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Note

Sum-perfect graphs ☆

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ABSTRACT

Inspired by a famous characterization of perfect graphs due to Lovász, we define a graph G to be *sum-perfect* if for every induced subgraph H of G, $\alpha(H)+\omega(H) \geq |V(H)|$. (Here α and ω denote the stability number and clique number, respectively.) We give a set of 27 graphs and we prove that a graph G is sum-perfect if and only if G does not contain any of the graphs in the set as an induced subgraph.

1. Introduction

All graphs in this article are simple, finite, and undirected. If G is a graph, then $\overline{G}$ denotes its complement. If G and H are graphs, then $G+H$ denotes the disjoint union of G and H. We write nG for the disjoint union of n copies of G, with $n \geq 1$. For $n \geq 1$, by P_n we denote the path on n vertices and K_n denotes the complete graph on n vertices. For $n \geq 3$, we let C_n denote the cycle on n vertices.

Define a graph G to be *sum-perfect* if for every induced subgraph H of G, $\alpha(H)+\omega(H) \geq |V(H)|$. Here, α is the stability number, i.e., the maximum size of a stable set. The parameter ω denotes the clique number; it is the maximum size of a clique. Clearly, a graph G is sum-perfect if and only if $\overline{G}$ is sum-perfect. In Theorem 1.1, we give a characterization of sum-perfectness in terms of forbidden induced subgraphs. If $\mathcal{L}$ is a set of graphs, we say that G is *$\mathcal{L}$-free* if G does not contain any graph in $\mathcal{L}$ as an induced subgraph. In this paper we prove the following.

Theorem 1.1. *A graph G is sum-perfect if and only if it is $\mathcal{F}$-free, where the set $\mathcal{F} := \{H_1, \ldots, H_{27}\}$ is depicted in Fig. 1.*

The set $\mathcal{F}$ of Theorem 1.1 consists of the 5-cycle H_1, all bipartite graphs $H_2, \ldots, H_{13}$ on 6 vertices containing a perfect matching, their complements $H_{14}, \ldots, H_{25}$, and two complementary graphs H_{26} and H_{27} on 7 vertices.

The motivation for studying sum-perfect graphs comes from the following characterization of perfect graphs. Recall that a graph G is perfect if $\chi(H) = \omega(H)$ for all induced subgraphs H of G. Here, χ denotes the chromatic number. Lovász [9] proved that a graph G is perfect if and only if $\alpha(H)\omega(H) \geq |V(H)|$ for all induced subgraphs H of G. When multiplication is replaced by addition, we move from perfect graphs to sum-perfect graphs. Since the condition is strengthened, the class of sum-perfect graphs is a subclass of the class of perfect graphs.

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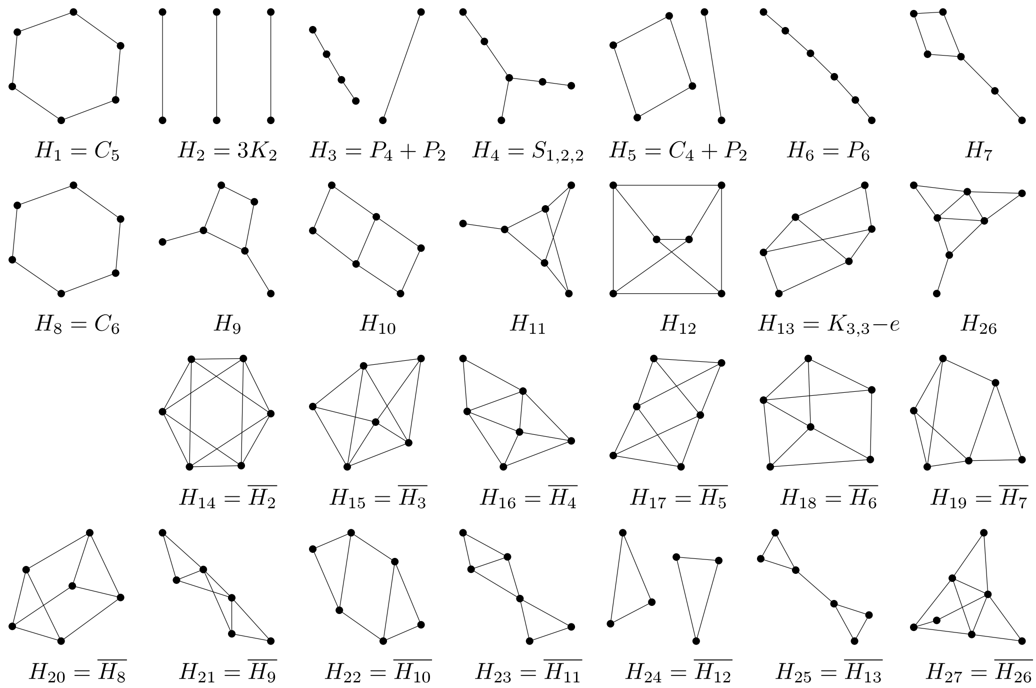

Fig. 1. The graphs in $\mathcal{F}$. For each graph, also the complement is contained in $\mathcal{F}$. For $i \in \{14, \ldots, 25\}$, we have $H_i := \overline{H_{i-12}}$, and we have $H_{27} := \overline{H_{26}}$.

