Book of Abstracts

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Multigrid Methods for Nonconforming Finite Elements

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It is an old problem whether convergence proofs for multigrid methods provide good informations on the quality of multigrids algorithms. This question arises in particular when one sees the proofs for nonconforming methods since in these cases the main ideas are often hidden behind technical details.

We believe, however, that one can learn something if one concentrates on the essential points. In this way we understand e.g. why the algorithm for the Morley element by Brenner is superior to the early version of the algorithm by Peisker and the author.

The analysis of multigrid schemes often refers to a *smoothing property*

$$\|S^\nu u_h - u_h^*\|_X \le c(\nu) h^{-p} \|u_h - u_h^*\|_Y$$

with $c(\nu) \to 0$ for $\nu \to \infty$ and an *approximation property*

$$\|v_h - P v_{2h}\|_Y \le c_0 h^p \|v_h\|_X .$$

Here S denotes the smoothing operator, v_{2h} is the coarse grid approximation to the error v_h. The projector

$$P : S_{2h} \to S_h \quad \text{or} \quad P : S_{2h} \oplus S_h \to S_h$$

describes the transfer of the finite element functions between the grids. It may be abandoned if $S_{2h} \subset S_h$ and the natural injection is used.

Typically, Y and X are endowed with norms $\|\cdot\|_{\ell,h}$ and $\|\cdot\|_{m,h}$ where $\ell < m$ and $m - l = p$. The spaces may be considered as discrete analoga of $H^\ell(\Omega)$ and $H^m(\Omega)$, resp. In the simplest case, i.e., when differential equations are treated by conforming elements, then usually $\|\cdot\|_Y$ equals $\|\cdot\|_{0,h} = \|\cdot\|_0$ or $\|\cdot\|_{1,h}$ and $\|\cdot\|_X = \|\cdot\|_{2,h}$.

We note that 2 modifications are typical for the treatment of nonconforming elements.

1. It is shown that the the transfer operator P is bounded with respect to the norm $\|\cdot\|_Y$.

   $$\|P v_{2h}\|_Y \le c \|v_{2h}\|_Y .$$

   In particular, the boundedness is easily verified if P is an orthogonal projector for some equivalent norm. In this way, the Crouzeix-Raviart element can be

easily treated, see e.g. Braess and Verfürth, and also the Morley element, see Brenner.
After having the boundedness of P, it is enough to establish the approximation property for $v_h - v_{2h}$ instead of $v_h - Pv_{2h}$.

2. Usually the approximation property is established via a duality argument. We will also use duality techniques although in a different way. In principle, following the author and Verfürth we write

$$\|v_h - v_{2h}\|_Y \leq \|v_h - \phi\|_Y + \|v_{2h} - \phi\|_Y,$$

where ϕ is the solution of some auxiliary variational problem. Moreover v_h and v_{2h} are finite element approximations of ϕ. Since the terms with v_h and v_{2h} are now separated, it is no longer a problem that the elements are nonconforming.

When the Crouzeix-Raviart element is treated in this framework, we intuitively expect that the pairing $\|\cdot\|_Y = \|\cdot\|_{0,h}$ and $\|\cdot\|_X = \|\cdot\|_{2,h}$ is optimal. With this the difference of the orders equals 2, i.e. the order of the differential equations. Unfortunately, the situation is not so simple. The proof contains the assumption that there are *sufficiently many smoothing steps*. If, on the other hand, a stepsize control is added in the "non-optimal" case $\|\cdot\|_Y = \|\cdot\|_{1,h}$, then one smoothing step turns out to be sufficient.

The situation for the Morley element seems to be more complicated if we look into the literature. In the paper by Peisker and the author, the pairing $\|\cdot\|_Y = \|\cdot\|_{1,h}$ and $\|\cdot\|_X = \|\cdot\|_{2,h}$ was chosen. A projector was chosen which is bounded for $\|\cdot\|_{2,h}$. It refers to the wrong norm. So a special transfer step was needed. In 1989, Brenner chose the same pairing. She could do without the transfer step, since she got the boundedness for the coarse space. Actually, she started from as projector with respect to the $\|\cdot\|_{0,h}$ norm. — When we now look once more at the general framework, we see that the proofs can be replaced by simpler ones.

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Multiscale Computation in Physics and Chemistry

A. Brandt

Abstract

Several kinds of multiscale computations become indispensable for advancing chemistry and physics. Often, several such kinds can indeed serve in the solution of one physical problem, and together they may cut its computational cost by many orders of magnitude.

Examples to be discussed: the calculation of nature's elementary forces and particles (quantum chromodynamics); simulations of many- body systems (molecular dynamics); and various aspects of the Schroediger wave equation.

A Generalized Multigrid Theory

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Multigrid methods are analyzed in the style of standard iterative methods. By rewriting standard multigrid algorithms in a nonstandard (but equivalent) form, a great simplification in multigrid theory can be achieved. A basic error bound is derived in terms of residuals on neighboring levels. The terms in this bound derive from the iterative methods used as smoothers on each level and the operators used to go from a level to the next coarser level. This bound is correct whether the underlying operator is symmetric or nonsymmetric, definite or indefinite, and singular or nonsingular. Any iterative method is allowed as a smoother or rougher in the multigrid cycle. While standard multigrid error analysis typically assumes a specific multigrid cycle (e.g., a V, W, or F cycle), analysis for arbitrary multigrid cycles is provided. This theory applies directly to aggregation-disaggregation methods used to solve arbitrary systems of linear equations.

One of the advantages of this theory is that all of the parameters are available during execution of a computer program. Hence, adaptively changing levels can be achieved with certainty of success. This is particularly important for solving problems in which there is no known useful convergence analysis. Two problems arising in modeling combustion problems (flame sheets and laminar diffusion flames with full chemistry) will be discussed.

While this theory is quite general, it is not always the correct approach when analyzing the convergence rate for a given problem. A discussion of when this theory is useful and when it is hopelessly nonsharp will be provided.

Multi-grid Methods for Calculation of Flows in Multi-Component Geometries

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We consider the flow of an incompressible fluid past a multi-component geometry. The different solid components may be stationary or may move freely relative to each other. In both cases one is faced with two non-trivial problems: One is the generation of a basic mesh of "good" quality; and the second issue is a numerical scheme that can solve efficiently the discrete approximation to the Navier-Stokes equations. Both issues are more accentuated when one or more of the objects move relative to each other: A new grid has to be generated in each time step, followed by a re-zoning step. Alternatively, one may use a moving grid system, such that grid points are atteched to the solid objects. In both situations (and in particular in the latter option) one often gets highly skewed grids with poor accuracy. Most "smoothing" operators are also considerably less efficient on such grids. These two aspects of numerical accuracy and efficiency are addressed in this paper.

We propose to use systems of overlapping grids, and locally refined subgrids, such that each sub-grid may move freely relative to the others. In this system, the shape of the grids is time independent, and motion is accounted for only through the motion(i.e. acceleration) of the grid. For accuracy reasons one may have to add locally refined grids at some instances. To enhance efficiency, one may not only add locally refined grids, but also may delete such sub-zones, once they are found to be unimportant for the global accuracy. The incompressible Navier-Stokes equations are discretized on the above mentioned grid system. The solution procedure is based on Multi-Grid (MG) iterations. The main issues associated with the grid system are the inter-zonal information exchange, and the impact of these 'internal boundary conditions' updates on the total efficiency of the MG process. Another issue that we consider is an "adaptive" relaxation scheme for situations where the flow/grid causes degeneration of the basic smoother. Such situation occur when the flow looses its isotropic character or the grid has cells with aspects ratio that differs considerably from unity.

Examples shall be given for flows with heat-transfer and IC-engine situations.

Some results concerning the (parallel) application of zonal grid solvers on a cluster of workstations shall also be given.

The Frequency Decomposition Multi-Grid Method

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In the past, many connections between the multi-grid method and the ideas of domain decomposition (more precisely: subspace decomposition) have been shown. Also the Frequency Decomposition Multi-Grid Method (FDMGM) is a special variant of the additive Schwarz iteration. Its characteristic is the use of more than one coarse grid during the coarse-grid correction.

The interpretation of FDMGM as a domain decomposition method allows new convergence proofs (at least in the positive definite case). Usually, standard multi-grid convergence proofs become complicated, when one is interested in robustness. In that case, the problem involves one or more parameters (besides the grid size), while the convergence speed should be guarenteed independently of these parameters.

In the lecture, we raise the question of uniform convergence of FDMGM for a whole class of problems. It will be a rather trivial result that having proved convergence for a set of fixed problems (i.e., for a set of certain positive definite matrices), we obtain uniform convergence estimates in the class consisting of all non-negative linear combinations of these positive definite matrices.

The FDMGM leads to a subspace decomposition, where the subspaces are orthogonal (with respect to the Euclidean norm, not with respect to the energy norm!). This property and the nestedness (level structure) are similar to the wellknown properties of wavelets.

Multigrid methods for computing propagators in lattice gauge theory

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ABSTRACT

Gauge theories, a particular kind of Quantum Field Theories, describe some basic interactions in nature: Quantum Chromodynamics (strong interactions) and Glashow-Salam-Weinberg model (weak and electromagnetic interactions). At present, the lattice regularization of these theories (lattice gauge theory) provides the only known nonperturbative tool to investigate some very important properties, e.g., the hadron mass-spectrum, weak interaction matrix elements etc. . These results are obtained by Monte Carlo simulations, requiring immense amounts of computer time. In state-of-the-art simulations of lattice QCD in $3+1$ dimensions, using more than 10,000 one-processor-hours on a CRAY-YMP, more than 90% of the CPU-time is spent computing propagators, i.e., inverting the Dirac operator, a very large matrix. Deterministic multigrid methods are promising candidates for accelerating this part of the simulation; in this way, more precise and more reliable physical predictions can be made using the same amount of computer time. Several competing strategies, proposed recently by different groups, will be presented and discussed.

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Adaptive multigrid on distributed memory computers

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The efficient use of parallel distributed memory machines for adaptive grid problems still requires substantial efforts in research and development. Very often current implementations on these machines are based on static grids generated in advance and distributed to the processors when the parallel application is started. With such static grids, the grid is an input into the computation and the user has a full control of it. It is therefore possible to optimize the parallel application based on the user's knowledge in regard to load balancing, which is most crucial for the overall efficiency of the application.

Generally, however, adaptive grids are a result of the computation. Dynamically mapping the work load to the processors and achieving load balance are tasks which have to be performed at run time. Careful strategies must be employed in order not to destroy the parallel efficiency through communication overhead. In this paper, we present results obtained for the multi-level adaptive techniques (MLAT), combined with the use of block-structured grids which are widely used in aerodynamic applications. Such grids allow the treatment of geometrically complex domains while keeping the local data structure (within a single process) regular. On each level of MLAT, the union of local refinement areas builds a new (not necessarily connected) block-structure.

We will discuss the communication and mapping strategies, the way in which local refinement areas are generated and embedded into block-structures and then distributed to the available processors. We will outline some features of the GMD Communications Library, which covers all the tasks necessary for the mapping, load balancing, communication (local as well as global, including communication between different levels), process generation, etc. This library is based on a portable interface, and can therefore be used on a wide variety of parallel machines.

Results will, in particular, be presented for the Euler equations. The numerical method adopted here is the fully implicit multigrid technique using Osher's shock-capturing scheme. The emphasis is laid on the adaptive refinement of the shock position. High parallel efficiencies (better than 70%) are already obtained for relatively small problems. Clearly, the smaller the numerical work per grid point, the more critical it becomes to obtain high efficiencies. We will demonstrate corresponding effects for scalar problems with singularities induced by the shape of the domain (re-entrant corners). Based on some simple analysis, we will point out the crucial points of parallel adaptive multigrid and make some theoretical comparisons to different approaches (e.g. the asynchroneous fast adaptive composite grid method, AFAC).

Multicomputer Solution of Parabolic Problems

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Introduction. We shall discuss the numerical solution of parabolic partial differential equations on multicomputers. We identify the algorithms appropriate for solving problems on small-scale, medium-scale, large-scale and massively parallel systems. The talk will focus on *space-time concurrent multigrid waveform relaxation methods*, i.e., waveform relaxation techniques which operate concurrently in space and time. Timing results will be given obtained on four generations of Intel multicomputers (iPSC/2, iPSC/860, Touchstone Delta, Paragon) and the CM/2 Connection Machine.

Time-stepping. The numerical solution of a time-dependent partial differential equation is usually calculated by using a time-stepping method. This is an inherently sequential process; the computation proceeds from time level to time level. The computations on each time level can be parallelized by applying a spatial grid partitioning, see Fig. 1.

The parallel overheads associated with communication of intermediate results, with processor-idling and load imbalance seriously degrade parallel performance. This is illustrated in [1], where we studied the parallel performance of explicit, implicit and line-implicit time-stepping schemes.

Multigrid waveform relaxation with spatial grid decomposition. In [2, 3], we have investigated the use of a waveform relaxation algorithm for solving parabolic problems. The algorithm avoids some of the limitations of the standard schemes by calculating simultaneously the solution at many time levels or along a continuous time-window. It is based on *waveform relaxation*, a highly concurrent technique for solving large systems of ordinary differential equations, and *multigrid*. The resulting algorithm is a rapidly convergent iterative method for solving systems of differential equations obtained by semi-discretization of parabolic PDEs.

The method can be parallelized by using a spatial grid partitioning similar to the one applied in time-stepping schemes. However, each process is now responsible for updating functions, i.e., the approximations to the solutions of the ODEs associated with each grid point. In other words, each process is assigned to a block of unknowns in the combined space-time grid, see Fig. 1. Extensive timing results illustrate a significant performance gain when compared to concurrent implementations of classic solvers. This is due to a dramatically reduced communication complexity.

Space-time concurrent multigrid waveform relaxation. The waveform relaxation method with spatial grid partitioning performs satisfactorily on medium-scale parallel processors (with, say, up to 100 processors). On large-scale multicomputers (with, e.g., 1000 processors) a loss of efficiency is caused by several reasons. The number of nodes may exceed the number of spatial grid points, e.g., on coarse spatial grids encountered in a multigrid hierarchy. The resulting loss of efficiency is much more severe on large multicomputers than it is on small ones. Even on relatively fine grids the number of

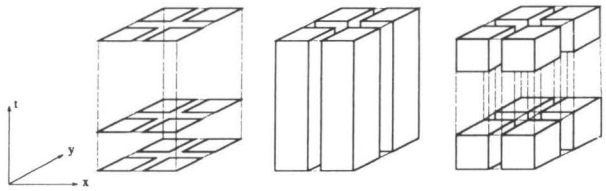

Figure 1: Grid partitioning on a rectangular domain: time-stepping (left), standard waveform relaxation (middle), space-time concurrent waveform relaxation (right).

spatial grid points may be too small to rely only on spatial grid partitioning, in which case good load balance is difficult to achieve.

