Asymptotic expansions of Jacobi polynomials and of the nodes and weights of Gauss-Jacobi quadrature for large degree and parameters in terms of elementary functions

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Abstract

Asymptotic approximations of Jacobi polynomials are given in terms of elementary functions for large degree n and parameters α and β. From these new results, asymptotic expansions of the zeros are derived and methods are given to obtain the coefficients in the expansions. These approximations can be used as initial values in iterative methods for computing the nodes of Gauss–Jacobi quadrature for large degree and parameters. The performance of the asymptotic approximations for computing the nodes and weights of these Gaussian quadratures is illustrated with numerical examples.

1 Introduction

This paper is a further exploration in our research on Gauss quadrature for the classical orthogonal polynomials; earlier publications are [3], [4], [5], [6]. Other recent relevant papers on this topic are [1], [7], [15].

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When we assume that the degree n and the two parameters α and β of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ are large, and we consider the variable x as a parameter that causes nonuniform behavior of the polynomial, it can be expected that, for a detailed and optimal description of the asymptotic approximation, we need a function of three variables. Candidates for this are the Gegenbauer and the Laguerre polynomial. The Gegenbauer polynomial can be used when the ratio α/β does not tend to zero or to infinity. When it does, the Laguerre polynomial is the best option.

It is possible to transform an integral of $P_n^{(\alpha,\beta)}(x)$ into an integral resembling one of the Gegenbauer or the Laguerre polynomial (and similar when we are working with differential equations). From a theoretical point of view this may be of interest, however, for practical purposes, when using the results for Gauss quadrature, the transformations and the coefficients in the expansions become rather complicated. In addition, computing the approximants, that is, large degree polynomials with large additional parameter and a variable in domains where nonuniform behavior of these polynomials may happen, gives an extra nontrivial complication.

Even when we use the Bessel functions or Hermite polynomials as approximants, these complications are still quite relevant. For this reason we consider in this paper expansions in terms of elementary functions, and we will see that to evaluate a certain number of coefficients already gives quite complicated expressions.

For large values of β with fixed degree n we have quite simple results derived in [5], which paper is inspired by [2]. Large-degree results valid near $x = 1$ are given in [14, §28.4], and for the case that β is large as well we refer to [14, §28.4.1].

2 Several asymptotic phenomena

To describe the behavior of the Jacobi polynomial for large degree and parameters α and β, with $x \in [-1, 1]$, it is instructive to consider the differential equation of the function

$$W(x) = (1-x)^{\frac{1}{2}(\alpha+1)}(1+x)^{\frac{1}{2}(\beta+1)} P_n^{(\alpha,\beta)}(x). \tag{2.1}$$

By using the Liouville-Green transformations as described in [11] uniform expansions can be derived for all combinations of the parameters n, α, β.

Let σ, τ and κ be defined by

$$\sigma = \frac{\alpha+\beta}{2\kappa}, \quad \tau = \frac{\alpha-\beta}{2\kappa}, \quad \kappa = n + \tfrac{1}{2}(\alpha+\beta+1). \tag{2.2}$$

Then $W(x)$ satisfies the differential equation

$$\frac{d^2}{dx^2}W(x) = -\frac{\kappa^2(x_+ - x)(x - x_-) + \frac{1}{4}(x^2+3)}{(1-x^2)^2} W(x), \tag{2.3}$$

where

$$x_{\pm} = -\sigma\tau \pm \sqrt{(1-\sigma^2)(1-\tau^2)}; \tag{2.4}$$

x_- and x_+ are called turning points. We have $-1 \le x_- \le x_+ \le 1$ when α and β are positive. When $\sigma^2 + \delta^2 = 1$, one of the turning points $x_\pm$ is zero.

When we skip the term $\frac{1}{4}(x^2+3)$ of the denominator in (2.3), the differential equation becomes one for the Whittaker or Kummer functions, with special case the Laguerre polynomial, and when we take $\alpha = \beta$ the equation becomes a differential equation for the Gegenbauer polynomial.

When κ is large we can make a few observations.

1. If $n \gg \alpha + \beta$, then $\sigma \to 0$ and $\tau \to 0$. Hence, $x_- \to -1$ and $x_+ \to 1$. This is the standard case for large degree, the zeros are spread over the complete interval $(-1, 1)$.

2. When α and/or β become large as well, the zeros are inside the interval (x_-, x_+). When, in addition, $\alpha/\beta \to 0$, the zeros shift to the right, when $\beta/\alpha \to 0$, they shift to the left. See also the limit in (2.9). The zeros become all positive when $x_- \ge 0$. In that case $\sigma^2 + \delta^2 \ge 1$.

3. When x is in a closed neighborhood around x_- that does not contain -1 and x_+, an expansion in terms of Airy functions can be given. Similar for x in a closed neighborhood around x_+ that does not contain x_- and 1. The points $x_\pm$ are called turning points of the equation in (2.3).

4. When $-1 \le x \le x_-(1+a) < x_+$, with a a fixed positive small number, an expansion in terms of Bessel functions can be given. Similar for $x_- < x_+(1-a) \le x \le 1$. The latter case corresponds to the limit

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(1 - \frac{x^2}{2n^2}\right) = \left(\frac{2}{x}\right)^{\alpha} J_\alpha(x). \tag{2.5}$$

Also, $\sqrt{x} J_\alpha\left(\alpha\sqrt{x}\right)$ satisfies the differential equation

$$\frac{d^2}{dx^2} w(x) = \left(\alpha^2 \frac{1-x}{4x^2} - \frac{1}{4x^2}\right) w(x), \tag{2.6}$$

in which $x = 1$ is a turning point when α is large.

