# Topological Price of Anarchy Bounds for Clustering Games on Networks

Pieter Kleer[1] and Guido Schäfer[1,2(✉)]

[1] Centrum Wiskunde & Informatica (CWI), Networks and Optimization Group, Amsterdam, The Netherlands
{kleer,schaefer}@cwi.nl

[2] Department of Econometrics and Operations Research, Vrije Universiteit Amsterdam, Amsterdam, The Netherlands

**Abstract.** We consider clustering games in which the players are embedded in a network and want to coordinate (or anti-coordinate) their choices with their neighbors. Recent studies show that even very basic variants of these games exhibit a large Price of Anarchy. Our main goal is to understand how structural properties of the network topology impact the inefficiency of these games. We derive *topological bounds* on the Price of Anarchy for different classes of clustering games. These topological bounds provide a more informative assessment of the inefficiency of these games than the corresponding (worst-case) Price of Anarchy bounds. As one of our main results, we derive (tight) bounds on the Price of Anarchy for clustering games on Erdős-Rényi random graphs, which, depending on the graph density, stand in stark contrast to the known Price of Anarchy bounds.

**Keywords:** Clustering games · Coordination games · Price of Anarchy · Random graphs · Nash equilibrium existence

## 1 Introduction

***Motivation.*** Clustering games on networks constitute a class of strategic games in which the players are embedded in a network and want to coordinate (or anti-coordinate) their choices with their neighbors. These games capture several key characteristics encountered in applications such as opinion formation, technology adoption, information diffusion or virus spreading on various types of networks (e.g., the Internet, social networks, biological networks, etc.).

Different variants of clustering games have recently been studied intensively in the algorithmic game theory literature, both with respect to the existence and the inefficiency of equilibria (see, e.g., [3,4,11,15,16,18,20,21]). Unfortunately, several of these studies reveal that the strategic choices of the players may lead to equilibrium outcomes that are highly inefficient. Arguably the most prominent notion to assess the inefficiency of equilibria is the *Price of Anarchy (PoA)* [19], which refers to the worst-case ratio of the optimal social welfare and the


I. Caragiannis et al. (Eds.): WINE 2019, LNCS 11920, pp. 241–255, 2019.
https://doi.org/10.1007/978-3-030-35389-6_18

social welfare of a (pure) Nash equilibrium. It is known that even the most basic clustering games exhibit a large (or even unbounded) Price of Anarchy (see below for details). These negative results naturally trigger the following questions: Is this high inefficiency inevitable in clustering games on networks? Or, can we trace more precisely what causes a large inefficiency? These questions constitute the starting point of our investigations: *Our main goal in this paper is to understand how structural properties of the network topology impact the Price of Anarchy in clustering games.*

In general, our idea is that a more fine-grained analysis may reveal topological parameters of the network which can be used to derive more accurate bounds on the Price of Anarchy; we term such bounds *topological Price of Anarchy bounds*. Given the many applications of clustering games on different types of networks, our hope is that such topological bounds will be more informative than the corresponding worst-case bounds. Clearly, this hope is elusive for a number of fundamental games on networks whose inefficiency is known to be *independent* of the network topology, the most prominent example being the selfish routing games studied in the seminal work by Rougharden and Tardos [22]. But, in contrast to these games, clustering games exhibit a strong *locality property* induced by the network structure, i.e., the utility of each player is affected only by the choices of her direct neighbors in the network. This observation also motivates our choice of quantifying the inefficiency by means of topological parameters (rather than other parameters of the game).

We derive topological bounds on the Price of Anarchy for different classes of clustering games. Our bounds reveal that the Price of Anarchy depends on different topological parameters in the case of symmetric and asymmetric strategy sets of the players and, depending on these parameters, stand in stark contrast to the known worst case bounds. As one of our primary benchmarks, we use Erdős-Rényi random graphs [13] to obtain a precise understanding of how these parameters affect the Price of Anarchy. More specifically, we show that the Price of Anarchy of clustering games on random graphs, depending on the graph density, improves significantly over the worst case bounds. To the best of our knowledge, this is also the first work that addresses the inefficiency of equilibria on random graphs.[1]

We note that the applicability of our topological Price of Anarchy bounds is not limited to the class of Erdős-Rényi random graphs. The main reason for using these graphs is that their structural properties are well-understood. In particular, our topological bounds can be applied to any graph class of interest (as long as certain structural properties are well-understood).

***Our Clustering Games.*** We study a generalization of the unifying model of *clustering games* introduced by Feldman and Friedler [11]: We are given an undirected graph $G = (V, E)$ on $n = |V|$ nodes whose edge set $E = E_c \cup E_a$ is partitioned into a set of *coordination* edges $E_c$ and a set of *anti-coordination*

[1] We note that Valiant and Roughgarden [23] study Braess' paradox in large random graphs (see Related Work).

edges $E_a$.[2] Further, we are given a set $[c] = \{1, \ldots, c\}$ of $c > 1$ colors and edge-weights $w : E \to \mathbb{R}_{\geq 0}$.[3] Each node $i$ corresponds to a player who chooses a color $s_i$ from her color set $S_i \subseteq [c]$. We say that the game is *symmetric* if $S_i = [c]$ for all $i \in V$ and *asymmetric* otherwise. An edge $e = \{i, j\} \in E$ is *satisfied* if it is a coordination edge and both $i$ and $j$ choose the same color, or if it is an anti-coordination edge and $i$ and $j$ choose different colors. The goal of player $i$ is to choose a color $s_i \in S_i$ such that the weight of all satisfied edges incident to $i$ is maximized.

We consider a generalization of these games by incorporating additionally: (i) individual player preferences (as in [21]), and (ii) different distribution rules (as in [3]): We assume that each player $i$ has a *preference function* $q_i : S_i \to \mathbb{R}_{\geq 0}$ which encodes her preferences over the colors in $S_i$. Further, player $i$ has a *split parameter* $\alpha_{ij} \geq 0$ for every incident edge $e = \{i, j\}$ which determines the share she obtains from $e$: if $e$ is satisfied then $i$ obtains a proportion of $\alpha_{ij}/(\alpha_{ij} + \alpha_{ji})$ of the weight $w_e$ of $e$. The utility $u_i(s)$ of player $i$ for choosing color $s_i \in S_i$ is then the sum of the individual preference $q_i(s_i)$ and the total share of all satisfied edges incident to $i$. We consider the standard utilitarian *social welfare* objective $u(s) = \sum_i u_i(s)$.

