

On the Complexity of Branching Proofs

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Abstract

We consider the task of proving integer infeasibility of a bounded convex K in \mathbb{R}^n using a general branching proof system. In a general branching proof, one constructs a branching tree by adding an integer disjunction $\mathbf{ax} \leq b$ or $\mathbf{ax} \geq b + 1$, $\mathbf{a} \in \mathbb{Z}^n$, $b \in \mathbb{Z}$, at each node, such that the leaves of the tree correspond to empty sets (i.e., K together with the inequalities picked up from the root to leaf is empty).

Recently, Beame et al (ITCS 2018), asked whether the bit size of the coefficients in a branching proof, which they named stabbing planes (SP) refutations, for the case of polytopes derived from SAT formulas, can be assumed to be polynomial in n . We resolve this question in the affirmative, by showing that any branching proof can be recompiled so that the normals of the disjunctions have coefficients of size at most $(nR)^{O(n^2)}$, where $R \in \mathbb{N}$ is the radius of an ℓ_1 ball containing K , while increasing the number of nodes in the branching tree by at most a factor $O(n)$. Our recompilation techniques works by first replacing each disjunction using an iterated Diophantine approximation, introduced by Frank and Tardos (Combinatorica 1986), and proceeds by “fixing up” the leaves of the tree using judiciously added Chvátal-Gomory (CG) cuts.

As our second contribution, we show that Tseitin formulas, an important class of infeasible SAT instances, have quasi-polynomial sized cutting plane (CP) refutations. This disproves a conjecture that Tseitin formulas are (exponentially) hard for CP. Our upper bound follows by recompiling the quasi-polynomial sized SP refutations for Tseitin formulas due to Beame et al, which have a special enumerative form, into a CP proof of the same length using a serialization technique of Cook et al (Discrete Appl. Math. 1987).

As our final contribution, we give a simple family of polytopes in $[0, 1]^n$ requiring exponential sized branching proofs.

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1 Introduction

A principal challenge in SAT solving is finding short proofs of unsatisfiability of SAT formulas. This task is particularly important in the automatic verification of computer programs, where incorrect runs of the program or bugs (e.g., divide by zero) can be encoded as satisfying



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assignments to SAT formulas derived from the program specification. In this case, the corresponding formula is an UNSAT instance if the corresponding program is correct, or at least devoid of certain types of bugs.

The study of how long or short such UNSAT proofs can be is the main focus of the field of proof complexity. Indeed, popular SAT algorithms, such as DPLL search, i.e. branching on variables combined with unit propagation, or Conflict Driven Clause Learning (CDCL), implicitly generate infeasibility proofs in standard proof systems such as Resolution or Cutting Planes. From the negative perspective, lower bounds on the length of UNSAT proofs in these systems automatically imply lower bounds on the running time of the corresponding SAT algorithms. On the positive side, understanding which UNSAT instances have short proofs can inspire the design of good heuristics and algorithms for trying to find such proofs automatically.

The analogous problem in the context of Integer Programming (IP) is that of showing that a linear system of inequalities has no integer solutions. This problem also encapsulates SAT: for a formula $\Phi(\mathbf{x}) := \bigwedge_{j \in [m]} C_j(\mathbf{x})$, where $C_j(\mathbf{x}) = \bigvee_{i \in L_j} x_i \bigvee_{i \in \bar{L}_j} \bar{x}_i$, $j \in [m]$, Φ is unsatisfiable if and only if the linear system

$$\begin{aligned} \sum_{i \in L_j} x_i + \sum_{i \in \bar{L}_j} (1 - x_i) &\geq 1, j \in [m] \\ 0 \leq x_i \leq 1, i &\in [n] \end{aligned} \tag{SAT-LP}$$

has no integer solutions (in this case $\{0, 1\}$). IP solvers such as CPLEX or Gurobi routinely produce such infeasibility proofs in the so-called proof of optimality phase of the solution process. More precisely, once a solver has found a candidate optimal solution \mathbf{x}^* to an integer linear program

$$\min \mathbf{c}\mathbf{x} \quad \text{subject to } \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n \tag{IP}$$

optimality is proved by showing that the linear system

$$\begin{aligned} \mathbf{c}\mathbf{x} &< \mathbf{c}\mathbf{x}^* \\ \mathbf{A}\mathbf{x} &\leq \mathbf{b} \end{aligned} \tag{IP-LP}$$

has no integer solutions. In practice, this is most often achieved by a mixture of Branch & Bound and Cutting Planes. We note that most applications are modelled using *mixed* integer linear programs (MIP), where a decision variable x_i can be continuous ($x_i \in \mathbb{R}$), binary ($x_i \in \{0, 1\}$) or general integer ($x_i \in \mathbb{Z}$), with binary and continuous variables being the most common.

1.1 Branching Proofs

For proving infeasibility of a SAT formula or an integer linear program, where we denote the continuous relaxation of the feasible region by $K \subseteq \mathbb{R}^n$ (e.g., (SAT-LP) or (IP-LP)), the most basic strategy is to build a search tree based on so-called variable branching. That is, we build a rooted binary tree \mathcal{T} , where at each internal node v we choose a “promising” candidate integer variable x_i and create two children v_l, v_r corresponding either side of the disjunction $x_i \leq b$ (left child v_l) and $x_i \geq b + 1$ (right child v_r), for some $b \in \mathbb{Z}$. The edge from the parent to its child is labelled with the corresponding inequality. If x_i is binary, one always sets $b = 0$, corresponding to branching on $x_i = 0$ or $x_i = 1$. To each node is associated its continuous relaxation K_v , corresponding to K together with the inequalities on the edges

of the unique path from the root to v in \mathcal{T} . To be a valid proof of integer infeasibility, we require that the continuous relaxation K_v be empty at every leaf node $v \in \mathcal{T}$. We then call the proof tree \mathcal{T} as above a *variable* branching proof of integer infeasibility for K . We will consider the length of branching proof, interpreted as the “number of lines” of the proof, to be equal to the number of nodes in \mathcal{T} , which we denote $|\mathcal{T}|$.

When applied to a SAT formula as in (SAT-LP), a variable branching tree \mathcal{T} as above is in correspondance with a run of DPLL search, noting that LP infeasibility of a node is equivalent to unit propagation (i.e., iteratively propagating the values of variables appearing in single literal clauses) yielding a conflict¹. Similarly when applied to an integer program as in (IP-LP) for which the optimal value is known, the above is equivalent to standard Branch and Bound.

Branching on General Integer Disjunctions. To obtain a more general proof strategy one may examine a richer class of disjunctions. Instead of branching only on variables as above, one may also branch on a general integer disjunction $\mathbf{ax} \leq b$ or $\mathbf{ax} \geq b + 1$, where $\mathbf{a} \in \mathbb{Z}^n$ and $b \in \mathbb{Z}$, noting that any integer point $\mathbf{x} \in \mathbb{Z}^n$ must satisfy exactly one of these inequalities. One may then define branching proofs of infeasibility for K using general integer disjunctions exactly as above, which we call *general* branching proofs. We note that in principle, the continuous relaxation K can be arbitrary, i.e. it need not be a polytope. In this work, we will in fact consider the case where K is a compact convex set in \mathbb{R}^n . Furthermore, it is easy to extend branching proofs to the case of *mixed* integer infeasibility, where we want to certify that $K \cap \mathbb{Z}^k \times \mathbb{R}^{n-k} = \emptyset$, that is, where only the first k variables are restricted to be integer. In this setting, one need only restrict the disjunctions $\mathbf{ax} \leq b$ or $\geq b$ to have support on the integer variables; precisely, we enforce $\mathbf{a} \in \mathbb{Z}^k \times \{0\}^{n-k}, b \in \mathbb{Z}$.

As formalized above, the attentive reader may have noticed that there is no mechanism to “certify” the emptiness of the leaf nodes of the tree. In many cases, such certificates can be appended to the leaves yielding a *certified* branching proof, however their exact form will differ depending on the representation of K (e.g., LP, SOCP or SDP). In the important case where the continuous relaxation is a polytope $K = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Cx} \leq \mathbf{d}\}$, emptiness of a leaf node can indeed be certified efficiently using a so-called Farkas certificate of infeasibility. Let \mathcal{T} be branching proof for K and let $v \in \mathcal{T}$ be a leaf node with $K_v = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Cx} \leq \mathbf{d}, \mathbf{A}_v \mathbf{x} \leq \mathbf{b}_v\}$, where $\mathbf{A}_v \mathbf{x} \leq \mathbf{b}_v$ represents all the inequalities induced by the branching decisions on the path from the root to v . Then, by Farkas’s lemma $K_v = \emptyset$ iff there exists multipliers $\boldsymbol{\lambda}_v := (\boldsymbol{\lambda}_{v,1}, \boldsymbol{\lambda}_{v,2}) \geq 0$, known as a *Farkas certificate*, such that $\boldsymbol{\lambda}_{v,1} \mathbf{C} + \boldsymbol{\lambda}_{v,2} \mathbf{A}_v = 0$ and $\boldsymbol{\lambda}_{v,1} \mathbf{d} + \boldsymbol{\lambda}_{v,2} \mathbf{b} < 0$. Therefore, for a polyhedral feasible region, we may certify the branching proof by labeling each leaf node $v \in \mathcal{T}$ with its Farkas certificate $\boldsymbol{\lambda}_v$.

For a variable branching proof \mathcal{T} , especially for $\{0, 1\}$ IPs, the tree size $|\mathcal{T}|$ is arguably the most important measure of the complexity of the proof. However, for a general branching proof \mathcal{T} , the tree size $|\mathcal{T}|$ ignores the “complexity” of the individual disjunctions. Note that we have not a priori set any restrictions on the size of the coefficients for the disjunctions $\mathbf{ax} \leq b$ or $\geq b + 1$ used in the nodes of the tree. To accurately capture this complexity, we will also measure the number of bits needed to write down the description of \mathcal{T} , which we denote by $\langle \mathcal{T} \rangle$. Here, $\langle \mathcal{T} \rangle$ includes the bit-size of all the disjunctions $\mathbf{ax} \leq b$ or $\geq b + 1$ used

¹ Note that if unit propagation finds a conflict at a node of the tree, the corresponding node LP (i.e. (SAT-LP) with some variables fixed to 0 or 1) is also infeasible. If unit propagation terminates without a conflict, then setting all non-propagated variables to 1/2 yields a feasible LP solution since every surviving clause has at least 2 literals.

in \mathcal{T} . For a certified branching proof, as introduced above, we also include the bit-length of the infeasibility certificates at the leaves to $\langle \mathcal{T} \rangle$. Understanding how large the coefficients need to be to ensure near-optimal tree size will be one of the principal interests of this work.

Applications of General Branching. While variable branching is the most prevalent in practice, due to its simplicity and ease of implementation, it is well-known that branching on general integer disjunctions can lead to much smaller search trees. In practice, general branching is used when certain simple constraints such as $\sum_{i=1}^n x_i = 1$, x_i binary, are present in the model, which is part of the family of specially ordered set constraints [3]. In this context, one may branch on $\sum_{i=1}^{n/2} x_i = 0$ or $\sum_{i=1}^{n/2} x_i = 1$ to get a more balanced search tree. A more recent idea of Fischetti and Lodi [11], known as local branching, is to branch on disjunctions which control the Hamming distance to the best incumbent solution \mathbf{x}^* , e.g. $\sum_{i:x_i^*=0} x_i + \sum_{i:x_i^*=1} (1 - x_i) \leq k$ or $\geq k + 1$. This provides a very effective way of controlling the search neighborhood, and allows one to find improving solutions more quickly.

From the theoretical side, a seminal result is that of Lenstra [17], who gave a fixed dimension polynomial time algorithm for Integer Programming based on basis reduction and general branching. Relating to branching proofs, his result directly implies that every integer free compact convex set admits a general branching proof of length $O(f(n)^n)$, where $f(n)$, the so-called flatness constant, is the supremum of the lattice width over integer free compact convex set in dimension n . It is known that $f(n) = \tilde{O}(n^{4/3})$ [2, 22] and $f(n) = \Omega(n)$. We note that already in \mathbb{R}^2 , there are simple integer free polytopes, e.g., $\{\mathbf{x} \in \mathbb{R}^2 : x_1 - x_2 = 1/2, 0 \leq x_1 \leq k\}$, for $k \in \mathbb{N}$, with arbitrarily long variable branching proofs. Inspired by Lenstra's result, there has been a line of work on the use of basis reduction techniques to reformulate IPs so that they become “easy” for variable branching. This approach has been successfully theoretically analyzed for certain classes of knapsack problems as well as random IPs (see [20] for a survey) and experimentally analyzed on various classes of instances [1, 16]. There has also been experimental work on how to come up with good general branching directions in practice using heuristic methods [19, 18, 14].

1.2 Cutting Planes

Another fundamental proof system, studied extensively within both the IP and SAT contexts are cutting planes (CP) proofs. The most fundamental class of cutting planes are so-called Chvátal-Gomory (CG) cuts, which are the principal class studied within SAT and one of the most important classes of cuts in IP [13].

CG cuts for a set $K \subseteq \mathbb{R}^n$ are derived geometrically as follows. Assume that the inequality $\mathbf{a}\mathbf{x} \leq r$, $\mathbf{a} \in \mathbb{Z}^n$, $r \in \mathbb{R}$, is valid for K , that is, $\mathbf{x} \in K \Rightarrow \mathbf{a}\mathbf{x} \leq r$. Then, the inequality $\mathbf{a}\mathbf{x} \leq \lfloor r \rfloor$ is valid for $K \cap \mathbb{Z}^n$, since $\mathbf{x} \in \mathbb{Z}^n$ implies that $\mathbf{a}\mathbf{x} \in \mathbb{Z}$. Given $\mathbf{a} \in \mathbb{Z}^n$, the strongest cut of this form one can derive for K is clearly $\mathbf{a}\mathbf{x} \leq \lfloor \sup_{\mathbf{z} \in K} \mathbf{a}\mathbf{z} \rfloor$. We therefore denote this cut to be the CG cut of K induced by \mathbf{a} , and we use the notation $\text{CG}(K, \mathbf{a}) := \{\mathbf{x} \in K : \mathbf{a}\mathbf{x} \leq \lfloor \sup_{\mathbf{z} \in K} \mathbf{a}\mathbf{z} \rfloor\}$ to denote applying the CG cut induced by \mathbf{a} to K . We may extend this to an ordered list $\mathcal{L} = (\mathbf{a}_1, \dots, \mathbf{a}_k)$, letting $\text{CG}(K, \mathcal{L})$ be the result of applying the CG cuts induced by $\mathbf{a}_1, \dots, \mathbf{a}_k$ to K one by one in this order (from left to right).

In terms of certifying such cuts, if $K = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}\mathbf{x} \leq \mathbf{d}\}$, $\mathbf{C} \in \mathbb{Q}^{m \times n}$, $\mathbf{d} \in \mathbb{Q}^m$, is a polyhedron, then by Farkas's lemma, every CG cut can be obtained as a conic combination of the constraints after rounding down the right hand side. That is, for each $\boldsymbol{\lambda} \geq 0$ such that $\boldsymbol{\lambda}\mathbf{C} \in \mathbb{Z}^n$, we have the corresponding CG cut $\boldsymbol{\lambda}\mathbf{C}\mathbf{x} \leq \lfloor \boldsymbol{\lambda}\mathbf{d} \rfloor$, and every CG cut for K can be derived in this way.

A cutting plane proof (CP) of integer infeasibility for $K \subseteq \mathbb{R}^n$ can now be described as a list $\mathcal{L} = (\mathbf{a}_1, \dots, \mathbf{a}_N)$, $\mathbf{a}_i \in \mathbb{Z}^n$, such that $\text{CG}(K, \mathcal{L}) = \emptyset$. In this context, the number of CG cuts N denotes the length of the CP proof. When $K = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}\mathbf{x} \leq \mathbf{d}\}$ is a polyhedron as above, to get a certified proof, we can augment \mathcal{L} with multipliers $\boldsymbol{\lambda}_1 \in \mathbb{R}_+^m, \boldsymbol{\lambda}_2 \in \mathbb{R}_+^{m+1}, \dots, \boldsymbol{\lambda}_{N+1} \in \mathbb{R}_+^{m+N}$ (we still refer to the length of \mathcal{L} as N in this case). Letting $\mathcal{L}_i := (\mathbf{a}_1, \dots, \mathbf{a}_i)$, $i \in [N]$, the multipliers $\boldsymbol{\lambda}_i \in \mathbb{R}_+^{m+i-1}$, $0 \leq i \leq N$, certify the cut $\mathbf{a}_i \mathbf{x} \leq \lfloor \sup\{\mathbf{a}_i \mathbf{z} : \mathbf{z} \in \text{CG}(K, \mathcal{L}_{i-1})\} \rfloor$, in the manner described in the previous paragraph, using the the original inequalities $\mathbf{C}\mathbf{x} \leq \mathbf{d}$ (the first m components of $\boldsymbol{\lambda}_i$) and the previous cuts $\mathbf{a}_j \mathbf{x} \leq \lfloor \sup\{\mathbf{a}_j \mathbf{z} : \mathbf{z} \in \text{CG}(K, \mathcal{L}_{j-1})\} \rfloor$, $j \in [i-1]$. Finally, $\boldsymbol{\lambda}_{N+1} \in \mathbb{R}_+^{m+N}$ provides the Farkas certificate of infeasibility for $\text{CG}(K, \mathcal{L})$, using the original system together with all the cuts.

As with branching proofs, it is important to be able to control the bit-size $\langle \mathcal{L} \rangle$ of a CP proof and not just its length (i.e., the number of cuts in the list \mathcal{L}). Here $\langle \mathcal{L} \rangle$ corresponds to the number of bits needed to describe $\langle \mathbf{a}_1, \dots, \mathbf{a}_N \rangle$, as well as $\langle \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{N+1} \rangle$ for a certified proof. When $K = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}\mathbf{x} \leq \mathbf{d}\}$ is a polyhedron as above, a fundamental theorem of Cook, Coullard and Turán [7] is that any CP proof \mathcal{L} of integer infeasibility for K can be recompiled into a *certified* CP proof \mathcal{L}' , such that $N := |\mathcal{L}| = |\mathcal{L}'|$ and $\langle \mathcal{L}' \rangle = \text{poly}(N, L)$, where $L := \langle \mathbf{C}, \mathbf{d} \rangle$ is the number of bits needed to describe the linear system defining K . Thus, for CP proofs on polyhedra, one can, without loss of generality, assume that the bit-size of a CP proof is polynomially related to its length and the bit-size of the defining linear system.

In terms of general complexity upper bounds, another important theorem of [7] is that every integer free rational polytope $K \subseteq \mathbb{R}^n$ admits a CP proof of infeasibility of length $O(f(n)^n)$, where $f(n)$ is the flatness constant. This bound was achieved by showing that a run of Lenstra's algorithm can effectively be converted into a CP proof.

To relate CP and branching proofs, there is a simple disjunctive characterization of CG cuts. Namely, $\mathbf{a}\mathbf{x} \leq b$, $\mathbf{a} \in \mathbb{Z}^n$, $b \in \mathbb{Z}$ is a CG cut for K iff $\{\mathbf{x} \in K : \mathbf{a}\mathbf{x} \geq b+1\} = \emptyset$. That is, if and only if the *right side* of the disjunction $\mathbf{a}\mathbf{x} \leq b$ or $\geq b+1$ is empty for K . From this observation, one can easily show that any CP proof of infeasibility can be converted into a branching proof of infeasibility with only an $O(1)$ factor blowup in length (see [5] for a formal proof).

1.3 Complexity of Branching Proofs

Despite its long history of study within IP, general branching has only recently been studied from the SAT perspective. In [5], Beame et al rediscovered the concept of general branching proofs in the context SAT, naming them stabbing planes (SP) refutations, and analyzed them from the proof complexity perspective. To keep with this nomenclature, we use the term stabbing planes (SP) refutations to refer specifically to a certified branching proofs of infeasibility for SAT formulas. In terms of results, they showed that SP refutations can size or depth simulate CP proofs and showed that they are equivalent to Krajíček's [15] tree-like R(CP) refutations. They further gave lower bounds and impossibility results, showing an $\Omega(n/\log n)$ lower bound on the depth of SP refutations and showed that SP refutations cannot be balanced.

Lastly, they provided upper bounds on the length of SP refutations, showing that any Tseitin formula has a quasi-polynomial sized SP refutation. We recall that a Tseitin formula is indexed by a constant degree graph $G = (V, E)$ and a set of parities $l_v \in \{0, 1\}$, $v \in V$, satisfying $\sum_{v \in V} l_v \equiv 1 \pmod{2}$. The variables $\mathbf{x} \in \{0, 1\}^E$ index the corresponding subset of edges where the assignment \mathbf{x} is a satisfying assignment iff $\sum_{e \in E: v \in e} x_e \equiv l_v \pmod{2}$,

$\forall v \in V$. Note that such a formula is clearly unsatisfiable, since the sum of degrees of any (sub)graph is even whereas $\sum_{v \in V} l_v$ is odd by assumption. For such formulas, Beame et al gave a $2^\Delta (n\Delta)^{O(\log n)}$ length SP refutations, where Δ is the maximum degree of G . A long standing conjecture [4, 5] is that Tseitin formulas are hard for cutting planes, and thus the above was seen as evidence that SP refutations are strictly stronger than CP. We note that exponential lower bounds for CP were first proven by Pudlák [21], who showed how to derive CP lower bounds from monotone circuit lower bounds. The corresponding monotone circuit problem for Tseitin formulas is easy however, and hence cannot be used for proving strong lower bounds.

