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The Adjoint of a Semigroup of Linear Operators

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THE ADJOINT OF A SEMIGROUP OF LINEAR OPERATORS

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Preface

The general theory of adjoint semigroups was initiated by Phillips [Ph2], whose results are presented in somewhat more generality in the book of Hille and Phillips [HPh], and was taken up a little later by de Leeuw [dL]. Before that, Feller [Fe] had already used adjoint semigroups in the theory of partial differential equations. After these papers almost no new results on adjoint semigroups were published, although the theory of strongly continuous semigroups continued to develop rapidly. A reason for this may have been the following. From the duality relation $\langle T^*(t)x^*, x\rangle = \langle x^*, T(t)x\rangle$ it follows that theorems on C_0-semigroups trivially translate into theorems on their adjoints, the difference being that the weak*-topology of X^* takes over the role of the strong topology of X. For example, $T^*(t)$ is a weak*-continuous semigroup, but not necessarily strongly continuous. From this point of view adjoint semigroups mirror in a rather bad sense the properties of their pre-adjoints and no interesting new phenomena seem to occur.

Recently the interest in adjoint semigroups revived however, due to many applications that were found e.g. to elliptic partial differential equations [Am], population dynamics [Ceal-5], [DGT], [GH], [GW], [In], control theory [Heij], approximation theory [Ti], and delay equations [D], [DV], [HV], [V]. This stimulated also renewed interest in the abstract theory of adjoint semigroups, e.g. [Pa1-3], [GNa] and [DGH]. As far as the abstract theory is concerned there are at least two points of view which lead to interesting new results.

The first one is to see whether certain results on C_0-semigroups can be improved under the assumption that the semigroup behaves 'well' with respect to taking adjoints, i.e. if it is $\odot$-reflexive. The main field where this idea was fruitful is perturbation theory: $\odot$-reflexive semigroups were discovered to admit a larger class of perturbations, viz. bounded linear maps from X into $X^{\odot *}$ (rather than into X). These perturbations were found to be the natural abstract setting to treat various problems.

The second one is the 'structure theoretical' point of view: (i) Knowing that the adjoint semigroup need not be strongly continuous, can one quantify 'how' non-strongly continuous it is and can one say something about the size and structure of the subspace $X^{\odot}$ on which it is strongly continuous, (ii) can one give conditions on the underlying Banach space which guarantee that less pathology occurs and (iii) can one obtain more detailed results if one restricts oneself to special classes of semigroups, such as positive semigroups? Thus the idea is to make the pathological properties of adjoint semigroups to the object of study themselves.

In this thesis we adopt the second point of view. In the first four chapters we set up the general abstract theory of adjoint semigroups and in the next four chapters we study some more special themes related to (i), (ii) and (iii). Let us describe in some more detail the contents of each chapter. In Chapter 1 the basic properties of $T^*(t)$ are proved and the canonical spaces $X^\odot$ and $X^{\odot\odot}$ associated with the adjoint semigroup are introduced. Already at this stage we treat the adjoints of certain semigroups arising in a natural way in connection with Schauder bases. The reason is the usefulness of these semigroups for providing counter-examples to many questions in later chapters. In Chapter 2 the $\sigma(X, X^\odot)$-topology is studied in detail. Many results show that this topology behaves much like the weak topology, although there are also some differences. In Chapter 3 a very simple proof of de Pagter's refinement of Phillips's characterization of $\odot$-reflexivity is given, along with a characterization in terms of the integrated semigroup and some stability results. In Chapter 4 the Favard class of a semigroup is discussed. We generalize the classical result that the Favard class is precisely the domain of the generator if X is reflexive by obtaining a characterization of those semigroups for which these two spaces are the same. As an application it is shown that these spaces cannot coincide for a C_0-semigroup on c_0, unless the semigroup is uniformly continuous. In Chapter 5 we introduce the natural embedding of $X^{\odot\odot}$ into X^{**} and study the second adjoint semigroup $T^{**}(t)$. As an application it is shown that one can quantify the non-strong continuity of an adjoint semigroup: if $X^\otimes$ denotes the subspace of X^* consisting of those elements whose orbits are strongly continuous for $t > 0$, then the quotient space $X^*/X^\otimes$ is either zero or non-separable. A modification of the proof is used to show that orbits in the quotient space $X^*/X^\odot$ are either identically zero for $t > 0$ or non-separable. In Chapter 6, after proving a Hahn-Banach type theorem and giving some applications, some Banach space geometry comes into play: it turns out that there are a number of connections between continuity of the adjoint semigroup and the Banach space X having the Radon-Nikodym property. For example, if X^* has the RNP, then $T^*(t)$ is strongly continuous for $t > 0$. In Chapter 7, which is based on joint work with Günther Greiner, we study the rather delicate problem to describe the semigroup dual of a tensor product of two semigroups in terms of the semigroup duals of the two semigroups. The special case where $T(t)$ is translation with respect to the first coordinate on $C_0(\mathbb{R} \times K)$ is discussed in detail. Finally, in Chapter 8, which is partly based on joint work with Ben de Pagter, we study adjoints of positive semigroups. The problem when the semigroup dual $X^\odot$ is a sublattice of X^* is discussed. Although, in general, this problem is difficult, there is very detailed information on the behaviour of the adjoint semigroup in the case where X is a $C(K)$-space or $T(t)$ is a multiplication semigroup. At the end of the thesis we state some open problems.

Chapter 0

Preliminaries

In this introductory chapter we discuss some standard material concerning Banach spaces and C_0-semigroups.

0.1. Banach spaces

The reader is assumed to be familiar with some elementary Banach space theory, such as is covered by the first four chapters of Rudin [Ru3]. In this section we recall some facts and fix our notation.

Unless stated otherwise, throughout this thesis all vector spaces can be real or complex.

The closed unit ball of a Banach space X, $\{x \in X : \ ||x|| \leq 1\}$, will be denoted by B_X and the dual space of X is denoted by X^*. The generic elements of X and X^* are denoted by x and x^* respectively. The notation $\langle x^*, x\rangle$ is used for the duality pairing between X^* and X. Let Y be a closed subspace of X and let $y^* \in B_{Y^*}$. The *Hahn-Banach theorem* asserts that there exists an $x^* \in B_{X^*}$ such that the restriction $x^*|_Y$ equals y^*.

The *weak topology* on X is the coarsest locally convex topology in which all elements of X^* are still continuous. It is generated by the family of seminorms $\{p_{x^*} : x^* \in X^*\}$, where

$$p_{x^*}(x) := |\langle x^*, x\rangle|.$$

It is a corollary of the Hahn-Banach theorem that every convex closed subset of X is actually weakly closed. The *Eberlein-Shmulyan* theorem asserts that a subset K of X is relatively weakly compact if and only if it is relatively weakly sequentially compact, i.e., every sequence in K contains a subsequence which is weakly convergent to some $x \in X$.

X can be naturally embedded into X^{**} by means of the map i defined by

$$\langle ix, x^*\rangle := \langle x^*, x\rangle.$$

The map i establishes an isometrical isomorphism between X and iX. By *Goldstine's theorem*, iB_X is weak*-dense in $B_{X^{**}}$. X is called *reflexive* if $iX = X^{**}$. This is the case if and only if B_X is weakly compact. Usually X is identified with iX.

The *weak*-topology* of X^* is the coarsest locally convex topology in which all elements of iX are still continuous as functionals on X^*. It is generated by the family of seminorms $\{p_x : x \in X\}$, where

$$p_x(x^*) := |\langle x^*, x\rangle|.$$

An element $x^{**} \in X^{**}$ belongs to iX if and only if it is weak*-continuous. The *Banach Alaoglu theorem* asserts that B_{X^*} is weak*-compact.

The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. Instead of $\mathcal{L}(X,X)$ we simply write $\mathcal{L}(X)$. If the graph $\{(x,Tx) : x \in X\}$ of a linear operator $T : X \to Y$ on X is closed in $X \times Y$ with the product topology, then T is bounded by the *closed graph theorem*. If $T \in \mathcal{L}(X,Y)$ is onto, then by the *open mapping theorem* T is open. If there are constants $0 < m \leq M$ such that for all $x \in X$ we have

$$m||x|| \leq ||Tx||_Y \leq M||x||,$$

then T is called an *embedding*. Note that a bounded map is an embedding if and only if it is injective and its range is a closed subspace. In particular, if T is 1-1 and onto, then T is an isomorphism. The *uniform boundedness theorem* asserts that if a sequence $(T_n) \subset \mathcal{L}(X,Y)$ has the property that $\sup_n ||T_n x|| < \infty$ for each $x \in X$, then $\sup_n ||T_n|| < \infty$. Applying this with Y equal to the scalar field, it follows in particular that weakly compact sets and weak*-compact sets are norm bounded.

A linear operator $T : X \to Y$ is weakly continuous if and only if it is bounded. An operator $S \in \mathcal{L}(Y^*, X^*)$ is weak*-continuous if and only if S is the adjoint T^* of some $T \in \mathcal{L}(X,Y)$.

An operator $T \in \mathcal{L}(X,Y)$ is called *weakly compact* if the set TB_X is relatively weakly compact in Y. By *Gantmacher's theorem* this is the case if and only if $T^{**}X^{**} \subset iY$.

Unless stated otherwise, topological concepts will always refer to the strong topology and vector integrals are always Bochner integrals (see the Appendix).

0.2. One-parameter semigroups of operators

Let X be a Banach space. A system $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X is called a *one-parameter semigroup of operators* (briefly, a *semigroup*) if the following two conditions are satisfied:

(S1) $T(0) = I$;
(S2) $T(s)T(t) = T(s+t), \quad \forall s, t \geq 0$.

Here I is the identity operator on X. In the sequel the notation $T(t)$ will be used instead of the more cumbersome notation $\{T(t)\}_{t\geq 0}$. A semigroup $T(t)$ is said to be *strongly continuous* (briefly, $T(t)$ is a C_0-*semigroup*) if it satisfies the additional condition

(S3) $\lim_{t\downarrow 0} \|T(t)x - x\| = 0, \quad \forall x \in X$.

More generally if τ is any locally convex topology on X, then $T(t)$ is τ-*continuous* if $\tau\text{-}\lim_{t\downarrow 0}(T(t)x - x) = 0$ for all $x \in X$.

A C_0-*group* is defined analogously, with the index t running over $\mathbb{R}$ instead of $[0, \infty)$.

We will now list some standard results about semigroups, which will be used in the following chapters. We start with the *weak semigroup theorem.*

Theorem 0.2.1. *A weakly continuous semigroup is strongly continuous.*

Proof: Put $X_0 := \{x \in X : \lim_{t\downarrow 0} \|T(t)x - x\| = 0\}$. We must show that $X_0 = X$. By applying the uniform boundedness theorem twice we see that $\|T(t)\|$ is bounded in a neighbourhood of $t = 0$ and hence by a standard $\epsilon/3$-argument X_0 is a closed subspace. Next, because of its weak right continuity, the range of each map $t \mapsto T(t)x$ is weakly separable, hence separable by a corollary of the Hahn-Banach theorem. Since these maps are weakly measurable, they are strongly measurable by Pettis's measurability theorem (Theorem A.4). In particular the Bochner integrals

$$y(t,x) := \frac{1}{t}\int_0^t T(\sigma)x \, d\sigma$$

exist for all $x \in X$ and $t > 0$. Fix $x \in X$ and $0 < t < 1$. If $0 < s < t$ then

$$\begin{aligned} \|T(s)y(t,x) - y(t,x)\| &= \frac{1}{t}\left\| \int_0^t T(s+\sigma)x \, d\sigma - \int_0^t T(\sigma)x \, d\sigma \right\| \\ &= \frac{1}{t}\left\| \int_t^{t+s} T(\sigma)x \, d\sigma - \int_0^s T(\sigma)x \, d\sigma \right\| \\ &\leq 2s \cdot \frac{1}{t}\Big(\sup_{0\leq\sigma\leq 2} \|T(\sigma)\| \Big) \, \|x\| \end{aligned}$$

shows that $y(t,x) \in X_0$. But then

$$\lim_{t\downarrow 0} \langle x^*, y(t,x) \rangle = \lim_{t\downarrow 0} \frac{1}{t}\int_0^t \langle x^*, T(\sigma)x \rangle \, d\sigma = \langle x^*, x \rangle$$

shows that x lies in the weak closure of X_0, hence in X_0. ////

A semigroup is said to be *strongly measurable* if for every $x \in X$ and $\tau > 0$ the map $t \mapsto T(t)x$ is strongly (Lebesgue) measurable on $[0,\tau]$ (cf. the Appendix). A semigroup is said to be *strongly continuous for* $t > 0$ (briefly, $T(t)$ *is* $C_{>0}$) if $\lim_{t\downarrow 0} \|T(s+t)x - T(s)x\| = 0$ holds for all $s > 0$ and $x \in X$. The following result is called the *measurable semigroup theorem.*

Theorem 0.2.2. *A strongly measurable semigroup is* $C_{>0}$.

Proof: Step 1. First we prove that $\|T(t)\|$ is bounded on each interval $[\alpha,\beta]$ with $\alpha > 0$. Suppose the contrary. By the uniform boundedness theorem there is a sequence $(\xi_n) \subset [\alpha,\beta]$ and an $x \in X$ such that $\xi_n \to \xi$ and $\|T(\xi_n)x\| > n$. On the other hand, because $t \mapsto \|T(t)x\|$ is measurable, there exists a constant M and a measurable subset $F \subset [0,\xi]$ of measure $m(F) > \xi/2$ such that $\|T(t)x\| < M$ on F. For each n the set $E_n := \{(\xi_n - t) : t \in F \cap [0,\xi_n]\}$ is measurable, and for n large enough $m(E_n) > \xi/2$. For $t \in F \cap [0,\xi_n]$ we have

$$n \leq \|T(\xi_n)x\| \leq \|T(\xi_n - t)\| \, \|T(t)x\| \leq M\|T(\xi_n - t)\|$$

and therefore $\|T(\eta)\| \geq n/M$ for all $\eta \in E_n$. Let $E := \cap_n \cup_{k \geq n} E_k$. Then $m(E) \geq \xi/2$ and for $\eta \in E$ it follows that $\|T(\eta)\| = \infty$, a contradiction (but see [HNS]).

Step 2. Fix $x \in X$ and $\xi > 0$ and choose numbers $0 < \alpha < \beta < \xi$. Now $T(\xi)x = T(\tau)T(\xi - \tau)x$ is independent of τ for all $\alpha < \tau < \beta$, hence certainly integrable on $[\alpha,\beta]$ with respect to τ. Therefore,

$$(\beta - \alpha)\,\big(T(\xi+\eta) - T(\xi)\big)x \;=\; \int_\alpha^\beta T(\tau)\big(T(\xi+\eta-\tau) - T(\xi-\tau)\big)x\,d\tau.$$

If M is such that $\|T(t)\| \leq M$ on $[\alpha,\beta]$, then the norm of the integrand does not exceed $M\|\big(T(\xi+\eta-\tau) - T(\xi-\tau)\big)x\|$, which is a measurable function of τ on $[\alpha,\beta]$. For $\eta \to 0$ this gives

$$(\beta - \alpha)\,\|\big(T(\xi+\eta) - T(\xi)\big)x\| \;\leq\; M \int_{\xi-\beta}^{\xi-\alpha} \|\big(T(\sigma+\eta) - T(\sigma)\big)x\|\,d\sigma \;\to 0,$$

the convergence being a direct application of [HPh, Thm. 3.8.3]. ////

From now on let $T(t)$ be a C_0-semigroup on X. By the uniform boundedness theorem, $T(t)$ is *locally bounded,* that is, $\|T(t)\|$ is bounded in some neighbourhood of 0. The semigroup property then easily implies that there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq Me^{\omega t}. \tag{0.1}$$

$T(t)$ is called a *contraction semigroup* if $\|T(t)\| \leq 1$ for all $t \geq 0$. If $T(t)$ is a *bounded* semigroup, i.e. if $\|T(t)\| \leq M$ for some M and all t, then

$$|x| := \sup_{t \geq 0} \|T(t)x\|$$

defines an equivalent norm on X with respect to which $T(t)$ is a contraction semigroup.

In (S3) we imposed right continuity on the maps $t \mapsto T(t)x$; but (0.1) easily implies that these maps are then automatically continuous on $[0, \infty)$. Similarly, a semigroup is $C_{>0}$ if and only if $t \to T(t)x$ is continuous on $(0, \infty)$ for all $x \in X$.

The *infinitesimal generator* (briefly, the *generator*) of a C_0-semigroup $T(t)$ is the linear operator A with domain $D(A)$ defined by

$$D(A) := \{x \in X : \lim_{t \downarrow 0} \frac{1}{t}\big(T(t)x - x\big) \text{ exists}\};$$

$$Ax := \lim_{t \downarrow 0} \frac{1}{t}\big(T(t)x - x\big)$$

The generator A is a densely defined closed linear operator which uniquely determines $T(t)$. For any $x \in X$ and $t > 0$ we have $\int_0^t T(\sigma)x \, d\sigma \in D(A)$ and

$$A \int_0^t T(\sigma)x \, d\sigma = T(t)x - x. \tag{0.2}$$

If $x \in D(A)$ we have $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$; moreover

$$A \int_0^t T(\sigma)x \, d\sigma = \int_0^t T(\sigma)Ax \, d\sigma. \tag{0.3}$$

A C_0-semigroup $T(t)$ is said to be *uniformly continuous* if

$$\lim_{t \downarrow 0} ||T(t) - I|| = 0.$$

A C_0-semigroup is uniformly continuous if and only if its generator is bounded. The following theorem characterizes generators of contraction semigroups.

Theorem 0.2.3. *(Hille-Yosida) A linear operator $(A, D(A))$ on X generates a contraction semigroup if and only if*

(i) A is densely defined and closed, and

(ii) $(0, \infty) \subset \varrho(A)$, and for every $\lambda > 0$

$$||R(\lambda, A)|| \leq \frac{1}{\lambda}.$$

Here $R(\lambda, A)$ is the resolvent of A, cf. Chapter 1. In the complex case, conditions (i) and (ii) automatically imply that the right-half plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ is contained in $\varrho(A)$. More generally, A generates a C_0-semigroup if and only if (i) and (ii)$'$ hold, with

(ii)$'$: $\varrho(A)$ contains (ω, ∞) for some ω, and there is an $M \geq 1$ such that for all $n = 1, 2, ..$ and $\lambda > \omega$ we have

$$||R(\lambda, A)^n|| \leq \frac{M}{(\lambda - \omega)^n}.$$

These results imply that $||\lambda R(\lambda, A)||$ is uniformly bounded for $\lambda \geq \lambda_0$ if $\lambda_0 > \omega$. It follows easily that

$$\lim_{\lambda \to \infty} \lambda R(\lambda, A)x = x, \quad x \in X. \tag{0.4}$$

The resolvent $R(\lambda, A)$ can be constructed from $T(t)$ by means of the *Laplace transform*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(\sigma)x \, d\sigma, \quad x \in X, \ \lambda > \omega. \tag{0.5}$$

Conversely, $T(t)$ can be recovered from $R(\lambda, A)$ by the *exponential formula*

$$T(t)x = \lim_{n \to \infty} \Big(\frac{n}{t} R(\frac{n}{t}, A)\Big)^n x, \quad x \in X. \tag{0.6}$$

Notes. The material on Banach spaces is classical and can be found in many textbooks on functional analysis. Especially Rudin [Ru3] is a beautiful introduction and contains most of the quoted results. More extensive treatments are given e.g. in [HPh], [DS], [Yo].

Also the results in Section 0.2 are classical. Some of the standard references are [HPh], [BB], [P], [Go], [Da2], [vC].

We defined a semigroup to be τ-continuous if it is τ-right continuous in the origin. If τ is the strong topology, then this already implies (two-sided) continuity of the maps $t \mapsto T(t)x$ at every $t \geq 0$, so one could also take this as a definition.

Theorem 0.2.1 is proved in [HPh, Thm. 10.6.5]. A similar proof is given in [Yo, Thm. IX.1]. Both proofs depend on Theorem 0.2.2. Our proof of Theorem 0.2.1, which seems to be new, is a simplification of the one indicated in [Go] and carries over to certain locally convex spaces.

Theorem 0.2.2 is the work of several mathematicians. We refer to [HPh], from which the above proof is taken, for a discussion of its history. We have included the proofs for reasons of self-containedness.

Chapter 1

The adjoint semigroup

If $T(t)$ is a C_0-semigroup on a Banach space X, then elementary examples show that the adjoint semigroup $T^*(t)$ need not be a C_0-semigroup. This gives rise to the basic problem of adjoint semigroup theory: what can one say about the strong continuity of the adjoint of a C_0-semigroup? The study of this problem is the subject matter of this thesis.

Although we will be primarily concerned with the adjoint theory of C_0-semigroups, in some cases we have to consider semigroups which are not necessarily strongly continuous. In order to avoid constant repetition of the phrase 'Let $T(t)$ be a C_0-semigroup on a Banach space X' in almost every result, throughout this thesis we adopt the following

Convention. *The symbol $T(t)$ will always denote a C_0-semigroup with generator A on a Banach space X. Whenever we are dealing with semigroups on X which are not assumed to be C_0, the notation $S(t)$ will be used.*

In this chapter the basic concepts of adjoint semigroup theory are introduced. In Section 1.1 we recall some results on unbounded linear operators. In Section 1.2 we study the adjoint of a C_0-semigroup $T(t)$. The main result is that it is weak*-generated by the adjoint of the generator of $T(t)$. In Section 1.3 the semigroup dual space is defined and its most important properties are derived. In Section 1.4 we study the spectrum of adjoint semigroups. Finally, in Section 1.5 we compute the semigroup dual of a class of semigroups modelled on Schauder bases. Such semigroups will be used later to construct various (counter)examples.

1.1. Unbounded linear operators

Let X be a Banach space. A *linear operator on* X is a pair $(A, D(A))$, where $D(A)$ is a linear subspace of X and $A : D(A) \to X$ is a linear map. Usually we will identify $(A, D(A))$ with the map A if it is clear that A is defined on $D(A)$ only. The space $D(A)$ is called the *domain* of A.

A linear operator A is said to be *closed* if the *graph*

$$G(A) := \{(x, Ax) \in X \times X : \quad x \in D(A)\}$$

of A is closed in $X \times X$ with respect to the product topology. The operator A is *densely defined* if $D(A)$ is dense.

We will associate with a densely defined linear operator A on X a linear operator A^* on X^*, called its *adjoint*, in the following way. Define $D(A^*)$ to be the set of all $x^* \in X^*$ with the property that there is a $y^* \in X^*$ such that

$$\langle y^*, x\rangle = \langle x^*, Ax\rangle, \qquad \forall x \in D(A).$$

Since $D(A)$ is assumed to be dense, y^*, if it exists, is unique and we define

$$A^* x^* := y^*.$$

Define $R : X \times X \to X \times X$ by $R(x, y) = (-y, x)$.

Proposition 1.1.1. *If A is a densely defined linear operator on X, then A^* is a weak*-closed operator.*

Proof: Define a pairing between $X^* \times X^*$ and $X \times X$ by putting

$$\langle (x^*, y^*), (x, y)\rangle := \langle x^*, x\rangle + \langle y^*, y\rangle.$$

By means of this pairing we can identify $X^* \times X^*$ with the dual $(X \times X)^*$. By definition of A^* we have $(x^*, y^*) \in G(A^*)$ if and only if

$$\langle (x^*, y^*), (-Ax, x)\rangle = 0, \quad \forall x \in D(A).$$

In other words, $G(A^*)$ is the annihilator of $R(G(A))$. Since annihilators of linear subspaces are weak*-closed, the result follows. ////

Note that in particular A^* is (norm) closed.

Proposition 1.1.2. *If A is a closed densely defined linear operator on X, then A^* is weak*-densely defined.*

Proof: $(X^*, weak^*)$ is a locally convex topological vector space whose dual is X. Hence if A^* is not weak*-densely defined, then by the Hahn-Banach theorem there is a non-zero $x \in X$ annihilating $D(A^*)$. Since $G(A)$ (and hence $RG(A)$) is closed in $X \times X$ and $(0, x) \notin G(A)$, by the Hahn-Banach theorem there is an $(x^*, y^*) \in X^* \times X^*$ annihilating $RG(A)$ and non-zero on $R(0, x) = (-x, 0)$. In other words,

$$\langle y^*, x\rangle = \langle x^*, Ax\rangle, \quad \forall x \in D(A),$$

and

$$\langle x^*, x\rangle \neq 0.$$

But the first equality implies that $x^* \in D(A^*)$, so the second one implies that x does not annihilate $D(A^*)$, a contradiction. ////

1.2. The adjoint semigroup

Let $S(t)$ be a semigroup on a Banach space X. The *adjoint semigroup* $S^*(t)$ is the semigroup on the dual space X^* which is obtained from $S(t)$ by taking pointwise in t the adjoint operators $S^*(t) := (S(t))^*$. It is elementary to see that $S(t)$ is a semigroup again. If $T(t)$ is a C_0-semigroup, then

$$|\langle T^*(t)x^* - x^*, x\rangle| = |\langle x^*, T(t)x - x\rangle| \leq ||x^*||\ ||T(t)x - x||$$

shows that $T^*(t)$ is weak*-continuous. But $T^*(t)$ need not be strongly continuous, as is shown by several examples at the end of Section 1.3.

Recall the convention that $T(t)$ always denotes a C_0-semigroup with generator A. Since A is closed and densely defined, the adjoint A^* is a weak*-densely defined, weak*-closed operator.

Proposition 1.2.1. *$D(A^*)$ is a $T^*(t)$-invariant subspace of X^*, and for all $x^* \in D(A^*)$ we have $A^*T^*(t)x^* = T^*(t)A^*x^*$.*

Proof: Let $x^* \in D(A^*)$ and $x \in D(A)$ be arbitrary. Then for any fixed $t \geq 0$ we have

$$\begin{aligned}\langle T^*(t)x^*, Ax\rangle &= \langle x^*, T(t)Ax\rangle = \langle x^*, AT(t)x\rangle \\ &= \langle A^*x^*, T(t)x\rangle = \langle T^*(t)A^*x^*, x\rangle.\end{aligned}$$

Therefore $T^*(t)x^* \in D(A^*)$ and $A^*T^*(t)x^* = T^*(t)A^*x^*$. ////

In the next lemma we use the concept of the *weak*-integral*. This integral, as well as some other types of integrals, is discussed in the Appendix.

Proposition 1.2.2. *weak** $\int_0^t T^*(\sigma)x^*\, d\sigma \in D(A^*)$ *for all* $t > 0$ *and* $x^* \in X^*$, *and*

$$A^*\Big(weak^* \int_0^t T^*(\sigma)x^*\, d\sigma\Big) = T^*(t)x^* - x^*.$$

If $x^* \in D(A^*)$, *then*

$$A^*\Big(weak^* \int_0^t T^*(\sigma)x^*\, d\sigma\Big) = weak^* \int_0^t T^*(\sigma)A^*x^*\, d\sigma.$$

Proof: Let $x \in D(A)$ be arbitrary. Using (0.2) and (0.3), the identities

$$\begin{aligned}\langle weak^* \int_0^t T^*(\sigma)x^*\, d\sigma, Ax\rangle &= \int_0^t \langle T^*(\sigma)x^*, Ax\rangle\, d\sigma = \int_0^t \langle x^*, T(\sigma)Ax\rangle\, d\sigma \\ &= \langle x^*, \int_0^t T(\sigma)Ax\, d\sigma\rangle = \langle x^*, A\int_0^t T(\sigma)x\, d\sigma\rangle \\ &= \langle x^*, T(t)x - x\rangle = \langle T^*(t)x^* - x^*, x\rangle\end{aligned}$$

show that $weak^* \int_0^t T^*(\sigma)x^* \, d\sigma \in D(A^*)$ and

$$A^*\Big(weak^* \int_0^t T^*(\sigma)x^* \, d\sigma\Big) = T^*(t)x^* - x^*.$$

The second formula follows from a similar calculation, using Proposition 1.2.1:

$$\int_0^t \langle T^*(\sigma)x^*, Ax\rangle \, d\sigma = \int_0^t \langle A^*T^*(\sigma)x^*, x\rangle \, d\sigma = \int_0^t \langle T^*(\sigma)A^*x^*, x\rangle \, d\sigma$$
$$= \langle weak^* \int_0^t T^*(\sigma)A^*x^* \, d\sigma, x\rangle.$$

////

Let $S(t)$ be a weak*-continuous semigroup on X^*. The *weak*-generator* of $S(t)$ is the linear operator B on X^* defined by

$$D(B) := \{x^* \in X^* : \ \ weak^*\text{-}\lim_{t\downarrow 0} \frac{1}{t}\big(S(t)x^* - x^*\big) \text{ exists } \};$$
$$Bx^* := weak^*\text{-}\lim_{t\downarrow 0} \frac{1}{t}\big(S(t)x^* - x^*\big), \quad x^* \in D(B).$$

In general it is not true that the weak*-generator of $T^*(t)$ uniquely determines $T^*(t)$ in the class of all weak*-continuous semigroups on X^*, cf. the notes at the end of this chapter. However, $T(t)$ is the unique C_0-semigroup on X whose adjoint is weak*-generated by A^*; this follows from Theorems 1.3.1, 1.3.3 and Corollary 1.3.7 below.

Theorem 1.2.3. *A^* is the weak*-generator of $T^*(t)$.*

Proof: Let B be the weak*-generator of $T^*(t)$ and fix $x^* \in D(A^*)$. For $x \in X$ arbitrary we have

$$\lim_{t\downarrow 0} \frac{1}{t}\langle T^*(t)x^* - x^*, x\rangle = \lim_{t\downarrow 0} \frac{1}{t}\langle A^*\Big(weak^* \int_0^t T^*(\sigma)x^* \, d\sigma\Big), x\rangle$$
$$= \lim_{t\downarrow 0} \frac{1}{t} \int_0^t \langle T^*(\sigma)A^*x^*, x\rangle \, d\sigma = \langle A^*x^*, x\rangle.$$

Hence weak*-$\lim_{t\downarrow 0} \frac{1}{t}(T^*(t)x^* - x^*)$ exists and equals A^*x^*. This shows that $x^* \in D(B)$ and $Bx^* = A^*x^*$, and therefore $A^* \subset B$. To prove the converse inclusion, fix $x^* \in D(B)$. Then for any $x \in D(A)$,

$$\langle Bx^*, x\rangle = \lim_{t\downarrow 0} \frac{1}{t}\langle T^*(t)x^* - x^*, x\rangle = \langle x^*, Ax\rangle.$$

This shows that $x^* \in D(A^*)$ and $A^*x^* = Bx^*$, proving that $B \subset A^*$. ////

1.3. The semigroup dual space

Let $S(t)$ be a semigroup on X. The *semigroup dual of X with respect to $S(t)$*, notation $X^{\odot}$ (pronounciation: X-sun), is defined as the linear subspace of X^* on which $S^*(t)$ acts in a strongly continuous way:

$$X^{\odot} := \{x^* \in X^* : \lim_{t \downarrow 0} \|S^*(t)x^* - x^*\| = 0\}.$$

It follows trivially from this definition that $X^{\odot}$ is $S^*(t)$-*invariant*, which by definition means that $S^*(t)X^{\odot} \subset X^{\odot}$ holds for all $t \geq 0$. It is easy to see that if $S(t)$ is locally bounded, then $X^{\odot}$ is a *closed* subspace of X^*. In particular this is the case for a C_0-semigroup.

Theorem 1.3.1. *Let $T(t)$ be C_0. Then $X^{\odot}$ is a closed, weak*-dense, $T^*(t)$-invariant linear subspace of X^*. Moreover $X^{\odot} = \overline{D(A^*)}$.*

Proof: We have already seen that $X^{\odot}$ is closed and $T^*(t)$-invariant. Weak*-denseness of $X^{\odot}$ follows from the weak*-denseness of $D(A^*)$ and $X^{\odot} = \overline{D(A^*)}$, which will be proved now.

Let $x^* \in D(A^*)$. Then for any $x \in X$ we have

$$\begin{aligned} |\langle T^*(t)x^* - x^*, x\rangle| &= |\langle A^*\Big(weak^* \int_0^t T^*(\sigma)x^*\, d\sigma\Big), x\rangle| \\ &= |\int_0^t \langle T^*(\sigma)A^*x^*, x\rangle\, d\sigma| \leq t \cdot \Big(\sup_{0 \leq \sigma \leq t} \|T(\sigma)\|\Big) \|A^*x^*\|\, \|x\|. \end{aligned}$$

Hence

$$\|T^*(t)x^* - x^*\| \leq t \cdot \Big(\sup_{0 \leq \sigma \leq t} \|T(\sigma)\|\Big) \|A^*x^*\|$$

which shows that $D(A^*) \subset X^{\odot}$. Since $X^{\odot}$ is closed, also the norm closure $\overline{D(A^*)}$ belongs to $X^{\odot}$.

For the converse inclusion let $x^{\odot} \in X^{\odot}$. Then for any $x \in X$ we have

$$\begin{aligned} |\langle \frac{1}{t}\int_0^t T^*(\sigma)x^{\odot}\, d\sigma - x^{\odot}, x\rangle| &= |\frac{1}{t}\int_0^t \langle T^*(\sigma)x^{\odot} - x^{\odot}, x\rangle\, d\sigma| \\ &\leq \Big(\sup_{0 \leq \sigma \leq t} \|T^*(\sigma)x^{\odot} - x^{\odot}\|\Big) \|x\|. \end{aligned}$$

Hence

$$\|\frac{1}{t}\int_0^t T^*(\sigma)x^{\odot}\, d\sigma - x^{\odot}\| \leq \sup_{0 \leq \sigma \leq t} \|T^*(\sigma)x^{\odot} - x^{\odot}\| \to 0 \quad \text{as } t \downarrow 0$$

since $x^{\odot} \in X^{\odot}$. But $\frac{1}{t}\int_0^t T^*(\sigma)x^{\odot}\, d\sigma \in D(A^*)$, and thus we have shown that $x^{\odot} \in \overline{D(A^*)}$. ////

If X is reflexive, then by Theorem 1.3.1 the subspace $X^{\odot}$ is weakly dense, hence norm dense by the Hahn-Banach theorem. Since $X^{\odot}$ is also closed we obtain:

Corollary 1.3.2. *If X is reflexive, then $T^*(t)$ is strongly continuous.*

This corollary shows that adjoint semigroup theory reduces to a triviality in reflexive Banach spaces.

Let $T^{\odot}(t)$ denote the restriction of $T^*(t)$ to the $T^*(t)$-invariant subspace $X^{\odot}$. Since $X^{\odot}$ is closed, $X^{\odot}$ is a Banach space and it is clear from the definition of $X^{\odot}$ that $T^{\odot}(t)$ is a strongly continuous semigroup on $X^{\odot}$. We will call $T^{\odot}(t)$ the *strongly continuous adjoint* of $T(t)$. Let its generator be $A^{\odot}$. The following theorem gives a precise description of $A^{\odot}$ in terms of A^*.

If $(B, D(B))$ is a linear operator on a Banach space Y and Z is a linear subspace of Y containing $D(B)$, then the *part of B in Z* is the operator B_Z defined by

$$D(B_Z) := \{y \in D(B) : \; By \in Z\};$$
$$B_Z\, y := By, \quad y \in D(B_Z).$$

Theorem 1.3.3. *$A^{\odot}$ is the part of A^* in $X^{\odot}$.*

Proof: Let B be the part of A^* in $X^{\odot}$. If $x^{\odot} \in D(A^{\odot})$, then

$$\lim_{t\downarrow 0} \frac{1}{t}\big(T^*(t)x^{\odot} - x^{\odot}\big) = \lim_{t\downarrow 0} \frac{1}{t}\big(T^{\odot}(t)x^{\odot} - x^{\odot}\big) = A^{\odot}x^{\odot},$$

where the limits are in the strong sense. Hence these limits exist also in the weak*-sense, so by Theorem 1.2.3 it follows that $x^{\odot} \in D(A^*)$ and $A^*x^{\odot} = A^{\odot}x^{\odot} \in X^{\odot}$. This proves that $A^{\odot} \subset B$.