We use $\bar{\alpha}_e$ to denote the *disparity* of an edge $e = \{i, j\}$, defined as $\bar{\alpha}_e = \max\{\alpha_{ij}/\alpha_{ji}, \alpha_{ji}/\alpha_{ij}\}$, and let $\bar{\alpha} = \max_{e \in E} \bar{\alpha}_e$ refer to the maximum disparity of all edges. We say that the game has the *equal-split distribution rule* if $\bar{\alpha} = \mathbf{1}$ (equivalently, $\alpha_{ij} = \alpha_{ji}$ for all $\{i, j\} \in E$).

Our clustering games generalize several other strategic games, which were studied extensively in the literature before, such as *max cut games* and *not-all-equal satisfiability games* [15], *max k-cut games* [16], *coordination games* [4], *clustering games* [11] and *anti-coordination games* [20].

***Main Contributions.*** We derive results for symmetric and asymmetric clustering games. Due to space restrictions, we elaborate on our main findings for symmetric clustering games only below; our results for the asymmetric case are discussed in Sect. 5. An overview of the bounds derived in this paper is given in Table 1.

**1. Topological Price of Anarchy Bound.** We show that the Price of Anarchy for symmetric clustering games is bounded as a function of the *maximum subgraph density* of $G$ which is defined as $\rho(G) = \max_{S \subseteq V}\{|E[S]|/|S|\}$, where $|E[S]|$ is the number of edges in the subgraph induced by $S$. More specifically, we prove that PoA $\leq 1 + (1 + \bar{\alpha})\rho(G)$ and that this bound is tight (even for coordination games). Using this topological bound, we are able to show that the Price of Anarchy is at most $4 + 3\bar{\alpha}$ for clustering games on planar graphs and $1 + 2\rho(G)$ for coordination games with equal-split distribution rule. We also derive a (qualitatively) refined bound of PoA $\leq 5 + 2\rho(G[E_c])$ for clustering games with equal-split distribution rule which reveals that the maximum

[2] The game is called a *coordination game* if all edges are coordination edges and an *anti-coordination game* (or *cut game*) if all edges are anti-coordination edges.

[3] In this paper, we use $[k]$ to denote the set $\{1, \ldots, k\}$ for a given integer $k \geq 1$.

**Table 1.** Overview of our topological Price of Anarchy bounds for symmetric and asymmetric clustering games. A "+" or "**1**" in the column "distr. $\alpha$" indicates whether the distribution rule $\alpha$ is positive or equal-split, respectively. $\bar{\alpha}$ is the maximum disparity, and $c$ is the number of colors. $\rho(G)$ and $\Delta(G)$ refer to the maximum subgraph density and the maximum degree of $G$, respectively. The stated bounds for random graphs hold with high probability.

| Symmetric clustering games | | | | | | |
|---|---|---|---|---|---|---|
| Graph topology | Coord. only | Indiv. pref. | Distr. $\alpha$ | Topological PoA (our bounds) | | PoA (prev. work) |
| Arbitrary | ✗ | ✓ | + | $1 + (1 + \bar{\alpha})\,\rho(G)$ | (Theorem 1) | $c$ [3,11] |
| Planar | ✗ | ✓ | + | $\leq 4 + 3\bar{\alpha}$ | (Corrollary 1) | |
| Arbitrary | ✗ | ✓ | **1** | $1 + 2\rho(G)$ | (Corrollary 2) | |
| Arbitrary | ✗ | ✓ | **1** | $\leq 5 + 2\rho(G_c)$ | (Theorem 2) | |
| Sparse random | ✓ | ✓ | **1** | $\Theta(1)$ | (Corrollary 3) | |
| Dense random | ✓ | ✗ | **1** | $\Omega(c)$ | (Theorem 3) | |
| Asymmetric clustering games | | | | | | |
| Graph topology | Coord. only | Indiv. pref. | Distr. $\alpha$ | $(\epsilon, k)$ -topological PoA (our bounds) | | $(\epsilon, k)$ -PoA (prev. work) |
| Arbitrary | ✓ | ✗ | **1** | $\leq 2\epsilon\Delta(G)$ | (Theorem 5) | |
| Arbitrary | ✓ | ✗ | **1** | $\geq \epsilon(\frac{\Delta(G)}{k-1} - 1)$ | (Theorem 5) | $\leq 2\epsilon\frac{n-1}{k-1}$ |
| Dense random | ✓ | ✗ | **1** | $\Omega(\epsilon n)$ | | $\geq 2\epsilon\frac{n-k}{k-1} + 1$ |
| Sparse random | ✓ | ✗ | **1** | $\Theta\left(\frac{\epsilon \ln(n)}{\ln\ln(n)}\right)$ | (Theorem 6) | [21] |
| + common color | ✓ | ✗ | **1** | $O(1)$ | (Theorem 7) | |

subgraph density with respect to the graph $G[E_c]$ (or simply $G_c$) induced by the *coordination edges* $E_c$ *only* is the crucial topological parameter determining the Price of Anarchy.

These bounds provide more refined insights than the known (tight) bound of PoA $\leq c$ (number of colors) on the Price of Anarchy for (i) symmetric coordination games with individual preferences and arbitrary distribution rule [3], and (ii) clustering games without individual preferences and equal-split distribution rule [11] (both being special cases of our model). An important point to notice here is that this bound indicates that the Price of Anarchy is unbounded if the number of colors $c = c(n)$ grows as a function of $n$. In contrast, our topological bounds are independent of $c$ and are thus particularly useful when this number is large (while the maximum subgraph density is small). Moreover, our refined bound of $5 + 2\rho(G[E_c])$ mentioned above provides a nice bridge between the facts that for max-cut (or anti-coordination) games the price of anarchy is known to be constant, whereas for coordination games the price of anarchy might grow large.

