Quantum asymptotic spectra of graphs and non-commutative graphs, and quantum Shannon capacities

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Abstract

We study quantum versions of the Shannon capacity of graphs and non-commutative graphs. We introduce the asymptotic spectrum of graphs with respect to quantum homomorphisms and entanglement-assisted homomorphisms, and we introduce the asymptotic spectrum of non-commutative graphs with respect to entanglement-assisted homomorphisms. We apply Strassen's spectral theorem (J. Reine Angew. Math., 1988) and obtain dual characterizations of the corresponding Shannon capacities and asymptotic preorders in terms of their asymptotic spectra. This work extends the study of the asymptotic spectrum of graphs initiated by Zuiddam (Combinatorica, 2019) to the quantum domain. We study the relations among the three new quantum asymptotic spectra and the asymptotic spectrum of graphs. The bounds on the several Shannon capacities that have appeared in the literature we fit into the corresponding quantum asymptotic spectra. In particular, we prove that the (fractional) complex Haemers bound upper bounds the quantum Shannon capacity, defined as the regularization of the quantum independence number (Mančinska and Roberson, J. Combin. Theory Ser. B, 2016), which leads to a separation with the Lovász theta function.

Keywords: quantum information theory, graphs, non-commutative graphs, entanglement, duality

1 Introduction

This paper studies quantum variations of the Shannon capacity of graphs via the theory of asymptotic spectra. The Shannon capacity of a graph G was introduced by Shannon in [Sha56] and is defined as

$$\Theta(G) := \sup_{n\geq 1} \sqrt[n]{\alpha(G^{\boxtimes n})} = \lim_{n\to\infty} \sqrt[n]{\alpha(G^{\boxtimes n})},$$

where $\alpha(G)$ denotes the independence number of G and where $G^{\boxtimes n}$ denotes the n-th strong graph product power of G. (All concepts used in this introduction will be defined in Section 2.) The definition of this graph parameter is motivated by the study of classical communication channels. One associates to a classical channel the *confusability graph* with vertices being the input symbols of the channel, and edges given by pairs of input symbols that may be mapped to the same output

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by the channel with a nonzero probability. The Shannon capacity then measures the amount of information that can be transmitted over the channel without error, asymptotically.

Deciding whether $\alpha(G) \geq k$ is NP-complete [Kar72], and Shannon capacity is not even known to be a computable function. A natural approach to study the Shannon capacity is to construct graph parameters that are upper bounds on Shannon capacity. Shannon himself introduced an upper bound in [Sha56], which is known as the *fractional packing number* or *Rosenfeld number*. In the seminal work of Lovász [Lov79], the *Lovász theta function* ϑ was introduced to upper bound the Shannon capacity. Remarkably, the theta function can be written as a semidefinite program that is efficiently computable. Using the theta function, Lovász proved that $\alpha(C_5^{\boxtimes 2})^{1/2} = \Theta(C_5) = \vartheta(C_5) = \sqrt{5}$, where C_n is the n-cycle graph. Lovász further conjectured that $\Theta(G) = \vartheta(G)$ for every graph G. This conjecture was shown to be false by Haemers. He introduced the Haemers bound $\mathcal{H}^{\mathbb{F}}$ as an upper bound on the Shannon capacity, and showed that $\mathcal{H}^{\mathbb{F}}(G) < \vartheta(G)$ when G is the complement of the Schläfli graph [Hae79]. For the odd cycle graphs C_{2k+1} with $k \geq 3$, it is still open whether $\Theta(C_{2k+1}) = \vartheta(C_{2k+1})$. For example, the currently best lower bound on $\Theta(C_7)$ is $\sqrt[5]{367} \approx 3.25787$ [PS18], whereas $\vartheta(C_7) \approx 3.31766$.

Recently, a *dual* characterization of the Shannon capacity was found by Zuiddam in [Zui19] via the *theory of asymptotic spectra*. This theory was developed by Strassen in [Str88], see also the exposition in [Zui18, Chapter 1]. In the general theory we are given a *commutative semiring* S with addition $+$, multiplication $\cdot$, and a preorder $\leq$ on S that satisfies the properties to be a "Strassen preorder". For $a \in S$, the rank $\mathrm{R}(a)$ is defined as the minimum number n such that $a \leq n$, and the subrank $\mathrm{Q}(a)$ is defined as the maximum number n such that $n \leq a$, where $n \in S$ stands for the sum of n times the element $1 \in S$. The asymptotic rank of a is defined as the regularization $\lim_{n\to\infty} \sqrt[n]{\mathrm{R}(a^n)}$ and the asymptotic subrank as the regularization $\lim_{n\to\infty} \sqrt[n]{\mathrm{Q}(a^n)}$. The asymptotic spectrum of S with respect $\leq$ is the set of all $\leq$-monotone semiring homomorphisms $S \to \mathbb{R}_{\geq 0}$. Strassen proves that the asymptotic rank of a equals the pointwise maximum and the asymptotic subrank equals the pointwise minimum, over the asymptotic spectrum. Strassen also defines the asymptotic preorder $\lesssim$ on S by $a \lesssim b$ if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\inf_n x_n^{1/n} = 1$ and such that for all $n \in \mathbb{N}$ holds $a^n \leq x_n \cdot b^n$. He proves that $a \lesssim b$ if and only if for every ϕ in the asymptotic spectrum holds $\phi(a) \leq \phi(b)$.

The theory of asymptotic spectra was originally motivated by the study of tensor rank and asymptotic tensor rank [Str86, Str87, Str88, Str91], which are the keys to understand the arithmetic complexity of matrix multiplication (see, e.g., [BCS97]). Here we let S be any family of isomorphism classes of tensors that is closed under direct sum and tensor product, and which contains the "diagonal tensors". We let $\leq$ be the *restriction preorder*, which in quantum information theory language is the preorder corresponding to convertibility by stochastic local operations and classical communication (SLOCC). The restriction preorder is a Strassen preorder, the rank as defined above equals tensor rank, and the asymptotic rank as defined above equals asymptotic tensor rank. Recently, Christandl, Vrana and Zuiddam in [CVZ18] constructed for the first time an infinite family of elements in the asymptotic spectrum of tensors over the complex numbers. A study of tensors with respect to local operations and classical communication was carried out in [JV18].

Let us return to the study of graphs as in [Zui19]. Here S is any family of isomorphism classes of graphs that is closed under the disjoint union and the strong graph product, and which contains the n-vertex empty graph $\overline{K_n}$ for all $n \in \mathbb{N}$. Let $\leq$ be the *cohomomorphism preorder*, which is defined by letting $G \leq H$ if there is a graph homomorphism from the complement of G to the complement of H. The cohomomorphism preorder is a Strassen preorder, the subrank of a graph equals the independence number, and the asymptotic subrank equals the Shannon capacity [Zui19]. Known elements in the asymptotic spectrum of graphs are the Lovász theta function [Lov79], the fractional Haemers bounds over all fields [Bla13, BC18], the complement of the projective rank [MR16] and the fractional clique cover number (see [Sch03, Eq. (67.112)]). The fractional Haemers bounds provide an infinite family of elements in the asymptotic spectrum due to the separation result in [BC18]. We note that the dual characterization is nontrivial in the sense that the asymptotic tensor rank of tensors and the Shannon capacity of graphs are not multiplicative. We note that Fritz in [Fri17] developed a theory for *commutative monoids* analogous to Strassen's theory of asymptotic spectra and that he applied this theory to graphs to obtain a dual characterization of Shannon capacity and of the asymptotic preorder $\lesssim$ in terms of $\leq$-monotone monoid-homomorphisms.

Quantum Shannon capacity of graphs

We now turn to the quantum setting. We consider two quantum variants of graph homomorphism. The first variant is characterized by the existence of perfect quantum strategies for the *graph homomorphism game* [MR16], which is defined as follows. Two players Alice and Bob are given two graphs G and H. During the game, the referee sends to Alice some vertex $g_A \in V(G)$ and to Bob some vertex $g_B \in V(G)$. Alice responds to the referee with a vertex $h_A \in V(H)$ and Bob respond to the referee with a vertex $h_B \in V(H)$. Alice and Bob win this instance of the (G, H)-homomorphism game, when their answer satisfy

$$\begin{aligned}&\text{if } g_A = g_B, \text{ then } h_A = h_B, \text{ and}\\&\text{if } \{g_A, g_B\} \in E(G), \text{ then } \{h_A, h_B\} \in E(H).\end{aligned}$$

Alice and Bob are not allowed to communicate with each other after having received their input from the referee, but they may together decide on a strategy beforehand. It is not hard to see that Alice and Bob can win the (G, H)-homomorphism game with a classical strategy (i.e. not sharing entangled states) if and only if there is a graph homomorphism from G to H. We say that there is a *quantum homomorphism* from G to H, and write $G \xrightarrow{q} H$, if there exists a perfect quantum strategy for Alice and Bob to win the (G, H)-homomorphism game. It is not hard to see that $G \to H$ implies $G \xrightarrow{q} H$. The *quantum cohomomorphism preorder* $\leq_q$ is defined by letting $G \leq_q H$ if $\overline{G} \xrightarrow{q} \overline{H}$. The *quantum independence number* of G is defined as the maximum number n such that $\overline{K_n} \leq_q G$ and the *quantum Shannon capacity* of G is defined as its regularization.

Entanglement-assisted Shannon capacity of graphs

The second quantum variant of graph homomorphism comes from the study of *entanglement-assisted zero-error capacity* of classical channels, which is a quantum generalization of Shannon's zero-error communication setting. In the zero-error communication model, Alice wants to transmit messages to Bob without error through some classical noisy channel. Shannon in [Sha56] showed that the maximum number of zero-error messages Alice can send to Bob equals the independence number of the confusability graph. In the entanglement-assisted setting, the maximum number of messages that can be sent with zero error turns out to be determined by the confusability graph and is called the *entanglement-assisted independence number* (of the confusability graph) [Bei10]. Similarly, the *entanglement-assisted Shannon capacity* is its regularization. Based on this definition, one naturally defines an *entanglement-assisted homomorphism* between graphs, denoted by $\xrightarrow{*}$ [CMR+14]. Let the *entanglement-assisted cohomomorphism preorder* $\leq_*$ be defined by letting $G \leq_* H$ if $\overline{G} \xrightarrow{*} \overline{H}$. The entanglement-assisted homomorphism has applications in the study of the *entanglement-assisted source-channel coding problem* [BBL+15, CMR+14]. It is easy to see that the *entanglement-assisted independence number* of G is the maximum number n such that $\overline{K_n} \leq_* G$.

It is not hard to see that $G \xrightarrow{q} H$ implies $G \xrightarrow{*} H$. It is believed that the reverse direction holds [MR16]. One may interpret $G \xrightarrow{q} H$ in the communication setting as restricting to use the maximally entangled state and projective measurements [MR16]. On the other hand, it is known that the entanglement-assisted Shannon capacity can be strictly larger than the Shannon capacity [LMM+12, BBG13]. We point out that these separation results also separate the quantum Shannon capacity from Shannon capacity (see the remarks in the proof of Theorem 21).

Entanglement-assisted Shannon capacity of non-commutative graphs

Finally, we consider the setting of sending classical zero-error messages through *quantum channels*. It turns out, analogous to the classical channel scenario, that the one-shot (entanglement-assisted) zero-error classical capacity of a quantum channel is characterized by the non-commutative graph associated with the channel [DSW13]. A *non-commutative graph*, or *nc-graph* for short, is a subspace S of the vector space of $n \times n$ complex matrices, satisfying $S^\dagger = S$ and $I \in S$. Duan in [Dua09] and Cubitt, Chen and Harrow in [CCH11] have shown that every such subspace S is indeed associated to a quantum channel. There are natural preorders $\leq$ and $\leq_*$ on nc-graphs such that the independence number $\alpha(S)$ and the entanglement-assisted independence number $\alpha_*(S)$, defined in [DSW13], equal the maximum number n such that $\overline{\mathcal{K}_n} \leq S$ and the maximum number n such that $\overline{\mathcal{K}_n} \leq_* S$ [Sta16], respectively. Here $\overline{\mathcal{K}_n}$ is the nc-graph associated to the n-message perfect classical channel (whose confusability graph is $\overline{K_n}$). The Shannon capacity Θ and the entanglement-assisted Shannon capacity Θ_* of nc-graphs are defined as the regularization of α and α_*, respectively (where the multiplication is the tensor product).

Overview of our results

In this paper, we extend the study of the asymptotic spectrum of graphs to the quantum domain. We introduce three new asymptotic spectra:

- the asymptotic spectrum of graphs with respect to the quantum cohomomorphism preorder
- the asymptotic spectrum of graphs with respect to entanglement-assisted cohomomorphism preorder
- the asymptotic spectrum of non-commutative graphs with respect to the entanglement-assisted cohomomorphism preorder.

We prove that the preorders in these scenarios are Strassen preorders. This allows us to apply Strassen's spectral theorem to obtain a dual characterization of the corresponding Shannon capacities and asymptotic preorders in terms of their asymptotic spectra, respectively. We then perform a study of the relations among the three new asymptotic spectra and the asymptotic spectrum of graphs of [Zui19]. The bounds on the several Shannon capacities that have appeared in the literature we fit into the corresponding asymptotic spectra.

Note that the fractional complex Haemers bound belongs to the asymptotic spectrum of graphs with respect to the quantum cohomomorphism preorder. It follows that the fractional Haemers bound upper bounds the quantum Shannon capacity. From the separation results of Haemers [Hae79] we know that the quantum Shannon capacity is different from the Lovász theta function. This observation connects two conjectures in quantum zero-error information theory. Namely, it is conjectured that the quantum Shannon capacity equals the entanglement-assisted Shannon capacity [MR16], and that the entanglement-assisted Shannon capacity equals the Lovász theta function [Bei10, CLMW11, LMM+12, DSW13, MSS13, CMR+14, WD18]. Since the fractional Haemers bound (over the complex numbers) is an upper bound on the quantum Shannon capacity, these two conjectures cannot both be true.

Organization of this paper

In Section 2 we cover the basic definitions of graph theory; the definition of the Lovász theta function, the fractional Haemers bounds, the projective rank and the fractional clique cover number; the theory of asymptotic spectra of Strassen; the known properties of the asymptotic spectrum of graphs; the definition of the quantum homomorphism; the definition of the entanglement-assisted homomorphism of graphs; and the definition of the (entanglement-assisted) Shannon capacity of non-commutative graphs. In Section 3 we study the quantum Shannon capacity and the entanglement-assisted Shannon capacity via the corresponding asymptotic spectra. In Section 4 we study the entanglement-assisted Shannon capacity of non-commutative graphs via the corresponding asymptotic spectrum.

