ZEIST

CONFERENTIE VAN NUMERIEK WISKUNDIGEN

CONFERENTIEOORD

WOUDSCHOTEN

10, 11 EN 12 OKTOBER 1977

STICHTING MATHEMATISCH CENTRUM

2e Boerhaavestraat 49, Amsterdam-1005

Telefoon (020) 947272

DOEL VAN DE BIJEENKOMST

Het doel van de bijeenkomst is hen die betrokken zijn bij onderzoek en/of onderwijs in de numerieke wiskunde de gelegenheid te geven kennis te nemen van de stand van zaken op enkele gebieden der numerieke wiskunde en de gelegenheid te geven met elkaar en met de buitenlandse sprekers intensief contact te hebben.

THEMA'S

De thema's zijn

- Numeriek oplossen van stelsels niet-lineaire vergelijkingen
- Numeriek oplossen van integraalvergelijkingen.

ORGANISATIE

De organisatie is in handen van de heren Dekker, Slagt, Spijker en Wetterling, en van het Mathematisch Centrum.

SPREKERS

P.M. ANSELONE, Universität Hamburg (tijdelijk), BRD.

C.T.H. BAKER, University of Manchester, United Kingdom.

E. BOHL, Universität Münster, BRD.

L.M. DELVES, University of Liverpool, United Kingdom.

P. DEUFLHARD, Technische Universität München, BRD.

F. ROBERT, University of Grenoble, France.

PROGRAMMA

maandag

10.00-11.15	aankomst en koffie	15.15-15.45	thee
11.15-12.30	opening, Anselone	15.45-16.45	Baker
12.45	lunch	16.45-17.45	Bohl
		18.15	diner

dinsdag

8.00	ontbijt	12.45	lunch
9.00-10.00	Delves	15.15-15.45	thee
10.00-10.30	koffie	15.45-16.45	Baker
10.30-11.30	Robert	16.45-17.45	Bohl
11.30-12.30	Deuflhard	18.15	diner

woensdag

8.00	ontbijt	12.45	lunch
9.00-10.00	Anselone	14.00-15.00	Robert
10.00-10.30	koffie	15.00	sluiting, thee, vertrek
10.30-11.30	Delves		
11.30-12.30	Deuflhard		

TITELS EN SAMENVATTINGEN VOORDRACHTEN

APPROXIMATE SOLUTIONS OF WEAKLY SINGULAR INTEGRAL EQUATIONS

P. M. Anselone

I. Introduction and Summary   II. Mathematical Analysis

ABSTRACT

Numerical integration is widely used, alone or with other techniques, to obtain approximate solutions of integral equations. A convergence and error analysis for such approximate solutions is given by the collectively compact operator approximation theory. In basic form, this theory concerns operators K and K_n on a Banach space X such that $K_n \to K$ pointwise as $n \to \infty$, K is compact, and $\{K_n\}$ is collectively compact. Thus, $\overline{KB}$ and $\overline{\bigcup_n K_n B}$ are compact for any bounded $B \subset X$. For example, $X = C[0,1]$ with the max norm, K is an integral operator with a continuous kernel, and the K_n are numerical-integral operators with a convergent quadrature rule.

Kernels with integrable singularities require special techniques. If the singularity is simple enough, the kernel factorization method of Atkinson applies. Our concern here is with double approximations K_{mn} obtained by bounded kernel approximation and numerical integration. With care, the collectively compact theory still applies. New results on the numerical integration of singular functions are developed for this approach. An extension of the double scheme covers the singularity subtraction method of Kantorovich and Krylov. We give applications to neutron transport and potential theory. The ideas carry over to problems in more than one dimension.

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3. Anselone, P. M., Approximate Solutions of Weakly Singular Integral Equations, in preparation.

4. Anselone, P. M. and A. G. Gibbs, Convergence and Error Bounds for Numerical Solutions of Integral Equations of Neutron Transport, submitted for publication.

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6. Atkinson, K. E., The Numerical Solution of Fredholm Integral Equations of the Second Kind, SIAM J. Num. Anal. 4 (1967), 337-348.

7. Davis, P. J. and P. Rabinowitz, Methods of Numerical Integration, Academic Press, 1975.

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Numerical treatment of Volterra integral equations

C.T.H. Baker

Abstract

We shall discuss numerical methods for Volterra integral equations considering, in the main, those of the type

$$f(x) - \int_0^x F(x,y;f(y))\,dy = g(x) \qquad (x \geq 0),$$

where $F(x,y;v)$, $g(x)$ are given and $f(x)$ is sought, but with some mention of equations of the type

$$\int_0^x F(x,y;f(y))\,dy = g(x) \qquad (x \geq 0)$$

(which may be ill-posed).

A wide class of numerical methods arise by discretizing the integral terms using quadrature rules, and the resulting numerical methods produce approximate values of the solution in a step-by-step or block-by-block fashion. (The methods thus bear similarities with methods employed in the treatment of ordinary differential equations or delay-differential equations.) The accuracy obtained depends both upon the local truncation errors and on the stability of the scheme. Some difficulties in devizing an adaptive algorithm will be noted.

