A bottleneck with randomly distorted arrival times

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ABSTRACT

We investigate the impact of random deviations in planned arrival times on user equilibrium in an extension of Vickrey's celebrated bottleneck model. The model is motivated by the fact that in real life, users can not exactly plan the time at which they depart from home, nor the delay they experience before they join the congestion bottleneck under investigation. We show that the arrival density advocated by the Nash equilibrium in Vickrey's model, is not a user equilibrium in the model with random uncertainty. We then investigate the existence of a user equilibrium for the latter and show that in general such an equilibrium can neither be a pure Nash equilibrium, nor a mixed equilibrium with a continuous density. Our results imply that when random distortions influence user decisions, the dynamics of standard bottleneck models are inadequate to describe such more complex situations. We illustrate with numerical analysis how the mechanics of a bottleneck with delayed arrivals are unstable for any continuous arrival strategy, thus shedding more light on the non-existence result.

CCS CONCEPTS

• **Theory of computation → Algorithmic game theory and mechanism design**; • **Applied computing → Transportation**; • **Mathematics of computing →** *Stochastic processes.*

KEYWORDS

traffic congestion, travel time uncertainty, Vickrey model, random arrival, pure equilibrium, mixed equilibrium

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1 INTRODUCTION

In the standard bottleneck traffic models, users choose a departure time with the goal of minimizing a cost function that takes into account waiting, earliness and tardiness penalties. The common assumption is that given a departure time the time it takes to reach the bottleneck is known and deterministic, and often scaled to zero. However, this may not be the case, as there may be randomness in the actual departure time or in the trip up until the bottleneck. This paper explores the implications of taking this uncertainty into account on the equilibrium behavior of the users. More specifically, people that arrive at the bottleneck have an intended time at which they want to arrive, but their actual arrival time will deviate from day to day. We investigate the effects of these deviations on the formation of the bottleneck. The users take into account the effect of deviation on the expected cost and therefore the formation of the bottleneck now depends on both the deviation distribution as well as on the cost structure.

In transportation literature, the $\alpha - \beta - \gamma$ bottleneck model is a popular approach to model congestion and user response in a tractable and isolated manner, we therefore use this model in our study. In brief summary, this bottleneck model was first introduced by Vickrey [16] and later adjusted by Arnott [2]. It considers a single bottleneck at which N travellers all want to arrive at their destination at a specified time denoted by t^*. These travellers are modeled as a fluid and are subject to costs of waiting, early and late departure (from the bottleneck), denoted by α, β, γ respectively. The response of travellers is captured by the assumption that they arrive at the bottleneck according to a Nash equilibrium, meaning that no traveller can improve its costs by unilaterally altering her/his departure time from the origin.

The Vickrey model has been extended in many directions, examples include the capacity drop observed at highways by Arnott [1], heterogeneity among travellers by [3, 11], and many more. A recent overview of these extensions was written by Small [14]. A variety of extensions that include the effects of stochasticity, have also been considered. However, this research mostly considered stochasticity at the bottleneck only.

Beyond the transportation literature, the response of travellers based on common preferences has been studied for a wide variety of applications that are closely related to the Vickrey model. These models use queuing theory in combination with game theory. The first model which uses a queueing approach was developed by Glazer and Hassin [5]. They consider a game where a population with a Poisson distributed size choose an arrival time with the aim of minimizing waiting time in the queue. Service times are assumed to be exponentially distributed. Many extensions have been studied with a broad range of application, such as a concert arrival game of Juneja et al. [9] and [10], at which the tardiness was added to the model, causing the order of arrivals to become relevant. In [8] the discrete stochastic queueing model is compared with the fluid approximation which is commonly used in the transportation literature, and also in this paper a fluid model is assumed. A review of this line of research can be found in Chapter 4.1 of [7].

Another model that is related to the problem we consider here is the meeting game of Fosgerau et al. [4] and also synchronization under uncertainty by Ostrovsky [12] who studies the optimal strategy of individuals that incur a cost for waiting until the last arrival occurrences. Both study the problem in a cooperative manner while taking into account possible random deviations between the intended time chosen by the players and the realized time of the event.

In this paper, we extend the bottleneck model by assuming that arrivals to the bottlenecks are not perfectly arriving at the planned time instants. Alternatively, we consider a system where people choose a time of arrival, but the actual time of arrival deviates by some predefined probability distribution. This uncertainty results in having a non-convex cost function. In [17] uncertainty on the road to the bottleneck is considered, which is similar to our case. However they include the travel time to the bottleneck in the cost function, whereas we are interested in the stochasticity effects in the departure time from home and therefore do not include this in the delay costs.

The goal of our analysis is to gain insight into the effects of uncertainty in the responses of travellers on equilibrium and the resulting queuing behavior at the bottleneck. We analyse the impact for various scenarios with respect to the cost function and the arrival time uncertainty. We then continue to investigate whether an equilibrium exists in our model. Our main result is that as opposed to most bottleneck models a continuous mixed strategy user equilibrium does not exist, and neither does a pure strategy equilibrium. This implies that equilibrium has a more elaborate form of a mixed strategy with multiple atoms, and therefore computing it probably requires a more algorithmic approach.