Beame et al [5] left open some very natural proof complexity theoretic questions about branching proofs, which highlighted fundamental gaps in our understanding of the proof system. Their first question relates to the relationship between bit-size $\langle \mathcal{T} \rangle$ and length $|\mathcal{T}|$ of an SP proof. Precisely, they asked whether one can always assume that the bit size of an SP refutation is bounded by a polynomial in the dimension and the length of the proof, that is, can an SP refutation be “recompiled” so that it satisfies this requirement without increasing its length by much. As mentioned previously, the corresponding result for CP refutations was already shown by Cook et al [7], though the techniques there do not seem to apply to SP. Their second question was whether one could show a separation between CP and SP, which would follow if Tseitin formulas are (say exponentially) hard for CP. Lastly, they asked whether one can prove super-polynomial lower bounds for SP.

1.4 Our Contributions

In this work, we give answers to many of the questions above. Firstly, we resolve Beame et al’s bit-size vs length question affirmatively. Secondly, we show that Tseitin formulas have quasi-polynomial size CP proofs, showing that they do not provide an exponential separation between CP and SP. Lastly, we give a very simple family of n -dimensional (mixed-)integer free polytopes for which any branching proof has size exponential in n . We describe these contributions in detail below.

Bit-size of Branching Proofs. As our first main contribution, we resolve Beame et al’s bit-size vs length question, by proving the following more general result:

► **Theorem 1.** *Let $K \subseteq \mathbb{R}^n$ be an integer free compact convex set satisfying $K \subseteq R\mathbb{B}_1^n$, where \mathbb{B}_1^n is the ℓ_1 ball and $R \in \mathbb{N}$. Let \mathcal{T} be a branching proof of integer infeasibility for K . Then, there exists a branching proof \mathcal{T}' for K , such that $|\mathcal{T}'| \leq O(n|\mathcal{T}|)$, and where every edge e of \mathcal{T}' is labeled by an inequality $\mathbf{a}'_e \mathbf{x} \leq b'_e$, $\mathbf{a}'_e \in \mathbb{Z}^n$, $b'_e \in \mathbb{Z}$, where $\max\{\|\mathbf{a}'_e\|_\infty, |b'_e|\} \leq (10nR)^{(n+2)^2}$. Moreover, $\langle \mathcal{T}' \rangle = O(n^3 \log_2(nR)|\mathcal{T}|)$.*

The above theorem says that at the cost of increasing the number of nodes in the branching tree by a factor $O(n)$, one can reduce the coefficients in the normals of the disjunctions to $(10nR)^{(n+2)^2}$. In particular, since \mathbf{a}'_e, b'_e are integral, they can be described with $O(n^3 \log_2(nR))$ bits. We note that the final bound on $\langle \mathcal{T}' \rangle$ ends up being better than $O(n^3 \log_2(nR)|\mathcal{T}'|) = O(n^4 \log_2(nR)|\mathcal{T}|)$, due to the fact that “extra” nodes we need in \mathcal{T}' use smaller disjunctions needing only $O(n^2 \log_2(nR))$ bits. In the context of SAT, the desired bound on the coefficients of SP proofs follows directly from the fact that any SAT polytope, as in (SAT-LP), is contained inside $[0, 1]^n \subseteq n\mathbb{B}_1^n$.

As mentioned in the previous subsection, one would generally want a branching proof to come with certificates of infeasibility for the leaf nodes. For a rational polytope K , the following corollary bounds the cost of extending the branching proof produced by Theorem 1

to a certified branching proof. Precisely, the bit-size of the final certified proof can be made proportional to the size of the original tree, the bit-encoding length of the defining system for K and a polynomial in the dimension.

► **Corollary 2.** *Let $K = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}\mathbf{x} \leq \mathbf{d}\}$ be rational polytope with $\mathbf{C} \in \mathbb{Q}^{m \times n}$, $\mathbf{d} \in \mathbb{Q}^m$ having bit-size $L := \langle \mathbf{C}, \mathbf{d} \rangle$. Let \mathcal{T} be a branching proof for K . Then there exists a certified branching proof \mathcal{T}' for K such that $|\mathcal{T}'| \leq O(n)|\mathcal{T}|$ and $\langle \mathcal{T}' \rangle = O(n^6 L)|\mathcal{T}|$.*

The bit-size $L := \langle \mathbf{C}, \mathbf{d} \rangle$ of K in Corollary 2 shows up for two related reasons. Firstly, we need L to upper bound the ℓ_1 circumradius R of K , which is in turn used to bound the bit-size of the disjunctions in Theorem 1. For a rational polytope K , R is in fact always upper bounded by $2^{O(L)}$. We stress that $2^{O(L)}$ more directly upper bounds the ℓ_1 norm of the vertices of K , which in turns upper bounds the ℓ_1 circumradius of K only under the assumption that K is indeed bounded (i.e., that K is polytope and not just a polyhedron). However, it is well known that for a rational polyhedron K , $K \cap \mathbb{Z}^n = \emptyset$ iff $K \cap 2^{O(L)}\mathbb{B}_1^n \cap \mathbb{Z}^n = \emptyset$ (see Schrijver [24] Chapter 17). Therefore, the boundedness assumption above is essentially without loss of generality. More precisely, one can simply add box constraints $-2^{O(L)} \leq x_i \leq 2^{O(L)}$, $i \in [n]$, to the description of K , which increases the description length by an $O(n)$ factor. The second reason for needing L is to bound the bit-complexity of the Farkas infeasibility certificates at the leaves of the modified branching tree. By standard bounds, such a certificate has bit-size bounded by $O(n)$ times the bit description length of a minimal infeasible subsystem (over the reals) at the corresponding leaf. By Helly’s theorem, a minimal infeasible subsystem has at most $n + 1$ inequalities consisting of a subset of the inequalities defining K and the inequalities from branching, where each of these inequalities has bit-size at most $O(n^3 L)$ by Theorem 1.

Sketch of Theorem 1. We now give some intuition about the difficulties in proving Theorem 1, which is technically challenging, and sketch the high level proof ideas.

We first note that any disjunction $\mathbf{a}\mathbf{x} \leq b$ or $\geq b + 1$, where \mathbf{a} has very large coefficients, only cuts off a very thin slice of K . In particular, the width of the band $b \leq \mathbf{a}\mathbf{x} \leq b + 1$ is exactly $1/\|\mathbf{a}\|_2$. Thus, it is perhaps intuitive that any “optimal” proof should use wide disjunctions instead of thin ones, and hence should have reasonably small coefficients. This intuition turns out to be false however, as the angle of a disjunction can in fact be more important than its width for a proof of optimal length.

The following simple 2 dimensional example shows that if one wishes to exactly preserve the length of a branching proof, then large coefficients are unavoidable even for sets of constant radius. Examine the line segment

$$K = \{(x_1, x_2) : Mx_1 + x_2 = 1/2, 0 \leq x_2 \leq 2\}$$

for $M \geq 1$. Clearly, branching on $Mx_1 + x_2 \leq 0$ or ≥ 1 certifies integer infeasibility in one step. Now let $\mathbf{a} \in \mathbb{Z}^2$ be any branching direction that also certifies infeasibility in one step. Then, the width of K with respect to \mathbf{a} must be less than one:

$$\max_{\mathbf{x} \in K} \mathbf{a}\mathbf{x} - \min_{\mathbf{x} \in K} \mathbf{a}\mathbf{x} = |2(a_2 - a_1/M)| < 1.$$

Now if $a_2 \neq 0$, then $|a_1| \geq M/2$, so $\|\mathbf{a}\|_\infty \geq M/2$. If $a_2 = 0$, then we should let $\mathbf{a} = (1, 0)$, since this choice yields the widest possible disjunctions under this restriction. Branching on $\mathbf{a} = (1, 0)$ cannot certify infeasibility in one step however, since $\mathbf{x} = (0, 1/2) \in K$ and $\mathbf{a}\mathbf{x} = 0$.

To recompile a proof \mathcal{T} using only small coefficients, we are thus forced to make do with a discrete set of disjunction angles that may force us to increase the length of the proof. Given an arbitrary branching direction \mathbf{a} , the standard tool for approximating the direction of \mathbf{a} using small coefficients is so-called *Diophantine* approximation (see Lemma 13). Thus, the natural first attempt would be to take every disjunction $\mathbf{a}\mathbf{x} \leq b$ or $\geq b + 1$ in \mathcal{T} and replace it by its small coefficient Diophantine approximation $\mathbf{a}'\mathbf{x} \leq b'$ or $\geq b' + 1$ to get \mathcal{T}' . As shown above, there are examples where any such small coefficient \mathcal{T}' will no longer be valid, namely some of the leaf nodes may become feasible.

Let $v \in \mathcal{T}$ be a leaf node with relaxation $K_v = \{\mathbf{x} \in K : \mathbf{A}\mathbf{x} \leq \mathbf{b}\} = \emptyset$ and corresponding approximation $v' \in \mathcal{T}'$ with $K_{v'} = \{\mathbf{x} \in K : \mathbf{A}'\mathbf{x} \leq \mathbf{b}'\} \neq \emptyset$. To transform \mathcal{T}' to a valid proof, we must therefore add branching decisions to \mathcal{T}' below v' to certify integer-freeness of $K_{v'}$. From here, the main intuitive observation is that since $P_v := \mathbf{A}_v\mathbf{x} \leq \mathbf{b}_v$ and $P_{v'} := \mathbf{A}'_v\mathbf{x} \leq \mathbf{b}'_v$ have almost the same inequalities, $P_{v'} \cap K$ should be very close to infeasible.

By inspecting a Farkas-type certificate of infeasibility for $K \cap P_v$ (see section 2.4), for a good enough Diophantine approximation $P_{v'}$ to P_v , one can in fact pinpoint an inequality of $P_{v'}$, say $\mathbf{a}'_{v,1}\mathbf{x} \leq b'_{v,1}$, such that replacing $b'_{v,1}$ by $b'_{v,1} - 1$ makes $K \cap P_{v'}$ empty. This uses the boundedness of K , i.e., $K \subseteq R\mathbb{B}_1^n$, and that the disjunctions induced by the rows of \mathbf{A}' are much wider than those induced by \mathbf{A} . Note that the emptiness of $\mathbf{a}'_{v,1}\mathbf{x} \leq b'_{v,1} - 1$ corresponds to saying that $\mathbf{a}'_{v,1}\mathbf{x} \geq b'_{v,1}$ is a valid CG cut for $K \cap P_{v'}$. Furthermore, this CG cut has the effect of reducing dimension by one since now $\mathbf{a}'_{v,1}\mathbf{x} = b'_{v,1}$.

Given the above, it is natural to hope than one can simply repeat the above strategy recursively. Namely, at each step, we try to find a new CG cut induced by a row of \mathbf{A}' which reduces dimension of $K \cap P_{v'}$ by one. Unfortunately, as stated, the strategy breaks down after one step. The main problem is that, after the first step, we have no “information” about $\mathbf{a}_{v,1}\mathbf{x} \leq b_{v,1}$ restricted to $\mathbf{a}'_{v,1}\mathbf{x} = b'_{v,1}$. Slightly more precisely, we no longer have a proxy for $\mathbf{a}_{v,1}\mathbf{x} \leq b_{v,1}$ in $P_{v'}$ that allows us to push this constraint “backwards” on the subspace $\mathbf{a}'_{v,1}\mathbf{x} = b'_{v,1}$. Since we must somehow compare $P_{v'}$ to P_v to deduce infeasibility, this flexibility turns out to be crucial for being able to show the existence of a dimension reducing CG cut.

To fix this problem, we rely on a more sophisticated iterated form of Diophantine approximation due to Frank and Tardos [12]. At a high level (with some simplification), for a disjunction $\mathbf{a}\mathbf{x} \leq b$ or $\geq b + 1$, $\mathbf{a} \in \mathbb{Z}^n$, $b \in \mathbb{Z}$, we first construct of sequence of Diophantine approximations $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{Z}^n$, containing \mathbf{a} in their span, which intuitively represent the highest to lower order bits of the direction of \mathbf{a} . From here, we carefully choose a sequence $b_1, \dots, b_k \in \mathbb{Z}$ indexing inequalities $\mathbf{a}_i\mathbf{x} \leq b_i$, $i \in [k]$, which allow us to get better and better approximations of $\mathbf{a}\mathbf{x} \leq b$. Since we are in reality replacing the disjunction $\mathbf{a}\mathbf{x} \leq b$ or $\geq b + 1$, we will in fact need a sequence that somehow approximates both sides of the disjunction at the same time. This will correspond to requiring that a “flipped” version of the sequence, namely $\mathbf{a}_i\mathbf{x} \geq b_i$, $i \in [k - 1]$, and $\mathbf{a}_k\mathbf{x} \geq b_k + 1$, gives improving approximations of $\mathbf{a}\mathbf{x} \geq b + 1$. Restricting attention to just the $\mathbf{a}\mathbf{x} \leq b$ side, we will show the existence of improving “error levels” $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_k = 0$, such that $\|\mathbf{x}\|_1 \leq R, \mathbf{a}_l\mathbf{x} \leq b_l, \mathbf{a}_i\mathbf{x} = b_i, i \in [l - 1] \Rightarrow \mathbf{a}\mathbf{x} \leq b + \gamma_l$. Furthermore, we will ensure that branching on $\mathbf{a}_l\mathbf{x} \leq b_l - 1$, not only reduces the error bound α_l , but in fact implies a far stronger inequality than $\mathbf{a}\mathbf{x} \leq b$. Precisely, we will require $\|\mathbf{x}\|_1 \leq R, \mathbf{a}_l\mathbf{x} \leq b_l - 1, \mathbf{a}_i\mathbf{x} = b_i, i \in [l - 1] \Rightarrow \mathbf{a}\mathbf{x} \leq b - n\gamma_l$. Hence, once we have learned the equalities $\mathbf{a}_i\mathbf{x} = b_i$, $i \in [l - 1]$, \mathbf{a}_l becomes a suitable proxy for \mathbf{a} which we can use to push the constraint $\mathbf{a}\mathbf{x} \leq b$ “backwards”. Note that if $l = k$, we have in fact fully learned $\mathbf{a}\mathbf{x} \leq b$ since $\alpha_k = 0$. If $l < k$ and $\mathbf{a}\mathbf{x} \leq b$ is the “closest inequality to infeasibility” in the current relaxation, corresponding to the inequalities in $P_{v'}$ for some leaf v' together with the additional equalities

as above, we will be able to guarantee that the CG cuts induced by \mathbf{a}_l and $-\mathbf{a}_l$ induce the new equality $\mathbf{a}_l \mathbf{x} = b_l$. Note that if we always manage to reduce dimension by at least 1, we will terminate with an infeasible node after adding at most $n + 1$ pairs of CG cuts. So far, we have discussed replacing a disjunction $\mathbf{a} \mathbf{x} \leq b$ or $\geq b + 1$ by sequence instead of a single disjunction, and the latter is what is actually needed. For this purpose, the new disjunction will have the form $\mathbf{a}' \mathbf{x} \leq b'$ or $\geq b' + 1$ where $\mathbf{a}' = \sum_{i=1}^k M^{k-i} \mathbf{a}_i$ and $b' = \sum_{i=1}^k M^{k-i} b_i$ for M chosen large enough. This is chosen to ensure that $\|\mathbf{x}\|_1 \leq R$, $\mathbf{a}' \mathbf{x} \leq b'$, $\mathbf{a}_i \mathbf{x} = b_i$, $i \in [l-1]$ “almost implies” $\mathbf{a}_l \mathbf{x} \leq b_l$, with a symmetric guarantee for the flipped sequence. The full list of $(a', b', k, a_1, b_1, \gamma_1, \dots, a_k, b_k, \gamma_k)$ satisfying the requisite properties is what we call a *valid substitution sequence* of $\mathbf{a} \mathbf{x} \leq b$ (see definition (24)). The main difficulty in constructing and analyzing the disjunction $\mathbf{a}' \mathbf{x} \leq b'$ or $\geq b' + 1$, is that each side of the disjunction should induce a valid substitution sequence for the corresponding side of $\mathbf{a} \mathbf{x} \leq b$ or $\geq b + 1$. That is, we need to work for “both sides” at once. As the remaining details technical, we defer further discussion of the proof to Section 3 of the paper.

As an interesting point of comparison, we note that in contrast to Theorem 1 the recompilation result of [7] does not give a *length independent bound* on the size of normals of the CG cuts it produces (e.g., depending only on the ℓ_1 radius of K). An interesting question is whether one can give length independent bounds for CP proofs based only on the bit-complexity L of the starting system. Perhaps one avenue for such a reduction, would be to first convert the CP proof to a branching proof and try to apply the techniques above. The main issue here is that first reduction phase above, which approximates each disjunction in the tree with a small coefficient one, need not preserve the CP structure. Namely, after the replacement, it is not clear how to guarantee that every disjunction in the replacement tree has at least one “empty” side (note that this problem is compounded by the approximation errors going up the tree).

Upper Bounds for Tseitin formulas. As our second contribution, we show that Tseitin formulas have quasi-polynomial CP proofs, refuting the conjecture that these formulas are (exponentially) hard for CP.

► **Theorem 3.** *Let $G = (V, E)$ be an n -vertex graph, $l_v \in \{0, 1\}$, for $v \in V$, be parities and Φ be the corresponding Tseitin formula. Then Φ has a CP refutation of length $2^\Delta (n\Delta)^{O(\log n)}$, where Δ is the maximum degree of G .*

To prove the theorem our main observation is that the quasi-polynomial SP proof of Beame et al [5] is of a special type, which we dub an *enumerative branching proof*, that can be automatically converted to a CP proof of the same size.

We define an *enumerative branching proof* for a compact convex set K to correspond, as before, to a tree \mathcal{T} with root r and root relaxation $K_r := K$. At every node $v \in \mathcal{T}$ with $K_v \neq \emptyset$, we choose a branching direction $\mathbf{a}_v \in \mathbb{Z}^n \setminus \{0\}$ and immediately branch on all possible choices $b \in \mathbb{Z}$ that intersect the current relaxation K_v . Note that tree \mathcal{T} need not be binary. Formally, we first label v with the bounds $l_v, u_v \in \mathbb{R}$ satisfying

$$\{\mathbf{a}_v \mathbf{x} : \mathbf{x} \in K_v\} \subseteq [l_v, u_v].$$

From here, we create a child node v_b , for every $b \in \mathbb{Z}$ such that $l_v \leq b \leq u_v$. The edge $e = \{v, v_b\}$ is now labeled with the *equality* $\mathbf{a}_v \mathbf{x} = b$ and the updated relaxation becomes $K_{v_b} = \{\mathbf{x} \in K_v : \mathbf{a}_v \mathbf{x} = b\}$. From here, each leaf node $v \in \mathcal{T}$ can be of two different types. Either $K_v = \emptyset$, or if $K_v \neq \emptyset$, the interval $[l_v, u_v]$ is defined and does not contain integer points, i.e., $[u_v] < l_v$. A tree \mathcal{T} satisfying the above properties is a valid enumerative branching proof of integer infeasibility for K .

34:10 On the Complexity of Branching Proofs

It is an easy exercise to check that any enumerative branching proof can be converted to a standard branching proof incurring only a constant factor blowup in the number of nodes. Theorem 3 follows directly from the observation that the Beame et al SP proof is enumerative together with the following simulation result.

► **Theorem 4.** *Let $K \subseteq \mathbb{R}^n$ be a compact convex set. Let \mathcal{T} be an enumerative branching proof of K . Then there exists $\mathcal{L} = (\mathbf{a}_1, \dots, \mathbf{a}_N) \in \mathbb{Z}^n$ such that $\text{CG}(K, \mathcal{L}) = \emptyset$ and $N \leq 2|\mathcal{T}| - 1$.*

While in the above generality the result is new, the main ideas (at least for rational polytopes) are implicit in Cook et al [7]. In particular, their proof that any integer free rational polytope admits a CP proof of length at most $O(f(n)^n)$ in effect treats Lenstra’s algorithm as an enumerative branching proof which they serialize to get a CP proof. Theorem 4 shows that their serialization technique is fully general and in fact can be applied to any enumerative branching proof. To get a certified CP proof of small bit-size from Theorem 4 for a rational polyhedron K , we note that it suffices to apply the recompilation technique of Cook et al [7] to the output of Theorem 4. While there is some technical novelty in the generalization to arbitrary compact convex sets, we feel the main contribution of Theorem 4 is conceptual. As evidenced by Theorem 3, the formalization of enumerative branching proofs and their relationship to CP can be a useful tool for constructing CP proofs.

We now sketch the main ideas for serializing an enumerative branching proof \mathcal{T} for K . We start from the root $r \in \mathcal{T}$, with branching direction $\mathbf{a}_r \in \mathbb{Z}^n$ and $\{\mathbf{a}_r \mathbf{x} : \mathbf{x} \in K\} \subseteq [l_r, u_r]$. The idea is to iteratively “push” the hyperplane $H_b = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_r \mathbf{x} = b\}$, with b initialized to u_r , backwards through K , until K is empty (i.e., iteratively decreasing b until it goes below l_r). The first push is given by the CG cut induced by \mathbf{a}_r which pushes H_{u_r} to $H_{\lfloor u_r \rfloor}$. That is, $b \leftarrow \lfloor b \rfloor$. Since b is now integral, we can no longer decrease b just using CG cuts induced by \mathbf{a}_r . At this point, we note that the subtree \mathcal{T}_{r_b} of \mathcal{T} rooted at the child r_b is a valid branching proof for $K \cap H_b$. We can thus apply the procedure recursively on $K \cap H_b$ and \mathcal{T}_{r_b} to “chop off” $K \cap H_b$. For this purpose, one crucially needs to be able to lift CG cuts applied to the face $K \cap H_b$ to CG cuts one can apply to K that have the same effect on $K \cap H_b$. Such a *lifting lemma* is classical for rational polyhedra [6] and was established more recently for compact convex sets in [10], a variant of which we use here. Applying the lifted CG cuts to K , we can thus guarantee that $K \cap H_b = \emptyset$. This allows us to push once more with the cut induced by \mathbf{a}_r , pushing H_b to H_{b-1} . The process now continues in a similar fashion until K is empty. We note that the enumerative structure is crucial here, as it allows one to keep the “action” on the boundary K throughout the entire proof.