To prove the converse inclusion, let $x^* \in D(B)$. This means that $x^* \in D(A^*)$ and $A^*x^* \in X^{\odot}$. But this implies that

$$\begin{aligned}\frac{1}{t}\big(T^{\odot}(t)x^* - x^*\big) &= \frac{1}{t}\big(T^*(t)x^* - x^*\big) = \frac{1}{t}A^*\Big(weak^* \int_0^t T^*(\sigma)x^*\, d\sigma\Big)\\ &= \frac{1}{t} weak^* \int_0^t T^*(\sigma)A^*x^*\, d\sigma = \frac{1}{t}\int_0^t T^*(\sigma)A^*x^*\, d\sigma.\end{aligned}$$

The integrand of the last integral being continuous since $A^*x^* \in X^{\odot}$, letting $t \downarrow 0$ gives

$$\lim_{t\downarrow 0} \frac{1}{t}\big(T^{\odot}(t)x^* - x^*\big) = A^*x^*.$$

This shows that $x^* \in D(A^{\odot})$ and $A^{\odot}x^* = A^*x^*$, that is, $B \subset A^{\odot}$. ////

Corollary 1.3.4. *A^* is the weak*-closure of $A^{\odot}$.*

Proof: Since A^* is a weak*-closed operator it suffices to prove that the graph of $A^\odot$ is weak*-dense in the graph of A^*. Let $x^* \in D(A^*)$. Since $D(A^*) \subset X^\odot$ we have $\frac{1}{t}\int_0^t T^*(\sigma)x^*\, d\sigma \in D(A^\odot)$ and $\lim_{t\downarrow 0} \frac{1}{t}\int_0^t T^*(\sigma)x^*\, d\sigma = x^*$. Moreover, taking the weak*-limit for $t \downarrow 0$ in

$$A^*\Big(\frac{1}{t}\int_0^t T^*(\sigma)x^*\, d\sigma\Big) = \frac{1}{t}(T^*(t)x^* - x^*),$$

from Theorem 1.2.3 it follows that

$$weak^*\text{-}\lim_{t\downarrow 0} A^*\Big(\frac{1}{t}\int_0^t T^*(\sigma)x^*\, d\sigma\Big) = A^*x^*.$$

////

Starting from the C_0-semigroup $T^\odot(t)$, the duality construction can be repeated. We define $T^{\odot *}(t)$ to be the adjoint of $T^\odot(t)$ and write $X^{\odot\odot}$ for $(X^\odot)^\odot$. Pronunciation X-sun-sun, or sometimes: X-*bosom.* $T^{\odot\odot}(t)$ and $A^{\odot\odot}$ are defined analogously. In order to relate $T(t)$ and $T^{\odot\odot}(t)$, we will now show that X can be identified with a closed subspace of $X^{\odot\odot}$. To this end, define the norm $\|\cdot\|'$ on X by

$$\|x\|' := \sup_{x^\odot \in B_{X^\odot}} |\langle x^\odot, x\rangle|,$$

where $B_{X^\odot}$ is the closed unit ball of $X^\odot$. Note that $\|x\|' \le \|x\|$ for all $x \in X$.

Theorem 1.3.5. $\|\cdot\|'$ *is an equivalent norm.*

Proof: Fix $\epsilon > 0$ and $x \in X$ arbitrary. Choose M such that $\|T(t)\| \le M$ for all t in some neighbourhood $[0, \delta)$ of 0. Choose $x^* \in B_{X^*}$ such that $|\langle x^*, x\rangle| > (1-\epsilon)\|x\|$. Choose $0 < t < \delta$ so small that $\|\frac{1}{t}\int_0^t T(\sigma)x\, d\sigma - x\| < \epsilon\|x\|$. Then

$$\begin{aligned}|\langle \frac{1}{t} weak^* \int_0^t T^*(\sigma)x^*\, d\sigma, x\rangle| &= |\langle x^*, \frac{1}{t}\int_0^t T(\sigma)x\, d\sigma\rangle| \\ &\ge |\langle x^*, x\rangle| - \epsilon\|x\| \ge (1-2\epsilon)\|x\|.\end{aligned}$$

Since $weak^* \int_0^t T^*(\sigma)x^*\, d\sigma \in X^\odot$ and $\|\frac{1}{t} weak^* \int_0^t T^*(\sigma)x^*\, d\sigma\| \le M$ it follows that $\|x\|' \ge M^{-1}(1-2\epsilon)\, \|x\|$. Since ϵ is arbitrary it follows that $\|x\|' \ge M^{-1}\|x\|$. ////

Note that we have actually shown a little bit more, viz. $\|\cdot\|' \le \|\cdot\| \le M\|\cdot\|'$, with

$$M = \limsup_{t\downarrow 0} \|T(t)\|.$$

Define a map $j : X \to X^{\odot *}$ by $\langle jx, x^\odot\rangle := \langle x^\odot, x\rangle$. Clearly $\|j\| \le 1$ and $j(X) \subset X^{\odot\odot}$. If $j(X) = X^{\odot\odot}$ then X is said to be $\odot$-*reflexive with respect to* $T(t)$.

Corollary 1.3.6. *j is an embedding, and $M^{-1} \le \|j\| \le 1$.*

Thus we can identify X isomorphically with the closed subspace jX of $X^{\odot\odot}$. One has to be careful here, since in general this isomorphism is not *isometric*. A counterexample is given in Section 2.3. The map j will be referred to as the *natural embedding of X into $X^{\odot *}$*. The following corollary says that $T^{\odot\odot}(t)$ and $A^{\odot\odot}$ can be regarded as extensions of $T(t)$ and A respectively.

Corollary 1.3.7. *$T^{\odot\odot}(t)$ is an extension of $jT(t)$ and $A^{\odot\odot}$ is an extension of jA. Moreover, $jD(A) = D(A^{\odot\odot}) \cap jX$.*

Proof: For $x \in X$ and $x^{\odot} \in X^{\odot}$ we have

$$\langle T^{\odot\odot}(t)jx, x^{\odot}\rangle = \langle jx, T^{\odot}(t)x^{\odot}\rangle = \langle T^{\odot}(t)x^{\odot}, x\rangle = \langle x^{\odot}, T(t)x\rangle,$$

so $T^{\odot\odot}(t)jx = jT(t)x$. That $A^{\odot\odot}j$ extends jA is proved similarly. In particular $jD(A) \subset D(A^{\odot\odot}) \cap jX$. If $jx \in D(A^{\odot\odot}) \cap jX$, then

$$\begin{aligned} j\lim_{t\downarrow 0}\frac{1}{t}(T(t)x - x) &= \lim_{t\downarrow 0}\frac{1}{t}(jT(t)x - jx) \\ &= \lim_{t\downarrow 0}\frac{1}{t}(T^{\odot\odot}(t)jx - jx) = A^{\odot\odot}jx \end{aligned} \tag{1.1}$$

shows that the left hand limit exists as an element of $X^{\odot\odot}$. Since jX is closed in $X^{\odot\odot}$ the limit belongs to jX. Applying j^{-1} to (1.1) shows that $x \in D(A)$. ////

We close this section with some simple examples.

Example 1.3.8. Let $T(t)$ be a uniformly continuous semigroup. From $\|T^*(t) - I\| = \|T(t) - I\|$ it is clear that also $T^*(t)$ is uniformly continuous, so in particular $T^*(t)$ is strongly continuous.

Example 1.3.9. Let $X = C_0(\mathbb{R})$, the Banach space of continuous functions on $\mathbb{R}$ vanishing at infinity with the sup-norm. The formula

$$T(t)f(y) := f(y+t)$$

defines a C_0-group on $C_0(\mathbb{R})$, called the *translation group*. In Chapter 7 it is shown (in much more generality) that $C_0(\mathbb{R})^{\odot} = L^1(\mathbb{R})$, where, by the Radon-Nikodym theorem, we identify absolutely continuous measures in $C_0(\mathbb{R})^*$ with their density functions. Moreover, $C_0(\mathbb{R})^{\odot\odot} = BUC(\mathbb{R})$, the Banach space of bounded uniformly continuous functions on $\mathbb{R}$ with the sup-norm.

Similarly one defines the *rotation group* $T(t)$ on $C(T)$, T the unit circle, by

$$T(t)f(e^{i\theta}) = f(e^{i(\theta+t)}).$$

Then $C(T)^{\odot} = L^1(T)$ and $C(T)^{\odot\odot} = C(T)$.

Example 1.3.10. Let $X = c_0$ or l^p, $1 \leq p < \infty$. Define $T(t)$ by

$$T(t)x_n := e^{-nt}x_n,$$

where x_n is the nth unit vector $(0, 0, .., 0, 1, 0, ..)$. This is a C_0-semigroup on X and we have $c_0^{\odot} = l^1$, $(l^1)^{\odot} = c_0$ and $(l^p)^{\odot} = l^q$ for $1 < p < \infty$, where $p^{-1} + q^{-1} = 1$.

1.4. The spectrum of $A^{\odot}$

For a linear operator $(A, D(A))$ on a Banach space X, define

$$\varrho(A) := \{\lambda : \text{ the inverse } (\lambda - A)^{-1} \text{ exists on } X \text{ and is bounded}\},$$

where λ ranges over the scalar field. The set $\varrho(A)$ is called the *resolvent set* of A and its complement $\sigma(A)$ the *spectrum*. If A is not closed, then $\varrho(A) = \emptyset$. Indeed, suppose $\lambda \in \varrho(A)$. Then $(\lambda - A)^{-1}$ is a bounded linear operator whose inverse $\lambda - A$ is easily seen to be closed. Hence A itself must be closed.

For $\lambda \in \varrho(A)$ we write $R(\lambda, A) := (\lambda - A)^{-1}$. The bounded linear operator $R(\lambda, A)$ is called the *resolvent* of A. We have the so-called *resolvent identity*: if $\lambda, \mu \in \varrho(A)$, then

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A). \tag{1.2}$$

Lemma 1.4.1. *If A is a densely defined closed operator on a Banach space X, then $\varrho(A) = \varrho(A^*)$ and for $\lambda \in \varrho(A)$ we have $R(\lambda, A)^* = R(\lambda, A^*)$.*

Proof: Suppose $\lambda \in \varrho(A)$. We will show that $\lambda \in \varrho(A^*)$. For any $x \in X$ and $x^* \in D(A^*)$ we have

$$\langle R(\lambda, A)^*(\lambda - A^*)x^*, x\rangle = \langle x^*, x\rangle$$

and consequently $R(\lambda, A^*)(\lambda - A^*)x^* = x^*$. From the definition of A^* it is easy to see that $R(\lambda, A)^*x^* \in D(A^*)$ for all $x^* \in X^*$, and for all $x \in D(A)$ we have

$$\langle(\lambda - A^*)R(\lambda, A)^*x^*, x\rangle = \langle x^*, x\rangle.$$

Since $D(A)$ is dense it follows that $(\lambda - A)R(\lambda, A)^*x^* = x^*$. We have shown that $R(\lambda, A)^*$ is a two sided inverse of $\lambda - A^*$, in other words $\lambda \in \varrho(A^*)$ and $R(\lambda, A^*) = R(\lambda, A)^*$.

Conversely, let $\lambda \in \varrho(A^*)$. We will show that $\lambda \in \varrho(A)$. Injectivity of $\lambda - A$ is proved as we did above for $\lambda - A^*$. We prove that the range of $\lambda - A$ is dense and closed. If the range were not dense, then there is a non-zero $x^* \in X^*$ such that $\langle x^*, (\lambda - A)x\rangle = 0$ for all $x \in D(A)$. Then $x^* \in D(A^*)$ and $(\lambda - A^*)x^* = 0$.

From $\lambda \in \varrho(A^*)$ it follows that $x^* = 0$, a contradiction. This proves denseness. To prove closedness, let $x \in D(A)$ be arbitrary and choose $x^* \in B_{X^*}$ such that $|\langle x^*, x\rangle| \geq \frac{1}{2}||x||$. Let $K := ||R(\lambda, A^*)||^{-1}$. Then

$$\begin{aligned} ||(\lambda - A)x|| &\geq K|\langle R(\lambda, A^*)x^*, (\lambda - A)x\rangle| \\ &= K|\langle(\lambda - A^*)R(\lambda, A^*)x^*, x\rangle| \geq \frac{K}{2}||x||. \end{aligned} \tag{1.3}$$

Now if (x_n) is a sequence such that $\lim_{n\to\infty}(\lambda - A)x_n = y$, then (1.3) implies that (x_n) is a Cauchy sequence, say with limit z. Since A is closed we have $z \in D(A)$ and $y = (\lambda - A)z$. ////

Let $R(\lambda, A)^\odot$ denote the restriction of $R(\lambda, A)^*$ to the $R(\lambda, A)^*$-invariant subspace $X^\odot$.

Theorem 1.4.2. *If A is the generator of a C_0-semigroup on X, then $\varrho(A) = \varrho(A^*) = \varrho(A^\odot)$ and $R(\lambda, A)^\odot = R(\lambda, A^\odot)$ for all $\lambda \in \varrho(A)$.*

Proof: The identity $\varrho(A) = \varrho(A^*)$ was proved in Lemma 1.4.1. Let $\lambda \in \varrho(A)$. As in the proof of 1.4.1 and by using Theorem 1.3.3 we have $R(\lambda, A)^\odot(\lambda - A^\odot)x^\odot = x^\odot$ for all $x^\odot \in D(A^\odot)$ and $(\lambda - A^\odot)R(\lambda, A)^\odot x^\odot = x^\odot$ for all $x^\odot \in X^\odot$. Hence $\lambda \in \varrho(A^\odot$ and $R(\lambda, A^\odot) = R(\lambda, A)^\odot$.

Conversely, let $\lambda \in \varrho(A^\odot)$. If $(\lambda - A)x = 0$ for some $x \in D(A)$, then for all $x^* \in D(A^*)$ we have

$$\langle(\lambda - A^*)x^*, x\rangle = \langle x^*, (\lambda - A)x\rangle = 0,$$

so x annihilates the range of $\lambda - A^*$. In particular x annihilates the range of $\lambda - A^\odot$, which equals $X^\odot$ since $\lambda \in \varrho(A^\odot)$. By the weak*-denseness of $X^\odot$ it follows that $x = 0$, so $\lambda - A$ is injective. Next, $\lambda - A$ has dense range: if not, then some non-zero $x^* \in X^*$ annihilates this range. Then $x^* \in D(A^*)$ and $(\lambda - A^*)x^* = 0$, so by Theorem 1.3.3 we have $x^* \in D(A^\odot)$ and $(\lambda - A^\odot)x^* = 0$, a contradiction to $\lambda \in \varrho(A^\odot)$. For the proof that the range of $\lambda - A$ is closed one can copy the argument in Lemma 1.4.1, the only difference being that now Theorem 1.3.5 must be invoked. ////

Remark 1.4.3. (i) For the point spectrum $\sigma_p(A^\odot)$ of $A^\odot$ we have $\sigma_p(A^\odot) = \sigma_p(A^*)$. This is an almost obvious consequence of Theorem 1.3.3.

(ii) Similar results hold for the spectra of $T(t)$, $T^*(t)$ and $T^\odot(t)$.

The following is a useful characterization of $X^\odot$ in terms of the resolvent.

Proposition 1.4.4. $x^* \in X^\odot$ *if and only if* $\lim_{\lambda\to\infty} ||\lambda R(\lambda, A^*)x^* - x^*|| = 0$.

Proof: If $\lim_{\lambda\to\infty} ||\lambda R(\lambda, A^*)x^* - x^*|| = 0$, then x^* lies in the closure of $D(A^*)$ since each $\lambda R(\lambda, A^*)x^* \in D(A^*)$. Therefore $x^* \in X^{\odot}$ by Theorem 1.3.1.

Conversely, let $x^* \in D(A^*)$. Since $\limsup_{\lambda\to\infty} ||\lambda R(\lambda, A)|| < \infty$ we have

$$\begin{aligned}\lim_{\lambda\to\infty} ||\lambda R(\lambda, A^*)x^* - x^*|| &= \lim_{\lambda\to\infty} ||R(\lambda, A^*)A^*x^*|| \\ &\le \Big(\limsup_{\lambda\to\infty} ||R(\lambda, A)||\Big)\ ||A^*x^*|| = 0.\end{aligned}$$

By the denseness of $D(A^*)$ in $X^{\odot}$, $\lambda R(\lambda, A^*)x^{\odot} \to x^{\odot}$ holds for every $x^{\odot} \in X^{\odot}$. ////

It is equally simple to prove that $x^* \in X^{\odot}$ if and only if

$$\lim_{t\downarrow 0} ||weak^* \int_0^t T^*(\sigma)x^*\ d\sigma - x^*|| = 0.$$

1.5. A class of examples

In this section we will associate to a Schauder basis in a given Banach space a natural class of C_0-semigroups, defined coordinatewise. The semigroup duals of these semigroups are the closed linear spans of the coordinate functionals. Since this allows us to carry over certain pathologies of Schauder bases to semigroups, we obtain a nice tool to construct counterexamples. We start with recalling some definitions.

A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called a *Schauder basis* (briefly, basis) if for every $x \in X$ there exists a unique sequence $\{\alpha_n\}_{n=1}^{\infty}$ of scalars such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$. The *coordinate functionals* $\{x_n^*\}_{n=1}^{\infty}$ defined by $\langle x_n^*, \sum_{k=1}^{\infty} \alpha_k x_k\rangle := \alpha_n$ are continuous. From this it is easy to see that the maps π_N and P_N defined by

$$\pi_N \sum_{n=1}^{\infty} \alpha_n x_n = \sum_{n=1}^{N} \alpha_n x_n, \qquad P_N \sum_{n=1}^{\infty} \alpha_n x_n = \alpha_N x_N$$

are projections with $C := \sup_N ||\pi_N|| < \infty$. If $x = \sum_{n=1}^{\infty} \alpha_n x_n$, then

$$||\sum_{n=M}^{N} \alpha_n x_n|| = ||(\pi_N - \pi_{M-1})x|| \le 2C||x||.$$

The constant C is called the *basis constant* of $\{x_n\}_{n=1}^{\infty}$.

The following proposition shows that a Banach space with a Schauder basis admits many 'multiplication' operators.

Lemma 1.5.1. *Let (γ_n) be a sequence of scalars such that*

$$\gamma := \lim_{n\to\infty} |\gamma_n| + \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty.$$

If $x = \sum_{n=1}^{\infty} \alpha_n x_n \in X$, then also $\sum_{n=1}^{\infty} \gamma_n \alpha_n x_n \in X$ and

$$||\sum_{n=1}^{\infty} \gamma_n \alpha_n x_n|| \leq \gamma C ||x||.$$

Proof: For each $N \geq 1$ we have

$$\begin{aligned}\sum_{n=1}^{N} \gamma_n \alpha_n x_n &= \sum_{n=1}^{N} \gamma_n \Big(\sum_{i=1}^{n} \alpha_i x_i - \sum_{i=1}^{n-1} \alpha_i x_i\Big)\\ &= \sum_{n=1}^{N} (\gamma_n - \gamma_{n+1}) \sum_{i=1}^{n} \alpha_i x_i + \gamma_{N+1} \sum_{i=1}^{N} \alpha_i x_i\end{aligned}$$

Now

$$\begin{aligned}||\sum_{n=1}^{N} (\gamma_n - \gamma_{n+1}) \sum_{i=1}^{n} \alpha_i x_i|| &\leq \sum_{n=1}^{N} |\gamma_n - \gamma_{n+1}|\ ||\sum_{i=1}^{n} \alpha_i x_i||\\ &\leq C||x|| \sum_{n=1}^{N} |\gamma_n - \gamma_{n+1}|,\end{aligned}$$

and

$$||\gamma_{N+1} \sum_{i=1}^{N} \alpha_i x_i|| \leq C||x||\ |\gamma_{N+1}| \leq C||x|| \Big(\lim_{n\to\infty} |\gamma_n| + \sum_{n=N+1}^{\infty} |\gamma_n - \gamma_{n+1}|\Big).$$

This shows that

$$||\sum_{n=1}^{N} \gamma_n \alpha_n x_n|| \leq \gamma C ||x||. \tag{1.4}$$

Applying this to the vector $x_M := \sum_{n=M}^{\infty} \alpha_n x_n$ it follows for $N \geq M$ that

$$||\sum_{n=M}^{N} \gamma_n \alpha_n x_n|| \leq \gamma C ||x_M||.$$

Since $\lim_{M\to\infty} ||x_M|| = 0$ it follows that the sum $\sum_{n=1}^{\infty} \gamma_n \alpha_n x_n$ indeed converges, and it is clear form (1.4) that

$$||\sum_{n=1}^{\infty} \gamma_n \alpha_n x_n|| \leq \gamma C ||x||.$$

////

Let $0 \le k_1 \le k_2 \le ... \to \infty$ and define operators $T(t)$ by

$$T(t)x_n := e^{-k_n t}x_n.$$

Then by the above lemma, $T(t)$ is a bounded operator for each t of norm $\le C$.

Let $[x_n^*]_{n=1}^\infty$ denote the closed linear span in X^* of the coordinate functionals $\{x_n^*\}_{n=1}^\infty$.

Theorem 1.5.2. *$T(t)$ is a C_0-semigroup on X with the following properties:*
(i) $||T(t)|| \le C$;
(ii) $X^\odot = [x_n^*]_{n=1}^\infty$;
(iii) $T^*(t)$ is $C_{>0}$;
(iv) X is $\odot$-reflexive with respect to $T(t)$.

Proof: (i) follows from the above remark. To show that $T(t)$ is C_0, fix $x \in X$ of norm 1, say $x = \sum_{n=1}^\infty \beta_n x_n$. Let $\epsilon > 0$ be arbitrary and take N such that

$$|| \sum_{n=N+1}^{\infty} \beta_n x_n|| \le \epsilon.$$

Let $t_0 > 0$ be so small that $1 - e^{-k_N t_0} \le \epsilon N^{-1}$. Since $0 \le k_1 < k_2 < ...$, also $1 - e^{-k_n t} \le \epsilon N^{-1}$ for all $1 \le n \le N$ and $0 \le t \le t_0$. Then for $0 \le t \le t_0$ we have, applying Lemma 1.5.1 to $\gamma_n := 1 - e^{-k_n t}$ and the vector $\sum_{n=N+1}^\infty \beta_n x_n$,

$$\begin{aligned}||T(t)x - x|| &\le ||\sum_{n=1}^{N}(1 - e^{-k_n t})\beta_n x_n|| + ||\sum_{n=N+1}^{\infty}(1 - e^{-k_n t})\beta_n x_n||\\ &\le N \cdot \frac{\epsilon}{N} \cdot \max_{1\le n\le N} ||\beta_n x_n|| + \Big(1 + \sum_{n=1}^{\infty} |e^{-k_n t} - e^{-k_{n+1}t}|\Big)C|| \sum_{n=N+1}^{\infty} \beta_n x_n||\\ &\le 2C\epsilon + 2C\epsilon.\end{aligned}$$

This shows that $T(t)$ is a C_0-semigroup on X.

Proof of (ii): It is obvious that $[x_n^*]_{n=1}^\infty \subset X^\odot$ since we have $T^*(t)x_n^* = e^{-k_n t}x_n^*$. To prove the reverse inclusion, let $x^* \in X^*$ be arbitrary. Putting $\alpha_n := \langle P_n^* x^*, x_n\rangle$ it is clear that

$$weak^*\text{-}\lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n x_n^* = x^*.$$

We claim that the weak*-convergent series $T^*(t)x^* = weak^* \sum_{n=1}^\infty e^{-k_n t}\alpha_n x_n^*$ is actually strongly convergent for every $t > 0$. Indeed, by Lemma 1.5.1 we have for every $x = \sum_{n=1}^\infty \beta_n x_n$ that

$$\begin{aligned}|\langle \sum_{n=M}^{N} e^{-k_n t}\alpha_n x_n^*, \sum_{n=1}^{\infty} \beta_n x_n\rangle| &= |\langle \sum_{n=M}^{N} \alpha_n x_n^*, \sum_{n=M}^{N} e^{-k_n t}\beta_n x_n\rangle|\\ &\le 2C||x^*|| \cdot e^{-k_M t}C||x||.\end{aligned}$$

Hence

$$\|\sum_{n=M}^{N} e^{-k_n t}\alpha_n x_n^*\| \le 2C^2 e^{-k_M t}\|x^*\|.$$

Since $k_M \to \infty$ as $M \to \infty$ we have shown that $(\sum_{n=1}^N e^{-k_n t}\alpha_n x_n^*)_{N=1}^\infty$ is a norm-Cauchy sequence in X^* for each $t > 0$. From this it follows that $T^*(t)x^* \in [x_n^*]_{n=1}^\infty$ for $t > 0$. Now if $x^* \in X^\odot$, then $x^* = \lim_{t\downarrow 0} T^*(t)x^*$ and by the closedness of $[x_n^*]_{n=1}^\infty$ it follows that we must have $x^* \in [x_n^*]_{n=1}^\infty$. This shows $X^\odot \subset [x_n^*]_{n=1}^\infty$.

That $T^*(t)$ is $C_{>0}$ follows from the above argument, and (iii) is proved.

Since the coordinate functionals of the Schauder basis $\{x_n^*\}_{n=1}^\infty$ of $X^\odot$ can be identified with $\{x_n\}_{n=1}^\infty$, it follows that $X^{\odot\odot} = jX$ and this is (iv). ////

Let us give some applications of this theorem.

Example 1.5.3. The *James space* J [Ja2, LT] consists af all sequences of scalars $x = (a_1, a_2, ...)$ for which $\lim_{n\to\infty} a_n = 0$ and

$$\|x\| := \sup\ [(a_{p_1} - a_{p_2})^2 + ... + (a_{p_{m-1}} - a_{p_m})^2 + (a_{p_m} - a_{p_1})^2]^{1/2} < \infty,$$

where the supremum is taken over all possible choices of integers m and $p_1 < p_2 < .... < p_m$. Let x_n denote the nth unit vector. The system $\{x_n\}_{n=1}^\infty$ is a Schauder basis for J and we have $J^* = [x_n^*]_{n=1}^\infty$. Define a C_0-semigroup $T(t)$ on J by

$$T(t)x_n = e^{-nt}x_n.$$

By Theorem 1.5.2 the adjoint $T^*(t)$ is strongly continuous on J^*. Moreover, also by Theorem 1.5.2, $(J^*)^\odot = J$. One can show that J has co-dimension one in J^{**}. Summarizing we have

$$J^\odot = J^*, \quad (J^*)^\odot = J, \quad \dim(J^*)^*/(J^*)^\odot = 1.$$

By regarding J^* as a co-dimension one subspace of J^{***}, the semigroup $S(t) := T^{**}(t)$ shows that the weak semigroup theorem (Theorem 0.2.1) cannot be improved:

Corollary 1.5.4. *There exists a Banach space X and a semigroup $S(t)$ on X with the following properties:*

(i) $S(t)$ is not strongly continuous;

(ii) $S(t)$ is $\sigma(X, Y)$-continuous for a co-dimension one subspace Y of X^*.

The phenomenon that $\dim(J^*)^*/(J^*)^\odot = 1$ will be discussed at length in Chapters 5 and 6.

We close this section with a partial converse of Corollary 1.3.2. For this we need the following concepts. A basis $\{x_n\}_{n=1}^\infty$ is called *shrinking* if the coordinate functionals $\{x_n^*\}_{n=1}^\infty$ form a basis of X^*. A basis $\{x_n\}_{n=1}^\infty$ is called *boundedly complete* if the following holds: whenever the sequence

$\{\|\sum_{n=1}^{N}\alpha_n x_n\|\}_{N=1}^{\infty}$ is bounded, then $\sum_{n=1}^{N}\alpha_n x_n$ actually converges to some $x \in X$ as $N \to \infty$. The concepts shrinking and boundedly complete are dual to each other in the sense that $\{x_n\}_{n=1}^{\infty}$ is boundedly complete if and only if $\{x_n^*\}_{n=1}^{\infty}$ is a shrinking basis for $[x_n^*]_{n=1}^{\infty}$, and $\{x_n\}_{n=1}^{\infty}$ is shrinking if and only if $\{x_n^*\}_{n=1}^{\infty}$ is a boundedly complete basis for $[x_n^*]_{n=1}^{\infty}$. A Banach space with a basis $\{x_n\}_{n=1}^{\infty}$ is reflexive if and only if $\{x_n\}_{n=1}^{\infty}$ is shrinking and boundedly complete. These results are due to R.C. James [Ja1].

If we consider all bases of X simultaneously we have the following theorem of M. Zippin [Zi].

Theorem 1.5.5. *Let X have a basis $\{x_n\}_{n=1}^{\infty}$. Then the following assertions are equivalent:*

(i) X is reflexive;

(ii) Every basis $\{y_n\}_{n=1}^{\infty}$ of X is shrinking;

(iii) Every basis $\{y_n\}_{n=1}^{\infty}$ of X is boundedly complete.

Corollary 1.5.6. *Let X be a non-reflexive Banach space with a Schauder basis. Then there exists a C_0-semigroup on X with $X^{\odot} \neq X^*$.*

This is not valid for arbitrary Banach spaces: if $T(t)$ is any C_0-semigroup on $X = L^\infty[0,1]$, then the following theorem, first proved by H.P. Lotz [Lo2] in the present generality, shows that $T(t)$ is uniformly continuous and consequently $X^{\odot} = X^*$.

Theorem 1.5.7. *Every C_0-semigroup on a Grothendieck space with the Dunford-Pettis property is uniformly continuous.*

A Banach space X has the *Grothendieck property* if every weak*-convergent sequence in X^* is weakly convergent, and it has the *Dunford-Pettis property* if every weakly compact operator on X maps relatively weakly compact sets into relatively compact sets. Examples of Grothendieck Dunford-Pettis spaces are $L^\infty[0,1]$ and l^∞. Every $L^1(\mu)$-space and every $C(K)$-space, K compact Hausdorff, has the Dunford-Pettis property. See e.g. [AB, S4] for the proofs. A partial converse of Theorem 1.5.7 is proved in [Ne7].

Notes. Adjoint semigroups were first studied systematically by Phillips in [Ph2].

The material in Sections 1.1 to 1.4 is standard. The presentation in Sections 1.1, 1.2 and 1.3 is based on [BB] (which in turn is based on [dL]), the main difference being the systematic use of the weak*-integral. The advantage of this is that we have a large supply of elements of $D(A^*)$. On the one hand, this makes the proofs more transparent and on the other hand it allows a completely elementary treatment of the basic theory. In this context it should be noted that all weak*-integrals used so far can be interpreted as Riemann weak*-integrals.

A different and somewhat more general approach is given in [HPh]. There $X^{\odot}$ is defined to be the closure of $D(A^*)$ and $A^{\odot}$ is defined to be the part of A^* in $X^{\odot}$.

One must then show that $T^*(t)$ leaves $X^\odot$ invariant, that the restrictions $T^\odot(t)$ to $X^\odot$ define a C_0-semigroup on $X^\odot$ with generator $A^\odot$, and that $X^\odot$ is maximal with respect to these properties. The proofs given in [HPh] are more difficult.

Another advantage of the present approach is that it generalizes without difficulty to the following situation. Let $\{T_n\}_{t\in\mathbb{N}}$ be a sequence of bounded linear operators on X with the following properties:

(i) $\lim_{n\to\infty} ||T_n x - x|| = 0, \quad \forall x \in X$;
(ii) $\lim_{n\to\infty} ||T_m(T_n - I)|| = \lim_{n\to\infty} ||(T_n - I)T_m|| = 0, \quad \forall m \in \mathbb{N}$.

Define $X^\odot := \{x^* \in X^* : \lim_{n\to\infty} ||T_n^* x^* - x^*|| = 0\}$. As an example, let $T(t)$ be a C_0-semigroup, let $\lambda_n \to \infty$ and put $T_n := \lambda_n R(\lambda_n, A)$. Then (i) holds and the resolvent identity (1.2) easily implies that (ii) also holds. Proposition 1.4.4 shows that the present definitions of $X^\odot$ agrees with the semigroup definition. Another example is provided by the projections $\{\pi_n\}_{n=1}^\infty$ associated with a Schauder basis.

By the uniform boundedness theorem and (i), the T_n are uniformly bounded and hence $X^\odot$ is closed. By (ii) the range of each T_n^* belongs to $X^\odot$ and it follows easily that $X^\odot$ induces an equivalent norm $||\cdot||'$ in X. Also $X^\odot$ is T_n^*-invariant for each n. Several results in the next chapters generalize to this setting, notably much of Chapter 2 and Chapter 3. In particular the analogue of Theorem 3.2.2 is valid.

We mention three more lines of generalization. Firstly, several results hold in certain locally convex spaces, e.g. Fréchet spaces. The norm topology of X^* then has to be replaced by the strong topology $\beta(X^*, X)$, see [S1] for the definition. Secondly, one can study a more general class of weak*-continuous semigroups. The problem that has to be overcome here is that an arbitrary weak*-continuous semigroup is not uniquely determined by its weak*-generator. For an example of this phenomenon as well as more results along this line, see [Cea5]. Thirdly, one can study the adjoints of more general classes of semigroups, where the strong continuity is relaced by weaker hypotheses. Some possibilities are discussed in [HPh]; see also Theorem 5.3.2 and [Fe]. In order to get any reasonable result it seems that one has to impose at least some weak measurability conditions. For example one has the following: *Let $S(t)$ be a locally bounded semigroup on X. Let $x^* \in X^*$ be such that for all $x \in X$ the map $t \mapsto \langle x^*, S(t)x\rangle$ is measurable. Then for all $t > 0$ we have*

$$weak^* \int_0^t S^*(\sigma)x^* \, d\sigma \in X^\odot.$$

The proof is direct and uses some estimates similar to the ones in Theorem 0.2.1. Even if $T(t)$ is weakly measurable with respect to all elements of X^* it can happen that $X^\odot = \{0\}$. Two examples are given in Section 8.2. For a detailed discussion of weak measurability of semigroups we refer to [Fe]. Weak measurability of adjoint semigroups is discussed in Chapters 5 and 8.

Corollary 1.3.2 is due to Phillips [Ph2].

The proof of Theorem 1.4. is along the lines of [HPh]. Theorem 1.5.2, Example 1.5.3 and Corollary 1.5.6 are taken from [Ne2]. The summation trick in Lemma 1.5.1 is taken from the proof of [Si, Theorem 1.5.2] and was shown to me by Ben de Pagter. Theorem 1.5.2 can be generalized to more general decompositions structures, e.g. Schauder decompositions. The optimality of the weak semigroup theorem, expressed in Corollary 1.5.4, was noticed in [Ne5].

A basis $\{x_n\}_{n=1}^{\infty}$ is called *unconditional* if for every $x \in X$ the expansion $\sum_{n=1}^{\infty} \alpha_n x_n$ of x converges unconditionally, that is, $\sum_{n=1}^{\infty} \alpha_{\sigma(n)} x_{\sigma(n)}$ converges for every permutation σ of the positive integers. In [Ne5] it is shown that if X has an unconditional basis, then X^* has the Radon-Nikodym property (the definition is given in Chapter 6) if and only if the adjoint of every C_0-semigroup on X^* is $C_{>0}$. The point of this is that a Banach space with unconditional basis is, up to an equivalent norm, a Banach lattice with order continuous norm (cf. Chapter 8). Indeed, the result just mentioned is valid for this class of spaces as well. It seems to be unknown whether it extends to a larger class of Banach spaces. We also ask whether Corollary 1.5.6 holds for arbitrary separable spaces (not every separable space has a basis; there is a famous counterexample of P. Enflo [En]). If so, then it holds for weakly compactly generated spaces (see Chapter 3 for the definition) as well: indeed, by the Eberlein Shmulyan theorem, if X is non-reflexive then there is a separable non-reflexive subspace Y of X and by the Amir-Lindenstrauss theorem [AL], Y is contained in a *complemented* separable (non-reflexive) subspace Z of X. In this context it is interesting to note the following theorem of Pelczynski [Pe]: *A Banach space is reflexive if and only if each of its subspaces with a basis is reflexive.*

Theorem 1.5.7 is proved in [Lo2] in the above setting of operators $\{T_n\}_{n\in\mathbb{N}}$ satisfying the above-mentioned conditions (i) and (ii). Partial results had been obtained earlier by several authors, notably Coulhon [Co], Kishimoto-Robinson [KR], and Lotz himself. A simple short proof in the semigroup case is given in [Na2], where also the following lemma is proved: if $T(t)$ is a C_0-semigroup on a Grothendieck space, then $X^{\odot} = X^*$ and $X^{\odot\odot} = X^{**}$. Of course, the first identity follows from Theorem 0.2.1. There is a partial converse of Lotz's theorem [Ne7]; see the notes of Chapter 8.

Chapter 2

The $\sigma(X, X^{\odot})$-topology

In this chapter we study the $\sigma(X, X^{\odot})$-topology on X, where $X^{\odot}$ is the semigroup dual of X with respect to a given C_0-semigroup $T(t)$ on X. In Section 2.1 we give various characterizations of $\sigma(X, X^{\odot})$-closed sets and in Section 2.2 we prove an Eberlein-Shmulyan type theorem for $\sigma(X, X^{\odot})$. In Section 2.3 we look at the $\|\cdot\|'$-norm induced by $X^{\odot}$.