2 MODEL DESCRIPTION

We first describe the classical Vickrey bottleneck model and then proceed to the extension with uncertainty in the individual arrival times of travellers at the bottleneck.

2.1 Standard bottleneck model

A fluid population with a volume of N travellers passes through a single bottleneck of capacity μ. Each traveller strategically decides when to arrive at the bottleneck in order to minimize his cost. It is assumed that each traveller wants to exit the bottleneck at time t^*, and incurs a penalty for deviations from this preference time and for delay. This penalty is captured by a linear cost function with coefficients α, β, γ, for waiting, early and late arrival respectively. The time dependent cost function is represented as follows:

$$c(t) = \alpha w(t) + \beta(t^* - (t + w(t)))^+ + \gamma(t + w(t) - t^*)^+, \quad (1)$$

where $w(t)$ is the waiting time within the bottleneck for an arrival at time t. It is assumed that travellers have full knowledge of each others behavior. The solution of the Vickrey model is given by a Nash equilibrium, meaning that no traveller can enhance his cost by altering his departure time. Throughout this paper we make the standard assumption that $\gamma > \alpha > \beta$, (see, e.g., [13]).

The integral of the resulting arrival rate function $a(t) > 0$ over a finite interval $t \in [t_a, t_b]$ should be equal to the total volume of travellers:

$$\int_{t_a}^{t_b} a(t)dt = N,$$

where N is the the total number of travellers.

Given a total amount of fluid N, we want to find an inflow curve $\lambda(t)$ such that no traveller can decrease its costs by altering its arrival time at the bottleneck. It has been shown (see, e.g., [15]) that such a Nash equilibrium is unique for $\gamma > \alpha > \beta$, and is given by

$$\lambda(t) = \begin{cases} r_1(t - t_a), & t \in [t_a, t_n) \\ r_1(t_n - t_a) + r_2(t - t_n), & t \in [t_n, t_b] \end{cases}, \quad (2)$$

where

$$r_1 = \mu + \frac{\beta\mu}{\alpha - \beta}, \quad r_2 = \mu - \frac{\gamma\mu}{\alpha + \gamma}, \quad (3)$$

$$t_a = t^* - \frac{\eta N/\mu}{1 + \eta}, \quad t_b = t^* + \frac{N/\mu}{1 + \eta}, \quad t_n = t^* - \frac{\delta N/\mu}{\alpha},$$

with $\eta = \frac{\gamma}{\beta}$ and $\delta = \frac{\beta\gamma}{\beta+\gamma}$. This arrival curve gives all travellers equal cost of

$$c = \delta\frac{N}{\mu}. \quad (4)$$

For $\alpha > \beta$ the inflow rate presented in (2) generates a single busy period, i.e., $w(t) > 0$ for all $t \in (t_a, t_b)$ [15]. In this equilibrium, the first and last traveller will only incur costs for early/late arrival, and experience no delay. The traveller leaving exactly at the preferred time t^* encounters costs consisting only of waiting.

2.2 Bottleneck model with arrival time uncertainty

In reality travellers do not necessarily arrive at their intended time. We therefore extend the above bottleneck model and assume that there is uncertainty about the actual time of arrival to the bottleneck. The deviation from the intended arrival time of each traveller is modelled by a continuous random variable $X \in (-\infty, +\infty)$ with density f, assuming the deviations of different users to be independent. If $a(t)$ is continuous then f acts as a smoothing kernel over the arrival function $a(t)$. The resulting arrival rate function is given by

$$\tilde{a}(t) = \int_{u=-\infty}^{t} f(u)a(t - u)du. \quad (5)$$

The time-dependent queue length at the bottleneck can be computed by the difference between the actual arrival rate of (5) and the departure rate μ,

$$q(t) = (\tilde{a}(t) - \mu)^+ \qquad Q(t) >= 0, \quad (6)$$

where $Q(t)$ is the queue length at time t.

As $q(0) = 0$, the waiting time of an arrival at time t can then be computed by

$$W(t) = \int_{u=-\infty}^{t} \frac{q(u)}{\mu} du. \tag{7}$$

By plugging (7) into (1) we compute the expected cost of an arrival at time t,

$$\tilde{C}(t) = \alpha W(t) + \beta\big(t^* - t - W(t)\big)^+ + \gamma\big(t + W(t) - t^*\big)^+. \tag{8}$$

Given a time-dependent arrival rate $\tilde{a}(t)$ we can compute the expected cost for a traveller that has an intended arrival time t by

$$\mathbb{E}[\tilde{C}(t)] = \int_{u=-\infty}^{\infty} \tilde{C}(t+u) f(u) du. \tag{9}$$

3 PRELIMINARY ANALYSIS

To gain insight on the impact of uncertainty about the exact arrival time, we demonstrate the cost over time given that travellers are unaware of this uncertainty. We compute the impact of uncertainty for a number of delay functions. For each, we show the impact for an increasing level of uncertainty.