Principal References:

1. Baker C.T.H. The Numerical Treatment of Integral Equations. Clarendon Press Oxford. (In press, appearing Fall 1977).

2. Delves L.M. and Walsh J.E. (editors) The Numerical Solution of Integral equations. Clarendon Press, Oxford 1974.

The above contain further references to the research journals.

3. Noble B. A bibliography on: Methods for solving integral equations. MRC Tech Summ Reports 1176&1177, Madison Wisconsin.

4. Jacobs, D.A.H. (Editor) The state of the art in Numerical Analysis. Academic 1977

Additional reference:

5. Baker, C.T.H. and Keech, M.S. Stability regions in the numerical treatment of Volterra integral equations. SIAM Journal on Numerical Analysis (in press).

The above paper contains a description of numerical methods as well as the corresponding stability results. An earlier version is obtainable as N.A. Technical Report No. 12, Department of Mathematics, University of Manchester.

The numerical solution of the eigenvalue problem for an integral equation

Abstract

Given a kernel $K(x,y)$ defined for $a \leqslant x, y \leqslant b$, the problem of interest is the solution, with $0 < \int_a^b |f_r(y)|^2 dy < \infty$, of

$$\int_a^b K(x,y) f_r(y) dy = \kappa_r f_r(x) \qquad (a \leqslant x \leqslant b) \quad .$$

With suitable conditions on $K(x,y)$, and with $|ab| < \infty$, at most a countable number of eigenvalues κ_r exist (counting according to either geometric or algebraic multiplication).

In the numerical solution of the problem we derive a matrix eigenvalue problem with a finite number of eigenvalues. The matrix problem is usually derived either by replacing integration by a suitable process of numerical integration or by an expansion method associated with an error distribution principle. (Rayleigh-Ritz methods, though based upon a variational principle, fall into the latter class.)

The nature of convergence results and error bounds, and the way these are derived, proves interesting, whilst the role of a condition number, akin to that of the algebraic eigenvalue problem, is of some significance.

Principal References:

(1) - (4) above and also

6. Anselone, P.M. Collectively compact operator approximation theory Prentice Hall 1971

7. Atkinson, K.E. The numerical solution of the eigenvalue problem for compact integral operators. Trans Amer Math Soc 129 (1967) ~~pp485~~ pp 458-465.

8. Bückner, H. Die Praktische Behandlung von Integralgleichungen. Springer 1952.

Abstracts for Lectures on

1) Numerical Methods for the solution of Fredholm Integral Equations.

2) The Design of a Package for solving integral and integro-differential equations.

L.M. Delves,

Department of Computational and Statistical Science,

University of Liverpool.

0) Overview

The integral equations most frequently discussed in the mathematical or numerical analysis literature are those which come under the classical heading of :

(a) Volterra equation of the first kind

$$0 = g(x) + \int_0^x K(x, y, f(y))\, dy \qquad (1.1)$$

(b) Volterra equations of the second kind

$$f(x) = g(x) + \int_0^x K(x, y, f(y))\, dy \qquad (1.2)$$

(c) Linear Fredholm equations of the first kind

$$0 = g(x) + \int_a^b K(x,y)\, f(y)\, dy \qquad (1.3)$$

(d) Linear Fredholm equations of the second kind

$$f(x) = g(x) + \int_a^b K(x,y)\, f(y)\, dy \qquad (1.4)$$

(e) Linear Fredholm equations of the third kind (eigenvalue equations)

$$f(x) = 0 + \lambda \int^b K(x,y)\, f(y)\, dy \qquad (1.5)$$

Here, $f(x)$ is the unknown function; $K(x,y)$ or $K(x,y,f(y))$ is the kernel of the equation; and $g(x)$ is a known driving term. A considerable literature exists describing and analysing methods for those classes of problems, under various assumptions on K and g. Unfortunately, with the exception of class (c), they seldom seem to arise in practice. The equations which do arise are very varied in form; and the provision of a manageably small set of routines capable of solving a reasonably wide class of pratical problem is correspondingly difficult. We discuss this in two stages :

Lecture 1

We take class (d) as the easiest available, and discuss the type of approximation schemes which can be used for this class. These include in particular

(i) Nystrom methods or quadrature methods : the integral in (1.3) is replaced by a p-point numerical quadrature rule with weights w_i and nodes x_i, $i = 1, 2 \ldots p$, and a set of algebraic equations set up and solved for the appropriate solution of (x_i), $i = 1 \ldots p$ at the nodes

(ii) Expansion methods

The solution is approximated by a linear expansion of the form

$$f(x) \approx f_p(x) = \sum_{i=1}^{N} a_i h_i(x)$$

and equations derived and solved for the coefficients a_i, $i = 1 \ldots p$.