**2. Price of Anarchy for Random Coordination games.** We derive the first price of anarchy bounds for coordination games on random graphs. We focus on the *Erdős-Rényi random graph model* [13] (also known as $G(n, p)$), where each graph consists of $n$ nodes and every edge is present (independently) with probability $p \in [0, 1]$. More specifically, we show that the Price of Anarchy is constant (with high probability) for coordination games on sparse random graphs (i.e., $p = d/n$ for some constant $d > 0$) with equal-split distribution rule. In contrast, we show that the Price of Anarchy remains $\Omega(c)$ (with high probability) for dense random graphs (i.e., $p = d$ for some constant $0 < d \leq 1$).

Note that our constant bound on the Price of Anarchy for sparse random graphs stands in stark contrast to the deterministic bound of PoA $= c$ [3,11] (which could increase with the size of the network). On the other hand, our bound for dense random graphs reveals that we cannot significantly improve upon this bound through randomization of the graph topology.

It is worth mentioning that all our results for random graphs hold against an *adaptive adversary* who can fix the input of the clustering game *knowing* the realization of the random graph. To obtain these results, we need to exploit some deep probabilistic results on the maximum subgraph density and the existence of perfect matchings in random graphs.

**3. Convergence of Best-Response Dynamics.** In general, pure Nash equilibria are not guaranteed to exist for clustering games with *arbitrary* distribution rules $\alpha$, even if the game is symmetric (see, e.g., [3]). While some sufficient conditions for the existence of pure Nash equilibria, or, the convergence of best-response dynamics (see also [3]) are known, a complete characterization is elusive so far.

In this work, we obtain a complete characterization of the class of distribution rules which guarantee the convergence of best-response dynamics in clustering games on a fixed network topology. Basically, we prove that best-response dynamics converge if and only if $\alpha$ is a *generalized weighted Shapley distribution rule* (Theorem 4). Our proof relies on the fact that there needs to be some form of *cyclic consistency* similar to the one used in [14].

Prior to our work, the existence of pure Nash equilibria was known for certain special cases of coordination games only, namely for symmetric coordination games with individual preferences and $c = 2$ [3], and for symmetric coordination games without individual preferences [11]. To the best of our knowledge, this is the first characterization of distribution rules in terms of best-response dynamics (which, in particular, applies to the settings in which pure Nash equilibria are guaranteed to exist for every distribution rule [3,11]).[4]

***Related Work.*** The literature on clustering and coordination games is vast; we only include references relevant to our model here. The proposed model above is a mixture of (special cases of) existing models in [3,4,11,21].

Anshelevich and Sekar [3] consider symmetric coordination games with individual preferences and (general) distribution rules. They show existence of *$\epsilon$-approximate k-strong equilibria*, $(\epsilon, k)$-equilibria for short, for various combinations; in particular, $(2, k)$-equilibria always exist for any $k$. Moreover, they show that the number of colors $c$ is an upper bound on the PoA. Apt et al. [4] study asymmetric coordination games with unit weights, zero individual preferences, and equal-split distribution rules. They derive an almost complete picture of the

[4] In the full version, we extend our ideas and provide a characterization of the existence of pure Nash equilibria in symmetric coordination games, complementing a result by Anshelevich and Sekar [3].

existence of $(1, k)$-equilibria for different values of $c$. Feldman and Friedler [11] introduce a unified framework (as introduced above) for studying the (strong) Price of Anarchy in clustering games with individual preferences set to zero and equal-split distribution rules. In particular, they show that the number of colors is an upper bound on the PoA and that $2(n-1)/(k-1)$ is an upper bound on the $(1, k)$-PoA. Rahn and Schäfer [21] consider the more general setting of polymatrix coordination games with equal-split distribution rule, of which our asymmetric coordination games with individual preferences are a special case. They show a bound of $2\epsilon(n-1)/(k-1)$ on the $(\epsilon, k)$-PoA and that an $(\epsilon, k)$-equilibrium is guaranteed to exist for any $\epsilon \geq 2$ and any $k$.

There is also a vast literature on different variants of anti-coordination (or cut) games, see, e.g., [16,18] and the references therein, which are also captured by our clustering games. In a recent paper, Carosi and Gianpiero [8] consider so-called *k-coloring games*. Moreover, clustering and coordination games were also studied on directed graphs [4,7]. Finally, certain coordination and clustering games can be seen as special cases of hedonic games [10]; we refer the reader to [6] for, in particular, a survey of recent literature on (fractional) hedonic games. Identifying topological inefficiency bounds for these type of games, as well as for clustering games on directed graphs, could be an interesting direction for future work.[5]

Regarding the study of the inefficiency of equilibria on random graphs, closest to our work seems to be the work by [23]. They study the Braess paradox on large Erdős-Rényi random graphs and show that for certain settings the Braess paradox occurs with high probability as the size of the network grows large. The study of randomness in games has also received some attention in other setting, see, e.g., [1,5]. These are mostly settings with small strategy sets and random utility functions, and are not comparable with ours.

Finally, our characterization results regarding the existence of pure Nash equilibria and convergence of best-response dynamics are conceptually similar to the work of Chen et al. [9] and Gopalakrishnan et al. [14].

## 2 Preliminaries

***Clustering Games.*** As introduced above, an instance of a *clustering game* $\Gamma = (G, c, (S_i), (\alpha_{ij}), w, q)$ is given by:

- an undirected graph $G = (V, E)$, where the set of edges $E = E_c \cup E_a$ is partitioned into coordination edges $E_c$ and anti-coordination edges $E_a$;
- a subset $S_i \subseteq [c]$ of colors available to player $i \in V$;
- a split parameter $\alpha_{ij} \geq 0$ for every player $i \in V$ and incident edge $\{i, j\} \in E$;
- a weight function $w : E \to \mathbb{R}_{\geq 0}$ on the edges;
- a vector $q = (q_i)_{i \in V}$ of individual preference functions $q_i : S_i \to \mathbb{R}_{\geq 0}$.

[5] Our results do not seem to extend to clustering games on directed graphs. One could model a directed edge $e = (i, j)$ by setting $\alpha_{ij} = 0$ and $\alpha_{ji} > 0$. E.g., Theorem 1 does not apply then as $\bar{\alpha} = \infty$ in this case.