2.1. $\sigma(X, X^{\odot})$-closed sets

Let Y be a weak*-dense subspace of the dual Banach space X^*. For each $y \in Y$,

$$p_y(x) := |\langle y, x\rangle|$$

is a seminorm on X. The collection $\{p_y : y \in Y\}$ induces a (Hausdorff) locally convex topology on X, called the *$\sigma(X, Y)$-topology*. Explicitly, the sets

$$V(y; \epsilon) := \{x \in X : |\langle y, x\rangle| < \epsilon\}, \qquad y \in Y, \ \epsilon > 0,$$

form a subbase for this topology at the origin.

If $T(t)$ is a C_0-semigroup on X, then $X^{\odot}$ is weak*-dense and therefore the $\sigma(X, X^{\odot})$-topology is a locally convex topology on X.

In this section we will study in detail which sets are $\sigma(X, X^{\odot})$-closed. For a subset $G \subset X$ put $G_0 := G$ and for $t > 0$ define $G_t := \{\frac{1}{t}\int_0^t T(\sigma)g\, d\sigma : g \in G\}$.

Proposition 2.1.1. *Let $G \subset X$ be arbitrary. Then*

$$\overline{G}^{\sigma(X,X^{\odot})} \subset \bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}^{weak}.$$

In particular, if $G = \bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}^{weak}$, then G is $\sigma(X, X^{\odot})$-closed.

Proof: Fix any $x \notin \bigcap_{t>0} \overline{\bigcup_{0\leq s\leq t} G_s}^{weak}$. We must show: $x \notin \overline{G}^{\sigma(X,X^{\odot})}$. By assumption there is a $t_0 > 0$ such that $x \notin \overline{\bigcup_{0\leq s\leq t_0} G_s}^{weak}$. Choose norm-1 functionals $x_1^*, ..., x_n^* \in X^*$ and $\epsilon > 0$ such that the weakly open set

$$V = V(x_1^*, ..., x_n^*; \epsilon; x) := \{y \in X : \ |\langle x_i^*, x - y\rangle| < \epsilon, \quad i = 1, ..., n\}$$

which contains x is disjoint from $\bigcup_{0\leq s\leq t_0} G_s$. By the strong continuity of $T(t)$ we may choose $0 < t_1 \leq t_0$ such that additionally we have

$$\|\frac{1}{t_1}\int_0^{t_1} T(\sigma)x \, d\sigma - x\| < \frac{\epsilon}{2}.$$

We claim that $\tilde{V} \cap G = \emptyset$, where

$$\tilde{V} = V(\frac{1}{t_1} weak^* \int_0^{t_1} T^*(\sigma)x_1^* \, d\sigma, ..., \frac{1}{t_1} weak^* \int_0^{t_1} T^*(\sigma)x_n^* \, d\sigma; \frac{\epsilon}{2}; x).$$

Indeed, fix any $g \in G$ and choose $i_0 \in 1, ..., n$ such that

$$|\langle x_{i_0}^*, x - \frac{1}{t_1}\int_0^{t_1} T(\sigma)g \, d\sigma\rangle| \geq \epsilon.$$

Such an i_0 exists since $V \cap G_{t_1} = \emptyset$. Then

$$\begin{aligned}
&|\langle \frac{1}{t_1} weak^* \int_0^{t_1} T^*(\sigma)x_{i_0}^* \, d\sigma, x - g\rangle| \\
&\qquad = |\langle x_{i_0}^*, \frac{1}{t_1}\int_0^{t_1} T(\sigma)x \, d\sigma - \frac{1}{t_1}\int_0^{t_1} T(\sigma)g \, d\sigma\rangle| \\
&\qquad \geq |\langle x_{i_0}^*, x - \frac{1}{t_1}\int_0^{t_1} T(\sigma)g \, d\sigma\rangle| - |\langle x_{i_0}^*, \frac{1}{t_1}\int_0^{t_1} T(\sigma)x \, d\sigma - x\rangle| \\
&\qquad \geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}.
\end{aligned}$$

This shows $\tilde{V} \cap G = \emptyset$ and the claim is proved. But $\frac{1}{t_1} weak^* \int_0^{t_1} T^*(\sigma)x_i^* \, d\sigma \in D(A^*)$. Therefore $\tilde{V}$ is $\sigma(X, X^{\odot})$-open, and we have $\tilde{V} \cap \overline{G}^{\sigma(X,X^{\odot})} = \emptyset$. Since $x \in \tilde{V}$ the proposition is proved. ////

Corollary 2.1.2. *Convex, closed and $T(t)$-invariant sets are $\sigma(X, X^{\odot})$-closed.*

In particular this applies to closed $T(t)$-invariant subspaces of X. In Section 2.2 we will single out a class of sets which are in general not $T(t)$-invariant, but do satisfy the condition of Proposition 2.1.1.

The following theorem asserts that *bounded* sets are in fact characterized by this property. Let us note here that a set G is bounded if and only if it is $\sigma(X, X^{\odot})$-bounded: regarding G as a subset of $X^{\odot *}$, by the uniform boundedness theorem G is bounded in $X^{\odot *}$. Since the natural map $j : X \to X^{\odot *}$ is an isomorphism into by Corollary 1.3.6, we see that G is bounded in X.

Theorem 2.1.3. *If G is a bounded set, then*

$$\overline{G}^{\sigma(X,X^{\odot})} = \bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}^{\sigma(X,X^{\odot})} = \bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}^{weak}.$$

Proof: In view of the inclusion proved in Proposition 2.1.1 we only have to prove the inclusion

$$\bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}^{\sigma(X,X^{\odot})} \subset \overline{G}^{\sigma(X,X^{\odot})}.$$

Suppose $x \notin \overline{G}^{\sigma(X,X^{\odot})}$. Then there are $x_1^{\odot}, ..., x_n^{\odot}$ in $X^{\odot}$ and $\epsilon > 0$ such that

$$V(x_1^{\odot}, ..., x_n^{\odot}; \epsilon; x) \cap G = \emptyset.$$

Since G is bounded there is a constant K such that $||g|| \le K$ for all $g \in G$. Choose $t_0 > 0$ such that for all $i = 1, ..., n$ and $0 \le s \le t_0$ we have

$$||\frac{1}{s}\int_0^s T^*(\sigma)x_i^{\odot}\, d\sigma - x_i^{\odot}|| < \frac{\epsilon}{2K}.$$

Let $g \in G$ be arbitrary and fixed. Choose $i_0 \in 1, ..., n$ such that $|\langle x_{i_0}^{\odot}, x-g\rangle| \ge \epsilon$. Then for $0 \le s \le t_0$

$$\begin{aligned} &|\langle x_{i_0}^{\odot}, x - \frac{1}{s}\int_0^s T(\sigma)g\, d\sigma\rangle| \\ &\quad\ge |\langle x_{i_0}^{\odot}, x - g\rangle| - |\langle x_{i_0}^{\odot}, g - \frac{1}{s}\int_0^s T(\sigma)g\, d\sigma\rangle| \\ &\quad\ge \epsilon - |\langle \frac{1}{s}\int_0^s T^*(\sigma)x_{i_0}^{\odot}\, d\sigma - x_{i_0}^{\odot}, g\rangle| \\ &\quad\ge \epsilon - \frac{\epsilon}{2K}K = \frac{\epsilon}{2}. \end{aligned}$$

It follows that for all $0 \le s \le t_0$ we have $\tilde{V} \cap G_s = \emptyset$, where $\tilde{V}$ is the set $V(x_1^{\odot}, ..., x_n^{\odot}; \frac{\epsilon}{2}; x)$. Since $\tilde{V}$ is $\sigma(X, X^{\odot})$-open, it follows that

$$\tilde{V} \cap \overline{\bigcup_{0\le s\le t_0} G_s}^{\sigma(X,X^{\odot})} = \emptyset.$$

Since $x \in \tilde{V}$ the proof is finished. ////

The boundedness assumption is essential, as Example 2.1.7 below shows.

Remark 2.1.4. If G is bounded, then one has

$$\overline{G}^{\sigma(X,X^{\odot})} = \bigcap_{t>0} \overline{\bigcup_{0\leq s\leq t} T(s)G}^{\sigma(X,X^{\odot})}.$$

The proof of this is similar to that of Proposition 2.1.1 and Theorem 2.1.3. Thus every bounded $\sigma(X, X^{\odot})$-closed set is 'infinitesimally invariant' with respect to $T(t)$ in the $\sigma(X, X^{\odot})$-topology. If the $\sigma(X, X^{\odot})$-topology is replaced by either the weak- or the norm topology, then the above formula is no longer true: in Section 2.3 we will construct a semigroup on c_0 for which

$$(2, 0, 0, ...) \in \bigcap_{t>0} \overline{\bigcup_{0\leq s\leq t} T(s)B_{c_0}}.$$

For convex sets, Theorem 2.1.3 assumes a particularly nice form. Let $\overline{\text{co}}G$ denote the closed convex hull of a set G.

Theorem 2.1.5. *If G is convex and bounded, then*

$$\overline{G}^{\sigma(X,X^{\odot})} = \bigcap_{t>0} \Big(\overline{\text{co}} \bigcup_{0\leq s\leq t} T(s)G\Big).$$

Proof: Since closed convex sets are weakly closed, for every set F we have $\overline{F}^{weak} \subset \overline{\text{co}}F$. On the other hand for every $0 \leq s \leq t$ we have

$$G_s \subset \overline{\text{co}} \bigcup_{0\leq s\leq t} T(s)G.$$

Together with Theorem 2.1.3 this proves the inclusion

$$\overline{G}^{\sigma(X,X^{\odot})} \subset \bigcap_{t>0} \Big(\overline{\text{co}} \bigcup_{0\leq s\leq t} T(s)G\Big).$$

For the converse inclusion, suppose $y \in \bigcap_{t>0} (\overline{\text{co}} \bigcup_{0\leq s\leq t} T(s)G)$. This means that there is a sequence of convex combinations

$$y_i = \sum_{n=1}^{N_i} \alpha_{in} T(t_{in}) g_{in}$$

converging to y strongly, with $g_{in} \in G$ and $\max_{n=1\ldots N_i} t_{in} < i^{-1}$. Put

$$z_i := \sum_{n=1}^{N_i} \alpha_{in} g_{in}.$$

Since G is convex we have $z_i \in G$ for all i. Since G is bounded, there is a $K < \infty$ such that $||g|| \leq K$ for all $g \in G$. For fixed $x^{\odot} \in X^{\odot}$ we have

$$\begin{aligned}
|\langle x^{\odot}, y_i - z_i\rangle| &= |\langle x^{\odot}, \sum_{n=1}^{N_i} \alpha_{in} T(t_{in}) g_{in} - \sum_{n=1}^{N_i} \alpha_{in} g_{in}\rangle| \\
&= |\sum_{n=1}^{N_i} \alpha_{in} \langle x^{\odot}, T(t_{in}) g_{in} - g_{in}\rangle| \\
&= |\sum_{n=1}^{N_i} \alpha_{in} \langle T^*(t_{in}) x^{\odot} - x^{\odot}, g_{in}\rangle| \\
&\leq K \sum_{n=1}^{N_i} \alpha_{in} ||T^*(t_{in}) x^{\odot} - x^{\odot}|| \quad \to 0 \qquad \text{as } i \to \infty,
\end{aligned}$$

since on the one hand $\max_{n=1\ldots N_i} t_{in} < \frac{1}{i}$ and on the other hand $||T^*(t)x^{\odot} - x^{\odot}|| \to 0$ as $t \downarrow 0$. This shows that $z_i - y_i$ converges to 0 in the $\sigma(X, X^{\odot})$-topology. But $y_i \to y$ strongly, hence $z_i \to y$ in the $\sigma(X, X^{\odot})$-topology. Since $z_i \in G$ for all i it follows that $y \in \overline{G}^{\sigma(X,X^{\odot})}$. ////

The weak closure of a convex set is just the norm closure; the above theorem can be regarded as an analogue for the $\sigma(X, X^{\odot})$-closure of bounded convex sets.

Weakly convergent sequences admit norm convergent convex combinations, cf. [Ru3]. For $\sigma(X, X^{\odot})$-convergent sequences we get the following analogue: if $x_n \to x$ in the $\sigma(X, X^{\odot})$-topology, then for every $\delta > 0$ and $\epsilon > 0$ there are numbers $t_n \in [0, \delta]$ and $\alpha_n \geq 0$, $n = 1, ..., N_n$ with $\sum_n \alpha_n = 1$ such that

$$||x - \sum_{n=1}^{N_n} \alpha_n T(t_n) x_n|| < \epsilon.$$

To see this, note that (x_n) is $\sigma(X, X^{\odot})$-bounded, hence bounded because of the remark preceding Theorem 2.1.3, and then apply Theorem 2.1.5. The following example shows what this means for the rotation group $T(t)$ on $C(T)$.

Corollary 2.1.6. *Let (f_n) be a bounded sequence in $C(T)$ which converges a.e. (with respect to the Lebesgue measure) to some $f \in C(T)$. Then for every $\delta > 0$ and $\epsilon > 0$ there are numbers $t_n \in [0, \delta]$ and $\alpha_n \geq 0$ with $\sum_n \alpha_n = 1$ such that*

$$||f - \sum_n \alpha_n T(t_n) f_n|| < \epsilon.$$

Proof: Since $C(T)^{\odot} = L^1(T)$ (cf. Example 1.3.9) and since (f_n) is bounded, the dominated convergence theorem shows that $f_n \to f$ pointwise a.e. implies that $f_n \to f$ in the $\sigma(C(T), C(T)^{\odot})$-topology. Now the conclusion follows from the preceding remarks. ////

The following example shows that Theorem 2.1.3 fails for arbitrary sets and that Theorem 2.1.5 fails for arbitrary convex sets (use the first inclusion in its proof).

Example 2.1.7. Let X be a Banach space with a Schauder basis $\{x_n\}_{n=1}^{\infty}$ and define the C_0-semigroup $T(t)$ by $T(t)x_n = e^{-(n-1)t}x_n$, cf. Theorem 1.5.2. Put $z_k := x_1 - k^2 x_k$, $k = 2, 3, \ldots$ and let Z be the linear span of $(z_k)_{k\geq 2}$. We claim that (i) and (ii) hold:

$$(i) \quad x_1 \in \bigcap_{t>0} \overline{\bigcup_{0<s\leq t} Z_s}^{weak},$$

$$(ii) \quad x_1 \notin \overline{Z}^{\sigma(X,X^{\odot})}.$$

Indeed, for arbitrary $t > 0$ we have

$$\lim_{k\to\infty} \frac{1}{t}\int_0^t T(\sigma)z_k d\sigma = \lim_{k\to\infty} x_1 - k^2 \frac{1 - e^{-(k-1)t}}{(k-1)t} x_k = x_1.$$

Hence

$$x_1 \in \overline{Z_t} \subset \overline{\bigcup_{0<s\leq t} Z_s}^{weak}.$$

This proves (i). Define $x^{\odot} \in X^*$ by

$$x^{\odot} := \sum_{n=1}^{\infty} \frac{1}{n^2} x_n^*.$$

By Theorem 1.5.2 actually $x^{\odot} \in X^{\odot}$. Since $\langle x^*, z_k\rangle = 0$ for all $k \geq 2$ it follows that $\langle x^{\odot}, z\rangle = 0$ for all $z \in Z$. On the other hand, $\langle x^{\odot}, x_1\rangle = 1$. This proves (ii).

Finally we can ask under what conditions the 'most convex' set of X, its closed unit ball, is $\sigma(X, X^{\odot})$-closed. There is a very simple answer: B_X is $\sigma(X, X^{\odot})$-closed if and only if the natural embedding $j : X \to X^{\odot *}$ is isometric. This will be proved in Section 2.3.

2.2. An Eberlein-Shmulyan type theorem for $\sigma(X, X^{\odot})$

In this section we continue the study of the $\sigma(X, X^{\odot})$-topology. The main result is an Eberlein-Shmulyan type theorem, stating that a set $G \subset X$ is $\sigma(X, X^{\odot})$-compact if and only if it is $\sigma(X, X^{\odot})$-sequentially compact.

We start by defining a class of sets which satisfy the condition of 'infinitesimal invariance' from Proposition 2.1.1. Let G be a subset of X. We will say that G is *equicontinuous with respect to a (semi)group* $T(t)$ (briefly, G is $T(t)$-*equicontinuous*) if the collection of maps $t \mapsto T(t)g$, where g ranges over G, is equicontinuous *at* $t = 0$. G will be called *weakly equicontinuous with respect to* $T(t)$ if for each $x^* \in X^*$ the collection of maps $t \mapsto \langle x^*, T(t)g\rangle$ is equicontinuous at 0. These definitions amount to a kind of '*uniform* (weak) infinitesimal invariance'.

If G is (weakly) $T(t)$-equicontinuous, so are $\overline{G}$, $\overline{\text{co}}G$ and hence also $\overline{G}^{weak}$. $T(t)$-Equicontinuous sets are weakly $T(t)$-equicontinuous, but the converse need not be true. For example consider the translation group on $C_0(\mathbb{R})$. Let f_n be the piecewise linear function defined by

$$f_n(x) = \begin{cases} 0, & x \leq n - \frac{1}{n}; \\ 1, & x = n; \\ 0, & x \geq n + \frac{1}{n}, \end{cases}$$

and which is linear on the intervals $[n - \frac{1}{n}, n]$ and $[n, n + \frac{1}{n}]$. The sequence (f_n) is equicontinuous in the classical sense but clearly not equicontinuous with respect to $T(t)$. We claim that (f_n) is weakly equicontinuous with respect to $T(t)$. This follows from the following proposition.

Proposition 2.2.1. *Let $T(t)$ be the translation group on $C_0(\mathbb{R})$. A bounded sequence (f_n) is weakly equicontinuous with resect to $T(t)$ if and only if (f_n) is equicontinuous (in the classical sense).*

Proof: If (f_n) is weakly $T(t)$-equicontinuous, then for each x the maps

$$t \mapsto \langle \delta_x, T(t)f_n\rangle = T(t)f_n(x) = f_n(x+t)$$

are equicontinuous. Hence (f_n) is equicontinuous in the classical sense. Conversely, suppose (f_n) is equicontinuous in the classical sense. Fix $\epsilon > 0$ arbitrarily and let K be such that $||f_n|| \leq K$ for all n. Let $\mu \in (C_0(\mathbb{R}))^*$ be arbitrary. By the Riesz representation theorem, μ is a regular Borel measure on $\mathbb{R}$. In particular, there is an $r > 0$ such that

$$|\mu|(\mathbb{R}\backslash[-r, r]) < \epsilon.$$

By the equicontinuity of (f_n), for each $x \in [-r, r]$ there is a $\delta(x) > 0$ such that $|x - y| < \delta(x)$ implies $|f_n(x) - f_n(y)| < \epsilon$ for all n. The open sets $B(x; \delta(x))$ form an open covering of the compact interval $[-r, r]$. Let $B_1, ..., B_N$ be some finite subcovering and let λ be its Lebesgue number. By definition this means that for each $x \in [-r, r]$ there is an $i \in 1, ..., N$ such that $B(x; \lambda) \subset B_i$. Note that if $y_1, y_2 \in B(x, \lambda)$ then $|f_n(y_1) - f_n(y_2)| < 2\epsilon$ for all n. For $|t| < \lambda$ we find

$$\begin{aligned} |\langle \mu, T(t)f_n - f_n\rangle| &= |\int_{-\infty}^{\infty} \Big(f_n(x+t) - f_n(x)\Big)\, d\mu(x)| \\ &\leq \Big(\int_r^{\infty} + \int_{-\infty}^{-r}\Big)|f_n(x+t) - f_n(x)|\, d\mu(x) + \int_{-r}^{r} |f_n(x+t) - f_n(x)|\, d\mu(x) \\ &\leq \epsilon \cdot 2K + 2\epsilon \cdot |\mu|([-r, r]) \leq 2\epsilon \cdot (K + ||\mu||). \end{aligned}$$

////

It is an easy consequence of the definition that for an $T(t)$-equicontinuous set G we have $\overline{G} = \bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}$. That this formula also holds with respect to the weak topology is the content of the following theorem.

Proposition 2.2.2. *If G is weakly $T(t)$-equicontinuous, then*

$$\overline{G}^{weak} = \bigcap_{t>0} \overline{\bigcup_{0\le s\le t} G_s}^{weak}.$$

Proof: Fix any $x \notin \overline{G}^{weak}$. We must show: $x \notin \overline{\bigcup_{0\le s\le t_0} G_s}^{weak}$ for some $t_0 > 0$. There are norm-1 functionals $x_1^*, ..., x_n^* \in X^*$ and $\epsilon > 0$ such that the weakly open set

$$V = V(x_1^*, ..., x_n^*; \epsilon; x) = \{y \in X : \ |\langle x_i^*, x - y\rangle| < \epsilon, \quad i = 1, ..., n\}$$

which contains x is disjoint from G. By the weak $T(t)$-equicontinuity of G we may choose $t_0 > 0$ such that for every $0 \le s \le t_0$, every $g \in G$ and $i = 1, ..., n$ we have

$$|\langle x_i^*, T(s)g - g\rangle| < \frac{\epsilon}{2}.$$

In particular we get for every $0 \le s \le t_0$, $g \in G$ and $i = 1, ..., n$

$$|\langle x_i^*, \frac{1}{s}\int_0^s T(\sigma)g \, d\sigma - g\rangle| < \frac{\epsilon}{2}.$$

Now the proof may be completed by using estimates similar to those in the proof of Theorem 2.1.3. ////

Corollary 2.2.3. *The weak- and the $\sigma(X, X^{\odot})$-closure of weakly $T(t)$-equicontinuous sets are equal. In particular weakly closed weakly $T(t)$-equicontinuous sets are $\sigma(X, X^{\odot})$-closed.*

For the proof, just combine Propositions 2.1.1 and 2.2.2. Since subsets of weakly $T(t)$-equicontinuous sets are weakly $T(t)$-equicontinuous, we obtain:

Corollary 2.2.4. *The relative weak- and $\sigma(X, X^{\odot})$-topology coincide on weakly equicontinuous sets.*

Proof: Let G be weakly $T(t)$-equicontinuous and suppose that $H \subset G$ is relatively weakly closed. Let $\tilde{H}$ be the weak closure of H in X. Then $\tilde{H} \cap G = H$. Moreover, $\tilde{H}$ is weakly $T(t)$-equicontinuous and therefore $\sigma(X, X^{\odot})$-closed by Corollary 2.2.3, so $H = \tilde{H} \cap G$ is relatively $\sigma(X, X^{\odot})$-closed in G. ////

Corollary 2.2.5. *A weakly $T(t)$-equicontinuous sequence in X is weakly convergent if and only if it is $\sigma(X, X^\odot)$-convergent.*

Proof: Suppose (x_n) is $\sigma(X, X^\odot)$-convergent to x. Put $G = \{x_n\}_{n=1}^\infty \cup \{x\}$. Then G is weakly $T(t)$-equicontinuous as well. Let V be a weakly open neighbourhood of x in X. Then $V \cap G$ is relatively weakly open in G, hence relatively $\sigma(X, X^\odot)$-open in G by Corollary 2.2.4. It follows that all but finitely many x_n lie in $V \cap G \subset V$, which was to be shown. ////

We now give a class of sets to which the preceding two corollaries apply.

Proposition 2.2.6. *If H is bounded then $\overline{R(\lambda, A)H}$ is $T(t)$-equicontinuous.*

For the proof, note that $T(t)R(\lambda, A)h - R(\lambda, A)h = \int_0^t T(\sigma)AR(\lambda, A)h \, d\sigma$ and use that the operator $AR(\lambda, A)$ is bounded. In particular if H is bounded and convex, then $\overline{R(\lambda, A)H}$ is $\sigma(X, X^\odot)$-closed.

As a corollary of these results we can prove an Eberlein-Shmulyan type theorem for the $\sigma(X, X^\odot)$-topology. We start by observing that a $\sigma(X, X^\odot)$-compact set G is norm bounded. Indeed, by regarding G as a subset of $X^{\odot *}$, G is weak*-compact, and the boundedness in $X^{\odot *}$ follows from the uniform boundedness theorem. Similarly every $\sigma(X, X^\odot)$-sequentially compact set is norm bounded.

Theorem 2.2.7. *A set is $\sigma(X, X^\odot)$-compact if and only if it is $\sigma(X, X^\odot)$-sequentially compact.*

Proof: Suppose G is $\sigma(X, X^\odot)$-compact and let (x_n) be a sequence in G. Since $R(\lambda, A)$ is continuous in the $\sigma(X, X^\odot)$-topology, also $R(\lambda, A)G$ is $\sigma(X, X^\odot)$-compact. By Corollary 2.2.4 $R(\lambda, A)G$ is weakly compact. Hence by the Eberlein-Shmulyan theorem there is a subsequence (x_{n_i}) and an $x \in G$ such that $R(\lambda, A)x_{n_i} \to R(\lambda, A)x$ weakly. So for every $x^* \in X^*$ we have

$$\langle R(\lambda, A^*)x^*, x_{n_i}\rangle = \langle x^*, R(\lambda, A)x_{n_i}\rangle \to \langle x^*, R(\lambda, A)x\rangle = \langle R(\lambda, A^*)x^*, x\rangle.$$

Since $R(\lambda, A^*)X^* = D(A^*)$ is norm-dense in $X^\odot$ and G is bounded it follows that $x_{n_i} \to x$ in the $\sigma(X, X^\odot)$-topology.

Conversely, assume that G is $\sigma(X, X^\odot)$-sequentially compact. Let $j : X \to X^{\odot *}$ be the natural embedding. Then jG is weak*-sequentially compact. Since jG is bounded it follows that the weak*-closure of jG up in $X^{\odot *}$ is weak*-compact. Therefore it suffices to show that we have $\overline{jG}^{weak^*} = jG$. Let $x^{\odot *}$ be any element of $\overline{jG}^{weak^*}$ and choose a net $x_\alpha \subset G$ such that jx_α is weak* convergent to $x^{\odot *}$. Consider the net $R(\lambda, A)x_\alpha$. Since the $\sigma(X, X^\odot)$-sequential compactness of G and the $\sigma(X, X^\odot)$-continuity of $R(\lambda, A)$ imply that also $R(\lambda, A)G$ is $\sigma(X, X^\odot)$-sequentially compact, it follows from Corollary 2.2.5 that $R(\lambda, A)G$ is weakly sequentially compact, hence weakly compact by the Eberlein-Shmulyan theorem. Hence the net $R(\lambda, A)x_\alpha$ has a weakly convergent subnet, say with limit $R(\lambda, A)x$. This forces that $jx = x^{\odot *}$ and the corollary is proved. ////

Implicitly we have proved the following result:

Corollary 2.2.8. *A bounded set G is $\sigma(X, X^{\odot})$-compact if and only if $R(\lambda, A)G$ is $\sigma(X, X^{\odot})$-compact.*

This corollary is no longer true if 'compact' is replaced by 'relatively compact'. The reader may check that a counterexample is given by $X = C(T)$, with $T(t)$ the rotation group and $G = B_X$. In this situation $R(\lambda, A)G$ is relatively $\sigma(X, X^{\odot})$-compact but G is not.

Theorem 2.2.9. *If $||T_n - T|| \to 0$ in the uniform operator topology and each T_n is $\sigma(X, X^{\odot})$-compact, then also T is $\sigma(X, X^{\odot})$-compact.*

Proof: Let (x_k) be a bounded sequence, say $||x_k|| \leq 1$ for all k. By Theorem 2.2.7 we must show that there is a subsequence (x_{k_i}) and a $y \in X$ such that $\langle x^{\odot}, Tx_{k_i} - y\rangle \to 0$ for all $x^{\odot} \in X^{\odot}$. Since each T_n is $\sigma(X, X^{\odot})$-compact, by Theorem 2.2.7 a diagonal argument produces a subsequence (x_{k_i}) such that for each n there is a $y_n \in X$ such that for all $x^{\odot} \in X^{\odot}$,

$$\lim_{i\to\infty} \langle x^{\odot}, T_n x_{k_i} - y_n\rangle = 0.$$

We claim that the sequence (y_n) is norm-Cauchy. Indeed, since for all i and $x^{\odot} \in X^{\odot}$ we have

$$|\langle x^{\odot}, y_n - y_m\rangle| \leq |\langle x^{\odot}, T_n x_{k_i} - T_m x_{k_i}\rangle| + |\langle x^{\odot}, T_n x_{k_i} - y_n\rangle| + |\langle x^{\odot}, T_m x_{k_i} - y_m\rangle|$$

it follows that for all $x^{\odot} \in X^{\odot}$,

$$|\langle x^{\odot}, y_n - y_m\rangle| \leq ||x^{\odot}|| ||T_n - T_m||.$$

But $||T_n - T_m|| \to 0$ as $n, m \to \infty$. Since $X^{\odot}$ induces an equivalent norm the claim follows. Let y be the norm-limit of (y_n) and fix some $x^{\odot} \in X^{\odot}$. Then for all n and i we have

$$|\langle x^{\odot}, Tx_{k_i} - y\rangle| \leq |\langle x^{\odot}, Tx_{k_i} - T_n x_{k_i}\rangle| + |\langle x^{\odot}, T_n x_{k_i} - y_n\rangle| + |\langle x^{\odot}, y_n - y\rangle|.$$

Let $\epsilon > 0$ be arbitrary. Then we may n_0 choose large enough such that for all i,

$$|\langle x^{\odot}, Tx_{k_i} - y\rangle| \leq |\langle x^{\odot}, T_{n_0} x_{k_i} - y_{n_0}\rangle| + 2\epsilon.$$

Hence

$$\limsup_{i\to\infty} |\langle x^{\odot}, Tx_{k_i} - y\rangle| \leq 2\epsilon$$

and the theorem is proved. ////

A different proof will be given in the next chapter.

If T and S are bounded operators with T $\sigma(X, X^{\odot})$-compact, then clearly TS is $\sigma(X, X^{\odot})$-compact. In other words, the $\sigma(X, X^{\odot})$-compact operators form a closed right ideal in the space of all bounded linear operators $\mathcal{L}(X)$. If S is $\sigma(X, X^{\odot})$-continuous, then ST is also $\sigma(X, X^{\odot})$-compact. These observations lead to the question whether the $\sigma(X, X^{\odot})$-compact operators form a two-sided ideal. The following simple example shows that this need not be the case.

Example 2.2.10. Let $X = X_1 \oplus X_2$, with $X_1 = X_2 = l^1$ and $||(x, y)|| = ||x||_{l^1} + ||y||_{l^1}$. Define $T(t) := T_1(t) \oplus I$, where the semigroup $T_1(t)$ on the first factor is given by $T_1(t)x_n := e^{-nt}x_n$. Here x_n is the nth unit vector of l^1. Then $X^{\odot} = X_1^{\odot} \oplus X_2^{\odot} = c_0 \oplus l^{\infty}$. Let $T : X \to X$ be the operator defined by

$$T((\alpha_1, \alpha_2, ...), (\beta_1, \beta_2, ...)) := ((\alpha_1, \beta_1, \alpha_2, \beta_2, ...), (0, 0, ...)).$$

Clearly $TB_X = B_{X_1}$. Now the $\sigma(X, X^{\odot})$-topology on X_1 is precisely the weak*-topology and consequently T is $\sigma(X, X^{\odot})$-compact. But T is left invertible with left inverse

$$S((\gamma_1, \gamma_2, ...), (\delta_1, \delta_2, ...) := ((\gamma_1, \gamma_3, ...), (\gamma_2, \gamma_4, ...)).$$

But $ST = I$ is certainly not $\sigma(X, X^{\odot})$-compact, since on X_2 the $\sigma(X, X^{\odot})$-topology agrees with the weak topology. This shows that the $\sigma(X, X^{\odot})$-compact operators do not form a left ideal.

2.3. The $||\cdot||'$-norm

In this section we study the equivalent norm $||\cdot||'$ introduced in Chapter 1. There we saw that $M^{-1}||\cdot|| \leq ||\cdot||' \leq ||\cdot||$, where $M \geq 1$ is any constant such that $||T(t)|| \leq Me^{\omega t}$ holds for some ω. In particular, if $T(t)$ is a contraction semigroup, then $||\cdot|| = ||\cdot||'$. The two norms always agree on $X^{\odot}$:

Proposition 2.3.1. *$||\cdot||$ and $||\cdot||'$ (the norm induced by $X^{\odot\odot}$ in $X^{\odot}$) agree on $X^{\odot}$.*

Proof: The original norm $||\cdot||$ on $X^{\odot}$ is obtained by norming with X but also by norming with $X^{\odot *}$. Therefore we have, noting that $||j|| \leq 1$,

$$\begin{aligned}||x^{\odot}|| = \sup_{x \in B_X} |\langle x^{\odot}, x\rangle| &\leq \sup_{jx \in B_{X^{\odot\odot}}} |\langle jx, x^{\odot}\rangle| \leq \sup_{x^{\odot\odot} \in B_{X^{\odot\odot}}} |\langle x^{\odot\odot}, x^{\odot}\rangle| \\ &= ||x^{\odot}||' \leq \sup_{x^{\odot *} \in B_{X^{\odot *}}} |\langle x^{\odot *}, x^{\odot}\rangle| = ||x^{\odot}||.\end{aligned}$$

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The following proposition gives a precise characterization of the semigroups for which the two norms agree. First we need some terminology. A subset G of Y is *circled* if $\lambda g \in G$ for all $g \in G$ and $|\lambda| \leq 1$. If G a subset of a locally convex space Y, then the *polar* of G is the set

$$G^{\bigcirc} := \{y^* \in Y^* : |\langle y^*, y\rangle| \leq 1, \quad \forall y \in G\}.$$

Similarly, if $H \subset Y^*$ then the polar of H is the set

$$H^{\bigcirc} := \{y \in Y : |\langle y^*, y\rangle| \leq 1, \quad \forall y^* \in H\}.$$

The *bipolar theorem* [S1, Thm. IV.1.5] asserts that $G^{\bigcirc\bigcirc} := (G^{\bigcirc})^{\bigcirc}$ is the convex, circled, $\sigma(Y, Y^*)$-closed hull of G, i.e., the smallest convex, circled, $\sigma(Y, Y^*)$-closed subset of Y which contains G.

Proposition 2.3.2. $||\cdot||' = ||\cdot||$ *if and only if* B_X *is* $\sigma(X, X^{\odot})$*-closed.*

Proof: It clearly suffices to show that $B_{(X,||\cdot||')} = \overline{B_X}^{\sigma(X,X^{\odot})}$. Consider B_X as a subset of the locally convex space $(X, \sigma(X, X^{\odot}))$. Writing out the definitions one sees that $B_X{}^{\bigcirc} = B_{X^{\odot}}$ and $B_X{}^{\bigcirc\bigcirc} = B_{(X,||\cdot||')}$. Hence the result is a consequence of the bipolar theorem. ////

The norms $||\cdot||$ and $||\cdot||'$ can indeed be different:

Example 2.3.3. Let x_n be the nth unit vector of c_0; put $y_n = \sum_{k=1}^{n} x_k$. It can easily be shown that $\{y_n\}_{n=1}^{\infty}$ is a Schauder basis for c_0 with basis constant 2. Often this basis is referred to as the *summing basis*. Define a semigroup $T(t)$ on c_0 by

$$T(t)y_n = e^{-(n-1)t}y_n.$$

By Theorem 1.5.2 this is a C_0-semigroup satisfying $||T(t)|| \leq 2$ for all $t \geq 0$. For later reference this semigroup will be called the *summing semigroup* on c_0.

We will show that for the summing semigroup on c_0 one has $||\cdot||' \neq ||\cdot||$ (In particular this implies that $c_0^{\odot} \neq l^1$). The claim is that

$$2y_1 = (2, 0, 0, 0, ...) \in \bigcap_{t>0} \overline{\bigcup_{0\leq s\leq t} T(s)B_{c_0}}.$$

In view of Theorem 2.1.5 this implies that $2y_1 \in \overline{B_{c_0}}^{\sigma(c_0,c_0^{\odot})}$, so in particular B_{c_0} is not $\sigma(c_0, c_0^{\odot})$-closed. Then we can apply Proposition 2.3.2. Indeed, put $z_k = 2y_1 - y_k$. Then $z_k \in B_{c_0}$ and for any $t > 0$ we have

$$\lim_{k\to\infty} T(t)z_k = \lim_{k\to\infty} (2y_1 - e^{-(k-1)t}y_k) = 2y_1.$$

The summing semigroup has the property that $\limsup_{t\downarrow 0} ||T(t)|| = 2$. We will now show that this is no coincidence.