Lower Bounds for Branching Proofs. As our final contribution, we give a simple family of n -dimensional (mixed-)integer free polytopes which require branching proofs of length exponential in n .

► **Theorem 5.** *The integer-free SAT polytope*

$$P_n := \{\mathbf{x} \in [0, 1] : \sum_{i \in S} x_i + \sum_{i \notin S} (1 - x_i) \geq 1, \forall S \subseteq [n]\}$$

requires branching proofs of length $2^n/n$.

The above example is due to Cook et al [7], which they used to give a $2^n/n$ lower bound for CP. In the above theorem, we show that their lower bound technique extends to branching proofs. As it is very simple and short, we give the full proof below.

Proof. The first observation is that P_n is “integer critical”, namely, removing any constraint from P_n makes the polytope integer feasible. In particular, removing $\sum_{i \in S} x_i + \sum_{i \notin S} (1 - x_i) \geq 1$, for any $S \subseteq [n]$, makes the vector $\mathbf{1}_{\bar{S}}$, the indicator of the complement of S , feasible.

Let \mathcal{T} denote any branching proof for P_n . For any leaf node v of \mathcal{T} , by Farkas’s lemma, the infeasibility of the continuous relaxation $(P_n)_v$ is certified by at most $n + 1$ constraints. Since P_n is non-empty, at most n of these constraints can come from the description of P_n . Letting N denote the number of leaves of \mathcal{T} , one can therefore certify the infeasibility of each leaf of \mathcal{T} using at most nN original constraints from P_n . If $nN < 2^n$, then \mathcal{T} would certify the integer infeasibility of P_n with at least one constraint removed. By integer criticality of P_n , this is impossible. Therefore $|\mathcal{T}| \geq N \geq 2^n/n$, as needed. ◀

One notable criticism of the above example is that it already has 2^n constraints. Thus, the length of the proof is simply proportional to the initial representation. Interestingly, P_n has a very simple extended formulation in \mathbb{R}^{2n} requiring only $O(n)$ constraints. In particular, a direct computation reveals that

$$\begin{aligned} P_n &= \{\mathbf{x} \in [0, 1]^n : \|(x_1 - 1/2, \dots, x_n - 1/2)\|_1 \leq n/2 - 1\} \\ &= \{\mathbf{x} \in [0, 1]^n : \exists \mathbf{y} \in [0, 1]^n, \sum_{i=1}^n y_i \leq n/2 - 1, \pm(x_i - 1/2) \leq y_i, i \in [n]\}. \end{aligned}$$

Combining the above with Theorem 5, we immediately get an exponential lower bound for proving the mixed-integer infeasibility of a compactly represented polytope. We note that in this setting, the lower bound is indeed exponential in the description length of P .

► **Corollary 6.** *Let $Q_n = \{(\mathbf{x}, \mathbf{y}) \in [0, 1]^{2n} : \sum_{i=1}^n y_i \leq n/2 - 1, \pm(x_i - 1/2) \leq y_i, i \in [n]\}$. Then any branching proof of mixed-integer infeasibility for Q_n , proving $Q_n \cap \mathbb{Z}^n \times \mathbb{R}^n = \emptyset$, has length at least $2^n/n$.*

To see the above, recall that a mixed-integer branching proof for Q_n only branches on integer disjunctions supported on the first n variables. Thus, it is entirely equivalent to a branching proof for the projection of Q_n onto these variables, namely, to a branching proof for P_n .

As a final remark, we note that in the extended space, Q_n does in fact have a very short proof of infeasibility using only n split cuts, which are perhaps the most important class of cutting planes in practice (in fact, the most generically effective cuts are the Gomory mixed-integer cuts (GMI), which are equivalent to split cuts for rational polyhedra [8]). Roughly speaking, a split cut here is any linear inequality that is *valid for both sides* $\mathbf{a}\mathbf{x} \leq b$ or $\geq b + 1$, $\mathbf{a} \in \mathbb{Z}^n$, $b \in \mathbb{Z}$, of an integer disjunction. In particular, $y_i \geq 1/2$ is a valid split cut for Q_n , for $i \in [n]$, since it is valid for $x_i \leq 0$ and $x_i \geq 1$. These n splits together imply that $\sum_{i=1}^n y_i \geq n/2$, and thus adding them to Q_n makes the system infeasible.

1.5 Conclusions

In this work, we have continued the proof complexity theoretic study of branching proofs started in [5], establishing analogues of the CP results in [7] for branching proofs. In the process, we have clarified basic properties of the branching proof system, including how to control the size of coefficients, how to simulate important classes of branching proofs using CP, and how to construct elementary lower bound examples for them. We hope that these results will help motivate a further study of this important proof system.

In terms of open questions, there are many. On the lower bound side, in the context of SAT, the example we use has exponentially many clauses. It would be much more interesting to find polynomial sized formulas with exponential sized branching proofs. In the context of integer programming, as mentioned previously, the best known algorithms for general integer programming require $n^{O(n)}$ time. A very interesting question is whether one can find an example of an integer free compact convex set $K \subseteq \mathbb{R}^n$, requiring branching proofs of size $n^{\Omega(n)}$. Such a lower bound would show that Lenstra-type algorithms for IP, which in fact yield enumerative branching proofs, cannot be substantially improved. We note that this still leaves open the possibility that so-called Kannan-type algorithms can do much better (see [9] Chapter 7 for a reference). In terms of upper bounds, a natural question is whether one can leverage the simulation of enumerative branching proofs by CP to give new upper bounds beyond Tseitin formulas. It was shown by Cook et al [7] that for SAT, CP can be simulated by extended resolution. A natural question is whether stabbing planes can also be simulated by extended resolution. Lastly, as mentioned previously, it would be interesting to establish length independent bounds for the coefficients of the normals in CP proofs.

1.6 Organization

In Section 2, we collect basic notation, formalize the definition of branching proofs and cover the necessary tools from Diophantine approximation. In Section 3, we present our branching proof recompilation theorem, which ensures that the bit-size of branching proofs can be polynomially bounded. In Section 4, we show how to simulate enumerative branching proofs via CP, and apply this simulation to get a quasi-polynomial CP bound for Tseitin formulas.

2 Preliminaries

Basic Notation. The natural numbers are denoted by \mathbb{N} , the reals and non-negative reals by \mathbb{R}, \mathbb{R}_+ respectively. For $m \in \mathbb{N}$, we denote the set $\{1, \dots, m\}$ by $[m]$. Vectors $\mathbf{x} \in \mathbb{R}^n$ are denoted in bold and scalars by $x \in \mathbb{R}$. The standard basis vectors of \mathbb{R}^n are denoted by $\mathbf{e}_i, i \in [n]$. Given two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we write $\mathbf{x}\mathbf{y} := \sum_{i=1}^n x_i y_i$ for their inner product. The ℓ_1 and ℓ_∞ norm of \mathbf{x} are $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$ and $\|\mathbf{x}\|_\infty = \max_{i \in [n]} |x_i|$ respectively. We denote the ℓ_1 ball in \mathbb{R}^n by $\mathbb{B}_1^n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_1 \leq 1\}$. For a vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we let $\lfloor \mathbf{x} \rfloor := (\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)$ denote the vector whose coordinates are those of \mathbf{x} rounded to the nearest integer.

Since we shall study convex bodies lying in the ℓ_1 ball of some radius $R \in \mathbb{N}$, it is helpful to define the following shorthand notation: for a set of linear inequalities $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and a vector \mathbf{c} , the expression $\mathbf{A}\mathbf{x} \leq \mathbf{b} \Rightarrow_R \mathbf{c}\mathbf{x} \leq d$ stands for

$$\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_1 \leq R, \mathbf{A}\mathbf{x} \leq \mathbf{b}\} \subseteq \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_1 \leq R, \mathbf{c}\mathbf{x} \leq d\}.$$

► **Definition 7 (Halfspace, Hyperplane).** For $\mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$, we define the halfspace $H_{\mathbf{a},b} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}\mathbf{x} \leq b\}$ and the hyperplane $H_{\mathbf{a},b}^- = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}\mathbf{x} = b\}$.

► **Definition 8 (Support Function).** Let $K \subseteq \mathbb{R}^n$. The support function $h_K : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as $h_K(\mathbf{a}) := \sup_{\mathbf{x} \in K} \mathbf{a}\mathbf{x}$. The support function is always convex and is continuous if K is non-empty and bounded. If K is non-empty and compact, the supremum in $h_K(\mathbf{a})$ is always attained. By convention, if $K = \emptyset$ we define $h_K(\mathbf{a}) = -\infty, \forall \mathbf{a} \in \mathbb{R}^n$.

For $K \subseteq \mathbb{R}^n$ non-empty and compact and $\mathbf{a} \in \mathbb{R}^n$, we define the supporting hyperplane of K induced by \mathbf{a} to be $H_{\overline{K}}(\mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}\mathbf{x} = h_K(\mathbf{a})\}$. We define the set of maximizers of \mathbf{a} in K to be $F_K(\mathbf{a}) := K \cap H_{\overline{K}}(\mathbf{a})$.

2.1 Bit-Sizes

► **Definition 9** (Bit-size). *The notation $\langle x \rangle$ is reserved for the number of bits required to express the object x , or the bit-size of x . We build up the precise definitions as follows:*

For $r \in \mathbb{Q}, r = p/q, p \in \mathbb{Z}, q \in \mathbb{Z}, q > 0, \langle r \rangle := 1 + \lceil \log(|p| + 1) \rceil + \lceil \log(q + 1) \rceil$. Next, for $\mathbf{c} \in \mathbb{Q}^n$ with $\mathbf{c} = (c_1, c_2 \dots c_n)$, $\langle \mathbf{c} \rangle := n + \sum_{i=1}^n \langle c_i \rangle$. Similarly for matrices $\mathbf{A} \in \mathbb{Q}^{m \times n}$, $\langle \mathbf{A} \rangle := mn + \sum_{i=1}^m \sum_{j=1}^n \langle A_{ij} \rangle$. $\langle A, B \rangle$ is simply $\langle A \rangle + \langle B \rangle$ when these terms are well-defined.

For a labeled rooted tree \mathcal{T} with n nodes and m edges $E[\mathcal{T}]$, and where edges $e \in E[\mathcal{T}]$ have labels L_e and nodes v have labels L_v , and if the labels belong to a class for which the bit-size has already been defined, then $\langle \mathcal{T} \rangle := n + m + \sum_{e \in E[\mathcal{T}]} \langle L_e \rangle + \sum_{v \in \mathcal{T}} \langle L_v \rangle$.

2.2 Branching Proofs

► **Definition 10** (Branching Proof). *A branching proof of integer infeasibility for a convex set $K \subseteq \mathbb{R}^n$ is represented by a rooted binary tree \mathcal{T} with root $r := r_{\mathcal{T}}$. Each node $v \in \mathcal{T}$ is labeled with (\mathbf{a}_v, b_v) , $\mathbf{a}_v \in \mathbb{Z}^n, b_v \in \mathbb{Z}$ and has two children nodes: the left child v_l and right child v_r . Since the inner product of two integer vectors is an integer, the integer lattice \mathbb{Z}^n can be partitioned into $\{\mathbf{x} \in \mathbb{Z}^n : \mathbf{a}_v \mathbf{x} \leq b_v\}, \{\mathbf{x} \in \mathbb{Z}^n : \mathbf{a}_v \mathbf{x} \geq b_v + 1\}$. This partition is referred to as the integer disjunction given by (\mathbf{a}_v, b_v) .*

Every edge $e \in E[\mathcal{T}]$ is labeled with an inequality $\mathbf{a}_e \mathbf{x} \leq b_e$. A left edge $e_l = \{v, v_l\}$ is labeled with $\mathbf{a}_v \mathbf{x} \leq b_v$. Thus we have $\mathbf{a}_e = \mathbf{a}_v, b_e = b_l$. However, a right edge $e_r = \{v, v_r\}$ is labeled with $\mathbf{a}_v \mathbf{x} \geq b_v + 1$, so that $\mathbf{a}_e = -\mathbf{a}_v, b_e = -b_l - 1$.

For each node $v \in \mathcal{T}$, we define $P_{\mathcal{T}}(v)$ to be the the unique path from the root r of \mathcal{T} to v . Also define for each v a polyhedron $P_v = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}_v \mathbf{x} \leq \mathbf{b}_v\}$ where the rows of \mathbf{A}_v are given by $\mathbf{a}_{v,e}, e \in E[P_{\mathcal{T}}(v)]$, and the coordinates of \mathbf{b}_v are $b_{v,e}, e \in E[P_{\mathcal{T}}(v)]$. Let $K_v := K \cap P_v$. Note that $K_r = K$.

For \mathcal{T} to be a proof of integer infeasibility for K , we require that every leaf $v \in \mathcal{T}$ (v is a leaf if its has no children) satisfies $K_v = \emptyset$.

We denote the length of the branching proof by $|\mathcal{T}|$, which is defined to be the number of nodes of \mathcal{T} . The size of a branching proof $\langle \mathcal{T} \rangle$ is simply its bit-size as a labeled rooted tree as given above in definition 9.

► **Definition 11** (Certified Branching Proof). *Suppose $K = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}\mathbf{x} \leq \mathbf{d}\}, \mathbf{C} \in \mathbb{Q}^{r \times n}, \mathbf{d} \in \mathbb{Q}^r$ belongs to the class of rational polyhedra. A certified branching proof of integer infeasibility for K is a standard branching proof \mathcal{T} of infeasibility of K , but where every leaf node v of \mathcal{T} is also labeled with a Farkas certificate $\boldsymbol{\lambda}_v \in \mathbb{Q}^{r+m_v}, \lambda_i \geq 0, \forall i \in [r+m_v]$, where now $K_v = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}\mathbf{x} \leq \mathbf{d}, \mathbf{A}_v \leq \mathbf{b}_v\}$, for $\mathbf{A}_v \in \mathbb{R}^{m_v \times n}, \mathbf{b}_v \in \mathbb{R}^{m_v}, m_v = |P_{\mathcal{T}}(v)|$. Let $\boldsymbol{\lambda}_v = (\boldsymbol{\lambda}_{v,1}, \boldsymbol{\lambda}_{v,2}), \boldsymbol{\lambda}_{v,1} \in \mathbb{Q}^r, \boldsymbol{\lambda}_{v,2} \in \mathbb{Q}^{m_v}$. The requirement that every $K_v = \emptyset$ for a leaf nodes v is certified by requiring $\boldsymbol{\lambda}_{v,1} \mathbf{C} + \boldsymbol{\lambda}_{v,2} \mathbf{A}_v = 0, \boldsymbol{\lambda}_{v,1} \mathbf{d} + \boldsymbol{\lambda}_{v,2} \mathbf{b}_v < 0$.*

The bit-size of a certified branching proof is its bit-size when viewed as a labeled rooted tree.

► **Definition 12** (Enumerative Branching Proof). *For a compact convex set K , an enumerative branching proof consists of a tree \mathcal{T} with root r and root relaxation $K_r := K$. Every node $v \in \mathcal{T}$ is labeled with (\mathbf{a}_v, l_v, u_v) , where $\mathbf{a}_v \in \mathbb{Z}^n, l_v, u_v \in \mathbb{Q}$ satisfying*

$$\{\mathbf{a}_v \mathbf{x} : \mathbf{x} \in K_v\} \subseteq [l_v, u_v].$$

There is a child of v denoted v_b for every $b \in \mathbb{Z}, l_v \leq b \leq u_v$, and the edge $e = \{v, v_b\}$ is labeled with the equality $\mathbf{a}_v \mathbf{x} = b$. The relaxation at K_{v_b} becomes $\{\mathbf{x} \in K_v : \mathbf{a}_v \mathbf{x} = b\}$.

\mathcal{T} is a valid enumerative branching proof of infeasibility if every leaf node $v \in \mathcal{T}$ satisfies $K_v = \emptyset$ or $K_v \neq \emptyset$ but $[l_v, u_v]$ contains no integer points, i.e., $\lfloor u_v \rfloor < l_v$.

$\langle \mathcal{T} \rangle$ is again simply the bit-size of \mathcal{T} as a labeled rooted tree.

2.3 Simultaneous Diophantine Approximation

The existence of a rational vector of small bit-size that well approximates an arbitrary real vector is of prime importance in this paper. For this purpose, we shall require standard tools from Diophantine approximation (see [23] for a reference). The following is a slightly adapted version of the Dirichlet's simultaneous approximation theorem, which will be convenient for our purposes. We provide a proof for completeness.

► **Lemma 13.** *Let $\mathbf{a} \in \mathbb{R}^n$ satisfy $\|\mathbf{a}\|_\infty = 1$ and let $N \geq 1$. Then, there exists a positive integer $l \leq N^n$ such that $\mathbf{a}' := \lfloor l\mathbf{a} \rfloor$ satisfies*

$$\|l\mathbf{a} - \mathbf{a}'\|_\infty < 1/N \quad \text{and} \quad \|\mathbf{a}'\|_\infty = l \geq 1.$$

Proof. Let $\mathcal{C} = \{I_{\mathbf{z}} : \mathbf{z} \in [N]^n\}$ denote the collection of N^n half-open cubes forming a partition of $[0, 1]^n$, where $I_{\mathbf{z}} = \times_{i=1}^n [(z_i - 1)/N, z_i/N)$ for $\mathbf{z} \in [N]^n$. For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, let $[\mathbf{x}] = \mathbf{x} - \lfloor \mathbf{x} \rfloor \in [0, 1)^n$ denote the fractional part of \mathbf{x} . Examine the sequence $[0\mathbf{a}], [1\mathbf{a}], \dots, [N^n\mathbf{a}]$. Since the sequence has length $N^n + 1$ and each element of the sequence lands in one of the cubes in \mathcal{C} , by the pigeonhole principle there must be distinct indices l_1, l_2 , $0 \leq l_1 < l_2 \leq N^n$ and $\mathbf{z} \in [N]^n$ such that $[l_1\mathbf{a}], [l_2\mathbf{a}] \in I_{\mathbf{z}}$. Since $I_{\mathbf{z}} - I_{\mathbf{z}} = (-1/N, 1/N)^n$, we note that $\|[l_1\mathbf{a}] - [l_2\mathbf{a}]\|_\infty < 1/N$. Let $l = l_2 - l_1$ and $\mathbf{a}' = \lfloor l\mathbf{a} \rfloor$, where we note that $1 \leq l \leq N^n$. For any $i \in [n]$, we have that

$$|la_i - \lfloor la_i \rfloor| = \min_{k \in \mathbb{Z}} |la_i - k| \leq |(l_1 - l_2)a_i - ([l_1a_i] - [l_2a_i])| = |[l_1a_i] - [l_2a_i]| < 1/N.$$

In particular, $\|l\mathbf{a} - \mathbf{a}'\|_\infty = \|l\mathbf{a} - \lfloor l\mathbf{a} \rfloor\|_\infty < 1/N$, as needed. We now show that $\|\mathbf{a}'\|_\infty = l$. By assumption on \mathbf{a} , there is a coordinate $i \in [n]$ such that $a_i = 1 = \|\mathbf{a}\|_\infty$. Thus, $a'_i = \lfloor la_i \rfloor = l$ and $\|\mathbf{a}'\|_\infty \geq l$. For any $j \in [n]$, also clearly have $la_j \in [-l, l] \Rightarrow a'_j = \lfloor la_j \rfloor \in [-l, l]$ since $l \in \mathbb{N}$. Thus, $\|\mathbf{a}'\|_\infty = l$ as needed. ◀

► **Remark 14.** For $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{a}' \in \mathbb{Z}^n$, $1 \leq l \leq N^n$ as above, observe that $a_i = 0 \Rightarrow a'_i = \lfloor la_i \rfloor = 0$. Furthermore, $\|\mathbf{a}'\|_\infty = l \leq N^n$.

► **Definition 15** (Diophantine Approximation of Precision N). *For a vector $\mathbf{a} \in \mathbb{R}^n \setminus \{0\}$ and $N \geq 1$, we say that \mathbf{a}' is a precision N Diophantine approximation of \mathbf{a} if \mathbf{a}' satisfies the conditions of Lemma 13 on inputs $\mathbf{a}/\|\mathbf{a}\|_\infty$ and N .*

We note that in our application, we will set $N = 10nR$, where R is an integer upper bound on the ℓ_1 radius of the convex set $K \subseteq \mathbb{R}^n$ whose branching proof we are modifying.

2.4 Farkas Certificates for General Convex Sets

A Farkas certificate $\boldsymbol{\lambda} \in \mathbb{R}_+^m$ certifies the infeasibility of the system $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ if $\boldsymbol{\lambda}^\top \mathbf{A} = 0$, $\boldsymbol{\lambda}^\top \mathbf{b} = -1$. It is possible to extend this definition to show a linear system is infeasible whenever $\mathbf{x} \in K$ for a compact convex set K .

► **Definition 16** (Generalized Farkas Certificate). *Let $K \subseteq \mathbb{R}^n$ be a compact convex set, and $P := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$. $\boldsymbol{\lambda} \in \mathbb{R}_+^m$ is a generalized Farkas certificate of infeasibility for $K \cap P$ if*

$$\min_{\mathbf{x} \in K} \boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} - \mathbf{b}) > 0$$

► **Lemma 17.** *With the notation of definition 16, $K \cap P = \emptyset$ if and only if there exists a generalized Farkas certificate $\boldsymbol{\lambda} \in \mathbb{R}_+^m$ of its infeasibility. Furthermore, if one generalized Farkas certificate exists, then so does one with at most $n + 1$ non-zero coordinates.*

Proof. That a generalized Farkas certificate implies infeasibility is trivial.