2 Preliminaries

2.1 Graphs, independence number, and Shannon capacity

In this paper we consider only finite simple graphs, so graph will mean finite simple graph. For a graph G, we use $V(G)$ to denote the vertex set of G and $E(G)$ to denote the edge set of G. We write $\{g, g'\} \in E(G)$ to denote an edge between vertex g and g'. Since our graphs are simple, $\{g, g'\} \in E(G)$ implies that $g \neq g'$. The *complement* of G is the graph $\overline{G}$ with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{\{g, g'\} : \{g, g'\} \notin E(G) \text{ and } g \neq g'\}$. (We emphasize that when we write $\{g, g'\} \notin E(G)$ we include the case that $g = g'$.) For $n \in \mathbb{N}$, the *complete graph* K_n is the graph with $V(K_n) = [n] := \{1, 2, \ldots, n\}$ and $E(K_n) = \{\{i, j\} : i \neq j \in [n]\}$. Thus $K_0 = \overline{K_0}$ is the empty graph and $K_1 = \overline{K_1}$ is the graph consisting of a single vertex and no edges. A *graph homomorphism* from G to H is a map $f : V(G) \to V(H)$, such that for all $g, g' \in V(G)$, $\{g, g'\} \in E(G)$ implies $\{f(g), f(g')\} \in E(H)$. We write $G \to H$ if there exists a graph homomorphism from G to H.

A *clique* of G is a subset C of $V(G)$, such that for any $g \neq g' \in C$ holds $\{g, g'\} \in E(G)$. The size of the largest clique of G is called the *clique number* of G and is denoted by $\omega(G)$. Equivalently,

(1) $$\omega(G) = \max\{n \in \mathbb{N} : K_n \to G\}.$$

An *independent set* of G is a clique of $\overline{G}$. The size of the largest independent set of G is called the *independence number* of G and is denoted by $\alpha(G)$. Equivalently,

(2) $$\alpha(G) = \max\{n \in \mathbb{N} : K_n \to \overline{G}\}.$$

Let G and H be graphs. The *disjoint union* $G \sqcup H$ is the graph with $V(G \sqcup H) = V(G) \sqcup V(H)$ and $E(G \sqcup H) = E(G) \sqcup E(H)$. The *strong graph product* $G \boxtimes H$ is the graph with

$$\begin{aligned} V(G \boxtimes H) &= V(G) \times V(H) := \{(g, h) : g \in V(G),\, h \in V(H)\} \\ E(G \boxtimes H) &= \{\{(g, h), (g', h')\} : (g = g' \text{ and } \{h, h'\} \in E(H)) \\ &\qquad \text{or } (\{g, g'\} \in E(G) \text{ and } \{h, h'\} \in E(H)) \\ &\qquad \text{or } (\{g, g'\} \in E(G) \text{ and } h = h')\}. \end{aligned}$$

We use $G^{\boxtimes N}$ to denote $\underbrace{G \boxtimes \cdots \boxtimes G}_{N}$. The *Shannon capacity* of G [Sha56] is defined as

(3) $$\Theta(G) := \lim_{N \to \infty} \sqrt[N]{\alpha(G^{\boxtimes N})}.$$

This limit exists and equals the supremum $\sup_N \sqrt[N]{\alpha(G^{\boxtimes N})}$ by Fekete's lemma.

2.2 Upper bounds on the Shannon capacity

For any $d \in \mathbb{N}$ and any field $\mathbb{F}$, let $M(d, \mathbb{F})$ be the space of $d \times d$ matrices with coefficients in $\mathbb{F}$. Let $I_d \in M(d, \mathbb{F})$ be the $d \times d$ identity matrix. Let $A \in M(d, \mathbb{C})$. Then $A^\dagger$ denotes the complex conjugate of A. The element A is called a *projector* if $A^\dagger = A$ and $AA = A$, i.e. A is *Hermitian* and *idempotent*.

Deciding whether $\alpha(G) \geq k$ is NP-hard [Kar72] and it is not known whether the Shannon capacity $\Theta(G)$ is a computable function. In the study of $\Theta(G)$, the following graph parameters have been introduced that upper bound $\Theta(G)$.

Lovász theta function $\vartheta(G)$

An *orthonormal representation* of a graph G is a collection of unit vectors $U = (u_g \in \mathbb{R}^d : g \in V(G))$ indexed by the vertices of G, such that non-adjacent vertices receive orthogonal vectors: $u_g^T u_{g'} = 0$ for all $g \neq g', \{g, g'\} \notin E(G)$. The celebrated Lovász theta function [Lov79], is defined as

$$(4) \qquad \vartheta(G) := \min_{c,U} \max_{g \in V(G)} \frac{1}{(c^T u_g)^2},$$

where the minimization goes over unit vectors $c \in \mathbb{R}^d$ and orthonormal representations U of G. Lovász proved that

$$\Theta(G) \leq \vartheta(G).$$

Equation (4) is a semidefinite program that is efficiently computable. There are several useful alternative characterizations of ϑ in the literature, see [Lov79].

Fractional Haemers bound $\mathcal{H}_f^{\mathbb{F}}(G)$

A *d-representation* of a graph G over a field $\mathbb{F}$ is a matrix $M \in M(|V(G)|, \mathbb{F}) \otimes M(d, \mathbb{F})$ of the form $M = \sum_{g,g' \in V(G)} e_g e_{g'}^\dagger \otimes M_{g,g'}$, such that $M_{g,g} = I_d$ for all $g \in V(G)$ and $M_{g,g'} = 0$ if $g \neq g', \{g, g'\} \notin E(G)$. Let $\mathcal{M}_{\mathbb{F}}^d(G)$ be the set of all d-representation of G over $\mathbb{F}$. The *fractional Haemers bound* [Bla13, BC18], as a fractional version of the Haemers bound [Hae79], is defined as

$$(5) \qquad \mathcal{H}_f^{\mathbb{F}}(G) := \inf\big\{\mathrm{rank}(M)/d : M \in \mathcal{M}_{\mathbb{F}}^d(G),\ d \in \mathbb{N}\big\}.$$

The original Haemers bound [Hae79] of a graph G can be fomulated as:

$$(6) \qquad \mathcal{H}^{\mathbb{F}}(G) = \min\big\{\mathrm{rank}(M) : M \in \mathcal{M}_{\mathbb{F}}^1(G)\big\}$$

and we have

$$\Theta(G) \leq \mathcal{H}_f^{\mathbb{F}}(G) \leq \mathcal{H}^{\mathbb{F}}(G).$$

Whether the (fractional) Haemers bound is computable remains unknown. Interestingly, for any field $\mathbb{F}$ of nonzero characteristic and $\epsilon > 0$, there exists an explicit graph $G = G(\mathbb{F}, \epsilon)$ so that if $\mathbb{F}'$ is any field with a different characteristic, $\mathcal{H}_f^{\mathbb{F}}(G) \leq \epsilon \mathcal{H}_f^{\mathbb{F}'}(G)$ [BC18, Theorem 19].

Projective rank $\xi_f(G)$

A *d/r-representation* of a graph G is a collection of rank-r projectors $(E_g \in M(d, \mathbb{C}) : g \in V(G))$, such that $E_g E_{g'} = 0$ if $\{g, g'\} \in E(G)$. The *projective rank* [MR16] is defined as

$$(7) \qquad \xi_f(G) := \inf\big\{d/r : G \text{ has a } d/r \text{ representation}\big\}.$$

The complement of the projective rank, $\overline{\xi}_f(G) := \xi_f(\overline{G})$, is an upper bound on the Shannon capacity,

$$\Theta(G) \leq \overline{\xi}_f(G).$$

Fractional clique cover number $\overline{\chi}_f(G)$

The fractional packing number can be written as a linear program (of large size), whose dual program is the fractional clique cover number (see, e.g., [Sch03] or [ADR+17, Eq. (A.16)]). Explicitly,

$$\begin{aligned}\overline{\chi}_f(G) &:= \min \sum_C s_C, \text{ s.t. } s_C \geq 0 \text{ for every clique } C, \sum_{C \ni g} s_C \geq 1 \text{ for every vertex } g \in V(G),\\ &= \max \sum_g t_g \text{ s.t. } t_g \geq 0 \text{ for every vertex } g \in V(G) \sum_{g \in C} t_g \leq 1 \text{ for every clique } C.\end{aligned} \tag{8}$$

where a clique C of G is an independent set of $\overline{G}$. It is known that

$$\Theta(G) \leq \overline{\chi}_f(G) = \overline{\chi}_f(G) = \lim_{n\to\infty} \sqrt[n]{\overline{\chi}(G^{\boxtimes n})} \text{ (e.g. see [Sch03]).} \tag{9}$$

Relations between graph parameters

We know the following inequalities among the graph parameters that we have just defined:

(10) $\Theta(G) \leq \vartheta(G) \leq \overline{\xi}_f(G) \leq \overline{\chi}_f(G)$

(11) $\Theta(G) \leq \mathcal{H}_f^{\mathbb{F}}(G) \leq \overline{\chi}_f(G)$

(12) $\mathcal{H}_f^{\mathbb{C}}(G) \leq \mathcal{H}_f^{\mathbb{R}}(G) \leq \overline{\xi}_f(G)$.

The inequalities in (10) can be found in [Lov79, MR16]. The inequalities in (11) follow from the work in [BC18]. The first inequality in (12) is actually an equality (cf. Prop. 24), and the argument that the real fractional Haemers bound is at most the complement of the real projective rank $\overline{\xi}^{\mathbb{R}}(G)$ is the following: We can obtain the definition of $\overline{\xi}_f^{\mathbb{R}}(G)$ from the definition of $\mathcal{H}_f^{\mathbb{R}}(G)$ by requiring the d-representations of G to be positive semidefinite, as implicitly shown in [HPRS17].

2.3 Asymptotic spectra and Strassen's spectral theorem

We present some fundamental abstract concepts and theorems from Strassen's theory of asymptotic spectra. For a detailed description, we refer the reader to [Str88, Zui18].

A *semiring* $(S, +, \cdot, 0, 1)$ is a set S equipped with a binary addition operation $+$, a binary multiplication operation $\cdot$, and elements $0, 1 \in S$, such that for all $a, b, c \in S$ holds

(13) $(a + b) + c = a + (b + c)$, $a + b = b + a$

(14) $0 + a = a$, $0 \cdot a = 0$, $1 \cdot a = a$

(15) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(16) $a \cdot (b + c) = a \cdot b + a \cdot c$.

A semiring $(S, +, \cdot, 0, 1)$ is *commutative* if for all $a, b \in S$ holds $a \cdot b = b \cdot a$. For any natural number $n \in \mathbb{N}$, let $n \in S$ denote the sum of n times the element $1 \in S$.

A *preorder* $\leq$ on S is a relation such that for any $a, b, c \in S$ holds that $a \leq a$, and that if $a \leq b$ and $b \leq c$, then $a \leq c$. A preorder $\leq$ on S is a *Strassen preorder* if for all $a, b, c \in S$, $n, m \in \mathbb{N}$ holds

(17) $n \leq m$ in $\mathbb{N}$ if and only if $n \leq m$ in S

(18) if $a \leq b$, then $a + c \leq b + c$ and $a \cdot c \leq b \cdot c$

(19) if $b \neq 0$, then there exists an $r \in \mathbb{N}$ such that $a \leq r \cdot b$.

Let $S = (S, +, \cdot, 0, 1)$ and $S' = (S', +, \cdot, 0, 1)$ be semirings. A *semiring homomorphism* from S to S' is a map $\phi : S \to S'$ such that $\phi(a + b) = \phi(a) + \phi(b)$, $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ for all $a, b \in S$, and $\phi(1) = 1$. Let $\mathbb{R}_{\geq 0} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$ be the semiring of non-negative real numbers with the usual addition and multiplication operations. The *asymptotic spectrum* $\mathbf{X}(S, \leq)$ of the semiring $S = (S, +, \cdot, 0, 1)$ with respect to the preorder $\leq$ is the set of $\leq$-monotone semiring homomorphisms from S to $\mathbb{R}_{\geq 0}$, i.e.

$$\mathbf{X}(S, \leq) \coloneqq \{\phi \in \mathrm{Hom}(S, \mathbb{R}_{\geq 0}) : \forall a, b \in S,\ a \leq b \ \Rightarrow\ \phi(a) \leq \phi(b)\}. \tag{20}$$

Let $a \in S$. The *subrank* of a is defined as $\mathrm{Q}(a) \coloneqq \max\{n \in \mathbb{N} : n \leq a\}$. The *rank* of a is defined as $\mathrm{R}(a) \coloneqq \min\{n \in \mathbb{N} : a \leq n\}$. The *asymptotic subrank* and *asymptotic rank* of a are defined as

$$\underset{\sim}{\mathrm{Q}}(a) \coloneqq \lim_{N\to\infty} \sqrt[N]{\mathrm{Q}(a^N)}, \text{ and } \underset{\sim}{\mathrm{R}}(a) \coloneqq \lim_{N\to\infty} \sqrt[N]{\mathrm{R}(a^N)}. \tag{21}$$

Fekete's lemma implies that the limits in (21) indeed exist and can be replaced by a supremum and an infimum, that is,

$$\underset{\sim}{\mathrm{Q}}(a) = \sup_N \sqrt[N]{\mathrm{Q}(a^N)}, \text{ and } \underset{\sim}{\mathrm{R}}(a) = \inf_N \sqrt[N]{\mathrm{R}(a^N)}.$$

Strassen proved the following dual characterizations of $\underset{\sim}{\mathrm{Q}}(a)$ and $\underset{\sim}{\mathrm{R}}(a)$ in terms of the asymptotic spectrum.

Theorem 1 ([Str88, Theorem 3.8], see also [Zui18, Cor. 2.14])**.** *Let S be a commutative semiring and let $\leq$ be a Strassen preorder on S. For any $a \in S$ such that $1 \leq a$ and $2 \leq a^k$ for some $k \in \mathbb{N}$, holds*

$$\underset{\sim}{\mathrm{Q}}(a) = \min_{\phi \in \mathbf{X}(S,\leq)} \phi(a), \text{ and } \underset{\sim}{\mathrm{R}}(a) = \max_{\phi \in \mathbf{X}(S,\leq)} \phi(a). \tag{22}$$

Besides asymptotic subrank and rank, the asymptotic spectrum of a commutative semiring with respect to a Strassen preorder $\leq$ also characterizes the asymptotic preorder $\lesssim$ associated to $\leq$. The *asymptotic preorder* $\lesssim$ associated to $\leq$ is defined by $a \lesssim b$ if there is a sequence of natural numbers $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\inf_n (x_n)^{1/n} = 1$ and such that for all $n \in \mathbb{N}$ holds $a^n \leq x_n \cdot b^n$. The dual characterization is that $a \lesssim b$ if and only if for all $\phi \in \mathbf{X}(S, \leq)$ holds $\phi(a) \leq \phi(b)$. See [Str88, Cor. 2.6] and see also [Zui18, Theorem 2.12].

Finally, we mention that the asymptotic spectrum is well-behaved with respect to subsemirings. Let S be a commutative semiring, let $\leq$ be a Strassen preorder on S, and let $T \subseteq S$ be a subsemiring, which means that $0, 1 \in T$ and that T is closed under addition and multiplication. Then clearly the restriction $\leq_T$ of $\leq$ to T is a Strassen preorder on T. For any $\phi \in \mathbf{X}(S, \leq)$ the restricted function $\phi|_T$ is clearly an element of $\mathbf{X}(T, \leq_T)$. The opposite is also true.

Theorem 2 ([Str88, Cor. 2.7], see also [Zui18, Cor. 2.17])**.** *Let S be a commutative semiring, let $\leq$ be a Strassen preorder on S, and let $T \subseteq S$ be a subsemiring. For every element $\phi \in \mathbf{X}(T, \leq|_T)$ there is an element $\psi \in \mathbf{X}(S, \leq)$ such that ψ restricted to T equals ϕ.*

We note that the proof of Theorem 2 is nonconstructive.

