CONFERENTIE
VAN NUMERIEK WISKUNDIGEN

5 oktober - 7 oktober 1992

CONFERENTIEOORD WOUDSCHOTEN

ZEIST

Werkgemeenschap Numerieke Wiskunde

CONFERENTIE
VAN NUMERIEK WISKUNDIGEN

Werkgemeenschap Numerieke Wiskunde

ZEVENTIENDE CONFERENTIE NUMERIEKE WISKUNDE

Doel van de conferentie

De Conferentie Numerieke Wiskunde wordt eenmaal per jaar gehouden onder auspiciën van de Werkgemeenschap Numerieke Wiskunde. Het doel van de conferentie is kennis te nemen van recente ontwikkelingen binnen de numerieke wiskunde. Hiertoe worden jaarlijks twee thema's vastgesteld. Enkele buitenlandse deskundigen worden uitgenodigd over deze thema's lezingen te houden.

Thema's

A. Meerdimensionale approximatie, wavelets en toepassingen in de beeldverwerking.

B. Adaptieve roostermethoden voor partiële differentiaalvergelijkingen.

Organisatie

De organisatie is in handen van de voorbereidingscommissie bestaande uit Th. J. Dekker (UvA)(voorzitter), P.W. Hemker (CWI/UvA), C.R. Traas (UT) en B.P. Sommeijer (CWI)(secretaris). Medewerking is verleend door S. van der Wolff (CWI) en het Centrum voor Wiskunde en Informatica.

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Themes and Speakers

Theme A. Multivariate approximation, wavelets and applications in image processing;

C.K. Chui, Texas A&M University (USA)
A. Cohen, CEREMADE, Université de Paris-Dauphine (France)
B. Mulansky, Technische Universität Dresden (Germany)

Theme B. Adaptive grid methods for solving partial differential equations;

J.E. Flaherty, Rensselaer Polytechnic Institute (USA)
C. Johnson, Chalmers Institute of Technology (Sweden)
D.F. Mayers, Oxford University (Great Britain)

Short presentations will be given by:

W.J. Layton, University of Nijmegen (Theme B)
M.J. van der Marel/M. Borsboom, Delft Hydraulics (Theme B)
Th. J. Ripmeester, University of Amsterdam (Theme A)
R.A. Trompert, Centre for Mathematics and Computer Science (Theme B)
P.A. Zegeling, Centre for Mathematics and Computer Science (Theme B).

Program and titles of lectures

Monday, October 5

10.00 - 11.10	**arrival, coffee**
11.10	**opening**
11.15	**C.K. Chui** 11.15 - 12.05: Introduction to Wavelets 12.05 - 12.15: discussion
12.30	**lunch**
13.45	**J.E. Flaherty** 13.45 - 14.35: Adaptive Methods for Time-Dependent Partial Differential Equations; Part I: Basic Strategies and Error Estimation 14.35 - 14.45: discussion
14.50	**A. Cohen** 14.50 - 15.40: Wavelets and Applications, I 15.40 - 15.50: discussion
15.50	**tea**
16.15	**C. Johnson** 16.15 - 17.05: Adaptive Finite Element Methods in Computational Mechanics, I 17.05 - 17.15: discussion
17.20	**P.A. Zegeling** 17.20 - 17.45: Moving-Grid Methods for Time-Dependent PDEs 17.45 - 17.50: discussion
18.15	**dinner**

Tuesday, October 6

08.00	**breakfast**
09.00	**B. Mulansky** 09.00 - 09.50: A Survey on Multivariate Shape Preserving Interpolation 09.50 - 10.00: discussion
10.05	**Th. J. Ripmeester** 10.05 - 10.30: Smooth Interpolation to Scattered Data by Bivariate Piecewise Polynomials of Odd Degree: Some Improvements 10.30 - 10.35: discussion
10.35	**coffee**
11.10	**D.F. Mayers** 11.10 - 12.00: Adaptive Mesh Refinement for Elliptic Problems, I 12.00 - 12.10: discussion
12.15	**W.J. Layton** 12.15 - 12.40: Adaptive Data Parallel Finite Element Methods 12.40 - 12.45: discussion
12.50	**lunch**
14.15	**J.E. Flaherty** 14.15 - 15.05: Adaptive Methods for Time-Dependent Partial Differential Equations; Part II: High-Order Methods and Parallel Computation 15.05 - 15.15: discussion
15.20	**R.A. Trompert** 15.20 - 15.45: Local Uniform Grid Refinement and Systems of Coupled Partial Differential Equations with and without Time Derivatives 15.45 - 15.50: discussion
15.50	**tea**
16.15	**C.K. Chui** 16.15 - 17.05: Characterization and Creation of Wavelet-Frames 17.05 - 17.15: discussion
17.20	**Meeting of the Dutch "Werkgemeenschap Numerieke Wiskunde"**
18.00	**dinner**

Wednesday October 7

08.00	**breakfast**
09.00	**C. Johnson** 09.00 - 09.50: Adaptive Finite Element Methods in Computational Mechanics, II 09.50 - 10.00: discussion
10.05	**M.J. van der Marel / M. Borsboom** 10.05 - 10.30: Development of an Adaptive Grid Algorithm that Compensates for Discretisation Errors, with an Application to the 1D Burgers Equation 10.30 - 10.35: discussion
10.35	**coffee**
11.10	**A. Cohen** 11.10 - 12.00: Wavelets and Applications, II 12.00 - 12.10: discussion
12.15	**lunch**
13.30	**B. Mulansky** 13.30 - 14.20: Shape Preserving Interpolation by Bivariate Quadratic Splines 14.20 - 14.30: discussion
14.35	**D.F. Mayers** 14.35 - 15.25: Adaptive Mesh Refinement for Elliptic Problems, II 15.25 - 15.35: discussion
15.35	**closure, tea, departure**

Introduction to Wavelets

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This is an introductory exposition of the subject of wavelet analysis. The objective is not to go into any details but only to discuss the basic ingredients of wavelets, comparing it with the established subject of Fourier analysis. Using spline functions as building blocks, the notion of multiresolution analysis is introduced, and the need of the complementary subspaces governed by a projection (approximation) operator is motivated. Time-frequency analysis is discussed, with the aid of modeling the cochlea of the human ear. In this regard, decomposition and reconstruction wavelet algorithms as well as wavelet-packet tree algorithms are demonstrated.

Examples of compactly supported wavelets are shown and a list of important applications is given. Demonstrations will include transient signal analysis, echo detection and cancellation, real- time noise removal, as well as image compression.

Characterization and Creation of Wavelet-Frames

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This talk reports on my joint work with X.L. Shi. When any finite energy signal is represented by a wavelet series, the coefficient sequence gives very valuable time-frequency information of the signal. However, since a wavelet basis is linearly independent, any error in the coefficient sequence has an effect on the signal representation. On the other hand, when a frame series is used for signal representation, due to the redundancy of the frame family, there is more room for errors. Any frame is a Bessel sequence. Hence, in order to construct frames, it is necessary to study the structure of Bessel sequences.

In this respect, criteria, in terms of certain smoothness and rate of decay, ofsquare-integrable functions that generate Bessel sequences are given. As a consequence, it is shown that frames are obtained by arbitrarily oversampling any of the well-known wavelets. On the other hand, for any integral scaling parameter a, it is also shown that nX oversampling of any frame preserves the frame bounds, provided that n and a are relatively prime; consequently, for tight frames, and more generally frames with duals, the frame series representations remain valid for such oversampling. Multivariate frames are also considered. Some of the tipics to be discussed are: Why frames? Necessary conditions for frames, sufficient conditions for frames, generating frames by oversampling, boundedness of affine operators, and dual frames.

WAVELETS AND APPLICATIONS

A. Cohen
(CEREMADE-ENSTA)

These two talks will describe the theory of wavelet bases and its potential applications, in particular for image processing and numerical analysis.

The first talk will be tutorial, following the approach of I. Daubechies in the construction of compactly supported wavelets ([1]). We shall describe their relations with multiresolution analysis, scaling functions and subband coding schemes, insisting on the properties that can be useful for the two previously mentionned applications.

The second talk will describe some more recent research (jointly made with I. Daubechies and P. Vial [2]) about the construction of wavelet bases adapted to an interval. The goal of this construction is to avoid the "border effects" in image processing and to control the limit conditions efficiently in numerical analysis.

References:

[1] I. Daubechies, "Ten lectures on Wavelets", SIAM 1992.
[2] A. Cohen, I. Daubechies and P. Vial, "Wavelets and Fast Wavelet Transform on an interval", preprint Bell Labs, 1992.

Adaptive Methods for Time-Dependent Partial Differential Equations

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Part I: Basic Strategies and Error Estimation

Abstract

Several of the "grand challenges" of scientific computation [Bro] involve the solution of complex multi-dimensional steady and transient partial differential equations. As mathematical models include more realistic effects, all of these problems exceed the capabilities of current computer systems and more massive levels of parallelism will be needed to address them. However, raw power alone is not sufficient to solve a complex problem. The results must be reliable, computations must be performed in an efficient manner, and the solutions must be applied only in relation to the limitations of the model. Adaptivity, with its associated error estimates, provide the appropriate measures of reliability and efficiency. Adaptive solutions often converge at rates that are much higher than those obtained by conventional methods using a single grid. Such algorithmic advances will provide the break throughs that will allow scientists and engineers to solve the most demanding problems with reasonable resources.

An adaptive method utilizes a preliminary solution of the partial differential equations obtained on a coarse mesh with a low-order numerical technique. The mesh, at this early stage of the process, is designed to reflect the spatial and temporal scales on which important events are expected to occur. Error indicators, which are frequently estimates of the local discretization error of the computational method, are used to identify spatial and temporal regions where resolution is inadequate. The solution space is enriched in these regions and the problem is recursively solved until the specified accuracy has been obtained. The basic adaptive enrichment strategies are (h-type) mesh refinement where the mesh is refined or coarsened in regions of low or high accuracy, (p-type) order variation where the order of the method is increased or decreased in different regions, and (r-type) mesh motion where the mesh is moved so as to follow and resolve evolving dynamic phenomena [FPS]. The most successful adaptive strategies use a combination of the basic techniques. The combination of h- and p-refinement, for example, can yield exponential convergence rates in certain situations [RB].

