ALGEBRAIC GROUPS WITH A COMMUTING PAIR OF INVOLUTIONS AND SEMISIMPLE SYMMETRIC SPACES

A.G. HELMINCK

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ALGEBRAISCHE GROEPEN MET EEN PAAR COMMUTERENDE INVOLUTIES
EN SEMISIMPELE SYMMETRISCHE RUIMTEN

(MET EEN SAMENVATTING IN HET NEDERLANDS)

PROEFSCHRIFT

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ALOYSIUS GERARDUS HELMINCK

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Introduction.

Let G be a connected reductive algebraic group defined over an algebraically closed field F of characteristic not 2. Denote the Lie algebra of G by $\mathfrak{g}$.

In this paper we shall classify the isomorphism classes of ordered pairs of commuting involutorial automorphisms of G. This is shown to be independent of the characteristic of F and can be applied to describe all locally semisimple symmetric spaces together with their fine structure.

Involutorial automorphisms of $\mathfrak{g}$ occur on several places in the literature. Cartan already showed that for $F=\mathbf{C}$, the isomorphism classes of involutorial automorphisms of $\mathfrak{g}$ correspond bijectively to the isomorphism classes of real semisimple Lie algebras, which correspond in their turn to the isomorphism classes of Riemannian symmetric spaces (see Helgason [11]). If one lifts this involution to the group G, then the present work gives a characteristic free description of these isomorphism classes. In a similar manner locally semisimple symmetric spaces correspond to pairs of commuting involutorial automorphisms of $\mathfrak{g}$. Namely let $(\mathfrak{g}_o,\sigma)$ be a locally semisimple symmetric pair, i.e. $\mathfrak{g}_o$ is a real semisimple Lie algebra and $\sigma\in\mathrm{Aut}(\mathfrak{g}_o)$ an involution. By a result of Berger [2], there exists for every involution $\sigma\in\mathrm{Aut}(\mathfrak{g}_o)$ a Cartan involution θ of $\mathfrak{g}_o$, such that $\sigma\theta=\theta\sigma$. These involutions induce a pair of commuting involutions of $\mathfrak{g}$. Conversely, if $\sigma,\theta\in\mathrm{Aut}(\mathfrak{g})$ are commuting involutions, then σ and θ determine two locally semisimple symmetric pairs. For if $\mathfrak{u}$ is a σ- and θ-stable compact real form with conjugation τ, then $(\mathfrak{g}_{\theta\tau},\sigma|_{\mathfrak{g}_{\theta\tau}})$ and $(\mathfrak{g}_{\sigma\tau},\theta|_{\mathfrak{g}_{\sigma\tau}})$ are locally semisimple symmetric pairs where

$$\mathfrak{g}_{\theta\tau} = \{X\in\mathfrak{g}|\theta\tau(X)=X\} \text{ and } \mathfrak{g}_{\sigma\tau} = \{X\in\mathfrak{g}|\sigma\tau(X)=X\}.$$

These pairs are called *dual*. To get a correspondence with these locally symmetric pairs we consider ordered pairs of involutions and let (θ,σ) correspond to the first and (σ,θ) to the second.

So let $\sigma,\theta\in\mathrm{Aut}(G)$ be commuting involutions and let G_σ resp. G_θ denote the group of fixed points of σ resp. θ. For a σ- and θ-stable torus T of G we write

$$T_\theta^+ = (T\cap G_\theta)^o \text{ and } T_\theta^- = \{t\in T|\theta(t) = t^{-1}\}^o.$$

The second torus is called a *θ-split* torus of G. Similarly we define T_σ^+ and T_σ^-. The torus $(T_\sigma^-)_\theta^- = \{t\in T|\sigma(t)=\theta(t)=t^{-1}\}^o$ is called *(σ,θ)-split* and is denoted by $T_{\sigma,\theta}^-$. Denote the set of characters, the set of roots and the Weyl group of T with respect to G by respectively, $X^*(T)$, $\Phi(T)$ and W(T). We use the notation $\mathcal{C}$ for the set of Int(G)-isomorphism classes of ordered pairs of commuting involutions of G and the notation $\mathcal{C}(T,W)$ for the set of W(T)-conjugacy classes of ordered pairs of commuting involutions of $(X^*(T),\Phi(T))$, where T is a maximal torus of G.

The idea is now to construct a map from $\mathcal{C}$ to $\mathcal{C}(T,W)$ (for a fixed maximal torus T) and to classify its image and the fibers. In order to construct such a map one could take in any class c of $\mathcal{C}$ a representative (σ,θ) such that T is σ- and θ-stable and consider the W(T)-conjugacy class of $(\sigma|_T,\theta|_T)$. However this leaves too much freedom for the choice of (σ,θ). Different representatives of the class c in $\mathcal{C}$, stabilizing T, can induce different classes in $\mathcal{C}(T,W)$. Hence we have to demand more properties of the representative.

In the case of a single involution (i.e. $\sigma = \theta$) one has two possible choices. Namely one can require of the representative (θ,θ) of c that T_θ^+ is a maximal torus of

G_θ or that T_θ^- is a maximal θ-split torus of G. Cartan used the first choice to classify the real semisimple Lie algebras. For Riemannian symmetric spaces however the second choice is more natural, because there is a natural notion of roots of symmetric spaces. These coincide with the non-zero restrictions of roots of $\Phi(T)$ to T_θ^-. Moreover this second characterization determines also the restricted root system together with multiplicities of the root spaces. Araki [1] followed this method to classify the Riemannian symmetric spaces.

To classify the locally semisimple symmetric pairs, Berger [2] made a choice analogous to Cartan, but did not obtain any results concerning the fine structure of those spaces. Our choice is similar to that of Araki. To be more specific, we call a pair (σ,θ) *normally related to* T if T is σ- and θ-stable and if $T_{\sigma,\theta}^-$, T_σ^-, T_θ^- are respectively maximal (σ,θ)-split, σ-split and θ-split. As in the case of a single involution, $\Phi(T_{\sigma,\theta}^-)$ is the natural root system of the corresponding symmetric pair. Every class in $\mathcal{C}$ contains a pair (σ,θ) which is normally related T (see (5.19)). Denoting the center of G by Z(G), we have furthermore:

5.16. Theorem. *Let* (σ_1,θ_1) *and* (σ_2,θ_2) *be pairs of involutorial automorphisms of* G, *normally related to* T. *Then* $(\sigma_1,\theta_1)|_T$ *and* $(\sigma_2,\theta_2)|_T$ *are conjugate under* W(T) *if and only if there exists* $\epsilon \in T_{\sigma,\theta}^-$ *with* $\epsilon^2 \in Z(G)$ *such that* (σ_2,θ_2) *is isomorphic to* $(\sigma_1,\theta_1 \mathrm{Int}(\epsilon))$.

The elements $\epsilon \in T_{\sigma,\theta}^-$ such that $\epsilon^2 \in Z(G)$ are called *quadratic elements of* $T_{\sigma,\theta}^-$. We can define now a mapping

$$\rho : \mathcal{C} \rightarrow \mathcal{Q}(W,T)$$

(see (5.19)). Denote the image of ρ by A and the fibers above $\rho((\sigma,\theta))$ by $\mathcal{Q}(\sigma,\theta)$. The ordered pairs of commuting involutions of $(X^*(T),\Phi(T))$, whose class in $\mathcal{Q}(T,W)$ is contained in A, are called admissible.

The W(T)-conjugacy classes of admissible pairs of commuting involutions of $(X^*(T),\Phi(T))$ can be described by a diagram, which can be obtained by gluing together two diagrams of admissible involutions under a combinatorical condition on the simple roots (see (7.11) and (7.16)). From this one obtains all the fine structure of the corresponding locally semisimple symmetric pair. As to the classification of the classes in $\mathcal{Q}(\sigma,\theta)$, it suffices to give a set of quadratic elements of a maximal (σ,θ)-split torus A of G, representing the classes in $\mathcal{Q}(\sigma,\theta)$. These quadratic elements can be described by using a basis of $\Phi(A)$. Namely let $\bar{\Delta}$ be a basis of $\Phi(A)$ and $\{\gamma_\lambda\}_{\lambda\in\bar{\Delta}}$ a dual basis in $X_*(A)$, the set of multiplicative one parameter subgroups of A. If $\epsilon_\lambda = \gamma_\lambda(-1)$, $\lambda\in\bar{\Delta}$, then $\epsilon_\lambda^2 = e$. There exists a subset $\Delta_1 \subset \bar{\Delta}$ such that $\{\epsilon_\lambda\}_{\lambda\in\Delta_1}$ is a set of quadratic elements representing the classes in $\mathcal{Q}(\sigma,\theta)$. This subset Δ_1 of $\bar{\Delta}$ is determined by the action of the restricted Weyl group $\underline{W}(A)$ on the group of quadratic elements of A and the signatures of the roots in $\bar{\Delta}$ (see (8.13) and (8.25)). This completes the classification.

Finally we note that every class $\mathcal{Q}(\sigma,\theta)$ contains a unique class of standard pairs (see (6.13)). This seems to be the natural class to start with in the analysis on these symmetric spaces. For example, if $\sigma=\theta$, then the standard pair in $\mathcal{Q}(\theta,\theta)$ is (θ,θ), which corresponds to a Riemannian symmetric space and the other pairs in $\mathcal{Q}(\theta,\theta)$ correspond to the K_ϵ-spaces described in [18].

A brief summary of the contents is as follows. After some preliminaries in section 1, we derive all the properties needed about root systems with involutions in section 2. The sections 3 and 4 deal with the classification of single involutorial automorphisms. The method of classification, presented here, simplifies the work of Araki [1] and Sugiura [22,appendix]. In section 5 we characterize the isomorphism classes of pairs of commuting involutions on a maximal torus as above. In section 6 we show that for a maximal (σ,θ)-split torus A of G, the set $\Phi(A)$ is a root system and introduce the standard pair. The classification of admissible pairs of commuting involutions of $(X^*(T),\Phi(T))$ is treated in section 7,where also all the fine structure is derived. A set of quadratic elements characterizing the classes in $\mathcal{C}(\sigma,\theta)$ is given in section 8. In section 9 we discuss the relations between ordered pairs of commuting involutions and locally semisimple symmetric spaces.

Recently Oshima-Sekiguchi [19] also determined the restricted root system of a locally semisimple symmetric pair, based on the classification of Berger. Some of this fine structure of a locally semisimple symmetric pair can be found also in Hoogenboom [13].

1. Preliminaries and recollections.

1.1. Let F denote an algebraically closed field of characteristic$\neq 2$. We use as our basic references for algebraic groups the books of Humphreys [14] and Springer [24] and we shall follow their notations and terminology. Throughout this paper G will denote a connected reductive linear algebraic group, defined over F. For any closed subgroup H of G, denote its Lie algebra by the corresponding (lower case) German letter $\mathfrak{h}$ and write H^o for the identity component. The center of H will be denoted by Z(H).

For a subtorus T of H let $X^*(T)$ denote the additively written group of rational characters of T and $X_*(T)$ the group of rational one-parameter multiplicative subgroups of T, i.e. the group of homomorphisms (of algebraic groups) : $GL_1 \to T$. $X^*(T)$ can be put in duality with $X_*(T)$ by a pairing $<.,.>$ defined as follows: if $\chi \in X^*(T)$, $\lambda \in X_*(T)$, then $\chi(\lambda(t)) = t^{<\chi,\lambda>}$ for all $t \in F^*$. The torus T acts on the Lie algebra $\mathfrak{h}$ of H by the adjoint representation. For $\alpha \in X^*(T)$ let $\mathfrak{h}_\alpha$ denote the weight space for the character α on $\mathfrak{h}$ and let $\Phi(T,H)$ denote *the set of roots of* H *with respect to* T, i.e. $\Phi(T,H)$ is the set of non-trivial characters $\alpha \in X^*(T)$ such that $\mathfrak{h}_\alpha \neq 0$. Set $W(T,H) = N_H(T)/Z_H(T)$, where

$$N_H(T) = \{x \in H \,|\, xTx^{-1} \subset T\},$$

$$Z_H(T) = \{x \in H \,|\, xt = tx \text{ for all } t \in T\}.$$

If H is connected, W(T,H) is called the *Weyl group of* H *relative to* T. It is a finite group, which acts on T, $X^*(T)$ and $X_*(T)$. Moreover the set of roots $\Phi(T,H)$ is stable under the action of W(T,H) on $X^*(T)$. In the case H = G we shall write $\Phi(T)$ for $\Phi(T,G)$ and W(T) for W(T,G).

If T is a torus of G such that $\Phi(T)$ is a root system in the subspace of $X^*(T)\otimes_{\mathbb{Z}}\mathbb{R}$ spanned by $\Phi(T)$ and if W(T) is the corresponding Weyl group, then for each $\alpha \in \Phi(T)$ the subgroup $G_\alpha = Z_G((\mathrm{Ker}\alpha)^o)$ is nonsolvable. If we choose now $n_\alpha \in N_{G_\alpha}(T) - Z_{G_\alpha}(T)$ and let s_α be the element of W(T) defined by n_α, then there exists a unique one parameter subgroup $\alpha^\vee \in X_*(T)$ such that $<\alpha,\alpha^\vee> = 2$ and $s_\alpha(\chi) = \chi - <\chi,\alpha^\vee>\alpha \quad (\chi \in X^*(T))$. We call $\alpha^\vee$ the *coroot* of α and denote the set of these $\alpha^\vee$ in $X_*(T)$ by $\Phi(T)^\vee$. We have a bijection of $\Phi(T)$ onto $\Phi(T)^\vee$.

For $x,y\in G$ denote the commutator $xyx^{-1}y^{-1}$ by (x,y). If A, B are subgroups of G, the subgroup of G generated by all (x,y), $x\in A$, $y\in B$ will be denoted by (A,B).

1.2. Involutorial automorphisms of G.

Let $\theta\in \mathrm{Aut}(G)$ be an involutorial automorphism of G, i.e. $\theta^2 = \mathrm{id}$. We denote the automorphism of $\mathfrak{g}$, induced by θ also by θ and write $K = G_\theta = \{x\in G \mid \theta(x) = x\}$ for the group of fixed points of θ. This is a closed, reductive subgroup of G (see Vust [31,§1]). If $F = \mathbb{C}$ then G/K is the complexification of a space $G(\mathbb{R})/K(\mathbb{R})$ with $G(\mathbb{R})$-invariant Riemannian structure. Here $G(\mathbb{R})$ (resp. $K(\mathbb{R})$) denotes the set of $\mathbb{R}$-rational points of G (resp. K.)

For a θ-stable subgroup H of G let $S_\theta(H) = \{h\theta(h)^{-1} \mid h\in H\}$. In the case $H = G$, we shall also write S_θ (or S) for $S_\theta(G)$. The group G acts transitively on S_θ by $g{\star}x = gx\theta(g)^{-1}$.

1.3. Proposition. *S_θ is a closed connected subvariety of G and the map $g\to g{\star}e$ induces an isomorphism of affine G-varieties: $G/K \to S_\theta$.*
This is proved in Richardson [20,2.4]

1.4. θ-split tori.

Let T be a θ-stable torus of G. (Recall that according to a result of Steinberg [27,7.5], there exists a θ-stable torus T of G.) If we write $T_\theta^+ = (T\cap K)^\circ$ and $T_\theta^- = \{x\in T \mid \theta(x) = x^{-1}\}^\circ$, then it is easy to verify that the product map

$$\mu\colon T_\theta^+ \times T_\theta^- \to T, \quad \mu(t_1,t_2) = t_1 t_2$$

is a separable isogeny. So in particular $T = T_\theta^+ . T_\theta^-$ and $T_\theta^+ \cap T_\theta^-$ is a finite group. (In fact it is an elementary abelian 2-group.) If T is a torus in a θ-stable subgroup H of G, then the automorphisms of $\Phi(T,H)$ and $W(T,H)$ induced by $\theta|_H$ will also be denoted by θ.

A torus A of G is called *θ-split* if $\theta(a) = a^{-1}$ for every $a\in A$. These tori are called θ-anisotropic in Vust [31] and Richardson [20]. We prefer the former terminology, because if $F = \mathbb{C}$, then A is a split torus, defined over $\mathbb{R}$, with respect to the real structure defined by $\theta\tau$, where τ is the complex conjugation with respect to a compact real form of G invariant under θ.

If $\theta \neq \mathrm{id}$, then non-trivial θ-split tori exist (see Vust [31,§1]), so in particular there are maximal ones. The following result can be found in Vust [31,§1]:

1.5. Proposition. *Let A be a maximal split torus of G. Then:*

(1) A is the unique θ-split torus of $Z_G(A)$;

(2) $(Z_G(A),Z_G(A))\subset K^\circ$ and $Z_G(A)$ is the almost direct product of $Z_K(A)^\circ$ and A;

(3) If T is a maximal θ-split torus of G, containing A, then T is θ-stable.

Moreover all maximal θ-split tori of G are conjugate under K° and so are all maximal tori of G containing a maximal θ-split torus of G.

1.6. Proposition. *Let A be a maximal θ-split torus of G and let* E_o *denote the vectorsubspace of* $X^*(A)\otimes_{\mathbb{Z}}\mathbb{R}$ *spanned by* $\Phi(A)$. *Then* $\Phi(A)$ *is a root system in* E_o *and the corresponding Weyl group is given by the restriction of* W(A) *to* E_o . *Moreover every element of* W(A) *has a representative in* $N_{K^\circ}(A)$.

For a proof, see Richardson [20,4.7].

Note that if T is a maximal torus of G containing A, then $\Phi(A)$ coincides with the set of restrictions of the elements of $\Phi(T)$ to A.

2. Involutions of root data.

To deal with the notion of root system in reductive groups it is quite useful to work with the notion of root datum (see Springer [23,§1]).

2.1. Root data

A *root datum* is a quadruple $\Psi = (X,\Phi,X^\vee,\Phi^\vee)$, where X and $X^\vee$ are free abelian groups of finite rank, in duality by a pairing $X\times X^\vee \to \mathbb{Z}$, denoted by $<.,.>$, Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$ with a bijection $\alpha\to\alpha^\vee$ of Φ onto $\Phi^\vee$. If $\alpha\in\Phi$ we define endomorphisms s_α and $s_{\alpha^\vee}$ of X and $X^\vee$, respectively, by

$$s_\alpha(\chi) = \chi - <\chi,\alpha^\vee>\alpha, \quad s_{\alpha^\vee}(\lambda) = \lambda - <\alpha,\lambda>\alpha^\vee$$

The following two axioms are imposed:

(1): If $\alpha\in\Phi$, then $<\alpha,\alpha^\vee> = 2$;

(2): if $\alpha\in\Phi$, then $s_\alpha(\Phi) \subset \Phi$, $s_{\alpha^\vee}(\Phi^\vee) \subset \Phi^\vee$.

It follows from (1), that $s_\alpha^2 = 1$, $s_\alpha(\alpha) = -\alpha$ and similarly for $s_\alpha^\vee$. Let Q be the subgroup of X generated by Φ and put $V = Q\otimes_{\mathbb{Z}}\mathbb{R}$, $E = X\otimes_{\mathbb{Z}}\mathbb{R}$. Consider V as a linear subspace of E. Define similarly the subgroup $Q^\vee$ of $X^\vee$ and the vector space $V^\vee$. If $\Phi \neq \varnothing$, then Φ is a non necessarily reduced root system in V in the sense of Bourbaki [5,Ch.VI, no. 1]. The rank of Φ is by definition the dimension of V. The root datum Ψ is called semi simple if $X\subset V$. We observe that $s_{\alpha^\vee} = {}^t s_\alpha$ and $s_\alpha(\beta)^\vee = s_{\alpha^\vee}(\beta^\vee)$ as follows by an easy computation (c.f. Springer[23,1.4]).
Let (.,.) be a positive definite symmetric bilinear form on E, which is Aut(Φ) invariant. Now the s_α $(\alpha\in\Phi)$ are Euclidean reflections, so we have

$$<\chi,\alpha^\vee> = 2(\alpha,\alpha)^{-1}.(\chi,\alpha) \qquad (\chi\in E,\ \alpha\in\Phi).$$

Consequently, we can identify $\Phi^\vee$ with the set $\{2(\alpha,\alpha)^{-1}\alpha \mid \alpha\in\Phi\}$ and $\alpha^\vee$ with $2(\alpha,\alpha)^{-1}\alpha$. If $\phi\in$Aut(X,Φ), then its transpose ${}^t\phi$ induces an automorphism of $\Phi^\vee$, so ϕ induces a unique automorphism in Aut(Ψ), the set of automorphisms of the root datum Ψ. We shall frequently identify Aut(X,Φ) and Aut(Ψ).

For any closed subsystem Φ_1 of Φ let W(Φ_1) denote the finite group generated by the s_α for $\alpha\in\Phi_1$.

2.1.1. Example. If T is a torus in a reductive group G, such that $\Phi(T)$ is a root system with Weyl group W(T), then the root datum associated to the pair (G,T) is: $(X^*(T),\Phi(T),X_*(T),\Phi^\vee(T))$, where $X^*(T)$, $\Phi(T)$, $X_*(T)$ and $\Phi^\vee(T)$ are as defined in (1.1).
In particular if T is a maximal torus of G or T = A a maximal θ-split torus of G, like in (1.6), then the above root datum exists.

2.2. Involutions

Let Ψ be a root datum with $\Phi \neq \emptyset$, as in (2.1), and let $\sigma,\theta \in \mathrm{Aut}(\Psi)$ be commuting involutions, i.e., $\sigma^2 = \theta^2 = id$, $\sigma\theta = \theta\sigma$. We now derive some properties of the set of restrictions of Φ to the common (-1)-eigenspace of σ and θ, which will play an important role in our classification.

Let $X_0(\sigma,\theta) = \{\chi \in X \mid \chi - \sigma(\chi) - \theta(\chi) + \sigma\theta(\chi) = 0\}$ and let $\Phi_0(\sigma,\theta) = \Phi \cap X_0(\sigma,\theta)$. Clearly $X_0(\sigma,\theta)$ and $\Phi_0(\sigma,\theta)$ are σ- and θ-stable and $\Phi_0(\sigma,\theta)$ is a closed subsystem of Φ. We denote the Weyl group of $\Phi_0(\sigma,\theta)$ by $W_0(\sigma,\theta)$ and identify it with the subgroup $W(\Phi_0(\sigma,\theta))$ of $W(\Phi)$. Put $W_1(\sigma,\theta) = \{w \in W(\Phi) \mid w(X_0(\sigma,\theta)) = X_0(\sigma,\theta)\}$, $\overline{X}_{\sigma,\theta} = X/X_0(\sigma,\theta)$ and let π be the natural projection from X to $\overline{X}_{\sigma,\theta}$. We frequently identify $\overline{X}_{\sigma,\theta}$ with $\{\chi \in X \mid \sigma(\chi) = \theta(\chi) = -\chi\}$, such that $\pi(\chi)$ corresponds to $\frac{1}{4}(\chi - \sigma(\chi) - \theta(\chi) + \sigma\theta(\chi))$. Every $w \in W_1(\sigma,\theta)$ induces an automorphism $\pi(w)$ of $\overline{X}_{\sigma,\theta}$ and $\pi(w\chi) = \pi(w)\pi(\chi)$ $(\chi \in X)$. If $\overline{W}_{\sigma,\theta} = \{\pi(w) | w \in W_1(\sigma,\theta)\}$, then $\overline{W}_{\sigma,\theta} \cong W_1(\sigma,\theta)/W_0(\sigma,\theta)$ (see Satake[22,2.1.3]). We call this the *restricted Weyl group* , with respect to the action of (σ,θ) on X. It is not necessarily a Weyl group in the sense of Bourbaki [5,Ch.VI,no.1].

Let $\overline{\Phi}_{\sigma,\theta} = \pi(\Phi - \Phi_0(\sigma,\theta))$ denote the set of *restricted roots of* Φ *relative to* (σ,θ). We shall mainly be concerned with the case that $\overline{\Phi}_{\sigma,\theta}$ is a root system with Weyl group $\overline{W}_{\sigma,\theta}$ (see e.g. (1.6), where $\sigma = \theta$).

2.3. Definition. An order $\succ$ on X is called a *(σ,θ)-order* if it has the following property:

$$\text{if } \chi \in X,\ \chi \succ 0 \text{ and } \chi \notin X_0(\sigma,\theta) \text{ then } \sigma(\chi) \prec 0 \text{ and } \theta(\chi) \prec 0$$

If $\succ$ is a (σ,θ)-order on X, then for $\chi \in X$ we have:

$$\chi \succ 0 \Leftrightarrow \text{either } \chi - \sigma(\chi) - \theta(\chi) + \sigma\theta(\chi) \succ 0 \text{ or } \pi(\chi) = 0 \text{ and } \chi \succ 0.$$

So a (σ,θ)-order on X induces orders on $X_0(\sigma,\theta)$ and $\overline{X}_{\sigma,\theta}$ and vice versa.
A basis Δ of Φ with respect to a (σ,θ)-order on X will be called a *(σ,θ)-basis* of Φ. We then write $\Delta_0(\sigma,\theta) = \Delta \cap \Phi_0(\sigma,\theta)$ and $\overline{\Delta}_{\sigma,\theta} = \pi(\Delta - \Delta_0(\sigma,\theta))$. (We call $\overline{\Delta}_{\sigma,\theta}$ *a restricted basis of* $\overline{\Phi}_{\sigma,\theta}$ with respect to Δ.) It is not hard to see that $\Delta_0(\sigma,\theta)$ is a basis of $\Phi_0(\sigma,\theta)$ and that a similar property holds for $\overline{\Delta}_{\sigma,\theta}$.

2.4. Lemma. *The elements of* $\overline{\Delta}_{\sigma,\theta}$ *are linearly independent. Moreover every* $\lambda \in \overline{\Phi}_{\sigma,\theta}$ *can be expressed uniquely in the form*

$$\lambda = \pm \sum_{\mu \in \overline{\Delta}_{\sigma,\theta}} m_\mu \mu \text{ with } m_\mu \in \mathbb{Z},\ m_\mu \geq 0.$$

For a proof see Satake [22,2.1.6]

Note that $W_1(\sigma,\theta)$ permutes the (σ,θ)-bases of Φ, i.e. if $w \in W_0(\sigma,\theta)$ and Δ is a (σ,θ)-basis of Φ, then $w(\Delta)$ is also a (σ,θ)-basis of Φ. Moreover $w \in W_0(\sigma,\theta)$ if and only if $\pi(w) = \mathrm{id}$. This is again equivalent to $\pi(w)(\overline{\Delta}_{\sigma,\theta}) = \overline{\Delta}_{\sigma,\theta}$ as is easily seen from the following useful result:

2.5. Lemma. *Let Δ,Δ' be (σ,θ)-bases of Φ such that $\overline{\Delta}_{\sigma,\theta} = \overline{\Delta}'_{\sigma,\theta}$. Then $\Delta' = w_o(\Delta)$, where $w_o \in W_o(\sigma,\theta)$ is the unique element such that $w_o(\Delta_o(\sigma,\theta)) = \Delta_o(\sigma,\theta)'$.*

For a proof see Satake [22,2.1.2]. The proof follows also immediately from the observation that a (σ,θ)-basis of Φ is completely determined by bases $\Delta_o(\sigma,\theta)$ resp. $\overline{\Delta}_{\sigma,\theta}$ of $\Phi_o(\sigma,\theta)$ resp. $\overline{\Phi}_{\sigma,\theta}$.

2.6. In case of a single involution we take $\sigma = \theta$ and we use the results stated above. Moreover we omit σ in the notations, i.e.we write $X_o(\theta)$, $\overline{X}_\theta$, $\Phi_o(\theta)$, $\overline{\Phi}_\theta$, $W_1(\theta)$, $\overline{W}_\theta$, $\Delta_o(\theta)$, $\overline{\Delta}_\theta$ instead of, respectively, $X_o(\theta,\theta)$, $\overline{X}_{\theta,\theta}$, $\Phi_o(\theta,\theta)$, $\overline{\Phi}_{\theta,\theta}$, $W_o(\theta,\theta)$, $W_1(\theta,\theta)$, $\overline{W}_{\theta,\theta}$, $\Delta_o(\theta,\theta)$, $\overline{\Delta}_{\theta,\theta}$. A (θ,θ)-order on X will be called a *θ-order* on X and a (θ,θ)-basis of Φ a *θ-basis* of Φ.

