\sum\{n_P(h-2) \mid n_P(g) \neq 0 \text{ for some } g \leq h-3\} \leq \frac{h-3}{2} \sum_{i \leq h-3} \binom{n}{i}$$
$$< \frac{k-2}{k-1}\binom{n}{h-2} < \binom{n}{h-2} - \binom{n}{h-3}$$

so that there are still some $(h-2)$-sets left.

As explained above we would like to make $\sum d_P$ zero by taking a suitable combination of partitions of types (β) and (γ).

If P is a 1-factor of type β, then $d_\beta := d_P = k - \frac{1}{2}(h-1)$; likewise $d_\gamma = -(k + \frac{1}{2}(h-3))$. Let a, b be a nonnegative integers such that $ad_\beta + bd_\gamma = -D$. The existence of such integers follows (by Lemma 3a) from

$$D := \sum_P (n_P(h) - (k-1)n_P(h-1)) \equiv k \sum_P (hn_P(h) + (h-1)n_P(h-1))$$
$$= k\sum_P \left(n - \sum_{g \leq h-3} gn_P(g)\right) \equiv -k \sum_{g \leq h-3} g \sum_P n_P(g)$$
$$= -k \sum_{g \leq h-3} g\binom{n}{g} = -kn \sum_{g \leq h-3} \binom{n-1}{g-1} \equiv 0 \pmod{(n, h-2)}$$

and

$$(d_\beta, d_\gamma) = (k - \tfrac{1}{2}(h-1), k + \tfrac{1}{2}(h+3)) = (2k-h+1, h-2) = (n, h-2) \mid D.$$

Hence if we take a partitions of type (β) and b partitions of type (γ), then $\sum d_P = 0$ where the sum is taken over all 1-factors chosen thus far.

Lemma 3a guarantees us that we can have

$$a + b \leq \max\left(\frac{-D + d_\beta}{d_\gamma}, \frac{-D + d_\gamma}{d_\beta}\right) + d_\beta - d_\gamma - 1$$

so that we need at most

$$(a+b)\frac{h+1}{2} \leq \left(\max\left(\frac{1}{k+\frac{1}{2}(h-3)}\binom{n}{h-3}, k + \frac{h-3}{2} + \frac{h-5}{2k}\binom{n}{h-4}\right)\right.$$
$$\left. + 2k - 2\right)\frac{h+1}{2} < \binom{n}{h-2}$$

$(h-2)$-sets, i.e., not more than was available. (Note that $h \geq 5$ implies $\binom{n}{h-3} > 4(k-1)^2$ and $h \geq 6$ implies $\binom{n}{h-4} > 4k(k-1)$.) At this moment the number of R of remaining $(h-2)$-sets is divisible by $n/(n, h-2)$ (for:

$$R(h-2) = \left(\binom{n}{h-2} - \sum_P n_P(h-2)\right)(h-2)$$

$$= n\binom{n-1}{h-3} - \sum_P \left(n - \sum_{g \neq h-2} g n_P(g)\right) \equiv 0 \pmod n$$

because $\sum_P g n_P(g) = n\binom{n-1}{g-1} \equiv 0 \pmod n$ for $g < h-2$ and $\sum d_P = 0$) and we cover all remaining $(h-2)$-sets by taking $-Rd_\gamma/n$ times a 1-factor of type (β) and Rd_β/n times a 1-factor of type (γ). Since this leaves $\sum d_P$ zero, the rest is done by 1-factors of type (α).

This settles the case $k \geq \frac{1}{2}(h+1)$. Now look at the case $k = \frac{1}{2}(h-1)$,

$$n = kh - 1 = 2k^2 + k - 1 = (k+1)(2k-1) = (k+1)(h-2).$$

We use partitions

(α) $\quad n = (k-1) * h + 1 * (h-1)$,

(β) $\quad n = (k+1) * (h-2)$

and for odd g, $g \in H$, $g \leq h-4$:

(γ) $\quad n = (k - \frac{1}{2}(g+1)) * h + \frac{1}{2}(g+1) * (h-2) + 1 * g$,

(δ) $\quad n = (k - \frac{1}{2}(g+3)) * h + 2 * (h-1) + \frac{1}{2}(g-1) * (h-2) + 1 * g$

and for even g, $g \in H$, $g \leq h-3$:

(ε) $\quad n = (k - \frac{1}{2}g - 1) * h + 1 * (h-1) + \frac{1}{2}g * (h-2) + 1 * g$,

(ρ) $\quad n = a * h + b * (h-2) + c * g$

where

$$c := \left\lceil \frac{h-1}{g} \right\rceil, \qquad a := h - 2 - \tfrac{1}{2}cg, \qquad b := \tfrac{1}{2}cg - \frac{h-1}{2}.$$

(Note that $cg \geq h-1$ and $\frac{1}{2}cg \leq \frac{1}{2}(h-1+g-1) \leq h-2$ so that a, b and c are nonnegative integers.)