To obtain the arrival rate over time at which travellers are unaware of the uncertainty function f, we compute the Nash equilibrium $\lambda(t)$ of the standard bottleneck model given by (2). Thus, the actual arrival rate $\tilde{a}(t)$ is computed by the convolution of the Nash equilibrium arrival rate of Equation (3), and the arrival uncertainty distribution of X. The actual arrival rate can be obtained by taking the convolution as defined in Equation (5). We then plug this rate into (8) to obtain the expected costs over time. Finally, the expected cost for a traveller that chooses time t is calculated by (9).

The numerical evaluation of the above described rates and cost function is carried out using the following approximating discretisation scheme. The interval of the bottleneck period is split into n small segments of length Δ where:

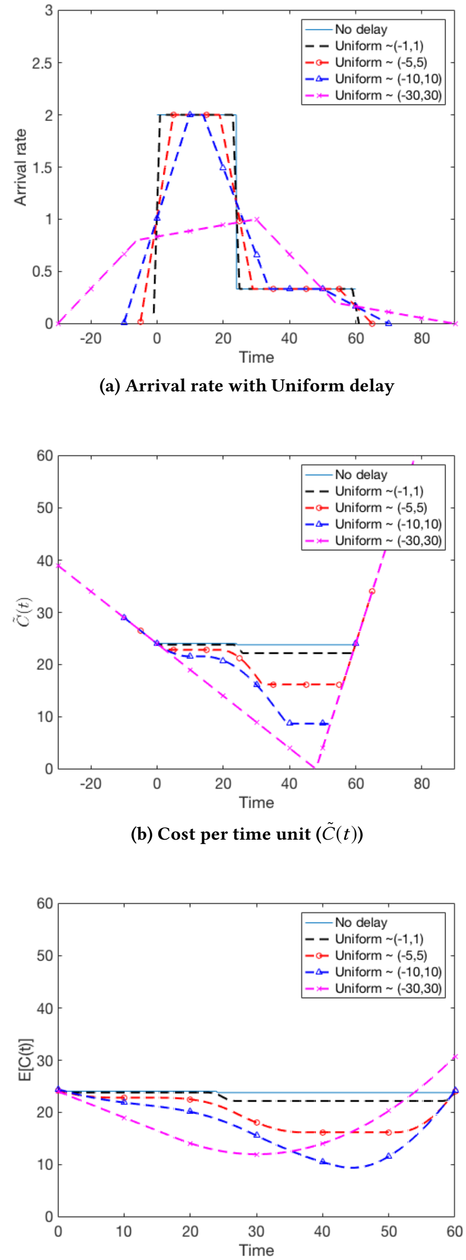
$$n = \left\lfloor \frac{t_{end} - t_{start}}{\Delta} \right\rfloor. \tag{10}$$

The probability volume of the k^{th} segment is obtained by

$$p_k = \mathbb{P}[X \le (k+1)\Delta + t] - \mathbb{P}[X \le k\Delta + t]. \tag{11}$$

First, we consider a uniform uncertainty distribution X. The cost function is taken equal to the standard values from [13] where $\beta/\alpha = 0.5$ and $\gamma/\alpha = 2$ ($\alpha = 1, \beta = 0.5, \gamma = 2$, N=60, s=1). In Figure 1 the results for $X \sim unif(\sigma, \tau)$ when $\tau \in \{0, 1, 5, 10, 30\}$ and $\sigma = -\tau$ are visualised. In these examples, the bottleneck period is extended to $t_{start} = t_a + \sigma$ and $t_{end} = t_b + \tau$, since deviations from the intended arrival times will cause users to arrive prior to t_a and later than t_b as well. The results of Figure 1c show that the expected cost is below that of the cost without any delay, as long as a queue exists. In the boundaries of the arrival interval, for which the delay is equal to the period of the bottleneck, there will be no queue at all: travellers will only incur earliness or lateness cost, depending on their arrival time.

Similar results appear for $X \sim exp(\mu)$, where the delay distribution is on an infinite support. For both functions the same observations are made: decrease in the average cost function for travellers while increasing the delay component. An important observation is that the expected cost is not constant throughout the arrival interval. *Conclusion: travellers will deviate from the Vickrey equilibrium arrival rate because they can reduce their cost.* With the same approach, this phenomenon can be displayed for other delay distributions.



(a) Arrival rate with Uniform delay

(b) Cost per time unit ($\tilde{C}(t)$)

(c) Expected cost for a traveller with intended time t ($\mathbb{E}C(t)$)

Figure 1: Impact of the arrival pattern and costs over time for a uniform delay.