H-refinement is by far the most popular of the three enrichment strategies. For time-dependent problems, it has been used within a method of lines framework [AFc, AFd], where temporal refinement is global and spatial refinement is local, or as a local refinement method where both spatial and temporal refinement are local [ArF, BO, MFa]. High-order methods with adaptive p- and hp-refinement are gaining in popularity. Spatial representation for continuous finite element solutions is simplified by use of a hierarchical basis [SB] where high-order approximations are obtained as corrections to lower-order ones. Discontinuous spatial approximations may be more appropriate for the solution of hyperbolic systems and these can be constructed using a basis of piecewise Legendre polynomials [CS, DF]. Local and global a posteriori

estimates of spatial errors may also be obtained by p-refinement. Thus, the error estimate is obtained by using a piecewise polynomial approximation of one degree higher than that used for the finite element solution. For parabolic systems, nodal superconvergence simplifies the process by allowing errors to be neglected at element vertices. A dichotomy principle of Babuska [Ba], stating that the errors of odd-order finite element approximations arise mainly near element edges and those of even-order approximations arise mainly in element interiors, further simplify the error estimation process [Ba, AFe, AFW]. Nodal superconvergence does not apply for hyperbolic systems; however, Devine and Flaherty [DF] have discovered other superconvergence points that may be used for the efficient construction of spatial error estimates.

Temporal integration of parabolic systems within a method of lines formulation is, perhaps, best performed using backward difference methods; however, one-step methods may be more suited for local refinement methods. The discontinuous Galerkin method (cf., e.g., Johnson [Jo]) employs a Legendre basis and has excellent stability properties. Singly implicit Runge-Kutta (SIRK) methods employ a Laguerre polynomial basis, also have excellent stability properties, and have efficiencies rivaling those of backward difference methods [MFb]. Techniques for balancing space and time errors involve maintaining a measure of the local error or local error per unit step at a fraction of the spatial error [LBD].

R-refinement has been a very successful adaptive method in one space dimension. Techniques are based on spatial equidistribution of a motion indicator [AFa, AFb] or a mixture of temporal and spatial equidistribution [DO]. Care must be exercised since most mesh motion procedures for dissipative partial differential systems are sensitive to small perturbations [CFL]. A rather complete treatment of the subject with detailed comparisons of three methods was recently completed [FVZ]. Multi-dimensional mesh motion is much more difficult and most successful techniques are limited to special applications.

Adaptive strategies that combine h-, p-, and r-refinement have the potential for the greatest efficiency. Specific techniques prefer p-refinement in regions where solutions are smooth and h-refinement near singularities [AFM].

Part II: High-Order Methods and Parallel Computation

Abstract

The combination of adaptive mesh (h-type) refinement and (p-type) order enrichment can produce numerical schemes that have exponential rates of convergence [RB]. This efficiency not withstanding, parallel computation will be necessary to solve the most demanding computational problems. On the other hand, algorithmic advances, such as adaptivity, and not evolutionary hardware advances will provide the break throughs necessary to solve the most demanding problems.

Quadtree and octree data structures are often used to manage the data associated with both mesh generation and adaptive mesh refinement [BWS]. This data structure has numerous parallel constructs that may be used to construct processor assignment strategies on shared- and distributed-memory computers. Coloring procedures separate contiguous spatial regions with a goal of avoiding critical sections on shared-memory computers. Coloring the quadrants or octants of a quadtree or octree, respectively, is far simpler than coloring the elements of the unstructured meshes that these procedures develop. Indeed, the quadrants of a quadtree structure can be colored using a maximum of six colors so that no neighboring quadrants have the same color [BBF]. The

procedure uses a depth-first traversal of the tree and, thus, has linear time complexity.

Performance indicates scalability when piecewise linear finite element approximations are used; however, parallel performance degrades with increasing polynomial degree and with adaptive p- or hp-refinement. Since hierarchical bases for continuous finite element approximations are mainly associated with element edges or interiors rather than with vertices, enhanced parallelization could occur with an edge- or face-coloring procedure. This is indeed the case, and a three-color procedure for separating the edges of triangular meshes had excellent scalability when used with high-degree finite element approximations [BFK].

The local Galerkin finite element method of Cockburn and Shu [CS] seems to be well suited to parallel computation on distributed-memory computers. The local projections only involve nearest-neighbor communications regardless of the degree of the piecewise polynomial approximation. Upwinding, projection limiting, and the discontinuous basis result in sharp resolution at shocks and other discontinuities without spurious oscillations or excessive diffusion. Computations on a ten-dimensional NCube-2 hypercube exhibited excellent scaled parallel performance when used with explicit time integration [DF]. An adaptive p-refinement procedure performed likewise.

The key problem with parallel adaptive strategies on distributed-memory computers is developing optimal metrics and strategies that include both computational and communications costs. At present, these are largely unexplored. Some preliminary strategies involve run-time task distribution [BFO].

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Adaptive Finite Element Methods in Computational Mechanics

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We present a general approach to adaptivity for finite element methods and give applications to linear convection-diffusion problems, linear elasticity, nonlinear elasto-plasticity, incompressible flow and nonlinear conservation laws including compressible flow.

1 Introduction

These lectures will review the general approach to adaptivity for finite element methods presented in [1] - [16]. We will present theoretical and computational results for a class of linear convection-diffusion problems ranging from diffusion-dominated elliptic and parabolic problems to convection-dominated hyperbolic problems,linear elasticity, nonlinear parabolic problems, incompressible flow, non-linear hyperbolic conservation laws including compressible flow, and some non-linear elliptic problems modelling elasto-plasticity and unilateral contact.

The basic problem in adaptivity for finite element methods may be formulated as follows: Suppose $\mathcal{P}$ is a given (initial)-boundary value problem with given data f and corresponding exact solution u. Let $\mathcal{P}_{h,p}$ be a finite element method for $\mathcal{P}$ based on piecewise polynomial approximation on a family $\{\mathcal{T}_{h,p}\}$ of meshes with mesh functions $h = h(\boldsymbol{x},t)$ and $p = p(\boldsymbol{x},t)$ giving the local mesh size $h(\boldsymbol{x},t)$ in space and time and local order $p(\boldsymbol{x},t)$ of the polynomial approximations as functions of space $\boldsymbol{x}$ and time t and let $\{u_{h,p}\}$ be the corresponding family of finite element solutions. Let further

$\|\cdot\|$ be a given norm and TOL > 0 a given tolerance. Then construct an algorithm $\mathcal{A}$ for finding a mesh $\mathcal{T}_{h,p}$ such that for the corresponding finite element solution $u_{h,p}$ we have

$$\|u - u_{h,p}\| \leq \text{TOL} \tag{1}$$

and the work to compute $u_{h,p}$ is nearly minimal. As a measure of computational work we shall below for definiteness use the total number of unknowns. We thus seek an algorithm $\mathcal{A}$ for solving (approximately) the following nonlinear optimization problem: Find a mesh $\mathcal{T}_{h,p}^{\text{opt}}$ with as few degrees of freedom as possible, referred to as an *optimal* mesh, under the side condition that $\|u - u_{h,p}\| \leq \text{TOL}$, where $u_{h,p}$ is the corresponding finite element solution. This is a complex problem which is not directly solvable since the exact solution u is not known and the dependence of $\|u - u_{h,p}\|$ on (h,p) is very implicit. In general, both the local mesh size h and the local degree p of the polynomial approximation are variable in space and time and are determined in the adaptive process. We refer to adaptive algorithms in this generality as (h,p)-methods. Adapative algorithms with p fixed or h fixed are referred to as h-methods and p-methods, respectively. In this note we mainly concentrate on h-methods, but our results also directly apply to the case of (h,p)-methods (if p is not large). Concerning h-adaptivity, note that in addition to the local mesh size also the local mesh orientation and stretching of the mesh (in space or space-time) may be adaptively controled. Thus, for simplicity of notation, we assume that the mesh parameter $h = h(x,t)$ in general represents both the mesh size in space and time, as well as the mesh orientation/stretching. The main emphasis below will be on methods without adaptive orientation/stretching, but cases including such features will also be discussed.

Clearly we require the adaptive algorithm $\mathcal{A}$ to be both *reliable* and *efficient*, in the sense that the desired error control should be guaranteed, and the computational work should be nearly minimal. The basic problem in adaptivity is to construct adaptive algorithms that are both reliable and efficient in the sense just given. Successful solutions of this problem will have a profound influence on the finite element software of tomorrow.

In our work ([1] - [16]) we have constructed reliable and efficient adaptive algorithms for a large class of problems, including elliptic, parabolic and hyperbolic problems, linear as well as non-linear problems. Typical applications concern convection-diffusion problems in the whole range from

diffusion-dominated to convection-dominated problems, stationary as well as time-dependent problems, linear wave equations, non-linear monotone elliptic problems modelling unilateral contact and elasto-plasticity, and also non-linear conservation laws.