2.7. Relations between (restricted) Weyl groups.

Assume that (σ,θ) is a pair of commuting involutions of Φ such that $\overline{\Phi}_\theta$ is a root system with Weyl group $\overline{W}_\theta$. Then $\sigma|_{\overline{X}_\theta}$ is an involution of $(\overline{X}_\theta,\overline{\Phi}_\theta)$, so we can also view $\overline{\Phi}_{\sigma,\theta}$ as the set of restricted roots of $\overline{\Phi}_\theta$ with respect to $\sigma|_{\overline{X}_\theta}$ Denote the restriction of σ to $\overline{X}_\theta$ also by σ and let $W_1^\theta(\sigma,\theta) = \{w \in W_1(\sigma,\theta) | w(X_o(\theta)) = X_o(\theta)\} = \{w \in W_1(\theta) | w\theta = \theta w\}$. Put $\overline{W}_\theta^\sigma = W_1^\theta(\sigma,\theta)/W_o(\sigma,\theta)$. It is not hard to show that $\overline{W}_\theta^\sigma$ is isomorphic to the restricted Weyl group of $\overline{\Phi}_{\sigma,\theta}$ with respect to the action of σ on $\overline{X}_\theta$.(See (2.2).) However this will not be needed in the sequel.
In case $\overline{\Phi}_\sigma$ is a root system with Weyl group $\overline{W}_\sigma$ we define $W_1^\sigma(\sigma,\theta)$ and $\overline{W}_\sigma^\theta$ similarly. In section 6 we shall encounter the situation that $\overline{W}_\theta^\sigma$, $\overline{W}_\sigma^\theta$ and $\overline{W}_{\sigma,\theta}$ coincide and are equal to the Weyl group of $\overline{\Phi}_{\sigma,\theta}$.

2.8. A characterization of θ on a θ-basis of Φ.

In the remaining part of this section we restrict ourselves to the situation of a single involution $\theta \in \mathrm{Aut}(X,\Phi)$. Let Δ be a θ-basis of Φ. Then $\theta(-\Delta)$ is also a θ-basis of Φ with the same restricted basis, so by (2.5) there is $w_o(\theta) \in W_o(\theta)$ such that $w_o(\theta)\theta(\Delta) = -\Delta$. Here $w_o(\theta)$ is the longest element of $W_o(\theta)$ with respect to $\Delta_o(\theta)$. Put $\theta^* = \theta^*(\Delta) = -w_o(\theta).\theta$. Then $\theta^*(\Delta) \in \mathrm{Aut}(X,\Phi,\Delta) = \{\phi \in \mathrm{Aut}(X,\Phi) | \phi(\Delta) = \Delta\}$, $\theta^*(\Delta)^2 = \mathrm{id}$ and $\theta^*(\Delta_o(\theta)) = \Delta_o(\theta)$.

2.9. Remarks. (1) θ^* can be described by its action on the Dynkin diagram of Δ. Notice that

(a). if Φ is irreducible, then θ^* is either the identity or a diagram automorphism of order 2.

(b). if $\Phi = \Phi_1 \amalg \Phi_2$ with Φ_1, Φ_2 irreducible and $\theta(\Phi_1) = \Phi_2$, then θ^* exchanges the Dynkin diagrams of Φ_1 and Φ_2. In particular $\Phi_o(\theta) = \varnothing$, so $w_o(\theta) = \mathrm{id}$ and $\theta = -\theta^*$

(2) If $\theta = \mathrm{id}$ and Δ is a basis of Φ, then $\theta^*(\Delta) = -w_o(\mathrm{id})$ is called the *opposition involution* of Δ. In this case we shall also write $\mathrm{id}^*(\Delta)$ for $\theta^*(\Delta)$.

(3) If Φ is irreducible and Δ a basis of Φ, then the opposition involution is non-trivial if and only if Φ is either of type A_l $(l \geqslant 2)$, D_{2l+1} $(l \geqslant 2)$ or E_6.

(4) The action of θ^* on $\Delta_o(\theta)$ is determined by $\Delta_o(\theta)$, because $\theta^*|_{\Delta_o(\theta)} = -w_o(\theta)$ is the opposition involution of $\Delta_o(\theta)$, which is uniquely determined on each irreducible

component of $\Phi_o(\theta)$.

The diagram automorphism θ^* relates the simple roots in Δ, which are lying above a restricted root in $\overline{\Delta}_\theta$:

2.10. Lemma. *Let Δ be a θ-basis of Φ and $\alpha, \beta \in \Delta$, $\alpha \neq \beta$ such that $\pi(\alpha) = \pi(\beta)$. Then $\alpha = \theta^*(\beta)$.*

Proof. Working in V, we have $\pi(\alpha) = \frac{1}{2}(\alpha - \theta(\alpha)) = \frac{1}{2}(\beta - \theta(\beta))$, so

$$\alpha - \beta = \theta(\alpha - \beta) = -\theta^*(w_o(\theta)(\alpha - \beta)) = \theta^*(\beta - \alpha - \delta)$$

for some $\delta \in \mathrm{Span}\,(\Delta_o(\theta))$. Since Δ is a basis of V and α, β, $\theta^*(\alpha)$, $\theta^*(\beta) \in (\Delta - \Delta_o(\theta))$, it follows that $\alpha = \theta^*(\beta)$, $\beta = \theta^*(\alpha)$ and $\delta = 0$.

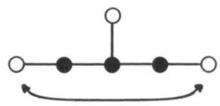
2.11. The index of θ.

Assume that the root datum Ψ is semisimple. If $\theta \in \mathrm{Aut}(\Psi)$ is an involution and Δ a θ-basis of Φ, then θ is determined by the quadruple $(X, \Delta, \Delta_o(\theta), \theta^*(\Delta))$, because $\theta = -\theta^*(\Delta) w_o(\theta)$. We call such a quadruple $(X, \Delta, \Delta_o(\theta), \theta^*(\Delta))$ an *index of θ*.

Two indices $(X, \Delta, \Delta_o(\theta_1), \theta_1^*(\Delta))$ and $(X, \Delta', \Delta_o'(\theta_2), \theta_2^*(\Delta'))$ are said to be *isomorphic* if there is a $w \in W(\Phi)$, which maps $(\Delta, \Delta_o(\theta_1))$ onto $(\Delta', \Delta_o'(\theta_2))$ and which satisfies $w\theta_1^*(\Delta)w^{-1} = \theta_2^*(\Delta')$.

2.12. Remarks. (1) The above index of θ is the same as the Satake diagram corresponding to an action of the finite group $\Gamma_\theta = \{\mathrm{id}, -\theta\}$ on (X, Φ) (See Satake [22,2.4]). Our terminology follows Tits [29].

(2) As in [29] we make a diagrammatic representation of the index of θ by colouring black those vertices of the ordinary Dynkin diagram of θ, which represent roots in $\Delta_o(\theta)$ and indicating the action of θ^* on $\Delta - \Delta_o(\theta)$ by arrows. An example in type E_6 is:



We omit the action of θ^* on $\Delta_o(\theta)$ because $\theta^*|_{\Delta_o(\theta)} = -w_o(\theta)$ is completely determined by the type of $\Phi_o(\theta)$ (See (2.9.4)).

(3) An index of θ may depend on the choice of the θ-basis of Φ, i.e. for two θ-bases Δ, Δ', the corresponding indices $(X, \Delta, \Delta_o(\theta), \theta^*(\Delta))$ and $(X, \Delta', \Delta_o'(\theta), \theta^*(\Delta'))$ need not be isomorphic. However this cannot happen if $\overline{\Phi}_\theta$ is a root system with Weyl group $\overline{W}_\theta$:

2.13. Lemma. *Let Ψ be semisimple and $\theta \in \mathrm{Aut}(\Psi)$ an involution such that $\overline{\Phi}_\theta$ is a root system with Weyl group $\overline{W}_\theta$. Let Δ, Δ' be two θ bases of Φ. Then $(X, \Delta, \Delta_o(\theta), \theta^*(\Delta))$ and $(X, \Delta', \Delta_o'(\theta), \theta^*(\Delta'))$ are isomorphic.*

Proof. Since $\overline{W}_\theta = W_1(\theta)/W_o(\theta)$ is the Weyl group of $\overline{\Phi}_\theta$, there is by (2.5) a unique element $w \in W_1(\theta)$ such that $w(\Delta) = \Delta'$. Then also $w(\Delta_o(\theta)) = \Delta_o'(\theta)$, so it suffices to show that $\theta^*(\Delta') = w\theta^*(\Delta)w^{-1}$.

Since $w_o(\theta)' = ww_o(\theta)\theta(w^{-1})$, where $w_o(\theta)'$,resp. $w_o(\theta) \in W_o(\theta)$, are as in (2.8), we get $w_o(\theta)' = \theta(w)(w_o(\theta)\theta)(\theta(w)^{-1})$, which implies the desired relation.

To classify the indices of involutions we note:

2.14. Lemma. *Let Δ be a basis of Φ, $\Delta_o \subset \Delta$ a subset and $\theta^* \in \mathrm{Aut}(X,\Phi,\Delta)$ such that $\theta^*(\Delta_o) = \Delta_o$, $(\theta^*)^2 = \mathrm{id}$. Let X_o be the $\mathbb{Z}$-span of Δ_o in X and $\Phi(\Delta_o) = \Phi \cap X_o$. Then there is an involution $\theta \in \mathrm{Aut}(X,\Phi)$ with index $(X,\Delta,\Delta_o,\theta^*)$ if and only if $\theta^* \,|_{\Delta_o} = \mathrm{id}^*(\Delta_o)$ (the opposition involution of Δ_o with respect to $\Phi(\Delta_o)$).*

Proof. "Only if" being clear, assume $\theta^* \,|_{\Delta_o} = \mathrm{id}^*(\Delta_o)$. Let w_o be the longest element of $W(\Phi(\Delta_o))$ with respect to Δ_o and let $\theta = -\theta^* w_o \in \mathrm{Aut}(X,\Phi)$.
Since $\theta|_{X_o} = \mathrm{id}$ it follows that θ^* and w_o commute, so θ is an involution. On the other hand, since $\theta^* \,|_{\Delta_o} = \mathrm{id}^*(\Delta_o)$ it follows that $\Delta_o = \Delta_o(\theta)$, so $(X,\Delta,\Delta_o,\theta^*)$ is an index of θ. This proves the result.

2.15. θ-normal root systems.

Let X, Φ and θ be as in (2.8) and let $\Phi' = \{\alpha \in \Phi \mid \tfrac{1}{2}\alpha \notin \Phi\}$ be the set of indivisible roots.

2.15.1. Definition. Φ is called *θ-normal* if for all $\alpha \in \Phi'$ with $\theta(\alpha) \neq \alpha$, we have $\theta(\alpha) + \alpha \notin \Phi$.

This definition is a generalization of the known definition of normality to non-reduced root systems (See Warner [32,1.1.3]).

2.16. Remark. If Φ is θ-normal, then $\overline{\Phi}_\theta$ is a root system with Weyl group $\overline{W}_\theta$ (see Warner [32,1.1.3.1]).

In the sequel we shall need the following results:

2.17. Lemma. *Assume Φ to be irreducible and let Δ be a θ-basis of Φ. Let $\mathrm{id}^*(\Delta) \in \mathrm{Aut}(X,\Phi,\Delta)$ be the opposition involution, as in (2.9.2). Then the following statements are equivalent:*

(1) $\mathrm{id}^*(\Delta)$ *and* $w_o(\theta)$ *commute.*

(2) $\Phi_o(\theta)$ *is stable under* $\mathrm{id}^*(\Delta)$.

Proof. The proof follows from the following equivalences:
(1) $\Leftrightarrow$ $\mathrm{id}^*(\Delta)$ and θ commute $\Leftrightarrow$ 1 and (-1)-eigenspaces of θ are $\mathrm{id}^*(\Delta)$-stable $\Leftrightarrow$ (2).

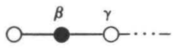
2.18. Note that in general $w_o(\theta)$ and id^* need not commute. For example, if Φ is of type A_2, then

is the index of an involution $\theta \in \mathrm{Aut}(X,\Phi)$, but clearly $\Phi_o(\theta)$ is not stable under id^*.
However when Φ is θ-normal, then the condition is satisfied.

2.19. Lemma. *Let Φ, θ, Δ and* id^* *be as in (2.16). If Φ is θ-normal, then $\Phi_o(\theta)$ is stable under* id^*.

Proof. We first note that we may assume that $\mathrm{id}^* \neq \mathrm{id}$. Then Φ is of type A_l, D_{2l+1} $(l \geqslant 2)$ or E_6.
We may also assume $\theta^* = \mathrm{id}$ (if not, we would have $\theta^* = \mathrm{id}^*$ and we are done). Now

$\Phi_o(\theta)$ must be a union of irreducible components, whose Weyl groups contain $-\mathrm{id}$. From the preceding remarks, it follows that $\Phi_o(\theta)$ is a union of a number of irreducible components of type A_1 and at most one component of type D_{2l} ($l \geqslant 2$).
If $\Phi_o(\theta)$ has an irreducible component of type D_{2l} ($l \geqslant 2$), then Φ is of type D_{2l+1} ($l \geqslant 2$) or E_6 and in both cases $\Phi_o(\theta)$ is stable under id^*. so we may assume that $\Phi_o(\theta)$ is of type $A_1 \times ..... \times A_1$. Say $\Delta_o(\theta) = \{\alpha_1, ..., \alpha_r\}$. Then $w_o(\theta) = s_{\alpha_1} ..... s_{\alpha_r}$.
If the index of θ would contain a subdiagram of the form

or


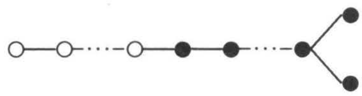
then Φ is not θ-normal, namely since $\theta = -w_o(\theta)$ we have $\theta(\beta+\gamma) = s_\gamma(-\beta-\gamma) = -\beta$, hence $\beta+\gamma+\theta(\beta+\gamma) = \gamma \in \Phi$.
It follows that the only possible indices of θ, with $\Phi_o(\theta)$ of type $A_1 \times ..... \times A_1$ are:

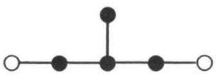
A_{2l+1}: 

In this case $\Phi_o(\theta)$ is obviously stable under id^*, which proves the result.

2.20. From this proof it also follows that the indices of involutions θ with Φ irreducible and θ-normal and $\mathrm{id}^* \neq \mathrm{id}$, $\theta^* = \mathrm{id}$ are:

A_{2l+1}: 

D_{2l+1}: 

E_6: 

3. A characterization of the conjugacy classes of involutorial automorphisms of G.

3.1. A realization of $\Phi(T)$ in G.

Let T be a maximal torus of G. If $\alpha \in \Phi(T)$, let x_α be the corresponding one-parameter additive subgroup of G defined by α. This is an isomorphism of the additive subgroup onto a closed subgroup U_α of G, normalized by T, such that

$$t x_\alpha(\xi) t^{-1} = x_\alpha(\alpha(t)\xi) \quad (t \in T,\ \xi \in F)$$

The x_α may be chosen such that

$$n_\alpha = x_\alpha(1) x_{-\alpha}(-1) x_\alpha(1)$$

lies in $N_G(T)$ for all $\alpha \in \Phi(T)$, as can be derived using a SL_2-computation. In that case we have

$$x_\alpha(\xi) x_\alpha(-\xi^{-1}) x_\alpha(\xi) = \alpha^\vee(\xi) n_\alpha \quad (\xi \in F),$$

here $\alpha^\vee \in X_*(T)$ is the coroot of α. Moreover $n_\alpha.T$ is the reflection $s_\alpha \in W(T)$ defined by α and $n_\alpha^2 = \alpha^\vee(-1) = t_\alpha$, $n_{-\alpha} = t_\alpha n_\alpha$, $t_{-\alpha} = t_\alpha$.

A family $\{x_\alpha\}_{\alpha\in\Phi(T)}$ with the above properties (3.1.1), (3.1.2) is called a *realization* of $\Phi(T)$ in G. Similarly the set of root vectors $X_\alpha = dx_\alpha(1)\in\mathfrak{g}_\alpha$ is called a *realization* of $\Phi(T)$ in $\mathfrak{g}$. We then have $Ad(t)X_\alpha = \alpha(t)X_\alpha \quad (t\in T)$. For these facts see Springer [24,11.2].

If $\alpha,\beta\in\Phi(T)$ are linearly independent (i.e. $\alpha \neq \pm\beta$) we have a formula:

$$(x_\alpha(\xi),x_\beta(\eta)) = \prod_{\substack{i\alpha+j\beta\in\Phi(T)\\ i,j>0}} x_{i\alpha+j\beta}(c_{\alpha,\beta;i,j}\xi^i\eta^j) \quad (\xi,\eta\in F),$$

the product being taken in a preassigned order. The elements $c_{\alpha,\beta;i,j}$ are called the *structure constants* of G for the given realization $\{x_\alpha\}_{\alpha\in\Phi(T)}$.

3.2. Let Δ be a basis of $\Phi(T)$. If $w\in W(T)$ and $w = s_{\alpha_1}.....s_{\alpha_k}$ is a shortest expression of w, the α_i being simple roots, then $\phi(w) = n_{\alpha_1}.....n_{\alpha_k}\in N_G(T)$ is a representative of $w\in W(T)$ in $N_G(T)$, depending only on w and not on the choice of the shortest expression (see Springer [24,11.2.9]). There exists a realization $\{x_\alpha\}_{\alpha\in\Phi(T)}$ such that

$$\phi(w)x_\alpha(\xi)\phi(w)^{-1} = x_{w(\alpha)}(\pm\xi)$$

for $\alpha\in\Phi(T)$, $w\in W(T)$, $\xi\in F$ and ϕ as above.

Moreover all the structure constants are of the form n.1 with $n\in\mathbb{Z}$. In particular if α, $\beta\in\Phi$, $\alpha+\beta\in\Phi$, $\alpha-c\beta\in\Phi$, $\alpha-(c+1)\beta\notin\Phi$, then $c_{\alpha,\beta;1,1} = \pm(c+1)$ and $c_{\alpha,\beta;1,1}c_{-\alpha,-\beta;1,1} = -(c+1)^2$. For more details see Springer [24,11.3.6]

3.3. θ-singular roots.

Let T be a maximal torus of G and $\{x_\alpha\}_{\alpha\in\Phi(T)}$ a realization of $\Phi(T)$ in G.

If $\phi\in Aut(G)$ such that $\phi(T) = T$, then there exists $c_{\alpha,\phi}\in F^*$ such that for $\xi\in F$

$$\phi(x_\alpha(\xi)) = x_{\phi(\alpha)}(c_{\alpha,\phi}\xi)$$

Now ϕ is an involution if and only if

$$(\phi\,|_T)^2 = id_T \text{ and } c_{\alpha,\phi}c_{\phi(\alpha),\phi} = 1 \text{ for all } \alpha\in\Phi(T).$$

Let $\theta\in Aut(G)$ be an involution stabilizing T. Then a root $\alpha\in\Phi(T)$ is called *θ-singular* if $\theta(\alpha) = \pm\alpha$ and $\theta\,|_{Z_G((Ker\alpha)^\circ)} \neq id$.

If $\theta(\alpha) = -\alpha$ we say that α is *real* with respect to θ.

If $\theta(\alpha) = \alpha$ and α is θ-singular, then α is also called *noncompact imaginary* with respect to θ. In that case $c_{\alpha,\theta} = -1$, as follows also by simple computation in SL_2.

If $\theta(\alpha) = \alpha$ and α is not θ-singular, then $c_{\alpha,\theta} = 1$. These roots are called *compact imaginary* with respect to θ.

3.4. Lemma. *Let* T *be a θ-stable maximal torus of* G. *Then* T_θ^- *is a maximal θ-split torus of* G *if and only if* $\Phi(T)$ *has no roots, which are noncompact imaginary with respect to θ, i.e.* $c_{\alpha,\theta} = 1$ *for all* $\alpha\in\Phi_0(\theta)$.

For a proof see [12] or [25].

3.5. Lemma. *Let* T *be a θ-stable maximal torus of* G, Δ *a θ-basis of* $\Phi(T)$ *and write* $\theta = -\theta^* w_o(\theta)$ *as in (2.8). Then for all* $t \in \bigcap_{\beta\in\Delta_o(\theta)} Ker(\beta)$ *such that* $\theta(t).t \in Z(G)$ *we have* $\theta^*(\alpha)(t) = \alpha(t)$ *for all* $\alpha \in \Phi(T)$.

Proof. If $t \in \bigcap_{\beta\in\Delta_o(\theta)} Ker(\beta)$ such that $\theta(t)t \in Z(G)$, then $\theta^*(\alpha)(t) = w_o(\theta)\alpha(\theta(t)^{-1}) = w_o(\theta)(\alpha)(t) = \alpha(t)\gamma(t)$ for some $\gamma \in Span(\Delta_o(\theta))$.
Since for all $\beta \in \Phi_o(\theta)$ we have $\beta(t) = 1$, it follows that $\theta^*(\alpha)(t) = \alpha(t)$.

Note that among others all elements of T_θ^- satisfy the above conditions.

3.6. Definition. Let T be a maximal torus of G. An automorphism θ of G of order $\leqslant 2$ is said to be *normally related* to T if $\theta(T) = T$ and T_θ^- is a maximal θ-split torus of G.

3.7. Theorem. *Let* $\theta_1, \theta_2 \in Aut(G)$ *be such that* $\theta_1^2 = \theta_2^2 = id$ *and assume* θ_1, θ_2 *are normally related to* T. *Then* θ_1 *and* θ_2 *are conjugate under* Int(G) *if and only if* $\theta_1|_T$ *and* $\theta_2|_T$ *are conjugate under* W(T).

Proof. If $\theta_2 = Int(g)\theta_1 Int(g^{-1})$ for some $g \in G$, then since all maximal θ_2-split tori are conjugate under $G_{\theta_2}^{o}$ and also all maximal tori containing them (See (1.5)), we may assume $g \in N_G(T)$. But then $\theta_1|_T$ and $\theta_2|_T$ are conjugate under W(T), which proves the "only if " statement.
Assuming that $\theta_1|_T$ and $\theta_2|_T$ are conjugate under W(T), it then suffices to consider the case that $\theta_1|_T = \theta_2|_T$. Henceforth we assume this and write θ for $\theta_i|_T$. By the isomorphism theorem (see Springer [24,11.4.3]), there is a $t \in T$ such that $\theta_1 = \theta_2 Int(t)$.
Since $\theta_1^2 = \theta_2^2 = id$, we get $Int(\theta(t)t) = id$, so $\theta(t)t \in Z(G)$. If $\alpha \in \Phi_o(\theta)$, then by (3.4) α is a compact imaginary root with respect to θ_1 as well as θ_2, so in particular $c_{\alpha,\theta_1} = c_{\alpha,\theta_2} = 1$, which implies $\alpha(t) = 1$.
Let Δ be a θ-basis of $\Phi(T)$ and let $\Delta_o(\theta)$, $\bar{\Delta}_\theta$ be as in (2.6). If $\gamma \in \bar{\Delta}_\theta$ and $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, such that $\pi(\alpha) = \pi(\beta)$, then by (2.10) $\beta = \theta^*(\alpha)$. So by (3.5) we have $\alpha(t) = \theta^*(\alpha)(t)$.
For each $\gamma \in \bar{\Delta}_\theta$, take now $\alpha \in \Delta$ such that $\gamma = \pi(\alpha) = \alpha|_{T_\theta^-}$ and choose $u_\gamma \in T_\theta^-$ such that $\lambda(u_\gamma) = 1$ for $\lambda \in \bar{\Delta}_\theta$, $\lambda \neq \gamma$ and $\gamma(u_\gamma^2) = \alpha(t)$. Let $u = \prod_{\gamma\in\bar{\Delta}_\theta} u_\gamma$. Then by (2.10) and (3.5) we find $\alpha(t.u^2) = 1$ for all $\alpha \in \Delta$. So $t.u^2 \in Z(G)$ and it follows that $Int(u)\theta_1 Int(u^{-1}) = \theta_2$. This proves the result.

3.8. Corollary. *Let* $\theta_1, \theta_2 \in Aut(G)$ *be as above. If* $\theta_1|_T = \theta_2|_T$, *then there is* $t \in T_\theta^-$ *such that* $\theta_1 = \theta_2 Int(t)$.

This follows from the proof of (3.7).

3.9. Definition. Let $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$ be a root datum with Φ a reduced root system and let $\theta \in Aut(\Psi)$ be an involution. Then θ is called *admissible* if there exists a reductive algebraic group G with maximal torus T and an involution $\tilde{\theta} \in Aut(G,T)$ such that Ψ is isomorphic to $(X^*(T), \Phi(T), X_*(T), \Phi^\vee(T))$, $\tilde{\theta}$ induces θ on Ψ and such that $T_{\tilde{\theta}}^-$ is a maximal θ-split torus of G.

If X is semisimple, then the indices of admissible involutions of Ψ shall be called *admissible indices.*

3.10. Remark. Let G, T be as in (3.1). If $\theta \in \mathrm{Aut}(X^*(T),\Phi(T))$ is an admissible involution, then by (1.6) $\overline{\Phi}_\theta = \Phi(T_\theta^-)$ is a root system with Weyl group $W_1(\theta)/W_o(\theta) \cong W(T_\theta^-)$. So if G is semisimple, then by (2.13) the $W(\Phi)$-conjugacy class of θ corresponds bijectively with the isomorphism class of the index of θ. We have obtained the following result.

3.11. Theorem. *Assume that* G *is semisimple and* T *is a maximal torus of* G. *Then there is a bijection of the set of* Int(G) *conjugacy classes of involutorial automorphisms of* G *and the isomorphism classes of indices of admissible involutions of* $(X^*(T),\Phi(T))$.

Proof. Since all maximal tori of G are conjugate under Int(G), every involutorial automorphism of G is conjugate to one which is normally related to T. The result follows now from theorem (3.7), lemma (2.13) and (3.10).

3.12. θ-normality of Φ(T).

For later use it is useful to note that an admissible involution $\theta \in \mathrm{Aut}(X^*(T),\Phi(T))$ implies θ-normality of the root system:

Lemma. *Let* G, T *be as in (3.1). If* $\theta \in \mathrm{Aut}(X^*(T),\Phi(T))$ *is an admissible involution, then* $\Phi(T)$ *is* θ*-normal.*

Proof. By (3.4) it suffices to show that if $\alpha \in \Phi(T)$ such that $\theta(\alpha) \neq \alpha$ and $\alpha+\theta(\alpha) \in \Phi(T)$, that then $\alpha+\theta(\alpha)$ must be non-compact imaginary. This last statement follows immediately by choosing a realization $\{X_\alpha\}_{\alpha \in \Phi(T)}$ of $\Phi(T)$ in $\mathfrak{g}$ such that $\theta(X_\alpha) = X_{\theta(\alpha)}$ and $[X_\alpha, X_{\theta(\alpha)}] = X_{\alpha+\theta(\alpha)}$. Then

$$\theta(X_{\alpha+\theta(\alpha)}) = [X_{\theta(\alpha)}, X_\alpha] = -X_{\alpha+\theta(\alpha)},$$

so $\alpha+\theta(\alpha)$ is non-compact imaginary.
For this see also Springer [25,2.6].

4. Classification of admissible involutions.

We discuss here the classification of involutorial automorphisms of G. It is quite similar to the classification of real forms of a complex semisimple Lie algebra, as is carried out by Araki [1]. See also section 9.

4.1. Lifting involutions of (X,Φ).

In this section we assume G to be semisimple.
Let T be a fixed maximal torus of G and write Φ for Φ(T), X for $X^*(T)$, W for W(T). Choose a realization of Φ in G like in (3.2). To determine whether an involution $\theta \in \mathrm{Aut}(X,\Phi)$ is admissible we need to determine first whether it can be lifted, i.e.

Definition. An involution $\theta \in \mathrm{Aut}(X,\Phi)$ can be *lifted* if there is an involutorial automorphism $\phi \in \mathrm{Aut}(G,T)$ inducing θ on (X,Φ).

Note that by (3.3) this is equivalent to $c_{\alpha,\phi} \cdot c_{\theta(\alpha),\phi} = 1$ for all $\alpha \in \Phi$. Moreover such a $\phi \in \mathrm{Aut}(G,T)$ is admissible if and only if $c_{\alpha,\phi} = 1$ for all $\alpha \in \Phi_0(\theta)$. On the other hand, it follows from the isomorphism theorem (see Springer [24,11.4.3]) that it also suffices to restrict to a basis of Φ:

4.2. Lemma. *Let Δ be a basis of Φ, $\theta \in \mathrm{Aut}(X,\Phi)$ an involution and $\phi \in \mathrm{Aut}(G,T)$ such that $\phi|_T = \theta$. Then ϕ is uniquely determined by the tuple $\{c_{\alpha,\phi}\}_{\alpha\in\Delta}$.*

This result is discussed in Seminaire C. Chevalley [7,17-08,09]

4.3. Definition. Let Δ be a fixed basis of Φ. For any involution $\theta \in \mathrm{Aut}(X,\Phi)$ let $\theta_\Delta \in \mathrm{Aut}(G,T)$ denote the unique automorphism of G such that

$$\theta_\Delta(x_\alpha(\xi)) = x_\alpha(\xi) \quad \text{for all } \alpha \in \Delta, \xi \in F.$$

It follows now from a result of Steinberg [26,Th.29] that $c_{\alpha,\theta_\Delta} = \pm 1$ for all $\alpha \in \Phi$ and moreover the constants c_{α,θ_Δ} do not depend on the characteristic of the field of definition F.