Take 1-factors of types (γ) and (δ) with frequencies

$$N_\gamma = \frac{k + \frac{1}{2}(g-1)}{2k-1}\binom{n}{g}$$

and

$$N_\delta = \frac{k - \frac{1}{2}(g+1)}{2k-1}\binom{n}{g} \quad \text{for each odd } g \in H,\ g \leq h-4.$$

(Note that $N_\gamma + N_\delta = \binom{n}{g}$ and $N_\gamma - N_\delta = (g/(2k-1))\binom{n}{g} = (k+1)\binom{n-1}{g-1}$ are integral, while $2k-1$ is odd; therefore N_γ and N_δ are integral.)

We have $d_\gamma = k - \frac{1}{2}(g+1)$ and $d_\delta = -(k + \frac{1}{2}(g-1))$ so that for these 1-factors $\sum d_P = N_\gamma d_\gamma + N_\delta d_\delta = 0$, i.e., the h- and $(h-1)$-sets remain in the correct proportion. Take 1-factors of types (ε) and (ρ) with frequencies

$$N_\varepsilon = \frac{2a}{cg+2a}\binom{n}{g}$$

and

$$N_\rho = \frac{g}{cg+2a}\binom{n}{g} \quad \text{for each even } g \in H,\ g \leq h-3.$$

(Note that $N_\varepsilon + cN_\rho = \binom{n}{g}$ and that

$$N_\rho = \frac{g}{2(h-2)}\binom{n}{g} = \frac{k+1}{2}\binom{n-1}{g-1}$$

is integral since g is even and h is odd.)

We have $d_\varepsilon = -\frac{1}{2}g$ and $d_\rho = a$ so that also for these 1-factors $\sum d_P = N_\varepsilon d_\varepsilon + N_\rho d_\rho = 0$. Cover the remaining h- and $(h-1)$-sets with 1-factors of type (α) and the remaining $(h-2)$-sets with 1-factors of type (β). (Note that $N_\beta \geq 0$: in the other partitions we used less than

$$\frac{h-3}{2}\sum_{i \leq h-3}\binom{n}{i} < \binom{n}{h-2}$$

$(h-2)$-sets. Also that it is impossible that at the end some $(h-2)$-sets are left: all sets together cover a number of points that is a multiple of n, and each partition takes away sets with a total size of n, so that as soon as the total drops below n it must have become zero.) This completes the proof of Theorem 1. □

More generally we proved:

Theorem 10. *Let $n = kh-1$, $k \geq \frac{1}{2}(h-1)$, h odd, $\{h-2, h-1, h\} \subset H$. Then K_n^H is 1-factorizable.* □

In order to give necessary and sufficient conditions for the case h even, $\{h, h-1, h-2\} \subset H$ we first need some definitions.

For an integer i, let $\bar{i}$ (the *buddy* of i) be the integer such that for some j (namely, $j = \lceil \frac{1}{2}i \rceil$) $\{i, \bar{i}\} = \{2j-1, 2j\}$.

For $H \subset \{1, 2, \ldots, n\}$ let $m(H) := \max\{i \in H \mid \bar{i} \notin H\}$ if there are $i \in H$ with $\bar{i} \notin H$, and put $m(H) = -1$ otherwise.

Now we can formulate

Theorem 11. *Let $n = kh-1$, $k \geq 3$, $\{h-2, h-1, h\} \subset H$. Then K_n^H is 1-factorizable if and only if*

(i) *$k \geq \frac{1}{2}h - 1$, and*

(ii) *$m(H)$ is odd.*

Proof. *Necessity*. The necessity of (i) is shown by Theorem 7. Let $m := m(H)$ be even. Then h is even (for if h is odd then $m(H) = h$), so that n is odd, and each 1-factor contains at least one set of odd size. Consequently the total number of partitions:

$$\frac{1}{n}\sum_{g\in H} g\binom{n}{g} = \sum_{g\in H}\binom{n-1}{g-1}$$

is at most the number of sets of odd size:

$$\sum_{\substack{g\in H\\ g \text{ odd}}}\binom{n}{g} = \sum_{\substack{g\in H\\ g \text{ odd}}}\left(\binom{n-1}{g-1}+\binom{n-1}{g}\right).$$

It follows that

$$\binom{n-1}{m-1} \leq \sum_{i\leq m-3}\binom{n-1}{i} < \binom{n-1}{m-2},$$

a contradiction.