4 OPTIMAL RESPONSES

We continue our analysis by considering the optimal response of travellers when they take into account the uncertainty function. We assume full information in the sense that all travellers are aware that arrival times deviate from the intended times according to f for themselves and all others. This will allow us to examine whether an equilibrium exists, and under which conditions. We explore both the options of pure and continuous mixed strategies, assuming a uniform delay function.

To explicitly study whether a pure or mixed equilibrium exists, we represent the departure delay by a uniformly distributed random variable $X \sim unif(0, 1)$

$$f(t) = \begin{cases} 1, & t \in [0, 1] \\ 0, & \text{otherwise} \end{cases}. \tag{12}$$

The results can be extended to general $X \sim unif(0, \tau)$ by rescaling time.

We first outline the simplifications in the model description that are due to the uniform delay assumption. Plugging Equation (12) into Equation (9) we obtain

$$\mathbb{E}C(t) = \int_{u=0}^{1} \tilde{C}(t + u)du.$$

Note that while the cost function is piecewise linear due to (1), and hence not smooth, the expected cost has a continuous derivative because the cost is continuous. Furthermore, as $\mathbb{E}C(t) \to \infty$ when $t \to \infty$ or $t \to -\infty$, a best response of a single customer is a global minimum that satisfies the necessary first order condition,

$$\frac{d\mathbb{E}C(t)}{dt} = \tilde{C}(t + 1) - \tilde{C}(t) = 0 \iff \tilde{C}(t + 1) = \tilde{C}(t). \tag{13}$$

However, $\mathbb{E}C(t)$ is not convex and there may be multiple local minima or saddle points, that satisfy the necessary condition.

4.1 Pure strategy equilibrium

We first investigate the conditions for a pure strategy equilibrium to exist, namely a time instant such that if all travellers arrive together then no single traveller can reduce the cost by choosing any other arrival time. We then show that these only hold in the degenerate case where the capacity is large enough for no queue to form, regardless of the strategies: $\mu > N$. For the general case we further illustrate numerically how the non-convex shape of the expected cost function makes a pure strategy equilibrium impossible.

We first compute the cost for a tagged traveller arriving at time $s \in \mathbb{R}$, given that a total volume of N decides to arrive at time $t \in \mathbb{R}$. We separate between two cases. When $\mu \geq N$, the tagged traveller encounters no waiting time and the moment of arrival of traveller s does not depend on the volume N at time t. The case where $\mu < N$, s does lead to waiting times when the actual arrival time overlaps with the interval of arrival of the volume N. Formally, A pure strategy Nash equilibrium is given by the fixed point $t \in \arg\min_{s \in \mathbb{R}} \mathbb{E}C_t(s)$.

Case $\mu \geq N$
We determine the cost for a traveller arriving at s by

$$\begin{aligned} C(s) &= \int_{s}^{t^*} \beta(t^* - u)du + \int_{t^*}^{s+1} \gamma(u - t^*)du \\ &= \beta[t^* u - 0.5u^2]\Big|_{s}^{t^*} + \gamma(\frac{u^2}{2} - t^* u)\Big|_{t^*}^{s+1} \\ &= \beta[t^*(t^* - s) - \frac{1}{2}((t^*)^2 - s^2)] + \gamma[\frac{(s+1)^2}{2} - \frac{(t^*)^2}{2} - t^* s]. \end{aligned}$$

To find the time instant that gives the best response we compute the solution of $\frac{dC}{ds} = 0$, yielding the unique solution

$$\begin{aligned} &\frac{dC}{ds} = -\beta t^* + s\beta + \gamma(s + 1 - t^*) = 0 \to \\ &s(\gamma + \beta) = (\gamma + \beta)t^* - \gamma \to \\ &s = t^* - \frac{\gamma}{\beta + \gamma}. \end{aligned}$$

Thus, the best response does not depend on the arrival of the fluid volume N. This is the same solution as in the model with no waiting costs by [6].

Case $\mu < N$
In case that $\mu < N$, a queue builds during the arrival interval of the volume N. For the sake of brevity, without loss of generality, we assume that the total volume of travellers is $N = 1$. Therefore, we need to consider the time of arrival of the volume, which is given by $\tilde{a}_t(u) = 1 \in [t, t+1]$. Including this in (6), we obtain the waiting time by

$$W_t(u) = \begin{cases} (\frac{1}{\mu} - 1)(u - t), & u \in [t, t+1] \\ \frac{1}{\mu} - (u - t), & u \in (t + 1, t + \frac{1}{\mu}] \\ 0, & \text{otherwise} \end{cases},$$

where u represents the intended arrival time.