2.4 Semiring of graphs and the dual characterization of Shannon capacity

Let $\mathcal{G}$ be the set of isomorphism classes of (finite simple) graphs. The *cohomomorphism preorder* $\leq$ on $\mathcal{G}$ is defined by $G \leq H$ if and only if $\overline{G} \to \overline{H}$, i.e. there is a graph homomorphism from the complement of G to the complement of H. Zuiddam proved in [Zui19] that $\mathcal{G} = (\mathcal{G}, \sqcup, \boxtimes, K_0, K_1)$ is a commutative semiring and that the cohomomorphism preorder $\leq$ is a Strassen preorder on $\mathcal{G}$. By definition, the *asymptotic spectrum of graphs* $\mathbf{X}(\mathcal{G}, \leq)$ consists of all maps $\phi : \mathcal{G} \to \mathbb{R}_{\geq 0}$ such that, for all $G, H \in \mathcal{G}$, holds

(23) $\phi(G \sqcup H) = \phi(G) + \phi(H)$

(24) $\phi(G \boxtimes H) = \phi(G) \cdot \phi(H)$

(25) $\phi(\overline{K_1}) = 1$

(26) $G \leq H \Rightarrow \phi(G) \leq \phi(H)$.

Note that the subrank of a graph G equals the independence number of G, since equation (2) is exactly

$$\alpha(G) = \max\{n \in \mathbb{N} : \overline{K_n} \leq G\}.$$

By Theorem 1, the Shannon capacity is dually characterized as

(27) $$\Theta(G) = \min_{\phi \in \mathbf{X}(\mathcal{G}, \leq)} \phi(G).$$

The known elements belonging to the asymptotic spectrum of graphs are: the *Lovász theta function* ϑ [Lov79], the *fractional Haemers bound* $\mathcal{H}_f^{\mathbb{F}}$ over any field $\mathbb{F}$ [BC18, Bla13], the *complement of projective rank* $\overline{\xi}_f$ [MR16, CMR+14] and the *fractional clique cover number* $\overline{\chi}_f$ [Sch03]. Note that there are infinitely many elements in $\mathbf{X}(\mathcal{G}, \leq)$, due to the separation result in [BC18] of the fractional Haemers bound over different fields.

Remark 3. We note that the fractional clique cover number is the pointwise largest element in $\mathbf{X}(\mathcal{G}, \leq)$. This is because the rank of a graph G equals the clique cover number and the asymptotic clique cover number equals the fractional clique cover number, see [Zui19].

2.5 Quantum variants of graph homomorphism

We present mathematical definitions of the two quantum variants of graph homomorphisms, arising from the *theory of non-local games* and from *quantum zero-error information theory*, respectively.

2.5.1 Quantum homomorphism

Definition 4 (Quantum homomorphism [MR16])**.** Let G and H be graphs. We say there is a *quantum homomorphism* from G to H, and write $G \xrightarrow{q} H$, if there exist $d \in \mathbb{N}$ and $d \times d$ projectors $E_g^h \in M(d, \mathbb{C})$ for every $g \in V(G)$ and $h \in V(H)$, such that the following two conditions hold:

(28) for every $g \in V(G)$ we have $\sum_{h \in V(H)} E_g^h = I_d$

(29) if $\{g, g'\} \in E(G)$ and $\{h, h'\} \notin E(H)$, then $E_g^h E_{g'}^{h'} = 0$.

Remark 5.

- The first condition implies $E_g^h E_g^{h'} = 0$ for all $g \in V(G)$ and $h \neq h' \in V(H)$. Namely, for a fixed $g \in V(G)$ and an arbitrary $h' \in V(H)$, $\sum_{h \in V(H)} E_g^h = I_d$ implies $\sum_{h \in V(H)} E_g^h E_g^{h'} = E_g^{h'}$. Since every E_g^h is a projector, we have $\sum_{h \neq h'} E_g^h E_g^{h'} = 0$. We conclude that $E_g^h E_g^{h'} = 0$ since projectors are also positive semidefinite.

- For every collection of complex projectors $(E_g^h \in M(d, \mathbb{C}) : g \in V(G), h \in V(H))$ satisfying the above two conditions, there exists a collection of real projectors which also satisfies the above two conditions. Namely, we take the collection of real matrices

$$(F_g^h = \begin{bmatrix} \mathrm{Re}(E_g^h) & \mathrm{Im}(E_g^h) \\ -\mathrm{Im}(E_g^h) & \mathrm{Re}(E_g^h) \end{bmatrix} \in M(2d, \mathbb{R}) : g \in V(G), h \in V(H)),$$

  where $\mathrm{Re}(E_g^h)$ and $\mathrm{Im}(E_g^h)$ denote the real part and the image part of E_g^h, respectively. Noting that $E_g^h E_g^h = (\mathrm{Re}(E_g^h)^2 - \mathrm{Im}(E_g^h)^2) + i(\mathrm{Re}(E_g^h)\,\mathrm{Im}(E_g^h) + \mathrm{Im}(E_g^h)\,\mathrm{Re}(E_g^h)) = (\mathrm{Re}(E_g^h) + i\,\mathrm{Im}(E_g^h)) = E_g^h$, we have

$$\begin{aligned} F_g^h F_g^h &= \begin{bmatrix} \mathrm{Re}(E_g^h)^2 - \mathrm{Im}(E_g^h)^2 & \mathrm{Re}(E_g^h)\,\mathrm{Im}(E_g^h) + \mathrm{Im}(E_g^h)\,\mathrm{Re}(E_g^h) \\ -\mathrm{Re}(E_g^h)\,\mathrm{Im}(E_g^h) - \mathrm{Im}(E_g^h)\,\mathrm{Re}(E_g^h) & \mathrm{Re}(E_g^h)^2 - \mathrm{Im}(E_g^h)^2 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Re}(E_g^h) & \mathrm{Im}(E_g^h) \\ -\mathrm{Im}(E_g^h) & \mathrm{Re}(E_g^h) \end{bmatrix} = F_g^h. \end{aligned}$$

  Moreover, it is easy to verify that $(F_g^h : g \in V(G), h \in V(H))$ satisfies the conditions in Definition 4 [MR16].

It is easy to see that $G \to H$ implies $G \xrightarrow{q} H$. The opposite direction is not true [MR16]. The *quantum cohomomorphism preorder* on graphs is defined by $G \leq_q H$ if and only if $\overline{G} \xrightarrow{q} \overline{H}$, and the *quantum independence number* as $\alpha_q(G) \coloneqq \max\{n \in \mathbb{N} : \overline{K_n} \leq_q G\}$. The *quantum Shannon capacity* is defined as $\Theta_q(G) \coloneqq \lim_{N\to\infty} \sqrt[N]{\alpha_q(G^{\boxtimes N})} = \sup_N \sqrt[N]{\alpha_q(G^{\boxtimes N})}$.

2.5.2 Entanglement-assisted homomorphism

Definition 6 (Entanglement-assisted homomorphism [CMR+14])**.** Let G and H be graphs. We say there is a *quantum homomorphism* from G to H, and write $G \xrightarrow{*} H$, if there exist $d \in \mathbb{N}$ and $d \times d$ positive semidefinite matrices ρ and $(\rho_g^h \in M(d, \mathbb{C}) : g \in V(G),\ h \in V(H))$, such that the following two conditions hold

(30) for every $g \in V(G)$ we have $\sum_{h \in V(H)} \rho_g^h = \rho$

(31) if $\{g, g'\} \in E(G)$ and $\{h, h'\} \notin E(H)$, then $\rho_g^h \rho_{g'}^{h'} = 0$.

Remark 7. We note that the positive semidefinite matrix ρ can be further restricted to be positive *definite*.

The *entanglement-assisted cohomomorphism preorder* is defined by $G \leq_* H$ if and only if $\overline{G} \xrightarrow{*} \overline{H}$. The entanglement-assisted independence number can be defined as $\alpha_*(G) = \max\{n \in \mathbb{N} : \overline{K_n} \leq_* G\}$. The *entanglement-assisted Shannon capacity* of G is defined as $\Theta_*(G) \coloneqq \lim_{N\to\infty} \sqrt[N]{\alpha_*(G^{\boxtimes N})} = \sup_N \sqrt[N]{\alpha_*(G^{\boxtimes N})}$.

It is easy to see that $G \xrightarrow{q} H$ implies $G \xrightarrow{*} H$. It remains unknown whether the reverse direction is true. In fact, as pointed out in [MR16], $G \xrightarrow{q} H$ can be interpreted in the zero-error communication setting by restricting to the use of maximally entanglement state and projective measurements.

2.6 (Entanglement-assisted) zero-error capacity of quantum channels

For related definitions in quantum information theory, we refer the reader to [NC10]. We use A and B to denote the (finite-dimensional) Hilbert spaces of the sender (Alice) and the receiver (Bob), respectively. Let $\mathcal{L}(A, B)$ be the space of linear operators from A to B. Let $\mathcal{L}(A) \coloneqq \mathcal{L}(A, A)$. The space $\mathcal{L}(A)$ is isomorphic to the matrix space $M(n, \mathbb{C})$ with $n = \dim(A)$. Let $\mathcal{D}(A) \subseteq \mathcal{L}(A)$ be the set of all (mixed) quantum states, i.e. all trace-1 positive semidefinite operators in $\mathcal{L}(A)$. A quantum state $\rho \in \mathcal{D}(A)$ is *pure* if it has rank 1, i.e. if it can be written as $\rho = |\psi\rangle\langle\psi|$ for some unit vector $|\psi\rangle \in A$. The *support* of a positive semidefinite matrix $P \in \mathcal{L}(A)$ is the subspace of A spanned by the eigenvectors with positive eigenvalues. A *quantum channel* $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$ can be characterized by a completely positive and trace-preserving (CPTP) map. This is equivalent to saying that $\mathcal{N}$ is of the form $\mathcal{N}(\rho) = \sum_i N_i \rho N_i^\dagger$ for some linear operators $\{N_i\}_i \subseteq \mathcal{L}(A, B)$, called the *Choi–Kraus operators* associated to $\mathcal{N}$, satisfying $\sum_i N_i^\dagger N_i = I_A$. (The Choi–Kraus operators are not unique, but they are unique up to unitary transformations, see, e.g., [NC10].)

We focus on the setting in which Alice and Bob use a quantum channel to transmit *classical* zero-error messages. To transmit k classical messages to Bob through the quantum channel $\mathcal{N}$, Alice prepares k pairwise orthogonal states $\rho_1, \ldots, \rho_k \in \mathcal{D}(A)$, where orthogonality is defined with respect to the Hilbert–Schmidt inner product $\langle \rho, \sigma \rangle = \mathrm{Tr}(\rho^\dagger \sigma)$. Bob needs to distinguish the output states $\mathcal{N}(\rho_1), \ldots, \mathcal{N}(\rho_k)$ perfectly, in order to obtain the messages without error. This is only possible when the output states are pairwise orthogonal. In this situation, without loss of generality, Alice may select the $\rho_i = |\psi_i\rangle\langle\psi_i|$ to be pure states for all $i \in [k]$. Note that $\mathcal{N}(|\psi\rangle\langle\psi|) \perp \mathcal{N}(|\phi\rangle\langle\phi|)$ if and only if $|\psi\rangle\langle\phi| \perp N_i^\dagger N_j$ for all $i \neq j$, where $\{N_i\}_i$ are the Choi–Kraus operators of $\mathcal{N}$. We now see that the number of messages one can transmit through the channel $\mathcal{N}$ is determined by the linear space of matrices $S = \mathrm{span}\{N_i^\dagger N_j\}_{i,j} \subseteq \mathcal{L}(A)$. Duan, Severini and Winter called the linear space S the *non-commutative graph* (*nc-graph*) of a quantum channel $\mathcal{N}$ [DSW13]. The nc-graphs may be thought of as the quantum generalization of confusability graphs of classical channels, mentioned in Section 2.5.2. In this analogy, for an nc-graph $S \subseteq \mathcal{L}(A)$ the density operators $\rho, \sigma \in \mathcal{D}(A)$ are input symbols of the channel, and they are "non-adjacent" in the nc-graph S if $|\phi\rangle\langle\psi| \perp S$ for all $|\phi\rangle$ and $|\psi\rangle$ in the support of ρ and σ, respectively. As in the classical setting, "non-adjacent vertices" are nonconfusable.

Note that for every quantum channel $\mathcal{N}$, the associated nc-graph S satisfies $S^\dagger = S$ and $I_A \in S$, where I_A is the identity operator in $\mathcal{L}(A)$. It is shown in [Dua09, CCH11] that any subspace $S \subseteq \mathcal{L}(A)$ that satisfies $S^\dagger = S$ and $I_A \in S$ is associated to some quantum channel. From now, we define a *non-commutative graph* or *nc-graph* as a subspace $S \subseteq \mathcal{L}(A)$ satisfying $S^\dagger = S$ and $I_A \in S$. We define the *independence number* $\alpha(S)$ as the maximum k such that there exist pure states $|\psi_1\rangle, \ldots, |\psi_k\rangle$ satisfying $|\psi_i\rangle\langle\psi_j| \perp S$ for all $i \neq j \in [k]$. The *Shannon capacity* is defined as $\Theta(S) := \lim_{N\to\infty} \sqrt[N]{\alpha(S^{\otimes N})}$, where $S_1 \otimes S_2 := \mathrm{span}\{E_1 \otimes E_2 : E_1 \in S_1, E_2 \in S_2\}$ denotes the tensor product of S_1 and S_2. One verifies that if S_1 and S_2 are the nc-graphs of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively, then the tensor product $S_1 \otimes S_2$ is the nc-graph of the quantum channel $\mathcal{N}_1 \otimes \mathcal{N}_2$. Then (the logarithm of) $\Theta(S)$ is exactly the *classical zero-error capacity* of quantum channels whose nc-graph is S [DSW13].

In the quantum setting, it will be more natural to consider that Alice and Bob are allowed to share entanglement to assist the information transmission. To make use of the entanglement, say $|\Omega\rangle \in \mathcal{L}(A_0 \otimes B_0)$, Alice prepares k quantum channels $\mathcal{E}_1, \ldots, \mathcal{E}_k : \mathcal{L}(A_0) \to \mathcal{L}(A)$ for encoding the classical messages. To send the ith message, Alice applies $\mathcal{E}_i$ to her part of $|\Omega\rangle$, and sends the output state to Bob via the quantum channel $\mathcal{N}$. Bob needs to perfectly distinguish the output states $\rho_i = ((\mathcal{N} \circ \mathcal{E}_i) \otimes \mathcal{I}_{B_0})(|\Omega\rangle\langle\Omega|)$ for $i \in [k]$. The following lemma shows that the maximum number of classical message which can be sent via the quantum channel $\mathcal{N}$ in the presence of entanglement can be also characterized by the nc-graph S, as also mentioned in [DSW13, Sta16].