Sufficiency. For odd h the sufficiency is shown by Theorem 10. Let h be even and choose some decomposition $H = \bigcup_{2i\in F}\{2i-1, 2i\} \cup G$ with $G = \emptyset$ or $m = m(H) = \max G$ odd. We use the following partitions (note that $(h-3)\in H$):

(α) $n = (k-1) * h + 1 * (h-1)$,

(β_0) $n = (k - \frac{1}{2}h + 1) * h + (\frac{1}{2}h - 1) * (h-2) + 1 * (h-3)$,

(γ_0) $n = (k - \frac{1}{2}h) * h + 1 * (h-1) + \frac{1}{2}h * (h-2)$,

(δ_0) $n = (k - \frac{1}{2}h) * h + 2 * (h-1) + (\frac{1}{2}h - 2) * (h-2) + 1 * (h-3)$,

furthermore for $f\in F$, $f\leq h-4$:

(β) $$n = \left(k - \frac{f}{(h,f)}\right) * h + \left(\frac{h}{(h,f)} - 1\right) * f + 1 * (f-1),$$

(γ) $$n = \left(k - \frac{f}{(h,f)} - 1\right) * h + 1 * (h-1) + \frac{f}{(h,f)} * f.$$

Note that $(h,f)\geq 2$ when $f\in F$ so that

$$k - \frac{f}{(h,f)} \geq k - \tfrac{1}{2}f \geq \tfrac{1}{2}(h-2-f) \geq 1 \quad \text{for} \quad f\leq h-4.$$

It follows that all coefficients are nonnegative except for the $k - \frac{1}{2}h$ in (γ_0) and (δ_0) in case $k = \frac{1}{2}h - 1$. Taking for $f\in F$ with $f\leq h-4$ partitions (β) exactly $\binom{n}{f-1}$ times and (γ) exactly

$$\binom{n+1}{f}\frac{(h,f)}{h} - \binom{n}{f-1} = \left(\frac{k}{f}(h,f) - 1\right)\binom{n}{f-1}$$

times (note that these numbers are positive integers) we cover all $(f-1)$- and f-sets.

If $G=\emptyset$ we do the same for $f=h-2$ (note that if $k=\frac{1}{2}h-1$, then $(k/f)(h,f)-1=0$ so that we do not take (γ_0) in this case) and finally cover the remaining h- and $(h-1)$-sets with partitions (α). [In fact this is Proposition 2 applied to Theorem 8a.]

If $G\neq\emptyset$ we need some $(h-2)$- and h-sets to accommodate the g-sets with $g\in G$: For g odd use partitions

$$(\varepsilon) \qquad n=(k-\tfrac{1}{2}(g+1))*h+\tfrac{1}{2}(g+1)*(h-2)+1*g.$$

For g even, $g\leq 2k-m-1$ use

$$(\rho) \qquad n=(k-\tfrac{1}{2}(m+g+1))*h+\tfrac{1}{2}(m+g+1)*(h-2)+1*m+1*g,$$

and for g even, $g\geq h-m-1$ use

$$(\eta) \qquad n=(k-\tfrac{1}{2}(m+g+1)+\tfrac{1}{2}(h-2))*h$$
$$+\tfrac{1}{2}(m+g+1-h)*(h-2)+1*m+1*g.$$

(Note that $g\leq m\leq h-5$ so that $2k-(m+g+1)+h-2>0$; next that we exhausted all possibilities: $2k-m-1<h-m-1$ is impossible for g even.)

Take for $g\in G$, $g\neq m$ exactly $\binom{n}{g}$ times one of these partitions, and then cover the remaining m-sets with partitions of type (ε) (with $g=m$). (If $h-m-1\leq 2k-m-1$ we take partitions of type (ρ).) Note that there are enough m-sets: each time we cover a g-set with $g<m$ we use only one m-set, and $\sum_{g\leq m-1}\binom{n}{g}<\binom{n}{m}$; also, that we do not use more than

$$k\sum_{i\leq h-5}\binom{n}{g}<\frac{k(h-4)}{kh-2h+8}\binom{n}{h-4}<\binom{n}{h-4}$$

of the h-sets or $(h-2)$-sets.