We insert the $W_t(u)$ in (8), and compute the cost for a traveller that intends to arrive at time u given that the volume N intends to arrive at time t by

$$\tilde{C}_t(u) = \alpha W_t(u) + \beta\big(t^* - (t + W_t(u))\big)^+ + \gamma\big(t + W_t(u) - t^*\big)^+.$$

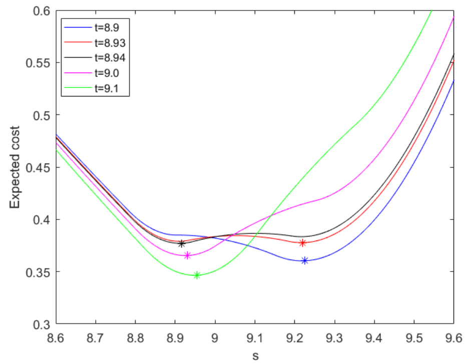
Finally, we compute the expected cost for a traveller that intends to arrive at time s by

$$\mathbb{E}C_t(s) = \int_{s}^{s+1} \tilde{C}_t(u)du. \tag{14}$$

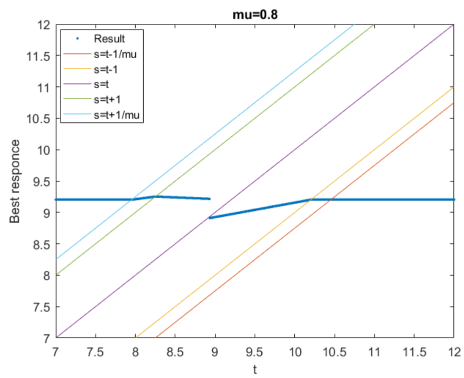
Proposition 1. *A pure Nash equilibrium does not exist for* $N > \mu$.

Before proving Proposition 1, we wish to provide intuition to the non-existence result by numerically illustrating the shape of the cost function. In Figure 2a, we plot the expected cost for a fixed t and observe that there are either two local minima or a single local minimum and a saddle point. Moving t affects the best response, as is illustrated in Figure 2b. The jump corresponds to the point where the global minimum changes from the earlier local minimum (on the left in Figure 2a) to the one at a later time (on the right in Figure 2a). This jump in the best response function is the reason why no fixed point exists.

Proof of Proposition 1. We firstly compute the expected cost for a traveller choosing intended arrival time s, given that the volume N intends to arrive at time t. Therefore, we split the integral



(a) Costs s for specific t



(b) Best response of s for range $t \in T$

Figure 2: Best response of s for $t \in T$ for $\mu = 0.8$.

of equation (14) into several cases. We make a division between the case where $s \leq t$ and $s \geq t$, and we separate between the point where the earliness cost changes to lateness cost denoted by $x^* = t + \mu(t^* - t)$.

For $s \leq t < s + 1 \leq x^*$,

$$\mathbb{E}C_t(s) = \int_s^t \beta(t^* - u)du + \int_t^{s+1} \alpha W_t(u) + \beta(t^* - u - W_t(u))du,$$

for $s \leq t < x^* < s + 1 \leq t + 1$,

$$\mathbb{E}C_t(s) = \int_s^t \beta(t^* - u)du + \int_t^{s+1} \alpha W_t(u)du$$
$$+ \int_t^{x^*} \beta(t^* - u - W_t(u))du + \int_{x^*}^{s+1} \gamma(u + W_t(u) - t^*)du,$$

and for $t \leq s < x^* < t + 1 \leq s + 1$,

$$\mathbb{E}C_t(s) = \int_s^{t+1} \alpha W_t(u)du + \int_s^{x^*} \beta(t^* - u - W_t(u))du$$
$$+ \int_{x^*}^{t+1} \gamma(u + W_t(u) - t^*)du + \int_{t+1}^{s+1} \alpha W_t(u)$$
$$+ \gamma(u + W_t(u) - t^*)du.$$

We excluded the cases where $s, t > x^*$, and also $s + 1, t + 1 < t^*$ which cannot be an equilibrium because the cost function of the agent at t can be trivially improved. To find the optimal time s when all others arrive at time t we solve the necessary condition (13) for each of the above cases.

Case 1: For $s \leq t < s + 1 \leq x^*$,

$$s = \frac{\alpha(\mu - 1 + t - \mu t) + \beta(1 - \mu - t + \mu t)}{\alpha(1 - \mu) - \beta(2\mu - 1)}$$
$$\downarrow s = t = t_e.$$
$$t_e = \frac{\alpha}{\beta}(1 - \frac{1}{\mu}) + \frac{1}{\mu} - 1$$

this will give a negative value for any $\mu < 1$ and $\alpha > \beta$.

Case 2: For $s \leq t < x^* < s + 1 \leq t + 1$,

$$s = \frac{\alpha(\mu + t - \mu t - 1) + \beta\mu t^* + \gamma(\mu t^* + t - \mu t - 1)}{\alpha + \gamma + \mu(\beta - \alpha)}$$
$$\downarrow s = t = t_e$$
$$t_e = t^* - \frac{\gamma}{\beta + \gamma}(\frac{1}{\mu}) - \frac{\alpha}{\beta + \gamma}(\frac{1}{\mu} - 1)$$

As t_e has to meet the criteria of $t_e \geq t^* - \frac{1}{\mu}$, we can only find a pure equilibrium when $\mu > \frac{\alpha - \beta}{\alpha}$.