Lemma 8 ([Sta16]). *Let $\mathcal{N}$, S, $\mathcal{E}_i$ and $|\Omega\rangle$ as above. Let $\{E_{i,l}\}_l$ and $\{E_{j,l'}\}_{l'}$ be the Choi–Kraus operators of $\mathcal{E}_i$ and $\mathcal{E}_j$, respectively, for $i \neq j \in [k]$. Then*

$$((\mathcal{N} \circ \mathcal{E}_i) \otimes \mathcal{I}_{B_0})(|\Omega\rangle\langle\Omega|) \perp ((\mathcal{N} \circ \mathcal{E}_j) \otimes \mathcal{I}_{B_0})(|\Omega\rangle\langle\Omega|)$$

is equivalent to

$$\operatorname{span}\{E_{i,l} \operatorname{Tr}_{B_0}(|\Omega\rangle\langle\Omega|)E_{j,l'}^\dagger\} \perp S.$$

Proof. Let $|\Omega\rangle = \sum_{i=1}^d \sqrt{\lambda_i}\, |i\rangle_{A_0} |i\rangle_{B_0}$ be the Schmidt decomposition of $|\Omega\rangle$. We have

$$\begin{aligned}
&\operatorname{Tr}\big(((\mathcal{N} \circ \mathcal{E}_i) \otimes \mathcal{I}_{B_0})(|\Omega\rangle\langle\Omega|)((\mathcal{N} \circ \mathcal{E}_j) \otimes \mathcal{I}_{B_0})(|\Omega\rangle\langle\Omega|)\big)\\
&= \sum_{\substack{x,y,z,w\\ l',l,m,n}} \sqrt{\lambda_x\lambda_y\lambda_z\lambda_w}\langle y|z\rangle \operatorname{Tr}\big(N_m E_{i,l}(|x\rangle\langle y|_{A_0})E_{i,l}^\dagger N_m^\dagger N_n E_{j,l'}(|z\rangle\langle w|_{A_0})E_{j,l'}^\dagger N_n^\dagger \otimes |x\rangle\langle w|_{B_0}\big)\\
&= \sum_{\substack{x,y\\ l',l,m,n}} \lambda_x\lambda_y \operatorname{Tr}\big(N_m E_{i,l}(|x\rangle\langle y|_{A_0})E_{i,l}^\dagger N_m^\dagger N_n E_{j,l'}(|y\rangle\langle x|_{A_0})E_{j,l'}^\dagger N_n^\dagger\big)\\
&= \sum_{\substack{x,y\\ l',l,m,n}} \lambda_x\lambda_y \, \langle y|_{A_0} E_{i,l}^\dagger N_m^\dagger N_n E_{j,l'} \, |y\rangle_{A_0} \langle x|_{A_0} \, E_{j,l'}^\dagger N_n^\dagger N_m E_{i,l} \, |x\rangle_{A_0}\\
&= \sum_{l',l,m,n} \big|\operatorname{Tr}\big(E_{j,l'} \operatorname{Tr}_{B_0}(|\Omega\rangle\langle\Omega|)E_{i,l}^\dagger N_m^\dagger N_n\big)\big|^2.
\end{aligned}$$

Thus, $((\mathcal{N} \circ \mathcal{E}_i) \otimes \mathcal{I}_{B_0})(|\Omega\rangle\langle\Omega|) \perp ((\mathcal{N} \circ \mathcal{E}_j) \otimes \mathcal{I}_{B_0})(|\Omega\rangle\langle\Omega|)$ is equivalent to $E_{i,l} \operatorname{Tr}_{B_0}(|\Omega\rangle \langle\Omega|)E_{j,l'}^\dagger \perp N_m^\dagger N_n$ for all possible l', l, m, n. ☐

We call $(|\Omega\rangle, \{\mathcal{E}_1, \ldots, \mathcal{E}_k\})$ a size-k *entanglement-assisted independent set* of S. Let $\alpha_*(S)$ be the maximum size of an entanglement-assisted independent set of S. The *entanglement-assisted Shannon capacity* of S is defined as $\Theta_*(S) \coloneqq \lim_{N\to\infty} \sqrt[N]{\alpha_*(S^{\otimes N})}$.

2.7 Semiring of non-commutative graphs and preorders

Recall that an nc-graph is a subspace $S \subseteq \mathcal{L}(A)$ satisfying $S^\dagger = S$ and $I_A \in S$. We point out that every classical graph G naturally corresponds to an nc-graph S_G. Namely, for any graph G, let $\{|g\rangle : g \in G\}$ be the standard orthonormal basis of $\mathbb{C}^{|V(G)|}$ and define

$$S_G \coloneqq \operatorname{span}\{|g\rangle\langle g'| : g = g' \in V(G) \text{ or } \{g, g'\} \in E(G)\} \subseteq \mathcal{L}(\mathbb{C}^{|V(G)|}).$$

For the nc-graphs corresponding to the complement of the complete graphs we use the notation

$$\overline{\mathcal{K}_n} \coloneqq S_{\overline{K_n}} = \operatorname{span}\{|i\rangle\langle i| : i \in [n]\} \subseteq \mathcal{L}(\mathbb{C}^n).$$

It is worth noting that $\overline{\mathcal{K}_n}$ is the nc-graph of the *n-message noiseless classical channel*, which maps $|m\rangle\langle m'|$ to $\delta_{m,m'} |m\rangle\langle m|$ for all $m, m' \in [n]$.

We say two nc-graphs S_1 and S_2 are *isomorphic* if they are equal up to a unitary transformation, i.e. if $S_2 = U^\dagger S_1 U$ for some unitary matrix U. Let $\mathcal{S}$ be the set of isomorphism classes of nc-graphs. Analogous to the operations in the semiring of graphs, for two nc-graphs $S_1 \subseteq \mathcal{L}(A_1)$ and $S_2 \subseteq \mathcal{L}(A_2)$, the "disjoint union" is their direct sum $S_1 \oplus S_2 \subseteq \mathcal{L}(A_1) \oplus \mathcal{L}(A_2) \subseteq \mathcal{L}(A_1 \oplus A_2)$ and the "strong graph product" is their tensor product $S_1 \otimes S_2 \subseteq \mathcal{L}(A_1) \otimes \mathcal{L}(A_2) \cong \mathcal{L}(A_1 \otimes A_2)$. The reader readily verifies the following.

Theorem 9. *$\mathcal{S} = (\mathcal{S}, \oplus, \otimes, \overline{\mathcal{K}_0}, \overline{\mathcal{K}_1})$ is a commutative semiring.*

Lemma 10. *The map $\mathcal{G} \to \mathcal{S} : G \mapsto S_G$ is an injective semiring homomorphism.*

In [Sta16], the *cohomomorphism preorder* and the *entanglement-assisted cohomomorphism preorder* on nc-graphs are defined as follows.

Definition 11. The *cohomomorphism preorder* $\leq$ is defined on $\mathcal{S}$ by, for any nc-graphs $S_1 \subseteq \mathcal{L}(A_1)$ and $S_2 \subseteq \mathcal{L}(A_2)$, letting $S_1 \leq S_2$ if there exists $E = \mathrm{span}\{E_i\}_i \subseteq \mathcal{L}(A_1, A_2)$ satisfying $\sum_i E_i^\dagger E_i = I_{A_1}$, such that $ES_1^\perp E^\dagger \perp S_2$, where $S_1^\perp := \{X \in \mathcal{L}(A_1) : \forall Y \in S_1 \ \mathrm{Tr}(X^\dagger Y) = 0\}$.

Definition 12. The *entanglement-assisted cohomomorphism preorder* $\leq_*$ is defined on $\mathcal{S}$ by, for any nc-graphs $S_1 \subseteq \mathcal{L}(A_1)$ and $S_2 \subseteq \mathcal{L}(A_2)$, letting $S_1 \leq_* S_2$ if there exist a positive definite $\rho \in \mathcal{D}(A_0)$ and $E = \mathrm{span}\{E_i\}_i \subseteq \mathcal{L}(A_1 \otimes \rho, A_2)$ satisfying $\sum_i E_i^\dagger E_i = I_{A_1 \otimes A_0}$, such that $E(S_1^\perp \otimes \rho)E^\dagger \perp S_2$.

Lemma 13. *If $S \leq T$, then $S \leq_* T$.*

Proof. Take the positive definite matrix ρ in the definition of $S \leq_* T$ to be the element 1. □

The idea behind the above definitions is as follows. Recall that $G \leq H$ if there exists a graph homomorphism from $\overline{G}$ to $\overline{H}$. In other words, there exists a *vertex map* $f : V(G) \to V(H)$ which maps non-adjacent vertices to non-adjacent vertices. Since we may view quantum states as vertices and matrices in the nc-graph as edges in nc-graphs, it is natural to adapt the "vertex map" among nc-graphs $S_1 \subseteq \mathcal{L}(A_1)$ and $S_2 \subseteq \mathcal{L}(A_2)$ as a CPTP map $\mathcal{E} : \mathcal{L}(A_1) \to \mathcal{L}(A_2)$, specified by the Choi–Kraus operators $\{E_i\}_i \subseteq \mathcal{L}(A_1, A_2)$. Now for "non-adjacent vertices" $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ in S_1, we require $\mathcal{E}(|\psi\rangle\langle\psi|)$ and $\mathcal{E}(|\phi\rangle\langle\phi|)$ are "non-adjacent" in S_2. The former is equivalent to $|\psi\rangle\langle\phi| \perp S_1$ and the latter is equivalent to $E_i\,|\psi\rangle\langle\phi|\,E_j^\dagger \perp S_2$ for all i, j. The definition of $S_1 \leq S_2$ is then obtained naturally.

To see that the above definitions are meaningful, Stahlke in [Sta16] also points out the following.

Lemma 14. *Let S be an nc-graph. Then*

(i) $\alpha(S) = \max\{n \in \mathbb{N} : \overline{\mathcal{K}_n} \leq S\}$

(ii) $\alpha_*(S) = \max\{n \in \mathbb{N} : \overline{\mathcal{K}_n} \leq_* S\}$.

Proof. We provide a detailed proof in Appendix A. □

3 Dual characterization of entanglement-assisted Shannon capacity and quantum Shannon capacity of graphs

In this section, we first prove that the entanglement-assisted capacity Θ_* and the quantum Shannon capacity $\Theta_q(\cdot)$ can be characterized by applying Strassen's theory of asymptotic spectra, and present elements in the corresponding asymptotic spectra. We also discuss the relations between two important conjectures in quantum zero-error information theory.

3.1 Entanglement-assisted Shannon capacity $\Theta_*(G)$ of a graph

We first prove that the entanglement-assisted cohomomorphism preorder $\leq_*$ (Definition 6) is a Strassen preorder on the semiring of graphs $\mathcal{G}$.

Lemma 15. *For any graphs G, H, K, L and any $n, m \in \mathbb{N}$, we have*

(i) $G \leq_* G$

(ii) *if* $G \leq_* H$ *and* $H \leq_* L$*, then* $G \leq_* L$

(iii) $\overline{K_m} \leq_* \overline{K_n}$ *if and only if* $m \leq n$

(iv) *if* $G \leq_* H$ *and* $K \leq_* L$ *then* $G \sqcup K \leq_* H \sqcup L$ *and* $G \boxtimes K \leq_* H \boxtimes L$

(v) *if* $H \neq \overline{K_0}$*, then there is an* $r \in \mathbb{N}$ *with* $G \leq_* \overline{K_r} \boxtimes H$*.*

Proof. (i) In general, if $G \leq H$, then $G \leq_* H$. It is clear that $G \leq G$. Therefore also $G \leq_* G$.

(ii) We adapt the proof of [MR16, Lemma 2.5] to show transitivity. Assume $G \leq_* H$ and $H \leq_* L$. Let ρ, $(\rho_g^h : g \in V(G), h \in V(H))$ and σ, $(\sigma_h^l : h \in V(H), c \in V(L))$ be corresponding positive semidefinite matrices, as in Definition 6. For $g \in V(G), l \in V(L)$, let $\tau_g^l = \sum_{h \in V(H)} \rho_g^h \otimes \sigma_h^l$. Note that τ_g^l is positive semidefinite for all $g \in V(G)$ and $l \in V(L)$. We have

$$\sum_{l \in V(L)} \tau_g^l = \sum_{\substack{h \in V(H) \\ l \in V(L)}} \rho_g^h \otimes \sigma_h^l = \sum_{h \in V(H)} \rho_g^h \otimes \sum_{l \in V(L)} \sigma_h^l = \rho \otimes \sigma. \tag{32}$$

For all $\{g, g'\} \notin E(G)$ and $\{l, l'\} \in E(L)$ or $l = l'$, we have

$$\rho_g^l \rho_{g'}^{l'} = \sum_{h, h' \in V(H)} \rho_g^h \rho_{g'}^{h'} \otimes \rho_h^l \rho_{h'}^{l'} = \sum_{\{h, h'\} \notin E(H)} \rho_g^h \rho_{g'}^{h'} \otimes \rho_h^l \rho_{h'}^{l'} = 0 \tag{33}$$

where the second equality holds since $\rho_g^h \rho_{g'}^{h'} = 0$ for all $\{g, g'\} \notin E(G)$ and $\{h, h'\} \in E(H)$ or $h = h'$, and the third equality holds since $\rho_h^l \rho_{h'}^{l'} = 0$ for all $\{h, h'\} \notin E(H)$ and $\{l, l'\} \in E(L)$ or $l = l'$. We conclude $G \leq_* L$.

(iii) We know that $m \leq n$ implies $\overline{K_m} \leq \overline{K_n}$, and thus $\overline{K_m} \leq_* \overline{K_n}$. To see that $\overline{K_m} \leq_* \overline{K_n}$ implies $m \leq n$, we note that $G \leq_* H$ implies $\vartheta(G) \leq \vartheta(H)$ [Bei10]. Thus $\overline{K_m} \leq_* \overline{K_n}$ implies $m = \vartheta(\overline{K_m}) \leq \vartheta(\overline{K_n}) = n$.

(iv) Assume that $G \leq_* H$ and $K \leq_* L$. Let $\rho, (\rho_g^h : g \in V(G), h \in V(H))$ and $\sigma, (\sigma_k^l : k \in V(K), l \in V(L))$ be corresponding positive semidefinite matrices, as in Definition 6. Let

$$\tau_u^v = \begin{cases} \rho_u^v \otimes \sigma & \text{if } u \in V(G), v \in V(H) \\ \rho \otimes \sigma_u^v & \text{if } u \in V(K), v \in V(L) \\ 0 & \text{otherwise.} \end{cases}$$

One readily verifies that τ_u^v is positive semidefinite for all $u \in V(G \sqcup K)$ and $v \in V(H \sqcup L)$. Moreover, for every $u \in V(G)$ we have $\sum_{v \in V(H \sqcup L)} \tau_u^v = \sum_{v \in V(H)} \tau_u^v + \sum_{v \in V(L)} \tau_u^v = \rho \otimes \sigma$, and for every $u \in V(K)$ we have $\sum_{v \in V(H \sqcup L)} \tau_u^v = \sum_{v \in V(H)} \tau_u^v + \sum_{v \in V(L)} \tau_u^v = \rho \otimes \sigma$. One verifies directly that $\tau_u^v \tau_{u'}^{v'} = 0$ for all $\{u, u'\} \in E(\overline{G \sqcup K})$ and $\{v, v'\} \notin E(\overline{H \sqcup L})$. We conclude $G \sqcup K \leq_* H \sqcup L$.

To prove that $G \boxtimes K \leq_* H \boxtimes L$, let $\tau_{(g,k)}^{(h,l)} = \rho_g^h \otimes \sigma_k^l$ for all g, h, k, l. One readily verifies that these operators satisfy the required conditions.