Routines are readily available which implement the Nystrom method with a wide variety of quadrature rules, or expansion methods with one of a variety of algorithms for determining the coefficient a_i, either automatically (the user specifies an accuracy, and the routine chooses p appropriately) or non-automatically (the user specifies p and the routine returns f_p and perhaps an error estimate.

For the user with a problem, the question is which routine is best to use. We discuss what "best" should mean, and how it is measured; and describe briefly, and give comparative results on the performance of, a number of different automatic and non-automatic algorithms for problems of class (d)).

Lecture (2)

Writing (and testing) routines to solve Fredholm equations of the second kind is an attractive game for numerical analysts, akin to solving Laplace's equation on a square : the problem is clearcut, numerically straightforward, and provides a gentle testing ground for new methods. Most practical problems are not of class (d); nor, alas, are they usually of any of the extended classes (a) - (e). For examples which do appear in practice, we cite the following, taken more or less at random from the literature :

(i) Population Competition (Downham & Shah (1976); Downham (1974))

$$\underline{f}(x) = \underline{g}(x) + \int_a^b K(\underline{x} - \underline{y})) \, h(\underline{y}) \, d\underline{y}; \quad \underline{f}^T = (f_1, f_2, f_3) \qquad - (2.1)$$

That is, a set of three coupled, nonlinear Fredholm second kind equations.

(2) Quantum Scattering: Close-Coupled Calculations (Horn and Fraser (1975))

$$f(x) = g(x) - \lambda \int_0^\infty K_1 (x,y) \int_0^\infty K_2 (y,z) \, f(z) \, dz \qquad - (2.2)$$

An equation of Fredholm origin but with infinite range and iterated kernel

(3) Problem area as (2), but a different system and an alternative formulation (Chan and Fraser (1973))

$$\left(\frac{d^2}{dx^2} + k^2\right) \underline{f}(x) = \underline{g}(x) + \int_0^\infty K(x,y)\, \underline{f}(y)\, dy \qquad - (2.3)$$

$$\underline{f}^T = (f_1, f_2); \quad K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} .$$

"Suitable" boundary conditions are supplied in addition. Equations (3) are integrodifferential, and coupled, with a linear Fredholm kernel.

(4) Currents in a superconducting strip (Rhoderick and Wilson (1962))

$$g(x) = \frac{1}{\pi} \int_0^1 \frac{t-x}{(t-x)^2+a^2}\, f(t)\, dt \qquad - (2.4)$$

A genuine, and not uncommon example of a first kind linear Fredholm equation.

(5) Flow round a Hydrofoil (Kershaw 1974))

$$f(x) = g(x) + \lambda \int_C f(y)\, K(x,y)\, dy \qquad - (2.5)$$

subject to $\int_C f(y)\, dy = 0$, C as a closed contour

This is an equation of Fredholm type, but with a subsidiary condition to be satisfied.

These examples illustrate the wide variety of equations encountered in practice; only (4) can be regarded as being of a "standard" type.

Additional difficulties are apparent when we look at the structure of the equations involved. Problems may be:

(i) Linear or non-linear

(ii) Coupled equations or single

(iii) Finite range or infinite

(iv) Well or ill-conditioned

(v) Smooth or nonsmooth (Kernels and driving terms)

(vi) Without or with side-conditions

This variety of problem is very difficult to service with a small number of "complete" routines; that is, routines which, on being called, return an approximate solution to the given problem. We discuss in this lecture an alternative approach which avoids this problem. The examples given above, although varied, can be seen to be made up from only three rather standard operators:

(a) $\int_a^b dy\, K\,(x,y)$

(b) $\dfrac{d^2}{dx^2}$

(c) I - the unit operator

provided that we allow K to be nonlinear (in the unknown function), matrix valued (for coupled equations) and to cover if necessary infinite intervals.

It is therefore possible in principle to provide a set of routines which perform a discretisation of these basic operators, together with a mechanism for

(a) Piecing together operations to form a complete equation

(b) Solving the equation

It can then be left to the user to piece the bits together and to

Such a set of routines provides the "package" of the lecture title; we describe briefly how suitable discretisations may be constructed, based on an expansion method; how the operations of adding and multiplying integral operators can be performed within the proposed scheme; and the economics of the process.

A package of this type is currently under construction written in Algol 68; a brief discussion of the language choice, and of the way in which the user will access the package, will be included if time permits.

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[1] Beyn, W.-J., Theorie und Anwendung eines iterativen Verfahrens zur Lösung von Operatorgleichungen Hammersteinschen Typs. PHD Thesis, University of Münster, W. Germany (1975).

[2] Beyn, W.-J., Das Parallelenverfahren für Operatorgleichungen und seine Anwendung auf nichtlineare Randwertaufgaben. ISNM 31, 9-33, Birkhäuser-Verlag (1976).

[3] Bohl, E., Monotonie: Lösbarkeit und Numerik bei Operatorgleichungen, Springer Tracts in Natural Philosophy, Bd 25, Springer-Verlag, Berlin (1974).