5. If $\alpha+\beta \gg n$, then $\sigma \to 1$ and the turning points x_- and x_+ coalesce at $-\tau$. When α and β are of the same order, the point $-\tau$ lies properly inside $(-1, 1)$, and this case has been studied in [10] to obtain approximations of Whittaker functions in terms of parabolic cylinder functions. In the present case the parameters are such that the parabolic cylinder functions become Hermite polynomials. This corresponds to the limit (see [9])

$$\lim_{\alpha,\beta\to\infty} \left(\frac{8}{\alpha+\beta}\right)^{n/2} P_n^{(\alpha,\beta)}\left(x\sqrt{\frac{2}{\alpha+\beta}} - \frac{\alpha-\beta}{\alpha+\beta}\right) = \frac{1}{n!} H_n(x), \tag{2.7}$$

derived under the conditions

$$x = \mathcal{O}(1), \quad n = \mathcal{O}(1), \quad \frac{\alpha - \beta}{\alpha + \beta} = o(1), \quad \alpha, \beta \to \infty. \tag{2.8}$$

6. If $\alpha \gg \beta$, then $\tau \to 1$, and x_- and x_+ coalesce at $-\sigma$; if $\beta/\kappa = o(1)$, then the collision will happen at -1. Approximations in terms of Laguerre polynomials can be given. This corresponds to the limit

$$\lim_{\alpha \to \infty} P_n^{(\alpha,\beta)}\bigl((2x/\alpha) - 1\bigr) = (-1)^n L_n^{(\beta)}(x). \tag{2.9}$$

   Similar for $\beta \gg \alpha$, in which case $L_n^{(\alpha)}(x)$ becomes the approximant.

As explained earlier, we consider in this paper the second case: new expansions of $P_n^{(\alpha,\beta)}(x)$, and its zeros and weights in terms of elementary functions. Preliminary results regarding the role of Gegenbauer and Laguerre polynomials as approximants can be found in [13].

3 An integral representation and its saddle points

The Rodrigues formula for the Jacobi polynomials reads (see [8, §18.15(ii)])

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!\, w(x)} \frac{d^n}{dx^n} \left(w(x)(1-x^2)^n\right), \tag{3.1}$$

where

$$w(x) = (1-x)^\alpha (1+x)^\beta. \tag{3.2}$$

This gives the Cauchy integral representation

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, w(x)} \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{w(z)(1-z^2)^n}{(z-x)^{n+1}}\, dz, \quad x \in (-1,1), \tag{3.3}$$

where the contour $\mathcal{C}$ is a circle around the point $z = x$ with radius small enough to have the points ± 1 outside the circle.

We write this in the form[1]

$$P_n^{(\alpha,\beta)}(x) = \frac{-1}{2^n\, w(x)} \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-\kappa\phi(z)} \frac{dz}{\sqrt{(1-z^2)(x-z)}}, \tag{3.4}$$

where

$$\kappa = n + \tfrac{1}{2}(\alpha + \beta + 1). \tag{3.5}$$

and

$$\phi(z) = -\frac{n+\alpha+\frac{1}{2}}{\kappa} \ln(1-z) - \frac{n+\beta+\frac{1}{2}}{\kappa} \ln(1+z) + \frac{n+\frac{1}{2}}{\kappa} \ln(x-z). \tag{3.6}$$

[1]The multi-valued functions of the integrand are discussed in Remark 3.1.

We introduce the notation

$$\sigma = \frac{\alpha+\beta}{2\kappa}, \quad \tau = \frac{\alpha-\beta}{2\kappa}, \tag{3.7}$$

and it follows that

$$\phi(z) = -(1+\tau)\ln(1-z) - (1-\tau)\ln(1+z) + (1-\sigma)\ln(x-z). \tag{3.8}$$

The saddle points $z_\pm$ follow from the zeros of

$$\phi'(z) = -\frac{(1+\sigma)z^2 + 2(\tau - x)z + 1 - \sigma - 2\tau x}{(1-z^2)(x-z)}, \tag{3.9}$$

and are given by

$$\begin{aligned} z_\pm &= \frac{x - \tau \pm iU(x)}{1+\sigma}, \\ U(x) &= \sqrt{1 - 2\sigma\tau x - \tau^2 - \sigma^2 - x^2} = \sqrt{(x_+ - x)(x - x_-)}, \end{aligned} \tag{3.10}$$

where (see also (2.4))

$$x_\pm = -\sigma\tau \pm \sqrt{(1-\sigma^2)(1-\tau^2)}. \tag{3.11}$$

In this representation we assume that $x_- \le x \le x_+$, in which x-domain the zeros of the Jacobi polynomial are located.

Remark 3.1. The starting integrand in (3.3) has a pole at $z = x$, while the one of (3.4) shows an algebraic singularity at $z = x$ and $\phi(z)$ defined in (3.6) has a logarithmic singularity at this point. To handle this from the viewpoint of multi-valued functions, we can introduce a branch cut for the functions involved from $z = x$ to the left, assuming that the phase of $z - x$ is zero when $z > x$, equals $-\pi$ when z approaches -1 on the lower part of the saddle point contour of the integral in (3.4), and $+\pi$ on the upper side. Because the saddle points $z_\pm$ stay off the interval $(-1, 1)$, we do not need to consider function values on the branch cuts for the asymptotic analysis. △

4 Deriving the asymptotic expansion

We derive an expansion in terms of elementary functions which is valid for $x \in [x_-(1+\delta), x_+(1-\delta)]$, where $x_\pm$ are the turning points defined in (3.11) and δ is a fixed positive small number. Also, we assume that $\sigma \in [0, \sigma_0]$ and $\tau \in [-\tau_0, \tau_0]$, where σ_0 and τ_0 are fixed positive numbers smaller than 1. The case $\sigma \to 1$ is explained in Case 5 of Section 2. A similar phenomenon occurs when $\tau \to \pm 1$.