The strong perfect graph theorem [3] asserts that a graph is perfect if and only if it neither has C_n nor $\overline{C_n}$ as an induced subgraph, for odd $n \geq 5$. We determine all the forbidden induced subgraphs for the class of sum-perfect graphs, i.e, graphs that are not sum-perfect but for which every proper induced subgraph is sum-perfect. As $\alpha(G) = \omega(\overline{G})$ and $\omega(G) = \alpha(\overline{G})$, the class of sum-perfect graphs is closed under taking the complement.

We give another reason why sum-perfect graphs are interesting. It is based on the following easy observation.

Let G be a graph, S a stable set and M a clique. Then $|S \cap M| \leq 1$. (1)

This implies that $\alpha(G) + \omega(G) \leq |V(G)| + 1$. In Theorem 1.2 we prove that the class of graphs all of whose induced subgraphs attain this upper bound, coincides with the class of threshold graphs (see Example 1.1 for the definition). This yields a characterization of threshold graphs that we could not find in the literature. Fixing $c \geq 0$, the class of graphs G for which $\alpha(H) + \omega(H) = |V(H)| - c$ for every induced subgraph H of G, is trivially empty. Indeed, an isolated vertex already does not satisfy this condition. Hence the natural generalization of threshold graphs from this point of view, is to consider graphs G for which $\alpha(H) + \omega(H) \geq |V(H)| - c$ for every induced subgraph H of G. For $c = 0$, we obtain the sum-perfect graphs.

Example 1.1. The following classes of graphs are examples of sum-perfect graphs: split graphs and (apex) threshold graphs. A graph is *split* if its vertex set can be partitioned into a clique and a stable set. A graph is *threshold* if there exists an ordering $(v_1, \ldots, v_n)$ of the vertices such that each vertex v_i is either adjacent or non-adjacent to all vertices $v_1, \ldots, v_{i-1}$. A graph is *apex-threshold* if it contains a vertex whose deletion results in a threshold graph.

Example 1.2. We give an example of a graph that is sum-perfect but neither split nor apex-threshold. Consider the graph G obtained from P_5 by adding a vertex adjacent to all vertices except for the vertex that is in the middle of P_5. Then G is sum-perfect since $\alpha(G) = \omega(G) = 3$ and $|V(G)| = 6$. It is not split since it contains C_4 as an induced subgraph. Finally, G is not apex-threshold as deleting any vertex results in a graph which still contains either C_4, or $2K_2$ or P_4 as an induced subgraph.

Theorem 1.2. *A graph G is threshold if and only if for every induced subgraph H of G, $\alpha(H) + \omega(H) = |V(H)| + 1$.*

Proof. Let G be a threshold graph. We argue by induction on $|V(G)|$. The base case is trivial. From the definition G either contains a dominating vertex or an isolated vertex. Suppose G has a dominating vertex v. Let $H := G - v$. By the induction hypothesis, $\alpha(H) + \omega(H) = |V(H)| + 1$. But $\alpha(G) = \alpha(H)$ and $\omega(G) = \omega(H) + 1$. Hence, $\alpha(G) + \omega(G) = \alpha(H) + \omega(H) + 1 = |V(H)| + 1 + 1 = |V(G)| + 1$. The case when there is an isolated vertex is similar.

For the converse, let G be a graph such that every induced subgraph H of G satisfies $\alpha(H) + \omega(H) = |V(H)| + 1$. It is easy to check that the three graphs P_4, C_4, $2K_2$ fail to satisfy the condition. Hence G contains none of P_4, C_4, $2K_2$ as an induced subgraph. But a well known theorem tells that a graph is threshold if and only if it contains none of P_4, C_4, $2K_2$ as an induced subgraph (see [4]). Hence G is threshold. □

Example 1.3. The class of sum-perfect graphs is contained in the class of weakly chordal graphs. A graph is *weakly chordal* if it contains neither a cycle of length at least 5 nor its complement as an induced subgraph.

With regard to complexity, we note that computing $\alpha(G) + \omega(G)$ of a graph G is NP-hard, as computing the stability number for triangle-free graphs is already NP-hard [10]. In Section 3, we briefly address some further optimization problems. Section 2 is devoted to the proof of Theorem 1.1.

2. Forbidden induced subgraphs

Let Forb denote the collection of forbidden induced subgraphs for the class of sum-perfect graphs and let $\mathcal{F}$ denote the set of 27 graphs from Fig. 1. If $\mathcal{L}$ is a collection of graphs, then for $n \geq 1$, let $\mathcal{L}_n$ denote the graphs in $\mathcal{L}$ that have n vertices. We define $\mathcal{L}_{\leq n} := \cup_{i=1}^{n} \mathcal{L}_i$. In Section 2.1 we show that $\text{Forb}_{\leq 7} = \mathcal{F}$. In Section 2.2 we prove that Forb_n is empty for $n \geq 8$. Combining both results yields Theorem 1.1.

2.1. Forbidden graphs with at most 7 vertices

With notation as above, we first determine $\text{Forb}_{\leq 5}$ and then show that the graphs in $\mathcal{F}$ with 6 vertices are forbidden.

Lemma 2.1. *We have that* $\text{Forb}_{\leq 5} = \{C_5\}$.