[4] Bohl, E., Iterative procedures in the study of discrete analogues for nonlinear boundary value problems. Istituto per le Applicazioni del Calcolo "Mauro Picone" (IAC), Pubb. S. III - N. 107 (1975).

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Parallel Chord Methods for Nonlinear Systems arising
from Difference Approximations to
Hammerstein Equations.

Erich Bohl

Consider the Hammerstein equation

HEQ: $$x(t) = \int_a^b K(t,x) f(s,x(s))\,ds$$

and let the kernel satisfy one of the following conditions

(i) $K(t,s)$ is symmetric;
(ii) $K(t,s)$ is symmetric and nonnegative;
(iii) $K(t,s)$ is the Green's function of a differential operator.

A difference approximation of HEQ on a grid $\Omega_h \subset [a,b]$ with the step width $h > 0$ leads to a system of the form

DHEQ: $$A_h x = B_h F_h x \quad \text{on } \mathbb{R}^{\Omega_h}$$

with real matrices A_h, B_h and the mapping F_h assigning to $x \in \mathbb{R}^{\Omega_h}$ the vector $F_h x \in \mathbb{R}^{\Omega_h}$ with the components

$$(F_h x)(t) = f(t,x(t)) \qquad (t \in \Omega_h).$$

To solve DHEQ we study parallel chord methods of the form

PC: $$(A_h - B_h D_h) x^{n+1} = B_h (F_h - D_h) x^n,$$

where the diagonal matrix D_h is chosen appropriately. We give global and local conditions for PC to converge globally. Systems with more then one solution are studied.

D.Y. Downham (1974) Models of Clines. Adv. in App. Prob. Vol. 6 No. 1 pp 7-10.

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Peter D e u f l h a r d

Technische Universität München
Institut für Mathematik

A Modified Newton Method for Numrically Sensitive Systems of Nonlinear Equations

(survey talk)

Let

(0.1) $F(x) = 0$

denote the nonlinear system to be solved by the modified Newton method. This method is, for given x^0, defined by the iteration $(k = 0,1,\ldots)$

(0.2) $$x^{k+1} := x^k + \lambda_k \cdot \Delta x^k \quad , \quad 0 < \lambda_k \leq 1$$

$$\Delta x^k := - F'(x^k)^{-1} F(x^k)$$

where $F'(x^k)$ denotes the *Jacobian matrix* (assumed to be nonsingular, for the time being), Δx^k the *Newton correction* vector and λ_k are determined by virtue of a *monotonicity test*, say

(0.3) $$\| SF(x^{k+1}) \|_2 < \| SF(x^k) \|_2 \qquad k = 0,1,\ldots$$

for a selected scaling matrix S.

In view of the construction of an efficient Newton algorithm, several quite recent results will be discussed.

(I) *Affine invariance* [6]. Obviously, problem (0.1) is equivalent to the problem

$$AF(x) = 0$$

for any nonsingular matrix A. Moreover, the ordinary Newton sequence ($\lambda_k = 1$ for all k) is independent of A. That is why both the underlying theory and the algorithm are required to be affine invariant.

(II) *Natural scaling* [1]. Concept (I) above and extensive theoretical investigations suggest to employ

$$(0.4) \qquad S = - F'(x^k)^{-1} \quad \text{in } (0.3)$$

as a natural choice. This means the additional computation of the *simplified Newton correction*

$$\overline{\Delta x}^{k+1} := - F'(x^k)^{-1} \; F(x^{k+1})$$

(III) *Relaxation Strategy* [2]. On a theoretical basis, the following strategy for determining λ_k is proposed:

$$(0.5) \qquad \lambda_k^{(o)} := \begin{cases} \mu^k & \text{for } \mu^k \leq 0.7 \\ 1 & \text{for } \mu^k > 0.7 \end{cases} \qquad k \geq 1$$

where

$$\mu^k := \frac{\| \Delta x^{k-1} \|_2}{\| \overline{\Delta x}^k - \Delta x^k \|_2} \cdot \lambda_{k-1} .$$

If the monotonicity test (0.3)/(0.4) fails to accept x^{k+1} with the choice (0.5), then $\lambda_k^{(o)}$ is reduced to $\lambda_k^{(1)}, \ldots$.

(IV) *Rank Strategy* [1], [2]. The inverse may be replaced by the Penrose pseudo-inverse

$$\left(F'(x^k)\; D_k\right)^{-1} \longrightarrow \left(F'(x^k)\; D_k\right)^{+}$$

where D_k denotes a diagonal scaling matrix (also to be used in (0.4)). Successive pseudo-rank reduction is employed at selected iterates.

(V) *Rank - 1 approximations* [2]. If the evaluation of $F'(x^k)$, say by finite difference approximation, requires the main bulk of computing costs, then $F'(x^k)$ is approximated due to BROYDEN (slightly modified scated version). A selection criterion will be given.