Summarizing, involutions of (X,Φ) which can be lifted, can be characterized as follows:

4.4. Proposition. *Let $\theta \in \mathrm{Aut}(X,\Phi)$ be an involution and Δ a basis of Φ. Then the following are equivalent:*

(i) θ can be lifted;
(ii) There is a $t \in T$ such that $\theta_\Delta \mathrm{Int}(t)$ is an involution;
(iii) There is a $t \in T$ such that $c_{\theta(\alpha),\theta_\Delta} = \alpha(\theta(t)t)$ for all $\alpha \in \Delta$;
(iv) There is a $t \in T_\theta^+$ such that $c_{\theta(\alpha),\theta_\Delta} = \alpha(t)$ for all $\alpha \in \Delta$.

This result follows immediately from the definition of θ_Δ, (4.2) and (3.3).

Note that if $t \in T_\theta^+$ such that $\theta_\Delta \mathrm{Int}(t)$ is an involution, then, since $c_{\alpha,\theta_\Delta} = \pm 1$ for all $\alpha \in \Phi$, we have by (iv) that $\alpha(t^4) = 1$ for all $\alpha \in \Phi$, hence $t^4 \in Z(G)$.

4.5. Corollary. *Let $\theta \in \mathrm{Aut}(X,\Phi)$ be an involution and let Δ be a θ-basis of Φ. Then θ is admissible if and only if there is a $t \in T$ such that*

(i) $c_{\theta(\alpha),\theta_\Delta} = \alpha(\theta(t)t)$ for all $\alpha \in \Delta - \Delta_o(\theta)$
(ii) $\alpha(t) = 1$ for all $\alpha \in \Delta_0(\theta)$

This follows from (4.4) and (3.4)

4.6. Proposition. *Assume that G, T, X, and Φ are as in (4.1). Whether an involution $\theta \in \mathrm{Aut}(X,\Phi)$ is admissible or not is independent of the field of definition F of G, if only char(F) $\neq 2$.*

Proof. An involution $\theta \in \mathrm{Aut}(X,\Phi)$ is admissible if for a fixed θ-basis Δ of Φ, there is a $t \in T_\theta^+$ such that the conditions (i) and (ii) of (4.5) are satisfied. But these conditions imply that $t^4 \in Z(G)$, so this can be verified independently of F, if only char(F) $\neq 2$.

4.7. The classification of conjugacy classes of involutorial automorphisms of G coincides now with the known classification over $\mathbb{C}$. For G of adjoint type this comes down to the classification of real forms of a semisimple Lie algebra over $\mathbb{C}$, as is carried out by Araki [1]. See also Sugiura [22,appendix] for a simplification of this method. Different treatments of the classification of real semisimple Lie algebras can be found for instance in Cartan [6], Gantmacher [9] (simplified by Murakami [17]), Helgason [11] and Freudenthal- de Vries [8].

On the other hand, with the above results it is not hard any more to obtain the classification. We will sketch this in the remainder of this section.

4.8. Reduction to restricted rank one.

Let G, T, X, Φ be as in (4.1).

The *restricted rank* of an involution $\theta\in\mathrm{Aut}(X,\Phi)$ is defined as the rank of the set of restricted roots $\overline{\Phi}_\theta$. If Δ is a θ-basis of Φ, then the restricted rank of θ is equal to $|\overline{\Delta}_\theta|$.

For each $\lambda\in\overline{\Phi}_\theta$ such that $\frac{1}{2}\lambda\notin\overline{\Phi}_\theta$ (i.e. $\lambda\in\overline{\Phi}'_\theta$, see (2.15)), let $\Phi(\lambda)$ denote the set of all roots $\beta\in\Phi$ such that the restriction of β to $\overline{X}_\theta$ is an integral multiple of λ. Then $\Phi(\lambda)$ is a θ-stable closed symmetric subsystem of Φ (See Borel-Tits [3,p.71]). Let $X(\lambda)$ denote the projection of X on the subspace of $E=X^*(T)\otimes_{\mathbb{Z}}\mathbb{R}$ spanned by $\Phi(\lambda)$.

4.9. Proposition. *Let* $\theta\in\mathrm{Aut}(X,\Phi)$ *be an involution and* Δ *a* θ*-basis of* Φ. *Then* θ *is admissible if and only if* $\theta|_{X(\lambda)}\in\mathrm{Aut}(X(\lambda),\Phi(\lambda))$ *is admissible for all* $\lambda\in\overline{\Delta}_\theta$.

This result is derived immediately from (4.5).(See also Satake [22]).

4.10. Classification of involutions of restricted rank one.

To determine the indices of involutions of restricted rank one we need a notion of irreducibility:

Definition. Let $\theta\in\mathrm{Aut}(X,\Phi)$ be an involution and Δ a θ-basis of Φ. An index $S=(X,\Delta,\overline{\Delta}_\theta,\theta^*)$ of θ is called *irreducible* If Δ is not the union of two mutually orthogonal θ^*-stable non-empty subsystems Δ_1 and Δ_2. The index is called *absolutely irreducible* if Δ is connected.

Clearly an absolutely irreducible index is irreducible. From (2.14) and (2.9) one easily deduces now:

4.11. Proposition. *Let* X *be of adjoint type. Then there exist 19 types of absolutely irreducible indices of non-trivial involutions of* (X,Φ) *of restricted rank one and one type of restricted rank one, which is irreducible but not absolutely irreducible. (See table 1).*

This result can also be found in Sugiura [22,appendix,prop.4]

4.12. To restrict this set of rank one indices Araki [1] and Sugiura [22,appendix] used the θ-normality of Φ (see (3.12)). One can also exclude these indices with Φ not θ-normal, using the following results:

Lemma. *Let* X *be of adjoint type,* $\theta\in\mathrm{Aut}(X,\Phi)$ *an involution of restricted rank one,* Δ *a* θ*-basis of* Φ *and* G, T, Φ_Δ *as in (4.3). If* $|\Delta-\Delta_0(\theta)|=1$, *then* θ *is admissible if and only if* θ_Δ *is an involution (i.e.* $c_{\theta(\alpha),\theta_\Delta}=1$ *for* $\alpha\in\Delta-\Delta_0(\theta)$).

Proof. The "if" statement being obvious, assume θ is admissible. By (4.5) there exists $t\in T_\theta^+$ such that $c_{\theta(\alpha),\theta_\Delta}=\alpha(t^2)$ for all $\alpha\in\Delta-\Delta_0(\theta)$ and $\alpha(t)=1$ for all $\alpha\in\Delta_0(\theta)$. So let $\alpha\in\Delta-\Delta_0(\theta)$. It suffices to show that $c_{\theta(\alpha),\theta_\Delta}=1$ or equivalently $\alpha(t^2)=1$. Since $|\Delta-\Delta_0(\theta)|=1$ we have $\theta(\alpha)=-w_0(\theta)(\alpha)=-(\alpha+\sum_{\beta\in\Delta_0(\theta)}m_\beta\beta)$

$m_\beta \in \mathbb{N}$, so it follows from (4.5) that $\theta(\alpha)(t) = \alpha(t)^{-1}$. On the other hand, since $t \in T_\theta^+$ we have $\theta(\alpha)(t) = \alpha(\theta(t)) = \alpha(t)$, hence $\alpha(t^2) = 1$. This proves the result.

4.13. Whether θ_Δ is an involution or not , is a matter determined by structure constants. This can be seen as follows.
Assume $\theta \in \mathrm{Aut}(X,\Phi)$ an involution of restricted rank one, Δ a θ-basis and $|\Delta - \Delta_0(\theta)| = 1$. Let $\alpha \in \Delta - \Delta_0(\theta)$. Then $\theta(\alpha) = -w_0(\theta)(\alpha)$. Since $w_0(\theta)$ is an involution in $W_0(\theta)$ we can write $w_0(\theta) = s_{\alpha_1} \ldots\ldots s_{\alpha_r}$, where $\alpha_1, \ldots\ldots, \alpha_r \in \Phi_0(\theta)$ are strongly orthogonal roots (i.e. for all i,j = 1,.....r we have $\alpha_i \pm \alpha_j \notin \Phi_0(\theta)$). (See e.g. Helminck [12]).
To determine $\theta(\alpha)$ we only need to consider those α_i such that $(\alpha,\alpha_i) \neq 0$. Note that if $\Phi_0(\theta) \neq \emptyset$, then we can choose α_i such that $(\alpha,\alpha_i) \neq 0$ and $\alpha_1 \in \Delta_0(\theta)$. Moreover there are at most 4 strongly orthogonal roots α_i such that $(\alpha,\alpha_i) \neq 0$. (See Helminck [12] or Kostant [15]).

Choose a realization of Φ in $\mathfrak{g}$ like in (3.1) and for α, $\beta \in \Phi$ let $N_{\alpha,\beta} \in F$ denote the corresponding structure constant (i.e. $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$). Let $\beta_1, \ldots\ldots, \beta_k$ be the set of those α_i for which $(\alpha,\alpha_i) \neq 0$. We can determine $c_{\theta(\alpha),\theta_\Delta}$ now by applying θ_Δ on the identity:

$$[\cdots[[X_{-\alpha}, X_{\beta_1}], X_{\beta_2}], \cdots, X_{\beta_k}] =$$
$$N_{-\alpha,\beta_1} \cdot N_{s_{\beta_1}(-\alpha),\beta_2} \cdots N_{s_{\beta_i} \cdots s_{\beta_1}(-\alpha),\beta_{i+1}} \cdots N_{s_{\beta_{k-1}} \cdots s_{\beta_1}(-\alpha),\beta_k} X_{\theta(\alpha)}$$

Note that it also follows from this identity that $c_{\theta(\alpha),\theta_\Delta}$ depends only on the structure constants. We can characterize this restricted rank one indices now as follows:

4.14. Lemma. *Let θ, Δ, θ_Δ, α and $w_0(\theta) = s_{\alpha_1} \ldots\ldots s_{\alpha_r}$ be as in (4.13). Then θ is admissible if and only if $\sum_{i=1}^{r} <\alpha, \alpha_i^\vee>$ is even.*

We give a proof for $\sum_{i=1}^{r} <\alpha,\alpha_i^\vee> = 1$. The other cases are left to the reader. So assume $\sum_{i=1}^{r} <\alpha,\alpha_i^\vee> = 1$. Say $<\alpha,\alpha_1^\vee> = 1$. Then

$$[X_{-\alpha}, X_{\alpha_1}] = N_{-\alpha,\alpha_1} X_{\theta(\alpha)}.$$

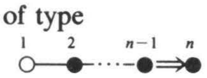
Applying θ_Δ on this identity gives:

$$N_{-\alpha,\alpha_1} \cdot c_{\theta(\alpha),\theta_\Delta} = N_{\theta(-\alpha),\alpha_1} = N_{\alpha-\alpha_1,\alpha_1}.$$

Since $N_{\alpha-\alpha_1,\alpha_1} = N_{\alpha,-\alpha_1}$, it follows $c_{\theta(\alpha),\theta_\Delta} = -1$, hence θ is not admissible.

It is not hard to determine $w_0(\theta)$ as a product of strongly orthogonal roots (see e.g. Helminck [12] or Kostant [15]). Here are two examples:

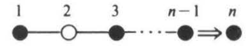
4.15. Examples. (1) Assume θ is of type



In this case θ is admissible. One sees this as follows.
$\Phi_0(\theta)$ is of type B_{n-1} and $\alpha = \alpha_1$, where $\Delta = \{\alpha_1, \ldots., \alpha_n\}$ is a basis of Φ corresponding to the above diagram. So if n = 2, then $w_0(\theta) = s_{\alpha_2}$ and $<\alpha_1,\alpha_2> = 2$. If n>2, then let β_1 be the longest root of $\Phi_0(\theta)$ with respect to $\Delta_0(\theta)$ and let

$\beta_2,\ldots\ldots,\beta_{n-2} \in \Phi_o(\theta)$ be such that α, $\beta_1,\ldots\ldots,\beta_{n-2}$ are strongly orthogonal. Now $w_o(\theta) = s_{\alpha_2} s_{\beta_1} \cdots s_{\beta_{n-2}}$, so $\sum_{i=1}^{r} <\alpha,\alpha_i^\vee> = 2$ and θ is admissible by (4.14).

(2) Similarly as in (1) one shows that the involution θ with index



is not admissible, because $w_o(\theta)$ contains s_{α_1} additional to the factors in (1).

4.17. There remain still 2 indices in table 1, which do not satisfy the conditions in (4.13) and (4.14). However in these cases one easily shows directly that the index is admissible. Summarizing we have obtained the following result:

Theorem. *Let* X *be of adjoint type. The absolutely irreducible indices of non-trivial admissible involutions of* (X,Φ) *are the ones given in table 2. The irreducible, but not absolutely irreducible indices are the ones given in table 3.*

We added in this tables some extra information which will be explained and used in section 7/8.

4.18. Passage to arbitrary G.

The classification for arbitrary groups G follows now easily from the above results. It is only a matter of checking whether a lattice X is θ-stable. Namely let Φ be a reduced root system and let Q, resp. P, denote the root lattice, resp. weight lattice, of Φ. If $\theta \in \mathrm{Aut}(Q,\Phi)$ is an admissible involution, then θ induces a (unique) involution $\tilde{\theta} \in \mathrm{Aut}(P,\Phi)$, which is also admissible by a result of Steinberg [27,9.16]. Now if X is any lattice such that $Q \subset X \subset P$, then θ may be lifted to an admissible involution of (X,Φ) if and only if X is $\tilde{\theta}$-stable.

5. Conjugacy classes of pairs of commuting involutorial automorphisms of G.

In this section we characterize conjugacy classes of pairs of commuting automorphisms of G in a manner similar to that of section 3.

5.1. Let $\sigma,\theta \in \mathrm{Aut}(G)$ be such that $\sigma^2 = \theta^2 = \mathrm{id}$ and $\sigma\theta = \theta\sigma$. Let $\mathfrak{g}_\sigma$, $\mathfrak{g}_\theta$, $\mathfrak{g}_{\sigma\theta}$ denote the Lie algebras of G_σ^o, G_θ^o, $G_{\sigma\theta}^o$ respectively. Write (for $\xi,\eta = \pm 1$) $\mathfrak{g}(\xi,\eta) = \{X \in \mathfrak{g} \mid \sigma(X) = \xi X,\ \theta(X) = \eta X\}$. Then,

$$\mathfrak{g}_\sigma = \mathfrak{g}(1,1) \oplus \mathfrak{g}(1,-1);$$
$$\mathfrak{g}_\theta = \mathfrak{g}(1,1) \oplus \mathfrak{g}(-1,1);$$
$$\mathfrak{g}_{\sigma\theta} = \mathfrak{g}(1,1) \oplus \mathfrak{g}(-1,-1).$$

Note that $\mathfrak{g}$ is the direct sum of the $\mathfrak{g}(\xi,\eta)$ $(\xi,\eta = \pm 1)$.

5.2. Definition. A torus A of G is called *(σ,θ)-split* if A is σ- and θ-split. A torus T of G, which is σ- and θ-stable shall be called *(σ,θ)-stable*. We then put $T_{\sigma,\theta}^- = \{t \in T \mid \sigma(t) = \theta(t) = t^{-1}\}^o$.

If G is an arbitrary reductive connected algebraic group and σ, $\theta \neq \mathrm{id}$, then non-

trivial (σ,θ)-split tori of G need not exist. One sees this in the example of a direct product $G = G_1 \times G_2$, with G_1, G_2 (reductive) groups and $\sigma(G_i) = G_i$, $\theta(G_i) = G_i$ $(i = 1,2)$, $\theta|_{G_1} = \mathrm{id}$, $\sigma|_{G_2} = \mathrm{id}$.
In (5.10) we shall see that if G is simple and σ, $\theta \neq \mathrm{id}$, then non-trivial (σ,θ)-split tori exist. In fact we shall show an equivalent statement that if G has no (σ,θ)-split tori, that then on each irreducible component of $\Phi(T)$ we have $\sigma = \mathrm{id}$ or $\theta = \mathrm{id}$. Here T is a (σ,θ)-stable maximal torus of G. To do so we first prove some results on (σ,θ)-stable tori.

5.3. Lemma. *The following statements are equivalent:*
(a) G *contains no non-trivial* (σ,θ)*-split tori;*
(b) $G^{\circ}_{\sigma\theta}$ *contains no non-trivial* σ*-split tori;*
(c) $G^{\circ}_{\sigma\theta} = G^{\circ}_{\sigma} \cap G^{\circ}_{\theta}$*;*
(d) $\mathfrak{g}(-1,-1) = 0$.

Proof. (a)⇔(b) is clear from the observation that the (σ,θ)-split tori of G are precisely the σ-split (or θ-split) tori of $G^{\circ}_{\sigma\theta}$.
(b)⇔(c) follows immediately from (1.4) and (c)⇔(d) follows from (5.1). Finally (d)⇔(a) is immediate from the observation that the Lie algebra of a (σ,θ)-split torus is contained in $\mathfrak{g}(-1,-1)$.

5.4. Proposition. *Let* $\sigma,\theta \in \mathrm{Aut}(G)$ *be a pair of commuting involutorial automorphisms of* G. *If* $\theta \neq \mathrm{id}$, *then there exists a maximal* θ*-split torus of* G, *which is* σ*-stable.*

Proof. Let A be a maximal (σ,θ)-split torus of G. It suffices to show that $Z_G(A)/A$ contains a σ-stable maximal θ-split torus.
If A is already maximal θ-split, we are done, so assume A is not maximal θ-split. Then passing to $Z_G(A)/A$, we may assume that G has no (σ,θ)-split tori and $\theta \neq \mathrm{id}$. Now $\theta|_{G^{\circ}_{\sigma}} \neq \mathrm{id}$, because if $G^{\circ}_{\sigma} \subset G^{\circ}_{\theta}$ then using (5.3) we get $\mathfrak{g}(1,-1) = \mathfrak{g}(-1,-1) = 0$, whence $\mathfrak{g}_{\theta} = \mathfrak{g}$, contradicting $\theta \neq \mathrm{id}$.
Let S be a maximal θ-split torus of G°_{σ}. Then, since G has no non-trivial (σ,θ)-split tori, the same holds for $Z_G(S)/S$. In other words S is a σ-stable maximal θ-split torus of G. This proves the result.

5.5. Corollary. *There exists a maximal torus of* G, *which is* (σ,θ)*-stable.*

Proof. Let T be a σ-stable maximal torus of $Z_G(A)$, where A is a σ-stable maximal θ-split torus of G. Then by (1.5) T is also θ-stable, hence the result.

5.6. Let T be a (σ,θ)-stable maximal torus of G, denote by $\Psi = (X^*(T), \Phi(T), X_*(T), \Phi^{\vee}(T))$ the corresponding root datum and write $A = T^{-}_{\sigma,\theta}$ (For the moment we do not yet assume that A is a maximal (σ,θ)-split torus of G). Using the notations of (2.2) we have the following identifications:

Lemma. *Let* T, Ψ, σ, θ *and* A *be as above. Then*
(i) $\underline{X}_o(\sigma,\theta) = \{\chi \in X^*(T) | \chi(A) = 1\}$*;*
(ii) $\underline{\Phi}_{\sigma,\theta} = \Phi(A)$*;*
(iii) $W_1(\sigma,\theta) = \{w \in W(T) \mid w(A) = A\}$ *and* $W_o(\sigma,\theta) = \{w \in W(T) \mid w_A = \mathrm{id}\}$*;*
(iv) $W(A) \cong W_1(\sigma,\theta)/W_o(\sigma,\theta) \cong \overline{W}_{\sigma,\theta}$.

Proof. (i) Note first that $X_o(\sigma) = \{\chi \in X^*(T) | \chi(T_\sigma^-) = 1\}$. Let $\chi \in X^*(T)$ be such that $\chi(A) = 1$. If $t \in T_\sigma^-$, then writing $t = t_1.t_2$, where $t_1 \in A$, $t_2 \in (T_\sigma^-)_\theta^+$, it follows that $\chi(t) = \chi(\theta(t))$, whence $\chi - \theta(\chi) \in X_o(\sigma)$. But then $\chi - \theta(\chi) = \sigma(\chi - \theta(\chi))$, in other words $\chi \in X_o(\sigma,\theta)$. On the other hand if $\chi \in X_o(\sigma,\theta)$, then $\chi - \theta(\chi) \in X_o(\sigma)$, so for all $t \in A$:

$$\chi(t) = \theta(\chi)(t) = \chi(t^{-1}),$$

hence $\chi(A) = 1$. This proves (i).
As for (ii), we only note that the roots of G with respect to the adjoint action of A on $\mathfrak{g}$ are exactly the restrictions of $\Phi(T)$ to A.
(iii) follows from the fact that $A = \{t \in T | \chi(t) = 1 \text{ for all } \chi \in X_o(\sigma,\theta)\}$.
Finally, using (iii), the proof of (iv) is as in Richardson [20,4.1]

5.7. Assume G does not contain a non-trivial (σ,θ)-split torus and let T be a (σ,θ)-stable maximal torus of G.

Lemma. $X^*(T) = X_o(\sigma,\theta)$, $\Phi(T) = \Phi_o(\sigma,\theta)$. *In particular if* $\chi \in X^*(T)$, $\sigma(\chi) = -\chi$, $\theta(\chi) = -\chi$, *then* $\chi = 0$.

Proof. Since G has no (σ,θ)-split tori $T_{\sigma,\theta}^- = \{e\}$, hence by (5.6), $X^*(T) = X_o(\sigma,\theta)$ and $\Phi(T) = \Phi_o(\sigma,\theta)$. But for $X_o(\sigma,\theta)$ the second assertion follows immediately.

5.8. Proposition. *Let* (G,T) *be as in (5.7). If* $\alpha \in \Phi(T)$, *then* $\sigma(\alpha) = \alpha$, $\mathfrak{g}_\alpha \subset \mathfrak{g}_\sigma$ *or* $\theta(\alpha) = \alpha$, $\mathfrak{g}_\alpha \subset \mathfrak{g}_\theta$.

Proof. Let $\alpha \in \Phi(T)$ and let $0 \neq X_\alpha \in \mathfrak{g}_\alpha$ denote a root vector. Since G has no (σ,θ)-split tori, we have $\mathfrak{g}(-1,-1) = 0$, by (5.3). So $(1-\sigma)(1-\theta)X_\alpha = 0$, whence

$$X_\alpha - \sigma(X_\alpha) - \theta(X_\alpha) + \sigma\theta(X_\alpha) = 0.$$

Now $\theta(X_\alpha) \in \mathfrak{g}_{\theta(\alpha)}$, $\sigma(X_\alpha) \in \mathfrak{g}_{\sigma(\alpha)}$ and $\sigma\theta(X_\alpha) \in \mathfrak{g}_{\sigma\theta(\alpha)}$. It follows that if $\theta(\alpha) \neq \alpha$, $\sigma(\alpha) \neq \alpha$, we must have $\sigma\theta(\alpha) = \alpha$. Since $X^*(T) = X_o(\sigma,\theta)$ we have $\chi = \alpha - \sigma(\alpha) = \alpha - \theta(\alpha)$. But then $\sigma(\chi) = -\chi = \theta(\chi)$, so, by (5.7), $\chi = 0$. This is a contradiction, hence the assertion has been shown.

5.9. Proposition. *Let* (G,T) *be as in (5.7) and let* $\Phi_1 \subset \Phi(T) = \Phi_o(\sigma,\theta)$ *be an irreducible component. Then* $\sigma|_{\Phi_1} = \mathrm{id}$ *or* $\theta|_{\Phi_1} = \mathrm{id}$.

Proof. Let Δ be a basis of Φ_1. Assume $\sigma|_{\Phi_1} \neq \mathrm{id}$ and $\theta|_{\Phi_1} \neq \mathrm{id}$. Then there are α, $\beta \in \Delta$ such that $\sigma(\alpha) = \alpha$, $\sigma(\beta) \neq \beta$, $\theta(\alpha) \neq \alpha$, $\theta(\beta) = \beta$. Since Φ_1 is irreducible, there is a string of simple roots $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_r = \beta$ connecting α and β. Moreover we can choose α, $\beta \in \Delta$ such that $\sigma(\alpha_i) = \theta(\alpha_i) = \alpha_i$ for $i = 2, \ldots, r-1$.
If now $\gamma = \alpha_1 + \ldots + \alpha_r \in \Phi_1$, then $\sigma(\gamma) \neq \gamma$, $\theta(\gamma) \neq \gamma$, what contradicts (5.8).

From (5.8) and (5.9) we conclude:

5.10. Corollary. *If* $\Phi(G,T)$ *is irreducible and* $\sigma \neq \mathrm{id}$, $\theta \neq \mathrm{id}$, *then non-trivial* (σ,θ)*-split tori exist.*

5.11. Corollary. *Let* A *be a maximal* (σ,θ)*-split torus of* G *and* A_1 *resp.* A_2 *maximal* σ*-split resp.* θ*-split tori of* $Z_G(A)$*. Then* A_1 *and* A_2 *commute.*

Proof. We may assume $G = Z_G(A)$. If $\overline{A}_i = (A_i \cap (G,G))^o$ $(i = 1,2)$, then it suffices to show that $\overline{A}_1$ and $\overline{A}_2$ commute in (G,G). But this follows from (5.9).

5.12. Lemma. *All maximal* (σ,θ)*-split tori of* G *are conjugate under* $(G_\sigma \cap G_\theta)^o$.

Proof. Let A_1, A_2 be maximal (σ,θ)-split tori of G. Since A_1 and A_2 are also maximal σ-split tori of $G^o_{\sigma\theta}$, they are conjugate under $(G^o_{\sigma\theta})^o_\sigma = (G_\sigma \cap G_\theta)^o$.

5.13. Proposition. *There exist* (σ,θ)*-stable maximal tori of* G *such that* $T^-_{\sigma,\theta}$ *is a maximal* (σ,θ)*-split torus of* G, T^-_σ *is a maximal* σ*-split torus of* G *and* T^-_θ *is a maximal* θ*-split torus of* G*. Moreover all such maximal tori of* G *are conjugate under* $(G_\sigma \cap G_\theta)^o$.

Proof. Let A be a maximal (σ,θ)-split torus of G and A_1 (resp. A_2) a maximal σ-split (resp. θ-split) torus of $Z_G(A)$. Since A_1 and A_2 commute (see (5.11)), the first assertion follows by taking a (σ,θ)-stable maximal torus T of $Z_G(A_1A_2)$
If $\tilde{T}$ is another maximal torus of G satisfying the above conditions, then by (5.12) we may assume that $A = T^-_{\sigma,\theta} = \tilde{T}^-_{\sigma,\theta}$. Moreover, passing to $Z_G(A)/A$, we may also assume that G has no non-trivial (σ,θ)-split tori. But then $T^-_{\sigma,\theta}$ and $\tilde{T}_{\sigma,\theta}$ are maximal $\sigma\theta$-split tori of G, hence by (1.5) there exists $g \in G^o_{\sigma\theta}$ such that $gTg^{-1} = \tilde{T}$. Since $G^o_{\sigma\theta} = (G_\sigma \cap G_\theta)^o$ (see (5.3)) the result follows.

The notion "normally related" is defined as in the case of one involution (see (3. 6)):

5.14. Definition. If (σ,θ) is a pair of commuting involutorial automorphisms of G and T is a maximal torus of G, then (σ,θ) is said to be *normally related* to T if $\sigma(T) = \theta(T) = T$ and $T^-_{\sigma,\theta}$, T^-_σ , T^-_θ are maximal (σ,θ)-split, σ-split, θ-split tori of G, respectively.

Note that in this case both σ- and θ are also normally related to T. Moreover using (5.9) on $Z_G(T^-_{\sigma,\theta})$ it follows that $\Phi(T)$ has an order which is simultaneously a σ- and θ-order. This will be used to represent such a pair of commuting involutions of $(X^*(T),\Phi(T))$ by a diagram. (See section 7).

5.15. Definition. Two pairs of involutorial automorphisms (σ_1,θ_1) and (σ_2,θ_2) of G are *isomorphic* if there exists a $g \in G$ such that $\mathrm{Int}(g)\sigma_1\mathrm{Int}(g^{-1}) = \sigma_2$ and $\mathrm{Int}(g)\theta_1\mathrm{Int}(g^{-1}) = \theta_2$.

The family of all pairs of commuting involutorial automorphisms of G will be denoted by $\mathfrak{F}$ and the set of isomorphism classes in $\mathfrak{F}$ by $\mathcal{C}$, i.e. $\mathcal{C} = \mathfrak{F}/\sim$.

Note that we only consider isomorphisms of *ordered* pairs of commuting involutions of G. We could also allow isomorphisms which map σ_1 onto θ_2 and θ_1 onto σ_2. Such an isomorphism identifies the isomorphism classes of (σ_1,θ_1) and (θ_1,σ_1). However when passing from pairs of commuting involutions to symmetric spaces (see section 9) it is more convenient to work with ordered pairs, because the pairs (θ,σ) and (σ,θ) will correspond to dual symmetric spaces.