Proof. Let $G \in \text{Forb}_{\leq 5}$. Since complete graphs and empty graphs are sum-perfect, we may assume that $\alpha(G), \omega(G) \geq 2$. Hence G must have 5 vertices. Note that $\alpha(G) = \omega(G) = 2$ (otherwise $\alpha(G) + \omega(G) \geq 5 = |V(G)|$, a contradiction to $G \in \text{Forb}$). So G does not contain a triangle, and is also not bipartite (otherwise there would be a triangle in the complement). But C_5 is the only non-bipartite graph on at most 5 vertices without triangles. □

Lemma 2.2. *We have that* $\mathcal{F}_6 \subseteq \text{Forb}$.

Proof. As Forb is closed under taking the complement, it suffices to prove that every 6-vertex bipartite graph with a perfect matching is in Forb. Let G be such a graph. By the previous lemma, every proper induced subgraph of G is sum-perfect. So it remains to prove that $\alpha(G) + \omega(G) < 6$. This is easy, since $\alpha(G) \leq 3$ (because G has a perfect matching) and $\omega(G) = 2$ (because G has no triangles). □

Note that there are exactly 12 bipartite graphs on 6 vertices containing a perfect matching. To see this, start with $3K_2$. There are at most 6 additional edges, giving rise to the graphs $H_2, \ldots, H_{13}$ in Fig. 1. One verifies that the 7-vertex graph H_{26} from Fig. 1 is in Forb. Its complement is then automatically forbidden as well. Hence, we have shown that $\mathcal{F} \subseteq \text{Forb}$.

We will derive some properties of minimal non-sum-perfect graphs. The following lemma is key in many of the proofs that will follow.

Lemma 2.3. *Let* $G \in \text{Forb}$. *Then for every* $v \in V(G)$, *we have* $\alpha(G - v) = \alpha(G)$ *and* $\omega(G - v) = \omega(G)$.

Proof. Let $v \in V(G)$. We know $H := G - v$ is sum-perfect, while G is not. Hence

$$\alpha(H) + \omega(H) \geq |V(H)| = |V(G)| - 1 \geq \alpha(G) + \omega(G).$$

As $\omega(H) \leq \omega(G)$ and $\alpha(H) \leq \alpha(G)$, this yields $\omega(H) = \omega(G)$ and $\alpha(H) = \alpha(G)$. □

Corollary 2.4. *Let* $G \in \text{Forb}$. *Then* $\alpha(G) + \omega(G) = |V(G)| - 1$.

Proof. As $G \in \text{Forb}$, we know that $\alpha(G) + \omega(G) < |V(G)|$. Let $v \in V(G)$ and set $H := G - v$. Then H is sum-perfect and we compute

$$|V(G)| > \alpha(G) + \omega(G) = \alpha(H) + \omega(H) \geq |V(H)| = |V(G)| - 1,$$

where in the first equality we use Lemma 2.3. The corollary now follows. □

Lemma 2.5. *Let* $G \in \text{Forb}_n$, *with* $n \geq 6$. *Then G is perfect.*

Proof. As G is in Forb and has at least 6 vertices, C_5 is not an induced subgraph of G. Assume G is neither P_6, C_6, nor their complements (otherwise we are done). Then it does not contain C_n or $\overline{C_n}$ as an induced subgraph, for $n \geq 7$. Indeed, C_n contains P_6 as an induced subgraph for $n \geq 7$, and $\overline{C_n}$ contains $\overline{P_6}$ as an induced subgraph for $n \geq 7$. Hence G contains neither a cycle of length at least 5 nor its complement as an induced subgraph, i.e., G is weakly chordal. It is well known that weakly chordal graphs are perfect [6], and so we are done. □

Let $G = (V, E)$ be a graph. A *vertex cover* of G is a subset $U \subseteq V$ such that $U \cap e \neq \emptyset$, for each $e \in E$. It is clear that $U \subseteq V$ is a vertex cover if and only if $V \setminus U$ is a stable set. Then

$$\alpha(G) + \tau(G) = |V(G)|, \tag{2}$$

where $\tau(G)$ denotes the *vertex cover number*, i.e., the minimum size of a vertex cover. Let $\nu(G)$ denote the *matching number*: it is the maximum size of a matching in G.

Lemma 2.6. *We have that* $\mathrm{Forb}_{\leq 6} = \mathcal{F}_{\leq 6}$.

Proof. The "$\supseteq$" inclusion follows from Lemmas 2.1 and 2.2. To show "$\subseteq$", let $G \in \mathrm{Forb}_6$. As G is in Forb, we have $\alpha(G), \omega(G) \geq 2$. By Corollary 2.4, without loss of generality we may assume that $\alpha(G) = 3$ and $\omega(G) = 2$. Eq. (2) gives $\tau(G) = |V(G)| - \alpha(G) = 3$. Lemma 2.5 implies that G is perfect. Hence $\chi(G) = \omega(G) = 2$, showing that G is bipartite. By König's theorem $\nu(G) = \tau(G) = 3$ (Theorem 2.1.1 in [5]), therefore $G \in \mathcal{F}_6$. □

Lemma 2.7. *The only disconnected members of* Forb *are* $3K_2, P_4 + K_2, C_4 + K_2, 2K_3 (= \overline{K_{3,3}})$. *In particular,* $G \in \mathrm{Forb}_n$*, with* $n \geq 7$*, implies that* G *is connected.*