Our algorithms are based on *a posteriori* error estimates of the form

$$\|u - u_{h,p}\| \leq E_1(h, p, u_{h,p}, f), \tag{2}$$

where the error $\|u - u_{h,p}\|$ is estimated in terms of a directly computable quantity $E_1(h, p, u_{h,p}, f)$ depending on the computed solution $u_{h,p}$, the mesh size $h(\boldsymbol{x}, t)$ and the degree $p(\boldsymbol{x}, t)$ of the polynomial approximation, and the data f of the problem. The dependence in E_1 on the computed solution $u_{h,p}$ typically occurs through the *residual* $R(u_{h,p})$ of $u_{h,p}$, which is basically the deviation from equality (properly evaluated) obtained inserting $u_{h,p}$ into the given continuous equation. For elliptic problems, (2) may typically take the form

$$\|u - u_{h,p}\| \leq C\|h^2 R(u_{h,p})\| \tag{3}$$

or

$$\|\nabla(u - u_{h,p})\| \leq C\|hR(u_{h,p})\|, \tag{4}$$

where $\|\cdot\|$ may be the L_2-norm or an L_q-norm, $1 \leq q \leq \infty$, or weighted such norms, and where C is a certain constant to be commented on below. For first order hyperbolic problems, (2) may typically take the form

$$\|u - u_{h,p}\|_{L_2} \leq C\|\min\left(1, h^{1/2}R(u_{h,p})\right)\|_{L_2}. \tag{5}$$

The adaptive algorithm producing a mesh $\mathcal{T}_{h,p}^{\text{adapt}}$ is constructed from the *a posteriori* error estimate by (approximately) minimizing the number of degrees of freedom under the condition

$$E_1(h, p, u_{h,p}, f) \leq \text{TOL}, \tag{6}$$

This is again a nonlinear minimization problem, but this problem is more directly solvable since knowledge of u is not required and the dependence of $E_1(h, p, u_{h,p}, f)$ on (h, p) is more explicit. Usually this nonlinear minimization problem may be solved iteratively by seeking to equidistribute the element contributions in the quantity $E_1(h, p, u_{h,p}, f)$. A very important feature of

$E_1(h, p, u_{h,p}, f)$ is that this quantity contains information on the *structure* of the error as a function of (h, p), which usually makes it possible to solve the minimization problem, i.e., construct the adaptive algorithm, rather easily. Without sufficient information on the structure of the error it appears difficult to design efficient adaptive algorithms, cf. the discussion below.

Clearly, adaptive algorithms using (6) in particular as stopping criterion will be reliable in the sense that, by the *a posteriori* error estimate (2), the error control (1) will be guaranteed. The adaptive algorithm will be efficient if $\mathcal{T}_{h,p}^{\text{adapt}}$ is close to $\mathcal{T}_{h,p}^{\text{opt}}$, which is the same as requiring that $\mathcal{T}_{h,p}^{\text{adapt}}$ is nowhere overly refined as compared to $\mathcal{T}_{h,p}^{\text{opt}}$. In the case without orientation/stretching, this is a question related to the quality of the *a posteriori* bound (2). To prove sharpness of (2), which is clearly required to give an efficient algorithm, we typically prove that

$$E_1(h, p, u_{h,p}, f) \leq C E_2(h, p, u), \tag{7}$$

where C is a constant and $E_2(h, p, u)$ is a sharp *a priori* error bound depending on h, p and the exact solution u satisfying

$$\|u - u_{h,p}\| \leq E_2(h, p, u). \tag{8}$$

If (8) is (reasonably) sharp and (7) holds, then it follows that $\mathcal{T}_{h,p}^{\text{adapt}}$ is not globally overly refined as compared to $\mathcal{T}_{h,p}^{\text{opt}}$, up to the constant C in (7), which indicates efficiency. To prove that $\mathcal{T}_{h,p}^{\text{adapt}}$ is not locally overly refined, which is the real test of efficiency in the case without orientation/stretching, localized forms of (7) may be used, cf. [2, 1].

Concerning the choice of norm $\|\cdot\|$ in the error control, note that in our approach a variety of different norms are possible, depending on the nature of the problem; we are not in general restricted to the use of energy norms as is the case in other approaches to adaptivity, cf. the discussion below. We recall that energy norms may be used for elliptic problems, while for parabolic and hyperbolic problems energy norms do not play the same role. For elliptic problems we may in addition to energy norms use L_q-norms, $1 \leq q \leq \infty$, for the solution itself or for derivatives of the solution, or weighted such norms. We may for instance use maximum norms for the displacements or the stresses in elasticity problems, which would give more precise error control of clear significance in applications than standard energy norm control, where the

stresses are only controled in a mean square sense. For time dependent problems we may use $L_\infty(L_q)$-norms, i.e., maximum-norm in time and L_q-norm in space, for the solution or derivatives thereof, or $L_2(L_2)$-norms in space-time. We emphasize the generality of our approach, giving the possibility of considering more general norms than energy norms and general problems, not only of elliptic type, but also parabolic and hyperbolic problems.

The proofs of the *a posteriori* error estimates underlying the adaptive algorithms typically have the following structure:

1. Representation of the error in terms of the residual of the finite element solution and the solution of a continuous (linearized) dual problem.

2. Use of the Galerkin orthogonality built in the finite element method.

3. Local interpolation estimates for the dual solution.

4. Strong stability estimates for the continuous dual problem.

Clearly, the *a posteriori* error estimates obtained in this way are residual-based. Critical issues are how to evaluate the residual and what norms to use when estimating the residual, cf. below. The data of the dual problem is the error itself in the case of L_2-norm control. In the case of energy norm control for elliptic problems, the solution of the dual problem coincides with the error itself, and the introduction of the dual problem may be avoided, cf. below. Notice that the error representation gives information on the structure of the error which is used in the design of the adaptive algorithm.

The proofs of the *a priori* error estimates have a similar structure:

1. Representation of the error in terms of the exact solution and a discrete linearized dual problem.

2. Use of the Galerkin orthogonality built in the finite element method to introduce the truncation error in the error representation.

3. Local interpolation estimates for the truncation error.

4. Strong stability estimates for the discrete dual problem.

In both cases the stability of the dual problem is the critical issue. Note that in the case of *a posteriori* error estimate, the stability of a continuous

dual problem enters, while in the case of an *a priori* error estimate, we are concerned with the stability of a discrete dual problem. In both cases the stability of the dual problem reflects the error propagation properties of the discretization procedure: In the case of the *a posteriori* error estimates, the error is connected to the residual through the error representation formula, involving the continuous dual solution. In the case of *a priori* estimates, the error is connected to the truncation error (which is the interpolation error for the exact solution) through the discrete dual solution. Thus, the solution of the (discrete or continuous) dual problem plays a fundamental role in any attempt to control the discretization error. More precisely, it is the stability properties of the dual solution that matter. A particular feature of our methodology is the use of (new) strong stability estimates for the dual problem, which makes it possible to derive sharp error estimates which result in efficient adaptive algorithms. These new strong stability estimates involve control of certain derivatives of the solution of the dual problem (typically the leading derivatives in the dual problem) in terms of the data of the dual problem (whereas standard stability estimates, except in the case of energy norms, involve weaker such control).

The possibility of using strong stability estimates, resulting in sharp error estimates, is closely connected to the orthogonality inherent in the Galerkin methods underlying the finite element methods. Roughly speaking, the strong stability estimates together with the Galerkin orthogonalities make it possible to obtain sharper results than through standard perturbation arguments, relying on standard (weak) stability estimates. In the case of energy norms, strong stability is basically the same as standard energy-norm stability (which involves certain derivative control), while, as already indicated, in other norms strong stability is indeed stronger than the standard stability. As particular cases where the use of strong stability gives new sharp results, we mention the results on long-time integration for parabolic problems in [3, 5] and the adaptive algorithms for hyperbolic problems of [6, 7, 16]. In both cases, it appears to be impossible to obtain sharp general results using classical stability concepts. The possibility of exploiting Galerkin orthogonalities, in combination with strong stability estimates, resulting in improved error estimates significantly adds to the list of advantages of Galerkin-based procedures.

In the *a posteriori* error estimates underlying the adaptive algorithms, two types of constants enter, one set of constants C^s related to the stabil-

ity estimates for the continuous dual problem, and one set of constants C^i related to local polynomial interpolation. These constants have to be determined approximatively in order to define the adaptive algorithm and control the error on the given tolerance level. The interpolation constants C^i depend on the shape of the elements and the local order of the polynomial approximation, but not (or only trivially so) on the particular problem $\mathcal{P}$ to be solved, and may thus be determined analytically or computationally once and for all for different classes of problems. The stability constants C^s, on the other hand, in general depend on the particular problem $\mathcal{P}$ (but not on the discretization, i.e., not on p and h), and it is less obvious how to estimate these constants with one important exception: In energy norms the problem is trivial, since by definition we have $C^s = 1$ (in which case the interpolation constants C^i in fact contain a dependence on the energy norm, which is easy to take into account). In other cases it may be possible to obtain analytical estimates of the stability constants fairly easily (e.g. for special classes of monotone parabolic problems). In general, however, we will have to rely on auxiliary computations to determine the actual values of the stability constants C^s, although of course it is still highly relevant to prove boundedness of these constants by analytical techniques, since this proves that the *a posteriori* error estimates have a correct form. Below we will discuss different possibilities of computational evaluation of the stability constants. We note that this situation cannot be avoided without seriously limiting the framework: Either we have to restrict ourselves to elliptic problems and use energy norms, in which case $C^s = 1$ by definition, or we restrict ourselves to a certain class of (simple) problems for which analytical estimates are possible to obtain, or we expand the framework to include more general problems and pay the prize (which usually is not large) required to estimate the stability constants computationally. We recall that the stability properties of the dual continuous problem gives us the bridge between the computable residual of the discrete solution and the error itself, and thus it is necessary to estimate the stability of the dual problem, one way or the other, for instance by estimating the stability constants C^s. There is no way we can get around this problem if we want to design algorithms for automatic error control based on rational arguments; the question is only how much computational work will be required for this purpose. It appears that in many cases this work is small compared to the total work.

We now briefly compare our approach to adaptivity with three other

well-known approaches presented in the literature, the one by Babuška et al. [18, 19] (see also Bank and Weiser [20] and Oden et al. [21]), the one by Zienkiewicz and Zhu [22, 23], and the one by Verfürth [24] concerning Stokes equations. Our *a posteriori* error estimates in energy norms for linear elliptic problems are of same form as those by Verfürth and are similar to those of Babuška. The difference between our energy-norm estimates and the ones by Babuška, is that the latter involve the solution of local problems with data related to the local residual (typically involving the jumps in stresses or fluxes across interelement boundaries), while our estimates involve the residual directly (again typically through the jumps in the indicated quantities), without having to solve local problems. We may view our *a posteriori* error estimates in energy norms as simplified versions of the ones by Babuška. The advantages of the simplification are considerable: The adaptive algorithms are easier and cheaper to implement and the proofs of the underlying *a posteriori* error estimates are much simpler and can be carried out in much greater generality. The estimates of Babuška may be more precise in certain cases, but this fact is countered by the greater simplicity and generality of our estimates. To sum up the comparison with adaptivity according to Babuška and Verfürth, we have that in energy norms for elliptic problems, our *a posteriori* error estimates are similar to those of Babuška and Verfürth, but our estimates have a greater generality, covering also other norms than energy norms and more general problems, e.g., of parabolic and hyperbolic type.