The available computing power of large-scale parallel processors can be harnessed by using waveform relaxation methods that operate concurrently on different time levels, [4]. In this talk we shall discuss several strategies that employ concurrency in space and in time simultaneously. Each processor is then responsible for the calculations in a block of space-time grid points as shown in Fig. 1. Most of the operations of the multigrid waveform relaxation algorithm can be parallelized straightforwardly in the time-direction. The only exception is a linear recursion, arising in the core of the ODE solver, which is parallelized by using a partitioning or cyclic reduction method.

Grid-point concurrent multigrid waveform relaxation. Multigrid waveform relaxation is very well suited for implementation on parallel systems of SIMD type. These are systems in which a very large number of (simple) processors operate in lock-step mode, i.e., executing an identical sequence of instructions on their local data. Assigning one processing element per grid point in the combined space-time domain leads to an algorithm with extremely low parallel complexity. It can be shown that grid-point concurrent multigrid waveform relaxation can solve a parabolic problem on an $N \times N \times N$ space-time grid (2 space and 1 time dimension) in only $O(\log^3(N))$ parallel steps.

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Multilevel Solution of Integral and Integro-differential Equations in Contact Mechanics and Lubrication

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This paper describes the development of a multilevel solver for the calculation of the pressure and film thickness in lubricated concentrated contacts as occurring in ball bearings and between gear wheels. Following an introduction into the subject, the contact model of two elastic surfaces in relative motion pressed together by a certain load is presented together with the equations from which pressure and film thickness are solved. The elastic deformation of the surfaces is described by a Fredholm integral of the first kind. For two reasons this multi-integral plays an essential role throughout the entire paper. Firstly from a relaxation point of view as special measures must to be taken to overcome stability problems and to obtain a relaxation that efficiently reduces high frequency components. Secondly because, when discretized on a given grid, its evaluation requires $O(n^2)$ operations if n is the total number of nodes. As a result, when using only the standard multilevel solution techniques, this evaluation will determine the overall computing time used by the solver, and its efficiency will be far removed from the usual multigrid efficiency. This problem was ovecome by the implementation of a recently developed multilevel technique named "multilevel multi integration". Exploiting the smoothness of the kernel in the multi-integral this technique allows evaluation in $O(n \ln n)$ operations without loss of accuracy. As $\ln n$ increases only slowly with increasing n this technique, when merged with the developed relaxation in the usual multigrid coarse grid correction cycle, yields a fully efficient multigrid solver, as is demonstrated by the calculational results. With the developed solver, lubricated concentrated contacts can be (and actually are) analysed in detail. In particular, its efficiency allows very detailed simulations with dense grids, which is a requirement to investigate the influence of local surface features on performance and operational life of the contact. The paper gives an example of the pressure and film thickness calculated for a specific set of contact conditions and concludes with some directions of future research.

On Robust Multi-Grid Methods

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Multi-grid methods are well known as very fast and efficient solvers for large sparse systems and are widely used in a large number of applications. However, it is also known that this efficiency strongly depends on the type of equations treated. Thus the application of multi-grid to singularly perturbed problems has to overcome typical difficulties. To scope with these problems we need robust multi-grid, i.e. a method yielding a convergence rate independent of the singular perturbation parameter.

This topic was first investigated by Wesseling in 1980. Since then robust multi-grid methods have been applied to many problems, in particular in CFD, and the first ideas have been theoretically as well as practically investigated and extended. Topics of special interest are: Robust methods for convection dominated problems, for systems of partial differential equations, robustness and adaptivity, robust methods for unstructured grids, and parallelization of robust multi-grid. The lecture will give an overview over robust multi-grid methods, the algorithmical and theoretical development, as well as applications.

Contributed papers

Contributed papers

1. X.S. Bai and L. Fuchs, *RIT Stockholm, Sweden*
2. A. Borzi, *SISSA, Trieste, Italy*
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51. Z.W. Zhu, C. Lacor and Chr. Hirsch, *Vrije Univ. Brussel, Belgium*

MULTI–GRID METHOD APPLIED TO CALCULATION OF TURBULENT REACTING FLOWS IN FURNACES

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The combustion process, such as hydrocarbon oxidation, often contains several hundreds steps of elementary reactions. These processes can be described by a system of N+1 Partial Differential Equations (PDEs), where N is the number of the elementary reactions. Often, N is very large so that an important aspect in the numerical computation of turbulent reacting flows is the choice of numerical schemes in order to obtain high numerical efficiency for engineering applications. Earlier work mainly utilized SIMPLE–type schemes with point– or line– relaxations [1–2]. The method is simple, however, the convergence rate of the iteration process is asymptotically slow. This behavior becomes more serious when the grid is refined. For numerical efficiency two issues are important. One is how to use less iterations to get a pre–defined error in solving the discrete equations; the other is how to use minimal mesh points to get a required level of accuracy. The *aims* of this paper are 1) to demonstrate the application of the Multi–grid method (MG) with a local grid refinement techniques, to the simulation of the turbulent reacting flows; 2) to apply the code to analyzing the flows in furnaces with respect to geometrical parameters.

The governing equations that we use consist of the Reynolds averaged Navier–Stokes equations together with a k–ε turbulent model and Eddy–Dissipation–Concept for modeling the turbulent reaction rates. A reduced (one– or four– step) chemical reaction scheme is used. The discretisation is done on a staggered grid. All terms with the possible exception of the convective terms are approximated by central differences. The convective terms in all the equations are discretized by the Hybrid scheme: Central differences if the local Peclet number is less than 2, otherwise upwind differences.

Currently, the system of the discretized equations is solved by a sequence of relaxation sweeps: (a) relaxing U_j (velocity components), p (pressure), (b) k (turbulent kinetic energy), ε (dissipation rate of k), μ_{eff} (effective viscosity), (c) h (enthalpy), m_α (mass fractions), f (mixture fractions), (d) for ρ (density), μ (laminar viscosity). This procedure is repeated until the iterations converge. In step (a) the coupling of U_j and p is handled by a Distributive Gauss Seidel relaxation. In steps (a), (b), (c) the relaxation is pointwise and lexicographically symmetric.

The pointwise relaxation is simple but slow. In order to accelerate the convergence, a MG method is used. The solution procedure starts on a global grid doing several V–cycle MG relaxations in the Full Approximate Storage (FAS) mode [3,4]. Locally refined sub–grids can be integrated into the MG V– cycling. In the MG procedure, The transfer to coarser grids of scalars is done by volume averaging, and for the velocity components by area (flux conserving) averaging. Mass flux conserving restriction is a necessary condition for coarse grid convergence. This can be done by applying a conventional area

restriction on the mass fluxes $\rho\mathbf{U}$ ($\mathbf{U}$ is the velocity vector) rather than the velocity components. The corrections are interpolated to fine grids by trilinear interpolations.

The numerical methods described above have been applied to furnace flow calculations. Our experience with the MG solver for reacting flow situations shows that the uncoupled (sequential) scheme that is described above, is non–optimal but still offers a considerable improvement over other single grid methods. Figure 1 depicts a typical speed–up of the MG method compared to a single–grid version of the same code. The speed–up depends on the grid–size. MG methods are being more and more efficient, relative to the single grid approach, as the mesh spacing is refined.

In the full paper we shall describe methods for improving MG efficiency for models with multiple step reactions.

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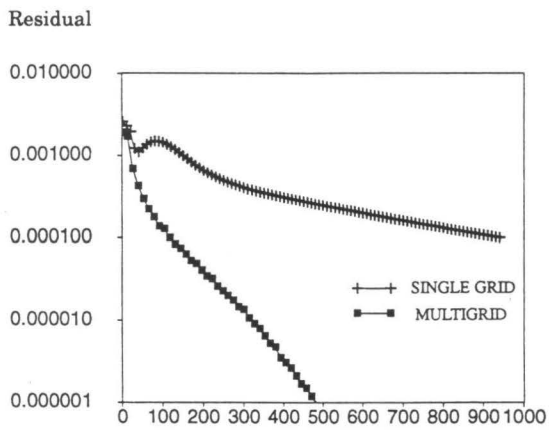

Figure 1. The convergence history for calculation of turbulent reacting flow in a single burner combustor.

Burgers Equation and Multi-Grid Techniques

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We analyze a discretization of the Burgers equation. For the resulting discrete problem we prove the existence of solutions for a class of initial-boundary conditions. Then we use this introductory evolution equation to present an algebraic multi-grid method for fluid flow problems. This method consists of a smoothing iteration, transfer of defects on a coarser space, and prolongation of obtained corrections to the previous space. But these operators are here interpreted in terms of a direct multi-grid substitution method. This fact seems to suggest that multi-grid methods are "approximating" versions of suitable chosen direct methods. This algorithm is compared with a standard multi-grid method by means of numerical experiments. It turns out that the algebraic algorithm provides the exact solution of the discrete elliptic problem to be solved at each time step.

A Multi-Grid Method for the Resolution of Thermodynamic Bethe Ansatz Equations

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Abstract

We present a multi-grid algorithm in order to solve numerically the thermodynamic Bethe ansatz equations. The solution of this system of non linear integral equations allows to extract any measurable quantity of the conformal field theory reached in the ultraviolet limit.

Program available from: CPC Program Library, Queen's University of Belfast, N. Ireland

Multigrid Parallelization of Parabolic Equations

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Algorithms for parabolic differential that exist now are ususlly sequential either in time or in space. In the first case, using values of an unknown function for given time step $(u(x,t))$, one calculates the values of $u(x,t+dt)$. Second possibility is using "waveform relaxation", i.e. replacing the poarabolic equaiton by the system of ODE.

We treat a discrete system of equation as entire system by performing 1-FMG algorithm. Red-black relaxation and semi-coarsening are used.

It will be shwon that the algorithm we present here is parallelizable in space and time. The convergence factor of sequence of V-cycle is below .14. By 1-FMG algorithm the algebraic error is reduced below the discretization error.

A MULTI-DIMENSIONAL SOLUTION ADAPTIVE MULTIGRID SOLVER FOR INVISCID CASCADES

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Two issues are playing an ever increasing role in current CFD research: the development of methods suitable for vector and parallel computers and the application of local refinement strategies on unstructured grids as a means for obtaining high quality results for complex flow problems at reasonable computer costs. Both of these items have increased the interest towards employing explicit schemes as smoothers in multigrid methods as well as developing space discretizations based on compact stencils.

In the last three years, an original procedure has been developed for optimizing the coefficients and time step of explicit multi-stage schemes, in order to design an efficient smoother to be used in conjunction with a multigrid method for solving multi-dimensional hyperbolic equations [1]. This approach, based on a two dimensional linear Fourier analysis, is not limited to space discretizations based on dimensional splitting so that it can be applied to some genuinely multi-dimensional upwind schemes characterized by low cross wind diffusion such as fluctuation splitting (FS) schemes [2,3]. This kind of space discretization has been crucial to the development of genuinely multi-dimensional methods for compressible inviscid flows based on a decomposition of the Euler equations into an equivalent set of scalar equations with solution dependent propagation directions, originally proposed by Roe [4]. Successful applications and improvements of this methodology [5,6] have been obtained also thanks to the development of a conservative linearization which can be performed analitycally [2], the flow variable gradients being defined uniquely on triangular grids. A major step towards making such a numerical technique robust and efficient has been achieved in [3,7], where the explicit multigrid (FAS) strategy has been extended to the Euler equations using the FS N-scheme and a new wave decomposition model. In this work, a local adaptive mesh refinement technique has been combined with the wave decomposition method and the multigrid acceleration strategy. Quadrilateral cells are subdivided into four *kid-cells*, if needed, using a quad-tree data structure developed at CWI [8]. The percentage of cells to be refined is assigned by the user, the sensor being based on the value either of the pressure gradient or of the divergence of the velocity. In this way, the present optimal smoother multigrid method applied to the FS N-scheme space discretization can be employed to solve complex flow configurations at very reasonable computer costs. Results will be presented for subsonic, transonic and supersonic flows in a channel with a circular bump and for subsonic and transonic flows through cascades of NACA-0012 and VKI LS59 profiles.

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ABSTRACT

"Multigrid acceleration of the 2D Euler equations applied to High Lift systems"

by

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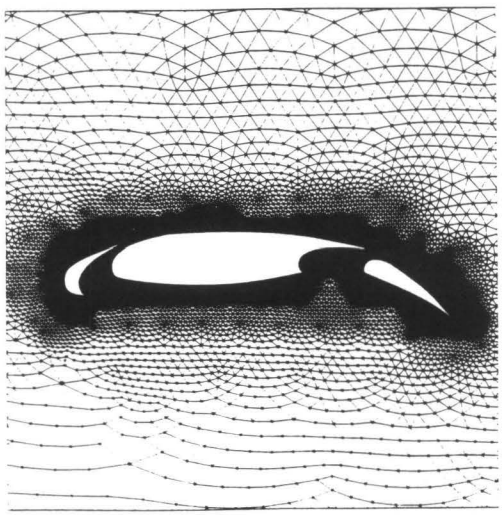
A 2D Euler solver on unstructured meshes has been developed at NLR (1991-1992) within a turn around time reduction feasibility study for complex configuration analysis.
Part of this work consisted of implementing a multigrid convergence acceleration algorithm, based on the defect correction technique.

In this paper the multigrid technique used to accelerate the convergence of the 2D Euler equations, discretised on unstructured meshes around complex High Lift devices, will be described.

A part of this description will treat the generation of grids suitable for the multigrid technique.
Further the multigrid method and its parts will be described. The consequences for adaption will be outlined.
Applications relevant for High Lift aerodynamics (see figure) will be shown.
Conclusions will be drawn.