Illustrating numerical examples will be presented - including real life applications.

Peter D e u f l h a r d

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A Stepsize Control for Discrete Continuation (or Homotopy, or Imbedding) Methods.

(Specialized talk)

A new stepsize control for continuation methods will be presented on a theoretical basis. The new method is helpful for the numerical solution of highly sensitive systems of nonlinear equations that permit a natural one-parameter imbedding taking into account the peculiarities of the problem to be solved. Compared with former empirical methods, there is a significant progress in both reliability and speed of the discrete continuation process. The improvement will be demonstrated by solving three realistic multiple shooting examples arising in mechanics (elastic deformation of a thin shallow spherical shell), quantum mechanics (singular Ginzburg-Landau model of superconductivity), and space flight engineering (optimal descent of the second stage of a space shuttle subject to reradiative heating constraints). In two of these examples, bifurcations occur: they are indicated by marked changes in the stepsize estimates.

The proceeding survey talk on the modified Newton method may be a helpful basis for understanding this topic.

R E F E R E N C E S (for both talks)

* [1] P. DEUFLHARD

A Modified Newton Method for the Solution of Ill-Conditioned Systems of Nonlinear Equations with Application to Multiple Shooting.

Numer. Math. 22, 289 - 315 (1974)

* [2] P. DEUFLHARD

A Relaxation Strategy for the Modified Newton Method.

Springer Lecture Notes 477, 59 - 73 (1975)

[3] P. DEUFLHARD, H.-J. PESCH, P. RENTROP

A Modified Continuation Method for the Numerical Solution of Nonlinear Two-Point Boundary Value Problems by Shooting Techniques.

Numer. Math. 26, 327 - 343 (1976)

[4] P. DEUFLHARD

A Stepsize Control for Continuation Methods with Special Application to Multiple Shooting Techniques.

Techn. Rep. TUM - MATH - 7627 (Dec. 76)

[5] P. DEUFLHARD, V. APOSTOLESCU

A Gauss-Newton Method for Nonlinear Least Squares Problems with Nonlinear Equality Constraints.

Techn. Rep. TUM-MATH-7607 (Dec. 76)

[6] P. DEUFLHARD, G. HEINDL

Affine Invariant Convergence Theorems for Newton's Method and Extensions to Related Methods.

Techn. Rep. TUM - MATH - 7723 (June 77)

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* : no reprints left

Meeting on numerical analysis
Zeist . Leiden
October 10-12 1977
F. Robert
University of Grenoble

Chaotic iterations for non linear fixed point equations in several variables.

(I) General survey and convergence results

Let $X_1 \ldots X_k$ be k sets and $X = \prod_1^k X_i$ their product (k large : 10 to 1000...)

Let $F: X \to X$. The problem is as follows:

> Find a fixed point ξ of F in X:
> $$\xi = F(\xi)$$
> that is:
> $$\xi_i = f_i(\xi_1 \ldots \xi_k) \quad k = 1, 2 \ldots k$$
>
> Questions: existence
> computation by 'chaotic iterations'

(1) Numerical context

$X_i = \mathbb{R}$ pointwise problems

$= \mathbb{R}^{\lambda_i}$ blockwise problems

$=$ Banach space

Basic examples: discretisation of partial differential equations on a grid.

① Discrete context (cf [19] [22])

X_i = a finite set ($i = 1, 2 \ldots k$) so that X is finite, too, but very large ($k = 100$ $|X_i| = 2$ → $|X| = 2^{100}$ for example)

Basic examples: evolution of 'cellular automata' in a discrete time (towards a stable configuration ?)

[Elementary properties of a mapping F of a finite set X into itself:
The graph of F is partitioned into connex components:

a) Some of them with a fixed point, then unique

b) Some of them without a fixed point

(a) (b)

case (a) convergence of successive approximations on F, in a finite number of steps

case (b) successive approximations is cyclic]

③ Chaotic iterations

For $J \subset \{1, 2 \ldots k\}$ define $F_J : X \to X$ by the following:

For $x = (x_1 \ldots x_k)$, $y = F_J(x) = (y_1 \ldots y_k)$ with

$$\begin{cases} y_i = x_i & \text{if } i \notin J \\ y_i = f_i(x_1 \ldots x_k) = f_i(x) & \text{if } i \in J \end{cases}$$

For computing $F_J(x)$, we relax in x only the components indexed by J. (of course, for $J = \{1, 2 \ldots k\}$, $F_J = F$)

if ξ is a fixed point of F, ξ is a fixed point of F_J $\forall J$

Example $k=4$ $J=\{1,3\}$

$$F_J(x)=\left|\begin{array}{l} f_1(x_1\ x_2\ x_3\ x_4)\\ x_2\\ f_3(x_1\ x_2\ x_3\ x_4)\\ x_4\end{array}\right.$$

A chaotic iteration on F (in order to compute a fixed point of F) is the following iterative scheme (very general!):