Whenever we refer to a *clustering game* below, we assume that all of the above input parameters are non-trivial; we specify the respective restrictions otherwise.

Each node $i \in V$ corresponds to a player whose goal is to choose a color $s_i \in S_i$ from the set of colors available to her to maximize her utility

$$u_i(s) = q_i(s_i) + \sum_{\{i,j\} \in E_c : s_i = s_j} \frac{\alpha_{ij}}{\alpha_{ij} + \alpha_{ji}} w_{ij} + \sum_{\{i,j\} \in E_a : s_i \neq s_j} \frac{\alpha_{ij}}{\alpha_{ij} + \alpha_{ji}} w_{ij}.$$

We call $\alpha = (\alpha_{ij}) \geq 0$ a *distribution rule*. We assume that $\alpha$ satisfies $\alpha_{ij} + \alpha_{ji} > 0$ for every edge $e = \{i,j\} \in E$; in particular, not both $i$ and $j$ have a zero split for edge $e$. We say that $\alpha$ is *positive* if $\alpha_{ij} > 0$ and $\alpha_{ji} > 0$ for all $e = \{i,j\} \in E$; we also write $\alpha > \mathbf{0}$. Further, $\alpha$ is called the *equal-split* distribution rule if $\alpha_{ij} = \alpha_{ji}$ for all $e = \{i,j\} \in E$; we also indicate this by $\alpha = \mathbf{1}$. The *disparity* of an edge $e = \{i,j\}$ is defined as $\bar{\alpha}_e = \max\{\alpha_{ij}/\alpha_{ji}, \alpha_{ji}/\alpha_{ij}\}$ and we use $\bar{\alpha} = \max_{e \in E} \bar{\alpha}_e$ to denote the maximum disparity.

We say that the clustering game is *symmetric* if $S_i = \{1, \dots, c\}$ for every player $i \in V$ and *asymmetric* otherwise. If we focus on symmetric clustering games, we omit the explicit reference of the strategy sets $(S_i)$ with $S_i = [c]$. A clustering game is called a *coordination game* if $E_a = \emptyset$ and an *anti-coordination game* (or *cut game*) if $E_c = \emptyset$. We use $n = |V|$ to refer to the number of players.

We consider the utilitarian *social welfare* objective $u(s) = \sum_{i \in V} u_i(s)$. The *Price of Anarchy* of an instance $\Gamma$ is defined as $\mathrm{PoA}(\Gamma) = \max_{s \in \mathrm{NE}(\Gamma)} u(s^*)/u(s)$, where $\mathrm{NE}(\Gamma)$ is the set of all pure Nash equilibria of $\Gamma$ and $s^*$ is a socially optimal strategy profile. Given a class of clustering games $\mathcal{G}$, the Price of Anarchy is defined as $\mathrm{PoA}(\mathcal{G}) = \sup_{\Gamma \in \mathcal{G}} \mathrm{PoA}(\Gamma)$.

***Random Clustering Games.*** In our probabilistic framework to study the Price of Anarchy of random clustering games, we use the well-known *Erdős-Rényi random graph model* [13], denoted by $G(n,p)$:[6] There are $n$ nodes and every (undirected) edge is present (independently) with probability $p = p(n) \in [0,1]$. We say that a random graph is *sparse* if $p = d/n$ for some constant $d > 0$, and it is *dense* if $p = d$ for some constant $0 < d < 1$. In this paper, we focus on random graph instances with equal-split distributions rules.[7]

Fix some probability $p = p(n) \in [0,1]$ and let $\beta = \beta(n, c(n))$ be a given function. Define $\mathcal{G}_{G_n}$ as the set of all clustering games on random graph $G_n \sim G(n,p)$. We say that the *Price of Anarchy for random clustering games is at most $\beta$ with high probability (*$\mathrm{PoA}(\mathcal{G}_{G_n}) \leq \beta$,*for short)* if $\mathbb{P}_{G_n \sim G(n,p)}\{\mathrm{PoA}\left(\mathcal{G}_{G_n}\right) \leq \beta\} \geq 1 - o(1)$. We use a similar definition if we want to lower bound the Price of Anarchy. Finally, for a constant $\beta$ (independent of $n$ and $c$) we say that the *Price of Anarchy for random clustering games is $\beta$ with high probability (*$\mathrm{PoA}(\mathcal{G}_{G_n}) \to \beta$*, for short)* if for all $\varepsilon > 0$ $\mathbb{P}_{G_n \sim G(n,p)}\left\{|\mathrm{PoA}\left(\mathcal{G}_{G_n}\right) - \beta| \leq \varepsilon\right\} \geq 1 - o(1)$. All our results for clustering games on random graphs hold with high probability.

[6] Although this model was first introduced by Gilbert, it is often referred to as the *Erdős-Rényi random graph model*.

[7] Some of our results naturally extend to more general distribution rules, but we omit the (technical) details here because they do not provide additional insights.

***Shapley Distribution Rules.*** We adapt the definition of Shapley distribution rules for resource allocation games [14] to our setting.

A distribution rule $\alpha$ corresponds to a *generalized weighted Shapley distribution rule* if and only if there exists a permutation $\sigma$ of the players in $V$ and weight vector $\gamma \in \mathbb{R}^V_{\geq 0}$ such that the following two conditions are satisfied for every edge $e = \{i, j\}$: (i) If $\alpha_{ij} = 0$, then $\sigma(i) < \sigma(j)$. (ii) If $\alpha_{ij} > 0$, then $\frac{\alpha_{ij}}{\alpha_{ij}+\alpha_{ji}} = \frac{\gamma_i}{\gamma_i+\gamma_j}$. If all weights are strictly positive, then the resulting distribution rule is a *weighted Shapley distribution rule*. If $\gamma_i = \gamma_j$ for all $i, j \in V$ the resulting distribution rule is an *unweighted Shapley distribution rule*. Note that this case corresponds to an equal-split distribution rule.

Due to space restrictions, some proofs below are omitted and will be given in the full version of the paper.