In the work by Babuška, a strong emphasis is put on the concept of the effectivity index θ defined to be the estimated total error divided by the true total error. Adaptive algorithms are sought with θ close to one. Ideally, one would like to construct algorithms for which it is possible to prove that θ would tend to one if the mesh size tends to zero. Certain results in this direction have been obtained by Babuška et al. for elliptic problems, using superconvergence on regular meshes. In our case the effectivity index would be given by

$$\theta = \frac{E_1(h, p, u_{h,p}, f)}{\|u - u_{h,p}\|}, \tag{9}$$

and with the constants in E_1 correctly estimated above we would have $\theta \geq 1$. The question is now how large the effectivity index θ defined by (9) would be in typical cases with our estimator E_1. The answer depends on the difficulty

of the problem and how much work we spend on (computationally) estimating the stability constants C^s. Our experience is that it is possible to obtain effectivity indices in the range from 1 to 2-3 for fairly general classes of problems. It appears to be difficult to guarantee effectivity indices close to one in general, but, on the other hand, from a practical point of view it could very well be acceptable with effectivity indices in the indicated range 1-3, at least for more difficult problems. Thus, we do not consider the question of having effectivity indices very close to one as so essential, and would rather trade generality for larger effectivity indices. Let us also remark that efficiency has no clear connection to the effectivity index being close to one. Even if the error is accurately estimated, so that the effectivity index is close to one, the underlying mesh may be very far from an optimal mesh, and there is no way we can detect this by simply looking at the effectivity index. Thus, in our opinion, the focus should shift from the question of whether the effectivity index is close to one or not, to the problem of efficiency of the adaptive algorithm, which is a more general question. An efficient adaptive algorithm necessarily has a reasonably small effectivity index, but, as indicated above, even an effectivity index very close to one does not in general imply efficiency of the adaptive algorithm.

We now turn to a comparison with the adaptive methods advocated by Zienkiewicz and Zhu [22], the so called Z^2-approach, which is fundamentally different from ours in spirit, although not necessarily so in practice in the case of energy norms as discussed below. The Z^2-approach is based on directly estimating the error in the solution $u_{h,p}$ by constructing from $u_{h,p}$, by suitable local averaging, a hopefully improved solution $u^*_{h,p}$, and taking the difference $e_{h,p} = u^*_{h,p} - u_{h,p}$ as an estimate of the true error $e = u - u_{h,p}$. The adaptive algorithm of Z^2 is then based on the estimate for $e_{h,p}$. To justify this type of algorithm, super-convergence results are required: The post-processed solution $u^*_{h,p}$ should be a better approximation than $u_{h,p}$. Such results may, however, only be expected to hold for regular or nearly regular meshes, and thus Z^2 does not appear to cover the general case of unstructured meshes. Another difficulty in this approach is the construction of the adaptive algorithm from the computation of $e_{h,p}$. In general, it is not correct to refine/unrefine according to the local size of $e_{h,p}$, since an error at one location may influence the error at other locations (e.g. through pollution in elliptic problems or convection mechanisms in convection-diffusion problems). However, for energy-norm control in elliptic problems, $e_{h,p}$ appears to

be a good indicator for local refinement/unrefinement, in fact also in the case of general meshes. An explanation for this phenomenon could be that the error indicator $e_{h,p}$ in the Z^2-approach in fact is close to the residual in our approach and the approach by Babuška and Verfürth (a circumstance which appears to have formed an initial motivation for Z^2, cf. [22] and [25]). Thus, the reason for the success of Z^2 for energy-norm control in elliptic problems is not necessarily a superconvergence phenomenon, requiring the postprocessed solution $u^*_{h,p}$ to be an improved approximation, but probably simply the fact that $e_{h,p}$ happens to give a good approximation of the residual of $u_{h,p}$. For elasticity problems, this would correspond to the difference between the computed and the postprocessed stress being close to the jump in the computed stress, which may be expected to be true on general meshes. Consequently, the Z^2-approach appears to be close to the other approaches discussed in the case of energy-norm control for elliptic problems. For other norms or other problems, it is likely that an adaptive procedure like Z^2, based on refining the mesh according to the size of the estimated local error, cannot be expected to generate efficient adaptive procedures in general, even if the estimate of the local error happens to be accurate. The simplest example of such a situation is given by the following linear ordinary differential equation: $\dot{u} = f(t)$ for $t > 0$, $u(0) = u_0$, with the given function $f(t) \sim 1$ except in a small interval $(1-\delta, 1+\delta)$ with $\delta > 0$, where $f \gg 1$. Solving this problem numerically by, e.g., the explicit Euler method, the rapid variation of u in $(1-\delta, 1+\delta)$ will cause an error $e(t)$ which will be large for all $t > 1+\delta$. In this case it is not correct to refine the mesh where $e(t)$ is large, i.e., in $(1+\delta, \infty)$, but only in $(1-\delta, 1+\delta)$, where the residual is large.

In this paper we shall, as model problems illustrating the general theory, consider first linear elasticity, then non-linear elasto-plasticity, and finally present some recent results on adaptive finite element methods for systems of non-linear conservation laws in one space dimension ([16]). The results on non-linear conservation laws, in particular, show the strength of the presented framework; as far as we know these are the first results to show that adaptive error control (in $L_2(L_2)$) based on *a posteriori* error estimates is possible for systems of conservation laws. The proofs of the *a posteriori* estimates use strong stability estimates for the dual problem, which (in one dimension) are proved by analytical techniques, coupled with Galerkin orthogonalities as indicated above. A particular feature of non-linear conservation laws is that the standard weak stability of the dual problem simply is not valid,

reflecting the fact that solutions of conservation laws may be unstable under L_2-perturbations. Thus, it appears that error control in L_2, using some classical perturbation technique, would be impossible in such a case. However, the dual problem in the case of conservation laws in one space dimension can be proved to satisfy nonstandard strong stability estimates, which indeed makes it possible to obtain (in fact seemingly optimal) *a posteriori* error control in L_2 in suitable Galerkin procedures. Note that the Kruzkow L_1-continuity, which may be used as basis for adaptive error control for scalar conservation laws, is not available for systems, and thus our L_2-approach seems to be the only possibility opened so far.

We now briefly outline the results obtained in [2]-[16]. In the series [2]-[5], we design and analyze adaptive finite element methods for parabolic problems in a fairly large generality. The mesh size in time and space may be variable in space and time; thus we have almost complete freedom concerning the mesh in space as well as time, and we also cover a class of non-linear parabolic problems. We prove optimal *a priori* and *a posteriori* error estimates in a variety of norms incuding $L_\infty(L_2)$ and $L_\infty(L_\infty)$. In particular we solve the problem of long-time integration for parabolic problems. In [2] we also give the basics of adaptivity for linear elliptic problems in the spirit described above. In [12, 13], we extend the framework for adaptivity for elliptic problems to some non-linear monotone elliptic problems modelling unilateral contact and elasto-plasticity. In [14], we prove *a posteriori* (and *a priori*) error estimates for finite element methods for second order linear wave equations, and design corresponding adaptive algorithms. In [6, 7] (see also [11]), we prove *a posteriori* error estimates for the streamline diffusion finite element method for linear convection-diffusion problems, stationary as well as time-dependent, and again design corresponding adaptive algorithms. These results are extended, as indicated above, in [16] to systems of non-linear conservation laws in one space dimension. In [15], we give computational results for such adaptive finite element methods applied to two-dimensional compressible flow. In [11], we give a survey of our results for linear convection-diffusion problems, including elliptic, parabolic and hyperbolic problems.

An outstanding open problem in adaptivity is error control for the equations of fluid mechanics, compressible or incompressible, in several space dimensions. As indicated, we have proved *a posteriori* error estimates and designed corresponding adaptive algorithms for compressible flow in one space dimension. These results can directly be extended to several dimensions, in-

cluding also incompressibel flow, except for the analytical proof of the strong stability estimate for the linearized continuous dual problem. Thus, the main question is whether or not the dual problem satisfies strong stability estimates in several dimensions. This may or may not be true, depending on the nature of the particular problem. If the problem itself is unstable, with chaotic or turbulent solutions, then most likely the dual problem will be unstable as well, while in a more stable case we could expect the dual problem to be stable in the strong sense discussed above. In general, it appears to be extremely difficult to (quantitatively) evaluate the stability of the dual problem by analytical techniques, but there may be a computational way out of this difficulty: by feeding in suitable data in the dual problem (linearized around a computed solution) and computing the corresponding dual solution numerically. This could yield a computational strategy for realizing quantitative error control also for complicated problems, such as those encountered in fluid mechanics. The general structure of the adaptive algorithm would then be as outlined above, together with computational evaluation of the strong stability of the dual problem. We have tested algorithms of this form for compressible and incompressible flow with good results, but more work is required to achieve closer control of the stability estimates involved.

2 Prospects for the future

We have outlined a general approach to adaptivity for finite element methods and indicated some concrete applications to model problems in computational mechanics. The presented methodology appears to have a strong potential and opens the possibility to efficient and reliable quantitative error control in a variety of norms for large classes of problems of elliptic, parabolic or hyperbolic type. A central component in our approach is the new concept of strong stability, which, when coupled with the Galerkin orthogonality inherent in the finite element method, makes it possible to obtain sharp error estimates of both *a priori* and *a posteriori* type. The problem of computational evaluation of the stability properties of the continuous linearized dual problem, connected with the *a posteriori* error estimates underlying the adaptive algorithms, is fundamental. This problem appears in principle to be possible to solve, even for, e.g., complicated flow problems, but more research is required to find cost effective solutions. Other important topics

concern algorithms for adaptive orientation/stretching of the mesh in space or space-time, which may yield substantial increase in efficiency, for instance for problems with boundary layers and shocks.