> Given: a starting point $x^0 \in X$
> a sequence $J_1, J_2 \ldots J_r \ldots$ of subsets of $\{1,2 \ldots k\}$
>
> compute
> $$x^{r+1} = F_{J_{r+1}}(x^r) \qquad (r=0,1,2\ldots)$$

Basic examples:

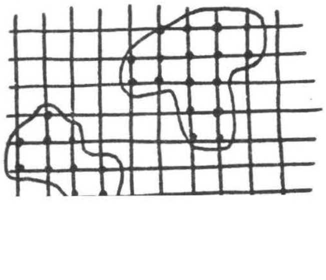
* $J_r=\{1,2 \ldots k\}$: successive approximations
* $J_r=\{r\}$ modulo k : general Gauss-Seidel iteration
* Southwell method for linear or nonlinear systems
* general chaotic iterations : given a fixed point equation

with a large number of variables (discretisation of a limit problem), 2 questions:

1. Convergence (mathematical analysis)
2. 'optimal' chaotic strategy.

The problem is to find which component (or group of components) are to be relaxed, in which order (strategy problem) so as to ensure 'fast convergence'

basic example on a grid:
(cf part II)



2 zones of slow convergence → to be relaxed more often.

④ Historical notes (cf Ⅲ; references)

Southwell method for linear systems. Convergence?

Ostrowski 'free steering' for H-systems (1955-1960)

Schechter (1959-1962)

Chazan-Miranker (1969) chaotic relaxation for linear systems (H-systems) with delays (→ application to multiprocessors)

→ Convergence studies for chaotic iterations (Robert (Grenoble), Miellou (Besançon) and pupils... from 1970 until now)

Recently: G. Baudet (Carnegie Mellon University)

Experimentation of chaotic iterations on C.m.m.p.

→ asynchronous iterations on multiprocessors.

To appear in JACM.

⑤ Convergence results for chaotic iterations

Available today! (cf Ⅲ: References):

Ⓐ Numerical context

α) global convergence (towards a unique fixed point) under a contraction hypothesis (with respect to a vectorial norm)

[7] [10] [12] [16] [17] [20]

β) monotone convergence (for monotone F)

[12] [23]

γ) local convergence (assuming existence of a fixed point ξ, and some condition on $F'(\xi)$)

[26]

(B) Discrete context

α') global convergence (towards a unique fixed point) under a contraction hypothesis (boolean matrices)

[18] [19]

β') monotone convergence

[22]

In what follows, we summarize α) only.

(6) global convergence under contraction hypothesis [13] [17] [20]

$X_i \quad \| \ \|_i$ Banach spaces $i = 1, 2 \ldots k$

$X = \prod_1^k X_i$ their product, with the canonical vectorial norm:

$$x = (x_1 \ldots x_k) \in X \longrightarrow p(x) = \begin{vmatrix} \|x_1\|_1 \\ \vdots \\ \|x_k\|_k \end{vmatrix} \in \mathbb{R}^k$$

(the corresponding topology is the usual product topology on X)

Now, we suppose F to be a contraction on X with respect to p:

$$\boxed{\begin{array}{l} \forall x, y \in X \quad p(F(x) - F(y)) \leq K\, p(x - y) \\ \qquad \text{where } K \text{ is a } \geq 0 \ (k, k) \text{ matrix} \\ \qquad \text{with spectral radius } \rho(K) < 1 \end{array}}$$

Example $X = \mathbb{R}^n \quad p(x) = |x| \quad (k = n)$

(*) $|F(x) - F(y)| \leq K\, |x - y| \qquad K \geq 0 \ \rho(K) < 1$

for F affine: $F(x) = Tx + h$, (*) is equivalent to

$\rho(|T|) < 1$ that is $I - T$ is a H-matrix

Then [13] [17] [20]

Theorem If F is a contraction (in the above sense)

1) F has a unique fixed point ξ in X

2) For any starting point x^0, and any sequence $J_1, \dots J_r \dots$, the chaotic iteration

$$x^{r+1} = F_{J_{r+1}}(x^r)$$

converges to a limit, which depends only on $J_1, J_2 \dots J_r \dots$:

α) if any component i $(i = 1, 2 \dots k)$ is relaxed infinitely many times, then the limit is the unique fixed point ξ we look for

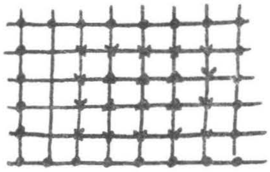
β) if not, the limit is only a partial fixed point η of F:

the relation $\eta_i = f_i(\eta_1 \dots \eta_k)$ is verified only for those i for which the corresponding component x_i has been relaxed infinitely many times.

For the case of convergence α), (costly) algorithms exist for bounding the error $p(x^r - \xi)$ [6] [17]

Comments * Contraction of F is a very strong hypothesis. It ensures existence, uniqueness of the fixed point, and convergence to it of any chaotic iteration, provided each component is relaxed infinitely many times.