# 3 Refined Bounds on the Price of Anarchy

In this section, we first establish our topological bound on the Price of Anarchy for symmetric clustering games and then use it to derive new bounds for some special cases as well as random clustering games.

## 3.1 Topological Price of Anarchy Bound

Our topological bound depends on the *maximum subgraph density* of $G$ which is defined as $\rho(G) = \max_{S \subseteq V}\{|E[S]|/|S|\}$, where $|E[S]|$ is the number of edges in the subgraph induced by $S$. Recall that $\bar{\alpha}$ refers to the maximum disparity.

**Theorem 1 (Density bound).** *Let $\Gamma = (G, c, \alpha, w, q)$ be a symmetric clustering game with $\alpha > 0$. Then $PoA(\Gamma) \leq 1 + (1 + \bar{\alpha})\,\rho(G)$ and this is tight.*

*Proof (upper bound).* Let $s$ and $s^*$ be a Nash equilibrium and a social optimum, respectively. Consider an edge $\{i, j\} \in E$ and assume without loss of generality that $u_i(s) \leq u_j(s)$. If $\{i, j\}$ is a coordination edge, then $u_i(s) \geq u_i(s_{-i}, s_j) \geq \alpha_{ij}/(\alpha_{ij}+\alpha_{ji})w_{ij}$, where $(s_{-i}, s_j)$ is the strategy profile in which player $i$ deviates to the color of player $j$ and all other players play according to $s$. Suppose $\{i, j\}$ is an anti-coordination edge. If $s_i \neq s_j$, then we trivially have $u_i(s) \geq \alpha_{ij}/(\alpha_{ij} + \alpha_{ji})w_{ij}$ by non-negativity of the weights and individual preferences. If $s_i = s_j$, then the same inequality holds by using the Nash condition for some arbitrary color which is not $s_j$. (We may assume that every player has at least two colors in her strategy set.) In either case, we conclude that

$$w_{ij} \leq \left(1 + \frac{\alpha_{ji}}{\alpha_{ij}}\right) u_i(s) \leq \left(1 + \max_{e \in E} \bar{\alpha}_e\right) u_i(s) = (1 + \bar{\alpha})\, u_i(s). \tag{1}$$

Moreover, by exploiting that $s$ is a Nash equilibrium and the non-negativity of the edge weights, we obtain for every $i \in V$, $u_i(s) \geq u_i(s_{-i}, s_i^*) \geq q_i(s_i^*)$.

Using that the sum of the weights of all satisfied edges in $s^*$ is at most the sum of all edge weights, we obtain

$$u(s^*) \leq \sum_{i \in V} q_i(s_i^*) + \sum_{e=\{i,j\} \in E} w_{ij} \leq \sum_{i \in V} u_i(s) + (1+\bar{\alpha}) \sum_{\{i,j\} \in E} \min\{u_i(s), u_j(s)\}.$$

If we can find a value $M$ such that

$$\sum_{\{i,j\} \in E} \min\{u_i(s), u_j(s)\} \leq M \cdot \sum_{i \in V} u_i(s) \tag{2}$$

then it follows that $u(s^*) \leq (1 + (1+\bar{\alpha}) \cdot M)\, u(s)$. We show that $M = \max_{S \subseteq V}\{|E[S]|/|S|\}$ satisfies (2).

Let $N(i) = \{j \in V : \{i,j\} \in E\}$ be the set of neighbors of $i$. Define $m_i = \big|\{j \in N(i) : u_i(s) < u_j(s) \text{ or } (u_i(s) = u_j(s) \text{ and } i < j)\}\big|$ and note that $\sum_{i \in V} m_i = |E|$. We can assume without loss of generality that $\sum_{i \in V} u_i(s) = 1$, since the expression in (2) is invariant under multiplication with a constant positive scalar. Moreover, the players may be renamed such that $u_1(s) \leq u_2(s) \leq \cdots \leq u_n(s)$.

We continue by showing that $M$ is an upper bound for the linear program below (in which $u_i = u_i(s)$ and the $m_i$ are considered constants).

$$\max \sum_{i \in V} u_i m_i \quad \text{s.t.} \quad \begin{array}{l} u_1 + u_2 + \cdots + u_n = 1 \\ 0 \leq u_1 \leq u_2 \leq \cdots \leq u_n \end{array}$$

The dual of this program is given by

$$\min z \quad \text{s.t.} \quad \begin{array}{l} -\pi_i + \pi_{i+1} + z = m_i, \quad i = 1, \dots, n-1 \\ -\pi_n + z = m_n \\ \pi_i \geq 0, \quad i = 1, \dots, n \\ z \in \mathbb{R} \end{array}$$

We now construct a feasible dual solution. Set $z^* = \max_{l \in V}\{\sum_{i=l}^{n-1} m_i/(n-l)\}$. We will often use that $(n-l)z^* \geq \sum_{i=l}^{n-1} m_i$ for any fixed $l$. In particular, with $l = n-1$, we find $z^* \geq m_n$, so that $\pi_n^* := z^* - m_n \geq 0$. Then we define $\pi_{n-1}^* := \pi_n^* + z^* - m_{n-1} = 2z^* - (m_{n-1} + m_n) \geq 0$. Using induction it then easily follows that $\pi_i^* := \pi_{i+1}^* + z^* - m_i \geq 0$ for all $i = 1, \dots, n-2$ as well. We have constructed a feasible dual solution with objective function value $z^*$. Using weak duality it follows that for any feasible primal solution $u = (u_1, \dots, u_n)$, we have

$$\sum_{\{i,j\} \in E} u_i m_i \leq \max_{l \in V} \left\{ \frac{\sum_{i=l}^{n-1} m_i}{n-l} \right\} \leq \max_{S \subseteq V} \left\{ \frac{|E[S]|}{|S|} \right\},$$

since the term in middle is precisely the density of the induced subgraph on the nodes $l, \dots, n$. This completes the upper bound proof. □

We use our topological bound to derive deterministic bounds on the Price of Anarchy for two special cases of clustering games. Note that these bounds cannot be deduced from [3,11].