Case 3: For $t \leq s < x^* < t + 1 \leq s + 1$, we obtain the same solution as in *Case 2* because the cost functions coincide when taking $s \uparrow t$ or $s \downarrow t$.

The above suggests that a pure equilibrium solution can only be at t_e. However, it can be shown that the cost of the tagged traveller arriving at time t_e when all others arrive at t_e, is not the global minimum. This is done by considering the derivative of the expected cost in the range $t \leq s < x^* < t + 1 \leq s + 1$ (i.e., *Case 3* above),

$$\frac{d\mathbb{E}_t C}{ds} = \frac{-\alpha(\mu - 1 + s - t) + \beta(s - \mu t^* + (\mu - 1)t) + \gamma(1 - t^*\mu + \mu t)}{\mu},$$

and observing that it is a decreasing function with s, as $\alpha > \beta$. This implies that t_e is a saddle point and the global minimum is at $t > t_e$. The global minimum is obtained in the range $t \leq s \leq x^* \leq t + 1 \leq t + \frac{1}{\mu} \leq s + 1$. The cost of s for this case is

$$\mathbb{E}C(s) = \int_s^{t+1} \alpha W(u)du + \int_s^{x^*} \beta(t^* - u - W(u))du$$
$$+ \int_{x^*}^{t+1} \gamma(u + W(u) - t^*)du + \int_{t+1}^{t+\frac{1}{\mu}} \alpha W(u)$$
$$+ \gamma(u + W(u) - t^*)du + \int_{t+\frac{1}{\mu}}^{s+1} \gamma(u - t^*)du,$$

with a first derivative $\frac{d\mathbb{E}_t C}{ds}$ that equals

$$\frac{-t^*(\beta + \gamma)\mu - \alpha s + \beta s - \alpha\mu s + \gamma\mu(1 + s) + \alpha t - \beta t - \alpha\mu t + \beta\mu t}{\mu}.$$

Solving the first order condition $\frac{d\mathbb{E}_t C}{ds} = 0$ yields the best response

$$s^* = \frac{t^*\mu(\beta + \gamma) - \gamma\mu + t(\alpha - \beta)(\mu - 1)}{\alpha(\mu - 1) + \beta + \gamma\mu}. \tag{15}$$

In Figure 3, we illustrate for a numerical example the saddle point at t_e with the global minimum $s^* > t_e$ corresponding to the

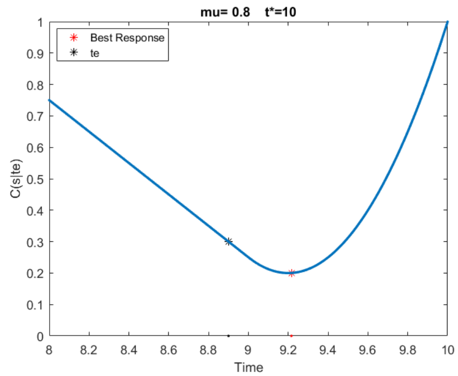


Figure 3: Best response time s of Equation (15) for a volume N intending to arrive at time t, where $t = t_e$.

best response. Finally, we conclude that a pure equilibrium does not exist. □

4.2 Continuous mixed strategy equilibrium

We continue our analysis by considering a continuous mixed arrival strategy. We formulate the conditions for such an equilibrium and show that there exists no solution for the uniform delay function. We further highlight why a continuous mixed strategy equilibrium fails by a numerical approximation. We focus here only on the non-degenerate case of $\mu < N = 1$.

A symmetric mixed arrival strategy is given by an arrival density $g(t)$ and cumulative distribution function $G(t)$ such that all travellers select their intended arrival time according to this distribution. Let $[t_a, t_b]$ be the support of the distribution g, and note that it is finite in equilibrium. For a traveller that intends to arrive at $t \in [t_a, t_b]$ the actual arrival time is $t + X$, where $X \sim unif(0, 1)$, as before. The arrival rate function of Equation (5) then simplifies to

$$\tilde{a}(t) = \int_{t_a \vee (t-1)}^{t_b \wedge t} g(t-u)du = G(t_b \wedge t) - G(t_a \vee (t-1)), \tag{16}$$

where $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$. This allows us to determine the waiting time when $q(s) \geq 0$ for $t_a \leq s \leq t$,

$$W(t) = \frac{1}{\mu} \int_{t_a}^{t} \left(\tilde{a}(u) - \mu\right)^+ du. \tag{17}$$

Let $\mathbb{E}_g C(t)$ denote the expected cost for a traveller with intended arrival time t when all others arrive according to g. The realized cost of a traveller arriving at t is then given by $\tilde{C}$ as defined in (8). A distribution g is a Nash equilibrium if both of the following conditions are satisfied for every t such that $g(t) > 0$

1. $\mathbb{E}_g C(t) \leq \mathbb{E}_g C(s)$ for every $s \in \mathbb{R}$,
2. the necessary condition (13): $\tilde{C}(t) = \tilde{C}(t+1)$ is satisfied.