Now let us suppose $K \cap P = \emptyset$. K is compact and convex by assumption, and P is clearly closed and convex. Therefore, there exists a strictly separating hyperplane $\mathbf{c}\mathbf{x} = d$ so that K and P lie on “opposite sides” of this hyperplane. More precisely, $\mathbf{c}\mathbf{x} - d > 0$ for $\mathbf{x} \in K$, and $\mathbf{c}\mathbf{x} - d < 0$ for $\mathbf{x} \in P$.

$\mathbf{c}\mathbf{x} < d$ for every $\mathbf{x} \in P$ means the system $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, $-\mathbf{c}\mathbf{x} \leq -d$ is infeasible. Let $(\boldsymbol{\lambda}, \gamma) \geq 0$ be a (conventional) Farkas certificate of the infeasibility of this system: $\boldsymbol{\lambda}^\top \mathbf{A} = \gamma \mathbf{c}$, $\boldsymbol{\lambda}^\top \mathbf{b} < \gamma d$. We now claim that $\boldsymbol{\lambda} \geq 0$ is a generalized Farkas certificate of infeasibility for $K \cap P$. Firstly, if $\gamma = 0$, we have that $\min_{\mathbf{x} \in K} \boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} - \mathbf{b}) = -\boldsymbol{\lambda}^\top \mathbf{b} > 0$. If $\gamma > 0$, then

$$\mathbf{x} \in K \Rightarrow \gamma(\mathbf{c}\mathbf{x} - d) > 0 \Rightarrow \boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} - \mathbf{b}) > 0.$$

In particular, $\min_{\mathbf{x} \in K} \boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} - \mathbf{b}) > 0$, noting that the minimum is indeed achieved since K is compact.

By Caratheodory’s theorem, there exists a generalized Farkas certificate of at most $n + 1$ non-zero coordinates whenever a generalized Farkas certificate exists. ◀

Although the correctness of a conventional Farkas certificate can be verified with simple matrix multiplication, this is not the case for a generalized Farkas certificate. In particular, one must exactly solve the (convex) minimization problem in definition 16 to verify the certificate. This is why the notion of a certified branching proof is sensible only for specific classes of compact convex sets, such as polyhedra.

The following lemma will be crucial for enabling us to deduce infeasibility information for “nearby” polyhedra. The proof relies upon the existence of generalized Farkas certificates as defined above.

▶ **Lemma 18.** *Let $K \subseteq \mathbb{R}^n$ be a compact convex set and let $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, be a polyhedron satisfying $P \cap K = \emptyset$. For $\boldsymbol{\varepsilon} \in \mathbb{R}^m$, define $P_\boldsymbol{\varepsilon} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b} + \boldsymbol{\varepsilon}\}$. Then, for any $\boldsymbol{\varepsilon} \in \mathbb{R}^m$, either $K \cap P_\boldsymbol{\varepsilon} = \emptyset$, or there exists $j \in [m]$ such that $\varepsilon_j > 0$ and $K \cap P_{\boldsymbol{\varepsilon} - (n+1)\varepsilon_j \mathbf{e}_j} = \emptyset$.*

Proof. We assume that $K \cap P_\boldsymbol{\varepsilon} \neq \emptyset$, since otherwise there is nothing to prove.

Let $\boldsymbol{\lambda} \in \mathbb{R}_+^m$ be a generalized Farkas certificate of infeasibility for $K \cap P$ with at most $n + 1$ non-zero coordinates as guaranteed by Lemma 17. Let $j_* = \arg \max_{j \in [m]} \varepsilon_j \lambda_j$. We claim that $\varepsilon_{j_*} \lambda_{j_*} > 0$. Assume not, then $\varepsilon_j \lambda_j \leq 0$ for all $i \in [m]$. In particular,

$$\min_{\mathbf{x} \in K} \boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} - \mathbf{b} - \boldsymbol{\varepsilon}) = \min_{\mathbf{x} \in K} \boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} - \mathbf{b}) - \boldsymbol{\lambda}^\top \boldsymbol{\varepsilon} > -\boldsymbol{\lambda}^\top \boldsymbol{\varepsilon} \geq 0. \quad (2.1)$$

Thus, $\boldsymbol{\lambda}$ is a generalized Farkas certificate of infeasibility for $K \cap P_\boldsymbol{\varepsilon}$. But this contradicts our assumption that $K \cap P_\boldsymbol{\varepsilon} \neq \emptyset$. Therefore, we must have that $\varepsilon_{j_*} \lambda_{j_*} > 0$. In particular, since $\boldsymbol{\lambda} \geq 0$, we have that $\varepsilon_{j_*} > 0$ and $\lambda_{j_*} > 0$.

We now show that $\boldsymbol{\lambda}$ is in fact a valid generalized Farkas certificate of infeasibility for $K \cap P_{\boldsymbol{\varepsilon} - (n+1)\varepsilon_{j_*} \mathbf{e}_{j_*}}$. Let $S = \{j \in [m] : \lambda_j > 0\}$, and note that by assumption $|S| \leq n + 1$. Using a similar calculation to (2.1), we see that

$$\begin{aligned} \min_{\mathbf{x} \in K} \boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} - \mathbf{b} - \boldsymbol{\varepsilon} + (n+1)\varepsilon_{j_*} \mathbf{e}_{j_*}) &> -\boldsymbol{\lambda}^\top \boldsymbol{\varepsilon} + (n+1)\varepsilon_{j_*} \lambda_{j_*} \\ &= -\sum_{j \in S} \varepsilon_j \lambda_j + (n+1)\varepsilon_{j_*} \lambda_{j_*} \geq (n+1 - |S|)\varepsilon_{j_*} \lambda_{j_*} \geq 0. \end{aligned}$$

Since $\boldsymbol{\lambda}$ is a valid certificate of infeasibility, we have that $K \cap P_{\boldsymbol{\varepsilon} - (n+1)\varepsilon_{j_*} \mathbf{e}_{j_*}} = \emptyset$, as needed. ◀

2.5 Chvátal-Gomory Cuts

► **Definition 19** (Chvátal-Gomory Cut). For $\mathbf{a} \in \mathbb{Z}^n$, the CG cut of K induced by \mathbf{a} is the halfspace $H_K^{\text{cg}}(\mathbf{a}) := H_{\mathbf{a}, \lfloor h_K(\mathbf{a}) \rfloor}$. We define $\text{CG}(K, \mathbf{a}) := K \cap H_K^{\text{cg}}(\mathbf{a})$ to be the result of applying the CG cut induced by \mathbf{a} to K .

This definition is extended to an ordered list $\mathcal{L} = (\mathbf{a}_1, \dots, \mathbf{a}_k)$ of integer vectors as $\text{CG}(K, \mathcal{L}) := \text{CG}(\text{CG}(K, \mathbf{a}_1), (\mathbf{a}_2, \dots, \mathbf{a}_k))$. That is, we first apply the CG cut induced by \mathbf{a}_1 to K yielding $\text{CG}(K, \mathbf{a}_1)$, then we apply the CG cut induced by \mathbf{a}_2 to $\text{CG}(K, \mathbf{a}_1)$ yielding $\text{CG}(K, (\mathbf{a}_1, \mathbf{a}_2))$, and so forth. By convention, $\text{CG}(K, \emptyset) = K$, that is, applying the empty list of CG cuts does nothing to K .

The following *lifting lemma*, adapted from [10], shows that CG cuts on a “rational face” F of K can be lifted to a CG cut of K having the same effect on the face. We note that lifting is also possible from “irrational faces” [10], however this requires intersecting multiple CG cuts to achieve the desired effect. The corresponding lemma for rational polyhedra is classical [6].

We include its proof for clarity and completeness. The proof follows the standard approach of adding a large integer multiple of the normal vector to F to the cut.

► **Lemma 20** (Lifting CG cuts). Let $K \subseteq \mathbb{R}^n$ be a non-empty compact set. Let $\mathbf{c} \in \mathbb{Z}^n$, $F := F_K(\mathbf{c})$ and assume that $h_K(\mathbf{c}) \in \mathbb{Z}$. Then for any $\mathbf{a} \in \mathbb{Z}^n$, there exists $N \geq 0$ such that

$$H_K^{\text{cg}}(\mathbf{a} + i\mathbf{c}) \cap H_K^{\overline{}}(\mathbf{c}) = H_F^{\text{cg}}(\mathbf{a}) \cap H_K^{\overline{}}(\mathbf{c}), \forall i \geq N.$$

For the proof, we will need the following technical lemma, which shows converge properties of a sequence of maximizing faces.

► **Lemma 21.** Let $K \subseteq \mathbb{R}^n$ be a non-empty compact set. Let $(\mathbf{a}_i)_{i=1}^{\infty} \in \mathbb{R}^n$ be a convergent sequence with $\mathbf{a}_{\infty} := \lim_{i \rightarrow \infty} \mathbf{a}_i$ and let $F_i := F_K(\mathbf{a}_i)$, $i \in \mathbb{N} \cup \{\infty\}$. Then, $\forall \varepsilon > 0$ there exists $N_{\varepsilon} \geq 1$ such that $\forall i \geq N_{\varepsilon}$, $F_i \subseteq F_{\infty} + \varepsilon \mathbb{B}_1^n$.

Proof. For the sake of contradiction, let us assume that there exists a sequence $(\mathbf{x}_i)_{i=1}^{\infty}$ and an $\varepsilon > 0$ such that $\mathbf{x}_i \in F_i$ and $\mathbf{x}_i \notin F_{\infty} + \varepsilon \mathbb{B}_1^n$. Letting $K' = \text{closure}(K \setminus (F_{\infty} + \varepsilon \mathbb{B}_1^n))$, we see that $K' \subseteq K$ is compact and that $K' \cap F_{\infty} = \emptyset$. Furthermore, $\mathbf{x}_i \in F_i \subseteq K'$, $\forall i \in \mathbb{N}$. Therefore, by compactness of K' there exists a convergent subsequence $(\mathbf{x}_{s_i})_{i=1}^{\infty}$ with limit point $\mathbf{y} := \lim_{i \rightarrow \infty} \mathbf{x}_{s_i} \in K'$. Note that by construction $\mathbf{y} \in K$ and $\mathbf{y} \notin F_{\infty}$. Since K is compact, its support function h_K is continuous. By continuity of h_K and the standard inner product, we conclude that

$$\begin{aligned} \mathbf{v}_{\infty} \mathbf{y} &= \lim_{i \rightarrow \infty} \mathbf{v}_{s_i} \mathbf{x}_{s_i} = \lim_{i \rightarrow \infty} h_K(\mathbf{v}_{s_i}) \quad (\text{since } \mathbf{x}_{s_i} \in F_{s_i}) \\ &= h_K(\mathbf{v}_{\infty}). \end{aligned}$$

But then $\mathbf{y} \in F_{\infty}$, a clear contradiction. The lemma thus follows. ◀

We now give the proof of the lifting lemma.

Proof of Lemma 20. Firstly, if $\mathbf{c} = 0$ then $F = K$ and the statement trivially holds for $N = 0$. Thus, we may assume that $\mathbf{c} \neq 0$.

Let $b = h_F(\mathbf{a})$ and recall that $H_F^{\text{cg}}(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}\mathbf{x} \leq \lfloor b \rfloor\}$. For $i \geq 0$, let $b_i := h_K(\mathbf{a} + N\mathbf{c}) - ih_K(\mathbf{c})$. From here, we see that

$$\begin{aligned} \mathbf{x} \in H_K^{\text{cg}}(\mathbf{a} + i\mathbf{c}) \cap H_K^{\overline{}}(\mathbf{c}) &\Leftrightarrow (\mathbf{a} + i\mathbf{c})\mathbf{x} = \lfloor h_K(\mathbf{a} + i\mathbf{c})\mathbf{x} \rfloor, \mathbf{c}\mathbf{x} = h_K(\mathbf{c}) \\ &\Leftrightarrow (\mathbf{a} + i\mathbf{c})\mathbf{x} \leq \lfloor b_i + ih_K(\mathbf{c}) \rfloor, \mathbf{c}\mathbf{x} = h_K(\mathbf{c}) \\ &\Leftrightarrow (\mathbf{a} + i\mathbf{c})\mathbf{x} \leq \lfloor b_i \rfloor + ih_K(\mathbf{c}), \mathbf{c}\mathbf{x} = h_K(\mathbf{c}) \\ &\quad (\text{since } ih_K(\mathbf{c}) \in \mathbb{Z}) \\ &\Leftrightarrow \mathbf{a}\mathbf{x} \leq \lfloor b_i \rfloor, \mathbf{c}\mathbf{x} = h_K(\mathbf{c}). \end{aligned}$$

Given the above, it suffices to show that there exists $N \geq 0$ such that $\lfloor b_i \rfloor = \lfloor b \rfloor$, $\forall i \geq N$. Since F is the set of maximizers of \mathbf{c} in K , note that

$$b_i = h_K(\mathbf{a} + i\mathbf{c}) - ih_K(\mathbf{c}) \geq h_F(\mathbf{a} + i\mathbf{c}) - ih_K(\mathbf{c}) = h_F(\mathbf{a}) = b, \forall i \geq 0.$$

Letting $\varepsilon_1 = \lfloor b + 1 \rfloor - b > 0$, note that $\lfloor b' \rfloor = \lfloor b \rfloor$ for $b' \in [b, b + \varepsilon_1)$. Given this, it now suffices to show the existence of $N \geq 0$ such that $b_i < b + \varepsilon_1$, for $i \geq N$. Let $F_i := F_K(\mathbf{a} + i\mathbf{c})$, for $i \in N$. Since $\mathbf{a}/i + \mathbf{c} \rightarrow \mathbf{c}$ as $i \rightarrow \infty$ and K is compact, by Lemma 21 for $\varepsilon_2 > 0$ there exists $N_{\varepsilon_2} \geq 0$ such that $F_i \subseteq F + \varepsilon_2\mathbb{B}_1^n$, for $i \geq N_{\varepsilon_2}$. For $i \geq N_{\varepsilon_2}$, we may thus choose $\mathbf{x}_i \in F_i$ and $\mathbf{y}_i \in F$ satisfying $\|\mathbf{x}_i - \mathbf{y}_i\|_1 \leq \varepsilon_2$. From here, for $i \geq N_{\varepsilon_2}$ we have that

$$\begin{aligned} b_i = h_K(\mathbf{a} + i\mathbf{c}) - ih_K(\mathbf{c}) &= (\mathbf{a} + i\mathbf{c})\mathbf{x}_i - ih_K(\mathbf{c}) \quad (\text{since } \mathbf{x}_i \in F_i) \\ &\leq \mathbf{a}\mathbf{x}_i + ih_K(\mathbf{c}) - ih_K(\mathbf{c}) = \mathbf{a}(\mathbf{x}_i - \mathbf{y}_i) + \mathbf{a}\mathbf{y}_i \quad (\text{since } \mathbf{x}_i \in K) \\ &\leq \|\mathbf{x} - \mathbf{y}\|_1 \|\mathbf{a}\|_{\infty} + h_F(\mathbf{a}) \leq \varepsilon_2 \|\mathbf{a}\|_{\infty} + b \quad (\text{since } \mathbf{y}_i \in F). \end{aligned}$$

Setting $\varepsilon_2 := \varepsilon_1 / (2\|\mathbf{a}\|_{\infty})$ and $N := N_{\varepsilon_2}$ yields the desired bound. The lemma thus follows. \blacktriangleleft

3 Bounding the coefficients of Branching Proofs

In this section, we show how to transform any branching proof \mathcal{T} for a compact convex set $K \subseteq \mathbb{R}\mathbb{B}_1^n$ into a branching proof \mathcal{T}' having small coefficients with length $|\mathcal{T}'| = O(n|\mathcal{T}|)$.

The construction of \mathcal{T}' is a two step process. In the first step, we substitute each integer disjunction given by (\mathbf{a}, b) by an approximation (\mathbf{a}', b') with coefficients of size $(nR)^{O(n^2)}$. This bounds $\langle \mathcal{T}' \rangle$ while keeping $|\mathcal{T}'| = |\mathcal{T}|$. We shall use the ‘‘iterated Diophantine approximation’’ technique introduced by Frank and Tardos [12] to construct \mathbf{a}', b' from \mathbf{a}, b .

It is possible that the new inequalities are ‘‘stronger’’; e.g., it is possible that for $\mathbf{a}'\mathbf{x} \leq b' \Rightarrow_R \mathbf{a}\mathbf{x} \leq b$ and $\mathbf{a}'\mathbf{x} \geq b' + 1 \Rightarrow_R \mathbf{a}\mathbf{x} \geq b + 1$. However, one cannot always ensure this, and in general we will only be able to guarantee that $\mathbf{a}'\mathbf{x} \leq b' \Rightarrow_R \mathbf{a}\mathbf{x} \leq b + \varepsilon$ and $\mathbf{a}'\mathbf{x} \geq b' + 1 \Rightarrow_R \mathbf{a}\mathbf{x} \geq b + 1 - \varepsilon$ for some ‘‘small’’ $\varepsilon > 0$. As explained in the introduction, the combined error from all the substitutions may render the continuous relaxations at the leaves nonempty. In a second step, we ‘‘fix-up’’ these newly feasible leaf nodes by adding $O(n)$ judiciously chosen CG cuts to arrive at infeasible sets, causing the $O(n)$ factor increase in $|\mathcal{T}'|$. These cuts will be derived from so-called valid substitution sequences (see Definition 24) of the original disjunctions in \mathcal{T} , which we construct together with the replacement disjunctions $\mathbf{a}'\mathbf{x} \leq b'$ or $\geq b' + 1$ described above.

From here until the end of subsection 3.1, we explain the first step, showing how to construct appropriate replacement disjunctions together with substitution sequences and how to compute the initial (partial) replacement tree \mathcal{T}' from \mathcal{T} . In subsection 3.2, we explain

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the second step, showing how to construct the requisite $O(n)$ -size CP proof of infeasibility for each leaf node of \mathcal{T}' . Finally, in subsection 3.3, we give the proof of Theorem 1 which combines both steps.

We begin with the following lemma, which collects the properties of Diophantine approximations we will need to construct the replacement disjunctions and substitution sequences. In particular, part (ii) of the lemma will be used to choose the vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$ and right hand sides b_1, \dots, b_k inducing the pair of inequality sequences (we think of one as the “flipped” version of the other) $\mathbf{a}_1\mathbf{x} \leq b_1, \dots, \mathbf{a}_k\mathbf{x} \leq b_k$ and $\mathbf{a}_1\mathbf{x} \geq b_1, \dots, \mathbf{a}_{k-1}\mathbf{x} \geq b_{k-1}, \mathbf{a}_k\mathbf{x} \geq b_k + 1$ respectively used to approximate the left side $\mathbf{a}\mathbf{x} \leq b$ and right side $\mathbf{a}\mathbf{x} \geq b + 1$ of an initial disjunction. In this context, the disjunction (\mathbf{a}, b) in the lemma will represent an initial disjunction (\mathbf{a}, b) after “projecting out” $\mathbf{a}_1\mathbf{x} = b_1, \dots, \mathbf{a}_i\mathbf{x} = b_i$, for some intermediate $i \in [k-1]$, and \mathbf{a}', b' will correspond to the next $(\mathbf{a}_{i+1}, b_{i+1})$ in the sequence.

► **Lemma 22.** *For any vector $\mathbf{a} \in \mathbb{R}^n \setminus \{0\}$, $b \in \mathbb{R}$, $R, N \in \mathbb{N}$, let \mathbf{a}' be a Diophantine approximation of \mathbf{a} of precision N , and let $\alpha = \frac{\|\mathbf{a}\|_\infty}{\|\mathbf{a}'\|_\infty}$. Then the following statements hold:*

- (i) $\forall b' \in \mathbb{R}, \mathbf{a}'\mathbf{x} \leq b' \Rightarrow_R \mathbf{a}\mathbf{x} \leq \alpha(b' + \frac{R}{N})$ and symmetrically, $\mathbf{a}'\mathbf{x} \geq b' \Rightarrow_R \mathbf{a}\mathbf{x} \geq \alpha(b' - \frac{R}{N})$.
- (ii) When $\frac{R}{N} < \frac{1}{4}, \alpha \geq 2$, we can uniquely set $b' \in \mathbb{Z}$ according to exactly one of following cases:
 - (non- R -dominating case): $-R\|\mathbf{a}\|_\infty - 1 < b < R\|\mathbf{a}\|_\infty$ and \exists unique $b' \in \mathbb{Z}, |b'| \leq R\|\mathbf{a}'\|_\infty$,

$$\text{such that } (b, b+1) \cap \left[\alpha \left(b' - \frac{R}{N} \right), \alpha \left(b' + \frac{R}{N} \right) \right] \neq \emptyset.$$

- (R -dominating case): \exists unique $b' \in \mathbb{Z}, -R\|\mathbf{a}'\|_\infty \leq b' \leq R\|\mathbf{a}'\|_\infty - 1$,

$$\text{such that } (b, b+1) \subseteq \left(\alpha \left(b' + \frac{R}{N} \right), \alpha \left(b' + 1 - \frac{R}{N} \right) \right),$$

or

$$b \geq R\|\mathbf{a}\|_\infty, b' = R\|\mathbf{a}'\|_\infty,$$

or

$$b+1 \leq -R\|\mathbf{a}\|_\infty, b' = -R\|\mathbf{a}'\|_\infty - 1.$$

Furthermore, in the R -dominating case we have that

$$\mathbf{a}'\mathbf{x} \leq b' \Rightarrow_R \mathbf{a}\mathbf{x} \leq b \text{ and } \mathbf{a}'\mathbf{x} \geq b' + 1 \Rightarrow_R \mathbf{a}\mathbf{x} \geq b + 1.$$

Proof.