**Corollary 1 (Planar clustering games).** *Let $\Gamma = (G, c, \alpha, w, q)$ be a symmetric clustering game on a planar graph $G$ with $\alpha > 0$. Then $PoA(\Gamma) \leq 4 + 3\bar{\alpha}$.*

*Proof.* By Euler's formula, $|E(H)|/|V(H)| \leq 3$ for any planar graph $H$. Further, any induced subgraph $H$ of a planar graph $G$ is again planar. Using this in Theorem 1 proves the claim. □

**Corollary 2 (Equal-split coordination games).** *Let $G$ be a given undirected graph, and let $\mathcal{G}_G$ be the set of all symmetric coordination games $\Gamma = (G, c, \mathbf{1}, w, q)$ with equal-split distribution rule on $G$. Then $PoA(\mathcal{G}_G) = 1 + 2\rho(G)$.*

We emphasize that the bound in Corollary 2 is tight on *every* fixed graph topology $G$, rather than only in the *value* of $\rho(G)$.

It is known that the Price of Anarchy of anti-coordination games is 2 (see, e.g., [18]), which is not reflected by our bound in Theorem 1. Intuitively, this suggests that a large Price of Anarchy is caused by the coordination edges of the graph. Theorem 2 reveals that this intuition is correct: it shows that the maximum subgraph density with respect to the *coordination edges only* is the determining topological parameter.

**Theorem 2 (Refined density bound).** *Let $\Gamma = (G, c, \mathbf{1}, w, q)$ be a symmetric clustering game with equal-split distribution rule. Then $PoA(\Gamma) \leq 5 + 2\rho(G[E_c])$, where $G[E_c]$ is the subgraph induced by the coordination edges $E_c$.*

Using a similar construction as in the proof of Corollary 2 we can also establish a lower bound of $1 + 2\max_{S \subseteq V} \{|E_c[S]|/|S|\}$.

Note that for anti-coordination games we obtain an upper bound of 5 which is inferior to the known (tight) bound of 2. It would be interesting to see whether our topological bound in Theorem 2 can be improved to match this bound.

### 3.2 Price of Anarchy for Random Coordination Games

We now turn to our bounds for random coordination games. Recall that for random graphs we consider equal-split distribution rules only. We first show that for sparse random graphs the Price of Anarchy is constant with high probability.

**Corollary 3 (Sparse random coordination games).** *Let $d > 0$ be a constant. Let $\mathcal{G}_{G_n}$ be the set of all symmetric coordination games $\Gamma = (G_n, c, \mathbf{1}, w, q)$ on graph $G_n \sim G(n, d/n)$ with equal-split distribution rule. Then there is a constant $\beta = \beta(d)$ such that $PoA\,(\mathcal{G}_{G_n}) \to \beta$.*

*Proof.* The maximum subgraph density of a random graph $G_n$ approaches a constant $\beta = \beta(d)$ with high probability [2] (see [17] for approximations of this constant). Combining this with the bound in Corollary 2 proves the claim. □

As we show in Theorem 3, the result of Corollary 3 does not hold for sufficiently dense random graphs if the number of available colors grows large.

**Theorem 3 (Dense random coordination games).** *Let $0 < d \leq 1$ be a constant and let $(c_n)_{n\in\mathbb{N}} \to \infty$ be a sequence of available colors. Let $\mathcal{G}_{G_n}(c_n)$ be the set of all symmetric coordination games $\Gamma = (G_n, c_n, \mathbf{1}, w, \mathbf{0})$ on graph $G_n \sim G(n,d)$ with $c_n$ colors, equal-split distribution rule and no individual preferences. Then there is a constant $\beta = \beta(d)$ such that $PoA\left(\mathcal{G}_{G_n}(c_n)\right) \geq \beta c_n$.*

We note that this lower bound holds even for coordination games without individual preferences (as studied in [11]). Basically, this bound implies that for dense graph topologies we cannot significantly improve upon the Price of Anarchy bound of $c$ by [3,11], even if we randomize the graph topology.

*Proof (Theorem 3).* We first construct a deterministic instance $\Gamma$ with Price of Anarchy $\Omega(c_n)$ and then show that we can embed this construction into a random graph with high probability.

Consider a graph $G = (V, E)$ and let $c$ be the number of available colors. Let $M = \{e_1, \dots, e_q\} \subseteq E$ be a matching of size at most $c$. Let $V_M$ be the set of nodes which are matched in $M$. Define the weight of an edge $e \in E$ as $w(e) = 2$ if $e \in M$, $w(e) = 1$ if precisely one of $e$'s endpoints is matched in $M$, and $w(e) = 0$ otherwise.

Consider the strategy profile $s$ in which the nodes adjacent to $e_i$ play color $i$, for $i = 1, \dots, q$. Note that this is possible because $q \leq c$ by assumption. All other nodes play an arbitrary color; these nodes are irrelevant as all the edges that they are adjacent to have weight zero. In a social optimum $s^*$ all players choose a common color. It follows that $\text{PoA}(\Gamma) \geq |E[V_M]|/(2q)$, where $|E[V_M]|$ is the number of edges in the induced subgraph of $V_M$. Note that all these edges have weight at least one.

Now, let $G_n = (V_n, E_n) \sim G(n, d)$ and assume without loss of generality that $V_n = \{1, \dots, n\}$. We claim that with high probability the induced subgraph on nodes $W_n = \{1, \dots, \lceil c_n/4 \rceil\}$ contains both $\Omega(c_n^2)$ edges and a perfect matching (if $\lceil c_n/4 \rceil$ is odd, we consider the first $\lceil c_n/4 \rceil + 1$ nodes).[8]

The first claim follows from standard arguments. Note that $\mu = \mathbb{E}\{E_n[W_n]\} = d\binom{\lceil c_n/4 \rceil}{2} = \Omega(c_n^2)$. Using Chernoff's bound, it follows that $\mathbb{P}\{E_n[W_n] < \mu/2)\} \leq \exp(-\mu/8) = \exp(-\Omega(c_n^2)/8) \to 0$ as $n \to \infty$ as $(c_n) \to \infty$. The second claim relies on the following result (see, e.g., [12]): For every fixed $0 < d \leq 1$ it holds that $\lim_{n\to\infty} \mathbb{P}_{G_n \sim G(n,d)}\{G_n \text{ contains a perfect matching}\} = 1$. By applying this result to the induced subgraph on $W_n$ and using that $c_n$ approaches infinity as $n \to \infty$, the claim follows.[9]

[8] One may focus on any set of $\lceil c_n/4 \rceil$ nodes. The important thing to note is that we need a set of nodes with many edges on its induced subgraph *and* a perfect matching (it is not sufficient to find two different sets each satisfying one of these properties). Moreover, if $c_n \geq 4n$, we consider $W_n = \{1, \dots, n\}$ and then the same argument works.