Global view of the fine grid around the NLR422 three element airfoil

Separation Techniques for Multilevel Eigenvalue Solvers applied to a Nonlinear Schrödinger Eigenvalue Problem

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Extended Abstract

Large scale eigenvalue problems arise in physics and engineering, e.g. by solving Schrödinger equations for solid state computations or for analysis of microwave components. These problems are often nonlinear. Several multilevel eigenvalue techniques have been previously developed, analyzed and implemented to solve such problems. Some major tasks, not fully investigated in previous multilevel approaches are: 1) the separation of eigenvectors by coarse level techniques, 2) the treatment of equal and close clustered eigenvalues, 3) the efficient treatment of nonlinearities, 4) the design of adaptive, robust solvers. To compute q eigenvectors of size N on finest level, previous approaches usually need $O(q^2N)$ operations, mainly due to fine level separation techniques (Ritz projection). Moreover, the nonlinear eigenvalue problems are usually solved by a relatively large number of selfconsistent iterations.

The approach presented in this paper uses an adaptive simultaneous FMG - FAS algorithm which performs the eigenvectors separation on coarse levels, thus reducing the work to compute the q eigenvectors to $O(qN)$ work. The many selfconsistent iterations used to solve nonlinear problems by previous algorithms are reduced to only a few, resulting in large savings of computational work. The algorithm is efficient for close and equal eigenvalues also, and its robustness is due to an adaptive identification and treatment of clusters on different levels coupled with robustness tests.

An analyzed test problem is the computation of the eigenvectors corresponding to the smallest

eigenvalues (in modulus) of a discretized Schrödinger Nonlinear Eigenvalue problem of Hartree-Fock type:

$$\Delta u_i - (V + \epsilon W)u_i = \lambda_i u_i \tag{0.1}$$

$$\Delta W = -c_1 \sum_{i=1}^{q} u_i^2 + c_2 \tag{0.2}$$

$$\|u_i\| = 1, \quad \int W = 0 \tag{0.3}$$

The first q eigenvectors $u_1, ..., u_q$ corresponding to the smallest (in modulus) eigenvalues $\lambda_1, ..., \lambda_q$ are sought, where eigenvectors in degenerate eigenspaces should be orthogonal. The problem is considered in 2-D and 3-D with periodic boundary conditions. V is a given linear potential, W is a nonlinear potential defined by (0.2), and ϵ is a constant. A multilevel algorithm to solve such problems was presented by Davstad 1992. Our main contribution is in the separation approach which is done on coarse levels, thus reducing the computational work to $O(qN)$.

The central features of the present approach can be outlined as follows. The algorithm is based on separation of eigenspaces and of eigenvectors in eigenspaces, simultaneously treated with the nonlinearity on all levels. The algorithm is efficient when the approximate eigenvectors and W are close to the exact solutions. This is achieved by a coarse level continuation, the FMG and the simultaneous processing of eigenvectors and W. The single level separation combines a Generalized Rayleigh Ritz Projection with a selfconsistent updating of W.

The results of the numerical tests for the problem (0.1) - (0.3), show that: 1) Coarse level separation was obtained for nonlinear eigenvalue problems admitting close clustered and equal eigenvalues. 2) The algorithm achieved the same accuracy, using same amount of work (per eigenvector) as the Poisson multigrid solver. Solutions for second order schemes using the 5-point and 9-point Laplaceian (in 2-D and 3-D) were obtained by 1-FMG-V(1,1) in $O(qN)$ work, for first q eigenvectors of size N on finest grid. The memory required is of order $O(qN)$, allowing to solve very large problems. 3) The simultaneous ML schemes, reduced the many selfconsistent iterations to solve the nonlinearity to a single simultaneous iteration.

Multigrid Methods for SIMD Machines

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ABSTRACT

We begin this talk with a discussion of parallel machines generally and of SIMD (single instruction multiple data) machines specifically, attempting to justify why these machines are needed for today's "grand challenge problems." We then make the case that the solution of implicit systems on these machines presents a real bottleneck and that multigrid methods are the natural choice for eliminating this bottleneck. We consider four specific multigrid methods which are suitable for problems of the type

$$-\nabla \cdot (D\nabla U) + \sigma U = F,$$

in a domain Ω of R^2 or R^3, where $D^i, i = 1, 2, [3], \sigma$, and F jump by orders of magnitude across internal interfaces and where D can be anisotropic; this type of problem arises in many practical contexts, including petroleum reservoir simulation, nuclear reactor safety modeling, etc. These four methods are semicoarsening and standard coarsening black box multigrid, semicoarsening multigrid with multiple semicoarsened grids and BHT (also known as the ray multigrid method or as the frequency decomposition multigrid method). The first two methods were designed to handle discontinuous and anisotropic coefficients. We show how to modify the third and fourth methods so that they are robust with respect to discontinuous and anisotropic coefficients. Finally, we attempt to compare the relative efficiency of these four methods on the CM-200 and the CM-5.

Multigrid methods for steady Euler equations based on multi-stage Jacobi relaxation.

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Upwind methods of flux-difference type used on steady Euler equations generate, in the first order accurate form, a set of discrete equations of so-called positive type. This set can be solved by any classical relaxation method in multigrid form. The set of discrete equations generated by a higher order accurate form cannot be solved in the same way. The common approach is then to use a defect correction procedure. This procedure proves to work well in many applications. The speed of convergence is however largely determined by the defect correction cycle which forms an outer iteration. Sometimes the convergence is found to be rather dissapointing, especially when the first order and the higher order solutions differ significantly. It can be expected that if the higher order approximation also could be used in the multigrid itself a better performance could be obtained. A second difficulty is that often convergence cannot be obtained unless a suitable initial flow field is specified, i.e. there is a risk of choosing an initial approximation which is out of the attraction region of the iterative method.

In principle, both difficulties can be avoided by using time stepping methods on the unsteady equations instead of relaxation methods on the steady equations. The higher order accurate discretization can then be used on any grid so that the defect correction becomes unnecessary and convergence is guaranteed starting from any initial field due to the hyperbolicity of the equations with respect to time. Many multi-stage time stepping methods with optimization strategies for the smoothing have been proposed for this purpose in recent years. We cite the methods of Van Leer et al. [1989], von Levante et al. [1990], Catalano and Deconinck [1991], among others.

The drawback of time stepping even with local time stepping, is that the smoothing only can be tuned well for the fastest wave components in the flowfield. This results in a rather poor multigrid performance. As a remedy to this, we propose to use Jacobi relaxation as a basic algorithm, equivalent to single stage time stepping, and to bring in multi-staging in the same way as single stage time stepping is transformed into multi-stage time stepping. This procedure has the advantage that all wave components are first scaled so that, so to speak, they move all with the same CFL-number. This guarantees optimal tuning for all wave components. Nevertheless the hyperbolicity with respect to the relaxation direction, i.e. the ficticious time, is not lost. The principle of combining Jacobi relaxation and time stepping was first suggested by Morano et al. [1991], but not worked out. A preliminary analysis of the possible multigrid performance was made by the authors at the Reading conference [1992].

Here, we present a more in-depth discussion of the possibilities of the multi-stage Jacobi relaxation method.

First, we compare several simple test-cases (structured grids, vertex-centred formulation) for multi-stage Jacobi relaxation and Gauss-Seidel relaxation, both in defect correction form. There, in terms of workunits, the Gauss-Seidel relaxation is much more performant. The Jacobi method, of course, allows easy vectorization and parallelization where the Gauss-Seidel does not. Second, we show several applications where the higher order operator is used in the multigrid method itself. We consider methods where the higher order operator is used on all grid levels and variants where it is only used on the finest grid. Bringing the higher order operator into the multigrid cycle can only be done for the multi-stage relaxation. The performance of these methods is close to the one of the defect correction method with Gauss-Seidel relaxation. In these applications, the Van Leer's linear $\kappa = 1/3$ higher order operator is used. A non-linear TVD second order operator cannot generate smoothing without an additional residual averaging. The performance of TVD-variants with residual averaging is also discussed.

A multigrid method for solving the Boussinesq approximation of the Navier-Stokes equations

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Abstract

We consider the numerical solution of the Boussinesq approximation of the stationary Navier-Stokes equations

$$\begin{aligned} -\frac{1}{Re}\Delta u + u\cdot\nabla u + \nabla p &= \frac{Ra}{Re^2 Pr} T\vec{g} =: f(T) \quad \text{in } \Omega, \\ \nabla\cdot u &= 0 \quad \text{in } \Omega, \\ -\frac{1}{Re\,Pr}\Delta T + \nabla\cdot T &= 0 \quad \text{in } \Omega, \end{aligned}$$

$$u_{|\Gamma} = 0, \qquad T_{|\Gamma_D} = T_D, \qquad \frac{\partial T}{\partial n}_{|\Gamma_N} = 0,$$

where Ω is a bounded twodimensional domain with Lipschitz continuous boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$. The unknown functions u_1 and u_2 for the velocity belong to the space $V = H_0^1(\Omega)$, T for the temperature belongs to the space $S = \{s \in H^1(\Omega) : s = T_D \text{ on } \Gamma_D\}$ and the pressure p to $Q = L_0^2(\Omega)$. For the discretization we use a finite element method based on the nonconforming

P_1/P_0 element of Crouzeix/Raviart. Let us denote the corresponding finite dimensional spaces by $V_h \sim V$, $Q_h \sim Q$ and $S_h \sim S$. Then, the discrete problem reads
Find $u_h \in V_h^2$, $p_h \in Q_h$, and $T_h \in S_h$ such that

$$\begin{aligned}
\frac{1}{Re}(\nabla u_h, \nabla v_h)_h + n_h(u_h, u_h, v_h) - (p_h, \nabla \cdot v_h)_h &= (f(T_h), v_h), \\
(q_h, \nabla \cdot u_h)_h &= 0, \\
\frac{1}{RePr}(\nabla T_h, \nabla s_h)_h + N_h(u_h, T_h, s_h) &= 0,
\end{aligned}$$

for all $v_h \in V_h^2$, $q_h \in Q_h$, $s_h \in V_h$, where $(.,.)_h$ is defined as the sum of the L^2 products over all finite elements. The forms n_h and N_h correspond special upwind discretizations of the convective terms $u \cdot \nabla u$ and $u \cdot \nabla T$, respectively. As a result of the upwind technique we obtain an improved stability of the method in the case of moderate and higher Reynolds numbers. Moreover, we have the M-property of some block matrices for meshes of weakly acute type.

An outer simple iteration is applied to the nonlinear problem, where each linear problem is solved by a full multigrid method. The components of the multigrid solver are the following:

- a smoother of blockwise Gauß-Seidel type , where in each step the 10 degrees of freedom per element are simultanuously updated and
- a prolongation and a restriction which are defined as L^2 projections.

For a benchmark problem we discuss the convergence and accuracy of the finite element discretization with respect to the mesh size h and the Rayleigh number Ra. Furthermore, the convergence rate of the nonlinear iteration and the multigrid method is studied.

Abstract

Renormalization and Multi-grid for Reservoir Simulation

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We present a new approach to multi-grid for the case of strongly varying coefficients in the pressure equation which arise in resevoir simulation. Renormalization is incorporated into the cell centred multigrid method of Wesseling and the new technique is applied to the pressure equation of reservoir simulation.

The flow equations of reservoir simulation consist of a coupled system of hyperbolic conservation laws (for fluid transport) and an elliptic/parabolic equation for the pressure. The coupling between the equations is provided by Darcys law, which defines the fluid velocity to be proportional to the product of medium permeability and pressure gradient. Spatial variations in the permeability field present a difficult challenge to any multigrid method.

Renormalization is a technique which defines effective or restricted cell-centred permeabilities on coarse grids by ensuring that each coarse grid interface flux is equal to the sum of the corresponding fine grid fluxes across a common interface. Renormalization thereby defines a cell-centred permeability restriction operator which delivers appropiate effective permeability fields for use with a discrete equation operator at each multigrid level.

A flux continuous approximation of the pressure equation at the cell interfaces is necessary to allow physically correct discontinuous pressure gradients at interfaces between cells with discontinuities in permeability. For diagonal tensor applications with a five point stencil, use of the harmonic mean of adjacent cell-centred permeabilities to define cell face coefficients ensures flux continuity is maintained on all levels of multi-grid in contrast to standard multi-grid formulations.

The cell-centred renormalization scheme is compared with the standard cell-centred multigrid of Wesseling et al based on a Galerkin formulation with piecewise constant prolongation and the adjoint of linear interpolation as the restriction operator. The Galerkin formulation preserves flux continuity only on the finest multi-grid level. For the renormalization scheme, which preserves flux continuity on all multi-grid levels the choice of solution prolongation and restriction operators is investigated.

Significant improvement in performance is obtained by combining multigrid with renormalization. Improved convergence is demonstrated between the new scheme and the standard Wesseling et al multigrid scheme, for the typical case of a randomly varying permeability field with a finite correlation length. The new scheme ensures flux continuity at all grid interfaces for all grid levels. Further 2D diagonal tensor applications on model and realistic reservoir examples drawn from statistical reservoir descriptions are presented.

A new nine point scheme is presented which is flux continuous both for the diagonal and non-diagonal tensor permeability cases. The performance of the new scheme with renormalization is compared to the standard Galerkin formulation for a variety of test cases.

A PRESSURE CORRECTION METHOD FOR COMPRESSIBLE VISCOUS FLOW FOR BOTH LOW AND HIGH MACH NUMBERS

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Abstract

The aim of this work is to develop a fast 3-D compressible Navier-Stokes (N-S) solver, able to solve both low and high Mach number steady and unsteady flows. The code must also be able to solve flows in complex geometries.

A 3-D N-S solver based on Runge-Kutta (R-K) time marching has been used as a starting point. This code uses a cell centered finite volume scheme to discretize the equations in space. Adaptive artificial viscosity of fourth order globally and of second order in regions with high pressure gradients is used to damp unphysical oscillations caused by the central scheme. This is in line with the work done by A. Jameson, see [1], [2] and [3]. The solver can handle multi block geometries. This along with the finite volume formulation will make it possible to handle complex geometries such as multi body configurations or combustion systems.To accelerate the solution towards a steady state solution local time steps and multigrid relaxations are used.

This time stepping approach has at least two major drawbacks. First, it is an explicit scheme with strong restrictions on the time step due to stability reasons. For the Euler equations the time step restriction is not so serious since the CFL condition is only proportional to a characteristic mesh size. But for viscous flows the time step is also limited by a maximal time step proportional to the square of the mesh size. This is a very severe condition, especially for flows where the mesh have to be refined to resolve the flow field, for instance near solid boundaries. This will result in small time steps in some areas of the flowfield thus resulting in a slow convergence. For unsteady flow calculations a local time step can no longer be used and the smallest time step have to be used in the whole domain. Secondly, for low Mach numbers the time marching scheme will also be slow since the CFL condition gives a time step restriction that goes to zero with the Mach number.