(v) For the cohomomorphism preorder it is not hard to see that for all $G, H \neq \overline{K_0}$, there is an $r \in \mathbb{N}$ with $G \leq \overline{K_r} \boxtimes H$ [Zui19, Lemma 4.2]. Therefore, $G \leq_* \overline{K_r} \boxtimes H$. □

Let $\mathbf{X}(\mathcal{G}, \leq_*)$ be the asymptotic spectrum of graphs with respect to the entanglement-assisted cohomomorphism preorder $\leq_*$, i.e.

$$\mathbf{X}(\mathcal{G}, \leq_*) = \{\phi \in \mathrm{Hom}(\mathcal{G}, \mathbb{R}_{\geq 0}) : \forall G, H \in \mathcal{G},\ G \leq_* H \ \Rightarrow\ \phi(G) \leq \phi(H)\}. \tag{34}$$

Together with Theorem 1, we obtain the following dual characterization of the entanglement-assisted Shannon capacity of graphs, $\Theta_*(G)$.

Theorem 16. *Let G be a graph. Then*

$$\Theta_*(G) = \min_{\phi \in \mathbf{X}(\mathcal{G}, \leq_*)} \phi(G).$$

Since $G \leq H$ implies $G \leq_* H$, we have $\mathbf{X}(\mathcal{G}, \leq_*) \subseteq \mathbf{X}(\mathcal{G}, \leq)$. As we mentioned already in the proof of Lemma 15, the Lovász theta function is $\leq_*$-monotone [Bei10]. This implies the following.

Theorem 17. *$\vartheta \in \mathbf{X}(\mathcal{G}, \leq_*)$.*

We have not found any other elements in $\mathbf{X}(\mathcal{G}, \leq_*)$. In fact, the following conjecture has been mentioned in [Bei10, CLMW11, LMM+12, DSW13, MSS13, CMR+14].

Conjecture 18. $\Theta_*(G) = \vartheta(G)$ for all graph G.

It would be interesting to show that this conjecture is true by proving that ϑ is the minimal element in $\mathbf{X}(\mathcal{G}, \leq_*)$, or even the only point in $\mathbf{X}(\mathcal{G}, \leq_*)$.

3.2 Quantum Shannon capacity

We begin by proving that the quantum cohomomorphism preorder $\leq_q$ (Definition 4) is a Strassen preorder on the semiring of graphs.

Lemma 19. *For any graphs G, H, K, L and any $n, m \in \mathbb{N}$, we have*

(i) $G \leq_q G$

(ii) *if $G \leq_q H$ and $H \leq_q L$, then $G \leq_q L$*

(iii) *$\overline{K_m} \leq_q \overline{K_n}$ if and only if $m \leq n$*

(iv) *if $G \leq_q H$ and $K \leq_q L$ then $G \sqcup K \leq_q H \sqcup L$ and $G \boxtimes K \leq_q H \boxtimes L$*

(v) *if $H \neq \overline{K_0}$, then there is an $r \in \mathbb{N}$ with $G \leq_q \overline{K_r} \boxtimes H$.*

Proof. (i) In general, if $G \leq H$, then $G \leq_q H$. It is clear that $G \leq G$. Therefore also $G \leq_q G$.

(ii) Quantum homomorphisms are known to be transitive in the sense that if $G \xrightarrow{q} H$ and $H \xrightarrow{q} L$, then $G \xrightarrow{q} L$ [MR16, Lemma 2.5]. Therefore, if $G \leq_q H$ and $H \leq_q L$, then $G \leq_q L$.

(iii) It is known that $K_m \xrightarrow{q} K_n$ if and only if $m \leq n$ [MR16, Lemma 2.6]. Thus $\overline{K_m} \leq_q \overline{K_n}$ if and only if $m \leq n$.

(iv) Assume $G \leq_q H$ and $K \leq_q L$. Let $(E_g^h : g \in V(G), h \in V(H))$ and $(F_k^l : k \in V(K), l \in V(L))$ be the corresponding collections of projectors, as in Definition 4. To prove $G \sqcup K \leq_q H \sqcup L$, let

$$D_u^v = \begin{cases} E_u^v \otimes I & \text{if } u \in V(G), v \in V(H) \\ I \otimes F_u^v & \text{if } u \in V(K), v \in V(L) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that D_u^v is a projector for every $u \in V(G \sqcup K)$ and $v \in V(H \sqcup L)$. Moreover, for every $u \in V(G)$ we have $\sum_{v \in V(H \sqcup L)} D_u^v = \sum_{v \in V(H)} D_u^v + \sum_{v \in V(L)} D_u^v = I \otimes I$, and for every $u \in V(K)$ we have $\sum_{v\in V(H\sqcup L)} D_u^v = \sum_{v\in V(H)} D_u^v + \sum_{v\in V(L)} D_u^v = I\otimes I$. It is also easy to verify that $D_u^v D_{u'}^{v'} = 0$ for all $\{u,u'\}\in E(\overline{G\sqcup K})$ and $\{v,v'\}\notin E(\overline{H\sqcup L})$. We conclude that $G\sqcup K \leq_q H\sqcup L$.

To prove $G\boxtimes K \leq_q H\boxtimes L$, let $D_{(g,k)}^{(h,l)} = E_g^h \otimes F_k^l$ for all g,h,k,l. One can also verify that these operators satisfy the required conditions to conclude $G\boxtimes K\leq_q H\boxtimes L$.

(v) For the cohomomorphism preorder it is not hard to see that for all $G,H\neq \overline{K_0}$, there is an $r\in\mathbb{N}$ with $G\leq \overline{K_r}\boxtimes H$ [Zui19, Lemma 4.2]. Therefore, $G\leq_q \overline{K_r}\boxtimes H$. □

Let $\mathbf{X}(\mathcal{G},\leq_q)$ be the asymptotic spectrum of graphs with respect to the quantum cohomomorphism preorder $\leq_q$, i.e.

(35) $\mathbf{X}(\mathcal{G},\leq_q) = \{\phi\in \mathrm{Hom}(\mathcal{G},\mathbb{R}_{\geq 0}) : \forall G,H\in\mathcal{G},\ G\leq_q H \ \Rightarrow\ \phi(G)\leq\phi(H)\}.$

Together with Theorem 1, we obtain the following dual characterization of the quantum Shannon capacity of graphs.

Theorem 20. *Let G be a graph. Then*

$$\Theta_q(G) = \min_{\phi\in\mathbf{X}(\mathcal{G},\leq_q)} \phi(G).$$

We know that if $G\leq H$, then $G\leq_q H$. It is also easy to see that $G\leq_q H$ implies $G\leq_* H$. Therefore, $\mathbf{X}(\mathcal{G},\leq_*)\subseteq\mathbf{X}(\mathcal{G},\leq_q)\subseteq\mathbf{X}(\mathcal{G},\leq)$.

Theorem 21. *We have*

$$\{\vartheta,\mathcal{H}_f^{\mathbb{C}},\mathcal{H}_f^{\mathbb{R}},\overline{\xi}_f\}\subseteq\mathbf{X}(\mathcal{G},\leq_q).$$

Moreover, we have a proper inclusion

$$\mathbf{X}(\mathcal{G},\leq_q)\subsetneq\mathbf{X}(\mathcal{G},\leq),$$

since for any odd prime p such that there exists a Hadamard matrix of size $4p$ holds

$$\mathcal{H}_f^{\mathbb{F}_p}\notin\mathbf{X}(\mathcal{G},\leq_q).$$

Proof. We know that $\{\vartheta,\mathcal{H}_f^{\mathbb{C}},\mathcal{H}_f^{\mathbb{R}},\overline{\xi}_f\}\subseteq\mathbf{X}(\mathcal{G},\leq)$, so to prove that $\{\vartheta,\mathcal{H}_f^{\mathbb{C}},\mathcal{H}_f^{\mathbb{R}},\overline{\xi}_f\}\subseteq\mathbf{X}(\mathcal{G},\leq_q)$, it remains to show that the functions $\vartheta,\mathcal{H}_f^{\mathbb{C}},\mathcal{H}_f^{\mathbb{R}},\overline{\xi}_f$ are $\leq_q$-monotone.

Mančinska and Roberson proved in [MR16] that the Lovász theta function ϑ and the complement of projective rank $\overline{\xi}_f$ are $\leq_q$-monotone.

We prove that $\mathcal{H}_f^{\mathbb{C}}$ is $\leq_q$-monotone. Suppose $G\leq_q H$. Let E_g^h be corresponding complex $d'\times d'$ projector for all $g\in V(G)$ and $h\in V(H)$. Let

$$M = \sum_{h,h'\in V(H)} e_h e_{h'}^\dagger \otimes M_{h,h'} \in M(|V(H)|,\mathbb{C})\otimes M(d,\mathbb{C})$$

be a d-representation of H over $\mathbb{C}$. We construct a dd'-representation of G over $\mathbb{C}$ as follows. Let

$$M' = \sum_{g,g'\in V(G)} e_g e_{g'}^\dagger \otimes M'_{g,g'} \in M(|V(G)|,\mathbb{C})\otimes M(dd',\mathbb{C}),$$

with

$$M'_{g,g'} = \sum_{h,h'\in V(H)} M_{h,h'}\otimes E_g^h E_{g'}^{h'} \in M(dd',\mathbb{C}).$$

To see that M' is a dd'-representation of G, we first show $M'_{g,g} = I_{dd'}$ for all $g \in V(G)$. Note that

$$M'_{g,g} = \sum_{h,h' \in V(H)} M_{h,h'} \otimes E_g^h E_g^{h'} = \sum_{h \in V(H)} M_{h,h} \otimes E_g^h E_g^h = I_d \otimes I_{d'},$$

where the second equality uses $E_g^h E_g^{h'} = 0$ for all $g \in V(G)$ and $h \neq h'$ (Remark 5), and the last equality uses the facts that $M_{h,h} = I_d$ for all $h \in V(H)$ and $\sum_{h \in V(H)} E_g^h E_g^h = \sum_{h \in V(H)} E_g^h = I_{d'}$. On the other hand, we show $M'_{g,g'} = 0$ if $g \neq g'$ and $\{g, g'\} \notin E(G)$. In this case, we have

$$M'_{g,g'} = \sum_{h,h' \in V(H)} M_{h,h'} \otimes E_g^h E_{g'}^{h'} = \sum_{\{h,h'\} \notin E(H) \text{ and } h \neq h'} M_{h,h'} \otimes E_g^h E_{g'}^{h'} = 0,$$

where the second equality use the fact that $E_g^h E_{g'}^{h'} = 0$ for all $\{g, g'\} \in E(\overline{G})$ and $\{h, h'\} \notin E(\overline{H})$, and the last equality holds since $M_{h,h'} = 0$ for all $h \neq h', \{h, h'\} \notin E(H)$. Thus M' is a dd'-representation of G over $\mathbb{C}$.

Next we prove that $\operatorname{rank}(M') \leq d' \operatorname{rank}(M)$. We factor M' as

$$\begin{aligned} M' &= \sum_{g,g'} e_g e_{g'}^\dagger \otimes \Big(\sum_{h,h'} M_{h,h'} \otimes E_g^h E_{g'}^{h'}\Big) \\ &= \Big(\sum_{g,h} e_g e_h^\dagger \otimes I_d \otimes E_g^h\Big)\Big(\sum_{h,h'} e_h e_{h'}^\dagger \otimes M_{h,h'} \otimes I_{d'}\Big)\Big(\sum_{g',h'} e_{h'} e_{g'}^\dagger \otimes I_d \otimes E_{g'}^{h'}\Big) \\ &= \Big(\sum_{g,h} e_g e_h^\dagger \otimes I_d \otimes E_g^h\Big)(M \otimes I_{d'})\Big(\sum_{g',h'} e_{h'} e_{g'}^\dagger \otimes I_d \otimes E_{g'}^{h'}\Big). \end{aligned}$$

Thus $\operatorname{rank}(M') \leq \operatorname{rank}(M \otimes I_{d'}) = d' \operatorname{rank}(M)$. Therefore,

$$\mathcal{H}_f^{\mathbb{C}}(G) \leq \frac{\operatorname{rank}(M')}{dd'} \leq \frac{d' \operatorname{rank}(M)}{dd'} \leq \frac{\operatorname{rank}(M)}{d}. \tag{36}$$

Since (36) holds for all d-representation M of H over $\mathbb{C}$, we conclude $\mathcal{H}_f^{\mathbb{C}}(G) \leq \mathcal{H}_f^{\mathbb{C}}(H)$.

To prove that $\mathcal{H}_f^{\mathbb{R}}$ is $\leq_q$-monotone, one follows the above proof with real instead of complex d-representations and one uses the fact that the projectors E_g^h can be chosen to be real matrices (Remark 5).

Finally, we point out that $\mathbf{X}(\mathcal{G}, \leq_q)$ is a proper subset of $\mathbf{X}(\mathcal{G}, \leq)$. It is known that $\Theta(G)$ can be strictly smaller than $\Theta_*(G)$ for some graph G [LMM+12, BBG13]. More precisely, Briët, Buhrman and Gijswijt proved in [BBG13] that for any odd prime p such that there exists a Hadamard matrix of size $4p$, there exists a graph G satisfying $\Theta(G) \leq \mathcal{H}^{\mathbb{F}_p}(G) < \Theta_*(G)$. We note that the proof in [BBG13] of $\mathcal{H}^{\mathbb{F}_p}(G) < \Theta_*(G)$ in fact shows that $\mathcal{H}^{\mathbb{F}_p}(G) < \Theta_q(G)$. The key observation is the following:

Observation 22. *If $\overline{G}$ has an orthonormal representation $U = (u_g \in \mathbb{R}^d : g \in V(G))$ in dimension d, and $\overline{G}$ has M disjoint d-cliques, then $\Theta_q(G) \geq M$.*

Proof. Let the cliques be denoted by $C_1, \ldots, C_M$. Take

$$E_i^g = \begin{cases} u_g u_g^T & \text{if } g \in C_i \\ 0 & \text{if } g \notin C_i. \end{cases}$$

It is easy to see that $(E_i^g : g \in V(G),\ i \in V(\overline{K_M}))$ satisfies the conditions for the inequality $\overline{K_M} \leq_q G$. Thus $\Theta_q(G) \geq \alpha_q(G) \geq M$. □

It is known that if n is odd and there exists a Hadamard matrix of size $n+1$, then there exists a graph G_n whose complement has an n-dimensional orthonormal representation and $|V(G_n)|/n^2$ disjoint cliques of size n [BBG13]. Thus $\Theta_q(G_n) \geq |V(G_n)|/n^2$. On the other hand, it has been proved that the Haemers bound over some finite field $\mathbb{F}_p$ on G_n, $\mathcal{H}^{\mathbb{F}}(G)$, can be strictly smaller than $|V(G_n)|/n^2$ [BBG13]. Since $\mathcal{H}_f^{\mathbb{F}}(G) \leq \mathcal{H}^{\mathbb{F}}(G)$ for any field $\mathbb{F}$, we conclude that $\mathcal{H}_f^{\mathbb{F}_p} \notin \mathbf{X}(\mathcal{G}, \leq_q)$ for such odd prime p. □

Remark 23. It is not hard to adjust the above proof to show that the fractional Haemers bound for any field extension of $\mathbb{R}$ belongs to $\mathbf{X}(\mathcal{G}, \leq_q)$. We show that these parameters are actually the same. Moreover, one may naturally define a real projective rank $\xi_f^{\mathbb{R}}$ by requiring that the projectors in the definition of ξ_f are real. Again, we show that $\xi_f^{\mathbb{R}}$ is equal to projective rank.