An identification of the isomorphism classes in $\mathfrak{F}$ under the action of the group of outer automorphisms of G can be deduced from the classification of the classes in

$\mathcal{F}$. Details are left to the reader.

5.16. Theorem. *Let* (σ_1,θ_1) *and* (σ_2,θ_2) *be pairs of involutorial automorphisms of* G, *normally related to* T. *Then* $(\sigma_1,\theta_1)|_T$ *and* $(\sigma_2,\theta_2)|_T$ *are conjugate under* W(T) *if and only if there exists* $\epsilon \in T^-_{\sigma,\theta}$ *with* $\epsilon^2 \in Z(G)$ *such that* (σ_2,θ_2) *is isomorphic to* $(\sigma_1,\theta_1 \mathrm{Int}(\epsilon))$.

Proof. We may assume that $\sigma_1|_T = \sigma_2|_T = \sigma$ and $\theta_1|_T = \theta_2|_T = \theta$. By the proof of (3.7) we see that after conjugation with a suitable element of T, we may assume that $\sigma_1 = \sigma_2$.
Since $\theta_1|_T = \theta_2|_T$, there exists $t \in T^-_\theta$ such that $\theta_2 = \theta_1 \mathrm{Int}(t)$ (see (3.8)). Write $t = t_1 t_2$ where $t \in (T^-_\theta)^+_\sigma$ and $t_2 \in T^-_{\sigma,\theta}$. Taking $c \in (T^-_\theta)^+_\sigma$ such that $c^2 = t_1$, we obtain:

$$\mathrm{Int}(c)\theta_2\mathrm{Int}(c)^{-1} = \theta_2\mathrm{Int}(c^{-2}) = \theta_2\mathrm{Int}(t_2) \text{ and } \mathrm{Int}(c)\sigma_1\mathrm{Int}(c)^{-1} = \sigma_1.$$

Since $t_2 \in T^-_{\sigma,\theta}$ and $t_2^2 \in Z(G)$ we are done.

5.17. For a torus S of G we call the elements $s \in S$ for which $s^2 \in Z(G)$ *quadratic elements* of S.

We can define again a notion of admissibility:

5.18. Definition. Let T be a maximal torus of G. A pair of commuting involutorial automorphisms of $(X^*(T),\Phi(T))$ is said to be *admissible* (with respect to G) if there exists a pair of commuting involutorial automorphisms $(\tilde{\sigma},\tilde{\theta})$ of G, normally related to T and such that $\tilde{\sigma}|_T = \sigma$, $\tilde{\theta}|_T = \theta$.

5.19. Denote the set of W(T)-conjugacy classes of admissible pairs of commuting involutions of $(X^*(T),\Phi(T))$ by $\mathcal{A}(T)$.
By (5.13) and the conjugacy of the maximal tori of G it follows that every pair of commuting involutorial automorphisms of G is isomorphic to one normally related to T, so we have a natural surjection $\rho : \mathcal{C} \to \mathcal{A}(T)$. For an admissible pair of commuting involutions (σ,θ) of $(X^*(T),\Phi(T))$ denote its W(T)-conjugacy class by $\overline{(\sigma,\theta)}$ and put $\mathcal{C}(\sigma,\theta) = \rho^{-1}(\overline{(\sigma,\theta)})$. This is an equivalence relation on $\mathcal{C}$, with equivalence classes the sets $\mathcal{C}(\sigma,\theta)$.
Denote the subset of $\mathcal{F}$ consisting of all pairs of commuting involutions, whose isomorphism classes are contained in $\mathcal{C}(\sigma,\theta)$ by $\mathcal{F}(\sigma,\theta)$. For $(\sigma,\theta) \in \mathcal{F}$ and a maximal (σ,θ)-split torus A of G let $\mathcal{F}_A(\sigma,\theta) = \{(\sigma,\theta\mathrm{Int}(\epsilon)) \mid \epsilon \in A, \epsilon^2 \in Z(G)\}$.
We have now the following result:

5.20. Theorem. *Let* T *be a maximal torus of* G. *There is a bijection between the* W(T)*-conjugacy classes of admissible pairs of commuting involutions of* $(X^*(T),\Phi(T))$ *and the classes* $\mathcal{C}(\sigma,\theta)$ *in* $\mathcal{C}$.

5.21. Remarks. (i) We see that the classification of pairs of commuting involutions of G consists of two parts. The classes in $\mathcal{A}(T)$ will be represented again by diagrams ((σ,θ)-indices, see section 7). To show that these diagrams are independent of the choice of the (σ,θ)-basis we will need some properties of the restricted root system and Weyl group of a maximal (σ,θ)-split torus A of G. This will be treated in the next section.

The restricted Weyl group acts on the quadratic elements of A. This action shall be used to derive in section 8 a set of quadratic elements representing the classes in $\mathcal{C}(\sigma,\theta)$. In fact to have the restricted Weyl group acting on the quadratic elements in A we need a kind of standard pair in each set $\mathcal{C}(\sigma,\theta)$ to start from. This will be defined in (6.11).
(ii) If $(\sigma,\theta)\in\mathcal{F}$, A a (σ,θ)-split torus of G and $\epsilon\in A$, $\epsilon^2\in Z(G)$, then the pairs $(\sigma,\theta\mathrm{Int}(\epsilon))$ and $(\sigma\mathrm{Int}(\epsilon),\theta)$ are isomorphic. Namely, take $c\in A$ such that $c^2=\epsilon$. Then conjugating by Int(c) gives the desired isomorphism.

6. The restricted root system of (σ,θ) and standard pairs.

6.1. Let (σ,θ) be a pair of commuting involutorial automorphisms of G and A a non-trivial maximal (σ,θ)-split torus of G.

In this section we shall prove that $\Phi(A)$ is a root system in the vector space $X^*(A)\otimes_{\mathbb{Z}}\mathbb{R}$ and that the corresponding Weyl group is W(A). Since A is also a maximal σ-split torus of $G^{\circ}_{\sigma\theta}$ (see (5.3)), we already know, by (1.6), that $\Phi(A,G^{\circ}_{\sigma\theta})$ is a root system. The relations between $\Phi(A)$ and $\Phi(A,G^{\circ}_{\sigma\theta})$ will be treated and moreover we shall show that there exists a pair $(\sigma,\theta)\in\mathcal{F}_A(\sigma,\theta)$ for which the Weyl groups of $\Phi(A)$ and $\Phi(A,G^{\circ}_{\sigma\theta})$ coincide. In particular in this case every $w\in W(A)$ has a representative in $(G_\sigma\cap G_\theta)^{\circ}$. This will be used for the classification of those quadratic elements of A, which represent an isomorphism class in $\mathcal{C}(\sigma,\theta)$.

6.2. Let T be a (σ,θ)-stable maximal torus of G and let $A=T^-_{\sigma,\theta}$. For the moment we do not yet assume that A is a maximal (σ,θ)-split torus of G. This will only be needed to obtain all the reflections in W(A) (see (6.11)).

For $\lambda\in\Phi(A)$ let $\mathfrak{g}(A,\lambda)$ be the corresponding root space. Since $\sigma\theta(\lambda)=\lambda$, we have $\sigma\theta(\mathfrak{g}(A,\lambda))=\mathfrak{g}(A,\lambda)$. Put

$$\mathfrak{g}(A,\lambda)^{\pm}_{\sigma\theta}=\{X\in\mathfrak{g}(A,\lambda)|\sigma\theta(X)=\pm X\},$$
$$m^{\pm}(\lambda,\sigma\theta)=\dim_F\mathfrak{g}(A,\lambda)^{\pm}_{\sigma\theta},$$
$$\Phi(T,\lambda)=\{\alpha\in\Phi(T)|\alpha\,|_A=\lambda\}\text{ and}$$
$$m(\lambda)=\dim_F\mathfrak{g}(A,\lambda)=m^+(\lambda,\sigma\theta)+m^-(\lambda,\sigma\theta)=|\Phi(T,\lambda)|.$$

6.3. Definition. For $\lambda\in\Phi(A)$ call $m(\lambda)$ the *multiplicity* of λ and $(m^+(\lambda,\sigma\theta),m^-(\lambda,\sigma\theta))$ the *signature* of λ.

6.4. Remark. If $a\in A$ is a quadratic element and $\lambda\in\Phi(A)$ is such that $\lambda(a)=-1$, then $m^+(\lambda,\sigma\theta)=m^+(\lambda,\sigma\theta\mathrm{Int}(a))$.
Whether a root of $\Phi(A)$ is contained in $\Phi(A,G^{\circ}_{\sigma\theta})$ can be detected from its signature:

6.5. Lemma. *Let* $\lambda\in\Phi(A)$*, then* $\lambda\in\Phi(A,G^{\circ}_{\sigma\theta})$ *if and only if* $m^+(\lambda,\sigma\theta)>0$.

6.6. Quadratic elements of A with respect to a basis of $\Phi(A)$

Let Δ be a (σ,θ)-basis of $\Phi(T)$ and let $\overline{\Delta}_{\sigma,\theta}$ denote the restricted basis of (2.2). The linear independence of the elements of $\overline{\Delta}_{\sigma,\theta}$ (see (2.3)) implies that for all $\lambda\in\overline{\Delta}_{\sigma,\theta}$ there exists $\gamma_\lambda\in X_*(A)$ such that $<\lambda,\gamma_{\lambda'}>=\delta_{\lambda,\lambda'}$ for $\lambda,\lambda'\in\overline{\Delta}_{\sigma,\theta}$.

For $\lambda\in\overline{\Delta}_{\sigma,\theta}$ put $\epsilon_\lambda=\gamma_\lambda(-1)$. Then $\epsilon_\lambda^2=\gamma_\lambda(-1)\gamma_\lambda(-1)=\gamma_\lambda(+1)=e$, hence ϵ_λ is a quadratic element of A. If $\Phi(T)$ has a (σ,θ)-basis, which is simultaneously a σ-

and θ-basis, then we can describe ϵ_λ also in terms of one-parameter subgroups of $X_*(T)$ (for this see section 8).

6.7. Lemma. *Let Δ be a (σ,θ)-basis of $\Phi(T)$. There exists $\epsilon\in A$ with $\epsilon^2 = e$ such that for $\lambda\in\bar{\Delta}_{\sigma,\theta}$*

$$m^+(\lambda,\sigma\theta\mathrm{Int}(\epsilon))\geqslant m^-(\lambda,\sigma\theta\mathrm{Int}(\epsilon)).$$

In particular we then have: $\bar{\Delta}_{\sigma,\theta}\subset\Phi(A,G^o_{\sigma\theta\mathrm{Int}(\epsilon)})$.

Proof. Taking ϵ to be the product of those ϵ_λ, $\lambda\in\bar{\Delta}_{\sigma,\theta}$, for which $m^+(\lambda,\sigma\theta)<m^-(\lambda,\sigma\theta)$, the result follows from (6.4) and (6.6).

Let $E=X^*(T)\otimes_{\mathbf{Z}}\mathbf{R}$ and let $E^-_{\sigma,\theta}$ be the common (-1)-eigenspace of σ and θ in E. Take a positive definite σ, θ and W(T)-invariant inner product (.,.) in E. We identify W(A) with its image in $GL(E^-_{\sigma,\theta})$ and, for $\lambda\in\Phi(A)$, let $s_\lambda\in GL(E^-_{\sigma,\theta})$ denote the reflection in the hyperplane $E^-_{\sigma,\theta}(\lambda)=\{x\in E^-_{\sigma,\theta}|(x,\lambda)=0\}$. So $s_\lambda(x)=x-2(\lambda,x)(\lambda,\lambda)^{-1}\lambda$. If A is a maximal (σ,θ)-split torus of G, then, by (1.6), $\Phi(A,G^o_{\sigma\theta})$ is a root system. Hence for every $\lambda\in\Phi(A)$ with $m^+(\lambda,\sigma\theta)\neq 0$, there exists a reflection s_λ in W(A). Combining this with (6.7) we obtain:

6.8. Lemma. *Let A be a maximal (σ,θ)-split torus of G. If A is not central, then $N_G(A)\neq Z_G(A)$. In particular for every $\lambda\in\Phi(A)$ there exists $n\in N_G(A)$ whose image in W(A) is s_λ.*

Proof. The first statement readily follows from the above remark and the second statement follows from this by considering $Z_G((\mathrm{Ker}\lambda)^o)$, in which A is not central. Then any $n\in N_G(A)\cap Z_G((\mathrm{Ker}\lambda)^o)$ such that $n\notin Z_G(A)$, represents the reflection s_λ in W(A).

Now that we have constructed the reflections in W(A), we can follow the proof of Springer [24,9.1.9] to show:

6.9. Lemma. *(i) W(A) is generated by the reflections s_λ, $\lambda\in\Phi(A)$;*
(ii) If $\lambda\in\Phi(A)$ and $\chi\in X^*(A)$ then $2(\lambda,\lambda^{-1})(\lambda,\chi)\in\mathbf{Z}$.
(See also Richardson [20,4.5]).

For $\lambda\in\Phi(A)$ define now the dual root as the unique $\lambda^\vee\in X_*(A)$ such that $<\chi,\lambda^\vee> = 2(\lambda,\lambda^{-1})(\lambda,\chi)\in\mathbf{Z}$ for all $\chi\in X^*(A)$ (i.e. $s_\lambda(\chi)=\chi-<\chi,\lambda^\vee>\lambda$). Then denoting the set of dual roots by $\Phi^\vee(A)$ we have proved:

6.10. Proposition. *Let A be a non-central maximal (σ,θ)-split torus of G. Then the quadruple $(X^*(A),\Phi(A),X_*(A),\Phi^\vee(A))$ is a root datum in the sense of (2.1). In particular $\Phi(A)$ is a root system in the subspace E′ of $X^*(A)\otimes_{\mathbf{Z}}\mathbf{R}$ spanned by $\Phi(A)$ and its Weyl group is given by the restriction of W(A) to E′.*

Put $\Phi(A)'=\{\lambda\in\Phi(A)|\tfrac{1}{2}\lambda\notin\Phi(A)\}$.

6.11. Standard pairs

For the remaining part of this section we assume A to be maximal (σ,θ)-split and non-central.
In order to have the Weyl group of A acting on the quadratic elements in a family $\mathcal{F}_A(\sigma,\theta)$ we need representatives in $(G_\sigma \cap G_\theta)^o$. In case that the Weyl groups of $\Phi(A)$ and $\Phi(A,G^o_{\sigma\theta})$ coincide, this condition is satisfied, because every element of $W(A,G^o_{\sigma\theta})$ has a representative in $(G^o_{\sigma\theta})^o_\sigma = (G_\sigma \cap G_\theta)^o$. This leads to the following definition:

Definition. A pair of commuting involutorial automorphisms (σ,θ) of G is called a *standard pair* if $m^+(\lambda,\sigma\theta) \geqslant m^-(\lambda,\sigma\theta)$ for any maximal (σ,θ)-split torus A of G and any $\lambda \in \Phi(A)'$.

6.12. Lemma. *Let* σ, θ *and* A *be as in (6.11) and let* $\bar{\Delta}(A)$ *be a basis of* $\Phi(A)$. *If* $m^+(\lambda,\sigma\theta) \geqslant m^-(\lambda,\sigma\theta)$ *for any* $\lambda \in \bar{\Delta}(A)$, *then* $m^\pm(\lambda,\sigma\theta) = m^\pm(w(\lambda),\sigma\theta)$ *for any* $\lambda \in \Phi(A)$, $w \in W(A)$. *In particular* (σ,θ) *is a standard pair.*

Proof. Since $\bar{\Delta}(A)$ is a basis of both $\Phi(A)$ and $\Phi(A,G^o_{\sigma\theta})$, their Weyl groups coincide. By (1.6) every $w \in W(A,G^o_{\sigma\theta})$ has a representative in $(G^o_{\sigma\theta})^o_\sigma = (G_\sigma \cap G_\theta)^o$, so we get $m^+(w(\lambda),\sigma\theta) = m^+(\lambda,\sigma\theta)$ for any $\lambda \in \Phi(A)$, $w \in W(A)$. However since also $m(w(\lambda)) = m(\lambda)$ for any $\lambda \in \Phi(A)$, $w \in W(A)$, we have $m^-(w(\lambda),\sigma\theta) = m^-(\lambda,\sigma\theta)$, which proves the first statement.
Finally, observing that for any $\lambda \in \Phi(A)'$, there exists $w \in W(A)$ such that $w(\lambda) \in \bar{\Delta}(A)$, the result is a consequence of (5.12).

Using (6.7) this lemma implies immediately:

6.13. Theorem. *Every family* $\mathcal{F}(\sigma,\theta)$ *contains a standard pair.*

We shall see later, as a consequence of the classification, that the standard pair in $\mathcal{F}(\sigma,\theta)$ is unique up to isomorphism.

6.14. Note that if G is of adjoint type and (σ,θ), $(\sigma,\theta \mathrm{Int}(\epsilon))$ ($\epsilon \in A$, $\epsilon^2 = e$) are standard pairs in $\mathcal{F}_A(\sigma,\theta)$, then ϵ is a product of a number of the ϵ_λ ($\lambda \in \bar{\Delta}(A)$), where $\bar{\Delta}(A)$ is a basis of $\Phi(A)$ and ϵ_λ is as defined in (6.6) (see also (8.11)). But since both pairs are standard we must have $m^+(\lambda,\sigma\theta\mathrm{Int}(\epsilon)) \geqslant m^-(\lambda,\sigma\theta\mathrm{Int}(\epsilon))$ and $m^+(\lambda,\sigma\theta) \geqslant m^-(\lambda,\sigma\theta)$ for all $\lambda \in \bar{\Delta}(A)$. It follows that ϵ is a product of those ϵ_λ for which $m^+(\lambda,\sigma\theta) = m^-(\lambda,\sigma\theta)$ (see also (6.4)). Thus, in order to show that the standard pair (σ,θ) is unique up to isomorphism we need to show that $(\sigma,\theta\mathrm{Int}(\epsilon_\lambda))$ and (σ,θ) are isomorphic if $m^+(\lambda,\sigma\theta) = m^-(\lambda,\sigma\theta)$. This will be proved in (8.14).

6.15. Corollary. *Let* (σ,θ) *be a standard pair. Then any* $w \in W(A)$ *has a representative in* $(G_\sigma \cap G_\theta)^o$.

6.16. Corollary. *Let* (σ,θ) *be a pair of commuting involutorial automorphisms of* G *(not necessarily standard) and* A *a maximal* (σ,θ)*-split torus of* G. *Then any* $w \in W(A)$ *has a representative in* $N_{G^o_\sigma}(A)$ *as well as in* $N_{G^o_\theta}(A)$.

Proof. By (6.7) there exists an $\epsilon \in A$, $\epsilon^2 = e$ such that $(\sigma,\theta\mathrm{Int}(\epsilon))$ is a standard pair. Since $(\sigma,\theta\mathrm{Int}(\epsilon))$ is isomorphic to $(\sigma\mathrm{Int}(\epsilon),\theta)$ (see (5.21(ii))) the result follows from (6.15).

6.17. If (σ,θ) is a pair of commuting involutorial automorphisms of G, normally related to a maximal torus T, then by (1.6) both $\overline{\Phi}_\theta$ and $\overline{\Phi}_\sigma$ are root systems with Weyl groups $\overline{W}_\theta = W(T_\theta^-)$ and $\overline{W}_\sigma = W(T_\sigma^-)$ respectively. Now if $A = T_{\sigma,\theta}^-$, then we can see $\Phi(A) = \overline{\Phi}_{\sigma,\theta}$ also as the set of restricted roots of $\overline{\Phi}_\theta$ with respect to σ (or of $\overline{\Phi}_\sigma$ with respect to θ). Now (6.16) implies that we can choose representatives in W(T), commuting with θ (resp. σ). So together with (2.7) we have obtained:

6.18. Proposition. *Let $(\sigma,\theta)\in\mathcal{F}$ be normally related to* T *and identify* $W(T_{\sigma,\theta}^-)$, $W(T_\theta^-)$ *and* $W(T_\sigma^-)$ *with* $\overline{W}_{\sigma,\theta}$, $\overline{W}_\theta$ *and* $\overline{W}_\sigma$ *respectively. Then* $\overline{W}_{\sigma,\theta} \cong \overline{W}_\theta^\sigma \cong \overline{W}_\sigma^\theta$, *where* $\overline{W}_\theta^\sigma$ *and* $\overline{W}_\sigma^\theta$ *are as defined in (2.7).*

7. Classification of admissible pairs of commuting involutions

In this section we shall classify the isomorphism classes of admissible pairs of commuting involutions. To do this we shall first show that this classification can be obtained from the classification of single admissible involutions (see section 4), by use of a simple (combinatorical) condition on a (σ,θ)-basis of Φ. Moreover, the pair of isomorphism classes (σ,θ) and (θ,σ) can be represented by a diagram.
We fix a maximal torus T of G and write Φ for $\Phi(T)$, X for $X^*(T)$ and W for W(T). Let $(\sigma,\theta)\in \mathrm{Aut}(X,\Phi)$ be a pair of commuting involutions.

7.1. A strong (σ,θ)-order on Φ

Definition. A (σ,θ)-order $\succ$ on Φ is called a *strong (σ,θ)-order* if it is simultaneously a σ- and θ-order of Φ. A basis of Φ with respect to a strong (σ,θ)-order will be called a *strong (σ,θ)-basis.*

A strong (σ,θ) order does not always exist. Another way to characterize such an order is given in the following result:

7.2. Proposition. *Let (σ,θ) be a pair of commuting involutions of* (X,Φ). *Then Φ has a strong (σ,θ)-order if and only if* $\Phi_0(\sigma,\theta) = \Phi_0(\sigma)\cup\Phi_0(\theta)$.

Proof. If $\Phi_0(\sigma,\theta)$ satisfies this condition, then $\Phi_0(\sigma,\theta)$ has a strong (σ,θ)-order, which we can extend to a strong (σ,θ)-order on Φ by choosing an arbitrary order on $\overline{\Phi}_{\sigma,\theta}$. So it suffices to prove the "only if" statement.
Assume $\succ$ is a strong (σ,θ)-order on Φ and let Φ^+ be the set of positive roots with respect to this order. Now the induced order on $\Phi_0(\sigma,\theta)$ is also a strong (σ,θ)-order of $\Phi_0(\sigma,\theta)$. Suppose that there is $\alpha\in\Phi^+\cap\Phi_0(\sigma,\theta)$ such that $\sigma(\alpha)\neq\alpha\neq\theta(\alpha)$. Then $\alpha\succ 0$, $-\sigma(\alpha)\succ 0$, $-\theta(\alpha)\succ 0$, $\sigma\theta(\alpha)\succ 0$.Hence $0=\alpha-\sigma(\alpha)-\theta(\alpha)+\sigma\theta(\alpha)\succ 0$, a contradiction. This proves the result.

7.3. Remarks. (1) As in (5.9) one shows that $\Phi_0(\sigma,\theta) = \Phi_0(\sigma)\cup\Phi_0(\theta)$ if and only if for each irreducible component Φ_1 of $\Phi_0(\sigma,\theta)$ we have $\sigma|_{\Phi_1} = \mathrm{id}$ or $\theta|_{\Phi_1} = \mathrm{id}$.
(2) If (σ,θ) is an admissible pair of commuting involutions, then it follows from (5.9) that Φ has a strong (σ,θ)-basis. These involutions satisfy even a stronger condition, as follows from the next results:

7.4. Lemma. *Let (σ,θ) be an admissible pair of commuting involutions of* (X,Φ) *and* Δ *a strong (σ,θ)-basis of Φ. Write $\theta = -\theta^* w_0(\theta)$ as in (2.8). Then $\Phi_0(\sigma)\cap\Phi_0(\theta)$ is invariant under $w_0(\theta)$.*

Proof. Since $\Phi_0(\theta)$ is σ-stable and σ is admissible it follows by (3.12) that $\Phi_0(\theta)$ is σ-normal. The result follows now from lemma (2.19) and remark (2.9).

7.5. Lemma. *Let (σ,θ), Δ be as in (7.4). Then σ, $w_0(\theta)$ and θ^* commute.*

Proof. Since θ^* and $w_0(\theta)$ commute it suffices to show that σ and $w_0(\theta)$ commute. For this we show that $\sigma w_0(\theta)\sigma(\Phi_0(\theta)^+) = \Phi_0(\theta)^-$. Let $\alpha\in\Phi_0(\theta)^+$. If $\alpha\in\Phi_0(\theta)\cap\Phi_0(\sigma)$ then $\sigma w_0(\theta)\sigma(\alpha) = \sigma(w_0(\theta)(\alpha)) = w_0(\theta)(\alpha)\in\Phi_0(\theta)^-$.
If $\alpha\in\Phi_0(\theta)-(\Phi_0(\theta)\cap\Phi_0(\sigma))$, then $\sigma(\alpha)\in\Phi_0(\theta)^-$, $\sigma(\alpha)\notin\Phi_0(\theta)\cap\Phi_0(\sigma)$. On the other hand by (7.4) we also have $w_0(\theta)\sigma(\alpha)\in\Phi_0(\theta)^+$, $w_0(\theta)\sigma(\alpha)\notin\Phi_0(\theta)\cap\Phi_0(\sigma)$, which implies $\sigma w_0(\theta)\sigma(\alpha)\in\Phi_0(\theta)^-$. It follows that σ and $w_0(\theta)$ commute, hence we are done.

7.6. Lemma. *Let (σ,θ), Δ be as in (7.4). Then $w_0(\sigma)$ and θ^* commute.*

Proof. Since, by (7.5), $\Phi_0(\sigma)$ is θ^*-stable, we have $\theta^* w_0(\sigma)\theta^*(\Phi_0(\sigma)^+) = \Phi_0(\sigma)^-$, hence $\theta^* w_0(\sigma)\theta^* = w_0(\sigma)$.

Summarizing (7.4), (7.5), (7.6) we have obtained the following result:

7.7. Theorem. *Let (σ,θ) be an admissible pair of commuting involutions of* (X,Φ) *and* Δ *a strong (σ,θ)-basis of Φ. Then $w_0(\theta)$, $w_0(\sigma)$, θ^* and σ^* mutually commute.*

7.8. Remark. Note that for the proof of this result it is only needed that Φ has a strong (σ,θ)-order and that Φ is both σ- and θ-normal. Under these conditions it is also possible to prove that $\overline{\Phi}_{\sigma,\theta}$ is a root system with Weyl group $\overline{W}_{\sigma,\theta}$.

7.9. Definition. A pair of commuting involutions (σ,θ) of (X,Φ) is called *basic* if Φ has a strong (σ,θ)-basis Δ for which $w_0(\theta)$, $w_0(\sigma)$, σ^* and θ^* mutually commute.

These basic pairs of commuting involutions suffice to obtain all the admissible pairs of commuting involutions of (X,Φ). We still need a characterization of the roots in Δ lying above a restricted root in $\overline{\Delta}_{\sigma,\theta}$.

7.10. Lemma. *Let (σ,θ) be a basic pair of commuting involutions of* (X,Φ) *and let* Δ *be a strong (σ,θ)-basis of Φ. If α, $\beta\in\Delta$ such that $\alpha\neq\beta$ and $\pi(\alpha)=\pi(\beta)$, then α equals $\theta^*(\beta)$ or $\sigma^*(\beta)$ or $\sigma^*\theta^*(\beta)$.*

Proof. Let V be the subspace of $X^*(T)\otimes_{\mathbf{Z}}\mathbf{R}$ spanned by Φ (see (2.1)). Arguing as in (2.10) we obtain:

$$\alpha+\theta^*(\alpha)+\sigma^*(\alpha)+\sigma^*\theta^*(\alpha) = \beta+\theta^*(\beta)+\sigma^*(\beta)+\sigma^*\theta^*(\beta)+\delta$$

with $\delta\in\operatorname{Span}\Delta_0(\sigma,\theta)$. From this we deduce, as in (2.10), that $\delta = 0$ and α equals $\theta^*(\beta)$ or $\sigma^*(\beta)$ or $\sigma^*\theta^*(\beta)$.