To remedy these difficulties a pressure correction method is used. Such methods have been applied previously for incompresible flows, see [4] and [5]. The basic idea of this type of methods is to first, in a predictor step, solve the momentum equations using a given pressure field. Then with the help of the continuity equation a possion type equation is derived for a correction to the pressure field. From the pressure correction a correction to the velocity field can be derived so that the residuals of the momentum equations will not be changed or at least remain smooth. For high Reynolds number flows the pressure correction equation will require more computational effort to solve with ordinary point or line relaxation methods than the momentum or the energy equations, due to its elliptic character. Applying multigrid acceleration to this system of equations will speed up the whole solution procedure considerably, especially the elliptic pressure correction equation that will no longer be the bottle-neck of this method. Thus, the proposed scheme works equally well for low as for high Mach numbers. Implicit time marching is used when solving time dependent flows.

The new multigrid algorithm using the pressure correction scheme as a smoother seems to be considerable superior to the conventional variant that uses R-K stepping for smoothing. Typical 3-D wing flows at high and low Mach numbers are to be presented.

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No title received

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A Multigrid method for solving the fully implicit differential equations and a semi-implicit formulation used to model multiphase fluid flow in porous media is given. The basic unsymmetric equations for two phases are

$$\begin{aligned} \nabla \cdot [\lambda_o(\nabla P_o - \gamma \nabla D)] &= -\phi \frac{\partial S_w}{\partial t} + q_o, \\ \nabla \cdot [\lambda_w(\nabla P_o - \nabla P_c \gamma_w \nabla D)] &= \phi \frac{\partial S_w}{\partial t} + q_w. \end{aligned}$$

Multigrid is used to solve the system of linear equations produced by Newton's method at each time step. Because of the hyperbolic nature of the mass conservation equations for the phases, shocks are produced in the solution, which, combined with the discontinuities in the coefficients, necessitates the use of a special operator dependent interpolation operator. It was found, however, that fixed weight restriction operators worked best.

The resulting system of equations is highly nonsymmetric, which seems to cause the need for a more sophisticated approach to the grid transfer operators. Time step control is discussed in the examples as a way of maintaining stability and providing for effective starting values for Newton's method. Multigrid can be used advantageously with Newton's method to reduce the total number of iterations necessary to reach convergence. The usual good multigrid convergence was observed.

On matrix data structures and the stability of multigrid algorithms

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A multigrid method is presented which is able to solve symmetric and certain nonsymmetric problems with rapidly varying coefficients in two and three space dimensions. It is semi-algebraic, that means, the coarse grid matrices are generated algebraically from the fine grid matrix using the knowledge of the coarse mesh structure. This method has been tested and optimized in different fields of applications.

The implementation of the method on tensor product meshes is based on a sparse matrix diagonal storage scheme which holds the operators on different grids. The canonical scheme with the storage of the assembled main diagonal of the stiffness matrix results in numerically unstable representations for the intergrid transfer and coarse grid operators and does not guarantee the propagation of the fine grid matrix properties onto the coarse grid in the floating point arithmetic. Depending on the row- or columnwise diagonal dominance of the discrete operator, a sparse matrix diagonal storage scheme with unassembled main diagonal is proposed which results in numerically stable formulae for the multigrid components.

Examples of simulation problems are discussed which show very different convergence behaviour depending on the data structure chosen.

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A Domain Decomposition Algorithm for the Drift-Diffusion Equations

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1 Introduction

We present an algorithm for the solution of the linear systems that arise from the application of Gummel's method to the drift-diffusion equations. The method that we propose is a Schur domain decomposition algorithm suitable for anisotropic behavior (introduced by our nonuniform grid) as well as highly variable coefficients.

2 The Drift-Diffusion Equations

The following formulation of the drift-diffusion equations is considered

$$\varepsilon \nabla^2 \psi + q[n_{ie} e^{-q\psi/kT} v - n_{ie} e^{q\psi/kT} u + N_D - N_A] = 0, \tag{1}$$

$$\nabla \cdot [\frac{kT}{q} \mu_n n_{ie} e^{q\psi/kT} \nabla u] + R = 0 \tag{2}$$

$$\nabla \cdot [\frac{kT}{q} \mu_p n_{ie} e^{q\psi/kT} \nabla v] + R = 0 \tag{3}$$

where ε is the scalar permittivity of the semiconductor, N_D is the density of donor impurities, N_A is the density of acceptors, μ_n and μ_p are the electron and hole mobilities, n_{ie} is the effective intrinsic carrier concentration, ψ is the electric potential, u and v are the Slotboom variables associated with electron and hole density, q is the elementary charge, k is the Boltzmann constant, and T is the temperature in Kelvin. A discrete approximation to the drift-diffusion equations is obtained by approximating each of the derivative terms with central differences. The resulting system of three discrete nonlinear equations is then solved using a Gauss/Seidel-like method known as the Gummel iterative technique.

3 Domain Decomposition

In this paper we focus on the solution of the linear systems that arise from Gummel's method. A conjugate gradient algorithm is applied to the reduced system corresponding to a Schur complement domain decomposition method. As preconditioner, we consider

$$M = M_{probe} + M_{coarse}.$$

where M_{probe} approximates local coupling (on the interface) and M_{coarse} approximates global coupling (between distant interfaces).

For interface preconditioners we use a variant of the probe preconditioner [1]. The basic idea for the probe preconditioner is to approximate the Schur compliment operator along a domain interface by a band matrix which is constructed using a few matrix-vector products. For anisotropic problems the coupling along an interface may not be the dominate coupling. That is, a point on an interface may be more strongly coupled to the interface point across the domain than to its neighboring points.

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To alleviate this problem we introduce a series of probe matrices to approximate the coupling on each line of the grid across the entire domain (in both the horizontal and vertical directions). Thus for a 32×32 grid we create 32 probe matrices corresponding to the x-lines and another 32 probe matrices corresponding to the y-lines. Each of the probe matrices correspond to solving a tridiagonal system. Thus the resulting preconditioner looks somewhat more like an alternating line relaxation procedure as found in multigrid methods. More details on the alternating probe-line scheme can be found in [3]

Coarse grid points consist of both cross points (where four interfaces join) and points where the interfaces intersect the boundary. The coarse grid preconditioner is given by

$$M_{coarse} = I_H^h A_H {I_H^h}^T$$

where I_H^h is an interpolation operator from the coarse grid to the fine grid interfaces and A_H is a coarse grid approximation of the discrete fine grid operator. Given the highly variable nature of the PDE coefficients care must be exercised when defining interpolation, projection and coarse grid discretization operators. As with the multigrid method, it is beneficial to define operator dependent interpolation and restriction operators. The essential idea is that we use the fine grid discretization operator, A_h, to define the interpolation weights. We omit the details here and simply state that for a 1-dimensional problem the resulting coarse grid problem is closely related to cyclic reduction for solving a tridiagonal system of equations. Further, instead of arithmetic averages, we use a form of geometric averages (also similar to those used in multigrid, see [4]) to obtain coefficients for the PDE discretization. The main difference between the operators used here and those used for multigrid is that the operators used for domain decomposition are defined for a variety of coarse grid sizes (not just a doubling of the mesh spacing when going from the fine grid to the coarse grid). The details of both the geometric averages as well as the operator dependent operators are presented in [2].

Domains:	4×4	8×8	16×16	32×32
Iterations:	37	39	56	37

TABLE 1

Number of iterations per linear solve for a MOSFET simulation on a 129×129 *nonuniform grid*

Table (1) illustrates the average number of iterations per linear solve that are required for a MOSFET simulation. As the results illustrate, the number of iteration is quite small and that the number of iterations does not change significantly as the number of domains is changed.

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Interpolation and related coarsening techniques for the algebraic multigrid method

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ABSTRACT

In this paper we present an algorithm for coarsening in the algebraic multigrid method, related to a special construction of the interpolation operators. We use for this construction a preconditioning of the initial system matrix.

In a 4-levels algorithm we initially used the Galerkin (variational) approach, i.e.

$$I_n^p \text{ – full rank}$$
$$I_n^p = (I_n^p)^t$$
$$A_p = I_n^p A_n I_p^n$$

where we denoted by $I_{p,}^{n}$ $I_{n,}^{p}$ A_p, A_n the interpolation, the restriction, the coarse grid matrix and the initial matrix respectively.

We observed for the matrices A_p, corresponding to the third and fourth levels, a process of fill-in (due to the special construction of the interpolation and of rounding errors). Then, accordingly to the ideea of "strong connections" (Brandt, Stuben and Ruge) combined with the fact that small elements (in size) will have (at least for some classes of problems) a small influence in the coarsening process, we truncated the A_p matrices of the above mentioned levels and we analysed the effects of this truncation to the average error reduction factor (e.g. Hackbusch).

Examples are presented for Dirichlet, Helmholtz, Poisson anisotropic, a finite element matrix and a linear system arrising from an electrical networks problem.

PARALLEL POINT-ORIENTED MULTILEVEL METHODS

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Abstract Recently, a new concept for the development of multigrid and BPX-like multilevel algorithms had been presented (see [1]). There, instead of a basis approach on the finest grid and the acceleration of the basic iteration by a MG-coarse grid correction or a BPX-type preconditioner a generating system was used to allow a non-unique level-wise decomposed representation of the solution. The degrees of freedom are associated to the nodal basis functions of all levels under consideration. Now, the Galerkin approach leads to a semidefinite linear system with unknowns on all levels. The generalized condition number of this system is of the order $O(1)$. Its solution is non-unique but in some sense equivalent to the unique solution of the standard problem on the finest grid.

Furthermore, it was shown that traditional iterative methods for the semidefinite system are equivalent to modern elaborated multilevel methods which exhibit optimal convergence properties. The conjugate gradient method (with appropriate diagonal scaling) for the semidefinite system is equivalent to the BPX-conjugate gradient method for the fine grid system. Gauss-Seidel type iterations for the semidefinite system are equivalent to certain multigrid methods. For details see [1]. This methods are level-oriented and can be considered as a level-block technique. An outer iteration switches from level to level and an inner iteration operates on the specific grid.

Now, we consider the semidefinite system from a different point of view. We group together all unknowns which belong to different levels but are associated to one grid point. This results in point-oriented methods and can be considered as a point-block technique. Now, an outer iteration switches form grid point to grid point. The local system that belongs to all basis functions of different levels centered in one grid point can be solved either directly or by an inner iteration that runs over all levels that are associated to the grid point under consideration. See [2] and [3]. Furthermore, grid points can be grouped together to form subdomains. In this sense we get some sort of simple domain decomposition method which exhibits MG-type convergence properties.

We report the results of numerical experiments regarding the convergence rates of these new algorithms. It turns out that we obtain convergence rates for the new point-block ML-method which are similar to that of standard ML-algorithms.

As the point-block method allows directly a interpretation in terms of domain decomposition its parallelization is straightforward. In contrast to the parallelization of a multilevel method where communication has to take place on *all* levels to maintain good convergence rates, our point-block approach needs substantially less communication due to its domain decomposition qualities. In this sense, our new method is superior to other parallel multigrid and multilevel methods.

We discuss the parallelization properties of the point-oriented method, compare it with conventional parallel multigrid methods and present the results obtained for an

implementation on different MIMD-computers.

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LDS Method for Linear Evolution Equations

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Abstract

Long time simulation of linear partial differential equations is a highly intensive computational task. In these problems a large scale system of linear equations has to be evaluated or solved at each time step. The solution of such problems if often smooth and can be spatially approximated on a coarse grid. This appealing smoothness property is hard to exploit in long time simulations. The fast accumulations of computational errors necessitates the use of either high order schemes or significantly finer grids; resulting in a substantial increase of the computational cost.

The Large Discretization Step (LDS) method for the acceleration of linear time dependent finite difference calculations is presented. The LDS method is a generalization of the frozen tau technique aimed at achieving a substantially more efficient technique not restricted to parabolic equations. In this approach, a set of equations satisfied by the fine-to-coarse grid correction terms (tau) are derived and solved on the coarse grids at each time step, resulting in a time varying tau. The method employs time stepping on different temporal and spatial scales visiting the finer grids once in many coarse level time steps. Although most of the time marching is performed on the coarse levels in time-space, the resulting solution is practically the fine grid solution. The proposed method is very general, simple to implement and may be used to accelerate many existing time marching schemes, either implicit or explicit.

Numerical examples are e given demonstrating the effectiveness of the method which reduces the computational time by more than an order of magnitude.

Parallel Multigrid Method Based on Wavelet Multiresolution

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The Wavelet bases are natural hierarchical bases. It is easy to construct the MG algorithms using wavelets. In this paper, we discuss some parallel wavelet-based MG algorithms for solving PDEs.

Considering the variational problem: $a(u,v) = f(v)$, we adapt the Ritz-Galerkin discretizing technique which utilizes wavelet functions as othonoral bases. Let $(\mathbf{V}_n : n = 1,\ldots,m)$ is the wavelet multiresolution of $L^2(R)$, on each space $\mathbf{V}_n$, the discrete matrix problem is $\mathbf{A_n u_n} = \mathbf{f_n}$. Because of $\mathbf{V_n} \subset \mathbf{V_{n+1}}$, it is convenient to apply the MG method to the computation of the equation: $\mathbf{A}_m\mathbf{u}_m = \mathbf{f}_m$.

One way for parallelization is from the viewpoint of splitting matrix. For each $\mathbf{A_n}$, we will do some kinds of matrix splitting to get several independent iterative subtasks that can be processed on a parallel computer system. In fact, the splitting procedure may be regarded as a smoothing operator. Wit respect to different problems and computer systems, the splitting skills are varied.

The other way for parallelization is similar to the parallel multilevel method. That is, let $\mathbf{u}_m^{\mathbf{k}}$ be an approximative solution to $\mathbf{A}_m\mathbf{u}_m = \mathbf{f}_m$. and $\mathbf{d}_m^{\mathbf{k}}$ be the residual, then project $\mathbf{d}_m^{\mathbf{k}}$ to each subspace $\mathbf{V_n}$, solve simultaneously the residual equations: $\mathbf{A_n e_n^{k+1}} = \mathbf{d_n^k}, n = 1,\ldots,m$, let $\mathbf{u}_m = \mathbf{F}(\mathbf{u}_m^k, \mathbf{e}_1^{k+1},\ldots,\mathbf{e}_m^{k+1})$ where $\mathbf{F}(\mathbf{x}_0,\ldots,\mathbf{x}_m)$ is a kind of combination of $\mathbf{x}_0,\ldots,\mathbf{x}_m$, for example one may choose F is a kind of combination of $\mathbf{x}_0,\ldots,\mathbf{x}_m$. This function is also correspond to the smoothing procedure.