First we consider contributions from the saddle point z_+ using the transformation

$$\phi(z) - \phi(z_+) = \tfrac{1}{2}w^2 \tag{4.1}$$

for the contour from $z = +1$ to $z = -1$ through z_+, with $\phi(z)$ and z_+ given in (3.8) and (3.10). This transforms the part of the integral in (3.4) that runs with $\Im z \geq 0$ into

$$P^+ = \frac{e^{-\kappa\phi(z_+)}}{2^n\, w(x)} \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\kappa w^2} f_+(w)\, dw, \tag{4.2}$$

where

$$f_+(w) = \frac{1}{\sqrt{(1-z^2)(x-z)}} \frac{dz}{dw}, \quad \frac{dz}{dw} = \frac{w}{\phi'(z)}. \tag{4.3}$$

We expand $f_+(w) = \sum_{j=0}^{\infty} f_j^+ w^j$, where

$$f_0^+ = \frac{1}{\sqrt{(1-z_+^2)(x-z_+)\phi''(z_+)}} = \frac{e^{\frac{1}{4}\pi i}}{\sqrt{2U(x)}}, \tag{4.4}$$

and $U(x)$ is defined in (3.10). Because the contribution from the saddle point z_- is the complex conjugate of that from z_+[2], we take twice the real part of the contribution from z_+ and obtain the expansion

$$P_n^{(\alpha,\beta)}(x) \sim \Re \frac{e^{-\kappa\phi(z_+)-\frac{1}{4}\pi i}}{2^n\, w(x)\sqrt{\pi\kappa U(x)}} \sum_{j=0}^{\infty} \frac{c_j^+}{\kappa^j}, \quad c_j = 2^j \left(\tfrac{1}{2}\right)_j \frac{f_{2j}^+}{f_0^+}. \tag{4.5}$$

Evaluating $\phi(z_+)$ we find

$$\begin{aligned}
\phi(z_+) &= -\ln 2 + \psi + \xi + i\chi(x), \\
\psi &= -\tfrac{1}{2}(1-\tau)\ln(1-\tau) - \tfrac{1}{2}(1+\tau)\ln(1+\tau) + \\
&\quad \tfrac{1}{2}(1+\sigma)\ln(1+\sigma) + \tfrac{1}{2}(1-\sigma)\ln(1-\sigma), \\
\xi(x) &= -\tfrac{1}{2}(\sigma+\tau)\ln(1-x) - \tfrac{1}{2}(\sigma-\tau)\ln(1+x), \\
\chi(x) &= (\tau+1)\arctan\frac{U(x)}{1-x+\sigma+\tau} + (\tau-1)\arctan\frac{U(x)}{1+x+\sigma-\tau} + \\
&\quad (1-\sigma)\,\mathrm{atan2}(-U(x), \tau + x\sigma).
\end{aligned} \tag{4.6}$$

In Figure 1 we show a graph of $\chi(x)$ on (x_-, x_+) for $\alpha = 90$, $\beta = 75$, $n = 125$. For these values, $\kappa = 208$, $\sigma = \frac{165}{416}$, $\tau = \frac{15}{416}$, $x_- = -0.931$, $x_+ = 0.903$. At the left endpoint we have $\chi(x_-) = -(1-\sigma)\pi = -1.896$.

Remark 4.1. The denominators of the first and second arctan functions of $\chi(x)$ in (4.6) are always positive on (x_-, x_+); this follows easily from the relations in (3.7). The function $\mathrm{atan2}(y, x)$ in the third term of $\chi(x)$ denotes the phase $\in (-\pi, \pi]$ of the complex number $x + iy$. Because $\tau + x\sigma$ may be negative on (x_-, x_+) we cannot use the standard arctan function for that term. △

[2] We assume that $x \in (x_-, x_+)$ and that α and β are positive.

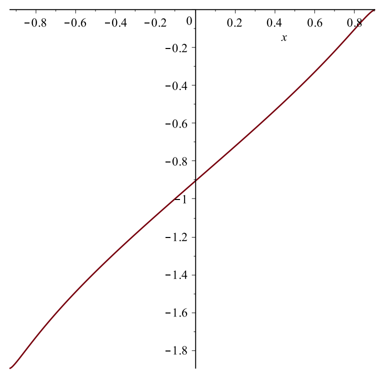

Figure 1: The quantity $\chi(x)$ defined in (4.6) for $x \in (x_-, x_+)$; $\alpha = 90$, $\beta = 75$, $n = 125$. For these values, $\kappa = 208$, $\sigma = \frac{165}{416}$, $\tau = \frac{15}{416}$, $x_- = -0.931$, $x_+ = 0.903$.

Observe that $e^{-\kappa\xi(x)} = \sqrt{w(x)}$, with $w(x)$ defined in (3.2). To compute x from $\chi(x)$, for example by using a Newton-procedure, it is convenient to know that

$$\frac{d\chi(x)}{dx} = \frac{U(x)}{(1-x^2)}. \tag{4.7}$$

We return to the result in (4.5) and split the coefficients of (4.5) in real and imaginary parts. We write $c_j^+ = p_j + iq_j$, and obtain

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{2^{\frac{1}{2}(\alpha+\beta+1)}e^{-\kappa\psi}}{\sqrt{\pi\kappa w(x)U(x)}}W(x), \\ W(x) &= \cos\left(\kappa\chi(x) + \tfrac{1}{4}\pi\right)P(x) + \sin\left(\kappa\chi(x) + \tfrac{1}{4}\pi\right)Q(x), \end{aligned} \tag{4.8}$$

with expansions

$$P(x) \sim \sum_{j=0}^{\infty} \frac{p_j}{\kappa^j}, \quad Q(x) \sim \sum_{j=0}^{\infty} \frac{q_j}{\kappa^j}. \tag{4.9}$$

Because $c_0^+ = 1$, we have $p_0 = 1$, $q_0 = 0$.

To evaluate the coefficients f_{2j}^+ of the expansion in (4.5), we need the coefficients z_j^+ of the expansion $z = z_+ + \sum_{j=1}^{\infty} z_j^+ w^j$ that follow from (4.1). The first values are

$$\begin{aligned} z_2^+ &= -\tfrac{1}{6}z_1^4\phi_3, \quad z_3^+ = \tfrac{1}{72}z_1^5\left(5z_1^2\phi_3^2 - 3\phi_4\right), \\ z_4^+ &= -\tfrac{1}{1080}z_1^6\left(9\phi_5 - 45z_1^2\phi_3\phi_4 + 40z_1^4\phi_3^2\right), \end{aligned} \tag{4.10}$$

where $z_1 = z_1^+ = 1/\sqrt{\phi''(z_+)}$ and ϕ_j denotes the jth derivative of $\phi(z)$ at the saddle point $z = z_+$ defined in (3.10).