- (i) By definition of \mathbf{a}' , $\|\frac{\mathbf{a}}{\alpha} - \mathbf{a}'\|_\infty < 1/N$. We have for any $b' \in \mathbb{Z}$:

$$\|\mathbf{x}\|_1 \leq R, \mathbf{a}'\mathbf{x} \leq b' \Rightarrow \frac{\mathbf{a}}{\alpha}\mathbf{x} \leq b' + \left(\frac{\mathbf{a}}{\alpha} - \mathbf{a}'\right)\mathbf{x} \leq b' + \left\| \frac{\mathbf{a}}{\alpha} - \mathbf{a}' \right\|_\infty \|\mathbf{x}\|_1 \leq b' + \frac{R}{N}.$$

Summarizing, we have that

$$\|\mathbf{x}\|_1 \leq R, \mathbf{a}'\mathbf{x} \leq b' \Rightarrow \mathbf{a}\mathbf{x} \leq \alpha \left(b' + \frac{R}{N} \right).$$

By a symmetric argument, we also have

$$\|\mathbf{x}\|_1 \leq R, \mathbf{a}'\mathbf{x} \geq b' \Rightarrow \mathbf{a}\mathbf{x} \geq \alpha \left(b' - \frac{R}{N} \right).$$

- (ii) When $\frac{R}{N} < \frac{1}{4}$, the intervals of the form $I(b') := [\alpha(b' - \frac{R}{N}), \alpha(b' + \frac{R}{N})]$, $b' \in \mathbb{Z}$, are pairwise disjoint. In fact, when $\alpha \geq 2$, they are more than unit distance apart. This implies that the interval $(b, b+1)$ cannot intersect more than one of the intervals $I(b')$, $b' \in \mathbb{Z}$.

Let us now suppose $-R\|\mathbf{a}\|_\infty - 1 < b < R\|\mathbf{a}\|_\infty$. We now show that only $b' \in [-R\|\mathbf{a}\|_\infty, R\|\mathbf{a}\|_\infty] \cap \mathbb{Z}$ need be considered in this case.

For $b' = -R\|\mathbf{a}'\|_\infty$, we have $I(b') = [R\alpha(\|\mathbf{a}'\|_\infty - \frac{1}{N}), R\alpha(\|\mathbf{a}'\|_\infty + \frac{1}{N})]$. The left end point $-R\|\mathbf{a}\|_\infty - \frac{R\alpha}{N}$ of $I(-R\|\mathbf{a}'\|_\infty)$ lies to the left of $b+1$ on the real line because

$$-R\|\mathbf{a}\|_\infty - \frac{R\alpha}{N} < -R\|\mathbf{a}\|_\infty < b+1.$$

Similarly the right end point $R\|\mathbf{a}\|_\infty + \frac{R\alpha}{N}$ of $I(R\|\mathbf{a}'\|_\infty)$ lies to the right of b as

$$R\|\mathbf{a}\|_\infty + \frac{R\alpha}{N} > R\|\mathbf{a}\|_\infty > b.$$

Thus, either $(b, b+1)$ intersects some $I_{b'}$ for $b' \in [-R\|\mathbf{a}'\|_\infty, R\|\mathbf{a}'\|_\infty] \cap \mathbb{Z}$ or it lies in between two such consecutive intervals $I_{b'}, I_{b'+1}$: these are the non-dominating and dominating cases respectively.

In the dominating case for $b \in (-R\|\mathbf{a}\|_\infty - 1, R\|\mathbf{a}\|_\infty)$, the fact that

$$(b, b+1) \subseteq \left(\alpha \left(b' + \frac{R}{N} \right), \alpha \left(b' + 1 - \frac{R}{N} \right) \right)$$

implies $b \geq \alpha \left(b' + \frac{R}{N} \right)$. Applying part (i) we see that

$$\mathbf{a}'\mathbf{x} \leq b' \Rightarrow_R \mathbf{a}\mathbf{x} \leq \alpha \left(b' + \frac{R}{N} \right) \Rightarrow \mathbf{a}\mathbf{x} \leq b.$$

On the other side, $b+1 \leq \alpha \left(b' + 1 - \frac{R}{N} \right)$ gives

$$\mathbf{a}'\mathbf{x} \geq b'+1 \Rightarrow_R \mathbf{a}\mathbf{x} \geq \alpha \left(b' + 1 - \frac{R}{N} \right) \Rightarrow \mathbf{a}\mathbf{x} \geq b+1.$$

Now let us consider the situation where $b \geq R\|\mathbf{a}\|_\infty$. Then $\forall \mathbf{x} \in R\mathbb{B}_1^n$ we have $\mathbf{a}\mathbf{x} \leq \|\mathbf{a}\|_\infty \|\mathbf{x}\|_1 \leq \|\mathbf{a}\|_\infty R \leq b$. Since this inequality holds for every vector in $R\mathbb{B}_1^n$, we have that $\mathbf{a}'\mathbf{x} \leq b' \Rightarrow_R \mathbf{a}\mathbf{x} \leq b \forall b' \in \mathbb{R}$ and in particular for $b' = R\|\mathbf{a}'\|_\infty$. Furthermore, for $b' = R\|\mathbf{a}'\|_\infty$, $\mathbf{a}'\mathbf{x} \geq b'+1$ does not hold for any $\mathbf{x} \in R\mathbb{B}_1^n$ and thus $\mathbf{a}'\mathbf{x} \geq b'+1 \Rightarrow_R \mathbf{a}\mathbf{x} \geq b+1$.

The symmetric reasoning applies to case $b+1 \leq -R\|\mathbf{a}\|_\infty$. Setting $b' = -R\|\mathbf{a}\|_\infty - 1$, we firstly have that $\mathbf{a}\mathbf{x} \geq b+1$ is a valid inequality for $R\mathbb{B}_1^n$ and hence $\mathbf{a}'\mathbf{x} \geq b'+1 \Rightarrow_R \mathbf{a}\mathbf{x} \geq b+1$ trivially. Secondly, the system $\mathbf{a}'\mathbf{x} \leq b', \|\mathbf{x}\|_1 \leq R$ is empty and hence $\mathbf{a}'\mathbf{x} \leq b' \Rightarrow_R \mathbf{a}\mathbf{x} \leq b$ trivially as well. \blacktriangleleft

► **Remark 23.** In the sequel, we will say that (\mathbf{a}', b') R -dominates or R -non-dominates (\mathbf{a}, b) when the corresponding case holds in Lemma 22. We will also drop label R - when R is clear from context.

When applying the above lemma to an initial disjunction $\mathbf{a}\mathbf{x} \leq b$ or $\geq b+1$, the R -dominating case above is a scenario in which the naive replacement of the disjunction $\mathbf{a}\mathbf{x} \leq b$ or $\geq b+1 \rightarrow \mathbf{a}'\mathbf{x} \leq b'$ or $\geq b'+1$ by the “first pass” Diophantine approximation does the job. Indeed, if every branching decision in \mathcal{T} was dominated by its first pass Diophantine approximation, then the naive replacements would be sufficient to obtain a branching proof with bounded coefficients.

Since domination does not always occur at the first level of approximation for an original disjunction $\mathbf{a}\mathbf{x} \leq b$ or $\geq b + 1$, we will require the use of an substitution sequence $\mathbf{a}_1\mathbf{x} \leq b_1, \dots, \mathbf{a}_k\mathbf{x} \leq b_k$ as described previously. The exact properties needed from this sequence as well as the algorithm to compute it are provided in the next subsection. At a high level, we continue creating additional levels of approximation until the dominating case occurs. More precisely, for every level $l \in [k]$, we will reduce the inequality $\mathbf{a}\mathbf{x} \leq b$ by subtracting non-negative combinations of the equalities $\mathbf{a}_i\mathbf{x} = b_i$, $i \in [l-1]$, to get a “remainder inequality” $\hat{\mathbf{a}}_l\mathbf{x} \leq \hat{b}_l$. The remainder is then given as input to Lemma 22 to get next level approximator $\mathbf{a}_l\mathbf{x} \leq b_l$. The final iteration k will correspond to the first time where the dominating case occurs (i.e., all previous iterations are non-dominating).

Since we are interested in approximating not just an inequality but a disjunction, it will be crucial that (non-)domination is well-behaved with respect to both sides of the disjunction. For this purpose, we will heavily make use of the following “flip-symmetry” in the definition of the R -domination and R -non-domination. Namely, if (\mathbf{a}', b') dominates (\mathbf{a}, b) , then $(-\mathbf{a}', -b' - 1)$ dominates $(-\mathbf{a}, -b - 1)$, and if (\mathbf{a}', b') non-dominates (\mathbf{a}, b) then $(-\mathbf{a}', -b')$ non-dominates $(-\mathbf{a}, -b - 1)$. One can easily check that these symmetries follow directly from simple manipulations of the definitions. These symmetries are what will allow us to conclude that the substitution sequence $\mathbf{a}_1\mathbf{x} \leq b_1, \dots, \mathbf{a}_k\mathbf{x} \leq b_k$ and its flipped version $\mathbf{a}_1\mathbf{x} \geq b_1, \dots, \mathbf{a}_k\mathbf{x} \geq b_k + 1$ will yield good approximations to $\mathbf{a}\mathbf{x} \leq b$ and $\mathbf{a}\mathbf{x} \geq b + 1$ respectively.

3.1 Step 1: Replacing Large Coefficient Branches by Small Coefficient Approximations

Given a branching proof \mathcal{T} of infeasibility for $K \subseteq RB_1^n$, where $R \in \mathbb{N}$, we begin the construction of the replacement proof \mathcal{T}' as follows:

We let \mathcal{T}' be a tree with vertex set V' containing a vertex v' for each $v \in V[\mathcal{T}]$, and an edge $e' = (v', w')$ for each $e = (v, w) \in E[\mathcal{T}]$. For each internal node $v \in V[\mathcal{T}]$ with children v_l, v_r , we compute through Algorithm 1 (see below) an approximation $(\mathbf{a}_{v'}, b_{v'})$ of the disjunction (\mathbf{a}_v, b_v) at v of precision $R, N := 10nR, M := (10nR)^{n+2}$. We label the left edge $e_l = (v', v'_l)$ in \mathcal{T}' by $\mathbf{a}_{v'}\mathbf{x} \leq b_{v'}$ and the right edge (v', v'_r) by $\mathbf{a}_{v'}\mathbf{x} \leq -b_{v'} - 1$ (equivalently, $\mathbf{a}_{v'}\mathbf{x} \geq b_{v'} + 1$).

In this first phase of construction, note that \mathcal{T}' retains the same tree structure as \mathcal{T} . Furthermore, note that the output of Algorithm 1 must serve equally well to approximate $\mathbf{a}_v\mathbf{x} \leq b_v$ and $\mathbf{a}_v\mathbf{x} \geq b_v + 1$ for $v \in \mathcal{T}$.

The required properties of the replacements of the form $\mathbf{a}\mathbf{x} \leq b \rightarrow \mathbf{a}'\mathbf{x} \leq b'$ are collected in the definition of a valid substitution sequence defined below. A valid substitution sequence of (\mathbf{a}, b) of precision R, N, M consists of, along with the approximations \mathbf{a}', b' , auxiliary information in the form of integers k, b_1, b_2, \dots, b_k , integer vectors $\mathbf{a}_1, \dots, \mathbf{a}_k$ and nonnegative reals $\gamma_1, \dots, \gamma_k$. While this auxiliary information is not included in the labels of \mathcal{T}' , it is computed by Algorithm 1 (and hence its existence is guaranteed). Furthermore, the existence of the valid substitution sequence is crucial for second part of the tree construction (see subsection 3.2), where the inequalities from the substitution sequences are used to construct CP proofs of infeasibility for the (possibly non-empty) leaves of \mathcal{T}' above.

► **Definition 24** (Valid Substitution Sequence). *We define a valid substitution sequence of an integer inequality $\mathbf{a}\mathbf{x} \leq b$, $\mathbf{a} \in \mathbb{Z}^n \setminus \{0\}$, $b \in \mathbb{Z}$ of precision $R, N, M \in \mathbb{N}$ to be a list*

$$(\mathbf{a}', b', k, \mathbf{a}_1, b_1, \gamma_1, \dots, \mathbf{a}_k, b_k, \gamma_k := 0),$$

where $k \in [n + 1]$ and $\mathbf{a}', \mathbf{a}_i \in \mathbb{Z}^n, b', b_i \in \mathbb{Z}, \gamma_i \in \mathbb{R}_+$, for $i \in [k]$, satisfying:

1. $\|\mathbf{a}'\|_\infty \leq N^n M^{n+1}, |b'| \leq RN^n M^{n+1}$ and
 $\|\mathbf{a}_i\|_\infty \leq 11nN^n, |b_i| \leq R\|\mathbf{a}_i\|_\infty + 1, i \in [k]$.
2. For $l \in [k-1]$, we have

$$\mathbf{a}'\mathbf{x} \leq b', \mathbf{a}_i\mathbf{x} = b_i, \forall i \in [l-1] \Rightarrow_R \mathbf{a}_l\mathbf{x} < b_l + 1.$$

3. For $l \in [k]$, we have

$$\mathbf{a}'\mathbf{x} \leq b', \mathbf{a}_i\mathbf{x} = b_i, \forall i \in [l-1] \Rightarrow_R \mathbf{a}\mathbf{x} \leq b + \gamma_l.$$

4. For $l \in [k-1]$, we have

$$\mathbf{a}_l\mathbf{x} \leq b_l - 1, \mathbf{a}_i\mathbf{x} = b_i, \forall i \in [l-1] \Rightarrow_R \mathbf{a}\mathbf{x} \leq b - n\gamma_l.$$

► **Remark 25.** In light of property 3, the terms γ_i are measures of precision for our approximation of $\mathbf{a}\mathbf{x} \leq b$. If $l_1 < l_2$, property 3 when applied to l_2 assumes more statements than when applied l_1 . Intuitively, this suggests that the implications should also be stronger (or at least not weaker). That is, one would expect $\gamma_{l_2} \geq \gamma_{l_1}$. Indeed, this assumption can be made without loss of generality. More precisely, if $(a', b', k, \mathbf{a}_1, b_1, \gamma_1, \dots, \mathbf{a}_k, b_k, \gamma_k := 0)$ is a valid substitution, then so is $(a', b', k, \mathbf{a}_1, b_1, \bar{\gamma}_1, \dots, \mathbf{a}_k, b_k, \bar{\gamma}_k)$, where $\bar{\gamma}_i := \min_{j \in [i]} \gamma_j$.

■ **Algorithm 1** LongToShort(\mathbf{a}, b, R, N, M).

Input: $\mathbf{a} \in \mathbb{Z}^n, b \in \mathbb{Z}, R, N, M \in \mathbb{N}$ such that $\frac{R}{N} < \frac{1}{4}$.

Output: $\mathbf{a}' \in \mathbb{Z}^n, b' \in \mathbb{Z}, k \in \mathbb{N}$, and $\mathbf{a}_i \in \mathbb{Z}^n, b_i \in \mathbb{Z}, \gamma_i \in \mathbb{R}_+, i \in [k]$, satisfying:

1. $(\mathbf{a}', b', k, \mathbf{a}_1, b_1, \gamma_1, \dots, \mathbf{a}_k, b_k, \gamma_k)$ is a valid substitution sequence of $\mathbf{a}\mathbf{x} \leq b$ of precision R, M, N .
2. $(-\mathbf{a}', -b' - 1, k, -\mathbf{a}_1, -b_1, \gamma_1, \dots, -\mathbf{a}_{k-1}, -b_{k-1}, \gamma_{k-1}, -\mathbf{a}_k, -b_k - 1, \gamma_k)$ is a valid substitution sequence of $-\mathbf{a}\mathbf{x} \leq -b - 1$ of precision R, M, N .

- 1 initialize $\hat{\mathbf{a}}_1 = \mathbf{a}, \hat{b}_1 = b, j = 1$;
- 2 while $\|\hat{\mathbf{a}}_j\|_\infty > 10nN^n$ do
 - 3 Set \mathbf{a}_j as a Diophantine approximation of $\hat{\mathbf{a}}_j$ of precision N ;
 - 4 Apply Lemma 22 part (ii) to $\hat{\mathbf{a}}_j, \hat{b}_j, \mathbf{a}_j, R, N$ to obtain b_j ;
 - 5 if (\mathbf{a}_j, b_j) dominates $(\hat{\mathbf{a}}_j, \hat{b}_j)$ then
 - 6 Set $k = j, \gamma_k = 0, \mathbf{a}' = \sum_{i=1}^k M^{k-i} \mathbf{a}_i$ and $b' = \sum_{i=1}^k M^{k-i} b_i$;
 - 7 return $\mathbf{a}', b', k, \mathbf{a}_i, b_i, \gamma_i, i \in [k]$;
 - 8 Set $\alpha_j = \frac{\|\hat{\mathbf{a}}_j\|_\infty}{\|\mathbf{a}_j\|_\infty}, \gamma_j = \frac{2\alpha_j}{5n}$;
 - 9 Set $\hat{\mathbf{a}}_{j+1} = \hat{\mathbf{a}}_j - \alpha_j \mathbf{a}_j, \hat{b}_{j+1} = \hat{b}_j - \alpha_j b_j$;
 - 10 Increment j ;
- 11 Set $k = j, \gamma_k = 0, \mathbf{a}_k = \mathbf{a} - \sum_{i=1}^{k-1} \lfloor \alpha_i \rfloor \mathbf{a}_i, \tilde{b}_k = b - \sum_{i=1}^{k-1} \lfloor \alpha_i \rfloor b_i$;
- 12 Set $b_k = \begin{cases} \tilde{b}_k & : -R\|\mathbf{a}_k\|_\infty - 1 < \tilde{b}_k < R\|\mathbf{a}_k\|_\infty \\ -R\|\mathbf{a}_k\|_\infty - 1 & : \tilde{b}_k \leq -R\|\mathbf{a}_k\|_\infty - 1 \\ R\|\mathbf{a}_k\|_\infty & : R\|\mathbf{a}_k\|_\infty \leq \tilde{b}_k \end{cases}$;
- 13 Set $\mathbf{a}' = \sum_{i=1}^k M^{k-i} \mathbf{a}_i$ and $b' = \sum_{i=1}^k M^{k-i} b_i$;
- 14 return $\mathbf{a}', b', k, \mathbf{a}_i, b_i, \gamma_i, i \in [k]$;

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► **Lemma 26.** *Algorithm 1 with input \mathbf{a}, b, R, N, M such that $N = 10nR, M = (10nR)^{n+2}$ terminates within $k \leq n + 1$ iterations and outputs valid substitution sequences of \mathbf{a}, b and $-\mathbf{a}, -b - 1$ of precision R, N, M .*

Proof. We begin by showing that the number of coordinates of $\hat{\mathbf{a}}_{k+1}$ that are zero is strictly greater than that of $\hat{\mathbf{a}}_k$. At iteration j , let p be such that $|(\hat{\mathbf{a}}_j)_p| = \|\hat{\mathbf{a}}_j\|_\infty$. As \mathbf{a}_j is the Diophantine approximation of $\hat{\mathbf{a}}_j$, we know that $|(\frac{\hat{\mathbf{a}}_j}{\alpha_j} - \mathbf{a}_j)_p| = |(\frac{\|\hat{\mathbf{a}}_j\|_\infty}{\|\hat{\mathbf{a}}_j\|_\infty} \hat{\mathbf{a}}_j)_p - (\mathbf{a}_j)_p| < 1/10nR$. By assumption, $(\frac{\hat{\mathbf{a}}_j}{\alpha_j})_p = \pm 1$. Thus $(\frac{\|\hat{\mathbf{a}}_j\|_\infty}{\|\hat{\mathbf{a}}_j\|_\infty} \hat{\mathbf{a}}_j)_p \in \mathbb{Z}$, and so $(\frac{\|\hat{\mathbf{a}}_j\|_\infty}{\|\hat{\mathbf{a}}_j\|_\infty} \hat{\mathbf{a}}_j)_p = (\mathbf{a}_j)_p$. As a result, $(\hat{\mathbf{a}}_{j+1})_p = 0$. As observed in remark 1.2, any zero entry of $\hat{\mathbf{a}}_j$ is also zero for $\hat{\mathbf{a}}_{j+1}$.

By this reasoning, either Algorithm 1 terminates with $k \leq n$, or $\hat{\mathbf{a}}_{n+1} = 0$. The while loop terminates as $\|\hat{\mathbf{a}}_{n+1}\|_\infty \leq 10nN^n$. This proves that Algorithm 1 terminates within $n + 1$ iterations.

We now show that Algorithm 1 outputs valid substitution sequences. For this purpose, we first prove below that $(\mathbf{a}', b', k, \mathbf{a}_1, b_1, \gamma_1, \dots, \mathbf{a}_k, b_k, \gamma_k)$ is a valid substitution sequence of $\mathbf{ax} \leq b$ of precision R, M, N . After this, we will argue that the flipped version of this sequence yields a valid substitution of $-\mathbf{ax} \leq -b - 1$ using the symmetries the algorithm.