For the soution of second linear elliptic PDEs problems, we prove that $\lim_{m\to\infty} ||u - \mathbf{u}_m|| = 0$, where u is the solution of $a(u,v) = f(v)$, and the parallel iterative schemes for solving the matrix equations is convergent by using the theory of algebrasic multigrid. Some numerical results from above algorithms show the efficiency of method.

SPECTRAL MULTIGRID METHODS FOR THE REFORMULATED STOKES EQUATIONS

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Abstract: We consider a reformulation of the Stokes equations where the continuity equation is replaced by a Poisson-like equation for the pressure. To guarantee full equivalence between the two systems of equations (provided all functions are sufficiently smooth) the continuity equation has to hold at the boundary of the domain. This treatment has been proposed in the famous article of Harlow and Welch. They proposed the MAC method which obtains an equation containing Δp (p denotes the pressure) by differentiating the time-dependent momentum equations and adding them. For finite difference discretizations this approach turned out to be very useful for the efficient solution with multigrid techniques. Furthermore this algorithm is not difficult to parallelize and to vectorize.

We were interested in the performance of spectral multigrid methods for the reformulated Stokes equations. Here the spectral discretization is accomplished by a pseudospectral (or collocation) method using Chebyshev polynomials. The collocation points are given by the standard Chebyshev Gauss-Lobatto nodes. In particular, we introduce a Chebyshev collocation method which yields the high spectral accuracy. If the continuity equation is directly discretized by a spectral method, the pressure is affected by spurious modes (parassite modes) which deteriorate the accuracy of the method. By our spectral discretization of the reformulated Stokes equations spurious modes (except the constant) are avoided. No staggered grids are needed and hence the method can be efficiently implemented by fast Fourier transforms (FFTs).

We also present an efficient finite difference preconditioner for the spectral system. The eigenvalues of the preconditioned spectral operator are complex and lie in a circle C which intersects the real axes in the points 1 and $\frac{\pi^2}{4}$. These eigenvalue bounds are already well known from similar considerations for the Poisson equation. Due to the good performance of finite difference preconditioning techniques we prefer a preconditioned Richardson relaxation for the iterative solution of the spectral systems. Finally we show the efficiency of our spectral multigrid method by numerical results.

ADAPTIVE MULTILEVEL FINITE ELEMENT METHODS FOR VARIATIONAL INEQUALITIES

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We consider the numerical solution of stationary variational inequalities of obstacle type associated with second order elliptic differential operators by adaptive finite element techniques. The adaptive algorithm features the multilevel iterative solution of the discretized problems based on the use of continuous, piecewise linear finite elements with respect to a nonuniform hierarchy of triangulations and error estimators for the global discretization error whose local contributions will serve as indicators for the local refinement of the triangulations. While the refinement strategy relies on the meanwhile standard process by Bank et al., the discretized problems are solved by an outer-inner iterative scheme with an outer active set strategy and inner preconditioned cg-iterations. In particular, we present four different multilevel preconditioners of hierarchical and BPX--type which are derived in the context of multilevel additive Schwarz iterations. The error estimation is based on a piecewise quadratic ansatz and leads to an implicit linear complementarity problem for which three different solution techniques are proposed. The refinement process and the performance of the multilevel preconditioners and the error estimators are illustrated by several numerical results for various obstacle type problems in elasto- and hydromechanics.

The convergence of Multigrid Methods for Solving Finite Element Equations in the Presence of Singularities

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April 13, 1993

ABSTRACT

We analyze the convergence of multigrid methods applied to finite element equations of second order with singularities caused by reentrant corners and abrupt changes in the boundary conditions. Provided much more weaker demands of classical multigrid proves,it is shown in this paper that,for symmetric and positive definite problems in the presence of singularities, multigrid algorithms with even one smoothing step converge at a rate which is independent of the number of levels or unknows. Furthermore,we extend this result to the nonsymmetric and indefinite problems.

Multigrid Waveform Relaxation with Spatial Finite Element Discretisation

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We shall discuss the numerical solution of parabolic partial differential equations by using multigrid waveform relaxation techniques on spatial finite element meshes. Application of a finite element Galerkin method to a parabolic PDE leads to a large system of ordinary differential equations of the following type:

$$B_h \dot{u_h} + A_h u_h = f_h.$$

Here u_h is the vector of unknowns; h is a measure of the mesh size; B_h and A_h are sparse mass and stiffness matrices. This formula differs from the standard finite difference discretisation of a parabolic PDE by the presence of the matrix B_h.

The Gauss-Seidel waveform relaxation method is an iterative procedure for calculating the solution to this system of ODEs. It is based on a splitting of the matrices B_h and A_h, and corresponds to the iteration

$$M_{B_h} \dot{u}_h^k + M_{A_h} u_h^k = N_{B_h} \dot{u}_h^{k-1} + N_{A_h} u_h^{k-1} + f_h,$$

with M_{B_h} and M_{A_h} the lower triangular parts of B_h and A_h, $N_{B_h} = M_{B_h} - B_h$ and $N_{A_h} = M_{A_h} - A_h$. The iteration index is denoted by k. This waveform iteration can be used efficiently as a smoother inside a multigrid procedure.

The talk will focus on theoretical and numerical results for a model problem: the one- and two-dimensional heat equation. We will compare the Gauss-Seidel waveform relaxation method to the multigrid acceleration and consider several combinations of space- and time-discretisation formulae. In particular, we will show that in many cases $\rho(\text{Gauss-Seidel}) = 1 - O(h^2)$, and $\rho(\text{multigrid}) \leq c \ll 1$, where c is a small constant independent of h.

A Full MultiGrid Method Applied to Turbulent Flow Using the SIMPLEC Algorithm Together with a Collocated Grid Arrangement.

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Abstract

An implementation of a multigrid method into a three-dimensional SIMPLEC code based on a collocated grid arrangement is presented. The multigrid algorithm is FMG-FAS using a V-cycle described by Brandt.

The coarse grid is obtained by merging eight fine grid cells in 3D and four in 2D. Restriction and prolongation of field quantities are made by a weighted trilinear interpolation, and restriction of residuals is made by a summation.

All equations are treated in the same way and that includes also the pressure correction equation. To prepare for local mesh refinements all variables are restricted, and therefore special care must be taken with regard to the implict calculation of mass fluxes (Rhie and Chow interpolation).

To stabilize the solution process a fraction of the multigrid sources is included in the diagonal coefficient a_p, and a damping function is used on negative corrections of the turbulent quantities to prevent them to be negative.

The multigrid method showed to be very stable, and the under relaxation parameters used are therefore 0.8 or 0.7 for all equations except for the pressure correction equation where 1.5 is used.

The V-cycle used is a 2-1-....-1-4-1-....-1-1 but the convergence rate was not greatly affected if other V-cycles were used. The convergence rate was also found to be insensitive to the choice of under relaxation parameters. The convergence rate of the multigrid was not much slower when the under relaxation was lower. Notice that with these under relaxation parameters the convergence rate of the single grid calculation was also slowed down.

Both turbulent and laminar calculations are presented for a 2D backward facing step, a 2D ventilated enclosure, and a 3D ventilated enclosure. The turbulent calculations are made with a two-equation low-Reynolds $k-\epsilon$ model, where the QUICK scheme is used for velocities, and the van Leer scheme for the turbulence equations. These calculations are also made with the Hybrid scheme and the required amount of WU's was of $O(1)$ for both discretizations. For the 2D turbulent ventilated room, with a 5-level V-cycle and 160x160 grid, the speedup factor was 150. This case was also calculated with a 6-level V-cycle and the estimated speedup is over 500 times.

A Subspace Decomposition Twogrid Method for Hyperbolic Equations

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Abstract

An efficient twogrid method for solving hyperbolic equations is presented. We analyse the model problem of a scalar advection equation:

$$a_1 \partial_x u + a_2 \partial_y u = 0 \quad .$$

The efficiency of the algorithm is based on two principles: An adequate discretisation of the continuous problem and a robust twogrid method for solving the discrete system.

A new finite volume discretisation based on a triangular grid is proposed. The resulting cell vertex scheme, $L u = f$, is overspecified. The problem is transformed into a discrete minimisation problem: $\|L u - f\| \rightarrow \min$. The system of normal equations, $A u = b$, is uniformly stable for all characteristic directions on triangulations based on equidistant grids. No upwinding or numerical dissipation is needed. The initial value problem has a unique solution which is second order accurate.

The twogrid iteration is based on a subspace decomposition method. These subspaces are defined by several prolongations which efficiently treat high frequency errors transverse to characteristic directions. The prolongations are based on a modification of the frequency decompositon approach of Hackbusch. The coarse grid corrections are defined by minimisation problems on these subspaces. Multiplicative Schwarz iteration is applied for the sequence of coarse grid corrections. (In parallel versions of the algorithm, additive Schwarz iteration could be used.) A gradient iteration is used for smoothing. Convergence rates close to $1/2$ are obtained for the twogrid iteration. The convergence rates are uniformly bounded for a class of hyperbolic problems. They are independent of the characteristic directions and a robust and efficient algorithm is obtained.

Parallel Steady Euler Calculations using Multigrid Methods and Adaptive Irregular Meshes

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The Euler equations, the limiting case of the Navier-Stokes equations for vanishing viscosity, constitute a nonlinear singular perturbation problem characterized by solutions with spatially localized features. In order to reduce the computation time required to solve these equations accurately, we have developed a solver that uses the following acceleration techniques :

- the use of *irregular meshes* and *adaptive mesh refinement*, so as to achieve a prescribed accuracy with a discrete problem that is as small as possible
- *multigrid methods*, because of their good convergence properties, in principle irrespective of the discrete problem size
- *distributed memory parallelism*, in order to achieve a high computation rate

A first order accurate discretization of the Euler equations is obtained with a finite volume method based on irregular polygonal volumes and van Leer flux vector splitting. A parallel mesh refinement strategy is used.

As solvers we consider both Newton-multigrid methods and nonlinear multigrid (FAS). In the former case we solve the Newton linearized system with a linear multigrid method based on damped collective Jacobi relaxation. Experiments indicate that Picard variants of this method may improve on the computation speed at the cost of a slight loss in convergence rate. For application in nonlinear multigrid methods, nonlinear smoothers based on pointwise linearization (nonlinear damped collective Jacobi) or complete linearization of the nonlinear system (implicit time-stepping) were constructed. We also use explicit time-stepping methods with certain smoothing properties (Forward Euler and multi-stage Runge-Kutta methods with local time-stepping. Both classes of multigrid methods were used within a nested iteration procedure.

On a distributed memory multiprocessor, each grid level must be partitioned and mapped so as to minimize the multigrid cycle execution time. In order to maintain a reasonable parallel efficiency for the multigrid cycle in spite of the run-time mesh adaptation, the use of appropriate grid partitioning and load (re)mapping strategies is advocated [1].

The performance of the resulting parallel adaptive multigrid algorithms has been evaluated on iPSC hypercubes. The comparison between Newton-multigrid and nonlinear multigrid methods made by Jespersen [2] could be extended to the irregular grid case, taking into account the effects of parallelization.

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Multigrid Solution of the 2-D Navier-Stokes Equations for Transonic Internal and External Flows

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The nonlinear FAS Multigrid Method is used to accelerate the solution process of the compressible twodimensional Navier-Stokes equations. Simulations of transonic internal and external flows are carried out achieving very good convergence rates. Results and residue histories are shown.

The basic solver integrates the 2D-Navier-Stokes equations in compressible, Reynolds averaged formulation. Finite volume discretisation and cell centered treatment of the flow variables is employed. An explicit four or five stage Runge Kutta local time stepping scheme advances the solution in space and time. The Baldwin-Lomax turbulence model is used to compute the eddy viscosity.

The boundary conditions are assumed to be no-slip at adiabatic walls with a zero wall normal pressure gradient. One dimensional Riemann invariants are used at the farfield and a non-reflecting boundary condition is applied at the outflow boundary of the internal flow.

The multigrid procedure coarsens the grids by omitting every second gridpoint in both directions of the mesh. The restriction of the flow variables is volume weighted and the residuals are summarized over four neighbouring cells. The corrections from the coarser meshes are prolongated by linear interpolation.

Computations of airfoil and channel flows are performed and the results for the RAE2822 airfoil with chamber correction using the case 9 freestream conditions from Cook et. al. are selected to be shown here. The grid consists of 257x65 points on the finest mesh level. Convergence histories of the maximal density residues are given in **Fig 1**, which shows the residue performances of one up to four grid levels. The four level multigrid computation converges 14 times faster than the single grid computation. The isobars are presented in **Fig 2** to have a global view of the computed flow field. The surface pressure coefficient is in good agreement with the experimental data indicated by triangles in **Fig 3**. The difference to the calculation is at least partly attributed to the chamber corrected airfoil.

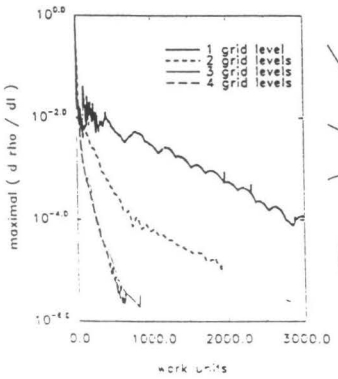

Figure 1: RAE2822 case 9 Convergence histories

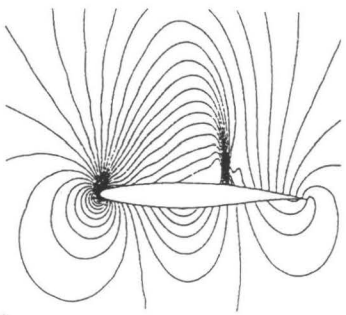

Figure 2: Isobars

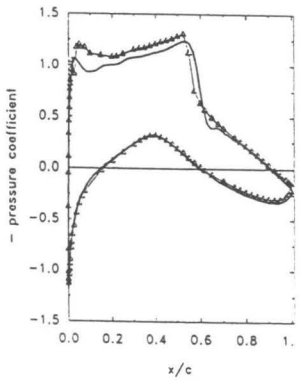

Figure 3: Surface pressure coefficient

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UNSTRUCTURED AGGLOMERATION MULTIGRID FOR DIFFUSION PROBLEMS

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The non expensive computation of the viscous flows involving boundary layer on unstructured meshes is still a challenge.
We study an unstructured Multigrid strategy for solving diffusion problems [KLD]. This strategy is adapted from the generalized **Finite Volume Agglomeration Multigrid** algorithm [LSD] that we have developed for the solution of the Euler equations. This agglomeration approach has the advantage to allow the fast automatic generation of all necessary coarser levels from only one mesh.
One important issue is the building of a consistent approximation of the diffusion terms on a rather exotic finite volume partition. We discuss one option, relying on a corrective factor, applied to the **algebrical coarsening** of the system : $a_{11}^{coarser} = a_{11}^{fine} + a_{12}^{fine} + a_{21}^{fine} + a_{22}^{fine}$.
The talk will involve the definition of the volume agglomeration method related to a variational formulation of the 2-D Poisson equation. Numerical results on the 2-D diffusion-convection model show the efficiency of this scheme.
The extension to compressive 2-D Euler and Navier-Stokes is also discussed.