7.11. Theorem. *Let* (σ,θ) *be a pair of commuting involutions of* (X,Φ)*. Then* (σ,θ) *is admissible if and only if* (σ,θ) *is basic and both* σ *and* θ *are admissible.*

Proof. This result is proved by using more or less the same arguments as in (3.7).
If (σ,θ) is admissible, then both σ and θ are admissible involutions and also (σ,θ) is basic by (7.7). So it suffices to show the "if" statement.
Assume (σ,θ) is basic and σ, θ are admissible involutions of (X,Φ). Let $\{x_\alpha\}_{\alpha\in\Phi(T)}$ be a realization of Φ in G as in (3.1) and let $\bar{\sigma},\bar{\theta}\in \mathrm{Aut}(G,T)$ be involutions inducing σ resp. θ on (X,Φ).
Since both $\overline{\sigma\theta}$ and $\overline{\theta\sigma}$ induce $\sigma\theta$ on (X,Φ) it follows from the isomorphism theorem (see Springer [24,11.4.3]) that there is a $t\in T$ such that $\overline{\sigma\theta} = \overline{\theta\sigma}\mathrm{Int}(t)$.
If $\alpha\in\Phi_0(\sigma,\theta)$ then, since $\Phi_0(\sigma,\theta) = \Phi_0(\theta)\cup\Phi_0(\sigma)$, we have by (3.4) $c_{\alpha,\bar{\theta}} = c_{\sigma(\alpha),\bar{\theta}} = 1$ or $c_{\alpha,\bar{\sigma}} = c_{\theta(\alpha),\bar{\sigma}} = 1$. But then

$$c_{\alpha,\bar{\theta}}c_{\theta(\alpha),\bar{\sigma}} = c_{\alpha,\bar{\sigma}}c_{\sigma(\alpha),\bar{\theta}}$$

which implies $\alpha(t) = 1$.
Let Δ be a strong (σ,θ)-basis of Φ and write $\sigma = -\sigma^* w_0(\sigma)$, $\theta = -\theta^* w_0(\theta)$ with respect to Δ (see (2.8)). Since $\bar{\theta}\bar{\sigma}\bar{\theta} = \bar{\sigma}\mathrm{Int}(t)$ is an involution, we get $\mathrm{Int}(\sigma(t)t) = \mathrm{id}$, hence $\sigma(t)t\in Z(G)$. Similarly we get $\theta(t)t\in Z(G)$. It follows now from (3.5) that for any $\alpha\in\Phi$ we have:

$$\alpha(t) = \theta^*(\alpha)(t) = \sigma^*(\alpha)(t) = \sigma^*\theta^*(\alpha)(t).$$

If $\gamma\in\bar{\Delta}_{\sigma,\theta}$ and $\alpha,\beta\in\Delta$, $\alpha\neq\beta$ such that $\pi(\alpha) = \pi(\beta) = \gamma$, then it follows from (7.10) that $\beta = \sigma^*(\alpha)$ or $\theta^*(\alpha)$ or $\sigma^*\theta^*(\alpha)$. Similarly as in (3.7), take now, for each $\gamma\in\bar{\Delta}_{\sigma,\theta}$, an $\alpha\in\Delta$ such that $\gamma = \pi(\alpha)$ and choose $u_\gamma\in T_{\bar{\sigma},\bar{\theta}}$ such that $\lambda(u_\gamma) = 1$ for $\lambda\in\bar{\Delta}_\theta$, $\lambda\neq\gamma$ and $\gamma(u_\gamma^4) = \alpha(t)$.
Take $u = \prod_{\gamma\in\bar{\Delta}_{\sigma,\theta}} u_\gamma$. Then by (7.10) and (3.5) we find $\alpha(tu^4) = 1$ for all $\alpha\in\Delta$. So $tu^4\in Z(G)$ and it follows that $\mathrm{Int}(u)^{-1}\bar{\sigma}\mathrm{Int}(u)\bar{\theta} = \bar{\theta}\mathrm{Int}(u)^{-1}\bar{\sigma}\mathrm{Int}(u)$. This proves the result.

7.12. Related involutions of (X,Φ)

Whether two involutions σ and θ of (X,Φ) are basic or not can be detected from their indices. To show this we need besides conditions to assure that σ^*, θ^*, $w_0(\theta)$ and $w_0(\sigma)$ commute, also an order on Φ, which is simultaneously a σ- and θ-order.

Definition. Two involutions σ,θ of (X,Φ) (not necessarily commuting) are said to be *related* if Φ has a basis Δ, which is simultaneously a σ- and θ-basis of Φ. In this case Δ is called the *relating basis* of Φ (relative to (σ,θ)).

Analogously to (2.11) we can define an index for a relating pair of involutions of (X,Φ):

7.13. (σ,θ)-indices

Assume that X is semisimple.
For a pair of related involutions (σ,θ) of (X,Φ) and a relating basis Δ of Φ, call the sextuple $(X,\Delta,\Delta_0(\sigma),\Delta_o(\theta),\sigma^*(\Delta),\theta^*(\Delta))$ an *index* of (σ,θ) (or (σ,θ)-index).
This (σ,θ)-index determines both σ and θ. If (σ,θ) is basic (resp. admissible) then we call this also a *basic* (resp. *admissible*) (σ,θ)-index. Two indices

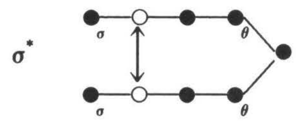
$(X,\Delta,\Delta_o(\sigma_1),\Delta_o(\theta_1),\sigma_1^*(\Delta),\theta_1^*(\Delta))$ and $(X,\Delta',\Delta_o'(\sigma_2),\Delta_o'(\theta_2),\sigma_2^*(\Delta'),\theta_2^*(\Delta'))$ are said to be *isomorphic* if there is a $w\in W(\Phi)$, which maps $(X,\Delta,\Delta_o(\sigma_1),\Delta_o(\theta_1))$ onto $(X,\Delta',\Delta_o'(\sigma_2),\Delta_o'(\theta_2))$ and which satisfies:

$$w\theta_1^*(\Delta)w^{-1} = \theta_2^*(\Delta') \text{ and } w\sigma_1^*(\Delta)w^{-1} = \sigma_2^*(\Delta').$$

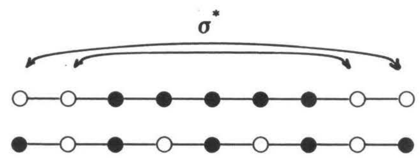
7.14. Remarks. (1) The above index of (σ,θ) determines the indices of both σ and θ and vice versa. When σ and θ commute, then this definition of (σ,θ)-index is an extension of the definition of the Satake diagram corresponding to the action of $\Gamma = \{\mathrm{id},-\sigma,-\theta,\sigma\theta\}$ on (X,Φ). In our situation we have additional actions of σ and θ on $\Phi_o(\sigma,\theta)$.

(2) We can make a diagrammatic representation of the (σ,θ)-index by colouring black those vertices of the ordinary Dynkin diagram of Φ, which represent roots in $\Delta_o(\sigma)\cup\Delta_o(\theta)$ and giving the vertices of $\Delta_o(\sigma)\cup\Delta_o(\theta)$ which are not in $\Delta_o(\sigma)\cap\Delta_o(\theta)$ a label σ or θ if $\sigma(\alpha)\neq\alpha$ or $\theta(\alpha)\neq\alpha$ respectively. The actions of σ^* and θ^* are indicated by arrows. Like in (2.12) we omit again the actions of σ^*, θ^* on $X_o(\sigma)$, $X_*(T)$ respectively.

Here is an example with Φ of type A_9:



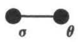
This (σ,θ)-index is obtained by gluing together the indices



of σ resp. θ with the above recipe

Note that such a diagram represents the indices of both (σ,θ) and (θ,σ).

(3) If σ,θ are related involutions of (X,Φ), then they need not commute. One can easily see this in the following example of a (σ,θ)-index, where Φ is of type A_2:



From (7.2) we see that σ and θ cannot commute.

(4) An index of (σ,θ) may depend again on the choice of the (σ,θ)-basis of Φ. Similarly to (2.13) we can prove:

7.15. Proposition. *Assume* X *is semisimple and let* (σ,θ) *be an admissible pair of commuting involutions of* (X,Φ). *Let* Δ,Δ' *be strong* (σ,θ)*-bases of* Φ. *Then* $(X,\Delta,\Delta_o(\sigma),\Delta_o(\theta),\sigma^*(\Delta),\theta^*(\Delta))$ *and* $(X,\Delta',\Delta_o'(\sigma),\Delta_o'(\theta),\sigma^*(\Delta'),\theta^*(\Delta'))$ *are isomorphic.*
In particular there is a bijection between the W*-isomorphism classes of admissible pairs of commuting involutions of* (X,Φ) *and the isomorphism classes of indices of basic pairs of admissible involutions of* (X,Φ).

Proof. Since $\overline{W}_\theta^\sigma$ corresponds to the Weyl group of $\overline{\Phi}_{\sigma,\theta}$ (see (6.18)), there is by (2.5) a unique element $w\in W_1^\theta(\sigma,\theta)$ such that $w(\Delta)=\Delta'$. Since $w\in W_1^\theta(\sigma,\theta)$ we have $w(\Delta_o(\sigma,\theta)) = \Delta_o'(\sigma,\theta)$ and $w(\Delta_o(\theta)) = \Delta_o'(\theta)$. But by (7.2) $\Delta_o(\sigma,\theta) = \Delta_o(\sigma)\cup\Delta_o(\theta)$, so

w maps $(X,\Delta,\Delta_o(\sigma),\Delta_o(\theta))$ onto $(X,\Delta',\Delta_o'(\sigma),\Delta_o'(\theta))$.
Similarly as in the proof of (2.13) one shows now that w satisfies: $w\theta^*(\Delta)w^{-1} = \theta^*(\Delta')$ and $w\sigma^*(\Delta)w^{-1} = \sigma^*(\Delta')$, which proves the first statement. The second statement follows immediately from this and theorem (7.11).

Whether two related involutions of (X,Φ) are basic or not can be detected now directly from their (σ,θ)-index:

7.16. Theorem. *Let σ,θ be related involutions of* (X,Φ) *and Δ a relating basis of Φ with respect to (σ,θ). Then (σ,θ) is basic if and only if*

(1) σ^* and θ^* commute,

(2) $\Delta_o(\theta)$ is σ^*-stable and $\Delta_o(\sigma)$ is θ^*-stable,

(3) for every connected component Δ_1 of $\Delta_o(\theta)\cup\Delta_o(\sigma)$ we have $\Delta_1\subset\Delta_o(\sigma)$ or $\Delta_1\subset\Delta_o(\theta)$.

Proof. If (σ,θ) is basic, then (1) and (2) are clear and (3) follows from (7.2), using the same arguments as in (5.9). So assume (1), (2) and (3) hold.
Then (2) implies that $w_o(\theta)$ and σ^* (resp. $w_o(\sigma)$ and θ^*) commute, because $\sigma^*w_o(\theta)\sigma^*(\Delta_o(\theta)) = -\Delta_o(\theta)$ (resp. $\theta^*w_o(\sigma)\theta^*(\Delta_o(\sigma)) = -\Delta_o(\sigma)$). So it suffices to show that $w_o(\sigma)$ and $w_o(\theta)$ commute or equivalently: $w_o(\sigma)w_o(\theta)w_o(\sigma)(\Delta_o(\theta)) = -\Delta_o(\theta)$.
If $\alpha\in\Delta_o(\sigma)\cap\Delta_o(\theta)$, then $w_o(\sigma)(\alpha) = -\sigma^*(\alpha)\in\Delta_o(\sigma)\cap\Delta_o(\theta)$ by (2). Similarly, since $\Delta_o(\sigma)$ is θ^*-stable, we have $\theta^*w_o(\sigma)(\alpha) = -\theta^*w_o(\sigma)(\alpha)\in\Delta_o(\sigma)\cap\Delta_o(\theta)$. Hence clearly $w_o(\sigma)w_o(\theta)w_o(\sigma)\in-(\Delta_o(\sigma)\cap\Delta_o(\theta))$.
If $\alpha\in\Delta_o(\theta)-(\Delta_o(\sigma)\cap\Delta_o(\theta))$, then let $\Delta_1\subset\Delta_o(\sigma)\cup\Delta_o(\theta)$ be the connected component such that $\alpha\in\Delta_1$. By (3) we have $\Delta_1\subset\Delta_o(\theta)$ and $w_o(\sigma)(\alpha) = \alpha + \sum_{\beta\in\Delta_o(\sigma)\cap\Delta_1} m_\beta\beta$ with $m_\beta\in\mathbb{Z}$, $m_\beta\geqslant 0$. Since, by (2), we have:

$-\theta^*(\alpha) = w_o(\theta)(\alpha)\in-(\Delta_o(\theta)-(\Delta_o(\sigma)\cap\Delta_o(\theta)))$,

it follows that $w_o(\sigma)w_o(\theta)w_o(\sigma)\in\Phi_o(\theta)^-$, which proves the result.

With the above result and (7.15) it became an easy exercise to obtain all the indices of basic pairs of admissible involutions of (X,Φ). Before we describe them, we need again a notion of irreducibility.

7.17. Definition. A (σ,θ)-index $S = (X,\Delta,\Delta_o(\sigma),\Delta_o(\theta),\sigma^*,\theta^*)$ is called *irreducible* if Δ is not the union of two mutually orthogonal σ^*- and θ^*-stable non-empty subsets Δ_1, Δ_2. S is called *absolutely irreducible* if Δ is connected.

Note that S is irreducible if and only if $\overline{\Delta}_{\sigma,\theta}$ is connected.

7.18. Classification of irreducible admissible (σ,θ)-indices

Assume that X is semisimple and of adjoint type. Let (σ,θ) be a basic pair of non-trivial admissible involutions of (X,Φ) with (σ,θ)-index $S = (X,\Delta,\Delta_o(\sigma),\Delta_o(\theta),\sigma^*,\theta^*)$. We assume that S is irreducible. The standard pair in $\mathcal{H}(\sigma,\theta)$ (see (6.11)) will also be denoted by (σ,θ). We shall use the Cartan notation to describe involutions, whose index is absolutely irreducible (see table II).
If S is absolutely irreducible, then we denote the pair (σ,θ) by $X_l^{p,q}$(type σ,type θ), where X denotes the type of Φ, i.e. one of A, B, ..., G and $l = \text{rank}\Phi$, $p = \text{rank}\overline{\Delta}_\sigma$, $q = \text{rank}\overline{\Delta}_\theta$. For example $A_{2l-1}^{2l-1,l}(\text{I},\text{III}_b)$ means that Φ is of type A_{2l-1}, σ is of type

AI, θ is of type AIII$_b$ and rank$\bar{\Delta}_\sigma = 2l-1$, rank$\bar{\Delta}_\theta = l$. We shall use the same notation for the isomorphism class of the standard pair within a family $\mathcal{F}(\sigma,\theta)$. To describe the other isomorphism classes in $\mathcal{F}(\sigma,\theta)$ we add the representing quadratic element in $T^-_{\sigma,\theta}$. So if we write $\epsilon_1, \cdots, \epsilon_p$ for the quadratic elements in $T^-_{\sigma,\theta}$ with respect to $\bar{\Delta}_{\sigma,\theta} = \{\lambda_1, \cdots, \lambda_p\}$ (see (6.5)), then in the above example $A^{2l-1,l}_{2l-1}(\mathrm{I}, \mathrm{III}_b)$ denotes the standard pair (σ,θ) in $\mathcal{F}(\sigma,\theta)$ and $A^{2l-1,l}_{2l-1}(\mathrm{I}, \mathrm{III}_b, \epsilon_l)$ denotes the pair $(\sigma,\theta\mathrm{Int}(\epsilon_l))$. For a classification of these quadratic elements, see section 8.
To make identifications with Berger's classification of affine symmetric spaces, it is sometimes useful to take $\epsilon_o = e$ and to denote the standard pair by $(\sigma,\theta\mathrm{Int}(\epsilon_o))$ (see table II).
In the classification of admissible irreducible (σ,θ)-indices with both σ and θ non-trivial, we have six cases:

7.18.1. Φ is irreducible and $\sigma = \theta$

In this case, the (σ,θ)-index equals the index of θ (and σ). If $F = \mathbb{C}$, then the standard pair corresponds to the complexification of a Riemannian symmetric pair and the quadratic elements give the K_ϵ-spaces as described in Oshima-Sekiguchi [18]. See also section 9. As for the signatures of the standard pair, we note that $m^-(\lambda,\sigma\theta) = 0$ for all $\lambda \in \bar{\Delta}_\theta$, so we have $m^+(\lambda,\sigma\theta) = m(\lambda)$.
In table II we list the (θ,θ)-index, the diagram of $\bar{\Delta}_\theta$, the multiplicities of the restricted roots in $\bar{\Delta}_\theta$, the quadratic elements in $T^-_{\sigma,\theta}$ representing the classes in $\mathcal{C}(\sigma,\theta)$ (see section 8) and the type of Int(ϵ_i) to determine the associated pair (see section 9). We have added also some information to identify these pairs with Berger's classification [2]. This will be explained in section 9.

7.18.2. Φ is irreducible and $\sigma \neq \theta$

The diagrams representing the indices of (σ,θ) and (θ,σ) are listed in table IV. We also give the type of $(\sigma,\theta\mathrm{Int}(\epsilon_i))$ as explained above, the diagram of $\bar{\Delta}_{\sigma,\theta}$ together with the signatures of the standard pair, the quadratic elements in $T^-_{\sigma,\theta}$ representing the classes in $\mathcal{C}(\sigma,\theta)$ (see section 8) and the type of $\sigma\tau$ according to the notation in table II.

7.18.3. $\Phi = \Phi_1 \amalg \Phi_2$ with Φ_1, Φ_2 irreducible, $\sigma = \theta$ and $\sigma(\Phi_1) = \Phi_2$

In this case the (σ,θ)-index equals again the indices of θ (and σ). Here σ^* and θ^* exchange the Dynkin diagrams of Φ_1 and Φ_2. We denote (σ,θ) by $(X_l \times X_l)$, where X_l denotes the type of Φ_i (i.e. one of A, ..., G). If $F = \mathbb{C}$ these pairs correspond to the symmetric pairs $(\mathfrak{g}_\mathbb{C}, \mathfrak{g})$, where $\mathfrak{g}$ is a real semisimple Lie algebra of inner type (i.e. $\mathfrak{g}$ contains a compact Cartan subalgebra) and $\mathfrak{g}_\mathbb{C}$ its complexification.
In table III we give the type of $(\theta,\theta\mathrm{Int}(\epsilon_i))$, the diagram of $\bar{\Delta}_\theta$, the multiplicities of the roots in $\bar{\Delta}_\theta$ and the quadratic elements representing a class in $\mathcal{C}(\sigma,\theta)$.

7.18.4. $\Phi = \Phi_1 \amalg \Phi_2$ with Φ_1, Φ_2 irreducible, $\sigma(\Phi_1) = \Phi_2$, $\theta(\Phi_1) = \Phi_2$, $\sigma^* \neq \theta^*$

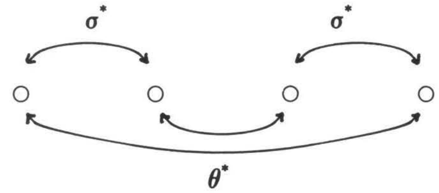
Since both $\sigma = -\sigma^*$ and $\theta = -\theta^*$, this can only occur if $\mathrm{Aut}(\Phi_i)$ (i = 1,2) contains a non-trivial diagram automorphism of order 2, i.e. Φ_i (i = 1,2) is one of A_l ($l \geq 2$), D_l ($l \geq 4$) or E_6. In this case (σ,θ) and (θ,σ) are isomorphic and we denote (σ,θ) by $({}^2X_l \times {}^2X_l)$, where ${}^2X_l$ denotes the twisted Dynkin diagram of Φ of type X_l (see (2.9)). If $F = \mathbb{C}$ these pairs correspond to the symmetric pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{G})$, where $\mathfrak{g}$ is of outer type.
In table V we list the type of (σ,θ), the (σ,θ)-index, the diagram of $\overline{\Delta}_{\sigma,\theta}$ together with the signatures of the standard pair and the quadratic elements representing the isomorphism classes in $\mathcal{Q}(\sigma,\theta)$.

7.18.5. $\Phi = \Phi_1 \amalg \Phi_2$ with Φ_1, Φ_2 irreducible, $\sigma(\Phi_1) = \Phi_2$, $\theta(\Phi_i) = \Phi_i$ (i = 1,2)

The diagram representing the indices of (σ,θ) and (θ,σ) is a double copy of the index of $\theta|_{\Phi_1}$ and the action of σ^* is described by arrows connecting both diagrams. Moreover $\overline{\Phi}_{\sigma,\theta} \cong \overline{\Phi}_{\theta|_{\Phi_1}}$ and for $\lambda \in \overline{\Phi}_{\sigma,\theta}$ we have $m^+(\lambda,\sigma\theta) = m^-(\lambda,\sigma\theta)$, which equals again the multiplicity of the corresponding root in $\overline{\Phi}_{\theta|_{\Phi_1}}$. All pairs in $\mathcal{F}(\sigma,\theta)$ are isomorphic (see section 8). In Berger these pairs are denoted by $\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ and $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g})$, where $\mathfrak{g}$ is a real semisimple Lie algebra and $\mathfrak{k}$ a maximal compact subalgebra of $\mathfrak{g}$. The pairs $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ are associated to the ones in (7.18.3) and (7.18.4). (See also (9.4).)

7.18.6. $\Phi = \Phi_1 \amalg \Phi_2 \amalg \Phi_3 \amalg \Phi_4$ with Φ_i (i = 1,2,3,4) irreducible, $\sigma(\Phi_1) = \Phi_2$, $\sigma(\Phi_3) = \Phi_4$, $\theta(\Phi_1) = \Phi_4$, $\theta(\Phi_2) = \Phi_3$

The diagram representing the index of (σ,θ) and (θ,σ) consists of four copies of the Dynkin diagram of Φ_1 and the actions of σ^* and θ^* are described by arrows. Here is an example of Φ_1 of type A_1:



In this case is $\overline{\Phi}_{\sigma,\theta}$ isomorphic to Φ_1 and $m^+(\lambda,\sigma\theta) = m^-(\lambda,\sigma\theta) = 2$ for all $\lambda \in \overline{\Phi}_{\sigma,\theta}$. All pairs in $\mathcal{F}(\sigma,\theta)$ are isomorphic (see section 8). These pairs correspond in the real case to symmetric pairs $(\mathfrak{g}_{\mathbb{H}}, \mathfrak{g})$, where $\mathfrak{g}$ is a real semisimple Lie algebra of compact type and $\mathfrak{g}_{\mathbb{H}}$ obtained by extension of the base field of $\mathfrak{g}$ to $F = \mathbb{H}$, the quaternions.

Summarizing we have obtained the following result:

7.19. Theorem. *Assume* X *is semisimple and of adjoint type. Then the irreducible indices of admissible pairs of commuting involutions* (σ,θ) *of* (X,Φ), *where* $\sigma,\theta \neq \mathrm{id}$, *are exhausted by the indices in (7.18.1-6).*

8. Classification of the quadratic elements representing the classes in $\mathcal{Q}(\sigma,\theta)$

In this section we shall determine a set of representatives for the isomorphism classes within a set $\mathcal{Q}(\sigma,\theta)$.

8.1. Let (σ,θ) be a pair of involutorial automorphisms of G, A a maximal (σ,θ)-split torus of G and $T\supset A$ a (σ,θ)-stable maximal torus of G such that T_θ^- resp T_σ^- is a maximal θ-split resp. σ-split torus of G (see (5.13)). We shall write Φ for $\Phi(T)$, X for $X^*(T)$ and W for W(T).

For a closed subgroup H of G, we call two pairs (σ_1,θ_1) and (σ_2,θ_2) in $\mathcal{F}$ *isomorphic* under H if there exists $h\in H$ such that $\mathrm{Int}(h)\sigma_1\mathrm{Int}(h)^{-1}=\sigma_2$ and $\mathrm{Int}(h)\theta_1\mathrm{Int}(h)^{-1}=\theta_2$.
In (5.16) we showed that any pair in $\mathcal{F}(\sigma,\theta)$ is isomorphic to a pair $(\sigma,\theta\mathrm{Int}(a))\in\mathcal{F}_A(\sigma,\theta)$. As for the possible isomorphisms between these pairs, we can restrict ourselves to $N_G(A)$:

8.2. Lemma. *Two pairs* $(\sigma,\theta\mathrm{Int}(a_1))$ *and* $(\sigma,\theta\mathrm{Int}(a_2))$ *in* $\mathcal{F}_A(\sigma,\theta)$ *are isomorphic under* G *if and only if they are isomorphic under* $N_G(A)$.

Proof. It suffices to show the "only if" statement. Assume $g\in G$ such that $\mathrm{Int}(g)\sigma\mathrm{Int}(g)^{-1}=\sigma$ and $\mathrm{Int}(g)\theta\mathrm{Int}(a_1g^{-1}) = \theta\mathrm{Int}(a_2)$. Since both gAg^{-1} and A are maximal $(\sigma,\theta\mathrm{Int}(a_2))$-split tori of G, there exists by (5.15) $h\in(G_\sigma\cap G_{\theta\mathrm{Int}(a_2)})^o$ such that $hg\in N_G(A)$. This proves the assertion.

8.3. Remark. The question whether two pairs in $\mathcal{F}_A(\sigma,\theta)$ are isomorphic under $N_G(A)$ can be reduced to the case where G is adjoint. Henceforth we assume this for the remaining part of this section.

8.4. Action of W(A) on $\mathcal{F}_A(\sigma,\theta)$

Let (σ,θ) be a standard pair. Then by (6.15) every $w\in W(A)$ has a representative $h\in(G_\sigma\cap G_\theta)^o$. So W(A) acts on the pairs in $\mathcal{F}_A(\sigma,\theta)$, namely if $(\sigma,\theta\mathrm{Int}(a))\in\mathcal{F}_A(\sigma,\theta)$, $w\in W(A)$ and $h\in(G_\sigma\cap G_\theta)^o$ a representative of w, then $\mathrm{Int}(h)\sigma\mathrm{Int}(h)^{-1}=\sigma$ and $\mathrm{Int}(h)\theta\mathrm{Int}(a)\mathrm{Int}(h)^{-1}=\theta\mathrm{Int}(hah^{-1})=\theta\mathrm{Int}(w(a))$.
Denote the set of quadratic elements of A by F(A). In (8.13) we shall describe a set of representatives of the W(A)-conjugacy classes in F(A). We are then left with the question when two pairs in $\mathcal{F}_A(\sigma,\theta)$ are isomorphic under $Z_G(A)$. We deal with the latter question first.

8.5. Proposition. *Let* T,A *be as in (8.1). Then two pairs* $(\sigma,\theta\mathrm{Int}(a_1))$ *and* $(\sigma,\theta\mathrm{Int}(a_2))$ *in* $\mathcal{F}_A(\sigma,\theta)$ *are isomorphic under* $Z_G(A)$ *if and only if there exists* $t\in T$ *such that* $\sigma(t)=t$ *and* $a_1a_2=\theta(t)t^{-1}$.

Proof. If $t\in T$ satisfies the above conditions, then Int(t) is the desired isomorphism. So assume there is $g\in Z_G(A)$ such that $\mathrm{Int}(g)\sigma\mathrm{Int}(g)^{-1}=\sigma$ and $\mathrm{Int}(g)\theta\mathrm{Int}(a_1g)^{-1}=\theta\mathrm{Int}(a_2)$. As in (1.4) let $T_{\sigma\theta}^-=\{t\in T|\sigma\theta(t)=t^{-1}\}$. Now $T_{\sigma\theta}^-$ and $g(T_{\sigma\theta}^-)g^{-1}$ are both maximal $\sigma\theta$-split tori of $Z_G(A)$, so by (1.5), (1.6) and (5.3) there is $h\in(Z_G(A)\cap G_\sigma\cap G_\theta)^o$ such that $hg\in Z_{Z_G(A)}(T_{\sigma\theta}^-)=Z_G(AT_{\sigma\theta}^-)=Z_G(T_\sigma^-T_\theta^-)$.
Now T and $hgTg^{-1}h^{-1}$ are maximal tori in $Z_G(AT_{\sigma\theta}^-)$ and since the derived group of $Z_G(AT_{\sigma\theta}^-)$ is contained in $Z_G(A)\cap G_\sigma\cap G_\theta)^o$, there exists $k\in Z_G(A)\cap G_\sigma\cap G_\theta)^o$ such

that $t = khg \in T$. Since $a_1, a_2 \in A$ and $kh \in (Z_G(A) \cap G_\sigma \cap G_\theta)^o$ we have $\sigma(t) = t$ and $a_2 = a_1\theta(t)t^{-1}$, which proves the result.

For a standard pair we can even prove a stronger result:

8.6. Corollary. *Assume* (σ,θ) *is a standard pair. Let* T, A *be as in (8.1) and* $a \in F(A)$. *Then* (σ,θ) *and* $(\sigma,\theta \mathrm{Int}(a))$ *are isomorphic if and only if there is* $t \in T$ *such that* $\sigma(t) = t$ *and* $a = \theta(t)t^{-1}$.

Proof. The result follows immediately from (8.5), (8.2) and the fact that by (6.15) any $w \in W(A)$ has a representative in $(G_\sigma \cap G_\theta)^o$.

It is possible to characterize these quadratic elements $\theta(t)t^{-1}$ occurring in (8.5) as a product of a quadratic element in $(T_\sigma^-)_\theta^+$ and one in $(T_\theta^-)_\sigma^+$. This is useful for checking whether in an explicit example two pairs in $\mathcal{F}_A(\sigma,\theta)$ are isomorphic under $Z_G(A)$. However we shall not use this result for the classification.