To sum up, the main features of adaptivity for finite element methods now seem to be visible, and the door is open to implementation of adaptivity in general purpose software. A great deal of activity in this direction is to be expected the next few years.

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ADAPTIVE DATA PARALLEL FINITE ELEMENT METHODS

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(joint work with P. Rabier and J. Maubach)

Elementwise parallel finite element solution procedures offer the ultimate in architectures, such as the connection machine. They are also ideal for adaptive methods since only the elemental data corresponding to the newly created elements need be recomputed. This talk will describe a new family of elementwise solution procedures.

These are provably convergent for any nonlinear monotone problem, including variational inequalities. Their rate of convergence is uniform in the relative size of the (possibly very small) symmetric part to the (possibly very large) nonsymmetric part of the underlying physical problem.

Development of an Adaptive Grid Algorithm that Compensates for Discretisation Errors, with an application to the 1D Burgers' Equation

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Abstract

In grid-based (Eulerian) discretisation methods, the numerical modelling error is a function of the number and the location of the grid points. For a fixed number of grid points this error can be minimised by finding the optimal location of the grid points. It is this approach that we are following in the development of our adaptive grid technique. A local error determination is necessary to solve the minimisation problem. Local error determination is generally not possible for standard discretisation techniques.

We have therefore developed a discretisation technique which permits to express the local discretisation errors in terms of local modelling errors in the numerical solution. It is a fully conservative, central finite-volume scheme that is based on uniquely defined, piecewise linear, discrete function approximations for all variables. An interesting feature is that the discretisation is compatible with the function approximations. The errors introduced by the piecewise linear approximation of the different variables, which we will call representation error, are of the same form and size as the discretisation error. Hence by using this compatible scheme we are able to compensate for the numerical modelling error by minimising for the representation error.

The representation error is also used to optimise the scheme for an artificial diffusion coefficient that reduces numerical errors but introduces a physical modelling error. Artificial diffusion is essential to assure a smooth solution which again is necessary to be able to neglect higher-order error terms. Hence in the grid adaptation procedure it is sufficient to minimise the leading second-order error terms only.

The local representation error is weighted in the L_1-norm. Only by using this norm the error is always consistent with the order of accuracy of the discretisation scheme, even in regions with steep gradients. Consequently there are no limiters necessary to constrain the grid point movement during the computation.

Subsequently, we have to minimise the representation error leading to an elliptic grid equation. Currently we are using an error equi-distribution algorithm, but we are developing a minimisation procedure in the L_1-norm which will be implemented in the near future.

We have applied the method to the 1D Burgers' equation, leading to a non-linear system of coupled equations for the solution and grid points which are solved iteratively using the pseudo time step technique. One physical time step requires a considerable amount of iterations but this seems to be largely compensated for by a significant increase in accuracy even when large time steps are used.

The results of our currently implemented algorithm are very promising. It is possible to obtain a fully converged solution and a fully converged grid at every physical time step. Although we do not minimise the numerical error yet, we have obtained already a substantial reduction in the overall error, as can be shown by comparison of the numerical results with analytical solutions for the Burgers' equation.

ADAPTIVE MESH REFINEMENT FOR ELLIPTIC PROBLEMS

D.F. MAYERS

We shall consider a boundary value problem for a partial differential equation of the form

$$Lu = f \quad \text{on } \Omega \tag{1}$$

$$Bu = g \quad \text{on } \partial\Omega, \tag{2}$$

where $\partial\Omega$ is the boundary of Ω. For simplicity we shall mainly restrict the discussion to two dimensions. For the moment we assume that L is a second order elliptic operator, such as the Laplacian, without being more specific. Typically the boundary conditions will be either Dirichlet or Neumann.

In order to obtain a numerical solution with prescribed accuracy we shall almost always have to use an a posteriori error estimator, so the simplest procedure will be:

(i) Choose an initial mesh to cover the region Ω.

(ii) Obtain a numerical solution on this mesh.

(iii) Compute a norm of an error estimate for this solution.

(iv) If this error norm is smaller than the prescribed tolerance, the computation is complete.

(v) If not, refine the mesh and start again from (i).

The most obvious way to refine the mesh in step (v) is to use a finer mesh everywhere, and this also has obvious advantages. If, for example, we have used a uniform square cartesian mesh with size h and a finite difference approximation which is known to give a solution with second-order accuracy, then a finer mesh with size $h/2$ should reduce the error by a factor of 4. Assuming that the error estimate in step (iii) is reliable we can therefore decide what mesh size will be needed in the refined mesh to give the required accuracy.

However this obvious approach is often much too inefficient. In many fluid flow problems involving shocks and discontinuities it may be very difficult

to devise a numerical method which is globally second order accurate. In this case, for a problem in three dimensions where our error estimate might, for example, be 10 times greater than the prescribed tolerance, the obvious finer mesh will require multiplying the number of points of the mesh by 1000. This will normally be quite impossible.

We therefore modify step (v) of the procedure,

(v) If not, compute an estimate of the error at each point of the mesh.

(vi) Using this error estimate, construct a new mesh which is refined only in part of the region Ω.

(vii) Return to (i).

The numerical procedure has now become more difficult. We need a more refined error estimator to use in (v), a procedure for deciding which parts of the mesh need refinement, and a numerical solution process which takes account of the partially refined mesh.

There are various possible ways of refining a mesh. Suppose that the original region is the unit square, and that our initial mesh is uniform and square, as in Fig.1.

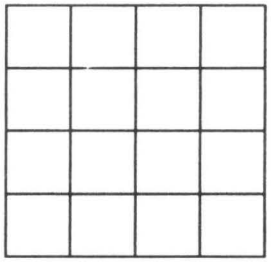

Figure 1: Initial uniform mesh.

Suppose that we now wish to refine the mesh in the top right corner. In Fig.2a four of the mesh points have been moved closer to the corner; the new mesh still consists of quadrilaterals in a structured array, but they no

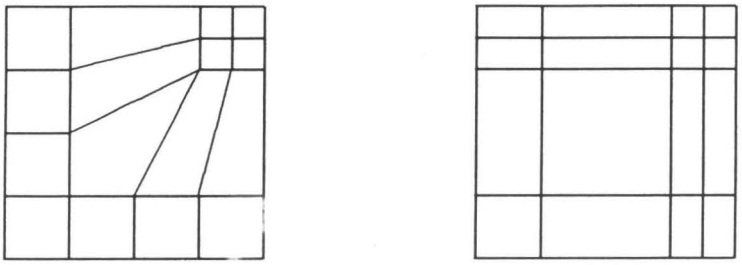

Figure 2: Adapted mesh by moving points twoards the top right corner. (a) On the left, moving 4 points only; (b) on the right, moving mesh lines to retain a rectangular mesh.

longer form a rectangular mesh. The second example in Fig.2b has moved mesh lines in such a way that the mesh becomes finer in the required region; the mesh is still rectangular, but not uniform, and it contains some long thin elements, and a larger square in the centre. Both these are examples of mesh adaptation, since the number of mesh points is the same as before, but they have been redistributed.

In Fig.3 we refine the mesh by introducing additional points. First in (a) we introduce two additional mesh lines; the mesh is now refined in the corner as required, but we have also introduced additional points in regions where they may not be needed. Finally in (b) we may refine the mesh only in the required region. We still have a basically square mesh, but it no longer has a simple structure; a more elaborate form of data structure is needed, and we also need to deal with the ***hanging nodes*** which are points on the boundary of the finer mesh which do not belong to the coarse mesh. Each of these methods of adaptation or refinement has advantages, and each leads to difficulties which we shall discuss.

We must begin by examining the question of how to decide which parts of the mesh need to be refined. First we require an ***indicator*** which is defined at each mesh point, or in each mesh element, and measures the local magnitude of the error. Then we can use this in two ways: taking the positive view, we can decide to refine the mesh where this indicator is larger than some tolerance, or taking the negative view we can refine the whole mesh, except for those regions where the indicator is less than some possibly different tol-

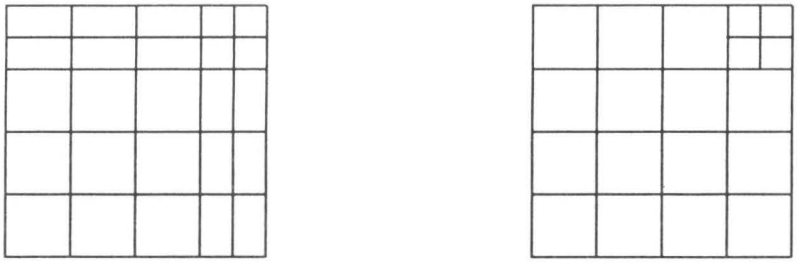

Figure 3: Mesh refinement. (a) On the left, additional mesh lines have been introduced, giving refinement near the corner, and near the two boundaries also; (b) on the right, additional points are introduced only near the corner.

erance. These different approaches may give the same result, but it appears that is often necessary to refine the mesh in a rather larger part of the region than the area where the error indicator is large. The effect of a partial mesh refinement is usually difficult to determine; it is certainly not true in general that a refinement over part of the region will reduce the error in that part in a ratio determined by the ratio of the refinement, leaving the error in the rest of the region unaltered. The effect is a great deal more complicated, and in particular will depend on the treatment of the interface between the regions with different mesh sizes, and the behaviour of the solution at these points.

A common approach is to consider the problem of adapting the mesh in such a way that the "error" is uniformly distributed, having approximately the same magnitude everywhere. It is not always certain that this is the best strategey, particularly for complicated systems. The real practical problem is to obtain a numerical solution with the required accuracy, and with minimum cost. Here the *cost* may be measured in terms of computer processor time, also perhaps of memory requirement; in an industrial situation the cost of the development of programs required to implement a new strategy will also probably be important.