Proposition 2. *A continuous mixed equilibrium does not exist.*

Proof. We will show that there is no density g for which the equilibrium condition of Equation (13) $\tilde{C}(t+1) - \tilde{C}(t) = 0$ holds for all $t \in [t_a, t_b]$. First observe that

$$\tilde{C}(t+1) - \tilde{C}(t) = \alpha(W(t+1) - W(t)) + At + B,$$

where A and B are nonzero constants. Therefore,

$$\frac{d}{dt}\left(\tilde{C}(t+1) - \tilde{C}(t)\right) = \alpha \frac{d}{dt}(W(t+1) - W(t)) + A,$$

and the equilibrium condition implies that this function is zero throughout the support. If $\frac{d}{dt}W(t) > 0$ for all $t \in [t_a, t_b]$, then by applying (17) we have that the equilibrium condition further implies that $\tilde{a}(t+1) - \tilde{a}(t) = -\frac{A}{\alpha}$. By (16) this is equivalent to

$$k(t) := G(t_b \wedge (t+1)) - G(t_a \vee t) - G(t_b \wedge t) + G(t_a \vee (t-1)) = -\frac{A}{\alpha} \neq 0.$$

Recall that $G(t_a) = 0$ and $G(t_b) = 1$. If $t_b \geq t_a + 1$ then

$$k(t_a) = G(t_a+1) - G(t_a) - G(t_a) + G(t_a) = G(t_a+1) > 0,$$

and

$$k(t_b) = G(t_b) - G(t_b) - G(t_b) + G(t_b - 1) = G(t_b - 1) - G(t_b) < 0.$$

If $t_b < t_a + 1$ then

$$k(t_a) = G(t_b) - G(t_a) - G(t_a) + G(t_a) = G(t_b) - G(t_a) = 1 > 0,$$

and

$$k(t_b) = G(t_b) - G(t_b) - G(t_b) + G(t_a) = G(t_a) - G(t_b) < 0,$$

and we conclude that both cases yield a contradiction to the assumption that $k(t)$ is constant. Note that this conclusion is unchanged if $\frac{d}{dt}W(t_a) = 0$ because as $\mu < 1$ it is not possible that $\frac{d}{dt}W(t) = 0$ for all $t \in [t_a, t_b]$, i.e., at some point the derivative must be nonzero. □

4.3 Approximation of continuous equilibrium

We propose a numerical procedure to obtain an arrival rate function for which the expected cost per traveller remains constant on most of the support. This method can be applied to non-uniform delays distributions and exhibits similar properties.

We want to obtain an arrival rate $\tilde{a}(t)$, for which the $\mathbb{E}C(t) \approx c$. To obtain a numerical solution, we discretise the functions of Section 4.2 by Equation (10) where $[t_{start}, t_{end}]$ denotes the interval including the support of the delay function and Δ is the stepsize. We can also compute the probability distribution of arrival as computed in Equation (11).

C_{target} is equal to Equation (4), the arrival rate $\tilde{a}(t)$ is captured in the vector $\bar{r} = (r_1, \ldots, r_n)$, and ϵ is a small value with which we increase the rate at the indicated location.

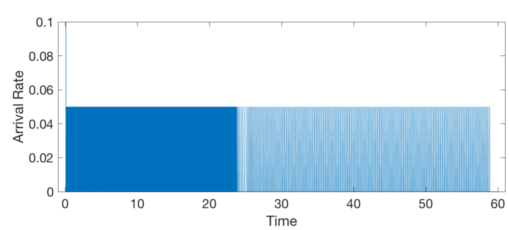


Figure 4: Arrival rate intensities over time.

Algorithm 1 shows the numerical procedure that results in an arrival rate function for which the costs over time remain relatively

Algorithm 1 Procedure to approximate a mixed equilibrium.

```
Inputs:
    n, t*, β, C_target, v, w, p_j
Initialize:
    i_Loc = (t* − βC_target − t_start) / Δ;
    r_i = 0 for i = 0, ..., n
while i_Loc ≠ ∅ do
    r_{i_Loc} = r_{i_Loc} + ϵ
    for i = 0, ... M do
        C_i = Σ_{j=0}^{i+w−v} p_j C̃_{j+v}
    end for
    i_Loc = arg min_i {i : C_i < C_target}
end while
```

constant. In summary, the procedure consists of the following steps. We first set a target cost denoted by C_{target}, which we want to keep constant. We search for the earliest moment of arrival such that this cost constraint is met. At this specific time instant we add a small arrival volume of rate ϵ. Given the updated arrival vector, we compute the new cost function over time. Again, we compute the earliest moment of arrival t such that $\mathbb{E}C(t) \leq C_{target}$. We continue this procedure until this condition can not be met anymore.