Theorem 2.3.4. *If $T(t)$ is a C_0-semigroup on c_0 satisfying $||T(t)|| \leq Me^{\omega t}$ with $M < 2$, then $c_0^{\odot} = l^1$.*

Proof: It follows from the Laplace transform (0.5) that $\limsup_{t\downarrow 0} ||\lambda R(\lambda, A)|| \leq M$. Choose $\epsilon > 0$ such that $M - 1 + \epsilon < 1$. Let $x_0 = \sum_n \alpha_n e_n \in l^1$ be arbitrary (where e_n denotes the nth unit vector of l^1); $||x_0|| = 1$. Let N be such that $||\sum_{n=N+1}^{\infty} \alpha_n e_n|| < \epsilon/5$. Choose $\lambda_1 > 0$ so large that $||\lambda_1 R(\lambda_1, A^*)x_0|| \leq M + \epsilon/5$ and $(|\lambda_1 R(\lambda_1, A^*)x_0 - x_0)_n| \leq \epsilon/(5N)$ $(n = 1, 2, ..., N)$. Such λ_1 exists by the weak*-continuity of the map $\lambda \to \lambda R(\lambda, A^*)x_0$ and by the above limsup estimate. We have

$$\sum_{n=1}^{N} |(\lambda_1 R(\lambda_1, A^*)x_0)_n| \geq \sum_{n=1}^{N} |(x_0)_n| - \sum_{n=1}^{N} |(\lambda_1 R(\lambda_1, A^*)x_0 - x_0)_n|$$
$$\geq 1 - \frac{\epsilon}{5} - N \cdot \frac{\epsilon}{5N} = 1 - \frac{2\epsilon}{5}.$$

Therefore

$$\begin{aligned} ||x_0 - \lambda_1 R(\lambda_1, A^*)x_0|| &= \sum_{n=1}^{N} |(\lambda_1 R(\lambda_1, A^*)x_0 - x_0)_n| \\ &\quad + \sum_{n=N+1}^{\infty} |(\lambda_1 R(\lambda_1, A^*)x_0 - x_0)_n| \\ &\leq \frac{\epsilon}{5} + \sum_{n=N+1}^{\infty} |(\lambda_1 R(\lambda_1, A^*)x_0)_n| + \sum_{n=N+1}^{\infty} |(x_0)_n| \\ &\leq \frac{\epsilon}{5} + (||\lambda_1 R(\lambda_1, A^*)x_0|| - (1 - \frac{2\epsilon}{5})) + \frac{\epsilon}{5} \leq M - 1 + \epsilon. \end{aligned}$$

Put $x_1 = x_0 - \lambda_1 R(\lambda_1, A^*)x_0$. In the same way, there is an $\lambda_2 > 0$ such that

$$||x_1 - \lambda_2 R(\lambda_2, A^*)x_1|| \leq (M - 1 + \epsilon)||x_1|| \leq (M - 1 + \epsilon)^2.$$

Put $x_2 = x_1 - \lambda_2 R(\lambda_2, A^*)x_1$. Proceed with the construction inductively in the obvious way. After n steps, we have $\lambda_1, \lambda_2, ..., \lambda_n > 0$ and vectors $x_1, x_2, ..., x_n$ such that $x_n = x_{n-1} - \lambda_n R(\lambda_n, A^*)x_{n-1}$ and

$$\begin{aligned} ||x_0 - \lambda_1 R(\lambda_1, A^*)x_0 - \lambda_2 R(\lambda_2, A^*)x_1 - ... - \lambda_n R(\lambda_n, A^*)x_{n-1}|| \\ = ||x_{n-1} - \lambda_n R(\lambda_n, A^*)x_{n-1}|| \leq (M - 1 + \epsilon)^n \end{aligned}$$

But $\lambda_i R(\lambda_i, A^*)x_{i-1} \in (c_0)^{\odot}$ for all $i = 1, 2, ...$ Since $(M - 1 + \epsilon)^n \to 0$ as $n \to \infty$ we have proved that x_0 is in the closure of $(c_0)^{\odot}$. But $(c_0)^{\odot}$ is closed and therefore $x_0 \in (c_0)^{\odot}$. Hence $(c_0)^* = l^1 = (c_0)^{\odot}$, as was to be shown. ////

The standard unit vector basis of c_0 is shrinking. Of course, this basis has basis constant $C = 1$. By Zippin's Theorem 1.5.5 there exists a non-shrinking basis for c_0, since c_0 is non-reflexive.

Corollary 2.3.5. *Every non-shrinking basis of c_0 has basis constant $C \geq 2$.*

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be non-shrinking basis of c_0 with basis constant C. Let $T(t)$ be a C_0-semigroup as in Theorem 1.5.2. Then $||T(t)|| \leq C$ and $T^*(t)$ is not strongly continuous. Now by Theorem 2.3.4 we must have $C \geq 2$. ////

Theorem 2.3.4 and Corollary 2.3.5 are optimal, as is shown by the summing basis and the summing semigroup. Another example of a semigroup on c_0 with the properties that $\limsup_{t\downarrow 0} ||T(t)|| = 2$ and that its adjoint is not strongly continuous is the convolution semigroup discussed Section 6.2.

Notes. The material of Sections 2.1 and 2.2 is taken from [Ne4], except for Examples 2.1.7 and 2.2.10 which are new.

Instead of working with the sets G_t one could also work with $G_\lambda := \lambda R(\lambda, A)G$. For a simpler proof of Corollary 2.1.2 see [Ne1].

Eberlein-Shmulyan theorems can be proved for certain locally convex spaces. To be precise there is the following result [S1, Theorem IV.11.4]: *Let X be a locally convex space which is quasi-complete in its Mackey topology $\tau(X, X^*)$. Then each weakly closed and countably compact subset of X is weakly compact.* Recall that a subset G of X is *countably compact* if every countable subset of G has a cluster point. In particular every sequentially compact set is countably compact. Hence Theorem 2.2.7 would be a special case of this if one can prove that the Mackey topology $\tau(X, X^{\odot})$ is quasi-complete.

The norm $||\cdot||'$ was introduced by [HPh]. There it is shown that it is an equivalent norm and Proposition 2.3.1 is proved. Also it is shown that $||T||' = ||T^{\odot}||$ holds for every operator T with the property that $T^{\odot} := T^*|_{X^{\odot}}$ leaves $X^{\odot}$ invariant. A characterisation of operators leaving $X^{\odot}$ invariant is given in Chapter 3. Proposition 2.3.2 is taken from [Ne4], where an elementary proof avoiding the bipolar theorem is given. It is clear that 2.3.2 holds for more general subspaces of X^* than only $X^{\odot}$. Theorem 2.3.4 is taken from [Ne2], where a somewhat more complicated argument was used. Elementary as the present proof is, it does not give any clue why this result is true. After its publication, I found the reference [GS] giving the following full explanation. For a closed subspace Y of a dual Banach space X^* define the *characteristic* $\rho(Y)$ of Y by

$$\rho(Y) := \inf_{x \in X, ||x||=1} ||x||_Y,$$

where $||x||_Y := \sup_{y \in B_Y} |\langle y, x\rangle|$. In other words, we norm X with Y and ask how bad this norm is. If $\alpha||\cdot|| \leq ||\cdot||_Y \leq ||\cdot||$, then by definition $\rho(Y) \geq \alpha$. Also, if Y is not weak*-dense, then $\rho(Y) = 0$. For more properties of $\rho(Y)$ we refer to [vD1]. Now $X := c_0$ (and more generally every space X that is a so-called *M-ideal* in its bidual, see [HL] for the definition) is shown to have the following property: *If Y is any proper closed subspace of X^*, then $\rho(Y) \leq \frac{1}{2}$.* If $||T(t)|| \leq Me^{\omega t}$ with $M < 2$, then for all $x \in c_0$ we have

$$\frac{1}{2}||x|| < \frac{1}{M}||x|| \leq ||x||' = ||x||_{c_0^{\odot}}$$

and it follows immediately that $c_0^{\odot}$ cannot be a *proper* subspace of l^1.

The space c_0 has some more remarkable features with respect to adjoint semigroups. For convenience, we list them here:

(i) If $T(t)$ is C_0 on c_0 such that $||T(t)|| \leq Me^{\omega t}$ with $M < 2$, then $c_0^{\odot} = l^1$ (Theorem 2.3.4) Moreover, the constant 2 is optimal (Example 2.3.3). In fact, with respect to the summing semigroup, $c_0^*/c_0^{\odot}$ is one-dimensional (Proposition 6.2.1).

(ii) If $T(t)$ is a positive C_0-semigroup on c_0, then $c_0^{\odot} = l^1$ (Corollary 8.1.9).

(iii) If $T(t)$ is a C_0-group on c_0, then $c_0^{\odot} = l^1$ (Corollary 6.2.6).

(iv) The Favard class of a C_0-semigroup on c_0 equals $D(A)$ if and only if A is bounded (Corollary 4.2.4).

Result (iv) seems to be related to Lotz's theorem, which applies in the bidual $l^\infty = c_0^{**}$.

Corollary 2.3.5 was proved first by Godun [Gd] for a somewhat broader class of spaces, and, unaware of that reference, I obtained it independently in [Ne2].

Chapter 3

$\odot$-*Reflexivity*

In this chapter we study the concept of $\odot$-reflexivity, introduced in Chapter 1. In Section 3.1 we characterize $\sigma(X, X^\odot)$-compact maps. In Section 3.2 various characterizations of $\odot$-reflexivity are given.

3.1. $\sigma(X, X^\odot)$-compact maps

Gantmacher's theorem asserts that a bounded operator T is weakly compact if and only if $T^{**}X^{**} \subset X$. We will prove an analogous statement for the $\sigma(X, X^\odot)$-topology. A bounded operator T is $\sigma(X, X^\odot)$-*compact* if TB_X is relatively $\sigma(X, X^\odot)$-compact.

We start by characterizing the operators which are continuous in the $\sigma(X, X^\odot)$-topology.

Proposition 3.1.1. *A bounded operator T on X is $\sigma(X, X^\odot)$-continuous if and only $T^*(X^\odot) \subset X^\odot$.*

Proof: Suppose $T^*(X^\odot) \subset X^\odot$. Choose $x^\odot \in X^\odot$ and $\epsilon > 0$ arbitrary and let $V(x^\odot; \epsilon)$ be as at the beginning of Section 2.1. Then

$$\begin{aligned} T^{-1}V(x^\odot; \epsilon) &= \{x \in X : \ |\langle x^\odot, Tx\rangle| < \epsilon\} \\ &= \{x \in X : \ |\langle T^*x^\odot, x\rangle| < \epsilon\} = V(T^*x^\odot; \epsilon), \end{aligned}$$

which is a $\sigma(X, X^\odot)$-open set. It follows that T is $\sigma(X, X^\odot)$-continuous.

Conversely, suppose $x^* = T^*x^\odot \notin X^\odot$ for some $x^\odot \in X^\odot$. Fix $0 < \epsilon < 1$ arbitrary. We claim that $T^{-1}V(x^\odot; \epsilon)$ is not $\sigma(X, X^\odot)$-open. For this it is clearly sufficient to show that for all finite collections $x_1^\odot, ..., x_n^\odot \in X^\odot$ and $\delta > 0$ we have

$$V(x_1^\odot, ..., x_n^\odot; \delta) := \cap_{i=1}^n V(x_i^\odot; \delta) \not\subset T^{-1}V(x^\odot; \epsilon).$$

Suppose the contrary. Then some $V(x_1^\odot, ..., x_n^\odot; \delta)$ is contained in $T^{-1}V(x^\odot; \epsilon)$. We may assume that $1 - \delta > \epsilon$. Since $x^* \notin X^\odot$, we may choose $x^{**} \in X^{**}$

such that $\langle x^{**}, x_i^{\odot}\rangle = 0$ for all i and $\langle x^{**}, x^*\rangle = 1$. Since X is weak*-dense in X^{**} there is an $x \in X$ satisfying $|\langle x_i^{\odot}, x\rangle| < \delta$ for all i and $|\langle x^*, x\rangle| > 1-\delta > \epsilon$. Then $x \in V(x_1^{\odot}, ..., x_n^{\odot}; \delta)$, but

$$|\langle x^{\odot}, Tx\rangle| = |\langle x^*, x\rangle| > \epsilon,$$

which shows that $x \notin T^{-1}V(x^{\odot}; \epsilon)$, a contradiction. ////

For a $\sigma(X, X^{\odot})$-continuous operator T we denote the restriction of T^* to $X^{\odot}$ by $T^{\odot}$. Let $r : X^{**} \to X^{\odot *}$ be the natural restriction map, given by $\langle rx^{**}, x^{\odot}\rangle := \langle x^{**}, ix^{\odot}\rangle$, where $i : X^{\odot} \to X^*$ is the inclusion map.

Theorem 3.1.2. *A $\sigma(X, X^{\odot})$-continuous operator T on X is $\sigma(X, X^{\odot})$-compact if and only if $T^{\odot *}X^{\odot *} \subset jX$.*

Proof: Suppose $T^{\odot *}X^{\odot *} \subset jX$. The set $K := T^{\odot *}B_{X^{\odot *}}$ is a weak*-compact subset of $jX \subset X^{\odot *}$, and $jTB_X = T^{\odot *}jB_X \subset K$. Hence the weak*-closure in $X^{\odot *}$ of jTB_X belongs to weak*-compact subset K of jX. Since the natural map $j : (X, \sigma(X, X^{\odot})) \to (jX, weak^*)$ is a homeomorphism it follows that the $\sigma(X, X^{\odot})$-closure of TB_X is $\sigma(X, X^{\odot})$-compact.

Conversely, suppose that T is $\sigma(X, X^{\odot})$-compact. On the one hand, jTB_X is weak*-dense in $\overline{jTB_X}^{weak^*}$, hence certainly $\overline{jTB_X}^{\sigma(X,X^{\odot})}$ is weak*-dense in $\overline{jTB_X}^{weak^*}$. But on the other hand $\overline{jTB_X}^{\sigma(X,X^{\odot})}$ is weak*-compact since $j : (X, \sigma(X, X^{\odot})) \to (X^{\odot *}, weak^*)$ is continuous. It follows that

$$\overline{j(TB_X)}^{\sigma(X,X^{\odot})} = \overline{jTB_X}^{weak^*}.$$

Now B_X is $\sigma(X, X^{\odot})$-dense in $B_{(X,\|\cdot\|')}$ $(= \overline{B_X}^{\sigma(X,X^{\odot})}$ by Proposition 2.3.2). By Goldstine's theorem, in turn $iB_{(X,\|\cdot\|')}$ is weak*-dense in $B_{(X,\|\cdot\|')^{**}}$, where $i : (X, \|\cdot\|') \to (X, \|\cdot\|')^{**}$ is the natural embedding. Hence iB_X is weak*-dense in $B_{(X,\|\cdot\|')^{**}}$. Since

$$riB_X = jB_X$$

it follows that jB_X is weak*-dense in $B_{X^{\odot *}} = rB_{(X,\|\cdot\|')^{**}}$. Hence we obtain

$$\begin{aligned} T^{\odot *}B_{X^{\odot *}} &= T^{\odot *}\overline{jB_X}^{weak^*} \subset \overline{T^{\odot *}jB_X}^{weak^*} \\ &= \overline{jTB_X}^{weak^*} = \overline{j(TB_X)}^{\sigma(X,X^{\odot})} \subset jX. \end{aligned}$$

////

Remark 3.1.3. With almost the same proof one shows that an arbitrary bounded operator T is $\sigma(X, X^{\odot})$-compact if and only if $r(T^{**}X^{**}) \subset jX$. In this way one obtains a new proof of Theorem 2.2.9.

Theorem 3.1.2 will be applied mostly to operators of the following class.

Proposition 3.1.4. *If T commutes with $T(t)$, then T is $\sigma(X, X^{\odot})$-continuous.*

Proof: By formula (0.5) T commutes with $R(\lambda, A)$. Hence if $x^* = R(\lambda, A^*)y^* \in D(A)^*$ we have $T^*x^* = R(\lambda, A^*)T^*y^* \in D(A^*)$. Hence T^* leaves $D(A^*)$, and therefore also $X^{\odot}$, invariant. ////

3.2. $\odot$-reflexivity

Recall that the Banach space X is said to be $\odot$-*reflexive* with respect to a C_0-semigroup $T(t)$ if $jX = X^{\odot\odot}$. If no confusion is possible we will just say that X is $\odot$-reflexive or even that $T(t)$ is $\odot$-reflexive. We encountered already a class of $\odot$-reflexive semigroups, viz. the multiplication semigroups on Schauder bases from Theorem 1.5.2. Also the rotation semigroup on $C(T)$ is $\odot$-reflexive: for $C(T)^{\odot} = L^1(T)$ and $C(T)^{\odot\odot} = C(T)$ by Example 1.3.9. The translation semigroup on $C_0(\mathbb{R})$ is not $\odot$-reflexive since $C_0(\mathbb{R})^{\odot\odot} = BUC(\mathbb{R})$.

Let us observe that it is *not* true that X is $\odot$-reflexive with respect to $T(t)$ if and only if B_X is (relatively) $\sigma(X, X^{\odot})$-compact, as one might hope. A simple counterexample is the rotation semigroup on $C(T)$. The reason why this is not true is simple. After renorming X with $||\cdot||'$ it is obvious that B_X is $\sigma(X, X^{\odot})$-compact if and only if the identity map on X is $\sigma(X, X^{\odot})$-compact. By Theorem 3.1.2 this is the case if and only if $jX = X^{\odot *}$. Note however that this argument shows the following.

Proposition 3.2.1. *If B_X is relatively $\sigma(X, X^{\odot})$-compact, then X is $\odot$-reflexive.*

For those who insist that any reasonable concept of reflexivity should be characterized by compactness of some unit ball we just mention the fact that X is $\odot$-reflexive if and only if B_{X^*} is $\sigma(X^*, kX^{\odot\odot})$-compact. Here k is a natural embedding of $X^{\odot\odot}$ into X^{**}, to be defined in Chapter 5. The following theorem gives a more useful characterization of $\odot$-reflexivity, due to Hille-Phillips and de Pagter.

Theorem 3.2.2. *The following are equivalent:*

(i) X is $\odot$-reflexive with respect to $T(t)$;
(ii) $R(\lambda, A)$ is weakly compact;
(iii) $R(\lambda, A)$ is $\sigma(X, X^{\odot})$-compact.

Proof: Assume (i). By Gantmacher's theorem we have to show that $R(\lambda, A)^{**}$ maps X^{**} into X. To this end fix $x^{**} \in X^{**}$. Then we have for all $x^* \in X^*$

$$\begin{aligned}\langle R(\lambda, A)^{**}x^{**}, x^*\rangle &= \langle x^{**}, R(\lambda, A^*)x^*\rangle = \langle rx^{**}, R(\lambda, A^*)x^*\rangle \\ &= \lim_{\mu\to\infty} \langle rx^{**}, \mu R(\mu, A^*)R(\lambda, A^*)x^*\rangle \\ &= \lim_{\mu\to\infty} \langle rx^{**}, R(\lambda, A^{\odot})\mu R(\mu, A^*)x^*\rangle \\ &= \lim_{\mu\to\infty} \langle R(\lambda, A^{\odot *})rx^{**}, \mu R(\mu, A^*)x^*\rangle \\ &= \langle R(\lambda, A^{\odot *})rx^{**}, x^*\rangle.\end{aligned}$$

The last identity follows from the fact that $R(\lambda, A^{\odot *})rx^{**} \in D(A^{\odot *}) \subset X^{\odot\odot} = jX$ and that weak*-$\lim_{\mu\to\infty} \mu R(\mu, A^*)x^* = x^*$. Therefore $R(\lambda, A)^{**}x^{**} = R(\lambda, A^{\odot *})rx^{**} \in X$.

The implication (ii)$\Rightarrow$(iii) is trivial.

Assume (iii). By Theorem 3.1.2 we have $\overline{D(A^{\odot *})} = \overline{R(\lambda, A^{\odot *})X^{\odot *}} \subset jX$. Since jX is closed in $X^{\odot *}$ it follows that $X^{\odot\odot} = \overline{D(A^{\odot *})} \subset jX$. ////

Note that the equivalence of (ii) and (iii) also follows at once from Corollary 2.2.4 and Proposition 2.2.6.

Remark 3.2.3. It follows from the resolvent identity that if $R(\lambda, A)$ is weakly compact for one $\lambda \in \varrho(A)$, then $R(\lambda, A)$ is weakly compact for all $\lambda \in \varrho(A)$. The same remark applies to the $\sigma(X, X^{\odot})$-compactness of $R(\lambda, A)$.

Recall from Section 1.5 that a Banach space X has the Dunford-Pettis property if every weakly compact operator on X takes relatively weakly compact sets into relatively compact sets. In particular, the square of a weakly compact operator on X is compact.

Corollary 3.2.4. *If X has the Dunford-Pettis property, then X is $\odot$-reflexive if and only if $R(\lambda, A)$ is compact.*

Proof: If $R(\lambda, A)$ is compact, then it is weakly compact and X is $\odot$-reflexive. Conversely, if X is $\odot$-reflexive, then $R(\lambda, A)$ is weakly compact and hence $R(\lambda, A)^2$ is compact. But it follows easily from the resolvent identity that

$$\lim_{\lambda\to\infty} ||(\lambda R(\lambda, A))^2 R(\mu, A) - R(\mu, A)|| = 0.$$

Since the ideal of compact operators is closed, it follows that $R(\mu, A)$ is compact for all $\mu \in \varrho(A)$. ////

Theorem 3.2.2 has a number of consequences. The first is that X is $\odot$-reflexive if and only if the *integrated semigroup* is weakly compact. For $t > 0$ define bounded operators S_t on X by $S_t x := \int_0^t T(\sigma)x \, d\sigma$.

Corollary 3.2.5. *The following are equivalent:*
(i) X is $\odot$-reflexive;
(ii) S_t is weakly compact;
(iii) S_t is $\sigma(X, X^\odot)$-compact.

Proof: Suppose X is $\odot$-reflexive. The formula

$$S_t x = \int_0^t T(\sigma)x\, d\sigma = R(\lambda, A)\Big(\lambda \int_0^t T(\sigma)x\, d\sigma - (T(t)x - x)\Big)$$

shows that $S_t B_X$ is contained in some multiple of the relatively weakly compact set $R(\lambda, A)B_X$. This proves that (i) implies (ii). The implication (ii)$\Rightarrow$(iii) is trivial. Finally assume (iii). For $x^* \in X^*$ we have

$$S_t^* x^* = weak^* \int_0^t T^*(\sigma)x^*\, d\sigma.$$

For $x^\odot \in X^\odot$ the integrand is norm continuous, so we have

$$S_t^\odot x^\odot = S_t^* x^\odot = \int_0^t T^*(\sigma)x^\odot\, d\sigma.$$

Hence

$$S_t^{\odot *} x^{\odot *} = weak^* \int_0^t T^{\odot *}(\sigma)x^{\odot *}\, d\sigma.$$

If S_t is $\sigma(X, X^\odot)$-compact then by Theorem 3.1.2 we have $S_t^{\odot *} X^{\odot *} \subset jX$. But for $x^{\odot\odot} \in X^{\odot\odot}$ we have $\lim_{t\downarrow 0} \frac{1}{t} S_t^{\odot *} x^{\odot\odot} = x^{\odot\odot}$ in norm, and it follows that $x^{\odot\odot}$ lies the the norm closure of jX. ////

Note that the equivalence of (ii) and (iii) also follows from Corollary 2.2.4 and the fact that $S_t B_X$ is contained in a multiple of the weakly equicontinuous set $R(\lambda, A)B_X$.

In general it is not true that $T(t)$ is $\odot$-reflexive if and only if $T(t)$ is weakly compact for $t > 0$; a counterexample is rotation on $C(\mathbf{T})$. If $T(t)$ is uniformly continuous for $t > 0$ however, e.g. if $T(t)$ is a compact or a holomorphic semigroup, then $T(t)$ is $\odot$-reflexive if and only if $T(t)$ is weakly compact for $t > 0$. This will be proved in Chapter 5. In order to obtain the same conclusion, one can also impose structural properties on the Banach space X, e.g. quasi-reflexivity. This will be done in Chapter 6. Recall that X is *quasi-reflexive* if X^{**}/X is finite-dimensional. An example of a quasi-reflexive space is James's space J from Section 1.5.

The next two corollaries are concerned with $\odot$-reflexivity of induced semigroups.

Let $S(t)$ be a semigroup on X, strongly continuous or not, and let Y be a closed, $S(t)$-invariant subspace. Let $q : X \to X/Y$ be the quotient map assigning to each $x \in X$ its coset $qx := x + Y$ in X/Y. On X/Y we define the operators $S_q(t)$ by

$$S_q(t)qx := q\big(S(t)x\big).$$

These operators are well-defined since Y is $S(t)$-invariant and form a semigroup on X/Y. If $T(t)$ is strongly continuous, then also $T_q(t)$ is strongly continuous. In that case the generator A_q is given by

$$D(A_q) = q\big(D(A)\big);$$
$$A_q(qx) = q(Ax)$$

and the resolvent satisfies $q(R(\lambda, A)x) = R(\lambda, A_q)qx$.

Corollary 3.2.6. *Let Y be a closed, $T(t)$-invariant subspace of a $\odot$-reflexive space X. Then both Y and X/Y are $\odot$-reflexive (with respect to the restricted semigroup and the quotient semigroup respectively).*

Proof: The resolvent of the restricted semigroup is the restriction of the resolvent, and the restriction of a weakly compact map is weakly compact. Hence Y is $\odot$-reflexive. Since $q(R(\lambda, A)x) = R(\lambda, A_q)qx$ and since $B_{X/Y}$ is contained in some multiple of qB_X (since q is open), also $R(\lambda, A_q)$ is weakly compact. ////

Remark 3.2.7. The converse is also true. The proof depends on the Hahn-Banach Theorem 6.1.1 and therefore it will be postponed to Chapter 6.

Corollary 3.2.8. *X is $\odot$-reflexive if and only if $X^{\odot}$ is $\odot$-reflexive (with respect to $T^{\odot}(t)$).*

Proof: If $R(\lambda, A)$ is weakly compact, then also $R(\lambda, A^*)$ is weakly compact, and hence also $R(\lambda, A^{\odot}) = R(\lambda, A)^{\odot}$. Conversely, if $X^{\odot}$ is $\odot$-reflexive, then by the above $X^{\odot\odot}$ is $\odot$-reflexive, and hence so is the closed $T^{\odot\odot}(t)$-invariant subspace jX. ////

A Banach space is said to be *weakly compactly generated* (WCG) if it is the closed linear span of one of its weakly compact subsets. Since

$$X = \overline{D(A)} = \overline{\text{linspan } R(\lambda, A)B_X},$$

each $\odot$-reflexive Banach space is necessarily WCG. The following consequence of Theorem 3.2.2 will be important in the context of positive semigroups.

Corollary 3.2.9. *Suppose a Banach space X is $\odot$-reflexive with respect to a C_0-semigroup. Then X does not contain a closed subspace isomorphic to l^∞.*

Proof: Suppose the contrary and let Y be a closed subspace of X which is isomorphic to l^∞. Since l^∞ is complemented in every Banach space containing it as a closed subspace [LT, Prop. I.2.f.2], it follows that Y is complemented in X. Since X is WCG and complemented subspaces of WCG spaces are trivially seen to be WCG again, we conclude that l^∞ is WCG, a contradiction. In fact, every weakly compact set of l^∞ is separable (e.g. note that l^∞ embeds into $L^\infty[0,1]$ and apply [DU, Thm. VIII.4.13]). ////

Notes. Lemma 3.1.1 is elementary, holds in more generality and is undoubtedly well-known. Theorem 3.1.2 is straightforward generalization of Gantmacher's theorem. Proposition 3.2.1 is from [Ne1]; more complete information is provided by Theorem 4.2.5 below. The characterization Theorem 3.2.2 is due to Hille-Phillips [HPh], who prove (i)$\Leftrightarrow$(iii) and de Pagter [Pa2], who proves (i)$\Rightarrow$ (ii). The simple proof of (i)$\Rightarrow$ (ii) is new. See also [Ne1], where a more complicated proof is given of which the present one is a simplification. Corollary 3.2.4 is from [Pa2]. Corollary 3.2.5 is from [Ne4] where a different proof is given, and the first part of Corollary 3.2.6 is from [Ne1]. Corollary 3.2.8 is from [HPh] and finally Corollary 3.2.9 was proved in [NP].

Chapter 4

The Favard class

In this chapter we study the Favard class of a semigroup on X. This is the subspace of all $x \in X$ whose orbits are locally Lipschitz continuous. The theory of the Favard class will be seen to be intimately related to duality.

In Section 4.1 we give a characterization of the Favard class of a C_0-semigroup and show that $\text{Fav}(T(t)) = D(A)$ is X is reflexive. In Section 4.2 we characterize the semigroups for which $\text{Fav}(T(t)) = D(A)$ holds.

4.1. The Favard class

Let $S(t)$ be a semigroup on X, strongly continuous or not. Define its *Favard class* by

$$\text{Fav}(S(t)) := \{x \in X : \limsup_{t \downarrow 0} \frac{1}{t} \|S(t)x - x\| < \infty\}.$$

In other words, $\text{Fav}(S(t))$ consists of those x whose orbits are Lipschitz continuous in a neighbourhood of $t = 0$. If $T(t)$ is a C_0-semigroup, then Corollary 4.1.7 below shows that the limes superior in this definition can be replaced by a limes inferior.

We start with a simple inclusion for C_0-semigroups.

Proposition 4.1.1. *$D(A) \subset \text{Fav}(T(t))$.*

Proof: For $0 < t \leq 1$ and $x \in D(A)$ we have

$$\frac{1}{t}\|T(t)x - x\| = \frac{1}{t}\left\| \int_0^t T(\sigma)Ax \, d\sigma \right\| \leq \Big(\sup_{0<\sigma\leq 1} \|T(\sigma)\| \Big) \|Ax\|.$$

////

For the adjoint of a C_0-semigroup we have the following nice result.

Theorem 4.1.2. *Fav*$(T^*(t)) = D(A^*) = \{x^* \in X^* : \liminf_{t\downarrow 0} t^{-1}\|T^*(t)x^* - x^*\| < \infty\}$.

Proof: From Proposition 1.2.2 and an estimate as in Proposition 4.1.1 it is clear that $D(A^*) \subset \text{Fav}(T^*(t))$. If $x^* \in \text{Fav}(T^*(t))$, then trivially the above limes inferior is finite. Finally let $x^* \in X^*$ be such that $\liminf_{t\downarrow 0} t^{-1}\|T^*(t)x^* - x^*\| < \infty$. This means that there is a sequence $t_j \downarrow 0$ and a finite constant C such that

$$\frac{1}{t_j}\|T^*(t_j)x^* - x^*\| \leq C, \quad \forall j.$$

Define a linear form y^* on $D(A)$ by

$$y^*(x) := \langle x^*, Ax\rangle, \qquad \forall x \in D(A).$$

From

$$\begin{aligned} |y^*(x)| = |\langle x^*, Ax\rangle| &= |\lim_{j\to\infty} \frac{1}{t_j}\langle x^*, T(t_j)x - x\rangle| \\ &= |\lim_{j\to\infty} \frac{1}{t_j}\langle T^*(t_j)x^* - x^*, x\rangle| \leq C\|x\| \end{aligned}$$

it follows that y^* is bounded. Hence by definition of A^* we have $x^* \in D(A^*)$, and $A^*x^* = y^*$. ////

Corollary 4.1.3. *Fav*$(T^\odot(t)) = D(A^*)$.

Proof: If $x^\odot \in \text{Fav}(T^\odot(t))$, then $x^\odot \in \text{Fav}(T^*(t))$ since $T^*(t)x^\odot = T^\odot(t)x^\odot$. Hence $x^\odot \in D(A^*)$. Conversely, if $x^* \in D(A^*)$, then $x^* \in X^\odot$, so $x^* \in \text{Fav}(T^\odot(t))$. ////

With this result we can describe the Favard class of $T(t)$.

Theorem 4.1.4. *Fav*$(T(t)) = D(A^{\odot *}) \cap X$.

Proof: From Corollary 4.1.3 we infer that $\text{Fav}(T^{\odot\odot}(t)) = D(A^{\odot *})$. But it is a trivial consequence of the definition of the Favard class that

$$\text{Fav}(T(t)) = \text{Fav}(T^{\odot\odot}(t)) \cap X,$$

where X is identified with its image jX in $X^{\odot *}$. ////

Corollary 4.1.5. *If X is $\odot$-reflexive, then Fav$(T(t)) = D(A^{\odot *})$.*

Corollary 4.1.6. *If X is reflexive, then Fav$(T(t)) = D(A)$.*

We will now show that the limes superior in the definition of the Favard class can be replaced by a limes inferior if the semigroup is C_0.

Corollary 4.1.7. *The following assertions are equivalent:*

(i) $x \in \text{Fav}(T(t))$;

(ii) $\liminf_{t\downarrow 0} \frac{1}{t}\|T(t)x - x\| < \infty$.

Proof: Suppose $\liminf_{t\downarrow 0} \frac{1}{t}\|T(t)x - x\| < \infty$. Applying Theorem 4.1.2 to the semigroup $T^\odot(t)$ gives that $jx \in D(A^{\odot *})$. Hence $x \in \text{Fav}(T(t))$ by Theorem 4.1.4. ////

4.2. When is Fav($T(t)$)= $D(A)$?

In Section 4.1 we saw that $\text{Fav}(T(t)) = D(A)$ if X is reflexive. In this section we will give a characterization of those semigroups for which $\text{Fav}(T(t)) = D(A)$ holds.

We start with a lemma, in which we identify X and jX. Recall from Section 1.3 that we defined $\|x\|' = \sup_{x^\odot \in B_{X^\odot}} |\langle x^\odot, x\rangle|$.

Lemma 4.2.1. *We have the following inclusions:*

$$\overline{R(\lambda, A)B_{(X,\|\cdot\|')}} \subset R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X \subset \bigcup_{n\in\mathbb{N}} n \cdot \overline{R(\lambda, A)B_{(X,\|\cdot\|')}}.$$

Proof: Since $B_{X^{\odot *}}$ is weak*-compact, $R(\lambda, A^{\odot *})B_{X^{\odot *}}$ is also weak*-compact, so in particular norm-closed. The first inclusion now follows easily from the fact that $j : X \to X^{\odot *}$ is an isometry from $(X, \|\cdot\|')$ into $X^{\odot *}$. The second inclusion follows from the equality

$$\frac{1}{t}\int_0^t T(\tau)x\, d\tau = R(\lambda, A)\Big(\frac{\lambda}{t}\int_0^t T(\tau)x\, d\tau - \frac{1}{t}(T(t)x - x)\Big).$$

Indeed, we have $R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X \subset D(A^{\odot *}) \cap X = \text{Fav}(T(t))$. Therefore if $x \in R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X$, then $\frac{\lambda}{t}\int_0^t T(\tau)x\, d\tau - \frac{1}{t}(T(t)x - x)$ remains bounded as $t \downarrow 0$ whereas the left hand side converges to x. ////

Since $B_X \subset B_{(X,\|\cdot\|')} \subset MB_X$ for some M, one may replace $\|\cdot\|'$ by $\|\cdot\|$ in Lemma 4.2.1. The reason for working with $\|\cdot\|'$ becomes apparent in the proof of Theorem 4.2.3.

The following theorem characterizes the Favard class in terms of approximation by elements of $D(A)$.

Theorem 4.2.2. *The following assertions are equivalent:*

(i) $x \in \text{Fav}(T(t))$;

(ii) There exists a bounded sequence $(y_n) \subset X$ *such that* $\lim_n R(\lambda, A)y_n = x$;

(iii) There exists a bounded sequence $(y_n) \subset X$ *and an integer* $n \in \mathbb{N}$ *such that* $\lim_n R(\lambda, A)^{n+1}y_n = R(\lambda, A)^n x$.

Proof: The implication (i)⇒(ii) is immediate from the above lemma. (ii)⇒(iii) is trivial.