With these coefficients we expand $f(w)$ defined in (4.4). This gives

$$
\begin{aligned}
c_1^+ = -&\frac{z_+}{8z_1(1-z_+^2)^2(x-z_+)^2}\Bigl(-6z_1^3z_+^2+3z_1^3-72z_1z_2z_+^2x+\\
&24z_1z_+z_2x^2-24z_1z_+^3z_2x^2-48z_3xz_+-48z_3z_+^2x^2+96z_3z_+^3x+\\
&24z_3z_+^4x^2-48z_3z_+^5x-12z_1z_+z_2+48z_1z_+^3z_2-48z_3z_+^4+\\
&24z_3z_+^6+12z_1z_2x-36z_1z_2z_+^5-4z_1^3xz_++8z_1^3z_+^2x^2-20z_1^3z_+^3x+\\
&4z_1^3x^2+15z_1^3z_+^4+24z_3x^2+24z_3z_+^2+60z_1z_2z_+^4x\Bigr),
\end{aligned}
\tag{4.11}
$$

where z_j denotes z_j^+. The coefficients p_1 and q_1 of the expansions in (4.9) follow from $c_1^+ = p_1 + iq_1$.

4.1 Expansion of the derivative

For the weights of the Gauss quadrature it is convenient to have an expansion of $\frac{d}{dx}P_n^{(\alpha,\beta)}(x)$. Of course this follows from using (4.8) with different values of α and β and the relation

$$
\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}\left(\alpha+\beta+n+1\right)P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{4.12}
$$

but it is useful to have a representation in terms of the same parameters.

By straightforward differentiation of (4.8) we obtain

$$
\begin{aligned}
\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = -\sqrt{\frac{\kappa}{\pi}}\,2^{\frac{1}{2}(\alpha+\beta+1)e^{-\kappa\psi}}\chi'(x)A(x)\,\times\\
\Bigl(\sin\left(\kappa\chi(x)+\tfrac{1}{4}\pi\right)R(x)-\cos\left(\kappa\chi(x)+\tfrac{1}{4}\pi\right)S(x)\Bigr),
\end{aligned}
\tag{4.13}
$$

where $\chi'(x)$ is given in (4.7) and

$$
\begin{aligned}
A(x) &= \frac{1}{\sqrt{w(x)U(x)}},\\
R(x) &= P(x) - \frac{1}{\kappa\chi'(x)}Q'(x) - \frac{A'(x)}{\kappa A(x)\chi'(x)}Q(x),\\
S(x) &= Q(x) + \frac{1}{\kappa\chi'(x)}P'(x) + \frac{A'(x)}{\kappa A(x)\chi'(x)}P(x).
\end{aligned}
\tag{4.14}
$$

We have the expansions

$$
R(x) \sim \sum_{j=0}^{\infty}\frac{r_j}{\kappa^j}, \quad S(x) \sim \sum_{j=0}^{\infty}\frac{s_j}{\kappa^j}, \tag{4.15}
$$

where the coefficients follow from the relations in (4.14). The first coefficients are $r_0 = p_0 = 1$, $s_0 = q_0 = 0$, and

$$
r_1 = p_1, \quad s_1 = q_1 + \frac{A'(x)}{A(x)\chi'(x)}. \tag{4.16}
$$

5 Expansion of the zeros

A zero x_ℓ, $1 \le \ell \le n$, of $P_n^{(\alpha,\beta)}(x)$ follows from the zeros of (see (4.8))

$$W(x) = \cos\left(\kappa\chi(x) + \tfrac{1}{4}\pi\right) P(x) + \sin\left(\kappa\chi(x) + \tfrac{1}{4}\pi\right) Q(x), \tag{5.1}$$

where $\chi(x)$ is defined in (4.8). For a first approximation we put the cosine term equal to zero. That is, we can write

$$\kappa\chi(x) + \tfrac{1}{4}\pi = \tfrac{1}{2}\pi - (n+1-\ell)\pi, \tag{5.2}$$

where ℓ is some integer. It appears that this choice in the right-hand side is convenient for finding the ℓth zero.

Because the expansions in (4.9) are valid for x properly inside (x_-, x_+), we may expect that the approximations of the zeros in the middle of this interval will be much better than those near the endpoints. We describe how to compute approximations of all n zeros by considering the zeros of $\cos\left(\chi(x)\kappa + \frac{1}{4}\pi\right)$.

We start with $\ell = 1$ and using (5.2) we compute $\chi_1 = \left(\frac{1}{4} - n\right)\pi/\kappa$. Next we compute an approximation of the zero x_1 by inverting the equation $\chi(x) = \chi_1$, where $\chi(x)$ is defined in (4.8). For a Newton procedure we can use $x_- + 1/n$ as a starting value.

Example 5.1. When we take $\alpha = 50$, $\beta = 41$, $n = 25$, we have $\kappa = 71$, $\sigma = 91/142$, $\tau = 9/142$. We find $\chi_1 = -1.095133$ and the starting value of the Newton procedure is $x = -0.7667437$. We find $x_1 \doteq -0.7415548$. Comparing this with the first zero computed by using the solver of Maple to compute the zeros of the Jacobi polynomial with Digits = 16, we find a relative error 0.00074.

For the next zero x_2, we compute χ_2 from (5.2) with $\ell = 2$, use x_1 as a starting value for the Newton procedure, and find $x_2 \doteq -0.682106$, with relative error 0.00032. And so on. The best result is for x_{13} with relative error 0.000013, and the worst result is for x_{25} with a relative error 0.0010. $\diamondsuit$

Remark 5.2. We don't have a proof that the found zero always corresponds with the ℓth zero, when we start with (5.2). In a number of tests we have found all agreement with this choice. $\triangle$

To obtain higher approximations of the zeros, we use the method described in our earlier papers. We assume that the zero x_ℓ has an asymptotic expansion

$$x_\ell = \xi_0 + \varepsilon, \quad \varepsilon \sim \frac{\xi_2}{\kappa^2} + \frac{\xi_4}{\kappa^4} + \ldots, \tag{5.3}$$

where ξ_0 is the value obtained as a first approximation by the method just described.