8.7. Corollary. *Let* σ, θ, T *and* A *be as in (8.5) and* $a \in F(A)$. *Then the following statements are equivalent:*

(1) There is a $t \in T$ *such that* $\sigma(t) = t$ *and* $a = \theta(t)t^{-1}$,

(2) $a = xy$ *where* $x \in (T_\sigma^-)_\theta^+$, $y \in (T_\theta^-)_\sigma^+$, $x^2 = y^2 = e$,

(3) there is a $t \in T$ *such that* $\theta(t) = t$ *and* $a = \sigma(t)t^{-1}$.

Proof. (1)⇒(2): Assume $t \in T$ such that $\sigma(t) = t$ and $a = \theta(t)t^{-1}$. Write $t = t_1t_2t_3t_4$ with $t_1 \in (T_\sigma^+)_\theta^+$, $t_2 \in (T_\theta^-)_\sigma^+$, $t_3 \in (T_\sigma^-)_\theta^+$, $t_4 \in A$. Then $t_2t_3t_4$ satisfies the same conditions, so we may assume $t = t_2t_3t_4$. From $\sigma(t) = t$ we see that $(t_3t_4)^2 = e$ and since $\theta(t_3t_4)(t_3t_4)^{-1} = t_4^{-2}$, we obtain $t_4^4 = e$ and also $t_3^4 = e$. Now $a = \theta(t)t^{-1} = t_2^{-2}t_4^{-2} = t_2^{-2}t_3^{-2}$, so it follows that $t_2^4 = e$. Taking $x = t_2^2$, $y = t_3^2$ the result follows.
(2)⇒(1): Assume now that $a = xy$ as in (2). Let $t_1 \in (T_\theta^-)_\sigma^+$ be such that $t_1^2 = x$ and let $t_2 \in (T_\sigma^-)_\theta^+$ be such that $t_2^2 = y$. Since $y \in T_\theta^- \cap T_\sigma^-$ and $T_\theta^- \cap T_\sigma^- = A$ (see also (8.9) below) there exists a $t_3 \in A$ such that $t_3^2 = y$. If $t = t_1t_2t_3$, then $\sigma(t) = t_1t_2^{-1}t_3^{-1} = t_1t_2t_3y^2 = t$ and $\theta(t)t^{-1} = t_1^{-2}t_3^{-2} = xy = a$, which proves (1).
The equivalence of (2) and (3) follows by symmetry.

8.8. A characterization of the quadratic elements of A

We can describe the quadratic elements of A as a product of quadratic elements of T. Let Δ be a strong (σ,θ)-basis of Φ and $\overline{\Delta}_{\sigma,\theta}$ the corresponding basis of $\Phi(A)$. Since the elements of both Δ and $\overline{\Delta}_{\sigma,\theta}$ are linearly independent, we can find for each $\alpha \in \Delta$ an $\omega_\alpha \in X_*(T)$ such that $<\alpha,\omega_\beta> = \delta_{\alpha,\beta}$ for α, $\beta \in \Delta$. Similarly for each $\lambda \in \overline{\Delta}_{\sigma,\theta}$ let $\gamma_\lambda \in X_*(A)$ be such that $<\lambda,\gamma_{\lambda'}> = \delta_{\lambda,\lambda'}$ (λ, $\lambda' \in \overline{\Delta}_{\sigma,\theta}$). In (6.6) we defined for $\lambda \in \overline{\Delta}_{\sigma,\theta}$ the quadratic elements $\epsilon_\lambda = \gamma_\lambda(-1) \in A$. Since $(\sigma,\theta)|_T$ is a pair of basic involutions of (X,Φ) we can describe ϵ_λ also in terms of the one parameter subgroups ω_α ($\alpha \in \Delta$).

8.9. Lemma. *Let* $\lambda \in \overline{\Delta}_{\sigma,\theta}$. *For* $\Phi(T,\lambda) \cap \Delta$ *we have the following possibilities:*

(1) $\Phi(T,\lambda) \cap \Delta = \{\alpha\}$ *with* $\alpha = \sigma^*(\alpha) = \theta^*(\alpha)$;

(2) $\Phi(T,\lambda) \cap \Delta = \{\alpha, \sigma^*\theta^*(\alpha)\}$ *with either* $\sigma^*(\alpha) = \alpha$ *or* $\theta^*(\alpha) = \alpha$;

(3) $\Phi(T,\lambda) \cap \Delta = \{\alpha, \sigma^*(\alpha), \theta^*(\alpha), \sigma^*\theta^*(\alpha)\}$.

In these three cases we have for ϵ_λ, *respectively,*

(1) $\epsilon_\lambda = \omega_\alpha(-1)$;

(2) $\epsilon_\lambda = (\omega_\alpha \omega_{\sigma^* \theta^*(\alpha)})(-1)$;

(3) $\epsilon_\lambda = (\omega_\alpha \omega_{\sigma^*(\alpha)} \omega_{\theta^*(\alpha)} \omega_{\sigma^* \theta^*(\alpha)})(-1)$.

Proof. The first statement follows immediately from lemma (7.10). As for the other statements note first that, since G is adjoint, $\{\omega_\alpha\}_{\alpha \in \Delta}$ is a basis of $X_*(T)$. But then every quadratic element of T is a product of the quadratic elements $\omega_\alpha(-1)$ $(\alpha \in \Delta)$. In particular there is a subset $\Delta' \subset \Delta$ such that $\epsilon_\lambda = \prod_{\alpha \in \Delta'} \omega_\alpha(-1)$.

Since for $\alpha \in \Delta$ we have $\alpha(\epsilon_\lambda) = \alpha(\gamma_\lambda(-1)) = (-1)^{<\pi(\alpha),\gamma_\lambda>}$, the result follows from the definition of the ω_α $(\alpha \in \Delta)$ and the first statement.

8.10. Remark. If (σ,θ) is normally related to T, then it follows from this result, (3.5) and the fact that $\Delta_o(\sigma,\theta) = \Delta_o(\sigma) \cup \Delta_o(\theta)$, that $T^-_{\sigma,\theta} = T^-_\sigma \cap T^-_\theta$. Since we won't need this in the sequel we leave the proof for the reader.

For arbitrary quadratic elements of A we note:

8.11. Lemma. *Let* $\epsilon \in F(A)$. *Then there exists a subset* Δ_1 *of* $\bar{\Delta}_{\sigma,\theta}$ *such that* $\epsilon = \prod_{\lambda \in \Delta_1} \epsilon_\lambda$. *In particular* F(A) *is completely determined by the set of indecomposible roots* $\Phi(A)'$.

This follows immediately from the fact that $\bar{\Delta}_{\sigma,\theta}$ is a $\mathbb{Z}$-basis of $X^*(A)$.

Since also W(A) is generated by the reflections s_λ with $\lambda \in \bar{\Delta}_{\sigma,\theta} \subset \Phi(A)'$, it follows that for determining a set of representatives of the W(A)-conjugacy classes in F(A) we may restrict ourselves to $\Phi(A)'$, which is reduced. Henceforth we will assume that $\Phi(A) = \Phi(A)'$.

8.12. Action of the affine Weyl group on F(A)

Assume G is semisimple, $\Phi(A)$ is reduced and Δ, $\bar{\Delta}_{\sigma,\theta}$ are as in (8.8). Write $X_*(A)$ additively and let $E = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. For $x \in E$, let t(x) denote the translation of E along the vector x and let Q denote the group of the translations t(v), where $v = \sum_{\lambda \in \bar{\Delta}_{\sigma,\theta}} m_\lambda \lambda^\vee$ with $m_\lambda \in \mathbb{Z}$ and $\lambda^\vee \in \Phi(A)^\vee$ a coroot.

If $W^a(A)$ denotes the affine Weyl group of $\Phi(A)$, then $W^a(A)$ is the semidirect product of W(A) and Q (see Bourbaki [5,Ch.VI,no.2.1]). For $n \in \mathbb{N}$ let

$$\Lambda_n = \{\frac{1}{n}(\sum_{\lambda \in \bar{\Delta}_{\sigma,\theta}} m_\lambda \gamma_\lambda) \mid m_\lambda \in \mathbb{Z}\}$$

and let $\xi_n \in F^*$ be a primitive n-th root of unity. Define now $\phi_n : \Lambda_n \to A$ by

$$\frac{1}{n}(\sum_{\lambda \in \bar{\Delta}_{\sigma,\theta}} m_\lambda \gamma_\lambda) \to \frac{1}{n}(\sum_{\lambda \in \bar{\Delta}_{\sigma,\theta}} m_\lambda \gamma_\lambda)(\xi_n).$$

Q acts transitively on the fibers of ϕ_n. In particular for $n = 2$ we get $\phi_2(\Lambda_2) = F(A)$. Moreover since $W^a(A) = Q.W(A)$, the orbits of Λ_2 under the action of $W^a(A)$ correspond one to one to the W(A)-conjugacy classes in F(A). Let C be the chamber of E with respect to $\bar{\Delta}_{\sigma,\theta}$ and P_o the unique fundamental region in C containing the origin in its closure. Denote the closures of C resp. P_o by $\bar{C}$ resp. $\bar{P}_o$. Now any $W^a(A)$ orbit in E meets $\bar{P}_o$ exactly once (see Bourbaki [5,Ch.VI,no.2.1]). So if $R = \Lambda_2 \cap \bar{P}_o$, then $\phi_2(R)$ is a set of representatives of the W(A)-conjugacy classes in

F(A). One easily sees that R consists of at most $|\bar{\Delta}_{\sigma,\theta}|$ vectors. Eventually after applying still a Weyl group element, we obtain the following result:

8.13. Proposition. *Assume* $\Phi(A)$ *is irreducible. Then any element of* F(A) *is conjugate under* W(A) *to one of the* ϵ_λ, $\lambda \in \bar{\Delta}_{\sigma,\theta}$, *as given in table VI.*

For more details on the proof we refer to Borel-Siebenthal [4] who also derived this result in a slightly different context. They work with compact groups, but this specific result depends only on the action of the affine Weyl group. See also Oshima-Sekiguchi [18].

We still need to determine which of these ϵ_λ in table VI give rise to isomorphic pairs in $\mathcal{F}_A(\sigma,\theta)$. This depends only on the signatures of the simple roots:

8.14. Theorem. *Assume* G *is semisimple and let* (σ,θ) *be a pair of commuting involutorial automorphisms of* G. *If* T, A, Δ *and* $\bar{\Delta}_{\sigma,\theta}$ *are as in (8.8), then, for any* $\lambda \in \bar{\Delta}_{\sigma,\theta}$ *with* $m^+(\lambda,\sigma\theta) = m^-(\lambda,\sigma\theta)$, *the pair* $(\sigma,\theta\mathrm{Int}(\epsilon_\lambda))$ *is isomorphic to* (σ,θ).

It is possible to prove this result by checking the condition (2) of (8.7) for all the irreducible (σ,θ)-indices in (7.18.1-6). We shall give another proof by proving five lemmas, which deal with all, except three, cases.
For (8.15)-(8.19) we assume that G, T, σ, θ, Δ are as in (8.14).

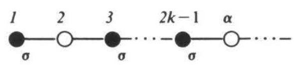
8.15. Lemma. *Assume* $\alpha \in \Delta - \Delta_0(\sigma,\theta)$ *such that* $\sigma^*\theta^*(\alpha) \neq \alpha$ *and either* $\sigma^*(\alpha) = \alpha$ *or* $\theta^*(\alpha) = \alpha$. *If* $\lambda = \pi(\alpha)$ *then* $(\sigma,\theta\mathrm{Int}(\epsilon_\lambda))$ *is isomorphic to* (σ,θ).

Proof. Let $t = \omega_\alpha(-1) \in T$. Then $\beta(t) = 1$ if $\beta \in \Delta - \{\alpha\}$ and $\alpha(t) = -1$. Let $u = t\sigma\theta(t^{-1})$. Then $u \in T^-_{\sigma\theta}$ (see (1.4)), so for $\beta \in \Delta$ we have $\beta(u) = \beta(t)\sigma^*\theta^*(\beta)(t^{-1})$, because $\sigma\theta(\beta) = \sigma^*\theta^*(\beta) + \gamma$, where γ lies in the $\mathbb{Z}$-span of $\Delta_0(\sigma,\theta)$.
Now since $\beta(u) = 1$ if $\beta \neq \alpha$ or $\sigma^*\theta^*(\alpha)$, it follows that $\mathrm{Int}(\epsilon_\lambda) = \mathrm{Int}(u)$, so by (8.5) we are done.

8.16. Lemma. *Assume* $\alpha \in \Delta - \Delta_0(\sigma,\theta)$, *such that* $\sigma^*\theta^*(\alpha) \neq \alpha$, $\sigma^*(\alpha) \neq \alpha$ *and* $\theta^*(\alpha) \neq \alpha$. *If* $\lambda = \pi(\alpha)$ *then* $(\sigma,\theta\mathrm{Int}(\epsilon_\lambda))$ *is isomorphic to* (σ,θ).

Proof. Let $t = \omega_\alpha(-1)$, $x = t\theta(t^{-1})$ and $u = x\sigma\theta(x^{-1}) = t\sigma(t^{-1})\sigma\theta(t^{-1})\theta(t)$. Similarly to (8.15) we have for $\beta \in \Delta$: $\beta(u) = \beta(t)\sigma^*(\beta)(t)\theta^*(\beta)(t^{-1})\sigma^*\theta^*(\beta)(t^{-1})$. So $\beta(u) = \alpha(t) = -1$ if β equals one of α, $\sigma^*(\alpha)$, $\theta^*(\alpha)$, $\sigma^*\theta^*(\alpha)$ and $\beta(u) = 1$ for the other roots in Δ. It follows that $\mathrm{Int}(u) = \mathrm{Int}(\epsilon_\lambda)$ and the result follows now from (8.5).

8.17. Lemma. *Assume* $\alpha \in \Delta - \Delta_0(\sigma,\theta)$ *such that* $\theta^*(\alpha) = \sigma^*(\alpha) = \alpha$ *and let* $\lambda = \pi(\alpha)$. *If* α *is contained in a subdiagram of the* (σ,θ)*-index of the form*



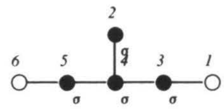
Then $(\sigma,\theta\mathrm{Int}(\epsilon_\lambda))$ *is isomorphic to* (σ,θ).

Proof. Assume that the roots of Δ are numbered as in the above diagram and that $\alpha = \alpha_{2k}$. For $i = 1,\ldots,k$ let $t_i = \omega_{\alpha_{2i-1}}(-1)$. Take $t = \prod_{i=1}^{k} t_i$ and let $u = t\sigma\theta(t^{-1})$. Then for $i = 1,\ldots,k$ we have $\alpha_{2i-1}(u) = \alpha_{2i-1}(t)^2 = 1$. Moreover for $i = 1,\ldots,k-1$ we have

$$\alpha_{2i}(u) = \alpha_{2i}(t)\sigma\theta(\alpha_{2i})(t^{-1}) =$$
$$(\alpha_{2i-1}\alpha_{2i}\alpha_{2i+1})(t^{-1}) = \alpha_{2i-1}(t^{-1})\alpha_{2i+1}(t^{-1}) = 1.$$

Since $\alpha_{2k}(u) = \alpha_{2k}(t)\sigma\theta(t^{-1}) = w_o(\sigma)w_o(\theta)(\alpha_{2k})(t^{-1}) = \alpha_{2k-1}(t^{-1}) = -1$ and $\beta(u) = 1$ for $\beta\in\Delta$, $\beta\neq\alpha_j$ $(j = 1,...,2k)$ it follows that $\mathrm{Int}(u) = \mathrm{Int}(\epsilon_\lambda)$, which proves the result.

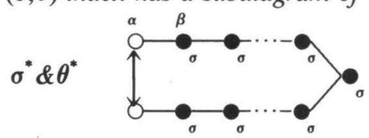
8.18. Lemma. *Assume that the (σ,θ)-index has a sub diagram of the form*



and $\Phi_o(\sigma) = \emptyset$, $\Phi_o(\theta)$ of type D_4. If $\lambda_1 = \pi(\alpha_1)$ and $\lambda_2 = \pi(\alpha_6)$, then for $i = 1,2$ $(\sigma,\theta\mathrm{Int}(\epsilon_{\lambda_i}))$ is isomorphic to (σ,θ).

Proof. Let $t_1 = \omega_{\alpha_1}(-1)\omega_{\alpha_2}(-1)$ and $t_2 = \omega_{\alpha_6}(-1)\omega_{\alpha_2}(-1)$. Put $u_1 = t_1\sigma\theta(t_1^{-1})$ and $u_2 = t_2\sigma\theta(t_2^{-1})$. Similarly as in (8.17) it follows that $\mathrm{Int}(u_1) = \mathrm{Int}(\epsilon_{\lambda_1})$ and $\mathrm{Int}(u_2) = \mathrm{Int}(\epsilon_{\lambda_2})$, hence the result follows from (8.5).

8.19. Lemma. *Assume the (σ,θ)-index has a subdiagram of the form:*



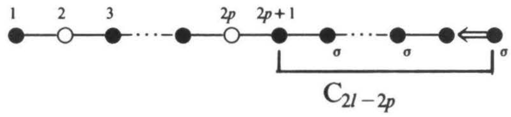
If $\lambda = \pi(\alpha)$, then $(\sigma,\theta\mathrm{Int}(\epsilon_\lambda))$ is isomorphic to (σ,θ).

Proof. Put $t_1 = \omega_\beta(-1)$, $t_2 = \omega_{\sigma^*(\beta)}(-1)$, $t = t_1t_2$ and $u = t\sigma\theta(t^{-1})$. Similarly as in (8.17) one verifies that $\mathrm{Int}(u) = \mathrm{Int}(\epsilon_\lambda)$, so the result follows by (8.5).

8.20. Proof of Theorem (8.14). Applying the lemmas (8.15-19) to the irreducible (σ,θ)-indices in (7.18.1-6), we are left with three cases, which do not satisfy any of the conditions in one of these lemmas. For each of them we shall give a quadratic element $u\in T_{\sigma\theta}^-$ such that $\mathrm{Int}(u) = \mathrm{Int}(\epsilon_\lambda)$, which will prove the result by (8.5). As an example we shall treat the following case in more detail.

(1) $C_{2l}^{l,p}(II_b,II_a)$ with $\alpha = \alpha_{2p}$.

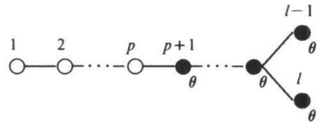
We number the roots according to the Dynkin diagram below.



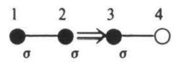
Let $\lambda = \pi(\alpha) = \lambda_p$, $t = \omega_{\alpha_{2p+1}}(-1)$ and $u = t\sigma\theta(t^{-1})$. Then $\mathrm{Int}(u) = \mathrm{Int}(\epsilon_\lambda)$, namely $w_o(\theta)(\alpha) = s_{\alpha_{2p-1}}w_1(\alpha)$, where w_1 is the longest element of $W(C_{2l-2p})$ with respect to $\Delta\cap C_{2l-2p}$. Let $\beta\in C_{2l-2p}$ be such that $-\beta$ is the highest root of C_{2l-2p} with respect to $\Delta\cap C_{2l-2p}$ (see Bourbaki [5,Ch.VI,no.1.8]). Then $\beta(t) = 1$.

Extend $\{\alpha_{2p+1},\beta\}$ to a maximal strongly orthogonal set of roots as in (4.13) and write w_1 as a product of the corresponding reflections. Then

$w_1(\alpha_{2p}) = \alpha_{2p}+\alpha_{2p+1}+\beta$. So $w_o(\theta)(\alpha_{2p}) = \alpha_{2p-1}+\alpha_{2p}+\alpha_{2p+1}+\beta$. Now $\sigma\theta(\alpha_{2p})(t) = w_o(\sigma)w_o(\theta)(\alpha_{2p})(t) = s_{\alpha_{2p+1}}s_{\alpha_{2p-1}}(\alpha_{2p-1}\alpha_{2p}\alpha_{2p+1}\beta)(t) = \beta(t) = -1$. So $\alpha_{2p}(u) = \beta(t) = -1$. Since clearly $\gamma(u) = 1$ for $\gamma\in\Delta-\{\alpha_{2p}\}$ we are done.
(2) $D_l^{p,l}(I_a,I_b)$ with $\alpha = \alpha_p$.



Let $\lambda = \pi(\alpha) = \lambda_p$, $t = \omega_{\alpha_{p+1}}(-1)\omega_{\alpha_{l-1}}(-1)$ and $u = t\sigma\theta(t^{-1})$. Then $\mathrm{Int}(u) = \mathrm{Int}(\epsilon_\lambda)$.
(3) $F_4^{4,1}(I,II)$ with $\alpha = \alpha_4$.



Let $\alpha = \alpha_4$, $\lambda = \pi(\alpha)$, $t = \omega_{\alpha_3}(-1)$ and $u = t\sigma\theta(t^{-1})$. Then $\mathrm{Int}(u) = \mathrm{Int}(\epsilon_\lambda)$. This completes the result.

8.21. Corollary. *The standard pair of a family $\mathcal{F}_A(\sigma,\theta)$ is unique up to isomorphism.*

Proof. The result follows immediately from (8.14), (6.13) and (6.14).

We are left with the problem which of the remaining ϵ_λ give rise to isomorphic pairs in $\mathcal{F}_A(\sigma,\theta)$. That they are not standard can be seen by looking at the corresponding rank one subgroups.

8.22. Restricted rank one subgroups

Let (σ,θ) be a standard pair and assume T, A are as in (8.1). For $\lambda\in\Phi(A)$ let $\Phi(\lambda) = \{\alpha\in\Phi(T) | \alpha|A = m\lambda,\ m\in\mathbb{Z}\}$. This is a closed symmetric subset of $\Phi(T)$. Let now $G(\lambda)$ denote the (closed) subgroup of G generated by T and the root subgroups U_β, with $\beta\in\Phi(\lambda)$. It follows from Borel-Tits [3,p.74, prop. 2.2 and p.65, 2.3 Remark], that $G(\lambda)$ is reductive and that $\Phi(\lambda) = \Phi(T,G(\lambda))$.
Since $\theta(\lambda) = \sigma(\lambda) = -\lambda$, we see that $\Phi(\lambda)$ and $G(\lambda)$ are σ- and θ-stable. Moreover if $\sigma_1 = \sigma|_{G(\lambda)}$, $\theta_1 = \theta|_{G(\lambda)}$, then (σ_1,θ_1) is normally related to T and $G(\lambda)$ has restricted (σ_1,θ_1)-rank one (i.e., $\mathrm{rank}\,\overline{\Phi(\lambda)}_{\sigma_1,\theta_1} = 1$).

8.23. Lemma. *Let (σ,θ) be a standard pair and* T, A *as in (8.1). Let Δ be a strong (σ,θ)-basis of Φ and $\overline{\Delta}_{\sigma,\theta}$ the corresponding basis of $\Phi(A)$. Let ϵ_λ $(\lambda\in\overline{\Delta}_{\sigma,\theta})$ be as in (8.8). If $\lambda\in\overline{\Delta}_{\sigma,\theta}$ is such that $m^+(\lambda,\sigma\theta) \neq m^-(\lambda,\sigma\theta)$ then, $(\sigma,\theta\mathrm{Int}(\epsilon_\lambda))$ and (σ,θ) are not isomorphic.*

Proof. Assume (σ,θ) and $(\sigma,\theta\mathrm{Int}(\epsilon_\lambda))$ are isomorphic $(\lambda\in\overline{\Delta}_{\sigma,\theta})$. From (8.6) it follows that also $(\sigma,\theta)|_{G(\lambda)}$ and $(\sigma,\theta\mathrm{Int}(\epsilon_\lambda))|_{G(\lambda)}$ are isomorphic. So we may assume $G = G(\lambda)$. Now

$$\mathfrak{g} = Z_{\mathfrak{g}}(A)\oplus\mathfrak{g}(A,\lambda)\oplus\mathfrak{g}(A,-\lambda)\oplus\mathfrak{g}(A,2\lambda)\oplus\mathfrak{g}(A,-2\lambda)$$

On the other hand $\sigma\theta|_{Z_G(A)} = \sigma\theta\mathrm{Int}(\epsilon_\lambda)|_{Z_G(A)}$ and

$$\sigma\theta(\mathfrak{g}(A,\pm m\lambda)) = \sigma\theta\mathrm{Int}(\epsilon_\lambda)(\mathfrak{g}(A,\pm m\lambda)) = \mathfrak{g}(A,\pm m\lambda),\ m = 1,2.$$

Comparing the dimensions of the eigenspaces of $\sigma\theta$ and $\sigma\theta\mathrm{Int}(\epsilon_\lambda)$ in $\mathfrak{g}$, we conclude that if (σ,θ) and $(\sigma,\theta\mathrm{Int}(\epsilon_\lambda))$ are isomorphic, we have $m^+(\lambda,\sigma\theta) = m^-(\lambda,\sigma\theta)$. The lemma is proved.

8.24. Lemma. *Let* (σ,θ) *be a standard pair. Let* $a \in F(A)$ *and* $\Delta' \subset \bar{\Delta}_{\sigma,\theta}$ *be such that* $a = \prod_{\lambda \in \Delta'} \epsilon_\lambda$. *Then* $(\sigma,\theta \mathrm{Int}(a))$ *is isomorphic to* (σ,θ) *if and only if* $m^+(\lambda,\sigma\theta) = m^-(\lambda,\sigma\theta)$ *for all* $\lambda \in \Delta'$.

Proof. Assume $\lambda \in \Delta'$ such that $m^+(\lambda,\sigma\theta) \neq m^-(\lambda,\sigma\theta)$. If $(\sigma,\theta\mathrm{Int}(a))$ and (σ,θ) are isomorphic, then by (8.6) also their restrictions to $G(\lambda)$ are isomorphic. But since $(\sigma,\theta\mathrm{Int}(a))|_{G(\lambda)} = (\sigma,\theta\mathrm{Int}(\epsilon_\lambda))|_{G(\lambda)}$ the result follows from (8.23).

We can prove now the following result:

8.25. Proposition. *Let* (σ,θ) *be a standard pair. Let* $\lambda_1,\lambda_2 \in \bar{\Delta}_{\sigma,\theta}$ *be such that* $m^+(\lambda_i,\sigma\theta) \neq m^-(\lambda_i,\sigma\theta)$ *(i=1,2). Then* $(\sigma,\theta\mathrm{Int}(\epsilon_{\lambda_1}))$ *and* $(\sigma,\theta\mathrm{Int}(\epsilon_{\lambda_2}))$ *are isomorphic if and only if* ϵ_{λ_1} *and* ϵ_{λ_2} *are conjugate under* $W(A)$.

Proof. The "if" statement being clear, assume $(\sigma,\theta\mathrm{Int}(\epsilon_{\lambda_1}))$ and $(\sigma,\theta\mathrm{Int}(\epsilon_{\lambda_2}))$ are isomorphic. By (8.4) there exists $w \in W(A)$ such that $(\sigma,\theta\mathrm{Int}(w\epsilon_{\lambda_1}))$ and $(\sigma,\theta\mathrm{Int}(\epsilon_{\lambda_2}))$ are isomorphic under $Z_G(A)$.

Let $a \in A$ be such that $\epsilon_{\lambda_2} = w(\epsilon_{\lambda_1})a$ and let $\Delta' \subset \bar{\Delta}_{\sigma,\theta}$ be such that $a = \prod_{\lambda \in \Delta'} \epsilon_\lambda$. Then by (8.5) (σ,θ) and $(\sigma,\theta\mathrm{Int}(a))$ are isomorphic. Hence by (8.24) $m^+(\lambda,\sigma\theta) = m^-(\lambda,\sigma\theta)$ for all $\lambda \in \Delta'$. We shall show now that a must equal e. Checking the signatures for the simple roots for the irreducible (σ,θ)-indices in (7.18.1-6) (see also tables II-V), it follows that only the following four cases occur for $\bar{\Phi}_{\sigma,\theta}$ irreducible:

(1) $m^+(\lambda,\sigma\theta) = m^-(\lambda,\sigma\theta)$ for all $\lambda \in \bar{\Delta}_{\sigma,\theta}$. Then by (8.14) $\mathcal{C}(\sigma,\theta)$ consists of a single isomorphism class.

(2) $m^+(\lambda,\sigma\theta) \neq m^-(\lambda,\sigma\theta)$ for all $\lambda \in \bar{\Delta}_{\sigma,\theta}$. Then $a = e$.

(3) $m^+(\lambda,\sigma\theta) \neq m^-(\lambda,\sigma\theta)$ for exactly one $\lambda \in \bar{\Delta}_{\sigma,\theta}$. In this case $\lambda_1 = \lambda_2$, so there is nothing to prove.

(4) $\bar{\Phi}_{\sigma,\theta}$ is of type B_n or BC_n and $m^+(\lambda,\sigma\theta) \neq m^-(\lambda,\sigma\theta)$ for all long roots in $\bar{\Delta}_{\sigma,\theta}$ and $m^+(\lambda,\sigma\theta) = m^-(\lambda,\sigma\theta)$ for the single short root $\mu \in \bar{\Delta}_{\sigma,\theta}$.