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A Multigrid Multiblock Solver for Compressible Turbulent Flow

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We describe a multigrid multiblock method for compressible turbulent flow simulations and present results obtained from calculations on a two-element airfoil. A vertex-based spatial discretization method and explicit multistage Runge-Kutta time-stepping are used. The smoothing properties of this relaxation method are further enhanced through the introduction of local time-stepping and implicit residual averaging in which the residuals at the block boundaries are kept fixed to their non-smoothed values.

Inviscid calculations on this test case have shown that a solution which is converged up to machine accuracy can be obtained with this multigrid method. A comparison with the single grid simulation method shows that a considerable reduction in calculation time was obtained with the multigrid method, although the convergence of the single grid method for inviscid calculations was already acceptable.

In the viscous calculations the single grid method was found to yield a well converged result. However, the convergence towards the steady state solution is extremely slow and makes the use of a multigrid approach essential. A comparison of the computational effort required in both methods shows that a total reduction with a factor of about 12.5 can be reached with the multigrid method. The numerical predictions obtained for the lift- and pressure coefficients compare well with experimental results and give confidence in the use of the Baldwin-Lomax model for this application. Moreover, the boundary layer thickness was found to agree well with the experiments. It is shown that the convergence of the multigrid process depends considerably on the ordering of the various loops. If the block loop is put inside the stage loop the process converges more rapidly than if the block loop is situated outside the stage loop in case a three-stage Runge-Kutta method is used. If a five-stage scheme is used the process does not converge in the latter block ordering. Apparently, the smoothing of the relaxation method becomes less effective as the number of stages between two 'updates' of the dummy-variables increases. This has some less favorable consequences for a possible parallel processing of the multigrid method. An optimal rate of data-exchange between the blocks should be determined in order to combine the high convergence rate of the multigrid method with specific requirements related to parallel processing. Finally, the process based on the five-stage method is about 60% more efficient than with the three-stage method, if the block loop is inside the stage loop.

Parallel Multilevel Methods with Adaptive Mesh Refinement — Results for Singular Perturbation Problems

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Adaptive mesh refinement matches the computational demands to the activity in the application's domain. In a flow problems this means that only regions of high local activity (shocks, boundary layers, etc) should receive enhanced resolution; and regions of little flow activity (or interest) should be more easily solved using coarser resolution. Based on our previous research on parallel fast adaptive composite methods (FAC, AFAC) for the solution of partial differential equations, we present an object oriented approach for complex parallel adaptive mesh refinement applications (e.g. fluid flow problems) on block structured grids.

With two C++ object class libraries we provide abstractions to separate the issues of developing parallel adaptive mesh refinement applications into those of parallelism, abstracted by P++, and adaptive mesh refinement, abstracted by AMR++. The AMR++/P++ environment allows for software development in the preferred serial environment, and such software efficiently runs, unchanged, on parallel architectures with distributed memory. Of course, high efficiency can only be obtained for algorithms with inherent parallelism as e.g. red black or zebra type relaxations. P++ is a parallel array class library to permit efficient development of architecture independent codes for 2D and 3D structured grid applications, and AMR++ provides support for adaptive mesh refinement for complex block structured applications based on P++. P++ is based on the Virtual Shared Grids (VSG) model. As opposed to general Virtual Shared Memory implementations, VSG, through its restriction to structured

grids, allows for obtaining similar parallel performance as for codes based on the traditionally used explicit Message Passing programming model. Using the AMR++/P++ libraries the application programmers' work is greatly simplified to primarily specifying the serial single grid application, and obtaining the parallel and adaptive mesh refinement code with minimal effort.

These tools are used in particular for the numerical solution of singular perturbation problems. It is well known that these problems arise frequently e.g. in computational fluid dynamics, and they have solutions with shocks and layers. This means that large problems of this kind are natural candidates for parallelism and adaptive refinement and thus typical candidates for implementation on the basis of P++ and AMR++.

On the other hand, it is well known that the efficient numerical solution of singular perturbation problems is extremely difficult. Apart from the simplest problems, multilevel methods might be useful but they lack their usual efficiency due to a convergence rate of $\rho \geq 0.5$ for the two level method which causes convergence rates approaching 1.0 with an increasing number of levels. The situation is even worse on (massively) parallel machines since flow direction dependant relaxations result in bad parallel efficiency. These conclusions are demonstrated by numerical examples.

Extensive and systematic numerical experiments have lead us to multigrid variants which show surprisingly nice convergence behaviour while being fully parallelizable. Numerical results will show this. The algorithm needs somewhat more artificial viscosity in the basic dicretization. Therefore, methods like defect correction (which will not be discussed here) and adaptive refinement become even more attractive. It is natural to implement these algorithms on the basis of the P++ and AMR++ tools. We will report on the results.

Solution of the stationary Euler equations by an improved GMRES algorithm and multigrid acceleration.

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The stationary Euler equations of fluid flow form a system of first order partial differential equations. The numerical solution of such equations is often obtained by an iterative method such as Runge-Kutta time-stepping (RK) [1] or the Generalized Minimum Residual method (GMRES) [2]. The convergence of these methods can be explained as a combination of two effects:

* wave propagation of smooth error modes out through the open boundaries,
* damping of the amplitude of oscillatory error modes.

The algorithms we advocate here enhance the wave propagation. Theoretical and numerical investigation of linear model problems support this view. Substantial improvements in convergence rate and increased robustness are obtained in numerical experiments.

A modified GMRES (mGMRES) method introduced in [3] is tested and compared with the original GMRES and the very successful RK. One advantage of GMRES, mGMRES and RK in comparison to other iterative schemes of Krylov subspace type is that the residual usually has a smooth behavior as the iterations proceed. Another advantage is that only residual evaluations are needed in each iterative step. GMRES and mGMRES also have a local optimality property. A disadvantage with GMRES and mGMRES is the extra storage requirements compared to RK. Another problem with GMRES is that sometimes the iteration stagnates and the residual ceases to decrease after a number of iterations. The remedy is to introduce mGMRES or a proper preconditioner. Both RK and mGMRES have a secured wave propagation property.

We consider two kinds of preconditioning of the Euler equations: residual smoothing and multigrid iteration. Both of them improve the propagation of smooth error waves. The residual smoothing technique in [4] is chosen. It contains the original one [5] as a special case and has better convergence properties and is less sensitive to parameters. The parameters in the multigrid acceleration are selected so that the wave speed of smooth error modes increases. The number of iterations on the coarse grids are increased and care is taken to ensure smooth prolongated corrections from the coarse grids.

The convergence of the iterative algorithms to the solution of the stationary Euler equations is studied in numerical experiments in 2D and 3D. The number of iterations and the total CPU-time are compared for three-stage RK, GMRES and mGMRES combined with residual smoothing and multigrid iteration. It is found that the convergence rate is improved considerably by techniques that enhance the propagation of smooth error modes out from the computational domain.

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A Study on Narrow Stencil Discretizations and Multigrid Solvers for the Generalized Stokes Equations

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Equations

In this study, the generalized Stokes equations

$$\left\{ \begin{array}{rcrcrcrcl} -\frac{1}{Re}\Delta u & + & a\,u_x & + & b\,u_y & + & p_x & = & f_1 \\ -\frac{1}{Re}\Delta v & + & a\,v_x & + & b\,v_y & + & p_y & = & f_2 \\ & & & & & & u_x + v_y & = & f_3 \end{array} \right\}$$

(which can be considered as linearization of the steady state incompressible Navier Stokes equations) are chosen as model problem for fluid flow.

Discretizations

The discretizations of the above equations are based on a flux difference splitting concept by Dick which, in this case, coincides with Osher's scheme. Depending on the definition of interior and exterior states at the boundaries of the control volumes, different discretizations of narrow stencil type are derived. Thus, a multidimensional discretization of the advection part is obtained.

Numerical results obtained for 1st order discretizations

As smoother in a multigrid context, pointwise Gauss Seidel relaxation with lexicographic ordering of gridpoints and alternating line relaxation (with relaxation parameters $\omega = 1.0$ and $\omega = 0.75$) are compared. Best results can be obtained in case of alternating line relaxation with $\omega = 0.75$, the convergence rates for the standard and some narrow stencil cases being not much different. With regard to accuracy, the discretization error of most narrow stencil discretizations proves to be of about a factor 2 better than in the standard case.

Numerical results obtained for 2nd order discretizations

In case of 2nd order discretizations, defect correction is used in order to solve the resulting equations. Here, some results obtained for the scalar advection equation are generalized. Again, the narrow stencil discretizations deliver more accurate results.

Numerical results obtained for 1st order discretizations
As smoother in a multigrid context, pointwise Gauss Seidel relaxation with lexicographic ordering of gridpoints and alternating line relaxation (with relaxation parameters $\omega = 1.0$ and $\omega = 0.75$) are compared. Best results can be obtained in case of alternating line relaxation with $\omega = 0.75$, the convergence rates for the standard and some narrow stencil cases being not much different. With regard to accuracy, the discretization error of most narrow stencil discretizations proves to be of about a factor 2 better than in the standard case. In general, the MOD2 variants are more accurate than the MOD1 variants.

Numerical results obtained for 2nd order discretizations
In case of 2nd order discretizations, defect correction is used in order to solve the resulting equations. Here, some results obtained for the scalar advection equation are generalized. Again, the narrow stencil discretizations deliver more accurate results.

A-posteriori Estimation of the Local Discretisation Error for Conservation Laws and Adaptive Local Grid Refinement

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Recently, the well-known technique to estimate the local discretisation error, based on Richardson extrapolation has been generalised for a locally refined grid [1]. The purpose of this paper is to present the construction of a sufficiently accurate estimate of the local discretisation error, for application in a refinement criterion for adaptive, local refinement of the grid in multigrid computations.

The main problem of the estimation of the local discretisation is that, in the context of local grid refinement, the local discretisation error on two consecutive levels of refinement cannot be represented with the same asymptotic expansion, at least not locally. Hence, application of the estimation technique in a straightforward fashion yields an error in the estimate which of the same order of magnitude as the estimate itself.

It appears to be possible to split the local error in two parts, each to be estimated in a different way. One part is the perturbation of the local discretisation error with respect to the regular discretisation scheme. This so-called non-uniform part can be readily found by estimation of the terms appearing in the expression resulting from the Taylor series analysis of the local discretisation error. Once this non-uniform part is known, the other (so-called uniform) part can then be found by application of Richardson extrapolation.

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A Critical Analysis of Concurrent and Standard Multigrid Methods on Massively Parallel Computers

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Abstract

The hierarchical nature of multigrid algorithms leaves domain parallel strategies with a deficiency of parallelism as the computation moves to coarser and coarser grids. To introduce more parallelism several strategies have been designed to project the original problem space into non-interfering subspaces, allowing all grids to relax concurrently. Our objective is to understand the potential efficiency of subspace parallel algorithms on existing and proposed massively parallel machines.

We study model problems on simple domains discretized with finite difference techniques on block structured meshes in two and three dimensions with up to 10^6 and 10^9 points, respectively. Performance of the standard domain-parallel V and F cycle schemes is compared to several proposed concurrent algorithms.

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The multigrid strategies are studied in several models of parallel computation, designed to reflect both the characteristics of existing machines and those of the proposed next generation of multicomputers. These include a SIMD fine-grain model which contains a large number ($10^4 - 10^6$) of small (bit-serial) processors as well as a SPMD medium-grain model with a more modest number (256–16,384) of powerful (single chip) processors, interconnected through a single-stage and multistage communication network, respectively.

Our analysis suggests that subspace parallelism is too expensive to be practical except under a very limited set of conditions. When compared to each other the relative performances of the concurrent methods depend critically on the degree of subspace interference. The closer the subspace decomposition is to the eigenfunctions of the relaxation matrix, the better the performance of the simple concurrent V-cycle with no intergrid communication as compared to a scheme such as the Gannon and Van Rosendale algorithm with more complex and frequent communication. When compared to optimized implementations of the V and F cycle methods, concurrent methods, which require more complex implementations, are outperformed unless convergence can be obtained in a small constant number of concurrent iterations. Comparing the performance of a concurrent methods across several models of computation confirms these results, and illustrates disparity between the motivation for concurrent methods and the architectural characteristics of the current generation of supercomputers.

MULTIGRID METHODS FOR MIXED FINITE ELEMENT DISCRETIZATIONS OF VARIATIONAL INEQUALITIES

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We consider stationary variational inequalities of obstacle type: Find $u \in K := \{v \in H^1(\Omega) \mid (v-g)|_{\partial\Omega} = 0, v \geq \psi \text{ a.e. in } \Omega\}$ such that

$$\langle a\nabla u, \nabla(u-v)\rangle \geq \langle f, u-v\rangle \quad , \quad v \in K$$

where $a = (a_{ij}(x))_{i,j=1}^{2}$, $x \in \Omega$, is assumed to be symmetric and uniformly positive definite. For strongly varying or even discontinuous coefficient functions the discretization by mixed finite element methods is considered as superior to the standard conforming techniques,. In particular, using the lowest order Raviart-Thomas elements and an appropriate post-processing technique the resulting scheme is equivalent to a nonconforming method of inverse average type based on the Crouzeix-Raviart elements augmented by suitable cubic bubble functions. This has been established by Arnold and Brezzi in the unconstrained case where for piecewise constant coefficient functions the components of the discrete solution associated with the nonconforming part and the bubbles totally decouple and thus can be computed independently. However, as we shall show, such a decoupling does not apply in case of variational inequalities, since there is an inherent global coupling of the nonconforming part and the bubbles caused by the constraints. Taking care of the constraints by piecewise constant Lagrangean multipliers and using static condensation we end up with a variational inequality in terms of the multipliers. This variational inequality is then solved by an outer-inner iterative scheme with an active set strategy in the outer iterations and suitable multigrid methods for the inner iterations. The proposed mixed finite element technique is compared with standard conforming methods and its efficiency is illustrated by some numerical results for variational inequalities arising from elastomechanical applications and stationary flow problems in porous media.