Proposition 24. *$\mathcal{H}_f^{\mathbb{R}}(G) = \mathcal{H}_f^{\mathbb{C}}(G)$ and $\xi_f^{\mathbb{R}}(G) = \xi_f(G)$ for all graph G.*

Proof. The following lemma is readily verified.

Lemma 25. *Let $E \in M(n, \mathbb{C})$. Define the real matrix*

$$R(E) = \begin{pmatrix} \operatorname{Re}(E) & \operatorname{Im}(E) \\ -\operatorname{Im}(E) & \operatorname{Re}(E) \end{pmatrix} \in M(2n, \mathbb{R}).$$

Then $\operatorname{rank}(R(E)) = 2\operatorname{rank}(E)$.

For the fractional Haemers bound, clearly $\mathcal{H}_f^{\mathbb{C}}(G) \leq \mathcal{H}_f^{\mathbb{R}}(G)$ since every real matrix is a complex matrix, and its rank over $\mathbb{R}$ equals the rank over $\mathbb{C}$. We prove $\mathcal{H}_f^{\mathbb{R}}(G) \leq \mathcal{H}_f^{\mathbb{C}}(G)$ by proving that, for every d-representation M of G, there exists a $2d$-representation M' of G, such that $\operatorname{rank}(M') \leq 2\operatorname{rank}(M)$. Assume G has n vertices. Write M in the block matrix form

$$M = \begin{bmatrix} M_{1,1} & \cdots & M_{1,n} \\ \vdots & & \vdots \\ M_{n,1} & \cdots & M_{n,n} \end{bmatrix} \in M(nd, \mathbb{C}), \tag{37}$$

where $M_{i,i} = I_d$ for $i \in [n]$, and $M_{i,j} = M_{j,i} = 0$ if $\{i,j\} \in E$. Let M' be the $2nd \times 2nd$ real matrix of the form $M' = \begin{bmatrix} \operatorname{Re}(M) & \operatorname{Im}(M) \\ -\operatorname{Im}(M) & \operatorname{Re}(M) \end{bmatrix}$. On the other hand, let $M'_{i,j} = \begin{bmatrix} \operatorname{Re}(M_{i,j}) & \operatorname{Im}(M_{i,j}) \\ -\operatorname{Im}(M_{i,j}) & \operatorname{Re}(M_{i,j}) \end{bmatrix}$ and denote

$$M'' = \begin{bmatrix} M'_{1,1} & \cdots & M'_{1,n} \\ \vdots & & \vdots \\ M'_{n,1} & \cdots & M'_{n,n} \end{bmatrix} \in M(2nd, \mathbb{R}). \tag{38}$$

It is clear that $2\operatorname{rank}(M) \geq \operatorname{rank}(M')$ and $M'' \in \mathcal{M}_{\mathbb{R}}^{2d}(G)$. We show that M'' can be transformed to M' by some row and column permutations of the blocks, which will not influence the rank. We first sort the columns, resulting that the first block row of the first n block columns is $\operatorname{Re}(M_{1,1}), \ldots, \operatorname{Re}(M_{1,n})$ and the last n block column is $\operatorname{Im}(M_{1,1}), \ldots, \operatorname{Im}(M_{1,n})$. Then we sort the rows, resulting that the first block column of the first n block rows is $\operatorname{Re}(M_{1,1}), \ldots, \operatorname{Re}(M_{n,1})$ and the last n block column is $\operatorname{Im}(M_{1,1}), \ldots, \operatorname{Im}(M_{n,1})$. Denote the matrix of these two permutations by S and T, it is easy to check that $SM''T = M'$ (In fact, $T = S^T$). Thus $\operatorname{rank}(M'') = \operatorname{rank}(M') \leq 2\operatorname{rank}(M)$, and $\mathcal{H}_f^{\mathbb{R}}(G) \leq \mathcal{H}_f^{\mathbb{C}}(G)$ follows.

For the projective rank, let $(E_g \in M(d,\mathbb{C}) : g \in V(G))$ be a d/r-representation of G. From Lemma 25 follows that $(R(E_g) : g \in V(G))$ is a $2d/2r$-representation. We conclude that $\xi_f^{\mathbb{R}}(G) \leq \xi_f(G)$. On the other hand, every real d/r-representation is also a complex d/r-representation. Therefore, $\xi_f(G) \leq \xi_f^{\mathbb{R}}(G)$. □

Recall that $\leq_q$ can be obtained from $\leq_*$ by restricting to the use of maximally entangled state and projective measurements in the zero-error information transmission setting [MR16]. An open problem in quantum zero-error information theory is to show maximally entangled state is also necessary to achieve the maximal entanglement-assisted Shannon capacity [MR16]. Namely,

Conjecture 26. $\Theta_q(G) = \Theta_*(G)$ for all graph G.

The original proof of Haemers [Hae79] shows that taking G to be the complement of the Schläfli graph, $\mathcal{H}^{\mathbb{R}}(G) \leq 7 < 9 = \vartheta(G)$. By Theorem 21, we know that Θ_q and ϑ are not the same parameters, which immediately implies the following:

Corollary 27. *Conjecture 18 and 26 cannot both be true. In other words, there exists a graph G, such that either $\Theta_*(G) < \vartheta(G)$ or $\Theta_q(G) \leq \Theta_*(G)$.*

4 Dual characterization of entanglement-assisted Shannon capacity of non-commutative graphs

In this section, we focus on the fully quantum setting: the entanglement-assisted Shannon capacity of nc-graphs. We discuss the (unassisted) Shannon capacity of nc-graphs in appendix B.

Recall that the map

$$\mathcal{G} \to \mathcal{S} : G \mapsto S_G \coloneqq \operatorname{span}\{|g\rangle\langle g'| : g = g' \in V(G) \text{ or } \{g, g'\} \in E(G)\}$$

is an injective semiring homomorphism. We prove that this homomorphism behaves well with respect to the entanglement-assisted cohomomorphism preorders on $\mathcal{G}$ and $\mathcal{S}$.

Lemma 28. *For any graphs $G, H \in \mathcal{G}$ holds $G \leq_* H$ if and only if $S_G \leq_* S_H$.*

Proof. Let $|V(G)| = n$ and $|V(H)| = m$.

($\Leftarrow$) Assume there exist a positive definite $\sigma \in \mathcal{D}(\mathbb{C}^d)$ and $E = \{E_i\}_i \subseteq \mathcal{L}(\mathbb{C}^n \otimes \mathbb{C}^d, \mathbb{C}^m)$ satisfying $\sum_i E_i^\dagger E_i = I_{nd}$ and $E(S_G^\perp \otimes \sigma)E^\dagger \perp S_H$. Let $\sigma = \sum_{x=1}^d \lambda_x |\psi_x\rangle\langle\psi_x|$ be the spectral decomposition of σ and let $|\Omega\rangle = \sum_{x=1}^d \sqrt{\lambda_x}\, |\psi_x\rangle\, |x\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ be a purification of σ. Let

$$(39) \quad \rho_g^h = \sum_i (\langle h| E_i \otimes I_d)(|g\rangle\langle g| \otimes |\Omega\rangle\langle\Omega|)(E_i^\dagger |h\rangle \otimes I_d) \in \mathcal{L}(\mathbb{C}^d)$$

for each $g \in V(G)$ and $h \in V(H)$. Let $\rho \coloneqq \sum_{x=1}^d \lambda_x |x\rangle\langle x|$. One verifies directly that every ρ_g^h is positive semidefinite. We first prove that $\sum_{h\in V(H)} \rho_g^h = \rho$ for all $g \in V(G)$. Note that

$$\begin{aligned}
\sum_{h\in V(H)} \rho_g^h &= \sum_{h\in V(H)} \sum_i \big(\langle h| E_i \otimes I_d\big)\big(|g\rangle\langle g| \otimes |\Omega\rangle\langle\Omega|\big)\big(E_i^\dagger |h\rangle \otimes I_d\big) \\
&= \sum_{h\in V(H)} \sum_{i,x,y} \sqrt{\lambda_x\lambda_y}\big(\langle h| E_i(|g\rangle\langle g| \otimes |\psi_x\rangle\langle\psi_y|)E_i^\dagger |h\rangle\big) \otimes |x\rangle\langle y| \\
&= \sum_{i,x,y} \sqrt{\lambda_x\lambda_y}\,\mathrm{Tr}\big(E_i(|g\rangle\langle g| \otimes |\psi_x\rangle\langle\psi_y|)E_i^\dagger\big) \otimes |x\rangle\langle y| \\
&= \sum_{i,x,y} \sqrt{\lambda_x\lambda_y}\big(\langle g| \otimes \langle\psi_y|\big)E_i^\dagger E_i\big(|g\rangle \otimes |\psi_x\rangle\big) \otimes |x\rangle\langle y| \\
&= \sum_x \lambda_x |x\rangle\langle x|,
\end{aligned} \tag{40}$$

where the last equality holds since $\sum_i E_i^\dagger E_i = I$ and $\langle\psi_y|\psi_x\rangle = 0$ for all $x \neq y \in [d]$. Now we are left to prove that $\rho_g^h\rho_{g'}^{h'} = 0$ for all $g \neq g', \{g,g'\} \notin E(G)$ and $\{h,h'\} \in E(H)$ or $h = h'$. Note that $\rho_g^h\rho_{g'}^{h'}$ equals

$$\begin{aligned}
\rho_g^h\rho_{g'}^{h'} &= \sum_{i,j}(\langle h| E_i \otimes I_d)(|g\rangle\langle g| \otimes |\Omega\rangle\langle\Omega|)(E_i^\dagger |h\rangle\langle h'| E_j \otimes I_d)(|g'\rangle\langle g'| \otimes |\Omega\rangle\langle\Omega|)(E_j^\dagger |h'\rangle \otimes I_d) \\
&= \sum_{\substack{i,j\\x,y,z,w}} \sqrt{\lambda_x\lambda_y\lambda_z\lambda_w}\,\langle h| E_i(|g\rangle\langle g| \otimes |\psi_x\rangle\langle\psi_y|)E_i^\dagger |h\rangle\langle h'| E_j(|g'\rangle\langle g'| \otimes |\psi_z\rangle\langle\psi_w|)E_j^\dagger |h'\rangle \otimes \langle y|z\rangle\, |x\rangle\langle w| \\
&= \sum_{\substack{i,j\\x,y,w}} \sqrt{\lambda_x\lambda_w}\lambda_y\,\langle h| E_i(|g\rangle\langle g| \otimes |\psi_x\rangle\langle\psi_y|)E_i^\dagger |h\rangle\langle h'| E_j(|g'\rangle\langle g'| \otimes |\psi_y\rangle\langle\psi_w|)E_j^\dagger |h'\rangle\, |x\rangle\langle w| \\
&= \sum_{i,j,x,w} \sqrt{\lambda_x\lambda_w}(\langle h| E_i(|g\rangle \otimes |\psi_x\rangle))(\mathrm{Tr}(E_j(|g'\rangle\langle g| \otimes \sigma)E_i^\dagger |h\rangle\langle h'|))((\langle g'| \otimes \langle\psi_w|)E_j^\dagger |h'\rangle)\, |x\rangle\langle w|,
\end{aligned}$$

where the last equality holds since

$$\sum_y \lambda_y(\langle g| \otimes \langle\psi_y|)E_i^\dagger |h\rangle\langle h'| E_j(|g'\rangle \otimes |\psi_y\rangle) = \mathrm{Tr}(E_j(|g'\rangle\langle g| \otimes \sigma)E_i^\dagger |h\rangle\langle h'|). \tag{41}$$

Recall that $E(S_G^\perp \otimes \sigma)E \perp S_H$, where $S_G^\perp = \{|g\rangle\langle g'| : g \neq g', \{g,g'\} \notin E(G)\}$ and $S_H = \{|h\rangle\langle h'| : \{h,h'\} \in E(H) \text{ or } h = h' \in V(H)\}$. Equation (41) equals 0 when $\{g,g'\} \notin E(G)$ and $h = h' \in V(H)$ or $\{h,h'\} \in E(H)$. We conclude that $\rho_g^h\rho_{g'}^{h'} = 0$ for $g \neq g', \{g,g'\} \notin E(G)$ and $h = h' \in V(H)$ or $\{h,h'\} \in E(H)$.

($\Rightarrow$) Assume $G \leq_* H$. There exist $d \in \mathbb{N}$, a positive definite matrix $\rho \in M(d,\mathbb{C})$ and positive semidefinite matrices $(\rho_g^h \in M(d,\mathbb{C}) : g \in V(G),\, h \in V(H))$ such that $\sum_{h\in V(H)} \rho_g^h = \rho$ for all $g \in V(G)$ and $\rho_g^h\rho_{g'}^{h'} = 0$ if $g \neq g', \{g,g'\} \notin E(G)$ and $\{h,h'\} \in E(H)$ or $h = h'$. We shall prove that there exist a positive definite $\sigma \in \mathcal{D}(\mathbb{C}^d)$ and $E = \mathrm{span}\{E_i\}_i \subseteq \mathcal{L}(\mathbb{C}^n \otimes \mathbb{C}^d, \mathbb{C}^m)$ satisfying $\sum_i E_i^\dagger E_i = I_{nd}$, such that $E(S_G^\perp \otimes \sigma)E^\dagger \perp S_H$. We need the following lemma, which we prove for the convenience of the reader.