In this case $a = e$ or $a = \epsilon_\mu$. Assume $a = \epsilon_\mu$. So $w(\epsilon_{\lambda_1}) = \epsilon_{\lambda_2}\epsilon_\mu$. Here $\lambda_1,\lambda_2 \in \bar{\Delta}_{\sigma,\theta} - \{\mu\}$. On the other hand $\epsilon_\mu\epsilon_{\lambda_2}$ is conjugate under $W(A)$ to ϵ_μ, what implies that ϵ_{λ_1} is conjugate under $W(A)$ to ϵ_μ. So by (8.23) and (8.14) we obtain a contradiction. Hence $a = e$.

Since (1)-(4) exhaust all the possibilities for $\bar{\Phi}_{\sigma,\theta}$ irreducible, the result is proved.

We shall say that $\mathcal{C}(\sigma,\theta)$ is irreducible if the index of the corresponding admissible pair of commuting involutions of (X,Φ) is irreducible (i.e. $\bar{\Phi}_{\sigma,\theta}$ is irreducible). Thus we have obtained the following characterization of the isomorphism classes in $\mathcal{C}$:

8.26. Theorem. *Assume* G *is semisimple and* T *a maximal torus of* G. *Then the classes* $\mathcal{C}(\sigma,\theta)$ *in* $\mathcal{C}$ *correspond bijectively to the isomorphism classes of the indices of the corresponding admissible pair of commuting involutions of* (X,Φ). *The isomorphism classes contained in* $\mathcal{C}(\sigma,\theta)$ *are represented by quadratic elements of a fixed maximal* (σ,θ)*-split torus* A *of* G. *For* $\mathcal{C}(\sigma,\theta)$ *irreducible these are given in the tables* II$-$IV.

Note that, in cases (7.18.5-6), $\mathcal{C}(\sigma,\theta)$ consists of a single isomorphism class of commuting involutorial automorphisms of G.

9. Classification of semisimple symmetric spaces

In this section we shall show that there is a bijection between the set of isomorphism classes of locally semisimple symmetric spaces and the isomorphism classes of (ordered) pairs of commuting involutions as treated in the sections 7-8. Moreover the fine structure as developed for pairs of commuting involutions, transfers directly to the corresponding symmetric space.

9.1. Let G_o be a real semisimple connected Lie group and denote its Lie algebra by $\mathfrak{g}_o$. Let $\sigma \in \mathrm{Aut}(G_o)$ be an involutorial automorphism and let H be a closed subgroup of G_o satisfying $(G_o)_\sigma^o \subset H \subset (G_o)_\sigma$. If $\mathfrak{h}$ denotes the Lie algebra of H (or $(G_o)_\sigma$), then the pair (G_o,H) is called an *semisimple symmetric pair* and $(\mathfrak{g}_o,\mathfrak{h})$ a *locally semisimple symmetric pair.* We shall write also $(\mathfrak{g}_o,\sigma)$ instead of $(\mathfrak{g}_o,\mathfrak{h})$. The symmetric space G_o / H is called a *affine symmetric space.* There is a bijection between the set of locally semisimple symmetric pairs and the set of affine symmetric spaces $G_o / (G_o)_\sigma^o$. We will restrict our analysis to the locally semisimple symmetric pairs.

Let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_o$ and let $G = \mathrm{Aut}(\mathfrak{g})^o$. An semisimple symmetric pair determines a pair of commuting involutions of $\mathfrak{g}$.

9.2. Proposition. *Let* $(\mathfrak{g}_o,\sigma)$ *be a locally semisimple symmetric pair. Then there exists a Cartan involution* θ *of* $\mathfrak{g}_o$ *such that* $\sigma\theta = \theta\sigma$.

This is proved in Berger [2].

If $\theta_1,\theta_2 \in \mathrm{Aut}(\mathfrak{g}_o)$ are Cartan involutions satisfying $\theta_i\sigma = \sigma\theta_i$ (i = 1,2), then there exists $Y \in \mathfrak{h}$ such that $\exp Y \theta_1 \exp -Y = \theta_2$ (see Matsuki [16]). In other words, if we lift σ, θ_1 and θ_2 to involutions of $\mathfrak{g}$, then the pairs (σ,θ_1) and (σ,θ_2) are isomorphic in the sense of (5.15). Conversely starting with a pair of commuting involutions of $\mathfrak{g}$, we obtain a locally semisimple symmetric pair. This follows from the following result.

9.3. Lemma. *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra and* $\theta_1,\cdots,\theta_n$ *commuting involutorial automorphisms of* $\mathfrak{g}$. *Then there exists a compact real form* $\mathfrak{u}$ *of* $\mathfrak{g}$, *with conjugation* τ, *such that* $\theta_i\tau = \tau\theta_i$ *for* $i = 1,....,n$.

This result is discussed in the thesis of B.Hoogenboom [13]. Another proof goes as follows. Let R denote the subgroup of $\mathrm{Aut}(\mathfrak{g})$ spanned by $\theta_1,\cdots,\theta_n$. Since R is a compact subgroup of $\mathrm{Aut}(\mathfrak{g})$, there exists a maximal compact subgroup U of $\mathrm{Aut}(\mathfrak{g})$ containing R. Since U is maximal compact, also $U \cap \mathrm{Aut}(\mathfrak{g})^o$ is a maximal compact subgroup of $\mathrm{Aut}(\mathfrak{g})^o$ and its Lie algebra $\mathfrak{u}$ satisfies the above properties.

9.4. Dual and associated pairs

Let (θ,σ) be a pair of commuting involutions of the complex Lie algebra $\mathfrak{g}$ and $\mathfrak{u}$ a (θ,σ)-stable compact real form of $\mathfrak{g}$ with conjugation τ. Denote $\theta\tau$ by $\bar{\theta}$ and $\sigma\tau$ by $\bar{\sigma}$. For a pair (θ,σ), the first involution determines a real form. Then $(\mathfrak{g}^{\bar\theta},\sigma|_{\mathfrak{g}^{\bar\theta}})$ is a locally semisimple symmetric pair, corresponding to (θ,σ). Here $\mathfrak{g}^{\bar\theta}$ is the set of fixed points of the conjugation $\bar{\theta}$ in $\mathfrak{g}$. The set of fixed points of σ in $\mathfrak{g}^{\bar\theta}$ will be denoted by $\mathfrak{g}_\sigma^{\bar\theta}$. It follows from Helgason [11,Cp.X,1.4] that the isomorphism class of $(\mathfrak{g}^{\bar\theta},\sigma|_{\mathfrak{g}^{\bar\theta}})$ does not depend on the choice of the (σ,θ)-stable compact real form $\mathfrak{u}$ of $\mathfrak{g}$.

The pair (σ,θ) is called the *dual pair* of (θ,σ) and the corresponding locally semisimple symmetric pair $(\mathfrak{g}^{\bar\sigma},\mathfrak{g}^{\bar\sigma}_{\theta})$ is called the *dual pair* of $(\mathfrak{g}^{\bar\theta},\mathfrak{g}^{\bar\theta}_{\sigma})$. Similarly the pair $(\theta,\sigma\theta)$ will be called the *associated pair* of (θ,σ) and $(\mathfrak{g}^{\bar\theta},\mathfrak{g}^{\bar\theta}_{\theta\sigma})$ the *associated symmetric pair* of $(\mathfrak{g}^{\bar\theta},\mathfrak{g}^{\bar\theta}_{\sigma})$.

9.5. Let (θ,σ), $\bar\theta$ and $\mathfrak{g}^{\bar\theta}$ be as in (9.4). We can lift (θ,σ) to a pair of commuting involutions of $G = \mathrm{Aut}(\mathfrak{g})^{\circ}$, which we denote also by (θ,σ). The pairs of commuting involutions of G correspond bijectively with the pairs of commuting involutions of $\mathfrak{g}$.

The tori occurring in section 1-8 correspond to the following subspaces of $\mathfrak{g}^{\bar\theta}$. Let $\mathfrak{g}^{\bar\theta} = \mathfrak{k}\oplus p$ be the usual decomposition in eigenspaces of θ (i.e. a Cartan decomposition of $\mathfrak{g}^{\bar\theta}$). Likewise let $\mathfrak{g}^{\bar\theta} = \mathfrak{h}\oplus\mathfrak{q}$ be the decomposition in eigenspaces of $\sigma|_{\mathfrak{g}^{\bar\theta}}$. Now θ-split (resp. σ-split and (σ,θ)-split) tori of G correspond to Cartan subspaces of $\mathfrak{p}$ (resp. $\mathfrak{q}$ and $\mathfrak{p}\cap\mathfrak{q}$).

The characterization of the pairs of commuting involutions of G in section 5-8 gives a characterization of locally semisimple symmetric pairs in terms of a (σ,θ)-stable Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}^{\bar\theta}$, such that $\mathfrak{t}\cap\mathfrak{p}$ (resp. $\mathfrak{t}\cap\mathfrak{q}$, resp. $\mathfrak{t}\cap\mathfrak{p}\cap\mathfrak{q}$) is maximal abelian in $\mathfrak{p}$ (resp. $\mathfrak{q}$, resp. $\mathfrak{p}\cap\mathfrak{q}$). These Cartan subalgebras of $\mathfrak{g}^{\bar\theta}$ are frequently used in the analysis on semisimple symmetric spaces (see [18]).

A symmetric pair is called *irreducible* if the adjoint representation of $\mathfrak{h}$ on $\mathfrak{q}$ is irreducible. This is equivalent to the notion of irreducibility defined in (7.17). From (8.26) we obtain now.

9.6. Theorem. *The isomorphism classes of the locally semisimple symmetric pairs* $(\mathfrak{g}_0,\sigma)$ *correspond bijectively to the isomorphism classes of pairs of commuting involutions* (θ,σ) *of* $\mathfrak{g}$ *or* $Aut(\mathfrak{g})^{\circ}$. *Here* $\mathfrak{g}$ *denotes the complexification of* $\mathfrak{g}_0$ *and* $\theta|_{\mathfrak{g}_0}$ *a Cartan involution of* $\mathfrak{g}_0$ *commuting with* σ. *In particular, a pair* (S,ϵ_λ), *where* S *is an admissible irreducible* (θ,σ)*-index and* ϵ_λ *one of the quadratic elements occurring with this* (θ,σ)*-index in (7.18.1-6), represents the isomorphism class of an irreducible locally semisimple symmetric pair.*

In order to identify these results with those of Berger [2], we listed in the tables II and VII the subalgebras $\mathfrak{g}^{\bar\sigma}_{\theta}(\epsilon_i)$, where $\mathfrak{g}^{\bar\sigma}_{\theta}(\epsilon_i)$ denotes the set of fixed points of $\theta\mathrm{Int}(\epsilon_i)$ in $\mathfrak{g}^{\bar\sigma}$. In the tables we added also the type of $\sigma\theta$, according to the notation in table II. Together with the type of σ (or θ) one easily determines now the associated pairs $(\sigma,\sigma\theta)$ and $(\theta,\sigma\theta)$. These are of importance in the Fourier analysis on symmetric spaces and also in descriptions of orbits of semisimple symmetric spaces under the action of minimal parabolic subgroups (see Matsuki [16]).

9.7. A pair (θ,σ) is called *self-dual* if (θ,σ) is isomorphic to (σ,θ) and *self-associated* if (θ,σ) is isomorphic to $(\theta,\sigma\theta)$ or equivalently if the associated dual pair is self-dual. These pairs can be characterized as follows.

9.8. Lemma. *Let* $(\theta,\sigma)\in\mathrm{Aut(G)}$ *be a pair of commuting involutions. Then the following are equivalent:*

(1) (θ,σ) *is self-dual,*

(2) θ *is isomorphic to* σ,

(3) there is a maximal θ*-split torus* A *of* G *and a quadratic element* $\epsilon\in$A *such that* $(\theta,\sigma)=(\theta,\theta\mathrm{Int}(\epsilon))$.

Proof. Since (3)⇒(1) follows immediately from (5.21.ii) and (1)⇒(2) is obvious it suffices to prove (2)⇒(3). If $g\in G$ such that $\mathrm{Int}(g)\sigma\mathrm{Int}(g^{-1})=\theta$, then $\theta=\sigma\mathrm{Int}(\sigma(g)g^{-1})$. By a result of Richardson [20,6.3] there is a maximal σ-split torus A of G such that $\sigma(g)g^{-1}\in A$. Since A is also maximal θ-split, the result is clear.

9.9. Remark. If $\theta\in\mathrm{Aut(G)}$ is an involution, A a maximal θ-split torus of G and $\epsilon\in A$, $\epsilon^2=e$, then the pair $(\theta,\theta\mathrm{Int}(\epsilon))$ corresponds to a symmetric pair of type K_ϵ, as introduced by Oshima-Sekiguchi [18]. From the above result it is now clear that a symmetric pair $(\mathfrak{g}^\theta,\sigma)$ is of type K_ϵ if and only if $\mathfrak{g}_\theta$ and $\mathfrak{g}_\sigma$ are isomorphic. This proves the inverse implication of Lemma 1.10 in Oshima-Sekiguchi [18]. In other words, the single exception $(\mathrm{so}^*(4l),\mathrm{su}(2l-i,i)+\sqrt{-1}\mathbb{R})$ (i odd), which occurs in Berger's lists, does not exist (see also Remark 1.15 in Oshima-Sekiguchi [18]).

9.10. Remark. Berger [2] gives a description how one can obtain the affine symmetric spaces from the locally semisimple symmetric pairs and the fundamental groups. Together with later, detailed descriptions of the fundamental groups (see Takeuchi [28] and Goto-Kobayashi [10]) a complete description of the global pairs can be obtained.

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Table 1

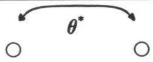
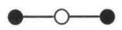
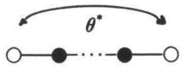
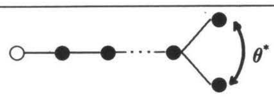
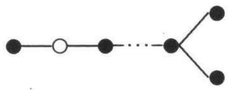
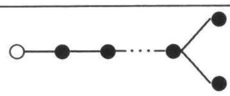
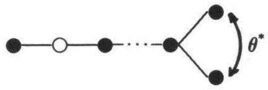
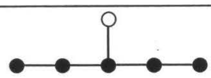
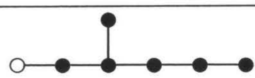
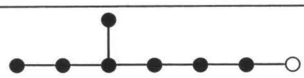
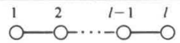
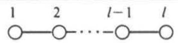
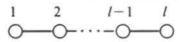
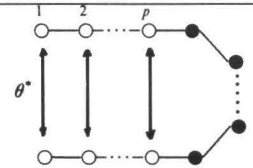
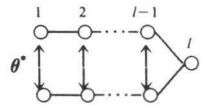
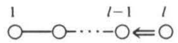
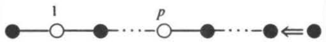
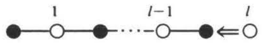
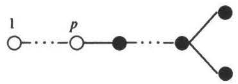
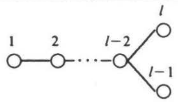
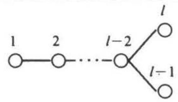
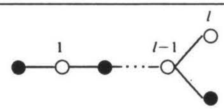
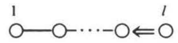
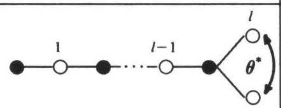
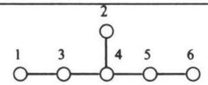
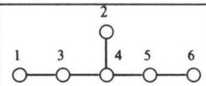
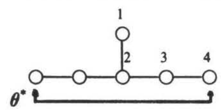
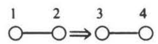
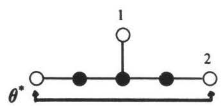
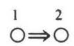
No.	Φ	θ-index	normal	admissible
1	$A_1 \times A_1$		+	+
2	A_1		+	+
3	A_2		−	−
4	A_3		+	+
5	A_l		+	+
6	B_l		+	+
7	$(l \geqslant 2)$		−	−
8	C_l		−	−
9	$(l \geqslant 3)$		+	+
10	D_{2l} $(l \geqslant 2)$		+	+
11			+	−
12	D_{2l+1} $(l \geqslant 2)$		+	+
13			+	−
14	E_6		+	−
15	E_7		+	−
16	E_8		+	−
17	F_4		−	−
18			+	+
19	G_2		−	−
20			−	−

Table 2

type θ Cartan not.	type $(\theta,\theta\mathrm{Int}(\epsilon_i))$	$(\theta,\theta)-index$	$\bar{\Delta}_\theta$	m(λ)	m(2λ)	quadratic elements	type $\mathrm{Int}(\epsilon_i)$	$\mathfrak{g}_\theta^{\bar\theta}(\epsilon_i)$
AI	$A_l^l(\mathrm{I},\epsilon_i)$			1	0	ϵ_i $(0\leqslant 2i\leqslant l+1)$	$A_l^i(\mathrm{III}_a)$	$\mathfrak{so}(l+1-i,i)$
AII	$A_{2l+1}^l(\mathrm{II},\epsilon_i)$			4	0	ϵ_i $(0\leqslant 2i\leqslant l+1)$	$A_{2i+1}^{2i}(\mathrm{III}_a)$	$\mathfrak{sp}(l+1-i,i)$
AIII$_a$ (AIV $(p=1)$)	$A_l^p(\mathrm{III}_a,\epsilon_i)$ $(1\leqslant p\leqslant l/2)$			2 $(i<p)$ 2$(l-2p+1)$ $(1=p)$	0 1	ϵ_i $(0\leqslant i\leqslant p)$	$A_l^{2i}(\mathrm{III}_a)$ $(l-2i>2i)$ $A_l^{l-2i}(\mathrm{III}_a)$ $(l-2i\leqslant 21)$	$\mathfrak{su}(l-p+1-i,i)+$ $\mathfrak{su}(p-i,i)+so(2)$
AIII$_b$	$A_{2l-1}^l(\mathrm{III}_b,\epsilon_i)$ $(l\geqslant 2)$			2 $(i<l)$ 1 $(i=l)$	0 0	ϵ_i $(0\leqslant 2i<l)$ ϵ_l	$A_{2l-1}^{2i}(\mathrm{III}_a)$ $A_{2l-1}^{2i}(\mathrm{III}_b)$	$\mathfrak{su}(l-i,i)+$ $\mathfrak{su}(l-i,i)+\mathfrak{so}(2)$ $\mathfrak{sl}(l,\mathbb{C})+\mathbb{R}$
BI (BII $(p=1)$)	$B_l^p(\mathrm{I},\epsilon_i)$ $(l\geqslant 2, 1\leqslant p\leqslant l)$			1 $(i<p)$ $2(l-p)+1$ $(i=p)$	0 0	ϵ_i $(0\leqslant i\leqslant p)$	$B_l^{2i}(\mathrm{I})$ $(2l+1>4i)$ $B_l^{2l+1-2i}(\mathrm{I})$ $(2l+1\leqslant 4i)$	$\mathfrak{so}(2l+1-p-i,i)$ $+\mathfrak{so}(p-i,i)$
CI	$C_l^l(\mathrm{I},\epsilon_i)$			1	0	ϵ_i $(0\leqslant 2i\leqslant l)$ ϵ_l	$C_l^i(\mathrm{II}_a)$ $C_l^l(\mathrm{I})$	$\mathfrak{u}(l-i,i)$ $\mathfrak{gl}(l,\mathbb{R})$
CII$_a$	$C_l^p(\mathrm{II}_a,\epsilon_i)$ $(l\geqslant 3)$ $(1\leqslant p\leqslant \frac{1}{2}(l-1))$			4 $(i<p)$ $4(l-2p)$ $(i=p)$	0 3	ϵ_i $(0\leqslant i\leqslant p)$	$C_l^{2i}(\mathrm{II}_a)$ $(21<l-2i)$ $C_l^{l-2i}(\mathrm{II}_a)$ $(2i\geqslant l-2i)$	$\mathfrak{sp}(l-p-i,i)+$ $\mathfrak{sp}(p-i,i)$

CII_b	$C^l_{2l}(II_b,\epsilon_i)$ $(l\geqslant 2)$			4 $(i<l)$ 3 $(i=l)$	0 0	ϵ_i $(0\leqslant 2i<l)$ ϵ_l	$C^{2i}_{2l}(II_a)$ $C^{2l}_{2l}(I)$	$\mathfrak{sp}(l-i,i)+\mathfrak{sp}(l-i,i)$ $\mathfrak{sp}(l,\mathbb{C})$
DI_a $(DII(p=1))$	$D^p_l(I_a,\epsilon_i)$ $(l\geqslant 4)$ $(1\leqslant p\leqslant l-1)$			1 $(i<p)$ $2(l-p)$ $(i=p)$	0 0	ϵ_i $(0\leqslant i\leqslant p)$	$D^{2i}_l(I_a)$ $(2l-2i>2i)$ $D^{2l-2i}_l(I_a)$ $(2l-2i\leqslant 2i)$	$\mathfrak{so}(2l-p-i,i)+$ $\mathfrak{so}(p-i,i)$
DI_b	$D^l_l(I_b,\epsilon_i)$ $(l\geqslant 4)$			1	0	ϵ_i $(0\leqslant 2i\leqslant l)$ $\epsilon_{l-1},\epsilon_l$	$D^{2i}_l(I_a)$ $D^{\frac{1}{2}l}_l(III_a)$ $(l$ *even*$)$ $D^{\frac{1}{2}(l-1)}_l(III_b)$ $(l$ *odd*$)$	$\mathfrak{so}(l-i,i)+\mathfrak{so}(l-i,i)$ $\mathfrak{so}(l,\mathbb{C})$
$DIII_a$	$D^l_{2l}(III_a,\epsilon_i)$ $(l\geqslant 2)$			4 $(i<l)$ 1 $(i=l)$	0 0	ϵ_i $(0\leqslant 2i\leqslant l)$ ϵ_l	$D^{4i}_{2l}(I_a)$ $D^l_{2l}(III_a)$	$\mathfrak{u}(2l-2i,2i)$ $\mathfrak{su}^*(2l)+\mathfrak{so}(l,l)$
$DIII_b$	$D^l_{2l+1}(III_b,\epsilon_i)$ $(l\geqslant 2)$			4 $(i<l)$ 4 $(i=l)$	0 1	ϵ_i $(0\leqslant i\leqslant l)$	$D^{4i}_{2l+1}(I_a)$ $(4i<4l+2-4i)$ $D^{4l+2-4i}_{2l+1}(I_a)$ $(4i\geqslant 4l+2-4i)$	$\mathfrak{u}(2l+1-2i,2i)$
EI	$E^6_6(I,\epsilon_i)$			1	0	ϵ_1 ϵ_2	EIII EII	$\mathfrak{sp}(2,2)$ $\mathfrak{sp}(4,\mathbb{R})$
EII	$E^2_6(III,\epsilon_i)$			1 $(i=1,2)$ 2 $(i=3,4)$	0 0	ϵ_1 ϵ_4	EII EIII	$\mathfrak{su}(3,3)+\mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{su}(4,2)+\mathfrak{su}(2)$
EIII	$E^2_6(III,\epsilon_i)$			6 $(i=1)$ 8 $(i=2)$	0 0	ϵ_1 ϵ_2	EII EIII	$\mathfrak{so}^*(10)+\mathfrak{so}(2)$ $\mathfrak{so}(8,2)+\mathfrak{so}(2)$

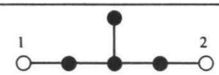
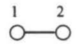
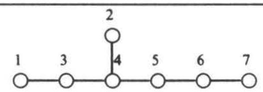
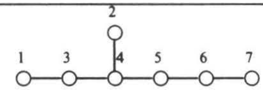
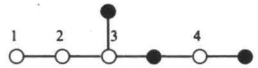
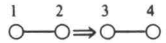
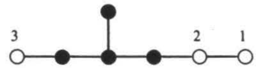
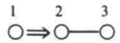
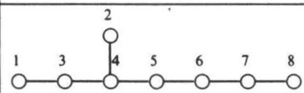
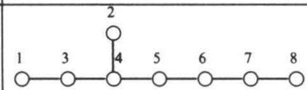
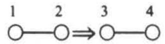
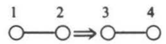
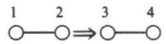
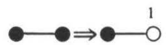
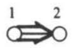
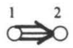
EIV	$E_6^2(\mathrm{IV},\epsilon_i)$			8	0	ϵ_1	EIII	FII
EV	$E_7^7(\mathrm{V},\epsilon_i)$			1	0	ϵ_1 ϵ_2 ϵ_7	EVI EV EVII	$\mathfrak{su}(4,4)$ $\mathfrak{sl}(8,\mathbf{R})$ $\mathfrak{su}^*(8)$
EVI	$E_7^4(\mathrm{VI},\epsilon_i)$			1 $(i=1,2)$ 4 $(i=3,4)$	0 0	ϵ_1 ϵ_4	EVI EVI	$\mathfrak{so}^*(12)+\mathfrak{sl}(2,\mathbf{R})$ $\mathfrak{so}(8,2)+\mathfrak{su}(2)$
EVII	$E_7^3(\mathrm{VII},\epsilon_i)$			1 $(i=1)$ 8 $(i=2,3)$	0 0	ϵ_1 ϵ_3	EVII EVI	EIV $+\,\mathfrak{so}(1,1)$ EIII $+\,\mathfrak{so}(2)$
EVIII	$E_8^8(\mathrm{VIII},\epsilon_i)$			1	0	ϵ_1 ϵ_8	EVIII EIX	$\mathfrak{so}(8,8)$ $\mathfrak{so}^*(16)$
EIX	$E_8^4(\mathrm{IX},\epsilon_i)$			1 $(i=1,2)$ 8 $(i=3,4)$	0 0	ϵ_1 ϵ_4	EIX EVIII	EVII $+\,\mathfrak{sl}(2,\mathbf{R})$ EVI $+\,\mathfrak{su}(2)$
FI	$F_4^4(\mathrm{I},\epsilon_i)$			1	0	ϵ_1 ϵ_4	FI FII	$\mathfrak{sp}(3,\mathbf{R})+\mathfrak{sl}(2,\mathbf{R})$ $\mathfrak{sp}(2,1)+\mathfrak{su}(2)$
FII	$F_4^1(\mathrm{II},\epsilon_i)$			8	7	ϵ_1	FII	$\mathfrak{so}(8,1)$
G	$G_2^2(\epsilon_i)$			1	0	ϵ_1	G	$\mathfrak{sl}(2,\mathbf{R})+\mathfrak{sl}(2,\mathbf{R})$