Fluid flow calculations by a nonlinear multigrid method

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Our aim is to present a nonlinear multigrid method for solving the problems governed by the systems of the steady state conservation equations in the three space dimensions. The systems under the consideration include the three momentum and mass conservation equations of the fluid mechanics, and a number of the scalar conservation equations (for the temperature, the concentrations of the chemical species, the turbulence energy and its dissipation, etc.).

At the present stage of this research we consider the flow problems inside the domains composed of a number of adjacent rectangular segments. A variety of conditions for the velocity, the pressure and other scalars, as well as for their normal derivatives and the flow rates can be imposed on the boundary.

The finite volume method on rectangular grids was used for the discretization. The grids may differ on various sub-segments of the domain. In particular, the grids need not match on the intersegmental boundaries.

A nonlinear multigrid method for solving such problems will be presented, with the convergence histories for various flow cases in complicated domains.

For this method, the results of the initial solution refinement procedure are put to a full use in each iteration of the subsequent multigrid process. Comparisons will be made with the classical nonlinear multigrid method.

A multigrid waveform relaxation method for time dependent incompressible Navier-Stokes equations.

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In the 80-ties several multigrid methods have been developed for the solution of time dependent equations. One of the approaches is based on the waveform relaxation methods. It is proposed by (amongst others) VandeWalle and described in detail in [?]. In waveform relaxation methods an approximation of an unknown in space is calculated along the whole time interval of interest. Instead of updating scalars time step by time step functions in time are updated in space. Multigrid waveform relaxation methods differ from parabolic multigrid methods only in the smoothing algorithm.
A multigrid waveform relaxation method to solve the incompressible Navier-Stokes equations has not been found in literature yet. Here, a multigrid waveform relaxation method is investigated for the incompressible Navier-Stokes equations in general coordinates. Due to the discretization for boundary-fitted grids flows in arbitrarily shaped domaines can be solved. Flux unknowns are used as primary unknowns together with the pressure in a staggered grid arrangement.
The smoothing method is called Symmetric Coupled Alternating Lines (SCAL). All velocity components and pressures in a line of cells are updated simultaneously. SCAL is robust; many steady problems in arbitrary domains, where cells with large aspect ratio occured have been solved succesfully ([?]). The SCAL waveform smoother is a zebra type smoother: First all horizontal "white" lines (which can be done in parallel on a parallel machine) are updated on all time levels $t_{n_1},...,t_{n_2}$. Then all horizontal "black" lines are updated on all time levels under consideration. After a waveform sweep along horizontal lines a sweep along vertical lines will be applied. A parallel algorithm is compared to a sequential one for several time dependent test problems. For the time discretization the second order accurate BDF(2)-scheme is used. We investigate the performance on a Convex 3820 computer on one processor. Efficiency on parallel machines is considered.

Grid-independent convergence based on preconditioning techniques

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Today numerical calculations are no longer restricted to a class of simple problems, but cope with complicated simulations and complex geometries. In many situations the accuracy of the numerical solution is determined by the limited amount of computer power and memory. Therefore much attention has been given to numerical methods for solving the large sparse system of equations $Ax = b$ obtained by discretizing some partial differential equation. Since direct methods require relatively much computer storage and CPU-time, a large variety of iterative methods has been derived. In this presentation we will focus on iterative methods like MICCG and algebraic multigrid. Gustafsson has shown that for several problems the CPU-time using MICCG is $O(N^{5/4})$ in 2 dimensions and $O(N^{7/6})$ for 3D-problems, where N is the total number of unknowns. Multigrid methods perform even better and for a large class of problems they have an optimal order of convergence: the amount of work and storage is proportial to the number of unknowns N. However, due to the required proper smoothers and the restriction and prolongation operators at each level, the implementation of multigrid for practical problems is much more complicated than that of MICCG. Therefore we have been looking for a special preconditioning technique by which we can construct a lower-triangular matrix L and an upper-triangular matrix U such that the preconditioned system

$$(LU)^{-1}Ax = (LU)^{-1}b$$

can be solved with the optimal computational complexity $O(N)$ with iterative methods such as CGSTAB or GMRES. The basic idea behind this is the same as in multigrid methods. Many iteration methods like SOR and CGSTAB can eliminate high-frequency errors very effectively, but they are inefficient at eliminating long wavelength error. Multigrid techniques use coarser grids in order to remove the low-frequency error effectively. Our new preconditioning technique uses a partition of the unknowns based on the different levels of multigrid. The ordering of the unknowns makes it possible to construct an incomplete LU-decomposition which can be used in eliminating effectively both high- and low-frequency errors.

The method is tested on a variety of well-known elliptic test problems described in the literature. These include strongly varying coefficients, advective terms and grid refinement. In all cases, the method is much cheaper than standard MICCG. This difference is more pronounced for the really difficult problems and increases with the dimension. The convergence behaviour is, in contrast to that of standard MICCG, always smooth, which is advantageous for the construction of stopping criteria when the linear solver is used as an inner-iteration method within some inexact Newton-method.

Multigrid with matrix–dependent transfer operators for convection–diffusion problems

by
Arnold Reusken

Abstract

In the first part of this paper we consider the 1D problem

$$-\varepsilon U''(x) + U'(x) = f(x) \quad \text{on }]0,1[,\ U(0) = U(1) = 0 \quad (\varepsilon > 0)$$

For discretization we use standard 3-point finite difference schemes for which the resulting matrix is an M–matrix. We consider a two–grid method with Jacobi or Gauss–Seidel smoothing and with a matrix–dependent prolongation and restriction as in [H], [W]. For the coarse grid matrix we use the Galerkin approach. For this two–grid method a complete convergence analysis is presented. An important part of this analysis is based on a close relation between the coarse grid matrix and the Schur complement of the fine grid matrix w.r.t. a red–black ordering of the nodes. The upper bound for the contraction number w.r.t. the maximum norm is a simple function of ε/h and ν (ε: perturbation parameter; h: mesh size; ν: number of smoothings), and shows some interesting phenomena. Moreover, numerical experiments indicate that the bound is sharp in a certain sense.
In the second part of the paper we generalize the 1D method to a method for a 2D convection–diffusion problem:

$$-\varepsilon \Delta U + \beta.\nabla U = f \quad \text{on } \Omega =]0,1[\times]0,1[\ ,u = 0 \quad \text{on } \delta\Omega \quad (\varepsilon > 0).$$

For discretization we use a uniform rectangular grid and 9–point finite difference approximations on this grid such that the resulting matrix is an M–matrix. As in the 1D case matrix–dependent transfer operators are used based on the Schur complement w.r.t. a red–black ordering of the fine grid nodes. For the coarse grid matrix we use the Galerkin approach. An important new part of the method in 2D is the use of *lumping* of 9–point stencils into 5–point stencils. This lumping is used to keep compact (9–point) stencils for the transfer operators and for the coarse grid operators.
Some theoretical results for this method will be given; for example, it can be shown that all coarse grid matrices are M–matrices, and using Fourier modes the effect of the lumping procedure can be analyzed. Numerical results will be presented which show the robustness of the method.

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Implementation aspects of the multigrid formulation of flux-difference splitting algorithms for steady Euler equations on unstructured adaptive grids.

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Multigrid methods were originally formulated on structured grids. Seeking an efficient implementation of the multigrid on unstructured meshes involves reformulation of the basic building blocks: the smoother, the injection, the restriction and the prolongation operators. The choice of grids and intergrid connections determines the amount of work needed for these operations.

To facilitate intergrid relations, coarser meshes can be constructed with some telescoping grid elements, telescoping nodes or cell faces. The coarser mesh is said to have telescoping nodes if all nodes appear in the finer mesh. In this case injection is straight forward, restriction of residuals needs a gather phase and prolongation needs a scatter phase over surrounding nodes. The way these gather and scatter operations can be implemented is discussed.

If there is no relation between the different grids, interpolation techniques must be applied during all intergrid operations, resulting in a higher cpu overhead. Of course in this case, grid generation of the different grids can be done independently.

Keeping the grid generation to a minimum by incorporating locally flow adapted meshes into the multigrid may lower the convergence speed. This happens when the coarse grids are not coarsened in the non-adapted regions. So-called grid independency of the multigrid may no longer be maintained. So, implementation choices of the multigrid method have great influence on the efficiency and on the mesh generation. The mesh generation must therefore be evaluated together with the multigrid.

The most cpu intensive part of a multigrid process is the smoother. We considered the vertex centred formulation of a flux-difference finite volume algorithm. Different types of smoothers were studied. The data structure used to calculate the fluxes and flux balances determines the efficiency and the ease of the construction of the elements of the smoother. Especially higher order formulations place an extra demand on memory and cpu resources.

Several alternatives, point-based, edge-based or cell-based data structures are possible. For every choice different possibilities exist to describe the connectivity information. An evaluation of these choices is given.

Finally for a given data structure type and a given algorithm, orderings can be imposed on the different lists of the data structure. These orderings can improve the speed of searching algorithms and the speed of execution by improving local use of data.

The present implementation is built on an edge-based data structure for the connectivity information. This means that two types of lists are maintained, edge lists and node lists. These lists are sufficient for formulation of the Euler solver. A third list contains the geometrical information. Since the connection between grids is based on geometrical information, this list is used for the construction of the intergrid operators.

Node list orderings of snake-walk type, reverse Cuthill-Mckee type, and others were tried. Ordering of the edge lists are based on the ordering of the node lists. Special orderings are used for vectorization. Timings are given for some test cases.

Multilevel, Extrapolation, and Sparse Grid Methods

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Abstract

Multigrid Methods are asymptotically optimal solvers for discretized partial differential equations (PDE). For the optimal solution of the PDEs, however, the quality of the discretization is of the same importance as the speed of the algebraic solution process. Especially for high accuracy requirements, high order discretizations become increasingly attractive.

In this presentation we describe higher order techniques, like *extrapolation* and *sparse grids* that are particularly interesting in the context of multilevel algorithms, because they are based on discretizing the problems on grids with different mesh sizes, as they naturally arize in multigrid.

There are many different variants of extrapolation for PDEs. The classical *Richardson extrapolation* can be extended and generalized in many ways. One generalization is to consider the mesh widths in the different coordinate directions as distinct parameters. This leads to the so-called *multivariate extrapolation* and the *combination* and *sparse grid* techniques. The combination technique has special advantages with respect to parallel computing, because it leads to completely independent subproblems that can be computed in parallel. In this respect sparse grids are particularly attractive for three dimensional problems.

Besides these *explicit* extrapolation methods that are based on linear

combinations of computed solutions there are also extrapolation methods, where an improved solution is defined *implicitly*. In this class are techniques that allow an especially efficient implementation within multigrid algorithms. Such techniques have been proposed and analyzed under the name τ-extrapolation. In the view of the above discussion, these methods can also be generalized by using multivariate combinations. In particular, we will show that the original sparse grid finite element method can be interpreted as an implicit combination technique.

Implicit extrapolation methods do not depend on the existence of global error expansions, but only on a *local* analysis. Thus it can also be shown that these techniques are closely related to the p- and h-p-version of the finite element method. In particular, they can be used adaptively in the mesh size and the order. This opens the possibility to use these methods in the context of local refinement algorithms and even on unstructured meshes. It is important to realize that implicit extrapolation methods can be generalized to arbitrary order, however, that these methods are unstable for extrapolation beyond a certain order (typically 4). The instability can be compensated by *stabilizing constraints*.

In the paper we will present a systematic overview of the algorithmic possibilities and will discuss their specific advantages and features with special emphasis to their compatibility with multigrid methods, parallelization, and adaptive techniques.

SOLUTION OF FULL POTENTIAL EQUATION ON AN AIRFOIL BY MULTIGRID TECHNIQUE

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A B S T R A C T

Multigrid technique is a method which accelerates the convergence and hence reduces the CPU time for a given flow problem. This technique has been used to solve the Full Potential Equation on a symmetrical airfoil in the cartesian grid. The Full potential equation is of non-conservative form. The present paper describes the multigrid technique solutions obtained upto three levels for zero angle of incidence and upto two levels for one degree of angle of incidence. Different grid levels considered have been shown and the CPU time obtained for the two cases have been tabulated. The table shows clearly that the CPU time is reduced considerably by this technique. The equivalent number of iterations for the converged solution decreases as the number of levels is increased. The solution did not converge in some cases because of the reason that the size of the lowest level grid is very coarse. Increase of angle of incidence might also have affected the multigrid convergence. The computed pressure distributions have also been presented.

Robustness of Multi-Grid applied to Anisotropic Boundary Value Problems on General Domains

Rob Stevenson

We will consider elliptic boundary value problems $L(\epsilon)u = f$ $(\epsilon > 0)$, where $L(0)$ is non-elliptic. When multi-grid is applied on discretizations of these singular perturbed problems, then generally the convergence rate of the method is not less than 1 uniform in h *and* ϵ, that is, the method is not *robust*. Since $L(0)$ is non-elliptic, for small ϵ and h there exist oscillating error functions that are mapped almost onto zero by the discretized differential operator. This means that a general smoother cannot be expected to reduce these errors significantly. Moreover, since these errors are oscillating, they are not reduced by the coarse-grid correction as well.

A well-known approach to obtain a robust multi-grid method is to apply a smoother that is an exact solver for the equation with $\epsilon = 0$. However, numerical experiments with ILU-smoothers show that this approach does not unconditionally yield a robust method. Further analysis is therefore necessary.

We will focus on the anisotropic equation $-(\epsilon\partial_1^2 + \partial_2^2)u = f$ on some domain Ω with homogeneous Dirichlet boundary conditions. For a square or a L-shaped domain Ω, we will give sufficient conditions for robustness of W-cycle multi-grid. We will show that these conditions are met for certain modifications of ILU-smoothers. For a square domain Ω, we even show robustness of the V-cycle.

From our analysis it will appear that we may expect problems in case the direction of the anisotropy is between both sides of the re-entrant corner. We will illustrate our findings by numerical results.