Now suppose $k\neq\frac{1}{2}h-1$, i.e., $k\geq\frac{1}{2}h$, so that partitions of type (γ_0) and (δ_0) are available. For $i\in H$ we denote by

$$r(i):=\binom{n}{i}-\sum n_P(i)$$

the number of i-sets not yet covered at the moment under consideration, and define

$$\Delta:=r(h)-(k-1)r(h-1)+(1-2h^{-1})r(h-2)-(k-1+2h^{-1})r(h-3).$$

Initially $r(i)=\binom{n}{i}$ and $\Delta=0$. Taking partitions (β) and (γ) in the stated proportions (for $f\leq h-4$) does not change Δ, while after having taken partitions (ε,ρ,η) as indicated we have $-\binom{n}{h-4}<\Delta\leq 0$ and $r(i)=0$ for $i<h-3$.

Taking partitions (α), (β_0) or (γ_0) also does not change Δ, while taking (δ_0) increases Δ by $2k-2h^{-1}$. Since

$$h\Delta\equiv hr(h)+(h-1)r(h-1)+(h-2)r(h-2)+(h-3)r(h-3)\equiv 0 \pmod n$$

and n is odd, $\Delta/(2k-2h^{-1})=h\Delta/2n$ is an integer. Hence take $(-\Delta)/(2k-2h^{-1})$ partitions of type (δ_0) in order to make Δ zero, and then take care of the remaining $(h-3)$-sets with partitions of type (β_0).

At this moment

$$r(h-2)>\binom{n}{h-2}-(\tfrac12 h-1)\binom{n}{h-3}-\binom{n}{h-4}$$
$$\geqslant\left(k-\tfrac12 h+1-\frac{1}{k-1}\right)\binom{n}{h-3}>0.$$

Also $\Delta=r(h-3)=0$ so that $2h^{-1}r(h-2)$ is an integer. Hence take $r(h-2)/\frac12 h$ partitions of type (γ_0) so as to make $r(h-2)=0$. Finally we use partions of type (α) for the remaining h- and $(h-1)$-sets. This completes the proof in case $k\neq\frac12 h-1$. □

The case $k=\frac12 h-1$, $n=2k^2+2k-1$ is treated along the same lines, but since we cannot use (γ_0) and (δ_0), we have to keep track of $D=\sum(n_P(h)-(k-1)n_P(h-1))$ and $D'=\sum(n_P(h-2)-kn_P(h-3))$ separately. (Note that $\binom{n}{h-2}=k\binom{n}{h-3}$.) We may assume $m>1$, since if $m=1$ then $G=\{1\}$, and after treating $H\backslash\{1\}$ we add the partition

(ι) $\quad n=n*1$

to complete the 1-factorization. This time we use the partitions (β) and (γ) for f-sets and $(f-1)$-sets with $f\in F$, $f\leqslant h-4$; the partitions (ε), (ρ) and (η) for the g-sets, $g\in G\backslash\{m\}$ and then use the following partitions for the i-sets with $i\in\{h,h-1,h-2,h-3,m\}$:

$$\begin{aligned}
(\alpha)\quad & n=(k-1)*h+1*(h-1)\\
(\beta_0)\quad & n=k*(h-2)+1*(h-3)\\
(\varepsilon_0)\quad & n=(k-\tfrac12(m+1))*h+\tfrac12(m+1)*(h-2)+1*m\\
(\theta)\quad & n=(k-\tfrac12(m-3))*h+2*(h-1)+\tfrac12(m-1)*(h-2)+1*m\\
(\kappa)\quad & n=1*(h-1)+(k-2)*(h-2)+2*(h-3)\\
(\lambda)\quad & n=1*h+(k-3)*(h-2)+3*(h-3)
\end{aligned}$$

In order to cover the g-sets for $g\in G\backslash\{m\}$ we used less than $\binom{n}{m}$ h-sets, and less than $\binom{n}{m}/k$ m-sets.

Initially $D=0$ and after use of the partitions (β), (γ), (ε), (ρ) and (η) we have $0\leqslant D<\binom{n}{m}$.

Taking a 1-factor (ε_0) increases D with $d_\varepsilon=k-\frac12(m+1)\geqslant 1$ and taking a 1-factor (θ) decreases D, adding $d_\theta:=-(k+\frac12(m-1))$ to it. We need no more than $\binom{n}{m}/k$ 1-factors (θ) to reduce D to about zero, and we have enough m-sets left to do so. After this we cover the remaining m-sets by taking a 1-factor (ε_0) when $D\leqslant 0$ and (θ) when $D>0$ until $r(m)=0$. We now have

$$-2k+3\leqslant 1-(k+\tfrac12(m-1))\leqslant D\leqslant k-\tfrac12(m+1)\leqslant k-2,$$

and

$$D' = \sum n_P(h-2) < k \sum_{i \leq m} \binom{n}{i} < \binom{n}{h-4}$$

and on the other hand

$$D' = \sum n_P(h-2) > \frac{k-1}{k} \binom{n}{m} > \frac{k-1}{k} \binom{n}{2} > n + 2k + 3.$$