The function $W(x)$ defined in (5.1) can be expanded at ξ_0 and we have

$$W(x_\ell) = W(\xi_0 + \varepsilon) = W(\xi_0) + \frac{\varepsilon}{1!}W'(\xi_0) + \frac{\varepsilon^2}{2!}W''(\xi_0) + \ldots = 0, \tag{5.4}$$

where the derivatives are with respect to x. We find upon substituting the expansions of ε and those of P and Q given (4.9), and comparing equal powers of κ, that the first coefficients are

$$
\begin{aligned}
\xi_2 &= \frac{\left(1-x^2\right) q_1(x)}{U(x)},\\
\xi_4 &= \frac{1}{6U(x)^4}\Bigl(3x^5q_1^2+3x^4q_1^2\sigma\tau-6x^3q_1^2-6x^2q_1^2\sigma\tau+3q_1^2x+3q_1^2\sigma\tau+\\
&\quad \left(6q_1'q_1x^4+6x^3q_1^2-12q_1'x^2q_1-6xq_1^2+6q_1'q_1\right)U(x)^2+\\
&\quad \left(6p_2x^2q_1+2q_1^3x^2+6q_3-6p_2q_1-6q_3x^2-2q_1^3\right)U(x)^3\Bigr),
\end{aligned}
\tag{5.5}
$$

where $U(x)$ is defined in (3.10), and x takes the value of the first approximation of the zero as obtained in Example 5.1.

When we take the same values $\alpha=50$, $\beta=41$, $n=25$ as in Example 5.1, and use (5.3) with the term ξ_2/κ^2 included, we obtain for the zero x_{13} a relative error 0.80×10^{-9}. With also the term ξ_4/κ^4 included we find for x_{13} a relative error 0.13×10^{-12}.

A more extensive test of the expansion is shown in Figure 2. The label ℓ in the abscissa represents the order of the zero (starting from $\ell=1$ for the smallest zero). In this figure we compare the approximations to the zeros obtained with the asymptotic expansion against the results of a Maple implementation (with a large number of digits) of an iterative algorithm which uses the global fixed point method of [12]. The Jacobi polynomials used in this algorithm are computed by using the intrinsic Maple function. As before, we use (5.3) with the term ξ_2/κ^2 included. As can be seen, for $n=100$ the use of the expansion allows the computation of the zeros x_ℓ, $10\le\ell\le90$, with absolute error less than 10^{-8}. When $n=1000$, an absolute accuracy better than 10^{-12} can be obtained for about 90% of the zeros of the Jacobi polynomials. The results become less accurate for the zeros near the endpoints ±1, as expected.

In Figure 3 we show the absolute errors for $n=100$ and $\alpha=50$, $\beta=41$ compared with $\alpha=150$, $\beta=141$. We see that the accuracy is slightly better for the larger parameters, and that the asymptotics is quite uniform when α and β assume larger values.

6 The weights of the Gauss-Jacobi quadrature

As we did in [6], and in our earlier paper [4] for the Gauss–Hermite and Gauss–Laguerre quadratures, it is convenient to introduce scaled weights. In terms of the derivatives of the Jacobi polynomials, the classical form of the weights of

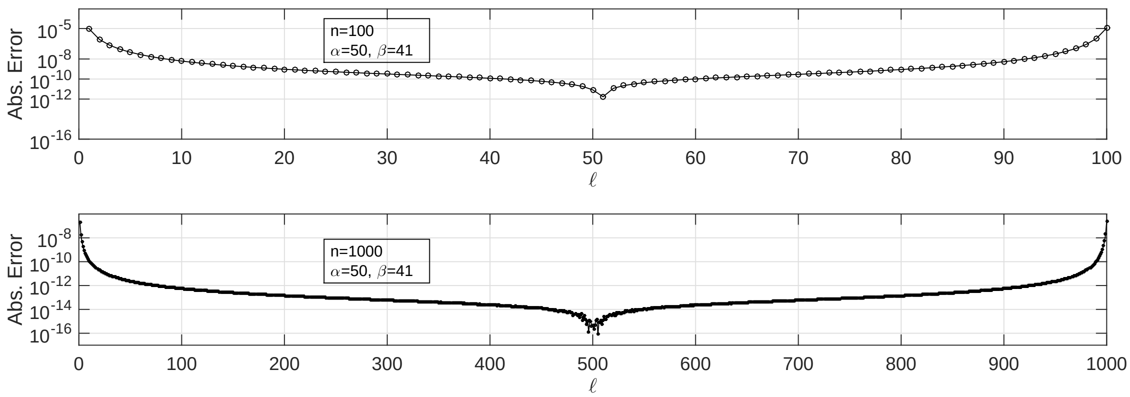

Figure 2: Performance of the asymptotic expansion for computing the zeros of $P_n^{(\alpha,\beta)}(x)$ for $\alpha = 50$, $\beta = 41$ and $n = 100$, 1000.

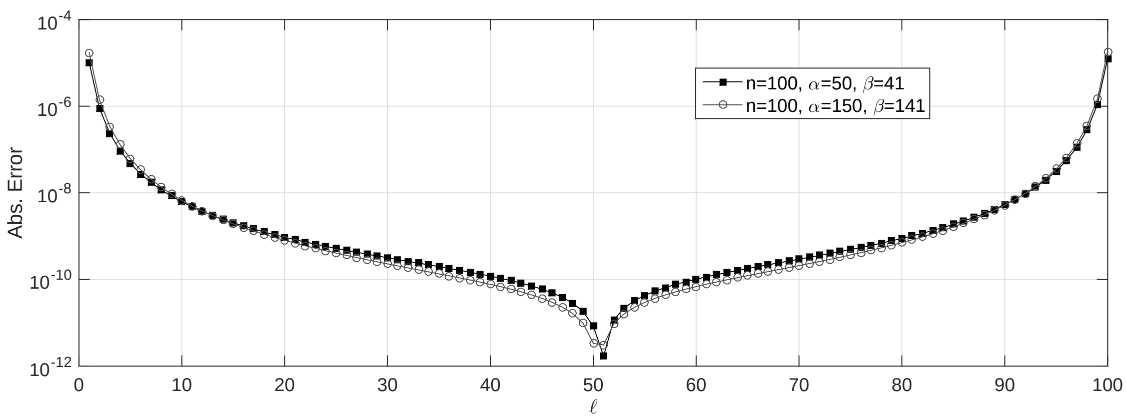

Figure 3: Performance of the asymptotic expansion for computing the zeros for $n = 100$ and $\alpha = 50$, $\beta = 41$ compared with $\alpha = 150$, $\beta = 141$.