Optimal Multigrid Solvers for Steady State Inviscid Flow Problems

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Abstract

A new approach to the solution of steady state inviscid flow problems is presented. It includes a new discretization methods and an optimal multigrid solver for it. Discretization is based on separating the hyperbolic part from the elliptic one (in subsonic and in incompressible flow problems), using upwind discretization for the hyperbolic part and central one for the elliptic part. This approach yields discretization with a different form of viscosity used before, and also to smaller amounts of viscosity compared to existing schemes (upwind or central differencing). In particular, no viscosity terms are added to the continuity equation.

The multigrid methods used for these dicscretizations were constructed using the same principles guided the discretization. That is, the elliptic part of the system is being treated in a different way than the hyperbolic part. The multigrid method presented here does not use the usual time dependent embedding of the steady state equations, but rather uses a relaxation that work directly on the steady state problem. A preconditioning matrix separates the two different types of subsystems (elliptic and hyperbolic) enabling the optimal treatment of each. Relaxation for the hyperbolic parts is a pointwise Gauss-Seidel in the stream direction while the elliptic part is relaxed by Kaczmarz relaxation. Residual transfers are modified to obtain optimal efficiencies.

Numerical experiments with four inviscid flow models (in the subsonic case) are presented for exterior and interior geometries, both for incompressible and compressible cases. Body fitted grids are used with the new conservative finite volume discretization. Convergence rates like that of the Poisson equation on the same grids are obtained. Nozzle calculations shows convergence rate of the order of .1 per multigrid cycle, and flow around a circle with convergence rate of .18 per multigrid cycle. Moreover, solutions for flow over symmetric bodies at angle of attack yield zero lift, unlike existing schemes which produce significant lift for such cases.

Multigrid techniques for a class of discretely divergence–free finite element spaces

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Abstract. We consider multigrid techniques for solving the Stokes problem

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f}\,,\ \nabla\cdot\mathbf{u} = 0\,, \quad \text{in } \Omega \quad , \quad \mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega\,, \tag{V}$$

where the pair $\{\mathbf{u}, p\}$ represents the velocity and the pressure respectively in a bounded region $\Omega \subset \mathbf{R}^2$ with prescribed boundary values on $\partial\Omega$ and a given force $\mathbf{f}$. For simplicity, we assume that Ω is a convex polygon and that the boundary values $\mathbf{g}$ are homogeneous.
Using a standard weak discrete formulation of problem (V) and taking discretely divergence–free finite element subspaces $\mathbf{H}_h^d \subset \mathbf{H}_h$, we achieve the simplified positive definite scheme for the velocity only:

Find $\mathbf{u}_h^d \in \mathbf{H}_h^d$, such that

$$a_h(\mathbf{u}_h^d, \mathbf{v}_h^d) = (\mathbf{f}, \mathbf{v}_h^d) \quad , \forall\, \mathbf{v}_h^d \in \mathbf{H}_h^d. \tag{V_h^d}$$

Here, $\mathbf{H}_h$ and the bilinear form $a_h(\cdot,\cdot)$ are discrete versions of $\mathbf{H}_0^1(\Omega)$ and $(\nabla\cdot, \nabla\cdot)_{L^2}$ respectively, and will be more precisely defined in the lecture, where also some examples for these spaces and some of their basic properties are given.
Beside the obvious advantages of the explicit construction of the subspaces $\mathbf{H}_h^d$ (elimination of the pressure, reduction of the number of unknowns, definite stiffness matrices) a disadvantage is the complicated implementation and the bad condition number of the system matrices. But, this can be overcome by an appropriate multigrid algorithm, with convergence rates independent of the mesh width. The theoretical problem is to prove the corresponding *smoothing* and *approximation properties* for the used smoothers and grid transfer routines. The convergence proof for the ***approximation property*** is similar to the proposed one by Brenner, while the *smoothing property* can be shown by a new technique, mainly using the basic properties of the finite element spaces. On the other hand we get the practical problem to implement in an efficient way our analyzed transfer operators which have to interpolate discretely divergence–free between two levels. This, too, can be solved by a very precise and optimized implementation such that the final numerical amount for solving the Stokes equations is about the same like solving a scalar Poisson equation with usual nonconforming spaces.
The subsequent numerical tests and the excellent experiences in a fully developed nonsteady Navier–Stokes code for high Reynolds numbers confirm the proposed theoretical results and show the high numerical applicability of this class of finite elements.

Different Approaches to Discretization within a Multilevel Algorithm

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The use of multilevel methods in Computational Fluid Dynamics (CFD) is now widely accepted. Their application has been extended to many fluid flows, and their implementation in commercial software has begun. However the underlying algorithms still require further development and new ideas in CFD need to be incorporated into the multilevel approach.

The authors have been concerned with the efficient and accurate finite volume solution of the primitive variable formulation of the Navier-Stokes equations for elliptic fluid flow problems. Solution of such problems has traditionally used a segregated solution technique such as SIMPLE [1] which solves throughout the domain of interest for each variable in turn, iterating until satisfactory convergence is achieved. This reduces convergence speed and stability. A further problem of traditional techniques has been the use of first order discretization which is marred by a numerically diffusive error term. Over the past ten years there has been much work done to improve on these. The authors have been involved in this in three main areas: multigrids, unsegregated solvers and high order discretizations[2, 3]. This has led to the production of a computer code that uses a multilevel version of the unsegregated solver BIM [2] to find solutions to the Navier-Stokes equations when using the high order bounded discretization SMART[4] via defect correction[5]. This code has been successfully applied to a number of problems of engineering interest and has also incorporated local grid refinement.

In this paper the flux-vector splitting method of Dick[6] is considered. His theory takes the ideas of Steger and Warming [7] for the inviscid Euler equations and applies them to the Navier-Stokes equations by using upwind differencing on a split differential operator. This has been implemented successfully for practical problems by Dick and other researchers[8]. This has demonstrated its wide applicability and stability, but it is still restricted to the use of upwind differencing. This paper will present a novel approach of implementing the method within the context of the authors existing work on unsegregated solvers with full operators and high order discretizations. This allows for a full comparison between these two approaches to discretization to be presented which should be of interest to many researchers. Detail is also

given about the use of the high order discretizations developed for full operators in the context of flux-difference split operators. This combines the accuracy of such discretizations with the stability of flux-differencing.

Solutions will be presented in three dimensions, making full use of the computing power and advanced graphics software available nowadays. The test problems presented will include ones with in/outflows, buoyancy effects and mass transfer.

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B-Wavelet MG Methods

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Multigrid methods are modern numerical methods for solving discrete equations $L_h U_h = F_h$ of elliptic P.D.E. with boundary value, and ablke to include some applies of following numerical techniques: 1) transforming the numerical informations between different levels, 2) smooting the high frequenc, 3) coarser-correction, 4) approaching of numerical solution by the refined grids, 5) changing construction of L_h, 6) applying different discrete methods for elliptic differential operator, etc... Multigrid methods can obtain the high efficiency characters: A) the computing complexity is of order $O(N)N$, is the variable number of the finest grid. B) the algorithm is numerical stable, G) the numerical solution to obtain has higher accuracy than that by other usual algorithms. We have summarized and classficated 6 kinds of multigrid methods according to the applied effects of 1)-6) and A)-G) in this paper, specially, the B-wavelet MG method presented in this paper is a new kind of multigrid methods.

Wavelet transform for solving P.D.E. has only very recently started to investigate, we will apply B-wavelet basis in finite element method and in multigrid methods, we call it as B-Wavelet MG method and represent it briefly in following.

Let $\Omega \subset R^2$ is the domain, Ω_{00}^* is the coarsest grid of $\Omega - [0,1]^2, \Omega_{00}^*$ can be refined dynanicly for n times respectively in x and y directions, also $\Omega_{00}^* = \cup\Omega_{ik}(i,k = 0,1,\ldots,n)$, $\dim(\Omega_{00}^*) = (M+1)^2, \dim(\Omega_{nn}) = (2^{n-1}M)^2$, $\dim(\Omega_{nn}^*) = (2^n M+1)^2 = N, \Omega_{nn}^*$ is the finest grid. We definit the B-wavelet basis $\Psi_{ijkl}(x,y)$ on Ω_{ik} and construct the conforming finite element function space V_{nn} and its wavelet subspaces W_{ik}:

$$\begin{aligned} V_{nn} &= \text{span}\ \{\Psi_{ijkl}^*(x,y) \ : \quad i,k = 0,1,\ldots,n(j,l) \in \Omega_{ik}\} \\ W_{ik} &= \text{span}\ \{\Psi_{ijkl}(x,y) \ : \quad (i,k) \in \Omega_{ik} i,k = 1,2,\ldots,n\} \\ V_{nn} &= \otimes W_{ik}, \qquad i,k = 0,1,\ldots,n \end{aligned}$$

for harmonic equations of $2m$-order with Dirichlet boundary value. Hence we can obtain the finite equations

$$\begin{aligned} &\sum_{ijkl} a(\Psi_{ijkl}, \Psi_{\hat{i},\hat{j},\hat{k},\hat{l}}) = (f, \Psi_{\hat{i},\hat{j},\hat{k},\hat{l}}) \\ &(j,l),(\hat{j},\hat{l}) \in \Omega_{ik}, \quad i,k = 0,1,\ldots,n. \end{aligned}$$

namely

$$K_{ik} U_{ik} = F_{ik}, \quad i,k = 0,1,\ldots,n.$$

B-wavelet hierarchical basis $\Psi_{ijkl}(x,y)$ definited above is semi-orhtogonal in the sense of the energy norm in Sobolev space $H^m(\Omega)$, therefore the stiff-matrix L_h is splitted to $(3n+1)$ diagonal

blocks, namely

$$L_h = \text{diag}\,(K_{00}, K_{01}, K_{10}, K_{11}, \ldots, K_{n-1,n}, K_{n,n-1}, K_{nn}).$$

Each block matrix can be the tridiagonal, the entries in each block are rapidly dencay on both sides. Every one of these blocks is strong positive definited and has a good condition number, spacially by using precondition algorithms the condition number of the handled matrix is $0(1)$. We can solve indepently $(3N+1)$ subsystems of the finite element equations not only by parallel algorithm but also by other highly efficient algorithm. It is obvious, the numerical process is numerical stable and the computing complexity work is of optimal order $0(N)$.

The B-wavelet MG method presented in this paper is a highly efficient algorithm for solving P.D.E., it can mix together with 6 aspects represented above in an organic whole, this method implicate more good algorithms. It connects with spline, wavelet and finite element approximation, with MG method and multiresoltuion analysis, with reduction method and the binary relation of the basis, with hierarchical basis method and the separation of frequence components in wavelet spaces, with splitting algorithm and the orthogonal resolution of function spaces.

MULTIGRID FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS ON STAGGERED GRIDS IN GENERAL COORDINATES

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Abstract

Galerkin coarse grid approximation (GCA) in multigrid methods is investigated for the incompressible Navier-Stokes equations in general coordinates. An efficient algorithm performing GCA is presented. The behavior of coarse grid matrices is studied under GCA with different transfer operators. For square and L-shaped driven cavity problems, the performance of the multigrid method using different combinations of transfer operators for the computation of coarse grid matrices and of coarse grid correction is investigated. Further computations are carried out in general coordinates for a channel flow problem with backward facing step in three dimensions.

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A MULTIGRID METHOD FOR LOW-MACH NUMBER COMPRESSIBLE NAVIER-STOKES EQUATIONS ON 2D UNSTRUCTURED GRIDS

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A wide class of non-isothermal viscous flows is characterized by low velocities and high temperature (and, hence, density) variations. An integration of the compressible Navier-Stokes equations for very low Much numbers presents severe problems in practice. On the other hand, the well-known Boussinesq approximation is invalid when the density variations are greater than the mean values. The low-Mach number Navier-Stokes ("hyposonic") equations seems to be the most adequate for problems in question. These equations follow from the full Navier-Stokes equations under the following assumptions: 1) Mach number is small; 2) the hydrostatic compressibility parameter is small; 3) the characteristic time is large compared to an acoustic time scale. In the case of an open domain (i.e. when the mean pressure is constant) the equations in dimensionless form are as follows [1-2]:

$$\frac{d\rho}{dt} + \rho\nabla\cdot V = 0 \qquad (1)$$

$$\rho\frac{dV}{dt} = -\nabla p^{(1)} + \frac{\rho-1}{Fr}\,j + \frac{1}{Re}\left[2\mathrm{Div}\left(\mu\dot{S}\right) - \frac{2}{3}\nabla\left(\mu\nabla\cdot V\right)\right] \qquad (2)$$

$$\rho\frac{dT}{dt} = \frac{1}{RePr}\nabla\cdot\left(\lambda\nabla T\right) \qquad (3)$$

$$\rho T = 1 \qquad (4)$$

Low-Mach number Navier-Stokes equations similarly to the incompressible flow equations do not describe acoustic phenomena. Hence, the numerical methods for hyposonic flows are usually developed by modification of incompressible flow solvers.

A number of numerical methods have been used to solve hyposonic equations in primitive variables on structured grids [1-3]. An aim of the present study is to develop a multigrid hyposonic flow solver on unstructured grids.Only explicit and point implicit schemes have been considered. Two space discretizations on Delaunay triangulation of two-dimensional domain are studied. The first one is storage scheme suggested by Nicolaides which generalizes the well-known MAC method to the triangular cells): in the second one both Cartesian components of velocity are stored at vertices.

As tests for methods' assessment the incompressible backward facing flow and the hyposonic flow in a plane channel with isothermal wall sections have been chosen. The presented methods have been applied to two practical problems: flow and heat transfer in epitaxial reactors used for growth of semiconductor materials [2] and flow around a number of cylinders in a plane channel.

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A New Residual Smoothing Method for Multigrid Acceleration Applied to the Navier-Stokes Equations

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A new residual smoothing method is proposed, based on a decomposition into forward and backward sweeping steps. It offers improved smoothing properties for both central and upwind schemes in Euler flow calculations. It is also computationally cheaper than central residual smoothing and vectorizes completely. The properties of the new residual smoothing method applied for viscous calculations are investigated by means of a Fourier analysis on the following model equations:

I) 2D convection equation with unity mesh aspect ratio,

II) 2D convection-diffusion equation with a high-aspect-ratio.

Both central and upwind space discretizations are considered together with explicit multi-stage Runge-Kutta time-stepping. In addition, different options for high-aspect-ratio meshes are compared and tested. The Fourier analysis shows the improvement of the high-frequency-damping, which plays an important role in the multigrid acceleration. The numerical results of 2D flat plate laminar flow calculations confirm the efficiency of the new residual smoothing approach.

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