1. When Algorithm 1 returns from line (7), we have $\|\mathbf{a}_i\|_\infty \leq N^n, i \in [k]$, since every \mathbf{a}_i is the result of Diophantine approximation of precision N . Furthermore, since every $b_i, i \in [k]$, is then the output of Lemma 22 part (ii), we also have $|b_i| \leq R\|\mathbf{a}_i\|_\infty + 1 \leq RN^n + 1, i \in [k]$. Therefore,

$$\|\mathbf{a}'\|_\infty \leq \sum_{i=1}^k M^{k-i} \|\mathbf{a}_i\|_\infty \leq N^n \sum_{i=1}^k M^{k-i} \leq N^n \sum_{i=0}^n M^i = N^n \frac{M^{n+1} - 1}{M - 1} \leq N^n M^{n+1}.$$

Similarly for b' , using $R, N \geq 1$ and $M \geq 3$,

$$|b'| \leq \sum_{i=1}^k M^{k-i} |b_i| \leq (RN^n + 1) \sum_{i=1}^k M^{k-i} = (RN^n + 1) \frac{M^{n+1} - 1}{M - 1} \leq RN^n M^{n+1}.$$

When Algorithm 1 returns from line (14), $\|\mathbf{a}_i\|_\infty \leq N^n$ for every $i \in [k - 1]$ and $\|\hat{\mathbf{a}}_k\| \leq 10N^n$. As in the previous case, we also have $|b_i| \leq R\|\mathbf{a}_i\|_\infty + 1, i \in [k]$. Note that for $i = k$, this is enforced on line (12) of the algorithm. Furthermore, b_k is indeed an integer since $R \in \mathbb{N}$ and $\tilde{b}_k, \|\mathbf{a}_k\|_\infty \in \mathbb{Z}$ by construction. To bound $\|\mathbf{a}_k\|_\infty$, we first note that

$$\|\mathbf{a}_k - \hat{\mathbf{a}}_k\|_\infty = \left\| \sum_{i=1}^{k-1} (\alpha_i - \lfloor \alpha_i \rfloor) \mathbf{a}_i \right\|_\infty \leq \sum_{i=1}^{k-1} \|(\alpha_i - \lfloor \alpha_i \rfloor) \mathbf{a}_i\|_\infty \leq (k-1)N^n \leq nN^n.$$

Since $\|\hat{\mathbf{a}}_k\|_\infty \leq 10nN^n$, we get that

$$\|\mathbf{a}_k\|_\infty \leq \|\hat{\mathbf{a}}_k\|_\infty + \|\mathbf{a}_k - \hat{\mathbf{a}}_k\|_\infty \leq 10nN^n + nN^n = 11nN^n.$$

To bound $\|\mathbf{a}'\|_\infty$, by the triangle inequality

$$\begin{aligned} \|\mathbf{a}'\|_\infty &\leq \sum_{i=1}^{k-1} M^{k-i} \|\mathbf{a}_i\|_\infty + \|\mathbf{a}_k\|_\infty \leq N^n \left(\sum_{i=1}^n M^i + 11n \right) \\ &= N^n \left(\frac{M^{n+1} - 1}{M - 1} + 11n - 1 \right) \leq N^n \left(\frac{2M^{n+1}}{M - 1} + 11n - 1 \right) \leq N^n M^{n+1}, \end{aligned} \quad (3.1)$$

where it can be easily be checked that the last inequality holds for $M = (10nR)^{n+2}$ and $R, n \geq 1$. The bound on $|b'|$ is computed in a manner similar to (3.1):

$$|b'| \leq \sum_{i=1}^k M^{k-i} (R \|\mathbf{a}_i\|_\infty + 1) \leq RN^n \left(\frac{2M^{n+1}}{M-1} + 11n - 1 \right) \leq RN^n M^{n+1}.$$

2. Let $l \in [k-1]$. When $\mathbf{a}_i \mathbf{x} = b_i, \forall i \in [l-1]$, we have that

$$\mathbf{a}' \mathbf{x} \leq b' \Leftrightarrow \sum_{i=l}^k M^{k-i} \mathbf{a}_i \mathbf{x} \leq \sum_{i=l}^k M^{k-i} b_i \Leftrightarrow \mathbf{a}_l \mathbf{x} + \sum_{i=l+1}^k M^{l-i} \mathbf{a}_i \mathbf{x} \leq b_l + \sum_{i=l+1}^k M^{l-i} b_i.$$

By the proof of part 1, $\|\mathbf{a}_i\|_\infty \leq N^n, b_i \leq R \|\mathbf{a}_i\|_\infty + 1$, for $i \in [k-1]$. Using these bounds, we get that

$$\mathbf{a}_l \mathbf{x} + \sum_{i=l+1}^k M^{l-i} \mathbf{a}_i \leq b_l + \sum_{i=l+1}^k M^{l-i} b_i \Rightarrow_R \mathbf{a}_l \mathbf{x} \leq b_l + \sum_{i=l+1}^k M^{l-i} b_i + R \left\| \sum_{i=l+1}^k M^{l-i} \mathbf{a}_i \right\|_\infty. \quad (3.2)$$

The error in the last term is bounded by

$$\sum_{i=l+1}^k M^{l-i} b_i + R \left\| \sum_{i=l+1}^k M^{l-i} \mathbf{a}_i \right\|_\infty \leq (2R+1)N^n \left(\sum_{i=1}^{k-l} M^{-i} \right) \leq \frac{(2R+1)N^n}{M-1} \leq \frac{1}{10n}, \quad (3.3)$$

where the last inequality is easily checked for $M = (10nR)^{n+2} = N^{n+2}$ and $n, R \in \mathbb{N}$. Combining (3.2) and (3.3), we conclude that

$$\mathbf{a}' \mathbf{x} \leq b', \mathbf{a}_i \mathbf{x} = b_i, i \in [l-1] \Rightarrow_R \mathbf{a}_l \mathbf{x} \leq b_l + \frac{1}{10n} < b_l + 1, \quad (3.4)$$

as needed.

3. We first deal with the case $l = k$. We have $\mathbf{a}_k + \sum_{i=1}^{k-1} M^{k-i} \mathbf{a}_i = \mathbf{a}'$ and $b_k + \sum_{i=1}^{k-1} M^{k-i} b_i = b'$. So if $\mathbf{a}_i \mathbf{x} = b_i, \forall i \in [k-1]$, we have that $\mathbf{a}' \mathbf{x} \leq b' \Leftrightarrow \mathbf{a}_k \mathbf{x} \leq b_k$. When Algorithm 1 returns from line (14), we have $\mathbf{a}_k + \sum_{i=1}^{k-1} \lceil \alpha_i \rceil \mathbf{a}_i = \mathbf{a}$ and $\tilde{b}_k + \sum_{i=1}^{k-1} \lceil \alpha_i \rceil b_i = b$ by construction. By the same argument as above, under the assumption $\mathbf{a}_i \mathbf{x} = b_i, \forall i \in [k-1]$, we have that $\mathbf{a}_k \mathbf{x} \leq \tilde{b}_k \Leftrightarrow \mathbf{a} \mathbf{x} \leq b$. It thus suffices to show that $\mathbf{a}_k \mathbf{x} \leq b_k \Rightarrow_R \mathbf{a}_k \mathbf{x} \leq \tilde{b}_k$, recalling that $\gamma_k = 0$. This proceeds in an analogous fashion to the analysis of the dominating case in Lemma 22 part (ii). Firstly, by our choice of b_k on line (12), if $-R \|\mathbf{a}_k\|_\infty - 1 < \tilde{b}_k < R \|\mathbf{a}_k\|_\infty$ then $b_k = \tilde{b}_k$, so this case is trivial. If $\tilde{b}_k \geq R \|\mathbf{a}_k\|_\infty$, then $b_k = R \|\mathbf{a}_k\|_\infty$ and $\mathbf{a}_k \mathbf{x} \leq \tilde{b}_k$ is valid inequality for $R\mathbb{B}_1^n$. Thus, the implication $\mathbf{a}_k \mathbf{x} \leq b_k \Rightarrow_R \mathbf{a}_k \mathbf{x} \leq \tilde{b}_k$ is again trivial. Lastly, if $\tilde{b}_k \leq -R \|\mathbf{a}_k\|_\infty - 1$, then $b_k = -R \|\mathbf{a}_k\|_\infty - 1$ and the system $\mathbf{a}_k \mathbf{x} \leq b_k, \|\mathbf{x}\|_\infty \leq R$ is empty. In particular, $\mathbf{a}_k \mathbf{x} \leq b_k \Rightarrow_R \mathbf{a}_k \mathbf{x} \leq \tilde{b}_k$, as needed.

Next, when the Algorithm 1 returns from line (7), by the guarantees of the R -dominating case in Lemma 22 part (ii), we have that

$$\mathbf{a}_k \mathbf{x} \leq b_k \Rightarrow_R \hat{\mathbf{a}}_k \mathbf{x} \leq \hat{b}_k.$$

Similarly to the previous case, under the assumption $\mathbf{a}_i \mathbf{x} = b_i, \forall i \in [k-1]$, we have that $\mathbf{a}' \mathbf{x} \leq b' \Leftrightarrow \mathbf{a}_k \mathbf{x} \leq b_k$ and $\mathbf{a} \mathbf{x} \leq b \Leftrightarrow \hat{\mathbf{a}}_k \mathbf{x} \leq \hat{b}_k$. The desired implication, $\mathbf{a}' \mathbf{x} \leq b', \mathbf{a}_i \mathbf{x} = b_i, \forall i \in [k-1] \Rightarrow_R \mathbf{a} \mathbf{x} \leq b$, thus follows.

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Now suppose $l \in [k-1]$. From (3.4) in part 2, we have that

$$\mathbf{a}'\mathbf{x} \leq b', \mathbf{a}_i\mathbf{x} = b_i, i \in [l-1] \Rightarrow_R \mathbf{a}_l\mathbf{x} \leq b_l + \frac{1}{10n}.$$

By lemma 22 part (i) applied to $\hat{\mathbf{a}}_{l+1}$ and \mathbf{a}_{l+1} ,

$$\mathbf{a}_l\mathbf{x} \leq b_l + \frac{1}{10n} \Rightarrow_R \hat{\mathbf{a}}_l\mathbf{x} \leq \alpha_l \left(b_l + \frac{1}{10n} + \frac{R}{N} \right). \quad (3.5)$$

Since $l \in [k-1]$, \mathbf{a}_l, b_l non-dominates $\hat{\mathbf{a}}_l, \hat{b}_l$ and thus by Lemma 22 part (ii),

$$(\hat{b}_l, \hat{b}_l + 1) \cap \left[\alpha_l \left(b_l - \frac{R}{N} \right), \alpha_l \left(b_l + \frac{R}{N} \right) \right] \neq \emptyset.$$

In particular, we get that

$$\alpha_l \left(b_l + \frac{R}{N} \right) = \alpha_l \left(b_l - \frac{R}{N} \right) + \alpha_l \left(\frac{2R}{N} \right) \leq \hat{b}_l + 1 + \alpha_l \left(\frac{2}{10n} \right). \quad (3.6)$$

Using $\alpha_l = \frac{\|\hat{\mathbf{a}}_l\|_\infty}{\|\mathbf{a}_l\|_\infty} \geq \frac{10nN^n}{N^n} = 10n$ combined with (3.6), we get that

$$\begin{aligned} \alpha_l \left(b_l + \frac{1}{10n} + \frac{R}{n} \right) &\leq \hat{b}_l + 1 + \alpha_l \left(\frac{3}{10n} \right) \leq \hat{b}_l + \alpha_l \left(\frac{1}{10n} + \frac{3}{10n} \right) \\ &= \hat{b}_l + \alpha_l \left(\frac{2}{5n} \right) = \hat{b}_l + \gamma_l, \end{aligned} \quad (3.7)$$

where the last equality follows by definition of γ_l . Finally, under the assumption that $\mathbf{a}_i\mathbf{x} = b_i, \forall i \in [l-1]$, observe that for any $\delta \in \mathbb{R}$, we have that

$$\hat{\mathbf{a}}_l\mathbf{x} \leq \hat{b}_l + \delta \Leftrightarrow \mathbf{a}_l\mathbf{x} \leq b_l + \delta. \quad (3.8)$$

The desired implication $\mathbf{a}'\mathbf{x} \leq b', \mathbf{a}_i\mathbf{x} = b_i, \forall i \in [l-1] \Rightarrow_R \mathbf{a}_l\mathbf{x} \leq b_l + \gamma_l$ now follows directly from the above combined with (3.4), (3.5) and (3.7).

4. Repeating the argument from part 3 starting from (3.5) with $\mathbf{a}_l\mathbf{x} \leq b_l - 1$, we have that

$$\mathbf{a}_l\mathbf{x} \leq b_l - 1 \Rightarrow_R \hat{\mathbf{a}}_l\mathbf{x} \leq \alpha_l \left(b_l - 1 + \frac{R}{N} \right). \quad (3.9)$$

Using (3.7) in part 3, we see that

$$\alpha_l \left(b_l - 1 + \frac{R}{N} \right) \leq \hat{b}_l - \alpha_l + \alpha_l \left(\frac{2}{5n} - \frac{1}{10n} \right) = \hat{b}_l - \alpha_l \left(1 - \frac{3}{10n} \right). \quad (3.10)$$

Recalling $\gamma_l := \alpha_l \left(\frac{2}{5n} \right)$, observe that $n\gamma_l = \alpha_l \left(\frac{2}{5} \right) \leq \alpha_l \left(1 - \frac{3}{10n} \right)$ since $n \geq 1$. Combining together with (3.9), (3.10) and (3.8) from part 3, we conclude that

$$\mathbf{a}_l\mathbf{x} \leq b_l - 1, \mathbf{a}_i\mathbf{x} = b_i, \forall i \in [l-1] \Rightarrow_R \mathbf{a}_l\mathbf{x} \leq b_l - \alpha_l \left(1 - \frac{3}{10n} \right) \leq b_l - n\gamma_l,$$

as needed.

We conclude the proof by showing that

$$(-\mathbf{a}', -b' - 1, k, -\mathbf{a}_1, -b_1, \gamma_1, \dots, -\mathbf{a}_{k-1}, -b_{k-1}, \gamma_{k-1}, -\mathbf{a}_k, -b_k - 1, \gamma_k)$$

is a valid substitution sequence of $-\mathbf{a}\mathbf{x} \leq -b - 1$ of precision R, M, N . We have already proved that Algorithm 1 correctly outputs a valid substitution sequence of \mathbf{a}, b , so we are done if we show that $(-\mathbf{a}', -b' - 1, k, -\mathbf{a}_1, -b_1, \gamma_1, \dots, -\mathbf{a}_{k-1}, -b_{k-1}, \gamma_{k-1}, -\mathbf{a}_k, -b_k - 1, \gamma_k)$ could have been output by Algorithm 1 upon input $-\mathbf{a}, -b - 1$.

If \mathbf{a}_i is a Diophantine approximation of $\hat{\mathbf{a}}_i$, then $-\mathbf{a}_i$ is a Diophantine approximation of $-\hat{\mathbf{a}}_i$. Referring to Remark 23 as our next step: when $j < k$, (\mathbf{a}_j, b_j) non-dominates $(\hat{\mathbf{a}}_j, \hat{b}_j)$, $(-\mathbf{a}_j, -b_j)$ also non-dominates $(-\hat{\mathbf{a}}_j, -\hat{b}_j - 1)$. As a ratio of norms, the α_j values are identical for both executions of Algorithm 1.

If Algorithm 1 with input \mathbf{a}, b returned from line (7), then the algorithm with input $-\mathbf{a}, -b - 1$ must also return from line (7). This is also a consequence of Remark 23: if (\mathbf{a}_k, b_k) dominates $(\hat{\mathbf{a}}_k, \hat{b}_k)$, then $(-\mathbf{a}_k, -b_k - 1)$ dominates $(-\hat{\mathbf{a}}_k, -\hat{b}_k - 1)$ and the algorithm returns with the expected valid substitution sequence of $-\mathbf{a}, -b - 1$.

If Algorithm 1 with input \mathbf{a}, b returned from line (14), this means $\|\hat{\mathbf{a}}_k\|_\infty \leq 10nN^n$. Running Algorithm 1 with input $-\mathbf{a}, -b - 1$ would give $-\hat{\mathbf{a}}_k$, also obviously of small norm. Line 11 would consequently give $-\mathbf{a} - \sum_{i=1}^{k-1} \lfloor \alpha_i \rfloor (-\mathbf{a}_i) = -\mathbf{a}_k$ and $-b - 1 - \sum_{i=1}^{k-1} \lfloor \alpha_i \rfloor (-b_i) = -\tilde{b}_k - 1$. It now suffices to check that output of line (12) given $-\tilde{b}_k - 1$ is $-b_k - 1$, recalling that b_k is the output of line (12) on input \tilde{b}_k . This follows by direct inspection, noting that it is analogous to the “flip-symmetry” of Lemma 22 part (ii). ◀

Replacing branches with large coefficients with their valid approximations reduces their bit-size, since valid approximations have coefficients of size $O(n^2) \log(nR)$. However, we are not yet done. We do not yet have a valid branching proof as the convex sets $K_{v'}$ associated to leaf nodes v' of \mathcal{T}' are not necessarily empty. We deal with this in Step 2.

3.2 Step 2: Adding Chvátal-Gomory (CG) Cuts to Trim the Leaves

We now show how to add CG cuts at each leaf of the current replacement tree \mathcal{T}' for \mathcal{T} , whose construction is described in the previous section, to ensure that all the leaf nodes in the final tree have empty continuous relaxations. The final tree will simply simulate the effect of the CG cuts applied to the leaves of \mathcal{T}' using additional branching decisions (see the proof of Theorem 1 in the next section).

Recall from the last section, that every leaf node $v \in \mathcal{T}$ has an associated leaf $v' \in \mathcal{T}'$ in the current replacement tree. The continuous relaxation for v' is $K_{v'} = P_{v'} \cap K$ (recall that unlike $K_v := P_v \cap K$, $K_{v'}$ need not be empty), where the inequalities defining $P_{v'}$ are derived from *valid substitution sequences* (as in Definition 24) of the original defining inequalities for P_v . Given this setup, our task is to add “low-weight” CG cuts to $P_{v'} \cap K$ to derive the empty set.

The main result of this section is a general procedure for deriving such CG cuts for any polyhedron P' induced by valid substitution sequences of the defining inequalities of a polyhedron P , where P satisfies $K \cap P = \emptyset$. The procedure will return a list of at most $2(n+1)$ CG cuts, which is responsible for the $O(n)$ factor blowup in the final tree size. Second, the normals of these CG cuts will all come from the substitution lists for the inequalities defining P , which ensures that they have low weight. The formal statement of this result is given below:

► **Theorem 27.** *Let $K \subseteq R\mathbb{B}_1^n$, $R \in \mathbb{N}$, be a compact convex set, and let $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, $\mathbf{A} \in \mathbb{Z}^{m \times n}$, $\mathbf{b} \in \mathbb{Z}^m$, be a polyhedron satisfying $P \cap K = \emptyset$. For each defining inequality $\mathbf{a}_i \mathbf{x} \leq b_i$ of P , for $i \in [m]$, let $(\mathbf{a}'_i, b'_i, k_i, \mathbf{a}_{i,1}, b_{i,1}, \gamma_{i,1}, \dots, \mathbf{a}_{i,k_i}, b_{i,k_i}, \gamma_{i,k_i})$ be a valid substitution sequence of precision R , $N := 10nR$, $M := (10nR)^{n+2}$. Let $P' = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}'\mathbf{x} \leq \mathbf{b}'\}$ be the corresponding “substitution” polyhedron, where $\mathbf{A}' \in \mathbb{Z}^{m \times n}$ has rows $\mathbf{a}'_1, \dots, \mathbf{a}'_m$ and $\mathbf{b}' \in \mathbb{Z}^m$ has rows b'_1, \dots, b'_m .*

Then, there exists an ordered list $\mathcal{L} := (\mathbf{a}_{j_1, p_1}, -\mathbf{a}_{j_1, p_1}, \dots, \mathbf{a}_{j_l, p_l}, -\mathbf{a}_{j_l, p_l}) \subseteq \mathbb{Z}^n$, where $j_r \in [m]$ and $p_r \in [k_{j_r} - 1]$, $r \in [l]$, satisfying $\text{CG}(K \cap P', \mathcal{L}) = \emptyset$ and $|\mathcal{L}| = 2l \leq 2(n+1)$.

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Proof. To prove theorem 27, we give a procedure to construct such a list \mathcal{L} in Algorithm 2. To prove the theorem, it thus suffices to prove the correctness of Algorithm 2.

■ **Algorithm 2** Generate CG Cuts.

Input: $K, P := \mathbf{A}\mathbf{x} \leq \mathbf{b}, P' := \mathbf{A}'\mathbf{x} \leq \mathbf{b}', R, M, N \in \mathbb{N}$,
 $(\mathbf{a}'_i, b'_i, k_i, \mathbf{a}_{i,1}, b_{i,1}, \gamma_{i,1}, \dots, \mathbf{a}_{i,k_i}, b_{i,k_i}, \gamma_{i,k_i})$, for $i \in [m]$, a valid substitution
sequence of $\mathbf{a}_i\mathbf{x} \leq b_i$ of precision R, M, N , as in theorem 27.

Output: An ordered list $\mathcal{L} := (\mathbf{a}_{j_1, p_1}, -\mathbf{a}_{j_1, p_1}, \dots, \mathbf{a}_{j_l, p_l}, -\mathbf{a}_{j_l, p_l})$ satisfying
 $\text{CG}(K \cap P', \mathcal{L}) = \emptyset$ and $0 \leq l \leq n + 1$.