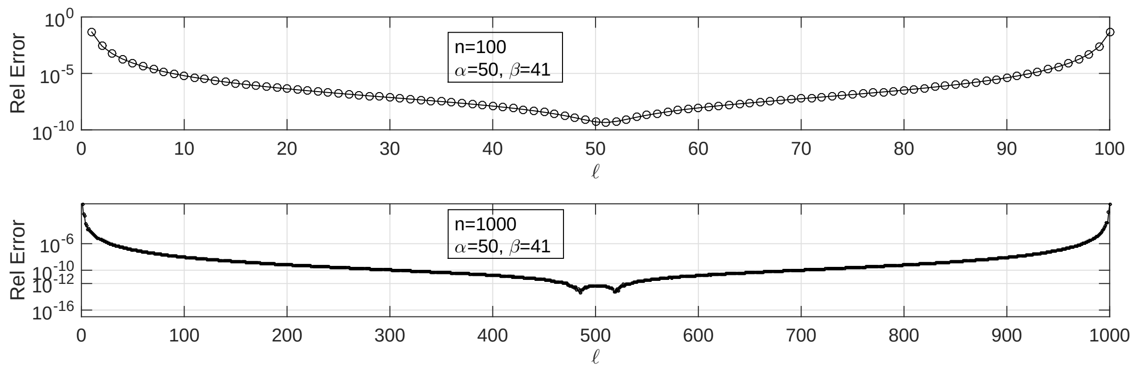

Figure 4: Performance of the computation of the weights w_ℓ by using the asymptotic expansion of the Jacobi polynomial for $\alpha = 50$, $\beta = 41$ and $n = 100,\, 1000$.

the Gauss-Jacobi quadrature can be written as

$$\begin{aligned} w_\ell &= \frac{M_{n,\alpha,\beta}}{(1-x_\ell^2)\, P_n^{(\alpha,\beta)\prime}(x_\ell)^2}, \\ M_{n,\alpha,\beta} &= 2^{\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}. \end{aligned} \tag{6.1}$$

In Figure 4 we show the relative errors in the computation of the weights w_ℓ defined in (6.1), with the derivative of the Jacobi polynomial computed by using the relation in (4.12). We have used the representation in (4.8), with the asymptotic series (4.9) truncated after $j = 3$ and the expansion (5.3) for the nodes with the term ξ_2/κ^2 included. The relative errors are obtained by using high-precision results computed by using Maple.

As an alternative we consider the scaled weights defined by

$$\omega_\ell = \frac{1}{v'(x_\ell)^2}, \tag{6.2}$$

where

$$v(x) = C_{n,\alpha,\beta}\,(1-x)^a(1+x)^b P_n^{(\alpha,\beta)}(x), \tag{6.3}$$

and we choose a and b such that $v''(x_\ell) = 0$; $C_{n,\alpha,\beta}$ does not depend on x, and will be chosen later. We have

$$\begin{aligned} v'(x) = C_{n,\alpha,\beta}\Bigl(&\bigl(-a(1-x)^{a-1}(1+x)^b + b(1-x)^a(1+x)^{b-1}\bigr)\, P_n^{(\alpha,\beta)}(x)\; + \\ &(1-x)^a(1+x)^b P_n^{(\alpha,\beta)\prime}(x)\Bigr). \end{aligned} \tag{6.4}$$

Evaluating $v''(x_\ell)$, we find

$$v''(x_\ell) = C_{n,\alpha,\beta}(1-x_\ell)^a(1+x_\ell)^b(1-x_\ell^2)\,\times \\ \Bigl((1-x_\ell^2)P_n^{(\alpha,\beta)\prime\prime}(x_\ell) + 2\,(b-a-(a+b)x_\ell)\,P_n^{(\alpha,\beta)\prime}(x_\ell)\Bigr)\,, \tag{6.5}$$

where we skip the term containing $P_n^{(\alpha,\beta)}(x_\ell)$, because x_ℓ is a zero.

The differential equation of the Jacobi polynomials is

$$\left(1-x^2\right)y''(x) + (\beta-\alpha-(\alpha+\beta+2)x)\,y'(x) + n(\alpha+\beta+n+1)y(x) = 0, \tag{6.6}$$

and we see that $v''(x_\ell) = 0$ if we take $a = \frac{1}{2}(\alpha+1)$, $b = \frac{1}{2}(\beta+1)$.

We obtain

$$v(x) = C_{n,\alpha,\beta}\,(1-x)^{\frac{1}{2}(\alpha+1)}(1+x)^{\frac{1}{2}(\beta+1)}P_n^{(\alpha,\beta)}(x), \tag{6.7}$$

with properties

$$v'(x_\ell) = C_{n,\alpha,\beta}\,(1-x_\ell)^{\frac{1}{2}(\alpha+1)}(1+x_\ell)^{\frac{1}{2}(\beta+1)}P_n^{(\alpha,\beta)\prime}(x_\ell), \quad v''(x_\ell) = 0. \tag{6.8}$$

The weights w_ℓ are related with the scaled weights ω_ℓ by

$$w_\ell = M_{n,\alpha,\beta}C_{n,\alpha,\beta}^2(1-x_\ell)^\alpha(1+x_\ell)^\beta\omega_\ell. \tag{6.9}$$

The advantage of computing scaled weights is that, similarly as described in [4], scaled weights do not underflow/overflow for large parameters. In additional, they are well-conditioned as a function of the roots x_ℓ. Indeed, introducing the notation

$$V(x) = \frac{1}{v'(x)^2}, \tag{6.10}$$

the scaled weights are $\omega_\ell = V(x_\ell)$ and $V'(x_\ell) = 0$ because $v''(x_\ell) = 0$. The vanishing derivative of $V(x)$ at x_ℓ may result in a more accurate numerical evaluation of the scaled weights.