The critical question is how we wish to measure the error of the numerical solution. If we require that some norm of the *truncation error* is less than a specified tolerance, the problem is not too difficult. A refinement of a uniform mesh over part of the domain, by a ratio k, will to a good approximation reduce the truncation error in that domain by the factor k^p, where p is the

order of the leading error term. But if we require to reduce some norm of the *global error* the situation is more difficult; the two are related by

$$L_h \eta = \tau, \quad \eta = L_h^{-1} \tau \tag{3}$$

and we need to understand the local behaviour of the inverse operator L_h^{-1}. We shall study this behaviour in some detail for simple model problems.

An additional complication arises in real practical problems. For example, consider the Navier-Stokes equations for the flow of air round a body. Here we have a system of 4 equations, in two dimensions, and the error has 4 components; a measure of the error therefore requires some form of weighting of these components. In many such problems the general behaviour of the solution is known in advance, and the mesh can be designed to take account of this. But the exact position of shocks, etc, will not be known, and the mesh must be adapted after a first solution has been found. In practice this is often done by using an indicator based on physical features of the flow, such as the gradient of the pressure. A number of variations of this procedure are used in practice; it is usually found that the use of different indicators of this type will lead to generally similar refined meshes, but a reliable mathematical foundation is still incomplete.

The refinement of a general triangulation of a region, as in a finite element solution, generally presents few problems. Methods for dividing a region into triangles are well understood, and solution algorithms suitable for a general triangulation are available[5].

When using a regular cartesian finite difference mesh, or a more general structured quadrilateral mesh, the problem is more difficult. The advantage of a structured mesh is that we can normally number the nodes of the mesh in an obvious manner, so that locating a particular node and its neighbours is a trivial matter. On a mesh which is partially refined in several different regions, with each region itself partially refined to several levels, the design of an efficient data structure is not trivial. The *Quadtree* structure involves a standard type of tree, where each of the quadrilateral elements of the mesh corresponds to a node of the tree, and a refined element corresponds to 4 branches from this node to the 4 nodes which refer to the small quadrilaterals coming from the subdivision. This structure is quite general, and allows for any level of refinement of any element. However, it requires a substantial amount of memory to store the tree, and may require a nontrivial number of operations to find which elements are neighbours of a given element.

We shall discuss an alternative approach which refines a *patch*, rather than a single element. A patch will have a prescribed minimum size, such as 8×8; the advantage is that a patch is determined by its size, and the coordinates of one corner, as in Quadtree, but within a patch the points are numbered in the usual regular fashion, and access to neighbours is immediate. The disadvantage, of course, is that we shall often have to refine a region which is rather larger than is strictly necessary. The result is rather like a form of *Domain Decomposition*, and may be appropriate for parallel computation.

The whole process may now be implemented within a *Multigrid* environment. Here an estimate of Truncation Error is quite easily available, by evaluating the residual of a solution using the operator on a coarser grid, and if we use Nested Multigrid an estimate of the global error is also available. Moreover a patch can be regarded in the multigrid hierarchy as a fine grid which covers only part of the whole domain. These ideas are discussed in considerable detail in McCormick[4].

We shall also mention briefly the ideas of h and $p-$refinement, where in certain regions we may either use a finer mesh (h-refinement) or a higher order scheme (p-refinement), or both.

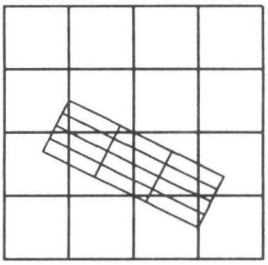

Figure 4: Refinement in an oblique direction

In many situations a single error estimate is inadequate, and we need a directional or anisotropic indicator. There are many cases where we should refine a cartesian mesh in the $x-$direction, but not in the $y-$direction, for example. This is easily done, provided we have an estimator which shows that it is necessary. The problem is more difficult where the solution has a special feature which is not aligned with the mesh lines, so that we should refine the

mesh in some oblique direction. Refinement of the cartesian mesh is then unsatisfactory, and it may be best to make a transformation of coordinates in some area, and use a regular mesh in this region[1]. The result is a new patch in the new coordinates which is not aligned with the original mesh, as in Fig.4.

In problems of this kind the conditions to be satisfied at the interface between the two meshes become very important. For hyperbolic equations we must try to ensure that the interface is *transparent*, so that a travelling wave reaching it passes through without any spurious reflection. If the equation represents the conservation of a physical quantity, such as the mass or the momentum of a fluid, it is important that the discrete equations also ensure conservation across the interface [2]. Some obvious simple treatments of the interface give rise to large errors; we shall discuss conditions which ensure good accuracy in simple model problems.

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A Survey on Multivariate Shape Preserving Interpolation

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1 Introduction

In many practical instances in data analysis, computer aided geometric design, and mathematical modeling, the following interpolation problem arises. Let $D = \{x^i\} \subset \mathbb{R}^s$ be a set of gridded or scattered data sites and let $F = \{f_i\} \subset \mathbb{R}$ be associated data values. The objective is to find a real-valued function f interpolating the given data, i.e., $f(x^i) = f_i$ for all i. Besides the interpolation of function values (Lagrange-interpolation), also other interpolation conditions may be imposed, e.g., conditions on derivative values (Hermite-Birkhoff-interpolation) or volume matching conditions (histopolation).

Usually, a certain degree of smoothness of the interpolating function is required. In many applications, also some additional shape properties such as nonnegativity, monotonicity, or convexity are desirable. This leads to the consideration of so-called constrained or *shape preserving interpolation* problems.

There are several reasons to impose shape constraints on the interpolant. Often, the interpolant is used to recover a unknown underlying function from its given discrete data. If some shape properties of this function are known a priori or are suggested by the data, then it is quite natural to take this qualitative information into account and to require the interpolant to satisfy the same properties. In many cases, the incorporation of the a priori information significantly improves the quality of the approximation. Furthermore, the constraints may be *dictated* by the physical nature of the practical problem, since the interpolant does not represent a meaningful solution unless the shape constraints are satisfied. In the design of curves and surfaces, e.g., in the automotive industry, shape constraints are usually imposed for aesthetic reasons. Unconstrained interpolants often contain wiggles and bumps, which can have unacceptable visual effects.

A vast amount of shape preserving interpolation methods have been devised in the univariate case ($s = 1$). But the problems become much more complicated in two or more variables, in particular for scattered data sites. Therefore, only a few algorithms for the solution of bivariate or even multivariate shape preserving interpolation problems are available at present.

In this lecture, we intend to give an *introduction* to the so-called *abstract variational approach* to shape preserving interpolation and its applications to multivariate problems.

A general framework of shape preserving interpolation problems can be described as follows. Let X be a real Banach space. In the applications we have in mind, X will usually be a suitable infinite-dimensional function space, such as a space of continuous functions or a

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Sobolew space. C will denote a convex set in X defining the desired shape properties (constraints), e.g., the cone of nonnegative, monotone, or convex functions. In the existence results of section 3, C is not required to be closed in order to include the case of strict constraints.

Furthermore, let $A \in L(X,Y)$ be a continuous linear operator from X into $Y = \mathbb{R}^n$. The interpolation conditions are given by $Ax = d$, where $x \in X$ and $d \in Y$ is a so-called data point.

We introduce the following notation. The set

$$A^{-1}(d) := \{x \in X \ : \ Ax = d\},$$

will denote a closed affine subspace (a flat) of X, and

$$A[C] := \{Ax \ : \ x \in C\} \subset Y,$$

the so called data set which consists of admissible data points.

The objective of shape preserving interpolation is, for any given admissible data point d, to find an element x from X satisfying

$$x \in K := C \cap A^{-1}(d),$$

or, in other words, to construct a shape preserving interpolation operator

$$I \ : \ A[C] \to C \ : \ d \mapsto I(d) \in C \cap A^{-1}(d).$$

The selection of $I(d)$ is usually based on the minimization of a suitable convex functional $\varphi : X \to \mathbb{R}$. Here, we will only consider the case that X is a Hilbert space, C is closed, $y \in X$, and φ is given by

$$\varphi(x) := \frac{1}{2}\|y - x\|^2.$$

Then we get

$$I(d) = P_K(y),$$

the best approximation to y from $K = C \cap A^{-1}(d)$.

The existence, uniqueness, characterization, and computational aspects of the solution of this extremal problem have been studied in [12, 4, 5]. The problem may also be considered as a special instance of abstract variational inequalities [9] or of partially finite convex programming in the sense of [2, 3].

2 Notations

The following notations will be used in the lecture.

In a Banach space X, for any convex subset K of X, $\mathrm{aff}(K)$ denotes the affine hull of K, $\overline{K}$ its closure, $\mathrm{int}(K)$ its interior, and $\mathrm{ri}(K)$ the relative interior of K, i.e., its interior with respect to $\overline{\mathrm{aff}(K)}$.

Now, let X, Y be Hilbert spaces, and let $A \in L(X,Y)$, i.e., A is a continuous linear operator from X into Y. The symbol $\mathrm{range}(A) := \{Ax : x \in X\}$ will denote the range of A, and $\ker(A) := \{x \in X : Ax = 0\}$ the kernel of A.

The adjoint A^* of A is defined by $(Ax, y) = (x, A^*y)$ for all $x \in X, y \in Y$, where $(\cdot,\cdot)$ denotes the inner product in X or Y, respectively.

Let us recall that a convex set K is called a convex cone if it satisfies $\lambda K \subset K$ for all $\lambda \geq 0$.

The conical hull of a nonempty set K, denoted by $\mathrm{cone}(K)$, is the intersection of all convex cones containing K. The closed conical hull of K is defined by $\overline{\mathrm{cone}}(K) := \overline{\mathrm{cone}(K)}$, and the dual cone of K is given by $K^0 := \{x \in X : (x, y) \leq 0 \text{ for all } y \in K\}$.