(iii)⇒(i): If $n = 0$ then (i) follows from Lemma 4.2.1. Therefore, suppose $n > 0$. From (iii) it follows that for any $x^\odot \in X^\odot$ we have

$$\lim_{n\to\infty} \langle R(\lambda, A^*)x^\odot, R(\lambda, A)^n y_n\rangle = \langle R(\lambda, A^*)x^\odot, R(\lambda, A)^{n-1}x\rangle.$$

By a density argument this implies that $R(\lambda, A)^n y_n \to R(\lambda, A)^{n-1}x$ in the $\sigma(X, X^\odot)$-topology. Repeating this argument it follows that $R(\lambda, A)y_n \to x$ in the $\sigma(X, X^\odot)$-topology. Therefore x belongs to the $\sigma(X, X^\odot)$-closure of $K R(\lambda, A)B_X$ for some constant K, which by the results of Section 2.2 is equal to the norm-closure of $K R(\lambda, A)B_X$. Hence $x \in \mathrm{Fav}(T(t))$ by Lemma 4.2.1. ////

Note that (ii) is equivalent to: there is a sequence $(x_n) \subset D(A)$ such that $x_n \to x$ and $\sup_n \|Ax\| < \infty$.

Theorem 4.2.3. *Fav*$(T(t)) = D(A)$ *if and only if* $R(\lambda, A)B_{(X,\|\cdot\|')}$ *is norm-closed.*

Proof: Suppose first that $\mathrm{Fav}(T(t)) = D(A)$. Let $y \in \overline{R(\lambda, A)B_{(X,\|\cdot\|')}}$, that is, $y = R(\lambda, A^{\odot *})x^{\odot *}$ for some $x^{\odot *} \in B_{X^{\odot *}}$, using Lemma 4.2.1. Since $\mathrm{Fav}(T(t)) = D(A^{\odot *}) \cap X$ by Theorem 4.1.4, $y \in \mathrm{Fav}(T(t))$ and hence by assumption there is an $x \in X$ such that $y = R(\lambda, A)x$. But $R(\lambda, A)x = R(\lambda, A^{\odot *})jx$ and since $R(\lambda, A^{\odot *})$ is injective, we have $jx = x^{\odot *}$. But j i- an isometry from $B_{(X,\|\cdot\|')}$ into $B_{X^{\odot *}}$ which forces $x \in B_{(X,\|\cdot\|')}$. Hence $y \in R(\lambda, A)B_{(X,\|\cdot\|')}$ as was to be shown.

Conversely, if $R(\lambda, A)B_{(X,\|\cdot\|')}$ is closed, then by Lemma 4.2.1 we have

$$R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X \subset \bigcup_{n \in \mathbb{N}} n \cdot R(\lambda, A)B_{(X,\|\cdot\|')} = D(A).$$

Since $\mathrm{Fav}(T(t)) = D(A^{\odot *}) \cap X$ it follows that $\mathrm{Fav}(T(t)) \subset D(A)$, as was to be shown. ////

Let $T(t)$ be a contraction semigroup on $X = l^1$ as in Example 1.3.10. Since $X^\odot = c_0$ and $X = X^{\odot *} = l^1$ we have $R(\lambda, A) = R(\lambda, A^{\odot *})$, so in particular $R(\lambda, A)$ is weak*-continuous. Since B_{l^1} is weak*-compact, so is $R(\lambda, A)B_{l^1}$. Hence $R(\lambda, A)B_{l^1}$ is closed. By Theorem 4.2.3 we have $\mathrm{Fav}(T(t)) = D(A)$ for this semigroup, although l^1 is non-reflexive.

Another class of semigroups where $\mathrm{Fav}(T(t)) = D(A)$ (trivially) holds is given by the uniformly continuous semigroups. On c_0 these are the *only* semigroups with this property, so in a sense the very opposite of Corollary 4.1.6 is true in this space. For the proof we need two results of Bourgain and Rosenthal [BR]. A bounded operator $T : X \to Y$ is called a *G_δ-embedding* if T is injective and TK is a G_δ set for all closed bounded K in X. T is called a *semi-embedding* if T is injective and TB_X is closed. (i) *If X is isomorphic to c_0 and $T : X \to Y$ is a G_δ-embedding, then T is an isomorphism into.* (ii) *If T is a semi-embedding and X is separable, then T is a G_δ-embedding.* By combining this with Theorem 4.2.3 we find:

Corollary 4.2.4. *If $T(t)$ is a C_0-semigroup on c_0 with the property that* *Fav*$(T(t)) = D(A)$, *then $T(t)$ is uniformly continuous.*

For then $R(\lambda, A)$ is an isomorphism.

One might ask whether the $||\cdot||'$-norm is essential in Theorem 4.2.3. For the 'if' part it is not: if $R(\lambda, A)B_X$ is closed, then $\mathrm{Fav}(T(t)) = D(A)$. This follows from the proof of Theorem 4.2.3 and the remark preceding it. The 'only if' part of Theorem 4.2.3 is not true for the $||\cdot||$-norm. This is not surprising since the closedness of $R(\lambda, A)B_X$ is an isometrical property, whereas the equality $\mathrm{Fav}(T(t)) = D(A)$ is an isomorphical property. The following counterexample is based on this observation.

Example 4.2.5. Let $T(t)$ be the semigroup on $X = l^1$ defined after Theorem 4.2.3. As we noted, for this semigroup we have $\mathrm{Fav}(T(t)) = D(A)$. Put

$$x_0^* := (1, 1, 1, ...) \in l^\infty$$

and define on l^1 an equivalent norm by

$$|x| := \frac{1}{2}||x|| + |\langle x_0^*, x\rangle|.$$

Since

$$\frac{1}{2}||x|| \le |x| \le \frac{1}{2}||x|| + ||x_0^*||\ ||x|| = \frac{3}{2}||x||$$

this is indeed an equivalent norm. We claim that $R(\lambda, A)B_{(l^1,|\cdot|)}$ is not closed. Indeed, let $y_1 := (1, 0, 0, ...)$ and for $n \ge 2$ put

$$y_n := (1, 0, 0, ..., 0, -1, 0, ...)$$

with the -1 occurring on the nth coordinate. Clearly, as $n \to \infty$ we have

$$R(\lambda, A)y_n = (\frac{1}{\lambda + 1}, 0, ..., 0, -\frac{1}{\lambda + n}, 0, ..) \to (\frac{1}{\lambda + 1}, 0, ...) = R(\lambda, A)y_1$$

in the $|\cdot|$-norm. But for each $n \ge 2$ we have

$$|y_n| = \frac{1}{2} \cdot 2 + 0 = 1,$$

so $R(\lambda, A)y_n \in R(\lambda, A)B_{(l^1,|\cdot|)}$, whereas

$$|y_1| = \frac{1}{2} \cdot 1 + 1 = \frac{3}{2},$$

so $R(\lambda, A)y_1 \notin R(\lambda, A)B_{(l^1,|\cdot|)}$.

Note however that $||\cdot||'$ may be replaced by $||\cdot||$ in Theorem 4.2.3 if $T(t)$ is a contraction semigroup or if $T(t) = S^{\odot}(t)$ for a C_0-semigroup $S(t)$.

Our next result describes the $\odot$-reflexive case.

Theorem 4.2.6. *Suppose X is $\odot$-reflexive with respect to $T(t)$. The following are equivalent:*

(i) Fav$(T(t)) = D(A)$;

(ii) j maps X onto $X^{\odot *}$;

(iii) $R(\lambda, A)B_{(X,\|\cdot\|')}$ is weakly compact;

(iv) $R(\lambda, A)B_{(X,\|\cdot\|')}$ is $\sigma(X, X^{\odot})$-compact;

(v) $B_{(X,\|\cdot\|')}$ is $\sigma(X, X^{\odot})$-compact.

Proof: (i)$\Leftrightarrow$(iii): Since $\|\cdot\|'$ is an equivalent norm, this follows from Theorems 3.2.2 and 4.2.3.

(iii)$\Rightarrow$(i): By assumption X is $\odot$-reflexive and $R(\lambda, A)B_{(X,\|\cdot\|')}$ is closed. Hence from Theorem 4.2.3 and from the inclusions $D(A^{\odot *}) \subset X^{\odot\odot} = X$ we have $D(A^{\odot *}) = D(A^{\odot *}) \cap X = \text{Fav}(T(t)) = D(A) = D(A^{\odot\odot})$. Since $A^{\odot\odot}$ is the part of $A^{\odot *}$ in $X^{\odot\odot}$, it follows that $X^{\odot\odot} = X^{\odot *}$. Since X is $\odot$-reflexive with respect to $T(t)$, this is the desired result.

(ii)$\Rightarrow$(v): $B_{X^{\odot *}}$ is weak*-compact. By assumption we may identify $B_{X^{\odot *}}$ with $B_{(X,\|\cdot\|')}$ and (v) follows.

(v)$\Leftrightarrow$(iv)$\Leftrightarrow$(iii): Combine the Corollaries 2.2.8, 2.2.5 and Proposition 2.2.6. ////

Remark 4.2.7. In the proof of Proposition 2.3.2 we saw that $\overline{B_X}^{\sigma(X,X^{\odot})} = B_{(X,\|\cdot\|')}$. Therefore (i)-(v) remain equivalent if in (v) one replaces $B_{(X,\|\cdot\|')}$ by B_X, provided 'compact' is replaced by 'relatively compact'. ////

If one of the equivalent hypotheses of Theorem 4.2.6 are satisfied, then X actually has the Radon-Nikodym property. This is proved in Chapter 6.

Finally we give a converse of Corollary 4.1.6 for Banach spaces with a Schauder basis, analogous to Corollary 1.5.6.

Theorem 4.2.8. *Let X be a non-reflexive Banach space with a Schauder basis $\{x_n\}_{n=1}^{\infty}$. Then there exists a C_0-semigroup on X with Fav$(T(t)) \neq D(A)$.*

Proof: Suppose X is non-reflexive. By Zippin's Theorem 1.5.5 there is a non-boundedly complete basis $\{y_n\}_{n=1}^{\infty}$ for X. Define $T(t)y_n = e^{-nt}y_n$. By Theorem 1.5.2 these operators extend to a C_0-semigroup on X. We claim that the Favard class of this semigroup is strictly larger than $D(A)$. Let $\{y_n^*\}_{n=1}^{\infty}$ be the coordinate functionals of $\{y_n\}_{n=1}^{\infty}$; they form a non-shrinking basis for their closed linear span $[y_n^*]$. Let $\{y_n^{**}\}_{n=1}^{\infty}$ be the coordinate functionals of this basis. By Theorem 1.5.2 X is $\odot$-reflexive with respect to $T(t)$. But since $\{y_n^*\}_{n=1}^{\infty}$ is non-shrinking, $[y_n^{**}] = X$ is strictly smaller than $[y_n^*]^* = X^{\odot *}$. Now apply Theorem 4.2.6. ////

Notes. A comprehensive study of the Favard class and generalizations of it is carried out in the book [BB]. One can define Favard classes of order (α, n) for each real number $\alpha > 0$ and $n \in \mathbb{N}$ by setting

$$X_{\alpha,n} := \{x \in X : \limsup_{t\downarrow} t^{-\alpha}\|(T(t) - I)^n\| < \infty\}.$$

See [BB] for the a detailed study of these spaces. Of course, $\mathrm{Fav}(T(t)) = X_{1,1}$.

The results of Section 4.1 are taken from [BB]. The proof of Theorem 4.1.2, due to de Leeuw [dL], is a simplification of the argument given in [BB, Thm. 2.1.4]. The results of Section 4.2, except Theorem 4.2.2, Corollary 4.2.4 and Example 4.2.5 which are new, are taken from [Ne4]. It follows from Lemma 4.2.1 and Theorem 4.1.4 that

$$\mathrm{Fav}(T(t)) = \bigcup_{n\in\mathbb{N}} \overline{nR(\lambda, A)B_X}.$$

In the language of interpolation theory this formula just expresses the well-known fact that $\mathrm{Fav}(T(t))$ is the Gagliardo- (or relative-) closure of $D(A)$ in X. The equivalence of (i) and (ii) in Theorem 4.2.2 is well-known, cf. [Wa] where this is proved in more generality.

Chapter 5

Dichotomy theorems

For a C_0-semigroup on X, define $X^{\otimes}$ to be the subspace of all $x^* \in X^*$ for which the map $t \mapsto T^*(t)x^*$ is strongly continuous for $t > 0$. This subspace is closed and contains $X^{\odot}$. The first main result of this chapter is the following dichotomy theorem: *The quotient space $X^*/X^{\otimes}$ is either zero or non-separable.* After having proved this, we show how the proof can be modified to obtain an 'orbitwise' generalization, which has the following striking consequence: *every orbit under the quotient semigroup on $X^*/X^{\odot}$ is either zero for $t > 0$ or non-separable.*

In Section 5.1 we introduce the natural embedding $k : X^{\odot\odot} \to X^{**}$. Then in Section 5.2 we study the semigroup $T_{\odot\odot}(t)$, which is another natural semigroup associated with $T(t)$. In Section 5.3 the dichotomy theorem is proved and finally in Section 5.4 we present the orbitwise result.

5.1. The natural embedding $k : X^{\odot\odot} \to X^{**}$

The main goal of this section is to introduce the natural embedding of $X^{\odot\odot}$ into X^{**}. Let $T(t)$ be a C_0-semigroup satisfying $||T(t)|| \leq Me^{\omega t}$.

Theorem 5.1.1. *The formula*

$$\langle kx^{\odot\odot}, x^* \rangle := \lim_{\lambda \to \infty} \langle x^{\odot\odot}, \lambda R(\lambda, A^*)x^* \rangle, \quad x^{\odot\odot} \in X^{\odot\odot}$$

defines a natural embedding $k : X^{\odot\odot} \to X^{**}$. Moreover, $1 \leq ||k|| \leq M$ and $kx^{\odot\odot}|_{X^{\odot}} = x^{\odot\odot}$ and if $i : X \to X^{**}$ is the natural map, then we have $i = kj$.

Proof: Firstly, it is clear that if $x^{\odot} \in X^{\odot}$ and $x^{\odot\odot} \in X^{\odot\odot}$, then we have $\langle kx^{\odot\odot}, x^{\odot} \rangle = \langle x^{\odot\odot}, x^{\odot} \rangle$, the first pairing being in (X^{**}, X^*) and the second one in $(X^{\odot *}, X^{\odot})$.

Secondly, let $x^{\odot\odot} \in D(A^{\odot *})$, say $x^{\odot\odot} = R(\mu, A^{\odot *})y^{\odot *}$. For any $x^* \in X^*$

we have

$$\lim_{\lambda\to\infty}\langle x^{\odot\odot}, \lambda R(\lambda, A^*)x^*\rangle = \lim_{\lambda\to\infty}\langle y^{\odot *}, \lambda R(\lambda, A^*)R(\mu, A^*)x^*\rangle$$
$$= \langle y^{\odot *}, R(\mu, A^*)x^*\rangle,$$

using the fact that $R(\mu, A^*)x^* \in X^\odot$. Since $\limsup_{\lambda\to\infty}\|\lambda R(\lambda, A)\| \leq M$ we have $\|kx^{\odot\odot}\| \leq M\|x^{\odot\odot}\|$. By a density argument $\lim_{\lambda\to\infty}\langle x^{\odot\odot}, \lambda R(\lambda, A^*)x^*\rangle$ exists for *all* $x^{\odot\odot} \in X^{\odot\odot}$. It is clear that $\|k\| \leq M$.

Since $X^{\odot\odot}$ is normed by $X^\odot$ and $\langle kx^{\odot\odot}, x^\odot\rangle = \langle x^{\odot\odot}, x^\odot\rangle$ holds for $x^\odot \in X^\odot$, it follows that $\|k\| \geq 1$. ////

Example 5.1.2. Let $T(t)$ be the translation group on $X = C_0(\mathbb{R})$. Then by Example 1.3.9 we have $X^{\odot\odot} = BUC(\mathbb{R})$ and for $f \in X^{\odot\odot}$ and a Borel measure $\mu \in X^*$ one can check that

$$\langle kf, \mu\rangle = \int_{\mathbb{R}} f\, d\mu.$$

Let $r : X^{**} \to X^{\odot *}$ be the restriction map. The following lemma states that $R(\lambda, A)^{**}$ is a natural map from X^{**} into $kX^{\odot\odot}$. It generalizes part of the argument in the proof of Theorem 3.2.2.

Lemma 5.1.3. *For all $x^{**} \in X^{**}$ we have $R(\lambda, A)^{**}x^{**} = kR(\lambda, A^{\odot *})rx^{**}$.*

Proof: For any $x^* \in X^*$ we have, taking into account Proposition 1.4.4 and the fact that $R(\lambda, A^*)x^* \in X^\odot$,

$$\langle R(\lambda, A)^{**}x^{**}, x^*\rangle = \lim_{\mu\to\infty}\langle x^{**}, \mu R(\mu, A^*)R(\lambda, A^*)x^*\rangle =$$
$$\lim_{\mu\to\infty}\langle rx^{**}, R(\lambda, A^\odot)\mu R(\mu, A^*)x^*\rangle =$$
$$\lim_{\mu\to\infty}\langle R(\lambda, A^{\odot *})rx^{**}, \mu R(\mu, A^*)x^*\rangle =$$
$$\langle kR(\lambda, A^{\odot *})rx^{**}, x^*\rangle.$$

////

We have the following characterization of $kX^{\odot\odot}$.

Theorem 5.1.4. $kX^{\odot\odot} = \{x^{**} \in X^{**} : \lim_{\lambda\to\infty}\lambda R(\lambda, A)^{**}x^{**} = x^{**}\}$.

Proof: One checks that $rk = I_{X^{\odot\odot}}$. Hence by Lemma 5.1.3, if $x^{**} = kx^{\odot\odot} \in kX^{\odot\odot}$, then

$$\lim_{\lambda\to\infty}\lambda R(\lambda, A)^{**}kx^{\odot\odot} = \lim_{\lambda\to\infty}\lambda kR(\lambda, A^{\odot *})rkx^{\odot\odot}$$
$$= \lim_{\lambda\to\infty}\lambda kR(\lambda, A^{\odot *})x^{\odot\odot}$$
$$= k\lim_{\lambda\to\infty}\lambda R(\lambda, A^{\odot *})x^{\odot\odot} = kx^{\odot\odot}.$$

Conversely, if $\lim_{\lambda\to\infty}\lambda R(\lambda, A)^{**}x^{**} = x^{**}$, then by Lemma 5.1.3 x^{**} belongs to the norm-closure of $kX^{\odot\odot}$. But $kX^{\odot\odot}$ is closed in X^{**}, so $x^{**} \in kX^{\odot\odot}$. ////

5.2. The semigroup $T_{\odot\odot}(t)$

Theorem 5.1.4 suggests to define

$$X_{\odot\odot} := \{x^{**} \in X^{**} : \lim_{t \downarrow 0} T^{**}(t)x^{**} = x^{**}\}.$$

In the notation of Chapter 1, $X_{\odot\odot} = (X^*)^{\odot}$, where on X^* we have the semigroup $T^*(t)$. In view of Proposition 1.4.4 and Theorem 5.1.4 one might hope that $X_{\odot\odot} = kX^{\odot\odot}$. Trivially this is true if $X^* = X^{\odot}$, since then both definitions coincide. Lemma 5.2.1 below shows that $kX^{\odot\odot} \subset X_{\odot\odot}$ always holds, but Example 5.2.2 shows that this inclusion can be proper. We will prove below for separable X that $X_{\odot\odot} = kX^{\odot\odot}$ holds if $T^*(t)$ is weakly Borel measurable, i.e. for each $x^{**} \in X^{**}$ the map $t \mapsto \langle x^{**}, T^*(t)x^*\rangle$ is Borel measurable. In particular this is the case if $T^*(t)$ is $C_{>0}$.

Lemma 5.2.1. *Let $T(t)$ be C_0. Then:*
(i) r maps $X_{\odot\odot}$ into $X^{\odot\odot}$;
(ii) k maps $X^{\odot\odot}$ into $X_{\odot\odot}$.

Proof: (i) For $x^{\odot} \in X^{\odot}$ arbitrary we have

$$\begin{aligned} |\langle T^{\odot *}(t)rx_{\odot\odot} - rx_{\odot\odot}, x^{\odot}\rangle| &= |\langle rx_{\odot\odot}, T^{\odot}(t)x^{\odot} - x^{\odot}\rangle| \\ &= |\langle x_{\odot\odot}, T^{\odot}(t)x^{\odot} - x^{\odot}\rangle| = |\langle x_{\odot\odot}, T^*(t)x^{\odot} - x^{\odot}\rangle| \\ &= |\langle T^{**}(t)x_{\odot\odot} - x_{\odot\odot}, x^{\odot}\rangle| \le \|T^{**}(t)x_{\odot\odot} - x_{\odot\odot}\| \, \|x^{\odot}\|, \end{aligned}$$

and (i) follows by taking the supremum over all $x^{\odot} \in B_{X^{\odot}}$.

(ii) For $x^* \in X^*$ arbitrary we have

$$\begin{aligned} |\langle T^{**}(t)kx^{\odot\odot} - kx^{\odot\odot}, x^*\rangle| &= |\langle kx^{\odot\odot}, T^*(t)x^* - x^*\rangle| \\ &= \lim_{\lambda \to \infty} |\langle x^{\odot\odot}, T^{\odot}(t)\lambda R(\lambda, A^*)x^* - \lambda R(\lambda, A^*)x^*\rangle| \\ &= \lim_{\lambda \to \infty} |\langle T^{\odot *}(t)x^{\odot\odot} - x^{\odot\odot}, \lambda R(\lambda, A^*)x^*\rangle| \\ &= |\langle k\big(T^{\odot *}(t)x^{\odot\odot} - x^{\odot\odot}\big), x^*\rangle| \\ &\le \|k\| \, \|T^{\odot *}(t)x^{\odot\odot} - x^{\odot\odot}\| \, \|x^*\|, \end{aligned}$$

and (ii) follows by taking the supremum over all $x^* \in B_{X^*}$. ////

A trivial corollary of this is that $T^*(t)$ is $\sigma(X^*, kX^{\odot\odot})$-continuous.

Summarizing, we have the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{j} & X^{\odot\odot} & \longrightarrow & X^{\odot *} \\ & i \searrow & \downarrow k & & \uparrow r \\ & & X_{\odot\odot} & \longrightarrow & X^{**} \end{array}$$

Here $i : X \to X^{**}$ is the natural embedding of X into its bidual.

The following example shows that $kX^{\odot\odot}$ can be a proper subspace of X^{**}. It uses some terminology and elementary facts about Banach lattices which can be found in Chapter 8.

Example 5.2.2. Let $T(t)$ be the rotation group on $X = C(\mathbb{T})$. Then $X^{\odot\odot} = X = C(\mathbb{T})$. We will construct a non-zero $x_{\odot\odot} \in X_{\odot\odot}$ which annihilates $X^{\odot}$, so $x_{\odot\odot} \notin kX^{\odot\odot}$. Let $x^{**} \in X^{**}$ be any non-zero vector annihilating the subspace $L^1(\mathbb{T}) = X^{\odot}$ of X^*. Regarding X^{**} as a Banach lattice, by replacing x^{**} by its modulus $|x^{**}|$, which also annihilates $X^{\odot}$, we may assume without loss of generality that $x^{**} \geq 0$. Put

$$x_{\odot\odot} := \sup_{s\geq 0} T^{**}(s)x^{**}.$$

Since X^{**} is a $C(K)$-space by the Kakutani-Krein representation theorem and since the orbit of x^{**} is norm-bounded, this orbit is order bounded. Since X^{**}, being a dual space, is Dedekind complete it follows that the supremum indeed exists. Trivially $x_{\odot\odot} \geq x^{**}$, so $x_{\odot\odot} \neq 0$. Since for each $t \geq 0$ the operator $T^{**}(t)$ is an order continuous lattice homomorphism we have

$$T^{**}(t)x_{\odot\odot} = T^{**}(t)\Big(\sup_{s\geq 0} T^{**}(s)x^{**}\Big) = \sup_{t\geq 0} T^{**}(t+s)x^{**} = x_{\odot\odot},$$

using in the last identity that $T^{**}(t)$ is a periodic semigroup. Hence $x_{\odot\odot} \in X_{\odot\odot}$. On the other hand, since x^{**} annihilates $X^{\odot}$, so does each $T^{**}(t)x^{**}$. Since the annihilator of the projection band $X^{\odot}$ is a (projection) band in X^{**}, it follows that $x_{\odot\odot}$ annihilates $X^{\odot}$ as well.

In Lemma 5.2.1 a new operator is born: define $\pi : X_{\odot\odot} \to X_{\odot\odot}$ by $\pi x_{\odot\odot} := krx_{\odot\odot}$.

Lemma 5.2.3. *π is a projection onto $kX^{\odot\odot}$.*

Proof: Since $kX^{\odot\odot} \subset X_{\odot\odot}$ it suffices to prove that $\pi(kx^{\odot\odot}) = kx^{\odot\odot}$ holds for all $x^{\odot\odot} \in X^{\odot\odot}$. But this follows from the obvious fact that $rkx^{\odot\odot} = x^{\odot\odot}$ for all $x^{\odot\odot} \in X^{\odot\odot}$. ////

In order to identify the complement of $kX^{\odot\odot}$ in $X_{\odot\odot}$ we need some results on quotient semigroups. Let $S(t)$ be a semigroup on X, not necessarily C_0. Let Y be a $S(t)$-invariant subspace and $S_q(t)$ be the quotient semigroup on X/Y (see Chapter 3 for the definition). The dual space $(X/Y)^*$ is naturally isometrically isomorphic to the annihilator $Y^{\perp} := \{x^* \in X^* : \langle x^*, y\rangle = 0, \quad \forall y \in Y\}$ of Y; the isomorphism $m : (X/Y)^* \to Y^{\perp}$ is given by $\langle mz^*, x\rangle := \langle z^*, qx\rangle, \quad z^* \in (X/Y)^*$.

Lemma 5.2.4. *For any $z^* \in (X/Y)^*$ we have $\langle m(S_q^*(t)z^*) = S^*(t)(mz^*)$.*

Proof: Let $x \in X$ be arbitrary. Then

$$\begin{aligned}\langle m(S_q^*(t)z^*), x\rangle &= \langle S_q^*(t)z^*, qx\rangle = \langle z^*, S_q(t)qx\rangle \\ &= \langle z^*, q(S(t)x)\rangle = \langle mz^*, S(t)x\rangle = \langle S^*(t)(mz^*), x\rangle.\end{aligned}$$

////

The following lemma is a trivial corollary:

Lemma 5.2.5. *The map m induces an isometrical isomorphism $(X/Y)^{\odot} \simeq X^{\odot} \cap Y^{\perp}$.*

From now on, let $T(t)$ again be a C_0-semigroup.

Let $q : X^* \to X^*/X^{\odot}$ be the quotient map and let $T_q(t)$ be the quotient semigroup on $X^*/X^{\odot}$. From $||T_q(t)|| \leq ||T^*(t)||$ we see that $T_q(t)$ is locally bounded. Let $T_q^*(t)$ denote the adjoint $(T_q(t))^*$ of $T_q(t)$. Let $m : (X^*/X^{\odot})^* \to X^{\odot\perp}$ be the natural isomorphism. In this way $(X^*/X^{\odot})^{\odot}$ is identified with a closed subspace of X^{**}.

Theorem 5.2.6. $X_{\odot\odot} = kX^{\odot\odot} \oplus m(X^*/X^{\odot})^{\odot}$.

Proof: By Lemma 5.2.3 we have $X_{\odot\odot} = kX^{\odot\odot} \oplus Y$, where $Y = \ker \pi$. We claim that $\ker \pi = X_{\odot\odot} \cap X^{\odot\perp}$. Indeed, it is clear from $X^{\odot\perp} \subset \ker r$ and the definition of π shows that $X_{\odot\odot} \cap X^{\odot\perp} \subset \ker \pi$. Conversely, if $x_{\odot\odot} \in \ker \pi$, then we must have $rx_{\odot\odot} = 0$, since k is an isomorphism into. This means that $x_{\odot\odot} \in X^{\odot\perp}$. This proves the claim. From Theorem 5.2.5 we know that $m(X^*/X^{\odot})^{\odot} = (X^*)^{\odot} \cap X^{\odot\perp} = X_{\odot\odot} \cap X^{\odot\perp}$. Therefore $\ker \pi = m(X^*/X^{\odot})^{\odot}$. ////

In the next section we will derive an alternative representation of $X_{\odot\odot}$.

With this information we can characterize $kX^{\odot\odot}$ as those elements of $X_{\odot\odot}$ that commute with the weak*-integral of the adjoint semigroup.

Corollary 5.2.7. *An element $x_{\odot\odot} \in X_{\odot\odot}$ belongs to $kX^{\odot\odot}$ if and only if for all $t > 0$ and $x^* \in X^*$ we have*

$$\langle x_{\odot\odot}, weak^* \int_0^t T^*(\sigma)x^* \, d\sigma \rangle = \int_0^t \langle x_{\odot\odot}, T^*(\sigma)x^* \rangle \, d\sigma. \tag{5.1}$$

Proof: First suppose $x_{\odot\odot} = kx^{\odot\odot} \in kX^{\odot\odot}$. For any choice of t and x^* we have

$$\begin{aligned} &\langle kx^{\odot\odot}, weak^* \int_0^t T^*(\sigma)x^* \, d\sigma \rangle \\ &= \lim_{\lambda\to\infty} \langle x^{\odot\odot}, \lambda R(\lambda, A^*) weak^* \int_0^t T^*(\sigma)x^* \, d\sigma \rangle \\ &= \lim_{\lambda\to\infty} \langle x^{\odot\odot}, weak^* \int_0^t T^*(\sigma)\lambda R(\lambda, A^*)x^* \, d\sigma \rangle. \end{aligned}$$

But since $R(\lambda, A^*)x^* \in X^{\odot}$, the integrand of the last integral is norm continuous in σ, so the weak*-integral is actually a Bochner integral in $X^{\odot}$. Hence we may move the bounded linear functional $x^{\odot\odot}$ through the integral sign and obtain from the dominated convergence theorem

$$\lim_{\lambda\to\infty} \langle x^{\odot\odot}, w^* \int_0^t T^*(\sigma)\lambda R(\lambda, A^*)x^* \, d\sigma\rangle$$
$$= \lim_{\lambda\to\infty} \int_0^t \langle x^{\odot\odot}, T^*(\sigma)\lambda R(\lambda, A^*)x^*\rangle \, d\sigma = \int_0^t \langle kx^{\odot\odot}, T^*(\sigma)x^*\rangle \, d\sigma.$$

Conversely, let $x_{\odot\odot} \in X_{\odot\odot}$ such that (5.1) holds for all t and x^*. Write $x_{\odot\odot} = kx^{\odot\odot} + y$ with $y \in \ker \pi$. Let $x^* \in X^*$ be fixed. Since $y \in \ker \pi \subset X^{\odot\perp}$, for all $t > 0$ we have

$$\langle y, weak^* \int_0^t T^*(\sigma)x^* \, d\sigma\rangle = 0.$$

Since by the above (5.1) holds for $kx^{\odot\odot}$, it follows that (5.1) also holds for y. Thus for all $t > 0$,

$$\int_0^t \langle T^{**}(\sigma)y, x^*\rangle \, d\sigma = \int_0^t \langle y, T^*(\sigma)x^*\rangle \, d\sigma = \langle y, weak^* \int_0^t T^*(\sigma)x^* \, d\sigma\rangle = 0.$$

But $t \mapsto \langle T^{**}(t)y, x^*\rangle$ is a continuous function for $t \geq 0$ since $y \in X_{\odot\odot}$, so it must be identically zero. In particular $\langle y, x^*\rangle = 0$. Since x^* was arbitrary it follows that $y = 0$, as was to be shown. ////

Corollary 5.2.8. *$X_{\odot\odot} = kX^{\odot\odot}$ if and only if for all $t > 0$, $x^* \in X^*$ and $x_{\odot\odot} \in X_{\odot\odot}$ we have*

$$\langle x_{\odot\odot}, weak^* \int_0^t T^*(\sigma)x^* \, d\sigma\rangle = \int_0^t \langle x_{\odot\odot}, T^*(\sigma)x^*\rangle \, d\sigma.$$

Hence if $T^*(t)$ is *Pettis integrable*, i.e. for all $x^* \in X^*$ and $\tau > 0$ the map $t \mapsto T^*(t)x^*$ is Pettis integrable on $[0, \tau]$ (cf. the Appendix), then $X_{\odot\odot} = kX^{\odot\odot}$. In particular this holds if $T^*(t)$ is $C_{>0}$.

A semigroup $S(t)$ is *(weakly) compact* if for each $t > 0$ the operator $S(t)$ is (weakly) compact.

Corollary 5.2.9. *For a C_0-semigroup $T(t)$ the following assertions are equivalent:*

(i) $T(t)$ is weakly compact;

(ii) X is $\odot$-reflexive with respect to $T(t)$ and $T^{**}(t)$ is $C_{>0}$.

Proof: Suppose first that $T(t)$ is weakly compact. Then $T^{**}(t)X^{**} \subset X$ for each $t > 0$ by Gantmacher's theorem. Trivially this implies that $T^{**}(t)$ is $C_{>0}$. Fix $x^{\odot\odot} \in X^{\odot\odot}$. Then $T^{**}(t)kx^{\odot\odot} \subset X$ for all $t > 0$, and by letting $t \downarrow 0$ it follows from Lemma 5.2.1 that $kx^{\odot\odot} \in X$, so X is $\odot$-reflexive.

Conversely, suppose that (ii) holds. By the assumption on $T^{**}(t)$, $T^*(t)$ is weakly continuous for $t > 0$. Fix $t_0 > 0$ and $x^* \in X^*$. By applying Theorem 0.2.1 to the closed linear span of $\{T^*(t)x^* : t \geq t_0\}$ it follows that $T^*(t)$ is $C_{>0}$, so by Corollary 5.2.8 and $\odot$-reflexivity we have $X_{\odot\odot} = kX^{\odot\odot} = X$. Since $T^{**}(t)$ is $C_{>0}$ we have $T^{**}(t)X^{**} \subset X_{\odot\odot}$ for all $t > 0$. Hence $T^{**}(t)X^{**} \subset X$ for all $t > 0$, and the weak compactness of $T(t)$ follows from Gantmacher's theorem. ////

In particular, if $T(t)$ is uniformly continuous for $t > 0$ (e.g. if $T(t)$ is a holomorphic semigroup), then $T(t)$ is $\odot$-reflexive if and only if $T(t)$ is weakly compact. This follows more easily however by replacing the resolvents by the semigroup in the proof of Theorem 3.2.2.

Recall from [P, Chapter 2] that a C_0-semigroup is compact if and only if it is uniformly continuous for $t > 0$ and $R(\lambda, A)$ is compact. For $L^1(\mu)$ we have the following improvement:

Corollary 5.2.10. *A C_0-semigroup $T(t)$ on $L^1(\mu)$ is compact if and only if $T^*(t)$ is $C_{>0}$ and $R(\lambda, A)$ is weakly compact.*

Proof: 'If': Since $L^1(\mu)$ has the Dunford-Pettis property, in view of $T(t) = T(t/2)^2$ it is enough to prove that $T(t)$ is weakly compact. We will use that the dual of $L^1(\mu)$ has the Grothendieck property [AB]. Since $T^*(t)$ is $C_{>0}$, $T^{**}(t)$ is weak*- and hence, by the Grothendieck property, weakly continuous for $t > 0$. Hence for each $t_0 > 0$ and x^{**} the map $t \to T^{**}(t)x^{**}$ is strongly continuous for $t \geq t_0$ and the conclusion follows from Corollary 5.2.9. The 'only if' part is clear. ////

More generally, the same argument applies to every Dunford-Pettis space whose dual has the Grothendieck property.

Let K be a compact Hausdorff space. A function $\phi : K \to \mathbb{K}$, where $\mathbb{K}$ is the scalar field, is said to be *universally measurable* if it is μ-measurable (see the Appendix for the definition) for all finite positive regular Borel measures μ on K. A function $\psi : K \to X$, with X a Banach space, is *universally weakly measurable* if $\langle x^*, \psi(\cdot)\rangle$ is universally measurable for all $x^* \in X^*$. Note that every Borel measurable function is universally measurable. This implies that a weakly Borel measurable function is universally weakly measurable. Finally, ψ is called *universally Pettis integrable* if ψ is Pettis integrable with respect to every μ. The following theorem is a deep result of Riddle, Saab and Uhl [RSU].