When considering the representation of the Jacobi polynomials in (4.8), the function $v(x)$ can be written as

$$v(x) = \frac{2^{\frac{1}{2}(\alpha+\beta+1)}}{\sqrt{\pi\kappa}}\,C_{n,\alpha,\beta}e^{-\kappa\psi}Z(x)W(x), \quad Z(x) = \sqrt{\frac{1-x^2}{U(x)}}, \tag{6.11}$$

where $U(x)$ is defined in (3.10). For scaling $v(x)$ we choose

$$C_{n,\alpha,\beta} = 2^{-\frac{1}{2}(\alpha+\beta+1)}e^{\kappa\psi}. \tag{6.12}$$

This gives

$$v(x) = \frac{Z(x)W(x)}{\sqrt{\pi\kappa}}. \tag{6.13}$$

For the numerical computation of ψ defined in (4.6) for small values of σ or τ, we can use the expansion

$$(1-x)\ln(1-x)+(1+x)\ln(1+x)=\sum_{k=1}^{\infty}\frac{x^{2k}}{k(2k-1)}, \quad |x|<1. \tag{6.14}$$

For computing the modified Gauss weights it is convenient to have an expansion of the derivative of the function $v(x)$ of (6.13), with $W(x)$ defined in (4.8) and $Z(x)$ in (6.11).

We have

$$\frac{d}{dx}v(x)=-\sqrt{\frac{\kappa}{\pi}}\chi'(x)Z(x)\left(\sin\left(\kappa\chi(x)+\tfrac14\pi\right)M(x)-\cos\left(\kappa\chi(x)+\tfrac14\pi\right)N(x)\right), \tag{6.15}$$

where $\chi'(x)$ is given in (4.7) and

$$\begin{aligned} M(x)&=P(x)-\frac{1}{\kappa}p(x)Q'(x)-\frac{1}{\kappa}q(x)Q(x),\\ N(x)&=Q(x)+\frac{1}{\kappa}p(x)P'(x)+\frac{1}{\kappa}q(x)P(x), \end{aligned} \tag{6.16}$$

where

$$\begin{aligned} p(x)&=\frac{1}{\chi'(x)}=\frac{1-x^2}{U(x)},\\ q(x)&=\frac{Z'(x)}{Z(x)\chi'(x)}=\frac{(1-x^2)(x+\sigma\tau)-2xU^2(x)}{2U^3(x)}. \end{aligned} \tag{6.17}$$

We have the expansions

$$M(x)\sim\sum_{j=0}^{\infty}\frac{m_j}{\kappa^j}, \quad N(x)\sim\sum_{j=0}^{\infty}\frac{n_j}{\kappa^j}, \tag{6.18}$$

where the coefficients follow from the relations in (4.14). The first coefficients are $m_0=p_0=1$, $n_0=q_0=0$, and for $j=1,2,3,\ldots$

$$\begin{aligned} m_j&=p_j-p(x)q'_{j-1}-q(x)q_{j-1},\\ n_j&=q_j+p(x)p'_{j-1}+q(x)p_{j-1}. \end{aligned} \tag{6.19}$$

As an example, Figure 5 shows the performance of the asymptotic expansion (6.15) for computing the scaled weights (6.2) for $\alpha=50$, $\beta=41$ and $n=1000$. The computation of the non-scaled weights (6.1) is shown as comparison.

In Figure 6 and Figure 7 we compare the effect of computing the weights w_ℓ defined in (6.1) and the scaled weights ω_ℓ defined in (6.2) when we compute these weights with the asymptotic expansion of the zeros in (5.3) with the term ξ_4/κ^4 included or not included. From these computations it follows that the that the scaled weights are well-conditioned as a function of the nodes and therefore they are not so critically dependent on the accuracy of the nodes. Contrary the non-scaled weights have worse condition and the accuracy of the nodes is more important.

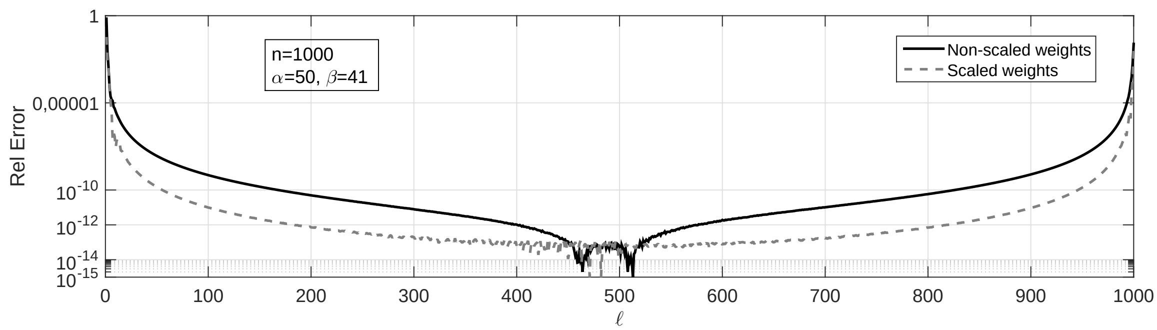

Figure 5: Comparison of the performance of the asymptotic expansions for computing non-scaled (6.1) and scaled (6.2) weights for $\alpha = 50$, $\beta = 41$ and $n = 1000$.

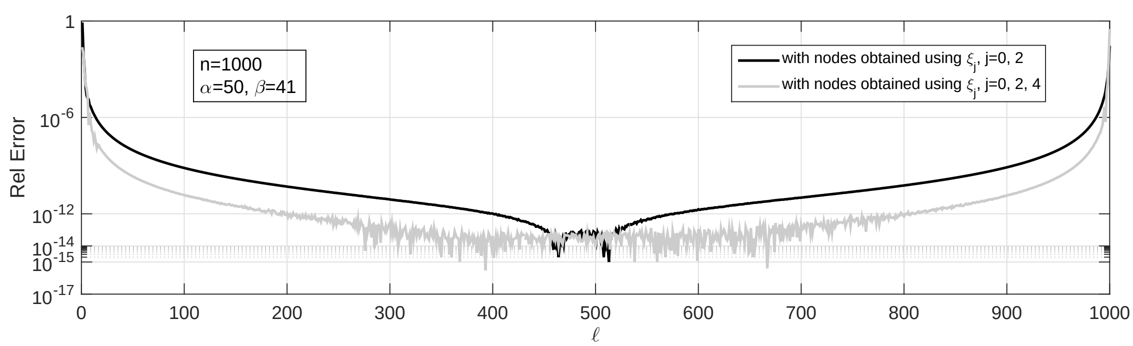

Figure 6: Performance of the computation of the weights w_ℓ defined in (6.1) by using the asymptotic expansion of the Jacobi polynomial for $\alpha = 50$, $\beta = 41$ and $n = 1000$. The comparison is between the expansion of the zeros in (5.3) with the term ξ_4/κ^4 included or not included.