* a comprehensive example of convergence to a partial fixed point in the case of the Dirichlet problem $\Delta u = 0$ on a grid :



u given on the boundary

7 Conclusions

A

1.1 on $\mathbb{R}^n$ $\rightarrow$ | global convergence of
- non linear Jacobi, G.S, S.O.R, etc.
  cf Ortega-Rheinboldt [9]
- linear Jacobi, G.S. S.O.R, etc.
  (free steering for H. systems, cf Ostrowski [2], [3])

block vectorial norms $\rightarrow$ | same results for block. methods (in particular, for block-H. matrices, see [8])

Then, the global convergence theorem provides a unified treatment of these iterative methods for the general non. linear case (in particular: non. linear Gauss. Seidel method: cf [13], [21].)

B Back to matrices:

Chaotic Stein Rosenberg theorem. see [14] [15]

C Open problem

Among all these converging chaotic iterative procedures, define a 'good' comparison criterium, and find 'efficient strategies'.

Difficult.

Until today, are available:

Only a few theoritical results (complicated)

A rather extensive experimental study

(cf part II)

II Experimental results

A. ABTROUN and Z. MAHJOUB 's thesis ([24] [25]) : experiments on chaotic iterations.

G. BAUDET 's work ([27]) : experiments on a multiprocessor (C.m.m.p.). Asynchronous iterations.

1 ABTROUN

'Given a linear system, find the optimal permutation on the variables, in order to get the better convergence of G.S.'

Very simple problem! But not easy to solve!

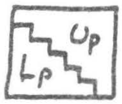
Abtroun results can be presented as follows: Given the Jacobi fixed point equation associated to the given system:

$$x = Jx + h$$

consider a permutation matrix P:

$$Px = \underbrace{PJP^t}\, Px + Ph$$

$$(P) \qquad y = J_p \; y + h_p$$

of course $\rho(J_p) = \rho(J)$. But, denoting $J_p =$  ,

G.S. fixed point equation on P is as follows:

$$y = \underbrace{(I - L_p)^{-1} U_p}_{G_p} \; y + (I - L_p)^{-1} h_p$$

The problem is:

<u>Compute P such that $\rho[(I - L_p)^{-1} U_p]$ minimum</u>

that is:

<u>P? $\quad$ Min $\rho(G_p)$</u>

Remarks *) if $J \geqslant 0$ with $\rho(J) < 1$

then $\forall P$: $\rho(G_P) \leq \rho(J_P) = \rho(J) < 1$ (Stein-Rosenberg)

*) Different cases where:

$$\forall P \quad \rho(G_P) = [\rho(J_P)]^2 = [\rho(J)]^2 = \text{cste}$$

In that case, nothing can be gained.

Abtroun's result: Under some general assumption H, generally verified:

greater the trace of $(I-L_P)^{-1}U_P$ is,

smaller (better) is $\rho((I-L_P)^{-1}U_P)$.

Then, the problem is:

$$\underline{P? \quad \text{Max trace}(G_P)}$$

A formal expression can be given of trace (G_P) which allows to see the effect of P on the trace, and to develop heuristics for the 'maximisation' of trace (G_P) by a 'good' choice of P.

Conclusion. 3 points:

* bounds for the spectral radius of G_P: can show why in some cases any choice of P is uneffective

* current numerical examples where, for example:

$$\frac{\log \rho(G_P)}{\log \rho(G)} = 2$$

(twice faster)

* we know how to construct numerical examples where, for instance: $\frac{\log \rho(G_P)}{\ldots} = 16$

② MAHJOUB has written rather sophisticated packages (using a display terminal) for free and useful experimentation of chaotic strategies.

Families of examples: some numerical families of linear and non linear problems; discretised linear elliptic boundary value problems (automatic discretisation)

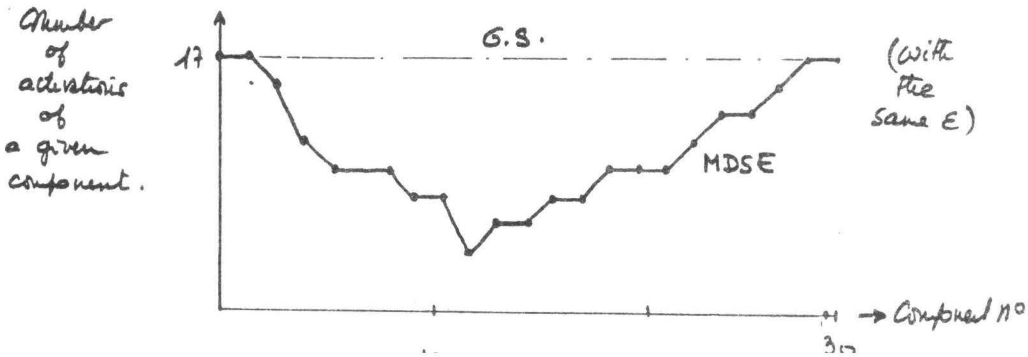
Very big experimentation, and practical conclusions: practical philosophy and 'handskill' for using chaotic iterations.