Theorem 5.2.11. *Let X be a separable Banach space and suppose $\psi : K \to X^*$ is a bounded, universally weakly measurable function. Then ψ is universally Pettis integrable.*

In particular, if $T^*(t)$ is *weakly Borel measurable*, i.e. for all $x^* \in X^*$ and $\tau > 0$ the map $t \mapsto T^*(t)x^*$ is weakly Borel measurable on $[0, \tau]$, it follows from this theorem that $T^*(t)$ is actually Pettis integrable. Combining this with Corollary 5.2.8 we obtain:

Corollary 5.2.12. *Suppose X is separable. If $T^*(t)$ is weakly Borel measurable, then $X_{\odot\odot} = kX^{\odot\odot}$.*

One might wonder whether Pettis integrability of $T^*(t)$ already implies strong continuity for $t > 0$. The answer is negative, as will be shown in Section 8.2.

Example 5.2.13. Let $T(t)$ be the rotation group on $X = C(T)$ or $L^1(T)$. In both cases it is known that $X_{\odot\odot} \neq kX^{\odot\odot}$. For $C(T)$ this will be proved in much more generality in Chapter 8, and for $L^1(T)$ this was proved by W. Rudin [Ru1]. Hence the adjoints of these two semigroups are not weakly Borel measurable.

By the Odell-Rosenthal theorem [OR], if X is separable and does not contain a closed subspace isomorphic to l^1, then each $x^{**} \in X^{**}$ is the weak*-limit of some *sequence* in X. If $T(t)$ is a C_0-semigroup on such a space, then $\langle x^{**}, T^*(t)x^* \rangle$ is the pointwise limit of the continuous functions $\langle T^*(t)x^*, x_n \rangle$ and hence Borel measurable. It follows that $X_{\odot\odot} = kX^{\odot\odot}$. Instead of applying Theorem 5.2.11 one could also use the dominated convergence theorem.

Motivated by this example, we recall that an element $x^{**} \in X^{**}$ is called a *Baire-1 functional* if it is the weak*-limit in X^{**} of a sequence $(x_n) \subset X$. The set of all Baire-1 functionals is a linear subspace of X^{**} and is denoted by $\mathcal{B}_1(X)$. A trivial but useful consequence of Corollary 5.2.7 and the dominated convergence theorem is the following.

Corollary 5.2.14. $X_{\odot\odot} \cap \mathcal{B}_1(X) \subset kX^{\odot\odot}$.

5.3. The dichotomy theorem

A measure for the non-strong continuity of the adjoint semigroup $T(t)$ is the size of the quotient $X^*/X^{\odot}$. But in some respect this measure is not very accurate: if $T(t)$ is the multiplication semigroup on $X = l^1$ from Example 1.3.10, then $X^*/X^{\odot} = l^\infty/c_0$ is a non-separable space, whereas $T^*(t)$ is $C_{>0}$ which is quite well-behaved. Therefore we introduce another canonical space associated with a C_0-semigroup. Define

$$X^{\otimes} := \{x^* \in X^* : \text{ the map } t \mapsto T^*(t)x^* \text{ is continuous for } t > 0\},$$

where of course equivalently we could impose right continuity for $t > 0$. This notation is adopted because $X^{\odot} \subset X^{\otimes} \subset X^*$ and the symbol '$\otimes$' is also something in between '$\odot$' and '$*$'. Note that $T^*(t)X^{\otimes} \subset X^{\odot}$ for all $t > 0$. If $T(t)$ extends to a C_0-group, then $X^{\otimes} = X^{\odot}$. Indeed, if $x^* \in X^{\otimes}$, then the map $t \mapsto T^*(t)x^*$ is continuous at $t = 1$ and hence

$$\lim_{t\to 0} T^*(t)x^* = \lim_{t\to 0} T^*(-1)T^*(1+t)x^* = x^*.$$

The size of $X^*/X^{\otimes}$ indicates to what extent $T^*(t)$ fails to be $C_{>0}$. For this quotient space we have the following dichotomy.

Theorem 5.3.1. *Let $T(t)$ be a C_0-semigroup on X. Then $X^*/X^{\otimes}$ is either zero or non-separable.*

We start with an analogue of Theorem 5.2.6 for the space $X^{\otimes}$. To be precise, we will show in Corollary 5.3.3 below that there are natural isomorphisms $X^{\odot\odot} \simeq X^{\otimes\odot}$ and $(X^*/X^{\odot})^{\odot} \simeq (X^*/X^{\otimes})^{\odot}$. We will not need this in the proof of Theorem 5.3.1 however.

For a locally bounded $C_{>0}$-semigroup $S(t)$ on X, define

$$X_0 := \{x \in X : \lim_{t \downarrow 0} \|S(t)x - x\| = 0\}.$$

Since $S(t)$ is locally bounded, this is a closed subspace of X and the restriction $S_0(t)$ of $S(t)$ to X_0 is a C_0-semigroup on X_0.

Theorem 5.3.2. *Let $S(t)$ be a locally bounded $C_{>0}$-semigroup on X. Then the inclusion map $i : X_0 \to X$ induces a natural isomorphism $X^{\odot} \simeq X_0^{\odot}$.*

Proof: On checks easily that i^* maps $X^{\odot}$ into $X_0^{\odot}$. This map is injective: if $i^* x^{\odot} = 0$, then for all $x \in X$ and $t > 0$ we have

$$\langle S^*(t)x^{\odot}, x\rangle = \langle x^{\odot}, S(t)x\rangle = \langle x^{\odot}, iS(t)x\rangle = \langle i^* x^{\odot}, S(t)x\rangle = 0,$$

since $S(t)x \in X_0$. Hence $S^*(t)x^{\odot} = 0$ for all $t > 0$. Letting $t \downarrow 0$ shows that $x^{\odot} = 0$, and injectivity is proved.

We will now show that i^* is also surjective. Let $M := \sup_{0 \le t \le 1} \|S(t)\|$. Fix an arbitrary $x_0^{\odot} \in X_0^{\odot}$. Define $x^* \in X^*$ by

$$\langle x^*, x\rangle := \lim_{t \downarrow 0} \langle x_0^{\odot}, S(t)x\rangle.$$

This limit indeed exists, for if $0 < t_1 \le t_2 \le 1$ then

$$\begin{aligned} |\langle x_0^{\odot}, S(t_2)x - S(t_1)x\rangle| &= |\langle x_0^{\odot}, S_0(t_2 - t_1)S(t_1)x - S(t_1)x\rangle| \\ &= |\langle S_0^*(t_2 - t_1)x_0^{\odot} - x_0^{\odot}, S(t_1)x\rangle| \\ &\le M\|S_0^*(t_2 - t_1)x_0^{\odot} - x_0^{\odot}\| \, \|x\|. \end{aligned}$$

The proof is finished if we show that $x^* \in X^{\odot}$ and $i^* x^* = x_0^{\odot}$. Firstly, for any $x \in X$ we have

$$\begin{aligned} |\langle S^*(s)x^* - x^*, x\rangle| = |\langle x^*, S(s)x - x\rangle| &= \lim_{t \downarrow 0} |\langle x_0^{\odot}, S(t+s)x - S(t)x\rangle| \\ &= \lim_{t \downarrow 0} |\langle S_0^*(s)x_0^{\odot} - x_0^{\odot}, S(t)x\rangle| \le M\|S_0^*(s)x_0^{\odot} - x_0^{\odot}\| \, \|x\|. \end{aligned}$$

Since x is arbitrary it follows that $x^* \in X^{\odot}$. Moreover, for any $x_0 \in X_0$,

$$\langle i^* x^*, x_0\rangle = \langle x^*, x_0\rangle = \lim_{t \downarrow 0} \langle x_0^{\odot}, S(t)x_0\rangle = \langle x_0^{\odot}, x_0\rangle,$$

which shows that $i^* x^* = x_0^{\odot}$. ////

Now let $T(t)$ be a C_0-semigroup on X again. The following corollary gives an alternative description of the decomposition of $X_{\odot\odot}$ in Theorem 5.2.6. Let $m : X_{\otimes}^{*} \to X^{\otimes\perp}$ be the natural isomorphism.

Corollary 5.3.3. $X_{\odot\odot} = kX^{\otimes\odot} \oplus m(X^{*}/X^{\otimes})^{\odot}$.

Proof: Let r be the restriction map $X^{**} \to X^{\odot *}$. By Lemma 5.2.3 and the proof of Theorem 5.2.6 we know that

$$X_{\odot\odot} = kX^{\odot\odot} \oplus \ \ker \pi = kX^{\odot\odot} \oplus (X_{\odot\odot} \cap X^{\odot\perp}). \tag{5.2}$$

Now by Lemma 5.2.5 we have

$$m(X^{*}/X^{\otimes})^{\odot} = (X^{*})^{\odot} \cap X^{\otimes\perp} = X_{\odot\odot} \cap X^{\otimes\perp}. \tag{5.3}$$

Theorem 5.3.2 and the identities (5.2) and (5.3) show that the theorem follows if we can prove that $X_{\odot\odot} \cap X^{\otimes\perp} = X_{\odot\odot} \cap X^{\odot\perp}$. The inclusion '$\subset$' is trivial. Suppose therefore that $x_{\odot\odot} \in X_{\odot\odot} \cap X^{\odot\perp}$. Then by definition of $X^{\otimes}$,

$$\langle T^{**}(t)x_{\odot\odot}, x^{\otimes}\rangle = \langle x_{\odot\odot}, T^{*}(t)x^{\otimes}\rangle = 0$$

for all $x^{\otimes} \in X^{\otimes}$. By letting $t \downarrow 0$ it follows that $\langle x_{\odot\odot}, x^{\otimes}\rangle = 0$. Therefore $x_{\odot\odot} \in X^{\otimes\perp}$. ////

Now we start with the proof of Theorem 5.3.1. The first lemma is trivial.

Lemma 5.3.4. *Suppose Z is a dense subspace of X and let (x_n^{*}) be a bounded sequence in X^{*} such that for each $z \in Z$ the limit $\lim_{n\to\infty}\langle x_n^{*}, z\rangle$ exists. Then*

$$\langle x^{*}, z\rangle := \lim_{n\to\infty} \langle x_n^{*}, z\rangle$$

defines an element $x^{*} \in X^{*}$ which satisfies $x^{*} = weak^{*} - \lim_{n\to\infty} x_n^{*}$.

The quotient $X^{*}/X^{\otimes}$ will be denoted by $X_{\otimes}$. On $X_{\otimes}$ one has a (locally bounded) quotient semigroup, which will be denoted by $T_{\otimes}(t)$. Usually we will identify $X_{\otimes}^{*}$ with $X^{\otimes\perp}$.

The following lemma is the key construction needed for the proof of Theorem 5.3.1.

Lemma 5.3.5. *If $X_{\otimes}$ is separable, then for every non-zero $x_{\otimes} \in X_{\otimes}$ there is a Baire-1 functional $x^{\otimes\perp} \in X^{\otimes\perp}$ such that $\langle x^{\otimes\perp}, x_{\otimes}\rangle \neq 0$.*

Proof: Fix a non-zero $x_{\otimes} \in X_{\otimes}$ and let $x^{*} \in X^{*}$ be a representative of $x_{\otimes}$. Since $x^{*} \notin X^{\otimes}$, there is an $\epsilon > 0$, $t > 0$ and a sequence $t_n \downarrow 0$ such that for all n,

$$||T^{*}(t + t_n)x^{*} - T^{*}(t)x^{*}|| > \epsilon.$$

Choose norm-1 elements $x_n \in X$ such that

$$|\langle T^{*}(t + t_n)x^{*} - T^{*}(t)x^{*}, x_n\rangle| > \epsilon$$

and put

$$z_n := T(t + t_n)x_n - T(t)x_n.$$

The sequence (z_n) is bounded and for all $x^{\otimes} \in X^{\otimes}$ we have

$$\begin{aligned}\lim_{n\to\infty} |\langle x^{\otimes}, z_n\rangle| &= \lim_{n\to\infty} |\langle T^*(t+t_n)x^{\otimes} - T^*(t)x^{\otimes}, x_n\rangle| \\ &\le \limsup_{n\to\infty} \|T^*(t+t_n)x^{\otimes} - T^*(t)x^{\otimes}\| = 0.\end{aligned} \tag{5.4}$$

Furthermore, since $X_{\otimes}$ is separable there is a countable set $F \subset X^*$ such that the linear span of $X^{\otimes} \cup F$ is norm dense in X^*. By a diagonal argument we find a subsequence of (z_n), labelled (z_n) again, such that $\lim_{n\to\infty}\langle f^*, z_n\rangle$ exists for all $f^* \in F$. Passing once more to a subsequence if necessary we also may assume that

$$\lim_{n\to\infty} |\langle x^*, z_n\rangle| = \lim_{n\to\infty} |\langle T^*(t+t_n)x^* - T^*(t)x^*, x_n\rangle| =: \epsilon_0 \ge \epsilon. \tag{5.5}$$

Regarding the z_n as elements of X^{**}, by Lemma 5.3.4 the sequence (z_n) is weak*-convergent to some $x^{\otimes\perp} \in X^{**}$. By (5.4) we have $x^{\otimes\perp} \in X^{\otimes\perp}$ and this together with (5.5) implies that

$$|\langle x^{\otimes\perp}, x_{\otimes}\rangle| = |\langle x^{\otimes\perp}, x^*\rangle| = \epsilon_0 > 0.$$

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Let $\Gamma \subset X^{**}$ be the linear subspace $X^{\otimes\perp} \cap \mathcal{B}_1(X)$. By Lemma 5.3.5, Γ separates the points of $X_{\otimes}$, but the problem is that we do not know whether Γ induces an equivalent norm on $X_{\otimes}$. This causes, however, only small technical complications with which we shall deal next. Define a norm on $X_{\otimes}$ by putting

$$|x_{\otimes}| := \sup_{x^{\otimes\perp}\in\Gamma, \|x^{\otimes\perp}\|\le 1} |\langle x^{\otimes\perp}, x_{\otimes}\rangle|.$$

This is indeed a norm, since by Lemma 5.3.5 $|x_{\otimes}| = 0$ implies $x_{\otimes} = 0$. Note that $|x_{\otimes}| \le \|x_{\otimes}\|$ for all $x_{\otimes} \in X_{\otimes}$. In this way $(X_{\otimes}, |\cdot|)$ becomes a normed linear space; denote its completion by $\overline{X_{\otimes}}$. Then $\overline{X_{\otimes}}$ is a Banach space and trivially each $x^{\otimes\perp} \in \Gamma$ of $\|\cdot\|$-norm ≤ 1 extends to a bounded linear functional on $\overline{X_{\otimes}}$ of $|\cdot|$-norm ≤ 1. Our next aim is to show that $T_{\otimes}(t)$ extends to a semigroup on $\overline{X_{\otimes}}$. This follows from a density argument and the following lemma, which uses the obvious fact that Γ is $T^{**}(t)$-invariant.

Lemma 5.3.6. $|T_{\otimes}(t)| \le \|T(t)\|$.

Proof: Let $|x_{\otimes}| = 1$. Then

$$\begin{aligned}|T_{\otimes}(t)x_{\otimes}| &= \sup_{x^{\otimes\perp}\in B_\Gamma} |\langle x^{\otimes\perp}, T_{\otimes}(t)x_{\otimes}\rangle| = \sup_{x^{\otimes\perp}\in B_\Gamma} |\langle T^{**}(t)x^{\otimes\perp}, x_{\otimes}\rangle| \\ &\le \sup_{z^{\otimes\perp}\in\|T^{**}(t)\|\cdot B_\Gamma} |\langle z^{\otimes\perp}, x_{\otimes}\rangle| = \|T(t)\|\cdot|x_{\otimes}|.\end{aligned}$$

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Lemma 5.3.7. *$T_\otimes(t)$ is $C_{>0}$ with respect to $|\cdot|$.*

Proof: Denote the extension of $T_\otimes(t)$ to $\overline{X_\otimes}$ by $\overline{T_\otimes(t)}$. Firstly, $\Gamma \subset (\overline{X_\otimes})^*$ is norming for $\overline{X_\otimes}$. Secondly, since $\Gamma \subset \mathcal{B}_1(X)$, for each $x^{\otimes\perp} \in X^{\otimes\perp}$ and $x_\otimes \in X_\otimes$ the function

$$t \mapsto \langle x^{\otimes\perp}, \overline{T_\otimes(t)} x_\otimes\rangle = \langle x^{\otimes\perp}, T_\otimes(t) x_\otimes\rangle$$

is the pointwise limit of a sequence of continuous functions

$$t \mapsto \langle T^*(t) x^*, x_n\rangle,$$

where $qx^* = x_\otimes$ and $x_n \to x^{\otimes\perp}$ weak*, hence in particular this function is measurable. Thirdly, since the topology in $X_\otimes$ induced by $|\cdot|$ is coarser than the one induced by $\|\cdot\|$, and since $X_\otimes$ is $\|\cdot\|$-separable, for each $x_\otimes \in X_\otimes$ the function $t \to T_\otimes(t)x_\otimes$ is $|\cdot|$-separably valued. Combining these three observations with Pettis's measurability theorem A.4 shows that $t \to \overline{T_\otimes(t)x_\otimes}$ is $|\cdot|$-strongly measurable for each $x_\otimes \in X_\otimes$. By density the semigroup $\overline{T_\otimes(t)}$ is $|\cdot|$-strongly measurable for each $x_\otimes \in \overline{X_\otimes}$ and therefore $C_{>0}$ by Theorem 0.2.2. ////

The functionals produced in Lemma 5.3.5 lie in $X^{\otimes\perp} = X_\otimes^*$ but in general do not lie in $X_\otimes^\odot$. By weak*-integration we can pull them into $X_\otimes^\odot$. The next lemma describes the result.

Lemma 5.3.8. *If $x^{\otimes\perp} \in X^{\otimes\perp} \cap \mathcal{B}_1(X)$, then for all $t > 0$*

$$weak^* \int_0^t T^{**}(\sigma) x^{\otimes\perp}\, d\sigma = 0.$$

Proof: First note that the weak*-integral is defined since $x^{\otimes\perp} \in \mathcal{B}_1(X)$. By a calculation as performed in the proof of Theorem 0.2.1 (cf. the Notes of Chapter 1), it is an element of $X_{\odot\odot}$. Also it belongs to $X^{\otimes\perp}$, hence to $X^{\odot\perp}$. On the other hand the facts that $x^{\otimes\perp} \in \mathcal{B}_1(X)$, the dominated convergence theorem and the definition of the weak*-integral imply that $weak^* \int_0^t T^{**}(\sigma)x^{\otimes\perp}\, d\sigma$ is in $\mathcal{B}_1(X)$ again. Therefore the integral belongs to $kX^{\odot\odot}$ by Corollary 5.2.14. We have shown that $weak^* \int_0^t T^{**}(\sigma)x^{\otimes\perp}\, d\sigma \in kX^{\odot\odot} \cap X^{\odot\perp} = \{0\}$. ////

Proof of Theorem 5.3.1: Suppose $X_\otimes$ is separable. If $X_\otimes$ were non-zero, i.e. if $T^*(t)$ were not $C_{>0}$, then there is a $t_0 > 0$ and a $z_\otimes \in X_\otimes$ such that $x_\otimes := T_\otimes(t_0)z_\otimes \neq 0$. Fix $x^{\otimes\perp} \in \Gamma$ such that

$$\langle x^{\otimes\perp}, x_\otimes\rangle =: \epsilon > 0.$$

By Lemma 5.3.7 we may choose $t > 0$ so small that

$$|x^{\otimes\perp}|\; |T_\otimes(\sigma)x_\otimes - x_\otimes| < \frac{\epsilon}{2}, \qquad \forall\, 0 \le \sigma \le t.$$

Here $|x^{\otimes\perp}|$ is the norm of $x^{\otimes\perp}$ regarded as a functional on $\overline{X_\otimes}$. From Lemma 5.3.8 we obtain

$$\begin{aligned}0 &= |\langle \frac{1}{t} weak^* \int_0^t T^{**}(\sigma)x^{\otimes\perp}\, d\sigma, x_\otimes\rangle| = |\frac{1}{t}\int_0^t \langle x^{\otimes\perp}, T_\otimes(t)x_\otimes\rangle\, d\sigma| \\ &\geq |\frac{1}{t}\int_0^t \langle x^{\otimes\perp}, x_\otimes\rangle\, d\sigma| - |\frac{1}{t}\int_0^t \langle x^{\otimes\perp}, T_\otimes(t)x_\otimes - x_\otimes\rangle\, d\sigma| \\ &\geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2},\end{aligned}$$

a contradiction.

Corollary 5.3.9. *If $X^*/X^\odot$ is separable, then $T^*(t)$ is $C_{>0}$.*

For then $X^*/X^\otimes$ is separable. If $T(t)$ is a C_0-group, then $X^\otimes = X^\odot$ and Theorem 5.3.1 reads:

Corollary 5.3.10. *If $T(t)$ is a C_0-group on X, then $X^*/X^\odot$ is either zero or non-separable.*

5.4. An orbitwise generalization

By Pettis's measurability theorem and the Measurable semigroup theorem, an $x^* \in X^*$ belongs to $X^\odot$ if and only if its orbit under $T^*(t)$ is separable. This observation, along with Theorem 5.3.1, leads to the following natural question: if $x^* \notin X^\otimes$, is it true that the orbit of $qx^* \in X^*/X^\otimes$ under the quotient semigroup $T_\otimes$ is non-separable? In other words, if we have a bad orbit and we divide out the good subspace, are we left with a bad orbit again? Of course, if this is true, then Theorem 5.3.1 follows trivially. It turns out that the answer to this question is affirmative.

Theorem 5.4.1. *If $x^* \notin X^\otimes$, then the orbit of qx^* in $X^*/X^\otimes$ is also non-separable.*

For the proof we use the following simple observation. Suppose the quotient orbit of some $x^* \in X^*$ is separable in $X_\otimes = X^*/X^\otimes$ and let $Y_\otimes$ be its closed linear span. Let $Y := q^{-1}Y_\otimes$. Then Y is a closed $T^*(t)$-invariant subspace of X^* containing $X^\otimes$ and the quotient $Y/X^\otimes \simeq Y_\otimes$ is separable. Then Theorem 5.4.1 can be proved by repeating verbatim the proof of Theorem 5.3.1, except that the roles of X^*, X^{**}, $X_{\odot\odot}$ and $X_\otimes$ are taken over by Y, Y^*, $Y^\odot$ and $Y_\otimes$ respectively. To be more precise, we use the following two facts:

(i) The separability of $Y/X^\otimes \simeq Y_\otimes$ allows us to prove an analogue of Lemma 5.3.5.

(ii) In the proof of Lemma 5.3.8 we did not use all information contained in Corollary 5.2.14. In fact, what we used is the following: If $y \in X^{\odot\perp} \cap X_{\odot\odot}$, then $y = 0$. If we let Y be as in Lemma 5.4.2, then what is needed for the proof of Theorem 5.4.1 is the following: if $y^* \in X^{\odot\perp} \cap Y^{\odot}$ is the $\sigma(Y^*, Y)$-limit of a sequence in X, then $y = 0$. This is proved precisely as in Corollary 5.2.7.

It follows that in $X^*/X^{\odot}$, every non-zero orbit is non-separable. By applying the natural quotient map $X^*/X^{\odot} \to X^*/X^{\otimes}$, this implies:

Corollary 5.4.2. *Every orbit in $X^*/X^{\odot}$ is either zero for $t > 0$ or non-separable.*

Corollary 5.4.3. *If the quotient semigroup on $X^*/X^{\otimes}$ is $C_{>0}$, then $X^* = X^{\otimes}$. If the quotient semigroup on $X^*/X^{\odot}$ is C_0, then $X^* = X^{\odot}$.*

Notes. The natural map $k : X^{\odot\odot} \to X^{**}$ was invented and studied in [Cea4] in the context of perturbation theory. The results of Section 5.1 are taken from there. In the same paper it is observed that $X_{\odot\odot} \neq kX^{\odot\odot}$ if $T(t)$ is the rotation group on $X = L^{(}T)$, but the systematic study of $X_{\odot\odot}$ was first undertaken in [Ne8]. The results of Sections 5.2 and 5.3 are taken form there.

That weakly compact semigroups are $\odot$-reflexive was obtained independently and by different methods by Kreulich [Kl].

The Riddle-Saab-Uhl theorem relies on results of Bourgain, Fremlin and Talagrand [BFT], Rosenthal's l^1-theorem [Ro] and weak Radon-Nikodym considerations. It deals with universally weakly measurable functions, hence in particular with weakly Borel measurable functions. It is a natural question whether Corollary 5.2.13 remains true if 'weakly Borel measurable' is replaced by 'weakly (Lebesgue) measurable'. A Banach space X is said to have the *Pettis integral property* (PIP) is every bounded weakly measurable function $f : [0, 1] \to X$ is Pettis integrable. Many spaces are known to have the PIP, e.g. all separable spaces and more generally all WCG spaces. In fact, assuming Axiom K, every Banach space has the PIP. Axiom K is known to be consistent with the Zermelo Fraenkel set theory ZFC. For more information, see [Ta] and the references given there.

In recent times Baire-1 functionals have been studied extensively by many authors.

Theorems 5.3.1 and 5.4.1 seem to be the answer to the following question by de Pagter (private communication) in 1989: If $T(t)$ is C_0 on X, is it true that $X^*/X^{\odot}$ is always zero or infinite-dimensional? In this form we know already that the answer is negative, but although the counterexample on the James space was known, the following variant of de Pagter's question remained open for some time: if $X^*/X^{\odot}$ is finite-dimensional (or separable), is it true that $T^*(t)$ is $C_{>0}$? In particular, if $T(t)$ extends to a C_0-group, is it true that $X^*/X^{\odot}$ is either zero or infinite-dimensional (or non-separable)? In view of Pettis's measurability theorem and Theorem 0.2.2 this should be true, but there were two problems: (i) the quotient semigroup on $X^*/X^{\odot}$, although living in a separable space, has at first sight no obvious measurability properties, and (ii) even if so, it is still not clear how to 'lift' strong continuity for $t > 0$ of the quotient semigroup to X^*. Before the above proof was found, the only partial result was Theorem 6.2.9 in the next chapter, which by Theorem 6.2.3 proves the case $X^{\odot\odot}/X$ separable.

Although there exist C_0-semigroups on spaces X for which $X^*/X^{\odot}$ is separable, no example is known of a *positive* semigroup with this property. On c_0 for example we have $c_0^{\odot} = l^1$ for all positive C_0-semigroups; this will be proved in Chapter 8. Compare this with Proposition 6.2.1, where it is shown that $\dim c_0^*/c_0^{\odot} = 1$ with respect to the summing semigroup.

Corollary 5.4.2 shows that the quotient semigroup on $X^*/X^{\odot}$ is very bad (cf. [LMM]) if it is non-trivial. Surprisingly it is possible that the quotient semigroup is weakly measurable even if it is non-trivial, see Example 8.2.2.

Corollary 5.4.3 solves two three space problems: given a Banach space X and a closed subspace Y such that both Y and X/Y have property P, does it follow that X has property P? It seems to be unknown whether in 5.4.3 the spaces X^* and $X^{\otimes}$ resp. $X^{\odot}$ can be repaced by an arbitrary Banach space and onbe of its closed $S(t)$-invariant subspaces, where $S(t)$ is, say, a locally bounded semigroup.

Chapter 6

Adjoint semigroups and the RNP

In Chapter 1 we saw that the adjoint of a C_0-semigroup on a reflexive space is C_0 again. In this chapter we study what happens if we replace reflexivity by the weaker condition that X has the Radon-Nikodym property. In order to deal later with non-separable spaces, we start in Section 6.1 with a Hahn-Banach theorem for $X^{\odot}$. In Section 6.2 the RNP is defined and it is shown that $T^*(t)$ is $C_{>0}$ if X^* has the RNP. Furthermore, we prove that if X^* does not have the RNP, then at least one of the quotients $X^{\odot\odot}/X$ or $X^*/X^{\odot}$ is non-separable, and that if X does not have the RNP, then $X^{\odot *}/X$ is non-separable.

6.1. The adjoint of the restricted semigroup

In this section we consider the problem of determining $Y^{\odot}$, where Y is a closed, $T(t)$-invariant subspace of X.

Denote the restriction of each operator $T(t)$ to Y by $T_Y(t)$. Then $T_Y(t)$ is a C_0-semigroup on Y. Let A_Y be its generator. It is easily checked that A_Y is precisely the part of A in Y. We begin with a Hahn-Banach extension theorem for the restricted semigroup.

Theorem 6.1.1. *Let $||T(t)|| \le Me^{\omega t}$. For each $\epsilon > 0$ and $y^{\odot} \in Y^{\odot}$ there is an element $x^{\odot} \in X^{\odot}$ such that $||x^{\odot}|| < (M+\epsilon)||y^{\odot}||$ and $x^{\odot}|_Y = y^{\odot}$. Moreover, if $y^{\odot} \in D(A_Y^*)$ then we may choose $x^{\odot} \in D(A^*)$.*

Proof: Fix $y^{\odot} \in D(A_Y^*)$ and $\epsilon > 0$. Since $\limsup_{\lambda\to\infty} ||\lambda R(\lambda, A)|| \le M$ and $\lim_{\lambda\to\infty}(I - \lambda^{-1}A_Y^*)y^{\odot} = y^{\odot}$ in the norm topology of Y^*, we can choose $\lambda = \lambda(y^{\odot})$ such that

$$||R(\lambda, A)||\ ||(\lambda I - A_Y^*)y^{\odot}|| = ||\lambda R(\lambda, A)||\ ||(I - \lambda^{-1}A_Y^*)y^{\odot}|| < (M+\epsilon)||y^{\odot}||.$$

Put $y^* = (\lambda I - A_Y^*)y^{\odot}$. Then $y^* \in Y^*$ and y^* can be extended to some $x^* \in X^*$ such that

$$|\langle x^*, x\rangle| \le ||y^*||\ ||x|| \quad \forall x \in X.$$

Put $x^{\odot} = R(\lambda, A^*)x^*$. Then $x^{\odot} \in D(A^*)$ extends $y^{\odot}$, and for all $x \in X$,

$$|\langle x^{\odot}, x\rangle| = |\langle x^*, R(\lambda, A)x\rangle| \le ||y^*|| \, ||R(\lambda, A)|| \, ||x|| < (M+\epsilon)||y^{\odot}|| \, ||x||.$$

Hence $||x^{\odot}|| < (M+\epsilon)||y^{\odot}||$. Now let $y^{\odot} \in Y^{\odot}$. Without loss of generality assume that $||y^{\odot}|| = 1$. Fix $k > 0$ so large that

$$\left(M + \frac{\epsilon}{2}\right)\left(1 + \frac{2}{k}\right) < M + \epsilon$$

and choose a sequence

$$(y_n^{\odot})_{n \ge 1} \to y^{\odot}, \quad y_n^{\odot} \in D(A_Y^*), \quad ||y_n^{\odot}|| = 1, \quad \forall n,$$

such that $||y_{n+1}{}^{\odot} - y_n^{\odot}|| \le 1/kn^2$, which is always possible since $Y^{\odot}$ is the closure of $D(A_Y^*)$. Choose $(z_n^{\odot})_{n \ge 0} \subset D(A^*)$, such that $z_0^{\odot}$ extends $y_1^{\odot}$, $z_n^{\odot}$ extends $y_{n+1}{}^{\odot} - y_n^{\odot}$ $(n \ge 1)$,

$$||z_0^{\odot}|| < M + \frac{\epsilon}{2}, \quad ||z_n^{\odot}|| < (M + \frac{\epsilon}{2})/kn^2 \quad (n \ge 1).$$

From this construction it follows that $\sum z_n^{\odot}$ converges to some $x^{\odot}$, which is in $X^{\odot}$ by the closedness of $X^{\odot}$. Since $\sum_{m=0}^{n-1} z_m^{\odot}$ is an extension of $y_n^{\odot}$, it follows that $x^{\odot}$ is an extension of $y^{\odot}$ which furthermore satisfies

$$||x^{\odot}|| < (M + \frac{\epsilon}{2})(1 + \sum_{n=1}^{\infty} \frac{1}{kn^2}) < M + \epsilon.$$

////

Corollary 6.1.2. *The topologies $\sigma(Y, Y^{\odot})$ and $\sigma(X, X^{\odot})$ agree on Y.*

The following example shows that the inequality in Theorem 6.1.1 cannot be sharpened to $||x^{\odot}|| \le M||y^{\odot}||$.

Example 6.1.3. Let $X = C_0[0, \infty)$, the space of continuous complex-valued functions vanishing at infinity, provided with the sup-norm. Then

$$T(t)f(x) = f(x+t)$$

defines a C_0-contraction semigroup, with $X^{\odot} = L^1[0, \infty)$, cf. Example 1.3.9. Put $Y = Y_1 \oplus Y_2$; $Y_1 = \{f \in X : f(x) = 0, \forall x \ge 1\}$, $Y_2 =$ the one-dimensional subspace spanned by the function e^{-x}. Y is closed and $T(t)$-invariant. Put

$$\langle y^{\odot}, f\rangle \quad = \quad f(1) \quad (f \in Y),$$

then it is easily verified that $y^{\odot} \in Y^{\odot}$ and $||y^{\odot}|| = 1$. Let $g \in L^1[0,\infty)$ be any extension of $y^{\odot}$. Since g vanishes on Y_1, it has support in $[1,\infty)$. Pick $\delta > 1$ such that

$$\int_1^{1+\delta} |g(x)|\, dx < ||g||.$$

Since g extends $y^{\odot}$, we have

$$\begin{aligned} e^{-1} = \langle y^{\odot}, e^{-x}\rangle &= \int_0^{\infty} g(x)e^{-x}\, dx \\ &= \int_{1+\delta}^{\infty} g(x)e^{-x}\, dx + \int_0^{1+\delta} g(x)e^{-x}\, dx \\ &\leq e^{-1}\int_1^{1+\delta} |g(x)|\, dx + e^{-(1+\delta)}\int_{1+\delta}^{\infty} |g(x)|\, dx < e^{-1}||g||. \end{aligned}$$

Hence $||g|| > 1 = ||y^{\odot}||$.

We will now give a series of simple applications of Theorem 6.1.1. Let $i : Y \to X$ be the inclusion map.

Corollary 6.1.4. *The map i^* induces a natural isomorphism $X^{\odot}/(X^{\odot} \cap Y^{\perp}) \simeq Y^{\odot}$.*

Indeed, by Theorem 6.1.1 the adjoint $i^* : X^* \to Y^*$ maps $X^{\odot}$ onto $Y^{\odot}$, and clearly the kernel of $i^*|_{X^{\odot}}$ equals $X^{\odot} \cap Y^{\perp}$. Combining this with Theorem 5.2.5 we get

Corollary 6.1.5. *There is a natural isomorphism $Y^{\odot} \simeq X^{\odot}/(X/Y)^{\odot}$.*

Next we prove that $\odot$-reflexivity is a three space property. This is the converse of Corollary 3.2.6.

Corollary 6.1.6. *Let Y be a closed $T(t)$-invariant subspace of X. If both Y and X/Y are $\odot$-reflexive, then X is $\odot$-reflexive.*

Proof: By Lemma 5.2.5, the natural isomorphism $m : (X/Y)^* \to Y^{\perp}$ induces an isomorphism $(X/Y)^{\odot} \simeq X^{\odot} \cap Y^{\perp}$. Let $x^{\odot\odot} \in X^{\odot\odot}$ be arbitrary and let $rx^{\odot\odot}$ be the restriction of $x^{\odot\odot}$ to the subspace $X^{\odot} \cap Y^{\perp}$. Then we can identify $rx^{\odot\odot}$ with an element of $(X/Y)^{\odot *}$ and it is easily checked that it actually belongs to $(X/Y)^{\odot\odot}$. Hence by $\odot$-reflexivity there is a $z_0 \in X/Y$ such that $z_0 = rx^{\odot\odot}$ (to keep notation simple we suppress all natural maps). Choose any representative $x_0 \in X$ of z_0 and for $y^{\odot} \in Y^{\odot}$ define

$$\langle y_0^{\odot\odot}, y^{\odot}\rangle := \langle x^{\odot\odot} - x_0, x^{\odot}\rangle,$$

where $x^{\odot} \in X^{\odot}$ is such that $x^{\odot}|_Y = y^{\odot}$. Such $x^{\odot}$ exists by Theorem 6.1.1. Since $x^{\odot\odot} - x_0 \in Y^{\perp\perp}$ this is well defined and gives rise to an element $y_0^{\odot\odot} \in Y^{\odot *}$, which again actually belongs to $Y^{\odot\odot}$. Therefore it can be identified with some $y_0 \in Y$, and it is clear that we must have $x^{\odot\odot} = x_0 + y_0$. ////

In the following section we will need the fact that the identity map $i : Y \to X$ induces an embedding of $Y^{\odot\odot}/Y$ into $X^{\odot\odot}/X$.