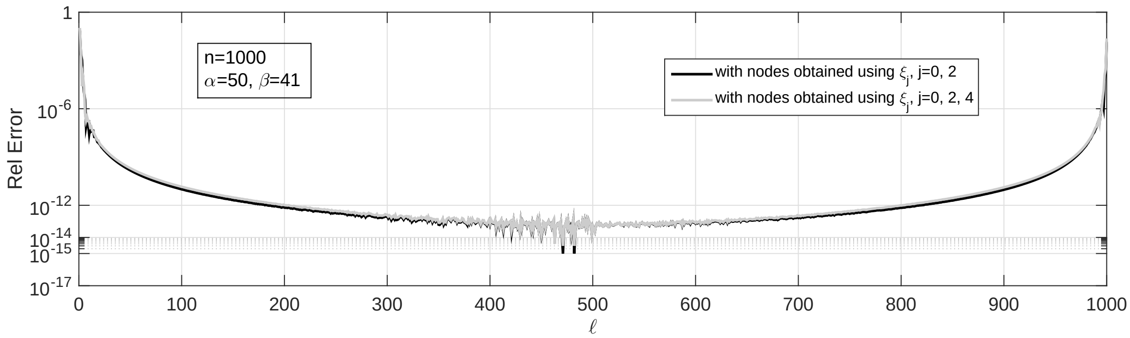

Figure 7: Same as in Figure 6 for the scaled weights ω_ℓ defined in (6.2).

6.1 About quantities appearing in the weights.

First we consider the term $e^{\kappa\psi}$, with ψ given in (4.6). Using the relations in (3.7), we have

$$\begin{aligned} \kappa(1+\tau) &= n+\alpha+\tfrac12, & \kappa(1-\tau) &= n+\beta+\tfrac12,\\ \kappa(1+\sigma) &= n+\alpha+\beta+\tfrac12, & \kappa(1-\sigma) &= n+\tfrac12, \end{aligned} \tag{6.20}$$

and this gives

$$\begin{aligned} e^{2\kappa\psi} &= \frac{\left(n+\alpha+\beta+\frac12\right)^{n+\alpha+\beta+\frac12}\left(n+\frac12\right)^{n+\frac12}}{\left(n+\alpha+\frac12\right)^{n+\alpha+\frac12}\left(n+\beta+\frac12\right)^{n+\beta+\frac12}}\\ &= \frac{\Gamma\left(n+\alpha+\beta+\frac12\right)\Gamma\left(n+\frac12\right)}{\Gamma\left(n+\alpha+\frac12\right)\Gamma\left(n+\beta+\frac12\right)}\ \frac{\Gamma^*\left(n+\alpha+\frac12\right)\Gamma^*\left(n+\beta+\frac12\right)}{\Gamma^*\left(n+\alpha+\beta+\frac12\right)\Gamma^*\left(n+\frac12\right)}\times\\ &\quad\sqrt{\frac{\left(n+\alpha+\beta+\frac12\right)\left(n+\frac12\right)}{\left(n+\alpha+\frac12\right)\left(n+\beta+\frac12\right)}}, \end{aligned} \tag{6.21}$$

where

$$\Gamma^*(z) = \sqrt{z/(2\pi)}\, e^z z^{-z}\Gamma(z), \quad \operatorname{ph} z \in (-\pi,\pi), \quad z\neq 0. \tag{6.22}$$

We have $\Gamma^*(z) = 1+\mathcal{O}(1/z)$ as $z\to\infty$.

It follows that (see (6.1) and (6.9))

$$\begin{aligned} M_{n,\alpha,\beta}C^2_{n,\alpha,\beta} &= \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)\,\Gamma\left(n+\alpha+\beta+\frac12\right)\Gamma\left(n+\frac12\right)}{\Gamma\left(n+\alpha+\frac12\right)\Gamma\left(n+\beta+\frac12\right)\Gamma(n+\alpha+\beta+1)\,\Gamma(n+1)}\times\\ &\quad\frac{\Gamma^*\left(n+\alpha+\frac12\right)\Gamma^*\left(n+\beta+\frac12\right)}{\Gamma^*\left(n+\alpha+\beta+\frac12\right)\Gamma^*\left(n+\frac12\right)}\sqrt{\frac{\left(n+\alpha+\beta+\frac12\right)\left(n+\frac12\right)}{\left(n+\alpha+\frac12\right)\left(n+\beta+\frac12\right)}}. \end{aligned} \tag{6.23}$$

Using $\Gamma\left(z+\frac12\right)/\Gamma(z)\sim z^{\frac12}$ as $z\to\infty$, we see that, in the case that α, β and n are all large, we have $M_{n,\alpha,\beta}C^2_{n,\alpha,\beta}\sim 1$, and that, when using more details on expansions of gamma functions and ratios thereof (see [14, §6.5]), we can obtain

$$M_{n,\alpha,\beta}C^2_{n,\alpha,\beta} \sim 1 + \frac{\sigma^2-\tau^2}{12(1-\sigma^2)(1-\tau^2)\kappa} + \frac{(\sigma^2-\tau^2)^2}{288(1-\sigma^2)^2(1-\tau^2)^2\kappa^2} + \cdots, \tag{6.24}$$

again, when α, β and n are all large.

As observed in the first lines of Section 4, in the present asymptotics we assume that σ and $|\tau|$ are bounded away from 1.

Acknowledgments

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