Proof. By Lemma 2.2 we have that $3K_2, P_4 + K_2, C_4 + K_2, 2K_3 \in$ Forb. Let G be a disconnected graph in Forb. Note that G has no isolated vertices. Suppose G has at least 3 components. Each component must have an edge, and hence G contains $3K_2$. Thus the only graph in Forb with more than 2 components is $3K_2$. Now, let G be a graph in Forb with exactly 2 components. Suppose one of the components is not a threshold graph. Then it contains either P_4, C_4, or $2K_2$ (see [4]). Together with an edge in the other component, we get one of $3K_2, P_4 + K_2, C_4 + K_2$. Hence both the components are threshold graphs. One of them must be a star, for otherwise we obtain $2K_3$. Hence G is the disjoint union of a threshold graph with a star, which is apex-threshold, and hence sum-perfect, a contradiction. □

Before we move on to the case $n = 7$, we need a lemma.

Lemma 2.8. *Let* $G \in \mathrm{Forb}_n$*, with* $n \geq 7$*. Then* $\omega(G) \neq 2$ *(and hence also* $\alpha(G) \neq 2$*).*

Proof. Assume that $\omega(G) = 2$. Then $\alpha(G) = n - 3$ by Corollary 2.4. By Lemma 2.5, G is perfect. Hence $\chi(G) = \omega(G) = 2$, showing that G is bipartite. Lemma 2.7 implies that G is connected. As $\alpha(G) = n - 3$, one of its color classes then has size $n - 3$ and the other has size 3. By König's theorem, $\nu(G) = 3$. As $n \geq 7$, removing any vertex not in the edges of a maximum size matching results in a graph G' with $\alpha(G') = n - 4$, contradicting Lemma 2.3. □

Lemma 2.9. *We have that* $\mathrm{Forb}_7 = \mathcal{F}_7$.

Proof. The "$\supseteq$" inclusion is immediately verified. To show "$\subseteq$", let $G = (V, E) \in \mathrm{Forb}_7$. By Lemma 2.7, G is connected. By going to the complement if necessary,

$$|E| \leq 10. \tag{3}$$

Then we must prove that $G = H_{26}$, from Fig. 1. Corollary 2.4 and Lemma 2.8 show that

$$\alpha(G) = \omega(G) = 3. \tag{4}$$

Hence there is a triangle in G. If there is a vertex $v \in V$ that is contained in all triangles of G, then $\omega(G - v) < \omega(G)$, contradicting Lemma 2.3. Hence

no vertex is contained in all triangles (hence by (4) there is no dominating vertex). (5)

If there are two vertex-disjoint triangles and if v denotes the vertex not in the vertex-disjoint triangles, then $\alpha(G - v) < \alpha(G)$, contradicting again Lemma 2.3. Hence

there are no two vertex-disjoint triangles. (6)

We introduce some notation. For $v \in V$, let $N(v)$ denote the set of neighbors of v. By $N[v]$ we denote the set $N(v) \cup \{v\}$. For $v \in V$, let $d_\Delta(v)$ denote the number of triangles in which v is contained and let $d(v)$ denote the degree of v.

Claim 1: For $v \in V$, $d_\Delta(v) \leq 3$.

Suppose $d_\Delta(v) = 4$. Clearly then $d(v) > 3$. If $d(v) = 4$, then $N(v)$ induces a 4-cycle and in $G \setminus N[v]$ there are two more vertices and at most two more edges. We see that v is in every triangle, contradicting (5). Suppose $d(v) = 5$. The graph induced by $N[v]$ has 9 edges, and again we see that v is in every triangle, a contradiction.

Suppose $d_\Delta(v) \geq 5$. Then $d(v) > 4$, so $d(v) = 5$ (as there is no dominating vertex by (5)). The graph induced by $N[v]$ has at least 10 edges, and so the vertex not in $N[v]$ is isolated, contradicting connectedness of G. This proves the claim.

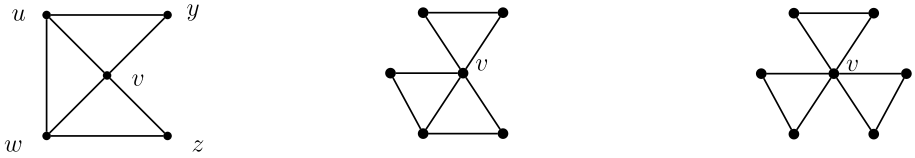

Fig. 2. From left to right: possibilities I, II and III.

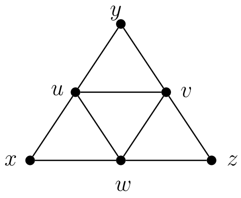

Fig. 3. The four triangles in G.

Claim 2*: For* $v \in V$, $d_\Delta(v) < d(v)$.

Let $d_\Delta(v) = i$. By claim 1 we know $i \leq 3$. If $i = 0$ the claim follows from the fact that G is connected. The cases $i = 1$ and $i = 2$ are clear. The case $i = 3$ follows from G being K_4-free (because of (4)), thus proving the claim.