Lemma 29 ([HJW93, SR02]). *Let $\rho_1, \ldots, \rho_l \in M(d,\mathbb{C})$ be a collection of positive semidefinite matrices which sum up to a positive definite matrix $\rho \in M(d,\mathbb{C})$. Then there exist $|\Omega\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ and a POVM $\{A_1, \ldots, A_l\} \subseteq M(d,\mathbb{C})$, i.e. a collection of positive semidefinite matrices that add up to the identity, such that $\rho_k = \mathrm{Tr}_1((A_k \otimes I)\,|\Omega\rangle\langle\Omega|)$. Namely, let $\rho = \sum_{i=1}^d \lambda_i |\psi_i\rangle\langle\psi_i|$ be the spectral decomposition of ρ, so $\lambda_i > 0$ for $i \in [d]$ and $\{|\psi_1\rangle, \ldots, |\psi_d\rangle\}$ forms an orthonormal basis of $\mathbb{C}^d$. Let $|\Omega\rangle = \sum_{i=1}^d \sqrt{\lambda_i}\,|\psi_i\rangle \otimes |\psi_i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ and let $A_k = \rho^{-1/2}\rho_k^T\rho^{-1/2}$.*

Proof. We have

$$A_k = \rho^{-1/2}\rho_k^T\rho^{-1/2} = \sum_{i,j=1}^{d} \frac{1}{\sqrt{\lambda_i\lambda_j}} |\psi_i\rangle\langle\psi_i| \rho_k^T |\psi_j\rangle\langle\psi_j| = \sum_{i,j=1}^{d} \frac{1}{\sqrt{\lambda_i\lambda_j}} \langle\psi_i| \rho_k^T |\psi_j\rangle |\psi_i\rangle\langle\psi_j|$$

for $k = 1, \dots, l$. Moreover,

$$\begin{aligned}\mathrm{Tr}_1((A_k \otimes I)|\Omega\rangle\langle\Omega|) &= \sum_{i,j,x,y=1}^{d} \frac{\sqrt{\lambda_x\lambda_y}}{\sqrt{\lambda_i\lambda_j}} \langle\psi_i| \rho_k^T |\psi_j\rangle \mathrm{Tr}_1((|\psi_i\rangle\langle\psi_j| \otimes I)(|\psi_x\rangle\langle\psi_y| \otimes |\psi_x\rangle\langle\psi_y|)) \\ &= \sum_{i,j} \langle\psi_i| \rho_k^T |\psi_j\rangle |\psi_j\rangle\langle\psi_i| = \rho_k.\end{aligned}$$

This proves Lemma 29. □

Following Lemma 29, we define the pure state $|\Omega\rangle = \sum_{i=1}^{d}\sqrt{\lambda_i}|\psi_i\rangle \otimes |\psi_i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, where $\rho = \sum_{i=1}^{d}\lambda_i |\psi_i\rangle\langle\psi_i|$ is the spectral decomposition of ρ. Then, for every $g \in V(G)$, there exists a POVM $(A_g^h = \rho^{-1/2}(\rho_g^h)^T\rho^{-1/2} : h \in V(H))$, indexed by $h \in V(H)$, such that $\rho_g^h = \mathrm{Tr}_1((A_g^h \otimes I)|\Omega\rangle\langle\Omega|)$. For $g \neq g', \{g, g'\} \notin E(G)$ and $\{h, h'\} \in E(H)$ or $h = h'$, $\rho_g^h\rho_{g'}^{h'} = 0$ implies $(\rho_g^h)^T(\rho_{g'}^{h'})^T = 0$, thus

$$A_g^h\rho A_{g'}^{h'} = (\rho^{-1/2}(\rho_g^h)^T\rho^{-1/2})\rho(\rho^{-1/2}(\rho_{g'}^{h'})^T\rho^{-1/2}) = \rho^{-1/2}(\rho_g^h)^T(\rho_{g'}^{h'})^T\rho^{-1/2} = 0. \tag{42}$$

Since A_g^h is positive semidefinite for all $g \in V(G)$ and $h \in V(H)$, there is a spectral decomposition $A_g^h = \sum_x \mu_x |\phi_x^{g,h}\rangle\langle\phi_x^{g,h}|$, with $\mu_x > 0$ for all possible x. By Equation 42, we know that $\langle\phi_x^{g,h}| \rho |\phi_y^{g',h'}\rangle = 0$ for all possible x, y and $g \neq g', \{g, g'\} \notin E(G)$ and $\{h, h'\} \in E(H)$ or $h = h'$. Let $M_g^h = \sum_x \sqrt{\mu_x}|\psi_x\rangle\langle\phi_x^{g,h}|$. We have $A_g^h = (M_g^h)^\dagger(M_g^h)$ and

$$M_g^h\rho(M_{g'}^{h'})^\dagger = 0, \ \forall g \neq g', \{g, g'\} \notin E(G) \text{ and } \{h, h'\} \in E(H) \text{ or } h = h'. \tag{43}$$

Let $E = \mathrm{span}\{E_{i,g,h} = |h\rangle(\langle g| \otimes \langle\psi_i| M_g^h) : g \in V(G),\ h \in V(H),\ i \in [d]\} \subseteq \mathcal{L}(\mathbb{C}^n \otimes \mathbb{C}^d, \mathbb{C}^m)$. Note that

$$\sum_{i,g,h} E_{i,g,h}^\dagger E_{i,g,h} = \sum_{i,g,h} |g\rangle\langle g| \otimes (M_g^h)^\dagger |\psi_i\rangle\langle\psi_i| M_g^h = \sum_{g,h} |g\rangle\langle g| \otimes A_g^h = I_n \otimes I_d,$$

where the second equality holds since $\{|\psi_1\rangle, \dots, |\psi_d\rangle\}$ forms an orthonormal basis of $\mathbb{C}^d$ and since $A_g^h = (M_g^h)^\dagger(M_g^h)$, the last equality holds since $\sum_{h\in V(H)} A_g^h = I_d$ for all $g \in V(G)$. We also claim that $E(S_G^\perp \otimes \rho)E^\dagger \perp S_H$. We have

$$E(S_G^\perp \otimes \rho)E^\dagger = \mathrm{span}\{\langle\psi_i| (M_g^h\rho M_{g'}^{h'})|\psi_j\rangle |h\rangle\langle h'| : \{g, g'\} \notin E(G),\ h, h' \in V(H),\ i, j \in [d]\}.$$

By Equation 43, we know that $E(S_G^\perp \otimes \rho)E^\dagger$ is at most spanned by those operators $|h\rangle\langle h'|$ with $h \neq h', \{h, h'\} \notin E(H)$. This immediately implies $E(S_G^\perp \otimes \rho)E^\dagger \perp S_H$, since $S_H^\perp = \{|h\rangle\langle h'| : \{h, h'\} \notin E(H)\}$. This proves Lemma 28. □

Now we prove that the entanglement-assisted cohomomorphism preorder $\leq_*$ (Definition 11) on nc-graphs is a Strassen preorder.

Lemma 30. *For any nc-graphs $S \subseteq \mathcal{L}(A)$, $S' \subseteq \mathcal{L}(A')$, $T \subseteq \mathcal{L}(B)$ and $T \subseteq \mathcal{L}(B')$ holds*

(i) $S \leq_* S$

(ii) *if* $S \leq_* T$ *and* $T \leq_* T'$, *then* $S \leq_* T'$

(iii) $\overline{\mathcal{K}_m} \leq_* \overline{\mathcal{K}_n}$ *if and only if* $m \leq n$

(iv) *if* $S \leq_* T$ *and* $S' \leq_* T'$, *then* $S \oplus S' \leq_* T \oplus T'$ *and* $S \otimes S' \leq_* T \otimes T'$

(v) *if* $T \neq \overline{\mathcal{K}_0}$, *then there is an* $r \in \mathbb{N}$ *with* $S \leq_* \overline{\mathcal{K}_r} \otimes T$.

Proof. (i) We know $S \leq T$ implies $S \leq_* T$ (Lemma 13). Clearly $S \leq S$ holds by taking $E = \operatorname{span}\{I\}$. Therefore, also $S \leq_* S$.

(ii) Let a positive definite $\rho \in \mathcal{D}(A_0)$ and $E = \operatorname{span}\{E_i\}_i \subseteq \mathcal{L}(A \otimes A_0, B)$ be given by $S \leq_* T$, and a positive definite $\sigma \in \mathcal{D}(B_0)$ and $F = \operatorname{span}\{F_j\}_j \subseteq \mathcal{L}(B \otimes B_0, B')$ be given by $T \leq_* T'$. To see $S \leq_* T'$, take $\tau = \rho \otimes \sigma \in \mathcal{D}(A_0 \otimes B_0)$ and $F' = \operatorname{span}\{F_j(E_i \otimes I_{B_0})\}_{i,j} \subseteq \mathcal{L}(A \otimes A_0 \otimes B_0, B')$. We have

$$F'(S^\perp \otimes \tau)F'^\dagger = F(E(S^\perp \otimes \rho)E^\dagger \otimes \sigma)F^\dagger \subseteq F(T^\perp \otimes \sigma)F^\dagger \perp T', \tag{44}$$

where the inequality holds since $E(S^\perp \otimes \rho)E^\dagger \perp T$ by $S \leq_* T$, and the orthogonality relation is given by $T \leq_* T'$.

(iii) By Lemma 28, $\overline{\mathcal{K}_n} \leq_* \overline{\mathcal{K}_m}$ is equivalent to $\overline{K_n} \leq_* \overline{K_m}$, which is equivalent to $m \leq n$ by Lemma 15.

(iv) Let a positive definite matrix $\rho \in \mathcal{D}(A_0)$ and $E = \operatorname{span}\{E_i\}_i \subseteq \mathcal{L}(A \otimes A_0, B)$ be given by $S \leq_* T$, and a positive definite matrix $\sigma \in \mathcal{D}(A'_0)$ and $F = \operatorname{span}\{F_j\}_j \subseteq \mathcal{L}(A' \otimes A'_0, B')$ be given by $S' \leq_* T'$. Let $E' = \operatorname{span}\{E_i \oplus 0, 0 \oplus F_j\}_{i,j} \subseteq \mathcal{L}((A \oplus A') \otimes (A_0 \oplus A'_0), B \oplus B')$, where $(E_i \oplus 0)(|\psi\rangle_A + |\psi'\rangle_{A'}) \otimes (|\phi\rangle_{A_0} + |\phi'\rangle_{A'_0}) = E_i(|\psi\rangle_A \otimes |\phi\rangle_{A_0})$ and $(0 \oplus F_j)(|\psi\rangle_A + |\psi'\rangle_{A'}) \otimes (|\phi\rangle_{A_0} + |\phi'\rangle_{A'_0}) = F_j(|\psi'\rangle_{A'} \otimes |\phi'\rangle_{A'_0})$ for all i,j and $|\psi\rangle_A \in A$, $|\psi'\rangle_{A'} \in A'$, $|\phi\rangle_{A_0} \in A_0$ and $|\phi'\rangle_{A'_0} \in A'_0$. One readily verifies that $E'((S \oplus S')^\perp \otimes (\rho \oplus \sigma))E'^\dagger \perp T \oplus T'$. To see $S \otimes S' \leq_* T \otimes T'$, Let $E' = \operatorname{span}\{E_i \otimes I_{A' \otimes A'_0}, I_{A \otimes A_0} \otimes F_j\}_{i,j} \subseteq \mathcal{L}((A \otimes A') \otimes (A_0 \otimes A'_0), B \otimes B')$. One readily verifies that $E'((S \otimes S')^\perp \otimes (\rho \otimes \sigma))E'^\dagger \perp T \otimes T'$.

(v) We show that for any $S, T \neq \overline{\mathcal{K}_0}$, there exists an $r \in \mathbb{N}$ such that $S \leq_* \overline{\mathcal{K}_r} \otimes T$. We first claim that $S \leq \mathcal{I}_n := \operatorname{span}\{I_n\} \subseteq \mathcal{L}(\mathbb{C}^n)$ for $n = \dim(A)$. This can be done by taking $E = \operatorname{span}\{E_i\}_i$, where $\sum_i E_i^\dagger E_i = I_n$ and $S = \operatorname{span}\{E_i^\dagger E_j\}_{i,j}$. Such E always exists by the results in [Dua09]. We then show that, for any $n \in \mathbb{N}$, $\mathcal{I}_n \leq_* \overline{\mathcal{K}_{n^2}}$. Let $E_{i,j} = |\Phi_{i,j}\rangle\langle i,j|$ for all $i,j \in \{0,\ldots,n-1\}$, where $\{|i\rangle \otimes |j\rangle : i,j \in \{0,\ldots,n-1\}\}$ is the computational basis of $\mathbb{C}^n \otimes \mathbb{C}^n$ and

$$|\Phi_{i,j}\rangle := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (X(i)Z(j)\,|k\rangle) \otimes |k\rangle\,, \tag{45}$$

where $X(i)\,|k\rangle = |i+k \mod n\rangle$ and $Z(j)\,|k\rangle = \exp(i2\pi jk/n)\,|k\rangle$, is the (i,j)-th element of the Bell basis of $\mathbb{C}^n \otimes \mathbb{C}^n$ (cf. [Wil17, page 114]). Take $\rho = I_n$ and $E = \{E_{i,j} : i,j \in \{0,\ldots,n-1\}\} \subseteq \mathcal{L}(\mathbb{C}^{n^2})$. Note that $\mathcal{I}_n^\perp = \{X \in \mathcal{L}(\mathbb{C}^n) : \operatorname{Tr}(X) = 0\}$, and $X \otimes I_n \perp |\Phi_{i,j}\rangle\langle\Phi_{i,j}|$ for all $i,j \in \{0,\ldots,n-1\}$, since

$$\operatorname{Tr}((X^\dagger \otimes I_n)\,|\Phi_{i,j}\rangle\langle\Phi_{i,j}|) = \operatorname{Tr}(X^\dagger \operatorname{Tr}_2(|\Phi_{i,j}\rangle\langle\Phi_{i,j}|)) = \operatorname{Tr}(X^\dagger) = 0.$$

This implies that $(\mathcal{I}_n^\perp \otimes I_n) \perp E^\dagger \overline{\mathcal{K}_{n^2}} E$, which is equivalent to $E(\mathcal{I}_n^\perp \otimes I_n)E^\dagger \perp \overline{\mathcal{K}_{n^2}}$. Thus we conclude that $S \leq_* \overline{\mathcal{K}_{N^2}}$ by transitivity. We derive that $S \leq_* \overline{\mathcal{K}_{N^2}} \otimes \overline{\mathcal{K}_1} \leq_* \overline{\mathcal{K}_{N^2}} \otimes T$ if $T \neq \overline{\mathcal{K}_0}$, which concludes the proof. □

Let $\mathbf{X}(\mathcal{S}, \leq_*)$ be the asymptotic spectrum of nc-graphs with respect to the entanglement-assisted cohomomorphism preorder, i.e.

(46) $$\mathbf{X}(\mathcal{S}, \leq_*) = \{\phi \in \mathrm{Hom}(\mathcal{S}, \mathbb{R}_{\geq 0}) : \forall S, T \in \mathcal{S},\ S \leq_* T \ \Rightarrow\ \phi(S) \leq \phi(T)\}.$$

Together with Theorem 1, we obtain the following dual characterization of the entanglement-assisted Shannon capacity of nc-graphs.

Theorem 31. *Let S be an nc-graph. Then*

$$\Theta_*(S) = \min_{\phi \in \mathbf{X}(\mathcal{S}, \leq_*)} \phi(S).$$

Recall that there exists an injective semiring homomorphism $\iota : \mathcal{G} \to \mathcal{S}$ mapping the graph G to the nc-graph S_G (Lemma 10) such that $G \leq_* H$ if and only if $S_G \leq_* S_H$ (Lemma 28). By Theorem 2 this implies that there exists a surjection from $\mathbf{X}(\mathcal{S}, \leq_*)$ to $\mathbf{X}(\mathcal{G}, \leq_*)$ via ι.

Theorem 32. *The map*

$$\mathbf{X}(\mathcal{S}, \leq_*) \to \mathbf{X}(\mathcal{G}, \leq_*) : \phi \mapsto \phi \circ \iota$$

is surjective.

Since $\vartheta \in \mathbf{X}(\mathcal{G}, \leq_*)$, we know by Theorem 32 that there exists a function in $\mathbf{X}(\mathcal{S}, \leq_*)$ that restricts to ϑ. Indeed, Duan, Severini and Winter in [DSW13] introduced the quantum Lovász theta function $\tilde{\vartheta}$, which has these properties. This is currently the only element in $\mathbf{X}(\mathcal{S}, \leq_*)$ that we know of.