Lemma 6.1.7. *The inclusion map $i : Y \to X$ induces an embedding of $Y^{\odot\odot}/Y$ into $X^{\odot\odot}/X$.*

Proof: Let $i^{\odot}$ be the restriction of i^* to $X^{\odot}$. By Theorem 6.1.1, $i^{\odot} : X^{\odot} \to Y^{\odot}$ is onto. Hence $i^{\odot *}$ is an embedding of $Y^{\odot *}$ into $X^{\odot *}$; it is easily seen that its restriction $i^{\odot\odot}$ to $Y^{\odot\odot}$ carries $Y^{\odot\odot}$ into $X^{\odot\odot}$. We claim that $i^{\odot\odot}Y^{\odot\odot} \cap X = Y$. By the bipolar theorem (cf. Section 2.3), the bipolar $Y^{\bigcirc\bigcirc}$ of Y is precisely the $\sigma(X, X^{\odot})$-closure of Y. But Y is $\sigma(X, X^{\odot})$-closed and consequently $Y^{\bigcirc\bigcirc} = Y$. Now for $x^{\odot} \in X^{\odot}$ to be an element of the polar $Y^{\bigcirc}$ means that

$$|\langle x^{\odot}, y\rangle| \leq 1, \qquad \forall y \in Y.$$

But since Y is a subspace of X, this is equivalent to

$$|\langle x^{\odot}, y\rangle| = 0, \qquad \forall y \in Y,$$

in other words, $x^{\odot} \in Y^{\perp}$. Thus we see that

$$Y^{\bigcirc} = Y^{\perp} \cap X^{\odot}.$$

By the same argument, an $x \in X$ belongs to $Y^{\bigcirc\bigcirc}$ if and only if

$$|\langle x^{\odot}, x\rangle| = 0, \qquad \forall x^{\odot} \in Y^{\perp} \cap X^{\odot}.$$

Now suppose $i^{\odot\odot}y^{\odot\odot} \in i^{\odot\odot}Y^{\odot\odot} \cap X$ and let $x^{\odot} \in Y^{\perp} \cap X^{\odot}$. Because the kernel of $i^* : X^* \to Y^*$ is precisely $Y^{\perp}$, the kernel of $i^{\odot} : X^{\odot} \to Y^{\odot}$ is $Y^{\perp} \cap X^{\odot}$. Hence

$$\langle i^{\odot\odot}y^{\odot\odot}, x^{\odot}\rangle = \langle y^{\odot\odot}, i^{\odot}x^{\odot}\rangle = \langle y^{\odot\odot}, 0\rangle = 0.$$

Thus $i^{\odot\odot}y^{\odot\odot}$, regarded as element of X, sits in $Y^{\bigcirc\bigcirc}$. But since $Y^{\bigcirc\bigcirc} = Y$ the claim is proved.
Since $i^{\odot\odot}Y \subset X$, the map $i^{\odot\odot}$ induces a map

$$\tilde{i}^{\odot\odot} : Y^{\odot\odot}/Y \to X^{\odot\odot}/X$$

sending $Y^{\odot\odot}/Y$ onto the closed subspace $(i^{\odot\odot}Y^{\odot\odot})/X$ of $X^{\odot\odot}/X$. By the claim, $\tilde{i}^{\odot\odot}$ is injective. Hence by the open mapping theorem, $\tilde{i}^{\odot\odot}$ is an isomorphism into. ////

6.2. Adjoint semigroups and the Radon-Nikodym property

In Chapter 5 we proved some results about the quotient $X^*/X^{\odot}$ (Corollaries 5.3.9 and 5.3.10). In this section we take up the study of this quotient.

In Example 1.5.3 we constructed a C_0-semigroup on the (quasi-reflexive) space $X = J^*$ with the property that $\dim X^*/X^{\odot} = 1$. In the next proposition we will show that also the summing semigroup (Example 2.3.5) has this property, as does the *convolution semigroup* on c_0, defined by

$$(Q(t)x)_n := \sum_{k=1}^{\infty} \binom{t}{k} x_{k+n}.$$

This is a C_0-semigroup on c_0, whose adjoint is given by

$$(Q^*(t))x_n = \sum_{k=0}^{n} \binom{t}{k} x_{n-k}.$$

Proposition 6.2.1. *Both with respect to the summing semigroup and to the convolution semigroup, $c_0^*/c_0^{\odot}$ is one-dimensional.*

Proof: First we prove this for the summing semigroup. Let $\{y_n\}_{n=1}^{\infty}$ be the summing basis of c_0 and let $\{y_n^*\}_{n=1}^{\infty}$ be its coordinate functionals. Since $y_n^* = x_n^* - x_{n+1}^*$, where $\{x_n\}_{n=1}^{\infty}$ is the unit vector basis of c_0, and since by Theorem 1.5.2 we have $c_0^{\odot} = [y_n^*]_{n=1}^{\infty}$, it follows that the closed linear span of $c_0^{\odot} \cup \{x_1^*\}$ is the whole $c_0^* = l^1$. Hence $c_0^{\odot}$ has at most co-dimension one. But if $c_0^{\odot} = c_0^*$ would imply that $\|\cdot\|' = \|\cdot\|$ on c_0, which is not the case by Example 2.3.5.

Similarly, with respect to the convolution semigroup one shows that $y_n^* := x_n^* + x_{n+1}^* \in c_0^{\odot}$ for all $n \in \mathbb{N}$. This follows by straightforward computation, using the estimate

$$|\binom{t}{k}| \le \frac{t}{k}, \quad 0 < t < 1.$$

The closed linear span Y of (y_n^*) has co-dimension one. Since it is known [Bu, Section 4.12] that $c_0^{\odot}$ is a proper subspace of c_0^*, it follows that $c_0^{\odot} = Y$. ////

By a theorem of Sobczyk [So], c_0 is complemented in every separable space containing it as a subspace. Since every $C(K)$-space, K compact Hausdorff, contains subspaces isomorphic to c_0, we see that every separable $C(K)$-space admits a C_0-semigroup whose semigroup dual has co-dimension one. Separability cannot be omitted, as the example l^{∞} shows: on the one hand this is an AM-space with unit, hence a $C(K)$-space by the Kakutani-Krein representation theorem (Chapter 8) and on the other hand it is a Banach space with the Grothendieck property and the Dunford-Pettis property. Hence by Theorem 1.5.7 every C_0-semigroup on l^{∞} is uniformly continuous and the adjoint of such a semigroup is clearly strongly continuous.

For $L^1(\mu)$-spaces the situation is different:

Theorem 6.2.2. *Suppose $T(t)$ is a C_0-semigroup on an $L^1(\mu)$-space X. If $X^\odot$ is complemented in X^*, then $T(t)$ is uniformly continuous.*

Proof: Since X is an $L^1(\mu)$-space, X^* has the Grothendieck- and the Dunford-Pettis property [AB]. Since $X^\odot$ is a complemented subspace of X^*, also $X^\odot$ has these properties. By Theorem 1.5.7, $T^\odot(t)$ is uniformly continuous. Hence $T^{\odot *}(t)$ is uniformly continuous and therefore also $T(t)$ is. ////

In particular, for $L^1(\mu)$-spaces it follows that $X^*/X^\odot$ is either zero or infinite-dimensional.

Both c_0^* and J^* have the Radon-Nykodym property, which will be defined below. This is no coincidence: we will see that if X is $\odot$-reflexive and $X^*/X^\odot$ is separable, then X^* must have the Radon-Nikodym property.

Let (Ω, Σ, μ) be a finite measure space. A Banach space X is said to have the *Radon-Nikodym property with respect to* (Ω, Σ, μ) if for every μ-continuous vector-valued measure $G : \Sigma \to X$ of bounded variation (see [DU] for the precise definitions) there exists $g \in L^1(\mu; X)$ such that

$$G(E) = \int_E g d\mu$$

for all $E \in \Sigma$. X has the *Radon-Nikodym property* (RNP) if it has the Radon-Nikodym property with respect to every finite measure space.
A bounded linear operator $S : L^1[0,1] \to X$ is called *representable* if there exists a $g \in L^\infty([0,1]; X)$ such that

$$Sf \;=\; \int_0^1 f(t)g(t)dt \qquad \text{for all } f \in L^1[0,1].$$

We will need the following geometric facts about the RNP [DU]:

(i) X has the RNP if and only if each bounded operator $S : L^1[0,1] \to X$ is representable;

(ii) Closed subspaces of spaces with the RNP have the RNP;

(iii) A dual space X^* has the RNP if and only if Y^* is separable for every separable closed subspace Y of X.

(iv) A Banach space has the RNP if and only if each of its separable closed subspaces has the RNP.

Theorem 6.2.3. *Let $S(t)$ be a semigroup on a dual Banach space X^* with the RNP. If $S(t)$ is weak*-continuous for $t > 0$, then $S(t)$ is $C_{>0}$.*

Proof: Fix an arbitrary $x^* \in X^*$. By the uniform boundedness theorem, for each $\delta > 0$ there exists a constant $M = M(\delta)$ such that $\|S(t)x^*\| \le M$ for all $t \in [\delta, 1]$. Define $S : L^1[0,1] \to X^*$ by

$$Sg \;=\; weak^* \int_\delta^1 g(t)S(t)x^*\, dt.$$

Since $\langle g(t)S(t)x^*, x\rangle \in L^1[\delta, 1]$ for all $x \in X$, the above integral is well-defined. S is bounded:

$$\begin{aligned} ||Sg|| &= \sup_{||x||=1} \left| \int_\delta^1 \langle g(t)S(t)x^*, x\rangle dt \right| \\ &\leq \sup_{||x||=1} \int_\delta^1 |g(t)|\, |\langle S(t)x^*, x\rangle| dt \leq M||g||_1. \end{aligned}$$

Since X^* has the RNP, by (i) above there is an $h \in L^\infty([0,1]; X^*)$ such that

$$Sg = \int_0^1 g(t)h(t)dt$$

for all $g \in L^1[0,1]$. For $\delta < t < 1$ and $\epsilon > 0$ small enough, let $E = [t, t+\epsilon]$ and put $g = \frac{1}{\epsilon}\chi_E$, where χ is the characteristic function. It follows that

$$weak^* \int_t^{t+\epsilon} \frac{1}{\epsilon} S(\tau)x^* d\tau = \int_t^{t+\epsilon} \frac{1}{\epsilon} h(\tau)d\tau.$$

By the Lebesgue differentiation Theorem A.2, for almost all $t \in (\delta, 1)$ the right-hand side converges to $h(t)$ as $\epsilon \to 0$. Hence, for such t we have

$$\frac{1}{\epsilon}\int_t^{t+\epsilon} \langle S(\tau)x^*, x\rangle d\tau \to \langle h(t), x\rangle \quad (\epsilon \to 0)$$

for all $x \in X$. But the integrand on the left-hand side is continuous, and therefore the integral converges to $\langle S(t)x^*, x\rangle$. So $S(t)x^* = h(t)$ a.e. In particular, $S(t)x^*$ is measurable on $[\delta, 1]$, hence on $[\delta, \infty)$. Since $\delta > 0$ is arbitrary, it follows from Theorem 0.2.2 that $T(t)$ is strongly continuous for $t > 0$. ////

It is classical result of Dunford and Pettis [DP] that separable duals have the Radon-Nikodym property. For such spaces Theorem 6.2.3 is trivial. Indeed, by Pettis's measurability theorem, for each $x^* \in X^*$ the map $t \mapsto S(t)x^*$ is strongly measurable and we can apply Theorem 0.2.2 directly.

For adjoints of $C_{>0}$-semigroups we have the following improvement.

Corollary 6.2.4. *Let $S(t)$ be a $C_{>0}$-semigroup on X. If $X^\odot$ has the RNP, then $S^*(t)$ is $C_{>0}$.*

Proof: The map S from the proof of Theorem 6.2.3 maps $L^1[0,1]$ into $X^\odot$. Observing this, the proof of Theorem 6.2.3 can be repeated. ////

Corollary 6.2.5. *Let $T(t)$ be a C_0-group on X. Then $X^\odot$ has the RNP if and only if X^* has the RNP.*

Proof: If X^* has the RNP then also its closed subspace $X^\odot$ has it. Suppose $X^\odot$ has the RNP. Then by Corollary 6.2.4 the adjoint $T^*(t)$ is $C_{>0}$. But since $T(t)$ is a group, actually $T^*(t)$ is C_0, so $X^\odot = X^*$. ////

We give two applications of these ideas.

Corollary 6.2.6. *Let $T(t)$ be a C_0-group on a Banach space X whose dual has the RNP (e.g. $X = c_0$). Then $X^\odot = X^*$.*

Of course, for c_0 the result already follows from Theorem 0.2.2 and Pettis's measurability theorem.

Corollary 6.2.7. *Suppose both X and X^* have the RNP. A C_0-semigroup $T(t)$ on X is $\odot$-reflexive with respect to $T(t)$ if and only if $T(t)$ is weakly compact.*

Proof: If $T(t)$ is weakly compact, then as in the proof of Corollary 5.2.7, $T(t)$ is $\odot$-reflexive. Conversely, suppose $T(t)$ is $\odot$-reflexive. By Corollary 5.2.7 it suffices to prove that $T^{**}(t)$ is $C_{>0}$. By Theorem 6.2.3, $T^*(t)$ is $C_{>0}$ and therefore by Corollary 5.2.8 we have $X_{\odot\odot} = kX^{\odot\odot} = X$. Since X has the RNP, Corollary 6.2.4, applied to the semigroup $T^*(t)$, shows that $T^{**}(t)$ is $C_{>0}$. ////

Examples of Banach spaces X such that X and X^* have the RNP are the quasi-reflexive spaces.

Our next goal is to prove an improvement of Corollary 5.3.9 for $\odot$-reflexive semigroups. We need one more lemma.

Lemma 6.2.8. *If $X^\odot$ is separable, then X is separable.*

Proof: Let $(x_n^\odot) \subset B_{X^\odot}$ be a countable dense set. Choose $(x_n) \subset B_X$ such that $|\langle x_n^\odot, x_n\rangle| > \frac{1}{2}$. Let Y be the closed subspace spanned by the set $\{T(t)x_n : n \in \mathbb{N}, t \geq 0\}$. Y is separable and $T(t)$-invariant. Suppose there is some $y \notin Y$. Then there is an element $x^* \in B_{X^*}$ that annihilates Y and is non-zero at y. Then for $t > 0$ sufficiently small, $x^\odot := weak^* \int_0^t T^*(\sigma)x^*\, d\sigma$ also annihilates Y and is non-zero at y. But then

$$\begin{aligned}\frac{1}{2} &\leq |\langle x_n^\odot, x_n\rangle| \leq |\langle \frac{x^\odot}{\|x^\odot\|} - x_n^\odot, x_n\rangle| + |\langle \frac{x^\odot}{\|x^\odot\|}, x_n\rangle| \\ &= |\langle \frac{x^\odot}{\|x^\odot\|} - x_n^\odot, x_n\rangle| \leq \|\frac{x^\odot}{\|x^\odot\|} - x_n^\odot\|,\end{aligned}$$

a contradiction to the density of $(x_n^\odot)$ in $B_{X^\odot}$. This shows $Y = X$ and hence X is separable. ////

Theorem 6.2.9. *If X^* does not have the RNP, then at least one of the quotients $X^{\odot\odot}/X$ and $X^*/X^\odot$ is non-separable.*

Proof: Suppose $X^*/X^\odot$ and $X^{\odot\odot}/X$ are separable. Let $Y_0 \subset X$ be an arbitrary separable closed subspace. To show that X^* has the RNP it suffices to show that Y_0^* is separable. Let Y be the smallest closed, $T(t)$-invariant subspace generated by Y_0. To show that Y_0^* is separable, it is enough to show that Y^* is separable. Since by assumption $X^*/X^\odot$ is separable, we can choose a separable subspace W in X^* such that the algebraic sum $X^\odot + W$ is norm-dense in X^*. Let $i : Y \to X$ be the inclusion map. i^* maps $X^\odot$ into $Y^\odot$, and therefore $Y^\odot + i^*W$ is norm-dense in $i^*X^* = Y^*$. Since W is separable, so is i^*W. Furthermore, by Lemma 6.1.7 we have that $Y^{\odot\odot}/Y$ is separable, being a subspace of the separable space $X^{\odot\odot}/X$. Since Y is separable it follows that $Y^{\odot\odot}$ is separable, hence by Lemma 6.2.8, $Y^\odot$ is separable. It follows that Y^* is separable and X^* has the RNP. ////

The following theorem is another application of Lemma 6.1.7.

Theorem 6.2.10. *If $X^{\odot *}/X$ is separable then $X^{\odot *}$ has the RNP.*

Proof: Let Z be an arbitrary separable closed subspace of $X^\odot$. We have to show that Z^* is separable. As in the proof of Theorem 6.2.9 we can assume without loss of generality that Z is $T^\odot(t)$-invariant. Choose a dense sequence (z_n) in Z. For each n, choose norm-1 vectors $x_n \in X$ such that

$$|\langle z_n, x_n\rangle| > \frac{||z_n||}{2}.$$

Let Y be the smallest closed $T(t)$-invariant subspace of X containing all the x_n and let $i : Y \to X$ be the inclusion map. Define a map

$$h : Z \to Y^*,$$
$$\langle hz, y\rangle := \langle z, iy\rangle.$$

It is easily verified that h actually sends Z into $Y^\odot$. We claim that h is an isomorphism into. Indeed, continuity of h is obvious. Furthermore, for given $z \in Z$ we can, by the denseness of (z_n), choose n such that

$$|\langle z, x_n\rangle| \geq \frac{||z||}{4}$$

and observe that

$$||hz|| \geq |\langle hz, x_n\rangle| = |\langle z, x_n\rangle| \geq \frac{||z||}{4}.$$

So we may regard Z as a closed subspace of $Y^\odot$. Since $X^{\odot\odot}/X$ is separable, by Lemma 6.1.7 also $Y^{\odot\odot}/Y$ is separable. But Y itself is separable, forcing $Y^{\odot\odot}$ to be separable. But note that by Theorem 6.1.1, $h^* : Y^{\odot *} \to Z^*$ maps $Y^{\odot\odot}$ onto $Z^\odot$. Hence $Z^\odot$ is separable, being a quotient of the separable space $Y^{\odot\odot}$.

Let $k : Z \to X^\odot$ be the inclusion map. Now by assumption $X^{\odot *}/X$ is separable, hence so is $X^{\odot *}/X^{\odot\odot}$, since X is a closed subspace of $X^{\odot\odot}$. Hence there is a separable subspace W of $X^{\odot *}$ such that $X^{\odot\odot} + W$ is norm-dense in $X^{\odot *}$. Thus $Z^\odot + k^*W = k^*X^{\odot\odot} + k^*W$ is dense in $k^*X^{\odot *} = Z^*$. This proves that Z^* is separable. ////

Since closed subspaces of spaces with the RNP have the RNP, we conclude:

Corollary 6.2.11. *If X does not have the RNP, then $X^{\odot *}/X$ is non-separable.*

Taking $T(t) := I$ in Theorems 6.2.9 and 6.2.10 we obtain the following result, which was first proved by Kuo [Ku]:

Corollary 6.2.12. *If X^{**}/X is separable, then both X^* and X^{**} have the RNP.*

We conclude this chapter with some remarks on $\odot$-reflexive groups.

Corollary 6.2.13. *Suppose X is $\odot$-reflexive with respect to a C_0-group $T(t)$. The following are equivalent:*
(i) X has the RNP;
(ii) $X = X^{\odot *}$;
(iii) $X^{\odot *}/X$ is separable.

Proof: Assume (i). By assumption, $X^{\odot\odot}(= X)$ has the RNP. By Corollary 6.2.4 and since $T(t)$ is a group we see that $T^{\odot *}(t)$ is strongly continuous, i.e. $X^{\odot\odot} = X^{\odot *}$ and we obtain (ii). Then implication (ii)$\Rightarrow$(iii) is trivial, and (iii)$\Rightarrow$(i) follows from Theorem 6.2.10. ////

The implication (ii)$\Rightarrow$ (i) holds for C_0-*semi*groups:

Proposition 6.2.14. *If $T(t)$ is $\odot$-reflexive and Fav$(T(t)) = D(A)$, then X has the RNP.*

Proof: Let Y be any separable closed subspace of X and let Z be the smallest closed $T(t)$-invariant subspace of X containing Y. Then also Z is separable. Clearly Z is $\odot$-reflexive and Fav$(T(t)) = D(A)$ holds for the restricted semigroup on Z, so by Theorem 4.2.6 we have $Z = Z^{\odot *}$. In other words, Z is a separable dual space and hence has the RNP by property (iii) preceding Theorem 6.2.3. Since Y is a closed subspace of Z, also Y has the RNP by property (ii). By property (iv), X has the RNP. ////

By applying Theorem 6.2.9 we get the following analogue of Corollary 6.2.13.

Corollary 6.2.15. *Suppose X is $\odot$-reflexive with respect to a C_0-group $T(t)$. The following are equivalent:*
(i) X^* has the RNP;
(ii) $X^\odot = X^*$;
(iii) $X^*/X^\odot$ is separable.

Hence either $X^*/X^\odot$ is non-separable or else $X^\odot = X^*$, and which of these two alternatives is fulfilled depends only on a geometrical property of the underlying Banach space. A similar remark applies to 6.2.13.

A Banach lattice is reflexive if and only if both X and X^* have the RNP [DU, p. 95]. Combining this with Theorems 6.2.9 and 6.2.10 we obtain:

Theorem 6.2.16. *Suppose $T(t)$ is a C_0-semigroup on a non-reflexive Banach lattice X. Then at least one of the quotients $X^*/X^\odot$, $X^{\odot *}/X^{\odot\odot}$ or $X^{\odot\odot}/X$ is non-separable.*

More generally, it follows that if $T(t)$ is a C_0-semigroup on a Banach space X such that either X or X^* does not have the RNP (this is the case for most of the classical non-reflexive spaces), then at least one of these three quotients is non-reflexive.

Notes. The standard reference for the RNP is [DU]; see also [Bo] and [vD2].

Theorem 6.1.1 and Example 6.1.3 are taken from [Ne1]. The rest of this section, with the exception of Lemma 6.1.7 which is from [Ne3], consists of simple ramifications of Theorem 6.1.1. The idea to use the bipolar theorem in 6.1.7 is due to Günther Greiner.

Convolution semigroups are studied in detail in [Bu]. The second part of Proposition 6.2.1 solves Problem 4.13.5 in [Bu].

In the famous paper [DFJP] it was shown that every weakly compact map $T : X \to Y$ factors through a reflexive Banach space Z. If one applies this factorization scheme to the (non-weakly compact) 'summing' map $T : l^1 \to c$ (the Banach space of all convergent sequences with the sup-norm) given by

$$T\Big((\alpha_n)_{n=1}^\infty\Big) := (\sum_{i=1}^n \alpha_i)_{n=1}^\infty,$$

then the space Z thus obtained satisfies $\dim Z^{**}/Z = 1$, cf [Wo, p. 56]. In other words, to the summing semigroup one can also associate 'canonically' a quasi-reflexive space!

A short proof of Sobczyk's theorem, extending it to WCG spaces is given by Veech [Ve]. Theorem 6.2.3 was proved in [Ne2] for weak*-continuous semigroups. The proof given here is only a slight modification. For *adjoint* semigroups, Theorem 6.2.3 is implicit in W. Arendt [Ar2], where it is obtained by an entirely different method of proof.

Corollary 6.2.7 is new. The rest of Section 6.2 is taken from [Ne3].

Chapter 7

Tensor products

In Chapter 1 we saw that $C_0(\mathbb{R})^{\odot} = L^1(\mathbb{R})$ holds with respect to the translation group. In other words, translation of a measure $\mu \in C_0(\mathbb{R})^*$ is continuous if and only if $\mu \in L^1(\mathbb{R})$. In Section 7.1 we prove the following generalization of this result for vector-valued measures: let X be a Banach space and let μ be an X-valued Borel measure of bounded variation on $\mathbb{R}$, then $\lim_{t\to 0} \|\mu - \mu_t\| = 0$ if and only if $\mu \in L^1(\mu; X)$. Here μ_t is the measure given by $\mu_t(E) = \mu(E+t)$. In particular, if $X = Y^*$ is a dual space, then $C_0(\mathbb{R}; Y)^{\odot} = L^1(\mathbb{R}; Y^*)$ holds with respect to the translation group on $C_0(\mathbb{R}; Y)$. We then specialize to the case $Y = C(K)$, where we have a natural isomorphism $C_0(\mathbb{R}; C(K)) \simeq C_0(\mathbb{R} \times K)$, and give some additional representations of $C_0(\mathbb{R} \times K)^{\odot}$.

Now both $C_0(\mathbb{R}; Y)$ and $L^1(\mathbb{R}; Y^*)$ can be written as certain tensor products, namely $C_0(\mathbb{R}; Y) = C_0(\mathbb{R})\tilde{\otimes}_\epsilon Y$ and $L^1(\mathbb{R}; Y^*) = L^1(\mathbb{R})\tilde{\otimes}_\pi Y^*$ (the injective and projective tensor product respectively), and the translation group on $C_0(\mathbb{R}; Y)$ can be regarded as the tensor product $T_0(t)\tilde{\otimes}_\epsilon I$, with $T_0(t)$ translation on $C_0(\mathbb{R})$. This suggests the following question: *Given two Banach spaces X, Y, a strongly continuous semigroup $T_0(t)$ on X and two 'dual' tensor products $\tilde{\otimes}_i$, $i = 1, 2$, when is it true that $(X\tilde{\otimes}_1 Y)^{\odot} = X^{\odot}\tilde{\otimes}_2 Y^*$ with respect to the semigroup $T_0(t)\tilde{\otimes}_1 I$?* This question will be addressed in Section 7.3 for the injective- and projective tensor product, after recalling some terminology on tensor products in Section 7.2. The results can be applied to the vector-valued function spaces $L^1(\mu; Y)$ and $C_0(\Omega; Y)$.

7.1. The translation group in $C_0(\mathbb{R}; Y)$

Let X be a Banach space and let $M(\mathbb{R}; X)$ denote the Banach space of all countably additive X-valued vector measures of bounded variation [DU] on $\mathbb{R}$. If X is the scalar field we simply write $M(\mathbb{R})$. For a $\mu \in M(\mathbb{R}; X)$, its *variation* $|\mu| \in M(\mathbb{R})$ is defined by

$$|\mu|(E) := \sup_{\pi}\{\sum_{A\in\pi} \|\mu(E \cap A)\|\},$$

where the supremum is taken over all partitions π of $\mathbb{R}$ into finitely many disjoints subsets. If $\mu \in M(\mathbb{R}; X)$ then $|\mu|$ is a finite positive measure. If $f \in C_0(\mathbb{R})$ and $\mu \in M(\mathbb{R}; X)$, then for $E \subset \mathbb{R}$ measurable the integral $\int_E f\, d\mu$ is defined in the natural way and we have

$$\| \int_E f\, d\mu\| \leq \int_E |f|\, d|\mu|.$$

Moreover, for every $x^* \in X^*$ the map

$$f \mapsto \langle x^*, \int_{\mathbb{R}} f\, d\mu\rangle$$

defines a bounded linear functional on $C_0(\mathbb{R})$, hence an element $\langle x^*, \mu\rangle \in M(\mathbb{R})$. Obviously for $E \subset \mathbb{R}$ measurable we have

$$\langle x^*, \int_E f\, d\mu\rangle = \int_E f(s)\, d\langle x^*, \mu\rangle(s).$$

The space $L^1(\mathbb{R}; X)$ of all Bochner integrable X-valued functions on $\mathbb{R}$ (cf. Appendix) can be identified with a closed subspace of $M(\mathbb{R}; X)$ in the following way: for $h \in L^1(\mathbb{R}; X)$ define $\mu_h \in M(\mathbb{R}; X)$ by

$$\mu_h(E) := \int_E h(t)\, dt.$$

Lemma 7.1.1. *Suppose $\mu \in M(\mathbb{R}; X)$ and $f \in C(\mathbb{R})$ with $\lim_{t\to-\infty} f(t) = 0$. Define*

$$F(r) := \int_{-\infty}^{r} f(s)\, d\mu(s).$$

Then F is strongly measurable.

Proof: In order to apply Pettis's measurability theorem, we must show that (i) F is weakly measurable, and (ii) F is essentially separably valued.

To prove (i) first let m be a measure in $M(\mathbb{R})$. Then $\tilde{F}$ defined by

$$\tilde{F}(r) := \int_{-\infty}^{r} f(s)\, dm(s)$$

is measurable. (To see this we may assume that μ and f are real-valued, split $f = f_+ - f_-$ and $m = m_+ - m_-$ and note that if f and m are *positive* then $\tilde{F}$

is monotone, hence measurable). Using this we see that for any $x^* \in X^*$ the function

$$r \mapsto \langle x^*, F(r)\rangle = \int_{-\infty}^{r} f(s)\, d\langle x^*, \mu\rangle(s)$$

is measurable. This proves (i).

To prove (ii) define

$$F_1(r) := \int_{-\infty}^{r} |f(s)|\, d|\mu|(s).$$

Since F_1 is monotone, F_1 is continuous except at a countable set E. For $r_0 \notin E$, $r \in \mathbb{R}$ we have

$$||F(r) - F(r_0)|| = ||\int_{r_0}^{r} f(s)\, d\mu(s)|| \leq \int_{r_0}^{r} |f(s)|\, d|\mu|(s) = |F_1(r) - F_1(r_0)|.$$

From this it follows that F is continuous on $\mathbb{R}\backslash E$ as well. Since moreover $\mathbb{R}\backslash E$ is separable it follows that $F(\mathbb{R}\backslash E)$ is separable. This proves (ii). ////

Let Y be a Banach space. On $C_0(\mathbb{R}; Y)$ the translation group $T(t)$ is defined by

$$T(t)f(s) = f(t+s), \qquad t \in \mathbb{R}.$$

For the calculation of the semigroup dual of this space we use the well-known fact (see [DU, p. 181-182]) that the dual of $C_0(\mathbb{R}; Y)$ may be identified in a natural way with $M(\mathbb{R}; Y^*)$. From the case where Y is the scalar field we see that there are at least two natural candidates for $C_0(\mathbb{R}; Y)^{\odot}$: the absolutely continuous Y^*-valued measures and $L^1(\mathbb{R}; Y^*)$. In general these two spaces are different; in fact they are the same if and only if Y^* has the RNP.

Theorem 7.1.2. *With respect to the translation group on $C_0(\mathbb{R}; Y)$ we have $C_0(\mathbb{R}; Y)^{\odot} = L^1(\mathbb{R}; Y^*)$.*

Proof: First we prove that $L^1(\mathbb{R}; Y^*) \subset C_0(\mathbb{R}; Y)^{\odot}$. Let $y^* \in Y^*$ and $f \in L^1(\mathbb{R})$. Define $f \otimes y^* \in L^1(\mathbb{R}; Y^*)$ by

$$(f \otimes y^*)(s) = f(s)y^*.$$

Since translation is continuous on $L^1(\mathbb{R})$ it is clear that $f \otimes y^* \in C_0(\mathbb{R}; Y)^{\odot}$. Since the linear span of such functions is dense in $L^1(\mathbb{R}; Y^*)$ (cf. Section 7.2), the inclusion $L^1(\mathbb{R}; Y^*) \subset C_0(\mathbb{R}; Y)^{\odot}$ follows. We now prove the reverse inclusion. Let A be the generator of $T(t)$. Since $C_0(\mathbb{R}; Y)^{\odot} = \overline{D(A^*)}$ it suffices to prove the inclusion $R(\lambda, A^*)M(\mathbb{R}; Y^*) \subset L^1(\mathbb{R}; Y^*)$. First we claim that

$$R(\lambda, A)f(s) = \int_0^{\infty} e^{-\lambda t} f(s+t)\, dt$$

holds for all $f \in C_0(\mathbb{R}; Y)$ and $s \in \mathbb{R}$. To see this, fix $y \in Y$ arbitrary and let $(f \otimes y)(s) := f(s)y$. For $f \otimes y$ the claim is obvious, and since the linear span of all such elements is dense in $C_0(\mathbb{R}; Y)$ (cf. Section 7.2), the claim follows. For $f \in C_0(\mathbb{R}; Y)$, $\mu \in M(\mathbb{R}; Y^*)$ we have by Fubini's theorem

$$\begin{aligned}\langle R(\lambda, A^*)\mu, f\rangle = \langle \mu, R(\lambda, A)f\rangle &= \int_{\mathbb{R}} \int_0^\infty e^{-\lambda t} f(s+t)\, dt\, d\mu(s) \\ &= \int_{\mathbb{R}} \int_s^\infty e^{\lambda(s-t)} f(t)\, dt\, d\mu(s) \\ &= \int_{\mathbb{R}} \int_{-\infty}^t e^{\lambda(s-t)} f(t)\, d\mu(s)\, dt \\ &= \int_{\mathbb{R}} f(t)F(t)\, dt,\end{aligned}$$

where

$$F(t) := e^{-\lambda t} \int_{-\infty}^t e^{\lambda s}\, d\mu(s).$$

We will show that $F \in L^1(\mathbb{R}; Y^*)$. By Lemma 7.1.1, F is strongly measurable. But then we have

$$\begin{aligned}\int_{\mathbb{R}} \|F(t)\|\, dt &= \int_{\mathbb{R}} e^{-\lambda t} \Big\| \int_{-\infty}^t e^{\lambda s}\, d\mu(s) \Big\|\, dt \\ &\le \int_{\mathbb{R}} \Big(\int_s^\infty e^{\lambda(s-t)}\, dt \Big)\, d|\mu|(s) \\ &= \frac{1}{\lambda} |\mu|(\mathbb{R}) < \infty.\end{aligned}$$

This proves that $F \in L^1(\mathbb{R}; Y^*)$. But since we had

$$\langle R(\lambda, A^*)\mu, f\rangle = \int_{\mathbb{R}} f(t)F(t)\, dt$$

for all f it is clear that $F = R(\lambda, A^*)\mu$ and the proof is finished. ////

For $\mu \in M(\mathbb{R}; X)$ and $t \in \mathbb{R}$ we define $\mu_t \in M(\mathbb{R}; X)$ by $\mu_t(E) = \mu(E+t)$, where $E \subset \mathbb{R}$ is measurable. According to Theorem 7.1.2 we have, in case X is a dual space, that $\|\mu_t - \mu\| \to 0$ as $t \to 0$ if and only if $\mu \in L^1(\mathbb{R}; X)$. This easily extends to the case where X is an arbitrary Banach space.

Corollary 7.1.3. *Let $\mu \in M(\mathbb{R}; X)$. Then $\lim_{t\to 0} \|\mu_t - \mu\| = 0$ if and only if $\mu \in L^1(\mathbb{R}; X)$.*

Proof: Suppose $\|\mu_t - \mu\| \to 0$. Regarding μ as an X^{**}-valued vector measure, it follows from Theorem 7.1.2 that $\mu \in L^1(\mathbb{R}; X^{**})$. But since μ takes its values in X, the same must be true for the density function h_μ representing μ. In fact, by the Lebesgue differentiation theorem (Theorem A.2) we have for almost all s,

$$h_\mu(s) = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_s^{s+\epsilon} h_\mu(\tau)\, d\tau = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \mu(s, s+\epsilon).$$

Since $\mu(s, s+\epsilon) \in X$ for all ϵ it follows that h_μ is X-valued. The converse assertion is clear. ////

In the scalar case it is well-known that $C_0(\mathbb{R})^{\odot\odot} = BUC(\mathbb{R})$, the Banach space of bounded, uniformly continuous functions on $\mathbb{R}$. As might be expected, in the vector-valued case we get $C_0(\mathbb{R}; Y)^{\odot\odot} = BUC(\mathbb{R}; Y^{**})$. This follows from Theorem 7.3.11 below.

We will now investigate the special case of Theorem 7.1.2 where $Y = C(K)$ with K compact Hausdorff (or $Y = C_0(\Omega)$ with Ω locally compact Hausdorff). The correspondence $(f(t))(s) \leftrightarrow f(t, s)$ defines a natural isometric isomorphism $C_0(\mathbb{R}; C(K)) \simeq C_0(\mathbb{R} \times K)$.

The following lemma is more or less standard. We use the notation $\nu \ll \mu$ to express that the measure ν is absolutely continuous with respect to the positive measure μ.