Claim 3*: G has at most* 4 *triangles.*

We have $20 \geq 2|E| = \sum_{v\in V} d(v) \geq \sum_{v\in V}(d_\Delta(v) + 1)$. Here we are using (3), the degree-sum formula and claim 2. We conclude that $3t(G) = \sum_{v\in V} d_\Delta(v) \leq 13$, where $t(G)$ is the number of triangles in G. Hence G has at most 4 triangles.

Claim 4*: G has at least* 4 *triangles.*

By (5) and (6) G cannot have precisely 1 triangle or precisely 2 triangles. Suppose there are precisely 3 triangles. Start with two triangles having a vertex v in common. A third triangle cannot contain the vertex v because of (5), and hence must contain a vertex x from one triangle, and a vertex y from the other triangle. But then we automatically create a fourth triangle $\Delta = vxy$, in contradiction with the fact that there were supposed to be only 3 triangles.

Next we consider the case that we start with 2 triangles having two vertices u and v in common. By (5) the third triangle cannot contain u or v, hence must contain a vertex x from one triangle and a vertex y from the other triangle. But then $\{x, y, u, v\}$ induces K_4, contradicting (4).

Claim 5*: There are two triangles in G that have exactly one vertex in common.*

Assume, to the contrary, that every pair of triangles in G has two vertices in common (here we use (5)). Let $\Delta_1 = abc$ and $\Delta_2 = abd$ be two triangles, with $a, b, c, d \in V$ and $c \neq d$, having the vertices a and b in common. By (4), $cd \notin E$. Then any triangle in G contains at most one of c, d. Therefore, by our assumption that every pair of triangles has two vertices in common, any triangle must contain both a and b. This contradicts (5). The claim follows.

By claims 3 and 4 there are exactly 4 triangles in G. By claim 5 there are two triangles that have exactly one vertex v in common. A third triangle must share a vertex with both triangles, leading (up to relabeling the vertex v) to the following three possibilities given in Fig. 2.

In possibilities II and III the fourth triangle must contain v, contradicting (5). In possibility I the fourth triangle cannot contain the vertex v, and hence must contain the vertices u and w. The fourth triangle does not contain y or z, since that would create a copy of K_4. Hence the picture looks like Fig. 3.

Let a denote the seventh vertex of G. By connectedness and by (3), a is adjacent to precisely one vertex. If a is adjacent to one of $\{u, v, w\}$, then $\alpha(G) = 4$, contradicting (4). Hence a is adjacent to one of $\{x, y, z\}$, yielding the graph H_{26} from Fig. 1. □

2.2. Non-existence of forbidden graphs with at least 8 vertices

In this section we complete the proof of Theorem 1.1. In order to do so, we show that Forb_n is empty for $n \geq 8$. If $G = (V, E)$ is a graph, $U \subseteq V$ a subset of vertices and $x \in V$, then $N_U(x) := \{v \in U \mid xv \in E\}$. We write $\deg_U(x) := |N_U(x)|$.

Lemma 2.10. *Let* $G \in \text{Forb}$. *Then* $\max\{\alpha(G), \omega(G)\} \leq 3$.

Proof. Let $G \in \text{Forb}$. Then $\alpha(G)+\omega(G) = |V(G)|-1$ by Corollary 2.4. We may assume, by going to the complement if necessary, that $\alpha(G) \geq \omega(G)$. We also may assume that $\alpha(G) \geq 4$, as there is nothing to prove if this were false. Then it follows that G is $\mathcal{F}$-free. Indeed, if G contains a graph H from $\mathcal{F}$ as an induced subgraph, then by minimality $G = H$, contradicting the fact that $\alpha(H) \leq 3$ for all $H \in \mathcal{F}$. We distinguish two cases:

1. There is a maximum size clique M and a maximum size stable set S such that $M \cap S = \emptyset$.
2. For every maximum size clique M and maximum size stable set S, $M \cap S \neq \emptyset$.

Case 1

We have $|M|+|S| = \omega(G)+\alpha(G) = |V(G)|-1$. Let x be the vertex in $V(G)\setminus(M\cup S)$. If x is not adjacent to any vertex in S, then $S\cup\{x\}$ is a stable set of size $\alpha(G)+1$. Hence, x is adjacent to a vertex in S. We consider the cases: $|N_S(x)| = 1$, $|N_S(x)| = 2$ and $|N_S(x)| \geq 3$, beginning with the latter.

1.1 $|N_S(x)| \geq 3$

Let $s \in S$. By the assumption on the cardinality of $N_S(x)$, a maximum size stable set in $G-s$ does not contain the vertex x. Hence, by Lemma 2.3 it must contain a vertex $m_s \in M$. This implies that m_s is not adjacent to any vertex in $S\setminus\{s\}$, but is adjacent to s (otherwise $S\cup\{m_s\}$ is a stable set of size $\alpha(G)+1$). This in turn implies that $m_s \neq m_{s'}$, if s and s' are distinct vertices in S. Hence, $\omega(G)\geq\alpha(G)$ and therefore $\alpha(G)=\omega(G)\geq 4$.

Let m be a non-neighbor of x in M. It exists, as otherwise $M\cup\{x\}$ induces a clique of size $\omega(G)+1$. Let $m' \in M\setminus\{m\}$. As $\deg_M(s) = 1$ for all $s \in S$, a maximum size clique in $G-m'$ cannot contain a vertex from S (here we use that $\omega(G)\geq 4$). As the vertex x is not adjacent to m, the vertices $(M\setminus m')\cup\{x\}$ do not form a clique. Then $\omega(G-m') < \omega(G)$, contradicting Lemma 2.3. Hence, this case cannot occur.