2 kinds of chaotic strategies:

① fully automatic strategies The program itself computes which components are to be relaxed at the next step. For example, a basic strategy is the following:

* G.S. is performed during some steps
* then, components are relaxed in the natural order ($1 \to n$, 'indefinitely', just as in G.S.) but some of them (and more and more) will be 'forgotten', for example when their relative variation falls under some fixed ε (MDSE: Methode de Disparition Successive des Eléments)

Example. On a problem with 30 components:



In other words, since in G.S all components are relaxed the same number of times, some unuseful work is performed on components which converge faster. In MDSE, this computing time is saved: just the necessary is made.

Linear and non linear problems.

Gain from $0 \to 45\%$ (time ratio: $\frac{0,65}{1}$)

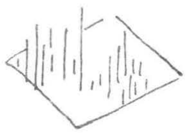
MDSE $\to$ G.S if $\varepsilon \to 0$ (of course)

Then, technique of a sequence of ε.

Problem of a partial fixed point.

(2) Interactive strategies. (for discretised problems on a grid)

Using a display terminal, zones of the grid are located, where convergence is slower, and a big lot of different techniques are availaible in order to perform more computation on these slow converging zones.



visualisation of the error
(or some estimation of it)

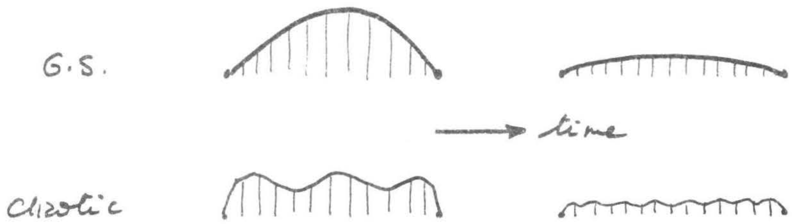
(Slides will be presented.)

Results. Very sophisticated techniques, to be used only for slow converging problems, especially when the differents components converge at very different speeds.

Then a significant improvement can be get, relative to G.S or S.OR. ($\sim 40\%$. time ratio $\frac{0,6}{1}$)

The idea, here again is to make not too much computation on components which converges sufficiently well, but to choose where to perform efficient computation.

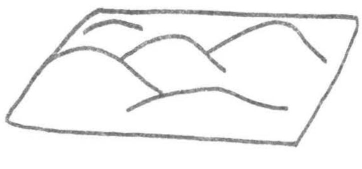
In fact, the main interest of such interactive techniques lies in the fact that the experimentator can *see* what happens and guide the computation step after step. Then, we get some handskill for practical driving of the computation. Using it, we see the error decreasing more uniformly:



In conclusion, consider the following picture:

'Given a bumped iron sheet (the error surface) and a hammer, we have to strike with the hammer in order to smooth the sheet (so that the error tends to 0)

Obviously the G.S. strategy is rather uninteresting. A good chaotic strategy consists in knowing where are the best places to strike on, in order to flat the sheet with a minimum of strokes of the hammer'



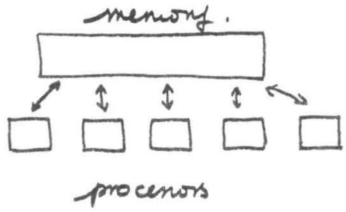
③ G. BAUDET is concerned with 'asynchronous iterations' for solving fixed-point equations in several variables on a multiprocessor:

$$x_i = f_i(x_1 \ldots x_k) \in \mathbb{R}^{\lambda_i}$$
$$i = 1, 2 \ldots k$$

Suppose we have k processors, working independantly, but connected to a common memory. The i^{th} block in the equation will be assigned to the i^{th} processor ($i = 1, 2 \ldots k$)

The computation starts with a given $x = (x^1, x_2 \ldots x_k)$

Then without any synchronisation between the processors, each unit computes 'indefinitely' new values for the components of his own block, putting them in the memory and picking values of the other components in the memory, too, :



This is the scheme of asynchronous iterations, more general than chaotic iterations.

Convergence is studied under the same contraction hypothesis

Experimentation has been possible on Cmmp (Carnegie Mellon multiprocessor)

On the model problem $\Delta u = 0$ discretised on a grid, various asynchronous iterations have been tested by

G. Baudet : Asynchronous Jacobi (A J)
Asynchronous G.S. (AGS)
Purely Asynchronous (P A) : (described just above.)

Here are interesting results :

	Jacobi	A J	AGS	PA
$k=1$	337	337	168	168
$k=2$	241	211	113	84
$k=3$	178	149	83	56
$k=4$	153	123	75	43
$k=6$	131	102	70	28

(Time required to divide the error by a factor of 10.)

<u>Experimentation confirms that the less the processors are synchronized, better is the convergence speed.</u>

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