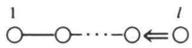
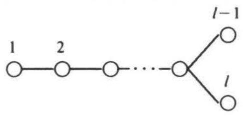
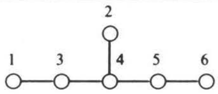
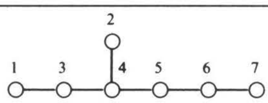
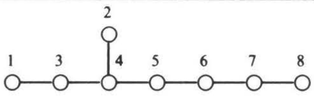
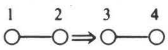
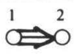
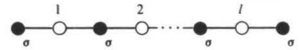
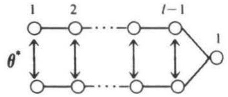
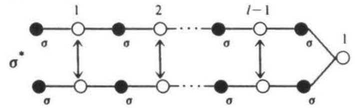
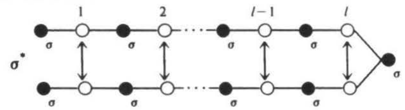
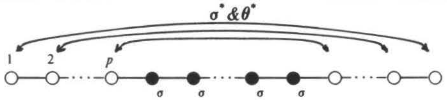
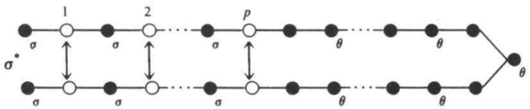
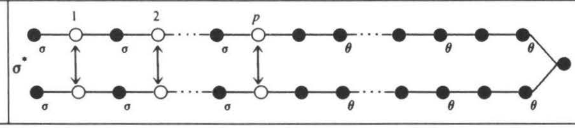
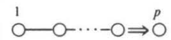
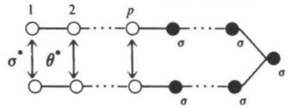
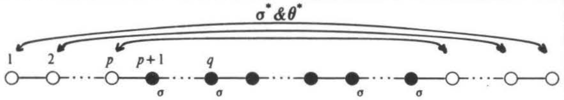
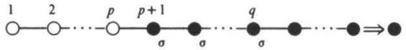
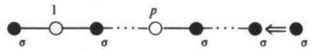
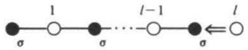
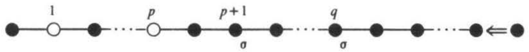
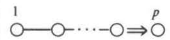
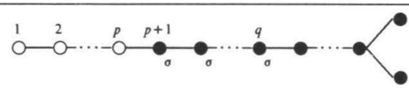
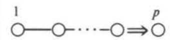
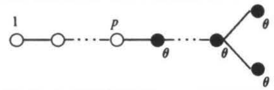
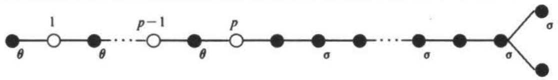
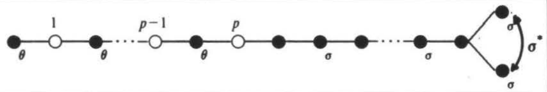
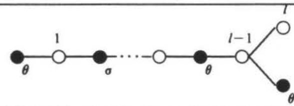
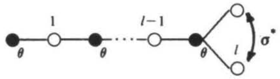
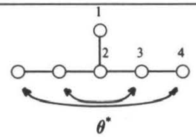
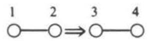
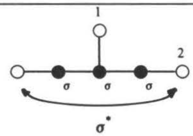
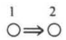
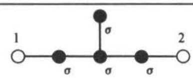
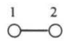
Table 3			
type $(\theta, \theta \mathrm{Int}(\epsilon_i))$	$\bar{\Delta}_\theta$	$m(\lambda)$	quadratic elements
$(A_l \times A_l)(\epsilon_j)$ $(l \geqslant 1)$		2	ϵ_j $(2j \leqslant l+1)$
$(B_l \times B_l(\epsilon_j)$ $(l \geqslant 2)$		2	ϵ_j $(j \leqslant l)$
$(C_l \times C_l)(\epsilon_j)$ $(l \geqslant 3)$		2	ϵ_j $(2j \leqslant l)$ ϵ_l
$(D_l \times D_l)(\epsilon_j)$ $(l \geqslant 4)$		2	ϵ_j $(2j \leqslant l)$ ϵ_{l-1} ϵ_l
$(E_6 \times E_6)(\epsilon_j)$		2	ϵ_1 ϵ_2
$(E_7 \times E_7)(\epsilon_j)$		2	ϵ_1 ϵ_2 ϵ_7
$(E_8 \times E_8)(\epsilon_j)$		2	ϵ_1 ϵ_8
$(F_4 \times F_4)(\epsilon_j)$		2	ϵ_1 ϵ_4
$(G_2 \times G_2)(\epsilon_j)$		2	ϵ_1

Table 4

type $(\sigma,\theta\mathrm{Int}(\epsilon_i))$	(σ,θ)-index	$\bar{\Delta}_{\sigma,\theta}$	$m^+(\lambda,\sigma\theta)$ $m^-(\lambda,\sigma\theta)$	$m^+(2\lambda,\sigma\theta)$ $m^-(2\lambda,\sigma\theta)$	quadratic elements	type $\sigma\theta$
$A_{2l+1}^{2l+1,l}(I,II)$ $(l\geqslant1)$			2 0 2 0			$A_{2l+1}^{l+1}(III_b)$
$A_{2l-1}^{2l-1,l}(I,III_b,\epsilon_l)$ $(l\geqslant2)$			1 0 1 0 $(i<l)$	1 0 0 0 $(i=l)$	ϵ_l	$A_{2l-1}^{l-1}(II)$
$A_{4l-1}^{2l,2l-1}(III_b,II\epsilon_l)$ $(l\geqslant1)$			4 0 4 0 $(i<l)$	3 0 1 0 $(i=l)$	ϵ_l	$A_{4l-1}^{4l-1}(I)$
$A_{4l+1}^{2l+1,2l}(III_b,II)$ $(l\geqslant1)$			4 0 4 0 $(i<l)$	4 1 4 3 $(i=l)$		$A_{4l+1}^{4l+1}(I)$
$A_l^{l,p}(I,III_a)$ $(1\leqslant p\leqslant \frac{1}{2}l)$			1 0 1 0 $(i<p)$	$(l-2p)$ 0 $(l-2p)$ 1 $(i=p)$		$A_l^l(I)$
$A_{4l-1}^{2p,2l-1}(III_a,II)$ $(1\leqslant 2p<4l-1)$			4 0 4 0 $(i<p)$	$4(2l-2p)$ 3 $4(2l-2p)$ 1 $(i=p)$		$A_{4l-1}^{2l-1}(II)$
$A_{4l+1}^{2p,2l}(III_a,II)$ $(1\leqslant 2p<4l+1)$			4 0 4 0 $(i<p)$	$4(2l-2p+1)$ 3 $4(2l-2p+1)$ 1 $(i=p)$		$A_{4l+1}^{2l}(II)$

$A^{l,p}_{2l-1}(III_b, III_a, \epsilon_i)$ $(1 \leq p \leq \frac{1}{2}(2l-1))$			2 0 0 0 $(i<p)$	$2(l-p)$ 1 $2(l-p)$ 0 $(i=p)$	ϵ_i $(1 \leq i \leq p-1)$	$A^{l-p}_{2l-1}(III_a)$
$A^{q,p}_l(III_a, III_a, \epsilon_i)$ $(1 \leq p < q \leq \frac{1}{2}l)$			2 0 0 0 $(i<p)$	$2(l-q-p+1)$ 1 $2(q-p)$ 0 $(i=p)$	ϵ_i $(1 \leq i \leq p)$	$A^{q-p}_l(III_a)$
$B^{q,p}_l(I, I, \epsilon_i)$ $(1 \leq p < q \leq l)$ $(l \geq 2)$			1 0 0 0 $(i<p)$	$(2l-q-p+1)$ 0 $(q-p)$ 0 $(i=p)$	ϵ_i $(1 \leq i \leq p)$	$B^{q-p}_l(I)$
$C^{l,p}_l(I, II_a)$ $(1 \leq p \leq \frac{1}{2}(l-1))$ $(l \geq 3)$			2 0 2 0 $(i<p)$	$2(l-p)$ 1 $2(l-p)$ 2 $(i=p)$		$C^p_l(II_a)$
$C^{2l,l}_{2l}(I, II_b, \epsilon_l)$ $(l \geq 2)$			2 0 2 0 $(i<p)$	2 0 1 0 $(i=l)$	ϵ_l	$C^l_{2l}(II_b)$
$C^{l,p}_l(II_a, II_a, \epsilon_i)$ $(1 \leq p < q \leq \frac{1}{2}(l-1))$ $(l \geq 3)$			4 0 0 0 $(i<p)$	$4(l-q-p)$ 3 $4(q-p)$ 0 $(i=p)$	ϵ_i $(1 \leq i \leq p)$	$C^{q-p}_l(II_a)$
$C^{l,p}_{2l}(II_b, II_a, \epsilon_i)$ $(1 \leq p \leq \frac{1}{2}(2l-1))$ $(l \geq 2)$			4 0 0 0 $(i<p)$	$4(l-p)$ 3 $4(l-p)$ 0 $(i=p)$	ϵ_i $(1 \leq i \leq p-1)$	$C^{l-p}_{2l}(II_a)$
$D^{q,p}_l(I_a, I_a, \epsilon_i)$ $(1 \leq p < q \leq l-1)$ $(l \geq 4)$			1 0 0 0 $(i<p)$	$(2l-q-p)$ 0 $(q-p)$ 0 $(i=p)$	ϵ_i $(1 \leq i \leq p)$	$D^{q-p}_l(I_a)$

$D_l^{p,l}(I_a,I_b,\epsilon_i)$ $(1\leqslant p\leqslant l-1)$ $(l\geqslant 4)$			1 0 0 0 $(i<p)$	$(l-p)$ 0 $(l-p)$ 0 $(i=p)$	ϵ_i $(1\leqslant i\leqslant p-1)$	$D_l^{l-p}(I_a)$
$D_{2l}^{l,2p}(III_a,I_a)$ $(1\leqslant p\leqslant l-1)$ $(l\geqslant 2)$			2 0 2 0 $(i<p)$	$4(l-p)$ 1 $4(l-p)$ 0 $(i=p)$		$D_{2l}^{l}(III_a)$
$D_{2l+1}^{l,2p}(III_b,I_a)$ $(1\leqslant p\leqslant l-1)$ $(l\geqslant 2)$			2 0 2 0 $(i<p)$	$2(2l-2p+1)$ 1 $2(2l-2p+1)$ 0 $(i=p)$		$D_{2l+1}^{l}(III_b)$
$D_{2l}^{l,2l}(III_a,I_b,\epsilon_i)$ $(l\geqslant 2)$			2 0 2 0 $(i<l)$	1 0 0 0 $(i=l)$	ϵ_l	$D_{2l}^{l}(III_a)$
$D_{2l+1}^{l,2l+1}(III_b,I_b)$ $(l\geqslant 2)$			2 0 2 0 $(i<l)$	2 0 2 1 $(i=l)$		$D_{2l+1}^{2l+1}(I_b)$
$E_6^{6,4}(I,II,\epsilon_i)$			1 0 0 0 $(i=1,2)$	1 0 1 0 $(i=3,4)$	ϵ_1	$E_6^{2}(IV)$
$E_6^{6,2}(I,III)$			3 0 3 0 $(i=1)$	4 0 4 1 $(i=2)$		$E_6^{6}(I)$
$E_6^{6,2}(I,IV)$			4 0 4 0			$E_6^{4}(II)$

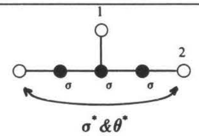
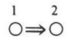
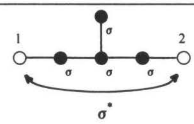
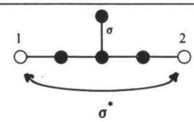
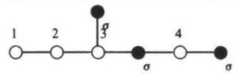
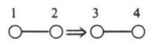
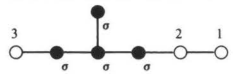
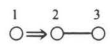
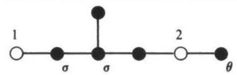
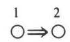
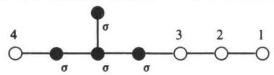
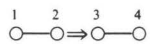
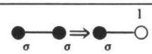
$E_6^{4,2}(\mathrm{II},\mathrm{III},\epsilon_i)$			4 0 4 1 2 0 4 0 $(i=1)$ $(i=2)$	ϵ_1	$E_6^2(\mathrm{III})$
$E_6^{4,2}(\mathrm{II},\mathrm{IV})$			8 3 8 5		$E_6^6(\mathrm{I})$
$E_6^{2,2}(\mathrm{III},\mathrm{IV})$			8 7 8 1		$E_6^2(\mathrm{IV})$
$E_7^{7,4}(V,VI,\epsilon_i)$			1 0 2 0 0 0 2 0 $(i=1,2)$ $(i=3,4)$	ϵ_1	$E_7^3(VII)$
$E_7^{7,3}(V,VII,\epsilon_i)$			1 0 4 0 0 0 4 0 $(i=1)$ $(i=3,4)$	ϵ_1	$E_7^4(VI)$
$E_7^{4,3}(VI,VII,\epsilon_i)$			6 0 8 1 2 0 8 0 $(i=1)$ $(i=2)$	ϵ_1	$E_7^7(V)$
$E_8^{8,4}(VIII,IX,\epsilon_i)$			1 0 4 0 0 0 4 0 $(i=1,2)$ $(i=3,4)$	ϵ_1	$E_8^4(\mathrm{IX})$
$F_4^{4,1}(\mathrm{I},\mathrm{II})$			4 3 4 4		$F_4^4(\mathrm{I})$

Table 5

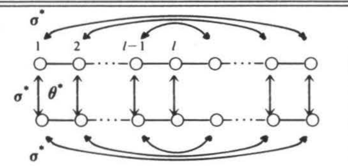
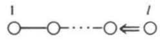
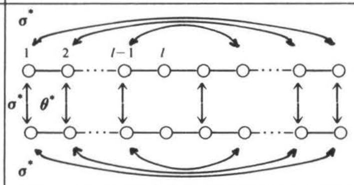
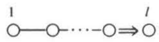
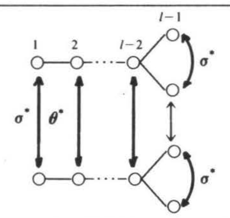
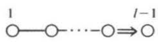
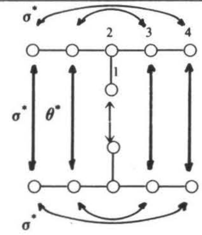
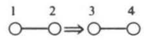
type $(\sigma,\theta\mathrm{Int}(\epsilon_i))$	(σ,θ)-index	$\bar{\Delta}_{\sigma,\theta}$	$m^+(\lambda,\sigma\theta)$ $m^-(\lambda,\sigma\theta)$	$m^+(2\lambda,\sigma\theta)$ $m^-(2\lambda,\sigma\theta)$	quadratic elements
$({}^2A_{2l-1}\times{}^2A_{2l-1})(\epsilon_j)$			2 0 2 0 $(j \leq l)$	2 0 0 0 $(j=l)$	ϵ_l
$({}^2A_{2l}\times{}^2A_{2l})$			2 0 2 0 $(j \leq l)$	2 0 2 2 $(j=l)$	
$({}^2D_l\times{}^2D_l)(\epsilon_j)$			2 0 0 0 $j \leq l-1)$	2 0 2 0 $(j=l)$	ϵ_j $(1 \leq j \leq l-2)$
$({}^2E_6\times{}^2E_6)(\epsilon_j)$			2 0 0 0 $(j=1,2)$	2 0 2 0 $(j=3,4)$	ϵ_1

Table 6

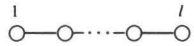
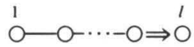
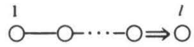
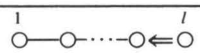
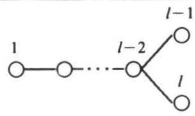
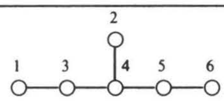
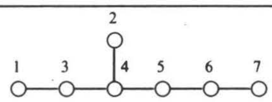
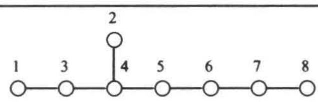
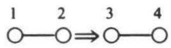
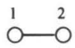
Φ	Dynkin diagram	quadratic elements
A_l $(l \geqslant 1)$		ϵ_j $(2j \leqslant l+1)$
B_l $(l \geqslant 2)$		ϵ_j $(j \leqslant l)$
BC_l $(l \geqslant 1)$		ϵ_j $(j \leqslant l)$
C_l $(l \geqslant 3)$		ϵ_j $(2j \leqslant l)$ ϵ_l
D_l $(l \geqslant 4)$		ϵ_j $(2j \leqslant l)$ ϵ_{l-1} ϵ_l
E_6		ϵ_1 ϵ_2
E_7		ϵ_1 ϵ_2 ϵ_7
E_8		ϵ_1 ϵ_8
F_4		ϵ_1 ϵ_4
G_2		ϵ_1

Table 7

(σ,θ)-index		$\mathfrak{g}^{\sigma}_{\theta}(\epsilon_i)$	$\mathfrak{g}^{\theta}_{\sigma}(\epsilon i)$
$A^{2l+1,l}_{2l+1}(I,II)$		$\mathfrak{sp}(l+1,\mathbf{R})$	$\mathfrak{so}^*(2l+2)$
$A^{2l-1,l}_{2l-1}(I,III_b,\epsilon_i)$	$i=0$	$\mathfrak{sl}(l,\mathbf{C})+\mathfrak{so}(2)$	$\mathfrak{so}^*(2l)$
	$i=l$	$\mathfrak{sl}(l,\mathbf{R})+\mathfrak{sl}(l,\mathbf{R})+\mathbf{R}$	$\mathfrak{so}(l,l)$
$A^{2l,2l-1}_{4l-1}(III_b,II,\epsilon_i)$	$i=0$	$\mathfrak{sp}(l,l)$	$\mathfrak{su}^*(2l)+\mathfrak{su}^*(2l)+\mathbf{R}$
	$i=l$	$\mathfrak{sl}(2l,\mathbf{R})$	$\mathfrak{sl}(2l,\mathbf{C})+\mathfrak{so}(2)$
$A^{2l+1,2l}_{4l+1}(III_b,II)$		$\mathfrak{sp}(2l+1,\mathbf{R})$	$\mathfrak{sl}(2l+1,\mathbf{C})+\mathfrak{so}(2)$
$A^{l,p}_{l}(I,III_a)$		$\mathfrak{sl}(p,\mathbf{R})+\mathfrak{sl}(l-p+1,\mathbf{R})+\mathbf{R}$	$\mathfrak{so}(p,l-p+1)$
$A^{2p,2l-1}_{4l-1}(III_a,II)$		$\mathfrak{sp}(p,2l-p)$	$\mathfrak{su}^*(2p)+\mathfrak{su}^*(4l-2p)+\mathbf{R}$
$A^{2p,2l}_{4l+1}(III_a,II)$		$\mathfrak{sp}(p,2l+1-p)$	$\mathfrak{su}^*(2p)+\mathfrak{su}^*(4l+2-2p)+\mathbf{R}$
$A^{l,p}_{2l-1}(III_b,III_a,\epsilon_i)$	$0\leqslant i\leqslant p-1$	$\mathfrak{su}(i,p-i)+\mathfrak{su}(l-i,l-p+i)+\mathfrak{so}(2)$	$\mathfrak{su}(i,l-i)+\mathfrak{su}(p-i,l-p+i)+\mathfrak{so}(2)$
$A^{q,p}_{l}(III_a,III_a,\epsilon_i)$	$0\leqslant i\leqslant p$	$\mathfrak{su}(l+1-q-i,q-p+i)+$ $\mathfrak{su}(i,p-i)+\mathfrak{so}(2)$	$\mathfrak{su}(i,l+1-q-i)+$ $\mathfrak{su}(p-i,q-p+i)+\mathfrak{so}(2)$
$B^{q,p}_{l}(I,I,\epsilon_i)$	$0\leqslant i\leqslant p$	$\mathfrak{so}(i,p-i)+\mathfrak{so}(2l+1-q-i,q-p+i)$	$\mathfrak{so}(i,2l+1-q-i)+\mathfrak{so}(p-i,q-p+i)$
$C^{l,p}_{l}(I,II_a)$		$\mathfrak{sp}(p,\mathbf{R})+\mathfrak{sp}(l-p,\mathbf{R})$	$\mathfrak{su}(p,l-p)+\mathfrak{so}(2)$
$C^{2l,l}_{2l}(I,II_b,\epsilon_i)$	$i=0$	$\mathfrak{sp}(l,\mathbf{C})$	$\mathfrak{su}(l,l)+\mathbf{R}$
	$i=l$	$\mathfrak{sp}(l,\mathbf{R})+\mathfrak{sp}(l,\mathbf{R})$	$\mathfrak{su}(l,l)+\mathfrak{so}(2)$
$C^{q,p}_{l}(II_a,II_a,\epsilon_i)$	$0\leqslant i\leqslant p$	$\mathfrak{sp}(i,p-i)+\mathfrak{sp}(l-q-i,q-p+i)$	$\mathfrak{sp}(i,l-q-i)+\mathfrak{sp}(p-i,q-p+i)$
$C^{l,p}_{2l}(II_b,II_a,\epsilon_i)$	$0\leqslant i\leqslant p-1$	$\mathfrak{sp}(i,p-i)+\mathfrak{sp}(l-i,l-p+i)$	$\mathfrak{sp}(i,l-i)+\mathfrak{sp}(p-i,l-p+i)$
$D^{q,p}_{l}(I_a,I_a,\epsilon_i)$	$0\leqslant i\leqslant p$	$\mathfrak{so}(i,p-i)+\mathfrak{so}(2l-q-i,q-p+i)$	$\mathfrak{so}(i,2l-q-i)+\mathfrak{so}(p-i,q-p+i)$
$D^{p,l}_{l}(I_a,I_b,\epsilon_i)$	$0\leqslant i\leqslant p-1$	$\mathfrak{so}(i,l-i)+\mathfrak{so}(p-i,l-p+i)$	$\mathfrak{so}(i,p-i)+\mathfrak{so}(l-i,l-p+i)$
$D^{l,2p}_{2l}(III_a,I_a)$		$\mathfrak{so}^*(2p)+\mathfrak{so}^*(4l-2p)$	$\mathfrak{su}(p,2l-p)+\mathfrak{so}(2)$
$D^{l,2p}_{2l+1}(III_b,I_a)$		$\mathfrak{so}^*(2p)+\mathfrak{so}^*(4l-2p+2)$	$\mathfrak{su}(p,2l+1-p)+\mathfrak{so}(2)$
$D^{l,2l}_{2l}(III_a,I_b,\epsilon_i)$	$i=0$	$\mathfrak{so}^*(2l)+\mathfrak{so}^*(2l)$	$\mathfrak{su}(l,l)+\mathfrak{so}(2)$
	$i=l$	$\mathfrak{so}(2l,\mathbf{C})$	$\mathfrak{sl}(2l,\mathbf{R})+\mathbf{R}$
$D^{l,2l+1}_{2l+1}(III_b,I_b)$		$\mathfrak{so}(2l+1,\mathbf{C})$	$\mathfrak{sl}(2l+1,\mathbf{R})+\mathbf{R}$
$E^{6,4}_{6}(I,II,\epsilon_i)$	$i=0$	$\mathfrak{su}^*(6)+\mathfrak{su}(2)$	$\mathfrak{sp}(3,1)$
	$i=1$	$\mathfrak{sl}(6,\mathbf{R})+\mathfrak{sl}(2,\mathbf{R})$	$\mathfrak{sp}(4,\mathbf{R})$
$E^{6,2}_{6}(I,III)$		$\mathfrak{so}(5,5)+\mathbf{R}$	$\mathfrak{sp}(2,2)$
$E^{6,2}_{6}(I,IV)$		FI	$\mathfrak{sp}(2,2)$
$E^{4,2}_{6}(I,III,\epsilon_i)$	$i=0$	$\mathfrak{so}^*(10)+\mathfrak{so}(2)$	$\mathfrak{su}(5,1)+\mathfrak{sl}(2,\mathbf{R})$
	$i=1$	$\mathfrak{so}(6,4)+\mathfrak{so}(2)$	$\mathfrak{su}(4,2)+\mathfrak{su}(2)$
$E^{4,2}_{6}(II,IV)$		FI	$\mathfrak{su}^*(6)+\mathfrak{su}(2)$
$E^{2,2}_{6}(III,IV)$		FII	$\mathfrak{so}(9,1)+\mathbf{R}$
$E^{7,4}_{7}(V,VI,\epsilon_i)$	$i=0$	$\mathfrak{so}^*(12)+\mathfrak{su}(2)$	$\mathfrak{su}(6,2)$
	$i=1$	$\mathfrak{so}(6,6)+\mathfrak{sl}(2,\mathbf{R})$	$\mathfrak{su}(4,4)$
$E^{7,3}_{7}(V,VII,\epsilon_i)$	$i=0$	EII $+\mathfrak{so}(2)$	$\mathfrak{su}(6,2)$
	$i=1$	EI $+\mathbf{R}$	$\mathfrak{su}^*(8)$
$E^{4,3}_{7}(VI,VII,\epsilon_i)$	$i=0$	EIII $+\mathfrak{so}(2)$	$\mathfrak{so}(10,2)+\mathfrak{sl}(2,\mathbf{R})$
	$i=1$	EII $+\mathfrak{so}(2)$	$\mathfrak{so}^*(12)+\mathfrak{su}(2)$
$E^{8,4}_{8}(VIII,IX,\epsilon_i)$	$i=0$	EVI $+\mathfrak{su}(2)$	$\mathfrak{so}(12,4)$
	$i=1$	EV $+\mathfrak{sl}(2,\mathbf{R})$	$\mathfrak{so}^*(16)$
$F^{4,1}_{4}(I,II)$		$\mathfrak{so}(5,4)$	$\mathfrak{sp}(2,1)+\mathfrak{su}(2)$

Samenvatting

De classificatie van de locaal semisimpele symmetrische ruimten, zoals gegeven door Berger in [2], is tamelijk onhandelbaar en geeft geen informatie over de fijnstructuur van deze ruimten zoals het restrictie wortelsysteem met de multipliciteiten.

In dit proefschrift worden de isomorfie klassen van geordende paren commuterende involutieve automorfismen van een reductieve algebraische group G geclassificeerd, waarbij G gedefinieerd is over een algebraisch afgesloten lichaam van karakteristiek ongelijk 2. Hieruit is een handelbare classificatie van de locaal semisimpele symmetrische ruimten af te leiden, met hun fijnstructuur.

Curriculum Vitae

De auteur van dit proefschrift werd op 10 januari 1954 te Rotterdam geboren. Na de middelbare school ging hij wis- en natuurkunde studeren aan de Rijksuniversiteit te Utrecht. Hij volgde colleges o.a. van de hoogleraren Van der Blij, Duistermaat, Ferrar, Mars, Springer en Veldkamp en verrichtte een groot experimenteel onderzoek op het Laboratorium voor Ruimteonderzoek te Utrecht.

Zijn afstudeerscriptie stond o.l.v. Prof. Dr. H. Duistermaat. In september 1980 legde hij het doctoraal examen af en op 1 december 1980 trad hij in dienst van het Mathematisch Centrum. Daar was hij werkzaam deels bij het project "Analyse op Lie groepen" o.l.v. T.H. Koornwinder, deels bij het project "Algebra" o.l.v. A.M. Cohen. De algemene supervisie over zijn onderzoek berustte bij Prof. Dr. T.A. Springer.

STELLINGEN

behorende bij het proefschrift

ALGEBRAIC GROUPS WITH A COMMUTING PAIR OF INVOLUTIONS
AND SEMISIMPLE SYMMETRIC SPACES

van

A.G. HELMINCK

In de volgende vijf stellingen zijn de notaties als in §1 van het proefschrift.

1. Zij A een maximaal θ-gespleten torus van G en T een maximale torus die A bevat met Weyl groep W(T) = W. Er is een bijectieve afbeelding van de verzameling van de G_θ^o-banen van θ-stabiele maximale tori naar de W-banen van involuties $w \in W$ met $T_w^- \subset A$.

2. Situatie als in stelling 1. Stel dat w_1, w_2 involuties zijn in W met $T_{w_i}^- \subset A$ (i = 1,2). Dan zijn w_1, w_2 geconjugeerd onder W dan en slechts dan als $w_1|_A$ en $w_2|_A$ geconjugeerd zijn onder W(A).

3. Zij A een maximaal (σ,θ)-gespleten torus van G en $A_1 \supset A$ een maximale θ-gespleten torus. Een involutie $w \in W(A_1)$ met $(A_1)_w^{-1} \subset A$ heet (σ,θ)-singulier als de commutator groep van $G_\sigma \cap Z_G((A_1)_w^+)^o$ een θ-split torus bevat van dezelfde dimensie als $(A_1)_w^-$. Er is een bijectie tussen de verzameling van $(G_\sigma \cap G_\theta)^o$-banen van σ-stabiele maximaal θ-gespleten tori en de $W(A_1)$-conjugatie klassen van (σ,θ)-singuliere involuties in $W(A_1)$.

4. Een parabolische ondergroep P van G heet (σ,θ)-gespleten als P zowel σ- als θ-gespleten is. Alle minimale (σ,θ)-gespleten parabolische ondergroepen zijn geconjugeerd onder $(G_\sigma \cap G_\theta)^o$.

5. Stel dat G gedefinieerd is over een grondlichaam $k \subset F$ en dat θ gedefinieerd is over k. De resultaten van Matsuki [16] voor $k = \mathbb{R}$, $F = \mathbb{C}$, kunnen worden gegeneraliseerd tot deze algemene situatie.

6. De resultaten van Matsuki [16] over de karakterisatie van banen van affien symmetrische ruimten onder de actie van minimale parabolische ondergroepen kunnen worden afgeleid uit werk van Rossmann [21].

7. Zij E een 7-dimensionale vectorruimte over een eindig lichaam en V de ruimte van de trilineaire alternerende vormen op E. Dan heeft GL(E) 11 banen in de projectieve ruimte behorend bij V.

8. Er is een constante m zodat voor elk eindig lichaam F het aantal projectieve GL(E)-banen van trilineaire alternerende vormen op de 8-dimensionale vectorruimte E over F ten hoogste m bedraagt.

9. Zij E een 8-dimensionele vectorruimte over een algebraisch afgesloten lichaam van karakteristiek 0 en V als in stelling 7. Voor $f \in V$ definieer

$$R(f) = \{x \in E \mid f(x, E, E) = 0\}.$$

Er zijn 13 banen van GL(E) in V van vormen f met $R(f) = 0$.

10. Door de as van een spektrometer onder een hoek β ($20° \leqslant \beta \leqslant 60°$) te plaatsen ten opzichte van de draaias van de goniometer kan het meetgebied vergroot worden.

11. Voor de situatie van de vorige stelling moet de "Bragg relatie" als volgt aangepast worden:

$$n\lambda = 2d_n \sin\theta \cos\beta$$

Hierbij is d_n de roosterconstante van het reflecterende kristal voor n^o-orde reflectie van een rö-emissielijn van golflengte λ.