[9] Note that here we implicitly use that the intersection of two probabilistic events which occur with high probability also occurs with high probability.

Combining this with the deterministic bound on the Price of Anarchy derived above concludes the proof. □

## 4 Convergence of Best-Response Dynamics

We provide a characterization of distribution rules that guarantee the convergence of best-response dynamics in symmetric clustering games.

**Theorem 4 (Best-response convergence).** *Let $\mathcal{G}_{G,c,\alpha}$ be the set of all symmetric clustering games $\Gamma = (G, c, \alpha, w, q)$ on a fixed graph $G$ with $c$ common colors and distribution rule $\alpha$. Then best-response dynamics are guaranteed to converge to a pure Nash equilibrium for every clustering game in $\mathcal{G}_{G,c,\alpha}$ if and only if $\alpha$ corresponds to a generalized weighted Shapley distribution rule.*

*Remark 1.* Theorem 4 remains valid also for various settings without individual preferences. For example, this holds for coordination games (corresponding to certain models in [3,11]) and for general clustering games with $c = 2$.[10]

For symmetric coordination games with $c \geq 3$ colors, we can strengthen the condition in Theorem 4 to "guaranteed existence of a pure Nash equilibrium", which complements the result in [3] (details will be given in the full version).

## 5 Results for Asymmetric Clustering Games

We give an overview of our results for asymmetric clustering games. We focus on coordination games with equal-split distribution rule and no individual preferences.

Apt et al. [4] show that the Price of Anarchy of coordination games is unbounded if $c \geq n + 1$; notably, this holds for arbitrary graph topologies. We slightly generalize this observation by showing that the Price of Anarchy is unbounded if and only if $c \geq \chi(G) + 1$, where $\chi(G)$ is the chromatic number of $G$. We exploit this insight to prove that if the number of colors $c$ is a constant then the Price of Anarchy is unbounded for sparse random graphs, while it is bounded by some constant for dense random graphs (details will be given in the full version).

Subsequently, we focus on the Price of Anarchy of $\epsilon$-*approximate* $k$-*strong equilibria*, called $(\epsilon, k)$-*equilibria* for short.[11] The Price of Anarchy naturally

[10] In general, this is not true if $c \geq 3$. For example, consider a cycle of length three with only anti-coordination edges.

[11] A strategy profile $s$ is an $(\epsilon, k)$-*equilibrium* with $\epsilon \geq 1$ and $k \in [n]$ if for every set of players $K \subseteq V$ with $|K| \leq k$ and every deviation $s'_K = (s'_i)_{i \in K}$, there is at least one player $j \in K$ such that $\epsilon \cdot u_j(s) \geq u_j(s_{-K}, s'_K)$. We turn to $(\epsilon, k)$-equilibria because pure Nash equilibria are not guaranteed to exist in asymmetric coordination games (see, e.g., [4]).

extends to the set of $(\epsilon, k)$-equilibria. It is known that the $(\epsilon, k)$-PoA of coordination games is between $2\epsilon(n-1)/(k-1)+1-2\epsilon$ and $2\epsilon(n-1)/(k-1)$ for $k \geq 2$ [21]. In particular, the Price of Anarchy grows like $\Theta(\epsilon n)$ if $k$ is a constant.

We derive a topological bound on the $(\epsilon, k)$-Price of Anarchy which depends on the maximum degree $\Delta(G)$ of the graph $G$.

**Theorem 5 (Degree bound).** *Let $\epsilon \geq 1$, $k \geq 2$, $c \geq 3$, and let $G$ be an arbitrary graph. Let $\mathcal{G}_G(c)$ be the set of all coordination games $\Gamma = (G, c, (S_i), \mathbf{1}, w, \mathbf{0})$ on graph $G$ with $c$ colors, equal-split distribution rule and no individual preferences. Then $\epsilon \cdot \max\{1, \Delta(G)/(k-1) - 1\} \leq (\epsilon, k)$-PoA$(\mathcal{G}_G(c)) \leq 2\epsilon \cdot \Delta(G)$.*

We use this result to bound the $(\epsilon, k)$-Price of Anarchy for random graphs. It is known that the maximum degree of a dense random graph is $\Theta(n)$ (see, e.g., [12]). So for these graphs the $(\epsilon, k)$-Price of Anarchy still grows like $\Omega(\epsilon n)$ (as in the worst case). In contrast, we obtain an improved bound for sparse random graphs.

**Theorem 6.** *Let $\epsilon \geq 1$, $k \geq 2$ and $d > 0$ be constants. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of integers with $c_n \geq 3$ for all $n$. Let $\mathcal{G}_{G_n}(c_n)$ be the set of all coordination games $\Gamma = (G_n, c_n, (S_i), \mathbf{1}, w, \mathbf{0})$ on graph $G_n \sim G(n, d/n)$ with $c_n$ colors, equal-split distribution rule and no individual preferences. Then $(\epsilon, k)$-PoA$(\mathcal{G}_{G_n}(c_n)) = \Theta(\epsilon \ln(n)/\ln\ln(n))$.*

If, in addition, the strategy sets are drawn according to a sequence of distributions that satisfy the so-called *common color property*, and all weights are equal to one (corresponding to the games studied in [4]), then we can even prove that the $(\epsilon, k)$-Price of Anarchy is bounded by a constant. Intuitively, the common color property requires that with positive probability any two players have a color in common in their strategy sets.[12] In particular, this condition is satisfied if we draw the strategy sets uniformly at random from $2^{[c]} \setminus \emptyset$.