Next we make D' small by taking 1-factors (κ) and (λ) in proportion $1:k-1$ until

$$n + 2k + 3 \leq D' \leq 2n + 2k + 2.$$

(Note that (κ) adds $d'_\kappa = -(k+2)$ to D' while $d'_\lambda = -(2k+3)$ so that taking (κ) once and (λ) $k-1$ times leaves D invariant and decreases D' by $(k-1)(2k+3) + k + 2 = n$.) Now make D' zero by taking an appropriate combination of 1-factors (κ) and (λ). (Note that $(k+2, 2k+3) = 1$ and $(k+1)(2k+2) = n + 2k + 3$ so that this is possible by Lemma 3.) We need no more than $2k+2$ partitions of type (κ) and no more than $\lfloor (2n+2k+2)/(2k+2) \rfloor = 2k$ partitions of type (λ) so that now

$$n < -(k-1)(2k+2) - 2k + 3 \leq D \leq 2k + k - 2 < n.$$

But $hD + (h-2)D' \equiv 0 \pmod{n}$, $D' = 0$ and $(h, n) = 1$ so that $D \equiv 0 \pmod{n}$ and therefore $D = 0$. Therefore we can cover the remaining $(h-2)$- and $(h-3)$-sets with 1-factors (β_0) and the remaining h- and $(h-1)$-sets with 1-factors (α). This finishes the proof of Theorem 11. □

Because of Theorem 5, statement (B) in the introduction is just a reformulation of Theorem 11.

6. Miscellaneous remarks

Up to now we concentrated on the case $(h-1) \in H$. It seems difficult to formulate a necessary and sufficient condition on H in order that K_n^H be 1-factorizable.

A plausible conjecture is that if $g \notin H$ and K_n^H is 1-factorizable, then so is $K_n^{H'}$, where $H' = \{i \in H \mid > g\}$ (assuming of course that $n > 2h$).

Looking at sets H with small cardinality we have that $K_n^{\emptyset}$ is 1-factorizable, and that $K_n^{\{h\}}$ is 1-factorizable if and only if $h \mid n$. The next step is provided by

Theorem 12. *Let $H = \{g, h\}$ with $0 < g < h$. K_n^H is 1-factorizable if and only if one of the following holds:*

(i) $n \equiv -1 \pmod{h}$ *and* $g = h - 1$,

(ii) $n \equiv 0 \pmod{h}$ *and* $n \geq gh/(g, h)$,

(iii) $n = g + h$.

Proof. If $n \leqslant 2h$ then by Theorem 2 K_n^H is 1-factorizable if and only if (ii) or (iii). If $n > 2h$ and K_n^H is 1-factorizable then by Theorem 3 and 5 either $n = kh$ or (i) holds. Moreover, when $n = kh$, then (ii) is necessary, since if $g \mid n$ then certainly $n \geqslant gh/(g, h)$ while if $g \nmid n$ then any partition $n = a * h + b * g$ must contain at least $h/(g, h)$ g-sets, hence again $n \geqslant gh/(g, h)$. Conversely, (i) is sufficient by Proposition 2, and (ii) is sufficient by Theorem 8a. □

Generalizing the necessary part of Theorem 11 we have that if $n = kh - 1$ then for a fixed prime $p \mid h$:

$$\# \text{ of partitions} = \sum_{g \in H} \binom{n-1}{g-1} \leqslant \# \text{ of sets with size not a multiple of } p = \sum_{\substack{p \nmid g \\ g \in H}} \binom{n}{g}.$$

For instance, $K_{39}^{\{10,9,5\}}$ is not 1-factorizable. (In fact if K_n^H is 1-factorizable and $H = \{g, h-1, h\}$, $n = kh - 1$, $g < h - 1$ then $(g, h) = 1$, and if g' is the smallest positive integer such that $gg' \equiv -1 \pmod h$ then $n \geqslant gg'$.)

As another example, $K_{29}^{\{10,9,3\}}$ is not 1-factorizable, this time because each partition must contain at least two 10-sets, but $2 \cdot \#$ of partitions $> \binom{29}{10}$. (What is wrong here is not so much that $(g, h-1) \neq 1$; one may verify that for $H = \{g, 9, 10\}$ with $g < 9$ we have that K_{49}^H is 1-factorizable exactly when $g = 1$ or 3 or 7.)

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