If K is convex and $x \in K$, the usual tangent cone to K at x is represented by

$$T(K, x) = \overline{\mathrm{cone}}(K - x).$$

Furthermore, the unique best approximation to $y \in X$ from a nonempty closed convex set $K \subset X$ will be denoted by $P_K(y)$.

3 Existence

Obviously, the existence of a shape preserving interpolant is equivalent to the admissibility of the given data point. Therefore, in the investigations of the extremal problem, d is often assumed to be admissible, i.e., such that $C \cap A^{-1}(d) \neq \emptyset$, or even a stronger condition. However, usually no attention is given to under what conditions d is admissible.

Let us mention here, that the set $A[C]$ of admissible data points is *not* necessarily closed, even if C is a closed convex cone.

The following sufficient conditions for the existence of shape preserving interpolants have been derived in [15]. The main idea consists in the imbedding of the considered problem into another one, for which the set of admissible data points can be easily obtained, and to guarantee the admissibility with respect to the original problem by denseness arguments.

Theorem 1 *If B is a dense convex subset of C, then*

$$\mathrm{ri}(A[C]) = \mathrm{ri}(A[B]) \neq \emptyset.$$

Therefore, the so-called *interior data points* $d \in \mathrm{ri}(A[C])$ with respect to C are also interior data points with respect to B. Actually, the theorem can be improved in the sense that interior data points can be not only interpolated but, at the same time, also approximated by elements from B.

Theorem 2 *Let B be a dense convex subset of C and $d \in \mathrm{ri}(A[C])$. Then $B \cap A^{-1}(d)$ is dense in $C \cap A^{-1}(d)$.*

This result generalizes the Singer-Yamabe theorem [10], which is obtained for $C = X$.

Usually, the set B is given as the intersection $C \cap S$ of C with a linear subspace S of X, e.g., a space of (piecewise) polynomials, a space of functions of certain smoothness, a Sobolew space, or a finite dimensional space as a result of discretization. In this case, the following result was established in [19] for so-called *Slater points* $d \in A[\mathrm{int}(C)]$.

Theorem 3 *Let S be a dense linear subspace of X and let $d \in A[\mathrm{int}(C)]$, i.e., $\mathrm{int}(C) \cap A^{-1}(d) \neq \emptyset$. Then $C \cap S \cap A^{-1}(d)$ is dense in $C \cap A^{-1}(d)$.*

This result is a weaker variant of Theorem 2 above, since each Slater point is an interior data point [4, 2] and the assumptions imply the denseness of $C \cap S$ in C. However, in practical situations, the assumption $\mathrm{int}(C) \neq \emptyset$ is often not satisfied, but nevertheless the denseness of $C \cap S$ in C can be verified directly.

We also want to mention a negative result concerning shape preserving interpolation in finite dimensional subspaces.

Theorem 4 *Let C be a closed convex cone and let S be a finite dimensional subspace of X. If there exists a $d \in A[C] \setminus A[C \cap S]$, then there is also a $d \in \operatorname{int}(A[C]) \setminus A[C \cap S]$.*

In [15], the results have been applied to multivariate monotonicity resp. convexity preserving interpolation by polynomials. It follows from the theorems stated above, that strictly monotone resp. convex data can be interpolated by monotone resp. convex polynomials. On the other hand, the required degree of the polynomial depends on the data values and can not be specified in advance. As an example, the convexity preserving interpolation will be considered in more detail in the lecture.

Remark 1 The existence results can be generalized to infinite dimensional spaces Y, i.e., to the case of an infinite number of interpolation conditions. One particular interesting problem, namely the existence of smooth monotone extensions to boundary data, is studied in [6].

4 Characterization

In this section, the known characterization results for the extremal problem posed in the introduction will be summarized. At first, we restate the problem.

The set-up is as follows. X is a Hilbert space, C is a closed convex set in X, $y \in X$, $Y = \mathbb{R}^n$, $A \in L(X,Y)$, $d \in A[C]$, and $K = C \cap A^{-1}(d)$.

To the primal problem

$$\begin{array}{rl} \min & \frac{1}{2}\|x - y\|^2 \\ \text{subject to} & Ax = d, \quad x \in C \subset X, \end{array} \tag{1}$$

we assign the dual problem

$$\max f(y + A^*\lambda) + d^T\lambda, \quad \lambda \in Y, \tag{2}$$

where the functional $f : X \to \mathbb{R}$ is defined by

$$f(x) := \frac{1}{2}\|P_C(x) - x\|^2 - \frac{1}{2}\|x\|^2.$$

The following Fenchel duality result is a particular case of Corollary 4.8. in [2] and was earlier presented in [12] (for $y = 0$).

Theorem 5 *If the constraint qualification $d \in \operatorname{ri}(A[C])$ is satisfied, i.e., if d is an interior data point, then the optimal values of the primal problem (1) and the dual problem (2) are equal. Furthermore, $\overline{\lambda}$ is optimal for (2) iff $AP_C(y + A^*\overline{\lambda}) = d$, and whenever $\overline{\lambda}$ is optimal for (2), then $\overline{x} = P_C(y + A^*\overline{\lambda})$ is the unique solution of (1).*

The second part of the theorem follows from the fact that the functional f is concave and differentiable, and the derivative at x is given by $-P_C(x)$ [12].

If the set C is a closed convex cone, then the functional f simplifies to $f(x) := -\frac{1}{2}\|P_C(x)\|^2$ due to $(x, P_C(x)) = \|P_C(x)\|^2$ for all $x \in X$ [12].

The optimality conditions of the theorem were also established in [4, 5] under slightly weaker assumptions. We want to sketch their approach.

As it is well known, for any $\overline{x} \in K$, $\overline{x} = P_K(y)$ is equivalent to $0 = P_{T(K,\overline{x})}(y - \overline{x})$ and characterized by $y - \overline{x} \in (K - \overline{x})^0 = (T(K, \overline{x}))^0$. In general, we only have $T(K, \overline{x}) \subset T(C, \overline{x}) \cap T(A^{-1}(d), \overline{x}) = T(C, \overline{x}) \cap \ker(A)$. It is therefore natural to impose the condition

$$T(K, x) = T(C, x) \cap \ker(A) \quad \text{for all } x \in K,$$

which is property CHIP (conical hull intersection property) of C and $A^{-1}(d)$ as defined in [4]. Since property CHIP is equivalent to

$$(T(K, x))^0 = \overline{(C - x)^0 + \text{range}(A^*)} \quad \text{for all } x \in K,$$

we have obtained

Theorem 6 *If C and $A^{-1}(d)$ satisfy property CHIP, then, for any $\overline{x} \in K$, $\overline{x} = P_K(y)$ iff $y - \overline{x} \in \overline{(C - \overline{x})^0 + \text{range}(A^*)}$.*

If, additionally, $(C - x)^0 + \text{range}(A^*)$ is assumed to be closed for all $x \in K$, we arrive at the optimality conditions of Theorem 5.

The connection is given by the observation, that $d \in \text{ri}(A[C])$ implies property CHIP as well as the closedness of $(C - x)^0 + \text{range}(A^*)$ for all $x \in K$ [5, 2].

Remark 2 Analogous results hold when d is in the boundary of the data set, at least when C is a closed convex cone [5]. In this situation, the characterization involves a certain subcone C_F of C, such that $d \in \text{ri}(A[C_F])$.

More general objective functionals in the primal problem (1) are considered in [13, 2, 3].

Useful applications of the abstract characterization to particular problems depend on a simple representation of the best approximation from C to elements of the form $y + A^*\lambda$. For multivariate problems, results in this direction have been obtained for equality constraint nonnegative L^2 approximation [5, 2], nonnegative volume matching [8], nonnegative thin plate splines [16, 12], and monotone thin plate splines [18, 17]. As an example, nonnegative thin plate splines will be discussed in more detail in the lecture.

Remark 3 Most of the results also hold true if Y is assumed to be an infinite dimensional Hilbert space [12, 4, 5, 11].

5 Computational Aspects

The dual problem (2) is a finite dimensional minimization problem without constraints, and the dual objective functional is continuously differentiable. Therefore, the dual problem is very attractive for the numerical solution of shape preserving interpolation problems, provided an efficient way to compute best approximations from C to elements of the form $y + A^*\lambda$ is available [12, 17, 3].

But the author is not aware of any numerical study of the application of this approach to a multivariate problem.

Another approach is based on the representation of C as the intersection of halfspaces, or, equivalently, by a infinite number of linear inequality constraints. Starting from the unconstraint solution, the shape preserving interpolant is constructed by successively incorporating the linear inequality constraints. This approach depends on the explicit solution of

the unconstrained interpolation problem, and therefore the range of its applicability is limited. The computation of the iterates requires the solution of linear systems, which can be accomplished by a rank-one perturbation technique [17]. This method has been implemented for nonnegative and monotone thin plate splines [16, 18].

In both approaches, the iterates will usually not correspond to primal feasible functions, i.e., the interpolation conditions resp. the shape constraints will not be satisfied during the iteration.

A third approach, which seems to be necessary for more complicated multivariate problems, consists in the discretization of the function space X, e.g. by finite elements [19, 1]. However, in shape preserving interpolation, the feasible region K should be discretized by a subset of K, since the constraints are usually more important than the minimized functional. In this way, we are led to the the consideration of shape preserving interpolation by piecewise polynomials. This problem will be the topic of the second lecture.

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Shape Preserving Interpolation by Bivariate Quadratic Splines

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In this lecture, the following problem of *shape preserving interpolation* by bivariate polynomial splines is considered. Let $D = \{x^i\} \subset \mathbb{R}^2$ be a set of gridded or scattered data sites and let $F = \{f_i\} \subset \mathbb{R}$ be associated data values. The objective is to find an interpolating piecewise polynomial C^1 function, which also preserves some shape properties suggested by the data, such as nonnegativity, monotonicity, or convexity.

Several shape preserving interpolation methods have been developed for data given on a rectangular grid, see [9] for a recent survey. But far less is known for scattered data sites.

Here, in particular, the nonnegative, monotone, resp. convex interpolation of scattered data by bivariate *quadratic* C^1 splines will be discussed. The presentation involves the following steps.