- 1 **initialize** $\mathcal{L} = \emptyset, V = \mathbb{R}^n, p(i) = 1$, for $i \in [m]$, and $\boldsymbol{\varepsilon} = (\gamma_{1, p(1)}, \dots, \gamma_{m, p(m)})$;
- 2 Define $P_{\boldsymbol{\varepsilon}} := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b} + \boldsymbol{\varepsilon}\}$;
- 3 **while** $K \cap P_{\boldsymbol{\varepsilon}} \neq \emptyset$ and $V \neq \emptyset$ **do**
- 4 Apply Lemma 18 to $K, P, \boldsymbol{\varepsilon}$ to obtain $j_* \in [m]$ satisfying $\varepsilon_{j_*} > 0$ and
 $K \cap P_{\boldsymbol{\varepsilon} - (n+1)\varepsilon_{j_*}\mathbf{e}_{j_*}} = \emptyset$;
- 5 Append vectors $\mathbf{a}_{j_*, p(j_*)}, -\mathbf{a}_{j_*, p(j_*)}$ to the list \mathcal{L} ;
- 6 Update $V \leftarrow V \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j_*, p(j_*)}\mathbf{x} = b_{j_*, p(j_*)}\}$;
- 7 **for** j from 1 to m **do**
- 8 Increment $p(j)$ to the largest integer $p \in [k_j]$ satisfying
 $V \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j, i}\mathbf{x} = b_{j, i}, 1 \leq i < p\}$;
- 9 $\varepsilon_j \leftarrow \gamma_{j, p(j)}$;

10 **return** \mathcal{L} ;

To begin, we first give a high level description of the algorithm and explain the key invariants it maintains. The algorithm proceeds in iterations, associated with runs of the while loop on line 3. At each iteration, we append the pair of CG cuts induced by $\mathbf{a}_{j,p}, -\mathbf{a}_{j,p}$, $j \in [m], p \in [k_j - 1]$, from one of our substitution lists to the end \mathcal{L} . These cuts are chosen so that after adding them to \mathcal{L} , we can guarantee that $\text{CG}(K \cap P', \mathcal{L})$ satisfies the equality $\mathbf{a}_{j,p}\mathbf{x} = b_{j,p}$.

We keep track of these learned equalities using the affine subspace $V \subseteq \mathbb{R}^n$, which is initialized as $V = \mathbb{R}^n$ at the beginning of the algorithm. The principal invariant needed to prove correctness of the algorithm is as follows: at the beginning of an iteration $l \geq 1$, \mathcal{L}, V satisfy

$$(i) \quad \text{CG}(K \cap P', \mathcal{L}) \subseteq V \quad \text{and} \quad \dim(V) \leq n - l + 1.$$

The condition on the dimension of V above will be achieved by ensuring that the new equality we add is not already implied by V . Precisely, the dimension of V will decrease by at least one at every iteration where we pass the while loop check. Using (i), at the beginning of iteration $l = n + 2$ (i.e., after $n + 1$ iterations) we will have that $\dim(V) \leq -1$ and hence $\text{CG}(K \cap P', \mathcal{L}) \subseteq V = \emptyset$. In particular, the while loop check $V \neq \emptyset$ will fail and we will correctly terminate. Thus, assuming (i) holds, the algorithm always terminates after at most $n + 1$ iterations. Since we add only 2 CG cuts per iteration, the total number of cuts in the list \mathcal{L} will be at most $2(n + 1)$ by the end the algorithm. To prove that (i) holds, we first introduce two other important invariants.

To keep track of the learned equalities in each substitution list, we keep a counter $p(j) \in [k_j]$, for $j \in [m]$. For the second invariant, the algorithm maintains that at the beginning of each iteration we have that

$$(ii) \quad V = \bigcap_{j \in [m]} \bigcap_{1 \leq i \leq p(j)-1} \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j, i}\mathbf{x} = b_{j, i}\},$$

and that each $p(j) \in [k_j], j \in [m]$, is maximal subject to the above equality. That is, for each $j \in [m]$, V satisfies all the equalities $\mathbf{a}_{j,i}\mathbf{x} = b_{j,i}$ for $i \in [p(j) - 1]$, and, if $p(j) < k_j$, V does not satisfy $\mathbf{a}_{j,p(j)}\mathbf{x} = b_{p(j)}$. The counters are initialized to $p(1) = \dots = p(m) = 1$ corresponding to $V = \mathbb{R}^n$ (i.e., we have not yet learned any equalities), which indeed yields a maximal choice. That the affine space V can be expressed in the above form is a simple consequence of how we update it on line (6). Namely, we only update V when we add the equality $\mathbf{a}_{j_*,p(j_*)}\mathbf{x} = b_{j_*,p(j_*)}$ to V on line (6). Note that since $\varepsilon_{j_*} > 0$ on line (4), we must have $p(j_*) < k_{j_*}$, since otherwise $\varepsilon_{j_*} = \gamma_{j_*,k_{j_*}} = 0$ (by definition of a valid substitution sequence). Thus, we only add an inequality $\mathbf{a}_{j,p}\mathbf{x} = b_{j,p}$, $j \in [m]$, to V if $1 \leq p < k_j$ and if V satisfies $\mathbf{a}_{j,i}\mathbf{x} = b_{j,i}$ for all $i \in [p - 1]$, as needed. Lastly, the required maximality is directly ensured by line (8). This proves that (ii) is indeed maintained. Note that under maximality, for each $j \in [m]$ such that $p(j) < k_j$, adding the equality $\mathbf{a}_{j,p(j)}\mathbf{x} = b_{p(j)}$ to V must reduce the dimension of V by at least one (more precisely, adding this equality either makes V empty or reduces its dimension by exactly 1). We will use this in the proof of (i).

With this notation, we may state the final invariant, which will be a direct consequence of the first two and the definition of a valid substitution sequence. Letting $\varepsilon := (\gamma_{1,p(1)}, \dots, \gamma_{m,p(m)})$ denote the “error level” for each constraint of P (note that this equality is maintained on line (9)), at the beginning of each iteration we maintain

$$(iii) \quad \text{CG}(K \cap P', \mathcal{L}) \subseteq P_\varepsilon,$$

where $P_\varepsilon := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leq \mathbf{b} + \varepsilon\}$. Crucially, invariant (iii) justifies the first termination condition $K \cap P_\varepsilon = \emptyset$, since if this occurs $\text{CG}(K \cap P', \mathcal{L}) \subseteq K \cap P_\varepsilon = \emptyset$. Note that for a constraint $j \in [m]$, with $p(j) = k_j$, the effective error level $\varepsilon_j = \gamma_{j,k_j} = 0$ (by definition of valid substitution). That is, we have effectively “learned” the defining constraint $\mathbf{a}_j\mathbf{x} \leq b_j$ for P for any $j \in [m]$ with $p(j) = k_j$. Clearly, once all the constraints of P have been learned, we will have $\text{CG}(K \cap P', \mathcal{L}) \subseteq K \cap P = \emptyset$, where the last equality is by assumption.

We now show that (iii) is a consequence of (i) and (ii). Let \mathcal{L}, V, p and ε be the state at the beginning of some iteration $l \geq 1$, and assume that (i) and (ii) hold. Then, for each $j \in [m]$, we have that

$$\text{CG}(K \cap P', \mathcal{L}) \subseteq K \cap P' \cap V \quad (\text{by (i) and } \text{CG}(K \cap P', \mathcal{L}) \subseteq K \cap P') \quad (3.11)$$

$$\subseteq R\mathbb{B}_1^n \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}'_j\mathbf{x} \leq b'_j, \mathbf{a}_{j,i}\mathbf{x} = b_{j,i}, \forall i \in [p(j) - 1]\} \quad (3.12)$$

$$(\text{by (ii) and } K \subseteq R\mathbb{B}_1^n)$$

$$\subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_j\mathbf{x} \leq b_j + \gamma_{j,p(j)}\} \quad (\text{by Definition 24 part 3.}).$$

Since the above holds for all $j \in [m]$, and $\varepsilon_j = \gamma_{j,p(j)}$, for $j \in [m]$, this proves invariant (iii).

Given the above, to prove correctness of algorithm it suffices to establish invariant (i). We now show that invariant (i) holds by induction on the iteration $l \geq 1$. Let \mathcal{L}, V, p and ε denote the state at the beginning of some iteration $l \leq 1$ for which (i) holds. Note that (i) trivially holds for the base case $l = 1$ since $V = \mathbb{R}^n$. By the reasoning in the previous paragraphs, we also have that invariant (ii) and (iii) hold at the beginning of l . We must now show that (i) holds at the beginning of iteration $l + 1$ under these assumptions. Clearly, we may assume that we pass the while loop check $K \cap P_\varepsilon \neq \emptyset$ and $V \neq \emptyset$, since otherwise there is nothing to prove.

Let $j_* \in [m]$ be the index satisfying $\varepsilon_{j_*} > 0$ and $K \cap P_{\varepsilon - (n+1)\varepsilon_{j_*}\mathbf{e}_{j_*}} = \emptyset$ as guaranteed by Lemma 18. This index indeed exists since we already checked that $K \cap P_\varepsilon \neq \emptyset$. As argued for (ii), we also know that $p(j_*) < k_{j_*}$, which will ensure we have access to the required inequalities from the valid substitution sequence of $\mathbf{a}_{j_*}\mathbf{x} \leq b_{j_*}$. Letting $P'_\mathcal{L} := \text{CG}(K \cap P', \mathcal{L})$, to prove that (i) holds for $l + 1$, it now suffices to show that

- (a) $\text{CG}(P'_{\mathcal{L}}, (\mathbf{a}_{j^*, p(j^*)}, -\mathbf{a}_{j^*, p(j^*)})) \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j^*, p(j^*)} \mathbf{x} = b_{j^*, p(j^*)}\}$,
 (b) $\dim(V \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j^*, p(j^*)} \mathbf{x} \leq b_{j^*, p(j^*)}\}) \leq \dim(V) - 1$.

As explained previously, (b) follows directly the maximality assumption in (ii). We may thus focus on (a). To begin, using (i) and (ii) and the same analysis as in (3.11), we see that

$$\begin{aligned} P'_{\mathcal{L}} &\subseteq R\mathbb{B}_1^n \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}'_{j^*} \mathbf{x} \leq b'_{j^*}, \mathbf{a}_{j^*, i} \mathbf{x} = b_{j^*, i}, \forall i \in [p(j^*) - 1]\} \\ &\subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j^*, p(j^*)} \mathbf{x} < b_{j^*, p(j^*)} + 1\}, \end{aligned}$$

where the last containment follows from Definition (24) part 2. In particular,

$$\sup_{\mathbf{x} \in P'_{\mathcal{L}}} \mathbf{a}_{j^*, p(j^*)} \mathbf{x} < b_{j^*, p(j^*)} + 1 \Rightarrow \lfloor \sup_{\mathbf{x} \in P'_{\mathcal{L}}} \mathbf{a}_{j^*, p(j^*)} \mathbf{x} \rfloor \leq b_{j^*, p(j^*)}, \quad (3.13)$$

since $b_{j^*, p(j^*)} \in \mathbb{Z}$. From (3.13), we conclude that

$$\text{CG}(P'_{\mathcal{L}}, \mathbf{a}_{j^*, p(j^*)}) \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j^*, p(j^*)} \mathbf{x} \leq b_{j^*, p(j^*)}\}. \quad (3.14)$$

From here, again using (i) and (ii), we have that

$$\begin{aligned} P'_{\mathcal{L}} \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j^*, p(j^*)} \mathbf{x} \leq b_{j^*, p(j^*)} - 1\} \\ \subseteq R\mathbb{B}_1^n \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j^*, p(j^*)} \mathbf{x} \leq b_{j^*, p(j^*)} - 1, \mathbf{a}_{j^*, i} \mathbf{x} = b_{j^*, i}, \forall i \in [p(j^*) - 1]\} \\ \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j^*} \mathbf{x} \leq b_{j^*} - n\varepsilon_{j^*}\}, \end{aligned} \quad (3.15)$$

where the last containment follows from Definition (24) part 4 and $\varepsilon_{j^*} = \gamma_{j^*, p(j^*)}$. Noting that

$$P_{\varepsilon} \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j^*} \mathbf{x} \leq b_{j^*} - n\varepsilon_{j^*}\} = P_{\varepsilon - (n+1)\varepsilon_{j^*} \mathbf{e}_{j^*}},$$

by the guarantees of Lemma 18, invariant (iii) and (3.15), we therefore have that

$$P'_{\mathcal{L}} \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_{j^*, p(j^*)} \mathbf{x} \leq b_{j^*, p(j^*)} - 1\} \subseteq K \cap P_{\varepsilon - (n+1)\varepsilon_{j^*} \mathbf{e}_{j^*}} = \emptyset.$$

In particular, we must have that

$$\sup_{\mathbf{x} \in P'_{\mathcal{L}}} -\mathbf{a}_{j^*, p(j^*)} \mathbf{x} < -b_{j^*, p(j^*)} + 1 \Rightarrow \lfloor \sup_{\mathbf{x} \in P'_{\mathcal{L}}} -\mathbf{a}_{j^*, p(j^*)} \mathbf{x} \rfloor \leq -b_{j^*, p(j^*)}, \quad (3.16)$$

since $-b_{j^*, p(j^*)} \in \mathbb{Z}$. From (3.16), we conclude that

$$\text{CG}(P'_{\mathcal{L}}, -\mathbf{a}_{j^*, p(j^*)}) \subseteq \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{a}_{j^*, p(j^*)} \mathbf{x} \leq -b_{j^*, p(j^*)}\}. \quad (3.17)$$

Property (a) now follows directly by combining (3.14) and (3.17). This concludes the proof of invariant (i) and the proof of correctness of the algorithm. \blacktriangleleft

3.3 Proof of Theorem 1

Let $N = 10nR$, $M = (10nR)^{n+2}$. Given \mathcal{T} , we construct \mathcal{T}' a labeled binary tree with the same structure as that of \mathcal{T} as described at the beginning of subsection 3.1. Recall that for each internal node $v \in \mathcal{T}$ with associated disjunction (\mathbf{a}_v, b_v) , we retrieve a pair of valid substitution sequences from $\text{LongToShort}(\mathbf{a}_v, b_v, R, N, M)$, yielding the precision R, N, M sequence $(\mathbf{a}'_v, b'_v, k_v, \mathbf{a}_{v,1}, b_{v,1}, \gamma_{v,1}, \dots, \mathbf{a}_{v,k}, b_{v,k}, \gamma_{v,k})$ for $\mathbf{a}_v \leq b_v$ and the corresponding flip

(as in the output description of Algorithm 1) for $-\mathbf{a}_v \mathbf{x} \leq -b_v - 1$. For the corresponding node $v' \in \mathcal{T}'$, we create two children v'_l, v'_r and label the left edge (v', v'_l) with $\mathbf{a}'_v \mathbf{x} \leq b'_v$ and the right edge (v', v'_r) with $-\mathbf{a}'_v \mathbf{x} \leq -b'_v - 1$.

From the properties of a valid substitution sequence, we have that $\|\mathbf{a}'_v\|_\infty \leq N^n M^{n+1}$ and $|b'_v| \leq RN^n M^{n+1}$. The choice of $N = 10nR$, $M = (10nR)^{n+2}$ gives

$$\|\mathbf{a}'_v\|_\infty \leq (10nR)^n (10nR)^{(n+2)(n+1)} = (10nR)^{n^2+4n+2} \quad \text{and} \quad |b'_v| \leq R(10nR)^{n^2+4n+2}.$$

Both of these quantities are upper bounded by $(10nR)^{(n+2)^2}$. For $\mathbf{x} \in \mathbb{Z}^n$, $\langle \mathbf{x} \rangle \leq n + n \log(1 + \|\mathbf{x}\|_\infty)$, so the bit-size of each inequality is $O(n^3 \log(nR))$.

Consider an arbitrary leaf node $v' \in \mathcal{T}'$ with associated leaf node $v \in \mathcal{T}$. Observe that by construction

$$P_{v'} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}'_e \mathbf{x} \leq b'_e, e \in E[P_{\mathcal{T}'}(v')]\}$$

satisfies the hypotheses of Theorem 27, so that there exists a list, say $\mathcal{L}_{v'}$ of integer vectors such that $\text{CG}(K \cap P', \mathcal{L}_{v'}) = \emptyset$. More precisely, $\mathcal{L}_{v'} = (\mathbf{a}_{j_1, p_1}, -\mathbf{a}_{j_1, p_1}, \dots, \mathbf{a}_{j_l, p_l}, -\mathbf{a}_{j_l, p_l})$, where $l \leq (n+1)$ and $\mathbf{a}_{j_r, p_r}, j_r \in [m_v], p_r \in [k_j - 1], r \in [l]$, are taken from the valid substitution sequences of precision R, N, M of the inequalities in the system $\mathbf{A}_v \mathbf{x} \leq \mathbf{b}_v$, $\mathbf{A}_v \in \mathbb{Q}^{m_v \times n}$, $\mathbf{b}_v \in \mathbb{R}^{m_v \times n}$, defining P_v . Note that by the properties of an R, M, N valid substitution sequence, $\|\mathbf{a}_{j_r, p_r}\|_\infty \leq 11nN^n, |b_{j_r, p_r}| \leq R11nN^n + 1, r \in [l]$, and hence via the same argument as above each $(\mathbf{a}_{j_r, p_r}, b_{j_r, p_r})$ can be described using $O(n^2 \log_2(nR))$ bits.

We now explain how to extend \mathcal{T}' to a valid branching proof. For each leaf node $v' \in \mathcal{T}'$, we will build a branching proof of infeasibility for $K_{v'}$ of length $O(n)$ which simulates the effect of the CG cuts in $\mathcal{L}_{v'}$. By appending these sub-branching proofs to \mathcal{T}' below each leaf node v' , the extended \mathcal{T}' clearly becomes a valid branching proof for K having length at most $|\mathcal{T}'| = O(n)|\mathcal{T}|$ by construction.

The construction of the subtree at v' using $\mathcal{L}_{v'} = (\mathbf{a}_{j_1, p_1}, -\mathbf{a}_{j_1, p_1}, \dots, \mathbf{a}_{j_l, p_l}, -\mathbf{a}_{j_l, p_l})$ (as above) proceeds as follows. Starting from v' , we create two children v'_l, v'_r , and label the edge (v', v'_l) with the inequality $\mathbf{a}_{j_1, p_1} \mathbf{x} \leq b_{j_1, p_1}$ and the edge (v', v'_r) with the inequality $\mathbf{a}_{j_1, p_1} \mathbf{x} \geq b_{j_1, p_1} + 1$. Recall that by the definition of a CG cut, the continuous relaxation $K_{v'_r}$ at the right child v'_r is now empty. The construction now proceeds inductively on v'_l using the sublist $(-\mathbf{a}_{j_1, p_1}, \dots, \mathbf{a}_{j_l, p_l}, -\mathbf{a}_{j_l, p_l})$. Note that for every cut in \mathcal{L}' , we add a left and right child to the current left-most leaf of the partially constructed subtree, for which the continuous relaxation of the newly added right child is always empty. At the end of the construction, it is easy to see that the left-most leaf of the constructed subtree has $\text{CG}(K_{v'}, \mathcal{L}')$ as its continuous relaxation, which is empty by assumption. From here, we immediately get that the constructed subtree yields a valid branching proof of infeasibility for $K_{v'}$, and that the number of nodes in the subtree distinct from v' is exactly $2|\mathcal{L}_{v'}| \leq 4(n+1)$. Furthermore, we may bound the bit-size of this subtree by $O(n^3 \log_2(nR))$, since it has $O(n)$ nodes and every edge is labeled with an inequality of bit-size $O(n^2 \log_2(nR))$.

To bound the total bit-size $\langle \mathcal{T}' \rangle$ of the final branching proof \mathcal{T}' , we combine the bit-size bound from the subtrees above together with the total bit-size of all the replacement disjunctions of the form $\mathbf{a}'_v \mathbf{x} \leq b'_v$ or $\geq b'_v + 1$ (as above) labeling the outgoing edges of nodes in \mathcal{T}' associated with internal nodes of \mathcal{T} . Given that each disjunction $\mathbf{a}'_v \leq b'_v$ or $\geq b'_v + 1$ requires $O(n^3 \log_2(nR))$ bits as explained above, their total bit-size is bounded by $O(n^3 \log_2(nR)|\mathcal{T}|)$. Furthermore, the bit-size contribution from all the subtrees in \mathcal{T}' associated with leaf nodes of \mathcal{T} is $O(n^3 \log_2(nR)|\mathcal{T}|)$, since the number of these subtrees is bounded by $|\mathcal{T}|$ and each has bit-size $O(n^3 \log_2(nR))$ as explained above. Thus, the total bit-size $\langle \mathcal{T}' \rangle = O(n^3 \log_2(nR)|\mathcal{T}|)$ as needed.

3.4 Proof of Corollary 2

The primary tools for this proof will be the following well-known facts pertaining to the bit-size of linear programs: (see [24] Chapter 10 for a thorough treatment):

► **Lemma 28.** *Let $A \in \mathbb{Q}^{m \times n}$, $\mathbf{b} \in \mathbb{Q}^m$, $\mathbf{c} \in \mathbb{Q}^n$.*

1. *For $\mathbf{c} \in \mathbb{Q}^n$, if $\max\{\mathbf{c}\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ is finite, then it has size at most $4(\langle \mathbf{A}, \mathbf{b} \rangle + \langle \mathbf{c} \rangle)$.*
2. *If the system $\lambda\mathbf{A} = 0$, $\lambda \geq 0$ and $\lambda\mathbf{b} < 0$ is feasible, then there exists a solution λ of with bit-size $\langle \lambda \rangle = O(n\langle A \rangle)$.*

► **Remark 29.** Part 2 of the above lemma in fact corresponds to a bound on the bit-size of a generator λ of the relevant extreme ray of the cone $\lambda\mathbf{A} = 0$, $\lambda \geq 0$, where we note that a Farkas certificate of infeasibility for $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ (if it exists) can always be chosen to be an extreme ray of this cone. This is also the reason why the bit-size bound does not in fact depend on $\langle \mathbf{b} \rangle$.