In Figure 4, a representation of the outcome of the approximation procedure of Algorithm 1 is visualised. The line density indicates the arrival rate intensity over time. We observe a large density in the beginning, followed by a reduced density at the peak moment x^* ($t = 24$), which increases again shortly after. The arrival rate over a specified period of time is given by the sum of the lines. We apply a moving average filter to obtain the arrival rate function over time.

In Figures 5 and 6, the results of these rates and the costs over time are visualised for uniform delay function. These figures show that for a larger uncertainty, obtaining a constant cost function becomes more difficult.

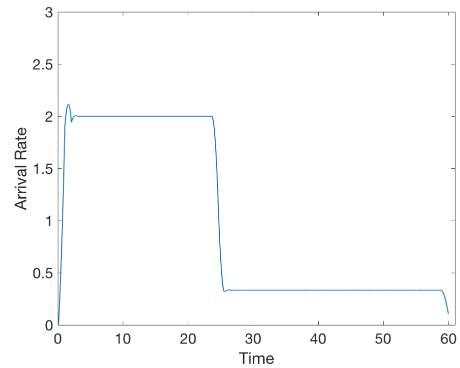
In Table 1, we observe that the total amount of travellers passing the bottleneck decreases with respect to delay, while fixing the expected cost to the value of Equation (4). Conclusions on the impact of arrival time uncertainty with respect to disutility can not be made, as the current results are not in equilibrium. However, this does suggest that uncertainty increases the disutility of individual travellers.

Table 1: Total amount of travellers (N) for fixed costs with varying delay function $f(t)$ and mean delay τ.

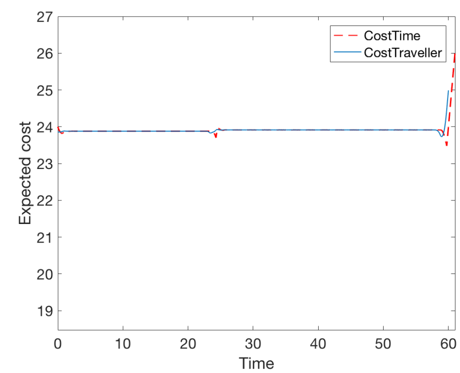
	Uniform	Exponential
$\tau = 0$	60	60
$\tau = 1$	59.6	59.2
$\tau = 5$	58.7	56.5
$\tau = 10$	57.5	54.1

5 CONCLUSION

In this paper we investigated the impact of uncertainty when timing arrival to a bottleneck. Our model allows a random deviation from the intended arrival times of users to a congestion bottleneck. Such a random distortion models the fact that the actual arrival time of users at a specific congestion point can not be completely controlled by the users. In reality it is common that the departure times from the points of origin, and the delays incurred before reaching the specific bottleneck can only be estimated up to a certain confidence range.



(a) Arrival rate

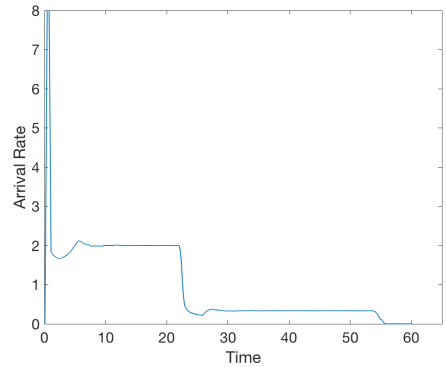


(b) Cost function

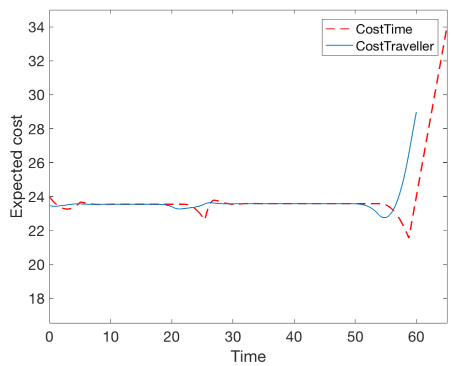
Figure 5: Approximated equilibrium function for $X \sim$ unif$(0, 1)$.

We have shown that the equilibrium rate of the Vickrey model without distorted arrivals is, in general, not a good approximation for a possible equilibrium in the model with distortions. The numerical analysis further showed that costs are reduced for almost the entire bottleneck period, while for highly variable uncertainty distributions, the costs spikes up at the end of the bottleneck period. This is due to the possibility of arrival after the end of the standard bottleneck period.

In fact, we have shown that standard user equilibrium solutions, namely pure strategy or continuous mixed strategy, do not exist for this model. We showed how the dynamics of delayed arrivals inhibits the existence of a constant expected cost function. This calls for future work seeking more elaborate equilibrium solutions that involve mixed strategies that are atomic and not continuous, i.e., may have multiple times with a volume of arrivals. A feasible



(a) Arrival rate



(b) Cost function

Figure 6: Approximated equilibrium function for $X \sim$ unif$(0, 10)$.

approach to do this may be by constructing a search algorithm that iteratively changes the volume at local minima points that appear in the cost function when the arrival strategy has masses (see Figure 2).

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