**Theorem 7.** *Let $\epsilon \geq 1$, $k \geq 2$ and $d > 0$ be constants. Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of integers with $c_n \geq 3$ for all $n$ and let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a sequence of strategy set distributions satisfying the common color property. Let $\mathcal{G}_{G_n,(S_i)}(c_n)$ be the set of all coordination games $\Gamma = (G_n, c_n, (S_i), \mathbf{1}, \mathbf{1}, \mathbf{0})$ on graph $G_n \sim G(n, d/n)$ with $c_n$ colors, strategy set $S_i \sim \mathcal{F}_n$ for every $i$, equal-split distribution rule, unit weights and no individual preferences. Then there exists a constant $\beta = \beta(d, \epsilon)$ such that $(\epsilon, k)$-PoA$(\mathcal{G}_{G_n,(S_i)}(c_n)) \leq \beta$.*

Theorem 7 does not hold for $k = 1$. To see this, consider the uniform distribution over strategy sets $\{s_0, s_1\}, \dots, \{s_0, s_n\}$. In the strategy profile where every player picks her color different from $s_0$, at most a constant number of edges will be satisfied with high probability. Thus, $(\epsilon, 1)$-PoA $\geq \beta n$ for some $\beta$ with high probability.

**Acknowledgements.** The first author thanks Remco van der Hofstad for a helpful discussion on random graph theory and, in particular, the results in [2].

[12] Note that in the deterministic setting the Price of Anarchy does not improve if all players have a color in common (see [21]).

## References


1. Amiet, B., Collevecchio, A., Scarsini, M.: Pure nash equilibria and best-response dynamics in random games. CoRR abs/1905.10758 (2019)
2. Anantharam, V., Salez, J.: The densest subgraph problem in sparse random graphs. Ann. Appl. Probab. **26**(1), 305–327 (2016)
3. Anshelevich, E., Sekar, S.: Approximate equilibrium and incentivizing social coordination. In: Proceedings of the 28th AAAI Conference on Artificial Intelligence, pp. 508–514 (2014)
4. Apt, K.R., de Keijzer, B., Rahn, M., Schäfer, G., Simon, S.: Coordination games on graphs. Int. J. Game Theory **46**(3), 851–877 (2017)
5. Bárány, I., Vempala, S., Vetta, A.: Nash equilibria in random games. Random Struct. Algor. **31**(4), 391–405 (2007)
6. Bilò, V., Fanelli, A., Flammini, M., Monaco, G., Moscardelli, L.: Nash stable outcomes in fractional hedonic games: existence, efficiency and computation. J. Artif. Intell. Res. **62**, 315–371 (2018)
7. Carosi, R., Flammini, M., Monaco, G.: Computing approximate pure nash equilibria in digraph k-coloring games. In: Proceedings of the 16th Conference on Autonomous Agents and Multi Agent Systems, pp. 911–919 (2017)
8. Carosi, R., Monaco, G.: Generalized graph $k$-coloring games. In: Wang, L., Zhu, D. (eds.) COCOON 2018. LNCS, vol. 10976, pp. 268–279. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-94776-1_23
9. Chen, H.L., Roughgarden, T., Valiant, G.: Designing network protocols for good equilibria. SIAM J. Comput. **39**(5), 1799–1832 (2010)
10. Drèze, J.H., Greenberg, J.: Hedonic coalitions: optimality and stability. Econometrica **48**(4), 987–1003 (1980)
11. Feldman, M., Friedler, O.: A unified framework for strong price of anarchy in clustering games. In: Halldórsson, M.M., Iwama, K., Kobayashi, N., Speckmann, B. (eds.) ICALP 2015. LNCS, vol. 9135, pp. 601–613. Springer, Heidelberg (2015). https://doi.org/10.1007/978-3-662-47666-6_48
12. Frieze, A., Karoński, M.: Introduction to Random Graphs. Cambridge University Press, Cambridge (2015)
13. Gilbert, E.N.: Random graphs. Ann. Math. Stat. **30**(4), 1141–1144 (1959)
14. Gopalakrishnan, R., Marden, J.R., Wierman, A.: Potential games are necessary to ensure pure nash equilibria in cost sharing games. Math. Oper. Res. **39**(4), 1252–1296 (2014)
15. Gourvès, L., Monnot, J.: On strong equilibria in the max cut game. In: Leonardi, S. (ed.) WINE 2009. LNCS, vol. 5929, pp. 608–615. Springer, Heidelberg (2009). https://doi.org/10.1007/978-3-642-10841-9_62
16. Gourvès, L., Monnot, J.: The max $k$-cut game and its strong equilibria. In: Kratochvíl, J., Li, A., Fiala, J., Kolman, P. (eds.) TAMC 2010. LNCS, vol. 6108, pp. 234–246. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-13562-0_22
17. Hajek, B.E.: Performance of global load balancing of local adjustment. IEEE Trans. Inf. Theory **36**(6), 1398–1414 (1990)
18. Hoefer, M.: Cost sharing and clustering under distributed competition. Ph.D. thesis (2007)
19. Koutsoupias, E., Papadimitriou, C.: Worst-case equilibria. In: Meinel, C., Tison, S. (eds.) STACS 1999. LNCS, vol. 1563, pp. 404–413. Springer, Heidelberg (1999). https://doi.org/10.1007/3-540-49116-3_38

20. Kun, J., Powers, B., Reyzin, L.: Anti-coordination games and stable graph colorings. In: Vöcking, B. (ed.) SAGT 2013. LNCS, vol. 8146, pp. 122–133. Springer, Heidelberg (2013). https://doi.org/10.1007/978-3-642-41392-6_11
21. Rahn, M., Schäfer, G.: Efficient equilibria in polymatrix coordination games. In: Italiano, G.F., Pighizzini, G., Sannella, D.T. (eds.) MFCS 2015. LNCS, vol. 9235, pp. 529–541. Springer, Heidelberg (2015). https://doi.org/10.1007/978-3-662-48054-0_44
22. Roughgarden, T., Tardos, E.: How bad is selfish routing? J. ACM **49**(2), 236–259 (2002)
23. Valiant, G., Roughgarden, T.: Braess's paradox in large random graphs. Random Struct. Algor. **37**(4), 495–515 (2010)