Proof of Corollary 2. Since $K = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}\mathbf{x} \leq \mathbf{d}\}$ is a polytope (i.e., bounded), we may invoke lemma 28 part 1 to conclude that

$$\max_{\mathbf{x} \in P} \|\mathbf{x}\|_1 = \max_{\mathbf{y} \in \{-1, 1\}^n} \max\{\mathbf{y}\mathbf{x} : \mathbf{C}\mathbf{x} \leq \mathbf{d}\} \leq 2^{4(L+2n)}$$

using that $\langle \mathbf{y} \rangle \leq 2n$, for $\mathbf{y} \in \{-1, 1\}^n$, and that $|a| \leq 2^{\langle a \rangle} \forall a \in \mathbb{Q}$. Therefore, $K \subseteq R\mathbb{B}_1^n$ for $R = 2^{4L+8n}$. Theorem 1 applied with $R = 2^{4L+8n}$ already gives us a \mathcal{T}' with $|\mathcal{T}'| \leq O(n)|\mathcal{T}|$, and the bound on $\|\cdot\|_\infty$ for integer vectors gives the bit-size of every inequality $\langle \mathbf{a}_e \mathbf{x} \leq b_e \rangle \leq O(n^3) \log(n(R+2)) = O(n^3L)$. As a result, the branching proof \mathcal{T}' has size $\langle \mathcal{T}' \rangle = n^{O(1)}L|\mathcal{T}|$. However, \mathcal{T}' is not yet a *certified* branching proof.

We are done if we can add to each leaf node $v' \in \mathcal{T}'$ a Farkas certificate $\lambda_{v'}$ of small size. Recall the continuous relaxation at v' in \mathcal{T}' is $K_{v'} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}\mathbf{x} \leq \mathbf{d}, \mathbf{A}_{v'} \leq \mathbf{b}'\}$. By assumption, we know that $K_{v'} = \emptyset$, and thus by Farkas's Lemma there exists $\lambda_{v'} := (\lambda_{v',1}, \lambda_{v',2}) \geq 0$ such that $\lambda_{v',1}\mathbf{C} + \lambda_{v',2}\mathbf{A}_{v'} = 0$ and $\lambda_{v',1}\mathbf{d} + \lambda_{v',2}\mathbf{b}' < 0$. Thus, by lemma 28 part 2, there exists a solution $\lambda_{v'}$ whose bit-complexity is upper bounded by $O(n\langle \mathbf{C}, \mathbf{A}_{v'} \rangle)$. This quantity may be large since we have not controlled the number of rows $\mathbf{A}_{v'}$. By Caratheodory's theorem however, there exists a solution $\lambda_{v'}$ with at most $n+1$ non-zero entries. Therefore, we can restrict our attention to a subset of rows of \mathbf{C} and $\mathbf{A}_{v'}$ of cardinality at most $n+1$. As argued above, by Theorem 1 each row of $\mathbf{A}_{v'}$ has bit-size at most $O(n^3 \log_2(nR)) = O(n^3L)$, and by assumption $\langle \mathbf{C} \rangle \leq L$. Thus, by restricting to the appropriate sub-system, the bit-length of the non-zero entries of $\lambda_{v'}$ can be bounded by $O(n((n+1)n^3L + L)) = O(n^5L)$, as needed.

Since the number of nodes in $|\mathcal{T}'| = O(n|\mathcal{T}|)$, the combined bit-size of the Farkas certificates above is at most $O(n^6L|\mathcal{T}|)$. This dominates the contribution of the disjunctions to the bit-size of \mathcal{T}' , which by Theorem 1 is $O(n^3L|\mathcal{T}|)$. Thus, the certified version of has size $\langle \mathcal{T}' \rangle = O(n^6L|\mathcal{T}|)$, as needed. ◀

4 Simulating Enumerative Branching Proofs by Cutting Planes

In this section, we prove that enumerative branching proofs can be simulated by CP, and give an application to Tseitin formulas (section 4.1).

To begin, we first extend the lifting lemma (Lemma 20) to a sequence of CG cuts. This will allow us to use induction on subtrees of an enumerative branching proof.

► **Lemma 30** (Lifting Sequences of CG cuts). *Let $K \subseteq \mathbb{R}^n$ be a non-empty compact set. Let $\mathbf{c} \in \mathbb{Z}^n$, $F := F_K(\mathbf{c})$ and assume that $h_K(\mathbf{c}) \in \mathbb{Z}$. Let $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{Z}^n$. Then, there exists $n_1, \dots, n_k \geq 0$ such that*

$$\text{CG}(K, (\mathbf{a}_1 + n_1\mathbf{c}, \dots, \mathbf{a}_k + n_k\mathbf{c})) \cap H_{\overline{K}}(\mathbf{c}) = \text{CG}(F, (\mathbf{a}_1, \dots, \mathbf{a}_k)) .$$

Proof. We prove the statement by induction on i . For $i = 0$, there are no CG cuts to apply and the statement becomes $K \cap H_{\overline{K}}(\mathbf{c}) = F$, which follows by definition. Now assume $1 \leq i \leq k$ with induction hypothesis

$$K_{i-1} \cap H_{\overline{K}}(\mathbf{c}) = F_{i-1} \tag{4.1}$$

where

$$K_{i-1} := \text{CG}(K, (\mathbf{a}_1 + n_1\mathbf{c}, \dots, \mathbf{a}_{i-1} + n_{i-1}\mathbf{c})) \quad \text{and} \quad F_{i-1} := \text{CG}(F, (\mathbf{a}_1, \dots, \mathbf{a}_{i-1})).$$

We must prove the existence of $n_i \geq 0$ such that (4.1) holds for i . Firstly, if $F_{i-1} = \emptyset$, then regardless of the choice of $n_i \geq 0$, both the sets F_i and $K_i \cap H_{\overline{K}}(\mathbf{c})$ will be empty since they are both contained in $F_{i-1} = \emptyset$. In particular, we may set $n_i = 0$ and maintain the desired equality.

So assume $F_{i-1} \neq \emptyset$. From here, since $\emptyset \neq F_{i-1} \subseteq F$, where we recall that F is the set maximizers of \mathbf{c} in K , and $F_{i-1} \subseteq K_{i-1} \subseteq K$, we have that

$$h_K(\mathbf{c}) \geq h_{K_{i-1}}(\mathbf{c}) \geq h_{F_{i-1}}(\mathbf{c}) = h_K(\mathbf{c}) \in \mathbb{Z}.$$

In particular, $H_{\overline{K_{i-1}}}(\mathbf{c}) = H_{\overline{K}}(\mathbf{c})$. Therefore, by the induction hypothesis (4.1)

$$K_{i-1} \cap H_{\overline{K_{i-1}}}(\mathbf{c}) = K_{i-1} \cap H_{\overline{K}}(\mathbf{c}) = F_{i-1} ,$$

that, F_{i-1} is the set of maximizers of \mathbf{c} in K_{i-1} . Furthermore, since K_{i-1} is the intersection of K with closed halfspaces and K is compact, K_{i-1} is also compact. Therefore, we may apply Lemma 20 to choose $n_i \geq 0$ satisfying

$$H_{\overline{K_{i-1}}}^{\text{cg}}(\mathbf{a}_i + n_i\mathbf{c}) \cap H_{\overline{K}}(\mathbf{c}) = H_{\overline{F_{i-1}}}^{\text{cg}}(\mathbf{a}_i) \cap H_{\overline{K}}(\mathbf{c}). \tag{4.2}$$

We use the above n_i to define

$$K_i := K_{i-1} \cap H_{\overline{K_{i-1}}}^{\text{cg}}(\mathbf{a}_i + n_i\mathbf{c}) = \text{CG}(K, (\mathbf{a}_1 + n_1\mathbf{c}, \dots, \mathbf{a}_i + n_i\mathbf{c})).$$

Intersecting both sides of (4.2) with K_{i-1} , we conclude that

$$\begin{aligned} H_{\overline{K_{i-1}}}^{\text{cg}}(\mathbf{a}_i + n_i\mathbf{c}) \cap K_{i-1} \cap H_{\overline{K}}(\mathbf{c}) &= H_{\overline{F_{i-1}}}^{\text{cg}}(\mathbf{a}_i) \cap K_{i-1} \cap H_{\overline{K}}(\mathbf{c}) \Leftrightarrow \\ K_i \cap H_{\overline{K}}(\mathbf{c}) &= H_{\overline{F_{i-1}}}^{\text{cg}}(\mathbf{a}_i) \cap F_{i-1} \Leftrightarrow \\ K_i \cap H_{\overline{K}}(\mathbf{c}) &= F_i, \end{aligned}$$

as needed. The lemma thus follows. ◀

We are now ready to prove the main result of this section, which shows that enumerative branching proofs can be simulated by CP.

Proof of Theorem 4. Our procedure for converting enumerative branching proofs to CP proofs is given by Algorithm 3. The proof of correctness of the procedure will yield the theorem:

▷ **Claim 31.** Given an enumerative branching proof \mathcal{T} of integer infeasibility for a compact convex set $K \subseteq \mathbb{R}^n$, Algorithm 3 correctly outputs a list $\mathcal{L} = (\mathbf{a}_1, \dots, \mathbf{a}_N) \in \mathbb{Z}^n$ satisfying $\text{CG}(K, \mathcal{L}) = \emptyset$ and $|\mathcal{L}| := N \leq 2|\mathcal{T}| - 1$.

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Proof.

Algorithm Outline. We first describe the algorithm at a high level and then continue with a formal proof. The procedure traverses the tree \mathcal{T} in order, visiting the children of each node from right to left. We explain the process starting from the root node $r \in \mathcal{T}$. To begin, we examine its branching direction $\mathbf{a}_r \in \mathbb{Z}^n$ and bounds $l_r \leq u_r$ satisfying

$$\{\mathbf{a}_r \mathbf{x} : \mathbf{x} \in K\} \subseteq [l_r, u_r],$$

recalling that r has a child r_b for each $b \in [l_r, u_r] \cap \mathbb{Z}$.

Starting at r , the procedure adds CG cuts to “chop off” the children of r moving from right to left. In particular, it alternates between adding the CG induced by \mathbf{a}_r to K , which will either make K empty or push the hyperplane $H_{\mathbf{a}_r, b}^-$, where $b := h_K(\mathbf{a}_r)$, to the next child of r , and recursively adding CG cuts induced by the subtree \mathcal{T}_{r_b} rooted at the child r_b of r . The cuts computed on the subtree \mathcal{T}_{r_b} will be used to chop off the face $K \cap H_{\mathbf{a}_r, b}^-$ from K , which will require lifting cuts from the face to K using Lemma 30. Once the face has been removed, we add the CG cut induced by \mathbf{a}_r to move to the next child. The process continues until all the children have been removed and K is empty.

■ **Algorithm 3** EnumToCP(K, \mathcal{T}).

Input: Compact convex set $K \subseteq \mathbb{R}^n$, Enumerative branching proof \mathcal{T} for K .
Output: List \mathcal{L} of CG cuts satisfying $\text{CG}(K, \mathcal{L}) = \emptyset$ and $|\mathcal{L}| \leq 2|\mathcal{T}| - 1$.

- 1 **initialize** $\mathcal{L} = \emptyset$, $r \leftarrow$ root of \mathcal{T} ;
- 2 **if** $K = \emptyset$ **then**
- 3 **return** \emptyset ;
- 4 Retrieve branching direction $\mathbf{a}_r \in \mathbb{Z}^n$ and bounds $l_r \leq u_r$;
- 5 $K \leftarrow \text{CG}(K, \mathbf{a}_r)$;
- 6 $\mathcal{L} \leftarrow (\mathbf{a}_r)$;
- 7 **while** $l_r \leq h_K(\mathbf{a}_r)$ **do**
- 8 $b \leftarrow h_K(\mathbf{a}_r)$;
- 9 $\mathcal{T}_{r_b} \leftarrow$ subtree of \mathcal{T} rooted at r_b ;
- 10 $N' \leftarrow \text{EnumToCP}(F_K(\mathbf{a}_r), \mathcal{T}_{r_b})$;
- 11 $N \leftarrow$ Lift CG cuts in N' from $F_K(\mathbf{a}_r)$ to K using Lemma 30;
- 12 $K \leftarrow \text{CG}(K, N)$;
- 13 $K \leftarrow \text{CG}(K, \mathbf{a}_r)$;
- 14 Append N, \mathbf{a}_r to \mathcal{L} ;
- 15 **return** \mathcal{L} ;

Analysis. We show correctness by induction on $|\mathcal{T}| \geq 1$. Let $r \in \mathcal{T}$ denote the root node with branching direction $\mathbf{a}_r \in \mathbb{Z}^n$ and bounds $l_r \leq u_r$.

We prove the base case $|\mathcal{T}| = 1$. If $K = \emptyset$, then no CG cuts are needed and clearly $0 \leq 2|\mathcal{T}| - 1 = 1$. If $K \neq \emptyset$, then letting r be the root node, we must have $[l_r, u_r] \cap \mathbb{Z} = \emptyset \Rightarrow \lfloor u_r \rfloor < l_r$. Since $h_K(\mathbf{a}_r) \in [l_r, u_r]$, the initializing CG cut we add on line 5 induced by \mathbf{a}_r will make K empty. This follows since after the cut $h_K(\mathbf{a}_r) \leq \lfloor u_r \rfloor < l_r$. The algorithm thus correctly returns $\mathcal{L} = (\mathbf{a}_r)$, where $|\mathcal{L}| = 1 = 2|\mathcal{T}| - 1$, as needed.

Now assume that $|\mathcal{T}| \geq 2$ and that the algorithm is correct for all smaller trees. If $K = \emptyset$ or we do not enter the while loop on line 7, the algorithm correctly returns by the above analysis. So we now assume that $K \neq \emptyset$ and that the algorithm performs at least one iteration of the while loop.

Let K_0 denote the state of K at the beginning of the algorithm. Let K_i, b_i, \mathcal{L}_i , for $i \geq 1$, denote the state of K, b, \mathcal{L} at the beginning of the i^{th} iteration of the while loop on line 7 and let $N_i, i \geq 1$, denote the state of \mathcal{L} at the end the i^{th} iteration. Let $T \geq 1$ denote the last iteration (i.e., which passes the check of the while loop). By design of the algorithm, it is direct to check that $K_i = \text{CG}(K_0, \mathcal{L}_i)$, $\mathcal{L}_{i+1} = (\mathcal{L}_i, N_i, \mathbf{a}_r)$ and that $b_i = h_{K_i}(\mathbf{a}_r)$, $\forall i \in [T]$, where we define \mathcal{L}_{T+1} to be the list of CG cuts returned by the algorithm, $K_{T+1} := \text{CG}(K_0, \mathcal{L}_{T+1}) = \emptyset$ and $b_{T+1} := h_{K_{T+1}}(\mathbf{a}_r) = -\infty$. Note also that $K_i \neq \emptyset, \forall i \in [T]$, since otherwise we would have terminated earlier.

To begin, we claim that

$$b_i \in [l_r, u_r] \cap \mathbb{Z}, \forall i \in [T]. \quad (4.3)$$

To see this, note first that for any compact convex set \mathcal{C} either $\text{CG}(\mathcal{C}, \mathbf{a}_r) = \emptyset$ and $h_{\text{CG}(\mathcal{C}, \mathbf{a}_r)}(\mathbf{a}_r) = -\infty$ or $\text{CG}(\mathcal{C}, \mathbf{a}_r) \neq \emptyset$ and $h_{\text{CG}(\mathcal{C}, \mathbf{a}_r)}(\mathbf{a}_r) = \lfloor h_{\mathcal{C}}(\mathbf{a}_r) \rfloor \in \mathbb{Z}$, where the latter claim follows from convexity and compactness of $\text{CG}(\mathcal{C}, \mathbf{a}_r)$. Since we apply the CG cut induced by \mathbf{a}_r to K directly before the while loop and at the end of every iteration, we immediately get that $b_i = h_{K_i}(\mathbf{a}_r) \in \mathbb{Z}, i \in [T]$. Furthermore, since $\emptyset \neq K_i \subseteq K_0, \forall i \in [T]$, and $\{\mathbf{a}_r \mathbf{x} : \mathbf{x} \in K_0\} \subseteq [l_r, u_r]$, we must also have $b_i \in [l_r, u_r]$.

Given (4.3), for each $i \in [T]$, we see that r_{b_i} is indeed a child of r . Let $\mathcal{T}_{r_{b_i}}, i \in [T]$ denote the subtree of \mathcal{T} rooted at r_{b_i} . We claim that

$$b_{i+1} \leq b_i - 1, i \in [T], \quad \text{and} \quad |N_i| \leq 2|\mathcal{T}_{r_{b_i}}| - 1, i \in [T]. \quad (4.4)$$

To see this, first recall that $\mathcal{T}_{r_{b_i}}$ is a branching proof for $K_0 \cap H_{\mathbf{a}_r, b_i}^-$. In particular, since $K_i \subseteq K_0$, $\mathcal{T}_{r_{b_i}}$ is also a valid branching proof for $K_i \cap H_{\mathbf{a}_r, b_i}^- = F_{K_i}(\mathbf{a}_r)$. By the induction hypothesis the call to EnumToCP ($F_{K_i}(\mathbf{a}_r), \mathcal{T}_{r_{b_i}}$) on line 10 therefore correctly returns a list N'_i of CG cuts satisfying $\text{CG}(F_{K_i}(\mathbf{a}_r), N'_i) = \emptyset$ and $|N'_i| \leq 2|\mathcal{T}_{r_{b_i}}| - 1$. Furthermore, by Lemma 30, the lifting N_i of N'_i to K computed on line 11 satisfies $|N_i| = |N'_i| \leq 2|\mathcal{T}_{r_{b_i}}| - 1$, as needed, and $\text{CG}(K_i, N_i) \cap H_{\mathbf{a}_r, b_i}^- = \text{CG}(F_{K_i}(\mathbf{a}_r), N'_i) = \emptyset$. Letting $K'_i = \text{CG}(K_i, N_i)$, by compactness of K'_i we therefore must have $h_{K'_i}(\mathbf{a}_r) < b_i$. Recalling that $K_{i+1} = \text{CG}(K'_i, \mathbf{a}_r)$, we see that $b_{i+1} = h_{K_{i+1}}(\mathbf{a}_r) \leq \lfloor h_{K'_i}(\mathbf{a}_r) \rfloor \leq b_i - 1$, as needed.

From the above, we see that the procedure clearly terminates in finite time and returns a list \mathcal{L}_{T+1} satisfying $\text{CG}(K_0, \mathcal{L}_{T+1}) = \emptyset$. It remains to bound the size of $|\mathcal{L}_{T+1}|$. Since we add 1 CG cut before the while loop, and at iteration $i \in [T]$, we add $|N_i| + 1$ CG cuts, the total number of cuts is

$$1 + \sum_{i=1}^T |N_i| + 1 \leq 1 + \sum_{i=1}^T 2|\mathcal{T}_{r_{b_i}}| \leq 2|\mathcal{T}| - 1,$$

where the last inequality follows since the sum is over subtrees rooted at distinct children of r , noting that $|\mathcal{T}| = 1 + \sum_{b \in [l_r, u_r] \cap \mathbb{Z}} |\mathcal{T}_{r_b}|$. This completes the proof. \triangleleft

4.1 Upper Bounds for Tseitin Formulas

Proof of Theorem 3. As explained in the introduction, given Theorem 4, it suffices to show that the Beame et al [5] SP refutation is in fact enumerative. We thus describe their refutation briefly to make clear that this is indeed the case.

We start with a Tseitin formula indexed by a graph $G = (V, E)$, of maximum degree Δ , together with parities $l_v \in \{0, 1\}, v \in V$, satisfying $\sum_{v \in V} l_v \equiv 1 \pmod{2}$. We recall that the variables $\mathbf{x} \in \{0, 1\}^E$ index the corresponding subset of edges where the assignment \mathbf{x} is a satisfying assignment iff $\sum_{e \in E: v \in e} x_e \equiv l_v \pmod{2}, \forall v \in V$.

The Beame et al refutation proceeds as follows. At the root node r , we first divide the vertex set $V = V_1^r \cup V_2^r$ arbitrarily into two parts of near-equal size. We then branch on the number of edges crossing the cut

$$x(E[V_1^r, V_2^r]) := \sum_{\{v_1, v_2\} \in E, v_1 \in V_1^r, v_2 \in V_2^r} x_{v_1, v_2} \in [0, |E[V_1^r, V_2^r]|].$$

Let c be the child with $x(E[V_1^r, V_2^r]) = b$. c chooses $i_c \in \{1, 2\}$ such that $\sum_{v \in V_{i_c}^r} l_v \not\equiv b \pmod{2}$, corresponding the set of vertices still containing a contradiction. From here, again c partitions $V_{i_c}^r = V_1^c \cup V_2^c$ into two near-equal pieces. We now branch twice: we first branch on the number of edges crossing the cut $x(E[V_1^c, V_2^c])$, creating corresponding children, and at each such child, we branch on number of edges crossing the cut $x(E[V_1^c, V \setminus V_1^c])$. From here, every child c' , two levels down from c , can decide which set of vertices V_1^c or V_2^c still contains a contradiction. The process continues in a similar way until we find a contradicting set corresponding to a single vertex v . At this point, one constructs a complete branching tree on all possible values of the edges outgoing from v . This completes the description.

It is clear from the description, that every branching decision is enumerative. As shown in Beame et al, the above SP refutation has length $2^\Delta (n\Delta)^{O(\log n)}$. Theorem 4 shows that one can convert it to a CP refutation of the same length. This completes the proof. ◀

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