1.2 $|N_S(x)| = 2$

Let $N_S(x) = \{a, b\}$ and let $s \in S\setminus\{a, b\}$. If a stable set in $G-s$ contains x, then it cannot contain a and b and hence it is not of maximum size. Therefore, a maximum size stable set in $G-s$ does not contain x, but contains a vertex $m_s \in M$. As in the previous subcase, m_s is non-adjacent to every vertex in $S\setminus\{s\}$, but is adjacent to s. Furthermore, $m_s \neq m_{s'}$, if s and s' are distinct vertices in S, showing that $\omega(G)\geq\alpha(G)-2$.

As b and x are adjacent, a maximum size stable set in $G-a$ necessarily contains a vertex $m_a \in M$. The vertex m_a is non-adjacent to every vertex in $S\setminus\{a, b\}$. Hence $m_a \neq m_s$, for each $s \in S\setminus\{a, b\}$. This gives $\omega(G)\geq\alpha(G)-1$ and also that

$$\deg_M(s) \leq 2, \tag{7}$$

for $s \in S\setminus\{a, b\}$. The vertex m_a is non-adjacent to at least one of b, x, as one of b, x is contained in a maximum size stable set in $G-a$ that contains m_a. This also implies that m_a is adjacent to a vertex in $\{a, b\}$ (otherwise $S\cup\{m_a\}$ is a stable set of size $\alpha(G)+1$). Without loss of generality, assume that m_a is adjacent to a. We make a case distinction.

1.2.1 m_a *is not adjacent to b, but it is adjacent to x*

Let S' be a maximum size stable set in $G-b$. Suppose S' contains x. Then it contains neither a, nor m_a (as it is assumed to be adjacent to x) nor any vertex in M of the form m_s, with $s \in S\setminus\{a, b\}$. Hence there must be an additional vertex $m_b \in M$ that is in S'. Then $S' = (S\setminus\{a, b\})\cup\{x, m_b\}$. Now consider the graph $G-m_a$. As m_s is non-adjacent to every vertex in $S\setminus\{s\}$, for $s \in S\setminus\{b\}$, the vertices a and b have no neighbors in $M\setminus\{m_a, m_b\}$. Furthermore, x is not adjacent to m_b. Observe that there is no clique of size $\omega(G)$ in $G-m_a$, contradicting Lemma 2.3.

Hence $x \notin S'$. Then $S' = (S\setminus\{b\})\cup\{m_b\}$, for some $m_b \in M$ that is neither m_a (as m_a is adjacent to a), nor m_s, for some $s \in S\setminus\{a, b\}$ (as m_s is adjacent to s). Hence $\omega(G)=\alpha(G)\geq 4$. It also follows that $N_M(v) = \{m_v\}$, for $v \in \{a, b\}$. Together with (7) this yields $\deg_M(s) = 1$, for all $s \in S$. This in turn implies that no vertex in S can be in a maximum size clique in $G-m_a$. A maximum size clique in $G-m_a$ also cannot contain x, as x has a non-neighbor in M that is not m_a, by assumption. Therefore $\omega(G-m_a) < \omega(G)$, contradicting Lemma 2.3.

1.2.2 m_a *is not adjacent to b, and not adjacent to x*

Take a vertex of the form $m_s \in M$, for some $s \in S\setminus\{a, b\}$. Either $m_sx \in E$, in which case $\{a, b, x, s, m_s, m_a\}$ induces H_9 from Fig. 1, contradicting the fact that G was supposed to be $\mathcal{F}$-free. Or $m_sx \notin E$, in which case $\{a, b, x, s, m_s, m_a\}$ induces P_6 (which is H_6 in Fig. 1).

1.2.3 m_a *is adjacent to b, but not adjacent to x*

Take a vertex of the form $m_s \in M$, for some $s \in S\setminus\{a, b\}$. Either $m_sx \in E$, in which case $\{a, b, x, s, m_s, m_a\}$ induces H_{11}. Or $m_sx \notin E$, in which case $\{a, b, x, s, m_s, m_a\}$ induces H_7.

1.3 $|N_S(x)| = 1$

Let $N_S(x) = \{a\}$ and let $s \in S\setminus\{a\}$. A maximum size stable set in $G-s$ either contains a or x, but not both. Hence, it contains exactly one vertex m_s from M. Then m_s is non-adjacent to every vertex in $S\setminus\{a, s\}$. We argue that m_s is adjacent to s. Assume that $m_ss \notin E$. Since $S\cup\{m_s\}$ is not a stable set, we must have that $m_sa \in E$. Also, $m_sx \in E$ (otherwise $(S\setminus\{a\})\cup\{m_s, x\}$ is a stable set of size $\alpha(G)+1$). Observe that there is no stable set of size $\alpha(G)$ in $G-s$, contradicting Lemma 2.3. Hence m_s is adjacent to s. The argument also shows that $m_s \neq m_{s'}$, if s and s' are distinct vertices in $S\setminus\{a\}$, implying that $\omega(G)\geq\alpha(G)-1\geq 3$. We consider the following possibilities.