Theorem 33 ([DSW13, Sta16])**.** *We have*

$$\tilde{\vartheta} \in \mathbf{X}(\mathcal{S}, \leq_*).$$

Moreover, $\tilde{\vartheta}(S_G) = \vartheta(G)$ for any graph G.

We conclude our knowledge for the asymptotic spectra of graphs and noncommutative graphs in Figure 1.

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A Proof of Lemma 14

Proof of Lemma 14. Let $S \leq \mathcal{L}(A)$ be an nc-graph.

(i) We show that $\overline{\mathcal{K}_n} \leq S$ if and only if there is a size-n independent set of S.

Suppose $|\psi_1\rangle\langle\psi_1|, \ldots, |\psi_n\rangle\langle\psi_n|$ is a size-n independent set of S. Let $E_i = |\psi_i\rangle\langle i|$ for $i = 1, \ldots, k$. Then $E_i\,|l\rangle\langle l'|\,E_j^\dagger = \delta_{i,l}\delta_{j,l'}\,|\psi_l\rangle\langle\psi_{l'}|$ for all $l \neq l' \in [n]$. We compute the inner product of $E_i\,|l\rangle\langle l'|\,E_j^\dagger$ and X. We have $\mathrm{Tr}(E_j\,|l'\rangle\langle l|\,E_i^\dagger X) = \delta_{i,l}\delta_{j,l'}\,\mathrm{Tr}(|\psi_{l'}\rangle\langle\psi_l|\,X)$, for all $i, j,\ l \neq l' \in [n]$ and $X \in S$.

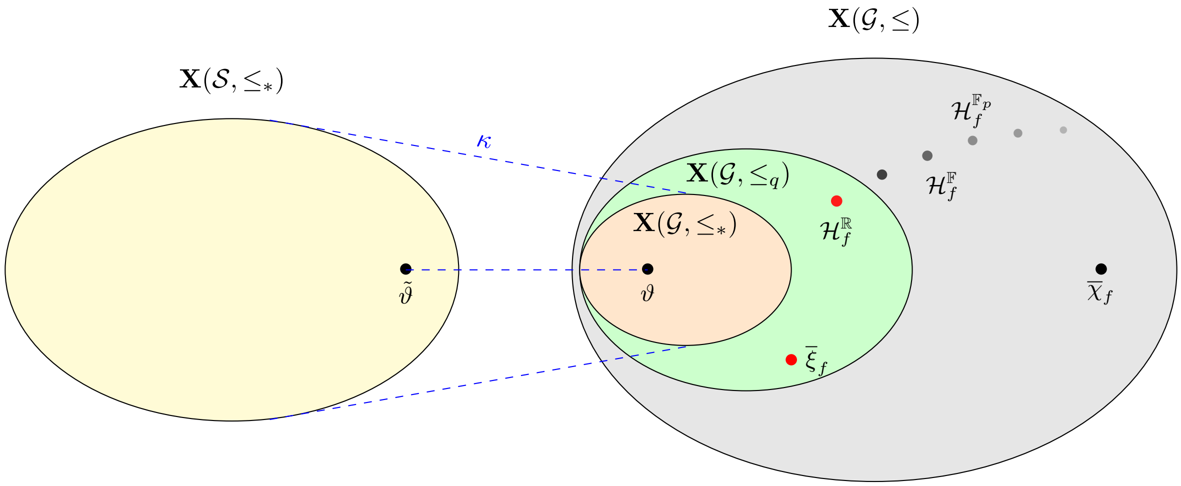

Figure 1: Relations among asymptotic spectra of graphs and non-commutative graphs with different preorder. The fractional Haemers bound provide an infinite family of elements in $\mathbf{X}(\mathcal{G}, \leq)$. We don't know whether the red elements belong to smaller asymptotic spectra or not. It is also open whether $\mathbf{X}(\mathcal{G}, \leq_*) = \mathbf{X}(\mathcal{G}, \leq_q)$.

For all $i \neq l$ or $j \neq l'$, the previous equation equals 0 since $\delta_{i,l} = 0$ or $\delta_{j,l'} = 0$, and otherwise $\mathrm{Tr}(|\psi_{l'}\rangle\langle\psi_l| X) = 0$ for all $X \in S$ as $\{|\psi_1\rangle\langle\psi_1|, \ldots, |\psi_k\rangle\langle\psi_k|\}$ forms an independent set of S. This concludes that $E_i |l\rangle\langle l'| E_j^\dagger \perp X$ for all i, j, $l \neq l' \in [n]$ and $X \in S$, which implies $\overline{\mathcal{K}_n} \leq S$.

On the other hand, suppose $\overline{\mathcal{K}_n} \leq S$. Then there exist $\mathcal{E} : \mathcal{L}(\mathbb{C}^n) \to \mathcal{L}(A)$ with Choi–Kraus operators $\{E_1, \ldots, E_l\} \subseteq \mathcal{L}(\mathbb{C}^n, A)$, such that $E\overline{\mathcal{K}_n}^\perp E^\dagger \perp S$. The condition $E\overline{\mathcal{K}_n}^\perp E^\dagger \perp S$ is equivalent to $E^\dagger S E \subseteq \overline{\mathcal{K}_n} = \mathrm{span}\{|i\rangle\langle i| : i \in [n]\}$. Since $I_A \in S$, we then have $E_j^\dagger E_j \in \mathrm{span}\{|i\rangle\langle i| : i \in [n]\}$ for any $j \in [l]$. Thus we know $E_j^\dagger E_j$ is diagonal. By the singular value decomposition, we can write $E_j = \sum_{x_j} \sqrt{\lambda_{x_j}^j} |\psi_{x_j}^j\rangle\langle v_{x_j}^j|$, where $\lambda_{x_j}^j > 0$ since $E_j^\dagger E_j$ is positive semidefinite, $|v_{x_j}^j\rangle \in \{|1\rangle, \ldots, |k\rangle\}$ for all possible x_j and $\langle\psi_{x_j}^j|\psi_{y_j}^j\rangle = 0$ for all possible $x_j \neq y_j$. Then for $X \in S$,

$$E_j^\dagger X E_l = \sum_{x_j, y_l} \sqrt{\lambda_{x_j}^j \lambda_{y_l}^l} \langle\psi_{x_j}^j| X |\psi_{y_l}^l\rangle |v_{x_j}^j\rangle\langle v_{y_l}^l| \in \mathrm{span}\{|i\rangle\langle i| : i \in [n]\},$$

which implies $\langle\psi_{x_j}^j| X |\psi_{y_j}^l\rangle = 0$ if $|v_{x_j}^j\rangle \neq |v_{y_l}^l\rangle$ for all $X \in S$. Note that $\sum_j E_j^\dagger E_j = I_{\mathbb{C}^n}$. Thus $\mathrm{span}\{|v_{x_j}^j\rangle\}_{j,x_j} = \mathbb{C}^n$. This then guarantees that we can find a size-n independent set of S.

(ii) We show that $\overline{\mathcal{K}_n} \leq_* S$ if and only if there is a size-n entanglement-assisted independent set of S.

Suppose $|\Omega\rangle \in A_0 \otimes B_0$ and $\mathcal{E}_1, \ldots, \mathcal{E}_k$ form an entanglement-assisted independent set of S. Let $\rho = \mathrm{Tr}_{B_0}(|\Omega\rangle\langle\Omega|)$ and $\mathcal{E} : \mathcal{L}(\mathbb{C}^n \otimes A_0) \to \mathcal{L}(A)$ be the quantum channel which maps $|i\rangle\langle i| \otimes \sigma$ to $\mathcal{E}_i(\sigma)$ for all $i = 1, \ldots, k$ and $\sigma \in \mathcal{D}(A_0)$. The Choi–Kraus operators of $\mathcal{E}$ can be written as $\{\langle i| \otimes E_{i,j}\}_{i \in [n], j}$, where $\{E_{i,j}\}_j$ are the Choi–Kraus operators of $\mathcal{E}_i$. We obtain that $E(\overline{\mathcal{K}_n}^\perp \otimes \rho)E^\dagger = \mathrm{span}\{E_{i,j}\rho E_{k,l}^\dagger : i \neq k \in [n],\ j, l\} \perp S$. We conclude $\overline{\mathcal{K}_n} \leq_* S$.

Suppose $\overline{\mathcal{K}_n} \leq_* S$. Then there exist a positive definite $\rho \in \mathcal{D}(A_0)$ and a quantum channel $\mathcal{E} : \mathcal{L}(\mathbb{C}^n \otimes A_0) \to \mathcal{L}(A)$ with Choi–Kraus operators $\{E_i\}_i$ such that $E(\overline{\mathcal{K}_n}^\perp \otimes \rho)E^\dagger \perp S$. Let $\mathcal{E}_i(\rho) = \mathcal{E}(|i\rangle\langle i| \otimes \rho)$ for $i \in [n]$ and let $|\Omega\rangle \in A_0 \otimes B_0$ be a purification of ρ. The Choi–Kraus operator of $\mathcal{E}_i$ can be written as $\{E_{i,j} = E_j(|i\rangle \otimes I_{A_0})\}_j \subseteq \mathcal{L}(A_0, A)$. Then for $i \neq i' \in [n]$ and

j, j', $E_{i,j}^\dagger \rho E_{i',j'} = E_j(|i\rangle\langle i'| \otimes \rho)E_{j'}^\dagger \in E(\overline{\mathcal{K}_n}^\perp \otimes \rho)E^\dagger$, thus $\operatorname{span}\{E_{i,j}^\dagger \operatorname{Tr}_{B_0}(|\Omega\rangle\langle\Omega|)E_{i',j'} : i \neq i' \in [n],\ j, j'\} \perp S$. We conclude $\{|\Omega\rangle, \mathcal{E}_1, \ldots, \mathcal{E}_n\}$ is an entanglement-assisted independent set of S. $\square$

B The unassisted Shannon capacity of nc-graphs

In this section, we discuss the unassisted Shannon capacity of nc-graphs. The unassisted Shannon capacity may not admit a dual characterization by its asymptotic spectrum. We first note that the cohomomorphism preorder on nc-graphs becomes the cohomomorphism preorder on graphs when restricting from nc-graphs to graphs.

Lemma 34 ([Sta16, Theorem 8])**.** *For any graphs $G, H \in \mathcal{G}$ holds $G \leq H$ if and only if $S_G \leq S_H$.*

The cohomomorphism preorder on nc-graphs has the following properties.

Lemma 35. *For any nc-graphs $S \subseteq \mathcal{L}(A)$, $S' \subseteq \mathcal{L}(A')$, $T \subseteq \mathcal{L}(B)$ and $T' \subseteq \mathcal{L}(B')$ and $n, m \in \mathbb{N}$, we have*

(i) $S \leq S$

(ii) *if $S \leq T$ and $T \leq T'$, then $S \leq T'$*

(iii) $\overline{\mathcal{K}_m} \leq \overline{\mathcal{K}_n}$ *if and only if $m \leq n$*

(iv) *if $S \leq T$ and $S' \leq T'$, then $S \oplus S' \leq T \oplus T'$ and $S \otimes S' \leq T \otimes T'$*

Proof. (i) We see that $S \leq S$ by taking $E = \operatorname{span}\{I_A\}$ in Definition 11.

(ii) Let $E = \operatorname{span}\{E_i\}_i \subseteq \mathcal{L}(A, B)$ be given by $S \leq T$, and $F = \operatorname{span}\{F_j\}_j \subseteq \mathcal{L}(B, B')$ be given by $T \leq T'$. To see $S \leq T'$, take $F' = \operatorname{span}\{F_j E_i\}_{i,j} \subseteq \mathcal{L}(A, B')$. We have

(47) $F' S^\perp F'^\dagger = F(ES^\perp E^\dagger)F^\dagger \subseteq FT^\perp F^\dagger \perp T'$,

where the inequality holds since $ES^\perp E^\dagger \perp T$ by $S \leq T$, and the last orthogonality relation is given by $T \leq T'$.

(iii) By Lemma 34, $\overline{\mathcal{K}_n} \leq \overline{\mathcal{K}_m}$ is equivalent to $\overline{K_n} \leq \overline{K_m}$, which is equivalent to $m \leq n$.

(iv) Let $E = \operatorname{span}\{E_i\}_i \subseteq \mathcal{L}(A, B)$ be given by $S \leq T$, and $F = \operatorname{span}\{F_j\}_j \subseteq \mathcal{L}(A', B')$ be given by $S' \leq T'$. Let $E' = \operatorname{span}\{E_i \oplus 0\}_i \cup \{0 \oplus F_j\}_j \subseteq \mathcal{L}(A \oplus A', B \oplus B')$, where $(E_i \oplus F_j)(|\psi\rangle_A \oplus |\psi'\rangle_{A'}) = E_i |\psi\rangle_A \oplus F_j |\psi'\rangle_{A'})$ for all i, j and $|\psi\rangle_A \in A$ and $|\psi'\rangle_{A'} \in A'$. One readily verifies that $E'(S \oplus S')^\perp E'^\dagger \perp T \oplus T'$. To see $S \otimes S' \leq_* T \otimes T'$, Let $E' = \operatorname{span}\{E_i \otimes I_{A'}, I_A \otimes F_j\}_{i,j} \subseteq \mathcal{L}(A \otimes A', B \otimes B')$. One readily verifies that $E'(S \otimes S')^\perp E'^\dagger \perp T \otimes T'$. $\square$

Recall the following property of the entanglement-assisted cohomomorphism preorder $\leq_*$. If $T \neq \overline{\mathcal{K}_0}$, then there is an $r \in \mathbb{N}$ with $S \leq_* \overline{\mathcal{K}_r} \otimes T$. The next example shows that the cohomomorphism preorder $\leq$ does not have this property, thus cannot be a Strassen preorder.

Example 36. Let $S = \mathcal{I}_2$ and $T = \overline{\mathcal{K}_1} = \mathbb{C}$. For any $r \in \mathbb{N}$ holds $S \not\leq \overline{\mathcal{K}_r} \otimes T$.

Proof. Assume $\mathcal{I}_2 \leq \overline{\mathcal{K}_r}$. Let $E = \operatorname{span}\{E_i\}_i \leq \mathcal{L}(\mathbb{C}^2, \mathbb{C}^r)$ satisfy $E\mathcal{I}_2^\perp E^\dagger \perp \overline{\mathcal{K}_r}$. Note that $ES^\perp E^\dagger \perp \overline{\mathcal{K}_r}$ implies $E^\dagger \overline{\mathcal{K}_r} E \subseteq S$, since $\operatorname{Tr}(E_i X^\dagger E_j^\dagger Y) = \operatorname{Tr}(E_i^\dagger Y^\dagger E_j X)$ implies $E_j^\dagger Y E_i \perp X$ for all $E_i, E_j \in E$, $X \in S^\perp$ and $Y \in \overline{\mathcal{K}_r}$. We obtain that $E_i^\dagger |j\rangle\langle j| E_i \in \mathcal{I}_2$ for all i and $j \in [r]$. This is impossible since $E \neq 0$ and since the nonzero elements in $\mathcal{I}_2$ have rank 2. $\square$

The reason why $\mathcal{I}_2 \not\leq \overline{\mathcal{K}_r}$ for every $r \in \mathbb{N}$ can be understood as: no classical channels can transmit even a single qubit. In the entanglement-assisted setting, this can be overcome by invoking the teleportation protocol, as mentioned in the proof of Lemma 30 (v).

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