Bezier-Bernstein representation and shape properties. We will shortly introduce the Bezier-Bernstein representation of a quadratic polynomial on a triangle, i.e., its representation as linear combination of quadratic Bernstein-polynomials in the barycentric coordinates with respect to the triangle [8]. The main advantage of this representation consists in the nice geometric interpretation of the boundary behaviour of the polynomial, in particular of its directional derivatives and of the (C^1) smoothness conditions between adjacent triangles.

Our methods are mainly based on the observation, that the nonnegativity, monotonicity with respect to a given direction, resp. convexity of the so-called B-net implies the same properties of the quadratic polynomial [3, 11]. However, the derivation of improved sufficient linear conditions by subdivision as well as the exact (nonlinear) conditions for the nonnegativity resp. convexity of quadratic polynomials in terms of the B-ordinates (i.e., the coefficients in the Bezier-Bernstein representation) will be also considered [3, 12, 13, 15].

Triangulation and Powell-Sabin interpolation. The approach is based on an admissible triangulation of the data sites, i.e., a partition of the convex hull of the data sites into nondegenerate triangles. Exactly the data sites occur as vertices of the triangles, and two different triangles are either disjoint or share a common edge or a common vertex. During the last years, the usefulness of so-called data-dependent triangulations was pointed out [7, 16]. Such triangulations take not only the data sites, but also the data values into account, in contrast to the well known Delaunay triangulation. The construction of an admissible triangulation of given data sites based on a certain (data-dependent) criterion turns out to be a particular problem of combinatorial optimization. Usually, numerous local optima occur, while we are looking for a global optimum. Therefore, the application of the simulated annealing algorithm was studied in [18].

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In order to define the so-called Powell-Sabin elements, each triangle (macrotriangle) of the triangulation is splitted into six microtriangles. Therefore, an interior split point is picked out in each triangle, and it is connected by straight lines to the vertices of the triangle and to the interior points of the neighbouring triangles. Usually, the incenters of the triangles are chosen.

Every piecewise quadratic C^1 function on the refined triangulation is uniquely defined by its values and gradients at the vertices of the macro-triangulation, as can be easily shown using the Bernstein-Bezier representation of the quadratic pieces on the microtriangles [8, 10]. In our application, the function values at the vertices are prescribed by the interpolation conditions, while we can dispose of the gradients as parameters in order to satisfy the required shape properties.

Nonnegative interpolation. The simple sufficient nonnegativity conditions, i.e., the nonnegativity of all B-ordinates, result in a system of linear inequalities for the gradients at the vertices. This system is separated with respect to the vertices. Therefore, introducing a suitable objective functional, the construction of a nonnegative Powell-Sabin interpolant reduces to the solution of independent quadratic optimization problems in two variables, namely the partial derivatives, for each vertex. In this way, a *local* nonnegativity preserving interpolation method for scattered data is obtained.

Monotone interpolation. Here, we only consider the monotonicity with respect to the natural semi-ordering in $\mathbb{R}^2$, which is equivalent to the monotonicity with respect to both coordinate directions. The data are called monotone, if there exists a monotone interpolating function. As shown in [17], this is equivalent to the condition that, for all pairs of data sites, $x^i \leq x^j$ (in the sense of the natural semi-ordering on $\mathbb{R}^2$) implies $f_i \leq f_j$. The monotonicity of the Powell-Sabin interpolant is equivalent to a large sparse linear inequality system for the gradients at the vertices. It has been proposed to solve the resulting quadratic optimization problem by an active set strategy [10] or by dualization [18].

However, the inequality system turns out to be inconsistent for some data sets, even if the data stem from a smooth monotone function. The situation may be improved, but not overcome, by adapting the triangulation to the data. For this purpose, several data-dependent triangulation criteria have been applied in [18].

Another idea consists in the application of a monotonicity preserving algorithm for gridded data, e.g., the Beatson-Ziegler algorithm [1]. To apply such a method to scattered data, we construct a rectangular grid containing all data sites, estimate the missing data values, and correct these values to assure the consistency with monotonicity [14, 17, 18].

Both approaches will be illustrated by numerical results obtained in [18].

Convex interpolation. The given data are called convex if there exists a convex interpolating function. The convexity of the data is equivalent to the existence of a suitable triangulation (called the convex triangulation) of the data sites such that the corresponding piecewise linear interpolant is convex [5, 16, 4]. The construction of this triangulation and its interpretation in the context of data-dependent triangulations have been discussed in [16].

In [4], the following construction of a convex Powell-Sabin interpolant on the convex triangulation was proposed. The projection of the intersection of the tangent planes at the three vertices of every triangle define the (variable) interior split point of this triangle. Since

these split points have to be situated in the interiors of the corresponding triangles, the problem to determine suitable gradients at the vertices arises, which is not always solvable.

We also want to mention the generation of a convex C^1 interpolating surface as the limit of a nonlinear geometric subdivision process, which can be described as a corner cutting algorithm in the dual [6, 2].

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Smooth Interpolation to Scattered Data by Bivariate Piecewise Polynomials of Odd Degree: Some Improvements

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Abstract. In reference [1] a numerical algorithm was presented which determines a bivariate smooth piecewise polynomial interpolant to function values given at points scattered in $\mathbb{R}^2$. The main idea consists in perturbing a piecewise polynomial initial interpolant of low smoothness by a minimum norm correction of a node vector defining the interpolant. This in order to achieve the desired smoothness. The iterative method proposed in [1] can be improved in several aspects.

1. In the most important cases, the initial interpolant can be chosen with a higher polynomial precision than was achieved in [1].
2. Also, one can construct with nearly no extra effort an initial interpolant that has already a higher number of globally continuous partial derivatives.
3. The construction of the final interpolant considered in [1] lacks covariance under affine domain transformations. A different choice of node vector and minimization norm reestablishes this covariance property.

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Local Uniform Grid Refinement and Systems of Coupled Partial Differential Equations with and without Time Derivatives *

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Local uniform grid refinement is an adaptive grid technique for computing solutions of time-dependent partial differential equations possessing local sharp spatial transitions.

The main feature of local uniform grid refinement is that integration takes place on a series of nested local uniform increasingly finer subgrids which are created up to a level of refinement where sufficient spatial accuracy is reached. These subgrids are automatically adjusted at discrete times in order to follow the movement of rapid spatial transitions. The refinement strategy which controls the refinement process is based on a space error analysis and aims at the domination of the global space error on each subgrid by the space discretization error at the finest subgrid. This means that the obtained spatial accuracy on all subgrids should be comparable with the accuracy we would get when the finest subgrid covers the entire domain.

In [2, 3] the space error was examined of the local uniform grid refinement method applied to systems of PDEs with time derivatives only, using implicit Euler and general Runge-Kutta schemes for time stepping, respectively. From this a refinement condition was deduced which underlies the refinement strategy. Due to this condition, a convergence result could be proved in certain model situations as if a single uniform grid was used. However, this refinement condition is not applicable for systems of coupled PDEs with and without time derivatives. This is due to the fact that the local and global space errors belonging to PDEs with and without time derivatives respectively, can exhibit a totally different behaviour in time. For this reason the space error analysis was reconsidered in [1] for these systems and a more general refinement condition was obtained with the same properties as the previous one.

The performance of the local uniform grid refinement method together with the new refinement strategy and refinement condition will be illustrated with some numerical example problems.

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Moving-Grid Methods for Time-Dependent PDEs

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Standard numerical methods to solve time-dependent partial differential equations (PDEs) integrate on a uniform grid that is kept fixed on the entire time interval. If the solutions have regions of high spatial activity, a standard fixed-grid method is computationally inefficient, since to afford an accurate approximation, it should contain a very large number of nodes. The grid then needs to be locally refined. If the regions of high spatial activity are, moreover, moving in time, like for steep moving fronts, then methods are needed that also adapt the grid in time.

Roughly spoken, one may distinguish two classes of time-dependent adaptive methods: 1. (class I) dynamic-regridding (moving-grid) methods and 2. (class II) static-regridding methods. In the latter class of methods, for which the adaptivity is also denoted by terms like 'local refinement' or '*h*-refinement', the grid is only adapted at discrete time levels. Methods from class I, sometimes characterized by the term '*r*-refinement', have the special feature to move the spatial grid continuously in the space-time domain while the discretization of the PDE and the grid selection procedure are intrinsically coupled. Both approaches have their advantages and disadvantages, depending, e.g., on the PDE model to be solved, the hardware used, the spatial domain in the model, etc..

The main advantage of class II methods is their conceptual simplicity and robustness in the sense, that they permit the tracking of a varying number of wave fronts. A drawback, however, is that interpolation must be used to transfer numerical quantities from the old grid to new grids. Also, numerical dispersion, appearing, e.g., when hyperbolic PDEs are numerically approximated, is not fully annihilated. Another disadvantage of static-regridding methods compared with methods using moving-grids is the fact that they produce no 'smoothing' in the time direction. For these methods the time-stepping accuracy therefore will demand, in general, smaller time steps than for moving-grid methods. Examples of these methods are found in [1,2].

Class I methods use a fixed number of grid points, without need of interpolation, and let them move with whatever fronts are present. In the case of several steep fronts acting in different regions of the spatial domain, this could give problems in the numerical computation, if the grid is following one wave front and another one arises somewhere else. Since the number of grid points is fixed throughout the entire course of the computation, no 'new' grid is created for the new wave, but rather the 'old' grid has to adjust itself abruptly. Another difficulty is of a topological nature, viz., the so-called 'mesh-tangling'. Moving-grid methods, therefore, often need a kind of regularization to cope with this phenomenon. This, unfortunately, involves, more or less, tuning of the extra regularization parameters. On the other hand, even though more computations per grid point are needed, the use of moving-grid

methods may work out very efficiently, since, in general, fewer spatial grid points will be necessary. Examples of these methods are found in [3,4,5].

This talk deals with moving-grid methods for time-dependent PDEs in one and two space dimensions.

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