1.3.1 $N_M(a) = \emptyset$ *and* $N_M(x) = \emptyset$

Let s_1, s_2 be two distinct vertices in $S\setminus\{a\}$ (they exist, as $\alpha(G)\geq 4$). Then $\{s_1, s_2, a, x, m_{s_1}, m_{s_2}\}$ induces $H_3(=P_4+K_2)$.

1.3.2 $N_M(a) = \emptyset$ *and* $N_M(x) \neq \emptyset$

Let m_x be a neighbor of x in M. All vertices in M have a neighbor in S, otherwise there is a stable set of size $\alpha(G)+1$. As a is non-adjacent to every vertex in M, we know that m_x has a neighbor $s_1 \in S\setminus\{a\}$. Let m_x' be a non-neighbor of x in M. Assume that the neighbor s_2 of m_x' in $S\setminus\{a\}$ is not s_1. Then $\{s_1, s_2, a, x, m_x, m_x'\}$ induces H_4.

Suppose now that $s_2 = s_1$. A maximum size clique in $G-m_x$ must contain x. Indeed, if it does not, then it must be of the form $(M\setminus\{m_x\})\cup\{s\}$, for some $s \in S\setminus\{a\}$. But $\omega(G)\geq 3$ and $\deg_{M\setminus\{m_x\}}(s)\leq 1$, for all $s \in S\setminus\{a\}$. So a maximum size clique in $G - m_x$ contains x, and then it necessarily does not contain m'_x. Then it must contain a vertex from S, namely a (as x is adjacent only to a in S). But a has no neighbors in M by assumption, a contradiction.

1.3.3 $N_M(a) \neq \emptyset$ *and* $N_M(x) = \emptyset$

This case is reduced to case 1.3.2 by interchanging the role of a and x.

1.3.4*a* $N_M(a) \cap N_M(x) \neq \emptyset$

Let m be a common neighbor of a and x in M. Observe that $m \neq m_s$, for $s \in S \setminus \{a\}$ (otherwise $(S \setminus \{s\}) \cup \{m_s\}$ is not a stable set). Hence $\omega(G) = \alpha(G) \geq 4$. Then a maximum size clique in $G - m$ uses no vertex of $S \setminus \{a\}$ (as $\deg_M(s) \leq 2$, for all $s \in S \setminus \{a\}$). As both a and x have a non-neighbor in M, a maximum size clique in $G - m$ is of the form $(M \setminus \{p, m\}) \cup \{a, x\}$. Here, p is a common non-neighbor of a and x in M and it is the only non-neighbor of a, and of x. But then $(M \setminus \{p\}) \cup \{a, x\}$ is a clique of size $\omega(G) + 1$ in G, a contradiction.

1.3.4*b* $N_M(a) \neq \emptyset$ *and* $N_M(x) \neq \emptyset$ *but* $N_M(a) \cap N_M(x) = \emptyset$

Let m_a be a neighbor of a in M and let m_x be a neighbor of x in M. Assume that $\omega(G) = 3$. Then there are $s, s' \in S \setminus \{a\}$ such that $m_s = m_a$ and $m_{s'} = m_x$. By the assumption that a and x have no common neighbor, $\{s, s', a, x, m_a, m_x\}$ induces H_9.

Hence $\omega(G) \geq 4$. Suppose a maximum size clique in $G - m_a$ contains a vertex from $S \setminus \{a\}$. Since $\deg_M(s) \leq 2$, for all $s \in S \setminus \{a\}$, it must also contain a or x (here we use that $\omega(G) \geq 4$). But both a and x have no other neighbors in $S \setminus \{a\}$. A maximum size clique in $G - m_a$ also cannot contain both a and x, as they do not have common neighbors in M. The set $(M \setminus \{m_a\}) \cup \{a\}$ is no clique as $m_x a \notin E$. Hence a maximum size clique in $M - m_a$ is of the form $(M \setminus \{m_a\}) \cup \{x\}$, implying that m_a is the only non-neighbor of x in M and also that $N_M(a) = \{m_a\}$. The same arguments as before show that a maximum size clique in $G - m_x$ neither contains both a and x, nor a vertex from $S \setminus \{a\}$. The set $(M \setminus \{m_x\}) \cup \{a\}$ is not a clique, as $N_M(a) = \{m_a\}$. But also $(M \setminus m_x) \cup \{x\}$ is not a clique, as $m_a x \notin E$. Hence $\omega(G - m_x) < \omega(G)$, a contradiction to Lemma 2.3.

Case 2

Let S be a maximum size stable set and M be a maximum size clique. By assumption $S \cap M \neq \emptyset$, hence $|S \cap M| = 1$. Let $x \in S \cap M$. Since $|V(G)| - 1 = \alpha(G) + \omega(G)$ there are exactly two vertices in $V(G) \setminus (S \cup M)$, say y, z. As we are in case 2, $(S \setminus \{x\}) \cup \{y, z\}$ cannot contain a stable set of size $\alpha(G)$. Hence each of y, z has a neighbor in $S \setminus \{x\}$. We consider two cases.

2.1 Both y, z *have exactly one neighbor in* $S \setminus \{x\}$, *which is a common neighbor.*

Let $a \in S \setminus \{x\}$ be the common neighbor of x and y. Note that in this case y is adjacent to z, for otherwise $(S \setminus \{a, x\}) \cup \{y, z\}$ is a stable set of size $\alpha(G)$ disjoint from M, which is impossible. Since $|S| \geq 4$, there is a vertex $s \in S \setminus \{x\}$ that is distinct from a. Now a stable set in $G - s$ can contain at most one vertex of M, and one of $\{a, y, z\}$. Thus $\alpha(G - s) \leq \alpha(G) - 3 + 1 + 1 = \alpha(G) - 1$, a contradiction to Lemma 2.3.

2.2 There exist distinct vertices $a, b \in S \setminus \{x\}$ *such that* a *is adjacent to* y *and* b *is adjacent to* z.

Since $|S| \geq 4$, there is a vertex $s \in S \setminus \{x\}$ that is distinct from a, b. Now a stable set in $G - s$ can contain at most one vertex of M, and at most one of a, y and at most one of b, z. Thus $\alpha(G - s) \leq \alpha(G) - 4 + 1 + 1 + 1 = \alpha(G) - 1$, contradicting Lemma 2.3. □

Lemma 2.11. *If* $G \in$ Forb, *then* $|V(G)| \leq 7$.

Proof. By Corollary 2.4 and Lemma 2.10 we have $|V(G)| = \alpha(G) + \omega(G) + 1 \leq 3 + 3 + 1 = 7$. □

Proof of Theorem 1.1. We have to prove that Forb $= \mathcal{F}$, or equivalently, that $\text{Forb}_n = \mathcal{F}_n$, for all $n \geq 1$. For $n \leq 7$, this is the content of Lemmas 2.6 and 2.9. By Lemma 2.11, Forb_n is empty for $n \geq 8$. Hence, we are done. □

3. Future directions

All but three graphs $H \in \mathcal{F}$ have the property that H or $\overline{H}$ is bipartite, has six vertices and has a perfect matching. It seems that the three graphs in $\mathcal{F}$ that do not satisfy this property (i.e., the 5-cycle $H_1 = C_5$ and the complementary seven-vertex graphs H_{26} and H_{27}) are not the most important excluded induced subgraphs for obtaining a high lower bound on $\alpha + \omega$. Write $\mathcal{B} := \mathcal{F} \setminus \{H_1, H_{26}, H_{27}\} = \{H_2, \ldots, H_{25}\}$. We believe that the following, which was verified by computer to be true for all graphs with at most 10 vertices, holds:

Conjecture 3.1. *Every* $\mathcal{B}$*-free graph* G *satisfies* $\alpha(G) + \omega(G) \geq |V(G)| - 1$.

Another problem that naturally arises from the main theorem is the following. For a positive integer c, let $\mathcal{H}_c$ denote the class of graphs G such that every induced subgraph H of G satisfies $\alpha(H) + \omega(H) \geq |V(H)| - c$. Is the list of forbidden induced subgraphs for $\mathcal{H}_c$ finite? (Note that $\mathcal{H}_0$ is the class of sum-perfect graphs.) For $c = 1$, the list of forbidden induced subgraphs with at most 8 vertices already contains $>$ 1000 members, as was found by computer.

From the definition of sum-perfect graphs and Lovász' characterization of perfect graphs, it is easy to see that sum-perfect graphs are perfect. A graph G is *strongly perfect* if every induced subgraph H of G contains a stable set that intersects all the maximal (with respect to inclusion) cliques of H (see [1]). Are sum-perfect graphs strongly perfect?

The class of sum-perfect graphs gives rise to interesting algorithmic questions. The problems STABLE SET, MAXCLIQUE, COLORING, CLIQUE COVER are all polynomial for sum-perfect graphs because sum-perfect graphs are weakly chordal, and fast algorithms are known for the latter (see [7,8]). Are there faster algorithms by exploiting the special structure of sum-perfect graphs?

A last algorithmic question is the problem of recognizing sum-perfect graphs. The main theorem of this paper implies that there is a $O(n^7)$ algorithm to recognize sum-perfect graphs with n vertices: just test whether any of the 27 graphs from $\mathcal{F}$ appears as an induced subgraph. Since the largest of these graphs has 7 vertices, we get an algorithm whose running time is $O(n^7)$. Is there a faster algorithm to recognize sum-perfect graphs? These questions are material for further research.

The class of sum-perfect graphs can be thought of as a generalization of split graphs. There are three other generalizations of split graphs in the literature which have similar forbidden induced subgraph characterization. This was pointed to us by one of the referees to whom we are very grateful.

- Superbrittle graphs: For definition of superbrittle graphs, see [11]. There the forbidden induced subgraph characterization for the class is given. The set consists of 7 graphs, each with at most 7 vertices.
- Split-perfect graphs: For definition of split-perfect graphs, see [2]. The forbidden induced subgraphs for the so-called "prime" graphs in this class consists of cycle of length at least 5, 8 graphs with at most 7 vertices, and their complements. Note that superbrittle graphs are split-perfect.
- Hereditary Satgraphs: For definition of hereditary satgraphs, see [12]. The forbidden induced subgraphs for this class consist of 21 graphs with at most 7 vertices.

The referee mentions that properties of these three generalizations of split graphs could possibly be helpful in understanding the structure of sum-perfect graphs. Determining the intersection of various combinations of these four classes will be a good first step.

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