Lemma 7.1.4. *Suppose $B \subset M(K)$ is separable. Then there is a positive $\mu \in M(K)$ such that $\nu \ll \mu$ for all $\nu \in B$.*

Proof: Let (ν_n) be a dense sequence in B and define $\mu := \sum_{n=1}^{\infty} 2^{-n} \|\nu_n\|^{-1} |\nu_n|$. Then $\nu_n \ll \mu$ for all n, so by closure also $\nu \ll \mu$ for all $\nu \in B$. ////

Identifying $C_0(\mathbb{R}; C(K))$ with $C_0(\mathbb{R}\times K)$ the translation group from above is given by

$$T(t)f(x, y) = f(x + t, y).$$

The following result gives an alternative representation of the semigroup dual of $C_0(\mathbb{R} \times K)$ with respect to this group. Lebesgue measure on $\mathbb{R}$ will be denoted by m; $\mu_1 \otimes \mu_2$ denotes the product measure of two measures μ_1, μ_2.

Lemma 7.1.5. $C_0(\mathbb{R} \times K)^{\odot} = \bigcup_{0 \le \mu \in M(K)} L^1(\mathbb{R} \times K, m \otimes \mu)$.

Proof: By Theorem 7.1.2 we have $C_0(\mathbb{R} \times K)^{\odot} = L^1(\mathbb{R}; M(K))$. But any $f \in L^1(\mathbb{R}; M(K))$ is essentially separably valued. Therefore, without loss of generality we may assume that $\{f(t) : t \in \mathbb{R}\}$ is a separable subset of $M(K)$. By Lemma 7.1.4, there is a positive $\mu \in M(K)$ such that $f(t) \ll \mu$ for all t. By the Radon-Nikodym theorem we may regard f as an element of $L^1(\mathbb{R}; L^1(K, \mu))$. By the Fubini theorem, the latter is isometric to $L^1(\mathbb{R} \times K, m \otimes \mu)$. This proves the inclusion $\subset$. For the reverse inclusion, let $\mu \ge 0$ and pick $f \in L^1(\mathbb{R} \times K, m \otimes \mu)$. Approximate f by a compactly supported $\tilde{f}$ in $C(\mathbb{R}\times K)$ and note that translation of $\tilde{f}$ is continuous in the L^1-norm. ////

By Lemma 7.1.5, any $\nu \in C_0(\mathbb{R} \times K)^{\odot}$ belongs to some $L^1(\mathbb{R} \times K, m \otimes \mu)$ with $\mu \geq 0$. We will now give an explicit description of a possible choice for μ. For $\nu \in M(\mathbb{R} \times K)$ positive, define $\pi\nu \in M(K)$ by $\pi\nu(F) := \nu(\mathbb{R} \times F)$. Then for $f \in C(K)$ we have

$$\int_K f(y)\ d\pi\nu(y) = \int_K \int_{\mathbb{R}} f(y)\ d\nu(x, y).$$

We need the following lemma.

Lemma 7.1.6. *Let λ, μ and ν be positive measures in $M(\mathbb{R})$, $M(K)$ and $M(\mathbb{R} \times K)$ respectively. If $\nu \ll \lambda \otimes \mu$ then $\nu \ll \lambda \otimes \pi\nu$.*

Proof: By assumption there is an $h \in L^1(\mathbb{R} \times K, \lambda \otimes \mu)$, $h \geq 0$ a.e., such that $d\nu = h\ d(\lambda \otimes \mu)$. Define

$$K_0 := \{y \in K : \int_{\mathbb{R}} h(x, y)\ d\lambda(x) = 0\};$$
$$K_1 := \{y \in K : \int_{\mathbb{R}} h(x, y)\ d\lambda(x) > 0\}.$$

By the Fubini theorem,

$$\nu(\mathbb{R} \times K_0) = \int_{K_0} \int_{\mathbb{R}} h(x, y)\ d\lambda d\mu = 0.$$

Now suppose $(\lambda \otimes \pi\nu)(A) = 0$. We have to show that $\nu(A) = 0$. But we have

$$\begin{aligned} 0 = (\lambda \otimes \pi\nu)(A) &= \int_K \int_{\mathbb{R}} \chi_A(x, y)\ d\lambda(x) d(\pi\nu)(y) \\ &= \int_K \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_A(x, y) h(z, y)\ d\lambda(x) d\lambda(z) d\mu(y) \\ &= \int_K \int_{\mathbb{R}} \chi_A(x, y) \Big(\int_{\mathbb{R}} h(z, y)\ d\lambda(z) \Big)\ d\lambda(x) d\mu(y) \\ &= \int_{K_1} \int_{\mathbb{R}} \chi_A(x, y) \Big(\int_{\mathbb{R}} h(z, y)\ d\lambda(z) \Big)\ d\lambda(x) d\mu(y) \end{aligned}$$

Since $\int_{\mathbb{R}} h(z, y)\ d\lambda(z) > 0$ for $y \in K_1$, we see that $A \cap (\mathbb{R} \times K_1)$ is a $\lambda \otimes \mu$-null set, hence also a ν-null set (since by assumption $\nu \ll \lambda \otimes \mu$). Therefore $A \subset (A \cap (\mathbb{R} \times K_1)) \cup (\mathbb{R} \times K_0)$ is a ν-null set. ////

Combination of Lemmas 7.1.5 and 7.1.6 gives the following intrinsic characterization of those ν belonging to $C_0(\mathbb{R} \times K)^{\odot}$.

Theorem 7.1.7. *$\nu \in C_0(\mathbb{R} \times K)^{\odot}$ if and only if $\nu \ll m \otimes \pi|\nu|$.*

One might wonder whether there is a more direct proof of Theorem 7.1.7. Indeed such a proof can be given, see [GNe]. What may be more surprising is that it is possible to re-deduce Theorem 7.1.2 from Theorem 7.1.7 and the Hahn-Banach Theorem 6.1.1.

Second proof of Theorem 7.1.2: Let Y be an arbitrary Banach space. By the Banach-Alaoglu theorem the dual unit ball $K := B_{Y^*}$ is weak*-compact. The map $i : Y \to C(K)$ defined by $iy(y^*) = \langle y^*, y \rangle$ is an isometric embedding. Let $\tilde{i} : C_0(\mathbb{R}; Y) \to C_0(\mathbb{R}; C(K)) = C_0(\mathbb{R} \times K)$ be the induced embedding. In this way we may regard $C_0(\mathbb{R}; Y)$ as a closed, translation invariant subspace of $C_0(\mathbb{R} \times K)$. Let $y^\odot \in C_0(\mathbb{R}; Y)^\odot$. We must show: $y^\odot \in L^1(\mathbb{R}; Y^*)$. By Theorem 6.1.1, $y^\odot$ can be extended to an element ν of $C_0(\mathbb{R} \times K)^\odot$. By Theorem 7.1.7 there is a density function $g \in L^1(\mathbb{R} \times K, m \otimes \pi|\nu|) = L^1(\mathbb{R}; L^1(K, \pi|\nu|))$ representing ν. We claim that $y^\odot = (\tilde{i})^* \nu$ can be regarded as an element of $L^1(\mathbb{R}; Y^*)$. To see this, let $f \in C_0(\mathbb{R}; Y)$ be arbitrary and note that

$$
\begin{aligned}
\int_{\mathbb{R}} f(\tau)\, dy^\odot(\tau) &= \langle y^\odot, f \rangle = \langle \nu, \tilde{i}(f) \rangle \\
&= \int_{\mathbb{R}} (\tilde{i}(f))(\tau)\, d\nu(\tau) = \int_{\mathbb{R}} g(\tau)\, (\tilde{i}(f))(\tau)\, d\tau \\
&= \int_{\mathbb{R}} g(\tau)\, i(f(\tau))\, d\tau = \int_{\mathbb{R}} i^*(g(\tau))\, f(\tau)\, d\tau.
\end{aligned}
$$

Hence $y^\odot$ can be represented by $\tilde{g}$, defined by $\tilde{g}(t) := i^*(g(t))$. Since $i^*(g(t)) \in Y^*$ for all $t \in \mathbb{R}$ we see that $y^\odot \in L^1(\mathbb{R}; Y^*)$ and the claim is proved.

7.2. Tensor products

Throughout this section X and Y will denote non-zero Banach spaces. Let $B(X,Y)$ denote the linear vector space of all bilinear forms on $X \times Y$. For each pair $(x,y) \in X \times Y$ the map $(x \otimes y)(\psi) := \psi(x,y)$ is a linear form on $B(X,Y)$, hence an element of the algebraic dual of $B(X,Y)$. The *(algebraic) tensor product* of X and Y is the linear hull in the algebraic dual of $B(X,Y)$ of all such $x \otimes y$. By definition, each $u \in X \otimes Y$ can be written in the form $u = \sum_{i=1}^n x_i \otimes y_i$. It is trivial to verify that for all $x_{(i)} \in X$, $y_{(i)} \in Y$ and scalars λ we have $\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y)$, $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$, and $(x_2 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$.

If X and Y are Banach spaces, one would also like to make $X \otimes Y$ into a Banach space. With this we will be concerned now. Everything in the remainder of this section can be found in [DU, Chapter 8].

Let $\mathcal{B}(X,Y)$ denote the Banach space of all continuous bilinear forms on $X \times Y$ under the norm $||\cdot||$ defined by

$$||\psi|| := \sup\{|\psi(x,y)| : x \in B_X, y \in B_Y\}.$$

Each $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$ naturally acts as a continuous bilinear form ψ_u on $X^* \times Y^*$ by

$$\langle \psi_u, (x^*, y^*) \rangle := \sum_{i=1}^{n} \langle x^*, x_i \rangle \langle y^*, y_i \rangle.$$

The *injective tensor norm* $||u||_\epsilon$ of $u \in X \otimes Y$ is defined as the norm of the bilinear form ψ_u on $X^* \times Y^*$. In other words,

$$||u||_\epsilon := \sup\{\psi_u(x^*, y^*) : x^* \in B_{X^*}, y^* \in B_{Y^*}\}.$$

By regarding the elements $\mathcal{B}(X, Y)$ as linear forms on $X \otimes Y$, the *projective tensor norm* $||u||_\pi$ of a $u \in X \otimes Y$ is defined as the norm induced by $\mathcal{B}(X, Y)$:

$$||u||_\pi := \sup\{|\psi(u)| : \psi \in B_{\mathcal{B}(X,Y)}\}.$$

For $x^* \in X^*$ and $y^* \in Y^*$, the element $x^* \otimes y^* \in X^* \otimes Y^*$ defines a linear form on $X \otimes Y$ by putting $\langle x^* \otimes y^*, x \otimes y \rangle := \langle x^*, x \rangle \langle y^*, y \rangle$. A *cross-norm* on $X \otimes Y$ is a norm $|\cdot|$ with the following properties:

(CN1) $|x \otimes y| \leq ||x|| \, ||y||$ for all $x \in X$ and $y \in Y$;

(CN2) For $x^* \in X^*$ and $y^* \in Y^*$, the element $x^* \otimes y^*$, regarded as an element of the normed space $(X \otimes Y, |\cdot|)^*$, has norm $\leq ||x^*|| \, ||y^*||$.

The following lemma is standard.

Proposition 7.2.1. *Suppose $|\cdot|$ is a cross-norm on $X \otimes Y$. Then:*

(i) $|x \otimes y| = ||x|| \, ||y||$ for all $x \in X$ and $y \in Y$;

(ii) $|x^* \otimes y^*| = ||x^*|| \, ||y^*||$ for all $x^* \in X^*$ and $y^* \in Y^*$.

(iii) $||\cdot||_\epsilon$ and $||\cdot||_\pi$ are cross-norms, and for all $u \in X \otimes Y$ we have

$$||u||_\epsilon \leq |u| \leq ||u||_\pi$$

Thus the injective- and the projective norm are the least- and the greatest cross-norms respectively.

The completions of $X \otimes Y$ with respect to the injective and projective tensor norm are called the *injective* and the *projective tensor product* respectively, notation $X \tilde{\otimes}_\epsilon Y$ and $X \tilde{\otimes}_\pi Y$. Sometimes these spaces are denoted by $X \check{\otimes} Y$ and $X \hat{\otimes} Y$ respectively.

The standard example for the ϵ-tensor product is as follows: let $X := C_0(\Omega)$, Ω locally compact, and Y be an arbitrary Banach space. Then there is a natural isometric isomorphism between $C_0(\Omega) \tilde{\otimes}_\epsilon Y$ and $C_0(\Omega; Y)$. The standard example for the π-tensor product is as follows: let $X := L^1(\mu)$, where μ is some positive measure, and Y an arbitrary Banach space. Then there is a natural isometric isomorphism between $L^1(\mu) \tilde{\otimes}_\pi Y$ and $L^1(\mu; Y)$. These isomorphisms have already been used implicitly in Theorem 7.1.2.

An element $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$ defines an operator $T_u \in \mathcal{L}(X^*, Y)$ by the formula

$$T_u x^* = \sum_{i=1}^{n} \langle x^*, x_i \rangle y_i.$$

The ϵ-norm on $X \otimes Y$ is precisely the norm induced by the operator norm on $\mathcal{L}(X^*, Y)$. Indeed, for $u = \sum_{i=1}^{n} x_i \otimes y_i$ the ϵ-norm is given by

$$\begin{aligned} \|T_u\| &= \sup\{ \Big\| \sum_{i=1}^{n} \langle x^*, x_i \rangle y_i \Big\| : x^* \in B_{X^*} \} \\ &= \sup\{ \Big| \sum_{i=1}^{n} \langle x^*, x_i \rangle \langle y^*, y_i \rangle \Big| : x^* \in B_{X^*}, y^* \in B_{Y^*} \} = \|u\|_\epsilon. \end{aligned}$$

It is well-known that dual spaces of tensor products can be identified with certain operator ideals. For $u^* \in (X \tilde{\otimes}_\epsilon Y)^*$ or $u^* \in (X \tilde{\otimes}_\pi Y)^*$, define $T_{u^*} \in \mathcal{L}(X, Y^*)$ by

$$\langle T_{u^*} x, y \rangle := \langle u^*, x \otimes y \rangle.$$

In this way it turns out that the dual of $X \tilde{\otimes}_\pi Y$ can be identified with the space $\mathcal{L}(X, Y^*)$. On the other hand, the dual of $X \tilde{\otimes}_\epsilon Y$ can be identified with the subspace of $\mathcal{L}(X, Y^*)$ of all *integral* operators $X \to Y^*$ (see [DU] for the definition), which we denote by $\mathcal{L}^i(X, Y^*)$. In what follows we will often identify u^* with the operator T_{u^*}.

A bounded linear operator $T \in \mathcal{L}(X)$ induces a linear operator $T \otimes I : X \otimes Y \to X \otimes Y$ by the formula

$$(T \otimes I)(x \otimes y) := Tx \otimes y.$$

The operator $T \otimes I$ is bounded for both the ϵ- and the π-norm. In fact, in both cases one has $\|T \otimes I\| = \|T\|$. The unique continuous extensions to $X \tilde{\otimes}_\epsilon Y$ and $X \tilde{\otimes}_\pi Y$ will be denoted by $T \tilde{\otimes}_\epsilon I$ and $T \tilde{\otimes}_\pi I$ respectively.

Lemma 7.2.2. *Suppose $|\cdot|$ is a cross-norm on $X \otimes Y$ with the additional property that every bounded linear operator $T : X \to Y$ extends to a bounded linear operator $T \tilde{\otimes} I$ on the completion $X \tilde{\otimes} Y$ of $X \otimes Y$ with respect to $|\cdot|$. Then $\sigma(T \tilde{\otimes} I) = \sigma(T)$.*

Proof: $\sigma(T \tilde{\otimes} I) \subset \sigma(T)$: Suppose $\lambda - T$ is invertible. Then $(\lambda - T)^{-1} \tilde{\otimes} I$ is a bounded operator on $X \tilde{\otimes} Y$ and it is obvious that on the dense subspace $X \otimes Y$, $(\lambda - T)^{-1} \otimes I$ is a two-sided inverse for $\lambda - (T \otimes I)$. By density it follows that $(\lambda - T)^{-1} \tilde{\otimes} I = (\lambda - (T \tilde{\otimes} I))^{-1}$, so $\lambda \in \varrho(T \tilde{\otimes} I)$.

$\sigma(T) \subset \sigma(T \tilde{\otimes} I)$: Suppose $\lambda \in \sigma(T)$. If $\lambda \in \sigma_{ap}(T)$, the approximate point spectrum of T (cf. [Na2]), then by definition we can choose an approximate eigenvector $(x_n)_{n=1}^{\infty}$, i.e., $\|x_n\| = 1$ for all n and

$$\lim_{n \to \infty} \|Tx_n - \lambda x_n\| = 0.$$

We claim that $(x_n \otimes y)_{n=1}^{\infty}$ is an approximate eigenvector of $T\tilde{\otimes}I$ for every norm-1 vector $y \in Y$. Indeed, we have $|x_n \otimes y| = ||x_n||\ ||y|| = 1$ and moreover

$$\begin{aligned}|(T\tilde{\otimes}I)(x_n \otimes y) - \lambda(x_n \otimes y)| &= |(Tx_n - \lambda x_n) \otimes y| \\ &= ||Tx_n - \lambda x_n||\ ||y|| \to 0, \qquad n \to \infty.\end{aligned}$$

Thus $\lambda \in \sigma(T\tilde{\otimes}I)$. If $\lambda \in \sigma(T)\backslash\sigma_{ap}(T)$ then the range of $\lambda - T$ cannot be dense. According to the Hahn-Banach theorem, $\lambda \in \sigma_p(T^*)$. Choose a norm-1 vector x^* such that $T^*x^* = \lambda x^*$. We claim that $\lambda \in \sigma_p((T\tilde{\otimes}I)^*)$ with eigenvector $x^* \otimes y^*$, where $y^* \neq 0$ is arbitrary in Y^*. Indeed, for any $x \otimes y$ we have

$$\begin{aligned}\langle (T\tilde{\otimes}I)^*(x^* \otimes y^*), x \otimes y\rangle &= \langle x^* \otimes y^*, Tx \otimes y\rangle \\ &= \langle x^*, Tx\rangle\langle y^*, y\rangle \\ &= \langle T^*x^*, x\rangle\langle y^*, y\rangle \\ &= \lambda\langle x^*, x\rangle\langle y^*, y\rangle \\ &= \lambda\langle x^* \otimes y^*, x \otimes y\rangle.\end{aligned}$$

The claim now follows from a density argument. Hence $\lambda \in \sigma((T\tilde{\otimes}I)^*) = \sigma(T\tilde{\otimes}I)$. The second inclusion is proved and the lemma follows. ////

7.3. The adjoint of $T_0(t) \otimes I$

Given a strongly continuous semigroup $T_0(t)$ on X with generator A_0 then $T(t) := T_0(t) \otimes I$ extends to a semigroup of bounded linear operators on $X\tilde{\otimes}_\epsilon Y$ and $X\tilde{\otimes}_\pi Y$ respectively. In fact it is easy to see that it is strongly continuous as well. Moreover, spectrum and resolvent can be described. We state these facts in the following proposition, in which $\tilde{\otimes}$ denotes either the ϵ- or the π-tensor product.

Proposition 7.3.1. *$T(t) = T_0(t)\tilde{\otimes}I$ is a C_0-semigroup. If we denote its generator by A then $\sigma(A) = \sigma(A_0)$. For λ in the resolvent set we have $R(\lambda, A) = R(\lambda, A_0)\tilde{\otimes}I$.*

Proof: By the spectral mapping formula (cf. [Na2]) we have

$$\sigma(R(\lambda, A_0))\backslash\{0\} = (\lambda - \sigma(A_0))^{-1}$$

and similarly for A. Hence, to prove the first assertion, we see that it suffices to show that $\sigma(R(\lambda, A)) = \sigma(R(\lambda, A_0)\tilde{\otimes}I)$, but this follows from the previous lemma. The second assertion is obvious (e.g. apply a density argument). ////

Our next aim is to give a description of the adjoints of $T(t)$ and $R(\lambda, A)$. In order to do this, we identify the dual spaces of $X\tilde{\otimes}_\pi Y$ and $X\tilde{\otimes}_\epsilon Y$ with $\mathcal{L}(X, Y^*)$ and $\mathcal{L}^i(X, Y^*)$ respectively. Given a bounded operator S on X, we want to determine the adjoint of $S\tilde{\otimes}I$, where $\tilde{\otimes}$ is either $\tilde{\otimes}_\epsilon$ or $\tilde{\otimes}_\pi$. Given $x \otimes y \in X \otimes Y$ and $S \in \mathcal{L}(X, Y^*)$ or $S \in \mathcal{L}^i(X, Y^*)$, then

$$\langle S, (T\tilde{\otimes}I)(x \otimes y)\rangle = \langle S, Tx \otimes y\rangle = \langle STx, y\rangle = \langle ST, x \otimes y\rangle.$$

This shows that we have $(T\tilde{\otimes}I)^*(S) = ST$. We summarize this observation in the following proposition.

Proposition 7.3.2. *The adjoint operators $T^*(t)$ and $R(\lambda, A^*) : \mathcal{L}(X, Y^*) \to \mathcal{L}(X, Y^*)$ are given as follows :*

$$T^*(t)(S) = ST_0(t), \qquad S \in \mathcal{L}(X, Y^*);$$
$$R(\lambda, A^*)(S) = SR(\lambda, A_0), \qquad S \in \mathcal{L}(X, Y^*).$$

The same assertions are valid for the $\tilde{\otimes}_\epsilon$ tensor product, with $\mathcal{L}(X, Y^*)$ replaced by $\mathcal{L}^i(X, Y^*)$.

Let us recall that the integral operators form a two-sided operator ideal, i.e. given $S \in \mathcal{L}^i(X, Y^*)$ and bounded linear operators $T_1 \in \mathcal{L}(X)$ and $T_2 \in \mathcal{L}(Y^*)$ then T_2ST_1 is integral as well and $||T_2ST_1||_i \leq ||T_2|| \cdot ||S||_i \cdot ||T_1||$. Here $|| \cdot ||_i$ is the norm induced by $(X\tilde{\otimes}_\epsilon Y)^*$.

Both dual spaces $\mathcal{L}(X, Y^*)$ and $\mathcal{L}^i(X, Y^*)$ contain $X^* \otimes Y^*$ as a subspace. In order to identify the closure of $X^* \otimes Y^*$ with appropriate subspaces of $\mathcal{L}(X, Y^*)$ and $\mathcal{L}^i(X, Y^*)$ respectively, we make for the rest of Section 7.3 the following assumption:

Assumption 7.3.3. *X^* has the approximation property.*

Recall that a Banach space X has the *approximation property (a.p)* if for each compact set $K \subset X$ and $\epsilon > 0$ there is a bounded finite rank operator T on X such that $||Tx - x|| \leq \epsilon$ holds for all $x \in K$. The Banach spaces l^p, $C_0(\Omega)$, $L^p(\mu)$ satisfy Assumption 7.3.3. In fact, only in 1973 Enflo [En] constructed a Banach space without the a.p. A dual Banach space X^* has the a.p. if and only if for all Banach spaces Y, every compact $T \in \mathcal{L}(X, Y)$ can be approxinated uniformly by finite rank operators [LT, Thm. I.1.e.5]. Moreover, if X^* has the a.p., then so does X.

The relevance of Assumption 7.3.3 for us is based on the following facts. If X^* has the a.p, then the closure of $X^* \otimes Y^*$ in $\mathcal{L}^i(X, Y^*)$ can be identified with $X^*\tilde{\otimes}_\pi Y^*$. Operators belonging to this closure are called *nuclear operators.* Also, the closure of $X^* \otimes Y^*$ in $\mathcal{L}(X, Y^*)$, which is $X^*\tilde{\otimes}_\epsilon Y^*$, is precisely the set of all compact operators from X into Y^*. The following is a result of Grothendieck: If X^* has the a.p. and S and T are bounded operators on X, S an integral operator and T weakly compact, then ST is nuclear. Since X^* has the a.p., every compact $T \in \mathcal{L}(X)$ can be approximated by finite-rank operators, hence, by the identifications discussed above, by elements of $X^* \otimes X$. The proofs can be found in [DU]; see also [LT, Chapter I.1.e].

Now we are going to show that in case of $\odot$-reflexivity the semigroup dual of the ϵ-tensor product can be described easily. We already noted in Chapter 3 that a semigroup is $\odot$-reflexive if and only if the resolvent of the generator is weakly compact.

We will use several times the obvious fact that $X^\odot$ is the norm closure of $R(\lambda, A^*)^2 X^*$.

Theorem 7.3.4. *Let X be $\odot$-reflexive with respect to $T_0(t)$. Then the semigroup dual of the semigroup $T(t)$ induced on $X\tilde{\otimes}_\epsilon Y$ is the closure in $X^*\tilde{\otimes}_\pi Y^*$ of $X^\odot \otimes Y^*$.*

Proof: For given $x^* \in X^*$ and $y^* \in Y^*$ we have $T^*(t)(x^* \otimes y^*) = (T_0^*(t)x^*)\otimes y^*$. It follows that

$$||T^*(t)(x^* \otimes y^*) - x^* \otimes y^*|| = ||(T_0^*(t)x^* - x^*)|| \cdot ||y^*||.$$

This shows that if $x^* \in X^\odot$ then $x^* \otimes y^* \in (X\tilde{\otimes}_\epsilon Y)^\odot$. Hence also the closed linear subspace of $X^*\tilde{\otimes}_\pi Y^*$ generated by $\{x^* \otimes y^* : x^* \in X^\odot, y^* \in Y^*\}$ is contained in $(X\tilde{\otimes}_\epsilon Y)^\odot$.

To prove the reverse inclusion, we first claim that $(X\tilde{\otimes}_\epsilon Y)^\odot \subset X^*\tilde{\otimes}_\pi Y^*$. For the rest of the proof fix $\lambda \in \varrho(A_0)$. For $S \in (X\tilde{\otimes}_\epsilon Y)^* = \mathcal{L}^i(X, Y^*)$ we have by Prop. 7.3.2 $R(\lambda, A^*)(S) = SR(\lambda, A_0)$. Since $R(\lambda, A_0)$ is weakly compact, from Grothendieck's theorem quoted above it follows that $SR(\lambda, A_0)$ is nuclear. Thus $R(\lambda, A^*)(S) \in X^*\tilde{\otimes}_\pi Y^*$ and the claim is proved.
Thus if we fix $S \in \mathcal{L}^i(X, Y^*)$, then for arbitrary $\epsilon > 0$ there exist $x_i \in X^*$, $y_i \in Y^*$ such that

$$||SR(\lambda, A_0) - \sum_{i=1}^{n} x_i^* \otimes y_i^*||_i < \epsilon.$$

It follows that

$$\begin{aligned} ||SR(\lambda, A_0)^2 - &\sum_{i=1}^{n} R(\lambda, A_0^*)x_i^* \otimes y_i^*||_i \\ &= ||S\Big(R(\lambda, A_0) - \sum_{i=1}^{n} x_i^* \otimes y_i^*\Big)R(\lambda, A_0)||_i \; < \epsilon \cdot ||R(\lambda, A_0)||. \end{aligned}$$

Since $R(\lambda, A_0^*)x_i^* \in X^\odot$ it follows that $R(\lambda, A^*)^2(S) = SR(\lambda, A_0)^2$ is in the closed linear subspace of $X^*\tilde{\otimes}_\pi Y^*$ generated by $\{x^* \otimes y^* : x^* \in X^\odot, y^* \in Y^*\}$. The conclusion follows. ////

We point out that, in contrast to the ϵ-tensor product, the π-tensor product is not injective, i.e. given a closed subspace X_1 of a Banach space X, then in general $X_1\tilde{\otimes}_\pi Y$ can not be identified with the closed linear subspace of $X\tilde{\otimes}_\pi Y$ generated by $\{x_1 \otimes y : x_1 \in X_1, y \in Y\}$. There are special cases where this is true however, e.g. if X_1 is complemented in X, or if Y is an AL-space (the definition is given in Chapter 8), in particular if $Y = M(\Omega)$ with Ω locally compact. Thus we have the following corollary.

Corollary 7.3.5. *If in addition $X^{\odot}$ is complemented in X^* or $Y = C_0(\Omega)$, Ω locally compact, then $(X\tilde{\otimes}_\epsilon Y)^{\odot} = X^{\odot}\tilde{\otimes}_\pi Y^*$.*

If $T_0(t)$ is a positive semigroup on a Banach lattice X whose dual has order continuous norm, then by Corollary 8.1.7 in the next chapter, $X^{\odot}$ is a projection band in X^*, hence an AL-space. This applies in particular to the case $X = C_0(\Omega)$ and we obtain from the Kakutani representation theorem (cf. Chapter 8):

Corollary 7.3.6. *Suppose $T_0(t)$ is a positive semigroup on $C_0(\Omega)$. Then there exists a measure space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ such that $C_0(\Omega; Y)^{\odot} = L^1(\tilde{\mu}; Y^*)$.*

Finally if $T_0(t)$ is translation on $C_0(\mathbb{R})$, then we recover Theorem 7.1.2.

Now we consider the projective tensor product. We are looking for conditions ensuring that the semigroup dual of $X\tilde{\otimes}_\pi Y$ can be identified with $X^{\odot}\tilde{\otimes}_\epsilon Y^*$. In contrast to Theorem 7.3.4, now $\odot$-reflexivity (weak compactness of the resolvent) is not sufficient as Example 7.3.9 below shows. If we require compactness of the resolvent however, then the semigroup dual can be described in a nice way.

Theorem 7.3.7. *Assume that the generator of the semigroup $T_0(t)$ on X has compact resolvent, then for the semigroup induced on $X\tilde{\otimes}_\pi Y$ we have $(X\tilde{\otimes}_\pi Y)^{\odot} = X^{\odot}\tilde{\otimes}_\epsilon Y^*$.*

Proof: As in the proof of Theorem 7.3.4, it can be shown that $X^{\odot}\tilde{\otimes}_\epsilon Y^*$ is contained in $(X\tilde{\otimes}_\pi Y)^{\odot}$. To prove the converse inclusion we observe that $R(\lambda, A_0)$ being compact implies that for $\epsilon > 0$ there exist $x_i \in X$ and $x_i^* \in X^*$ such that

$$||R(\lambda, A_0) - \sum_{i=1}^{m} x_i^* \otimes x_i|| < \epsilon.$$

Thus given $S \in \mathcal{L}(X, Y^*)$ then

$$\begin{aligned} ||SR(\lambda, A_0)^2 - &\sum_{i=1}^{m} R(\lambda, A_0^*)x_i^* \otimes Sx_i|| \\ &= ||S\Big(R(\lambda, A_0) - \sum_{i=1}^{m} x_i^* \otimes x_i\Big)R(\lambda, A_0)|| \ \le \epsilon||S||\ ||R(\lambda, A_0)||. \end{aligned}$$

It follows that $R(\lambda, A^*)^2(S)$ can be approximated with respect to the operator norm by elements of $X^{\odot} \otimes Y^*$. Since the operator norm induces the ϵ-norm it follows that $R(\lambda, A^*)^2(S) \in X^{\odot}\tilde{\otimes}_\epsilon Y^*$ for every $S \in \mathcal{L}(X, Y^*)$. Therefore $(X\tilde{\otimes}_\pi Y)^{\odot} \subset X^{\odot}\tilde{\otimes}_\epsilon Y^*$. ////

The case $X = L^1(\mu)$ was already proved in [Pal]. On spaces $C_0(\Omega)$, Ω locally compact, or spaces $L^1(\mu)$, a resolvent is weakly compact if and only it is compact (Corollary 3.2.4). Therefore the following corollary is an immediate consequence of Theorem 7.3.7.

Corollary 7.3.8. *Assume that X is either a space $L^1(\mu)$ or $C_0(\Omega)$, Ω locally compact. If the semigroup $T_0(t)$ is $\odot$-reflexive then $(X\tilde{\otimes}_\pi Y)^\odot = X^\odot\tilde{\otimes}_\epsilon Y^*$.*

In general weak compactness of the resolvent is not enough in Theorem 7.3.7, as the following example shows.

Example 7.3.9. Consider the semigroup of translations on $X = L^p(\mathbb{R})$. For $1 < p < \infty$ we have $L^p(\mathbb{R})^\odot = L^p(\mathbb{R})^* = L^q(\mathbb{R})$ with $1/p + 1/q = 1$ and the resolvent is weakly compact, X being reflexive. If we assume that $(L^p(\mathbb{R})\tilde{\otimes}_\pi Y)^\odot = L^q(\mathbb{R})\tilde{\otimes}_\epsilon Y^* = \{T \in \mathcal{L}(L^p(\mathbb{R}); Y^*) : T \text{ is compact}\}$ then from Proposition 7.3.2 we conclude that $SR(\lambda, A_0)$ is compact for every $S \in \mathcal{L}(L^p(\mathbb{R}); Y^*)$. Choosing $Y = L^q(\mathbb{R})$ and S the identity on $L^p(\mathbb{R})$ shows that $R(\lambda, A_0)$ has to be compact, which is not the case.

In case $p = 1$ the resolvent of the translation group even fails to be weakly compact and the conclusion of Theorem 7.3.7 again does not hold, as will be shown in Theorem 7.3.11.

The following lemma is taken from [Pal]. It uses the fact that every $f \in L^\infty(\mu, Y^*)$ can be identified in a natural way with an element of $L^1(\mu, Y)^* = (L^1(\mu)\tilde{\otimes}_\pi Y)^*$ by the formula

$$\langle f, g\rangle := \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu(\omega), \qquad f \in L^\infty(\mu; Y^*),\ g \in L^1(\mu; Y).$$

Lemma 7.3.10. *If T is a representable operator on $L^1(\mu)$, then $(T\tilde{\otimes}_\pi I)^*$ maps $L^1(\mu; Y)^*$ into $L^\infty(\mu; Y^*)$.*

Recall that representable operators were defined in Section 6.2.

Proof: We noted in Section 7.2 that $(X\tilde{\otimes}_\pi Y)^*$ is isometrically isomorphic to $\mathcal{L}(X, Y^*)$ by identifying $z^* \in (X\tilde{\otimes}_\pi Y)^*$ with the operator $T_{z^*} : X \to Y^*$ given by $\langle T_{z^*}x, y\rangle := \langle z^*, x \otimes y\rangle$. Taking $X = L^1(\mu)$ gives $L^1(\mu; Y)^* \simeq \mathcal{L}(L^1(\mu), Y^*)$. We claim that $z^* \in L^1(\mu, Y)^*$ corresponds to an element of $L^\infty(\mu; Y^*)$ if and only if T_{z^*} is representable. Suppose T_{z^*} is representable. Then T_{z^*} is given by $T_{z^*}f = \int_\Omega fg \, d\mu$ for all $f \in L^1(\mu)$ and some $g \in L^\infty(\mu; Y^*)$. Therefore

$$\langle z^*, f \otimes y\rangle = \langle \int_\Omega fg \, d\mu, y\rangle = \int_\Omega \langle g, f \otimes y\rangle \, d\mu$$

and hence $\langle z^*, F\rangle = \int_\Omega \langle g, F\rangle \, d\mu$ holds for all F in the dense subspace $L^1(\mu)\otimes Y$ of $L^1(\mu; Y)$. The converse follows from reversing the argument. This proves the claim. Now take some $T \in \mathcal{L}(L^1(\mu))$. For $z^* \in L^1(\mu; Y)^*$ we have

$$\langle (T\tilde{\otimes}_\pi I)z^*, f \otimes y\rangle = \langle z^*, Tf \otimes y\rangle = \langle T_{z^*}Tf, y\rangle$$

for all $f \in L^1(\mu)$ and $y \in Y$, which shows that the operator in $\mathcal{L}(L^1(\mu), Y^*)$ corresponding to $(T\tilde{\otimes}_\pi I)z^*$ is given by $T_{z^*}T$. Now suppose in addition that T is representable, so $Tf = \int_\Omega fg\, d\mu$ for all $f \in L^1(\mu)$ and some $g \in L^\infty(\mu, L^1(\mu))$. Then

$$(T_{z^*}T)f = \int_\Omega f(\omega)T_{z^*}(g(\omega))\, d\mu(\omega)$$

for all $f \in L^1(\mu)$, and hence $T_{z^*}T$ is representable by the function $T_{z^*} \circ g$. By this the lemma is proved. ////

Theorem 7.3.11. *If $T_0(t)$ is the translation group on $L^1(\mathbb{R})$ then $L^1(\mathbb{R}; Y)^\odot = BUC(\mathbb{R}; Y^*)$.*

Proof: First we claim that $R(\lambda, A_0)$ is representable. For almost all s we have

$$\begin{aligned}(R(\lambda, A_0)f)(s) &= \int_0^\infty e^{-\lambda t} f(s+t)\, dt \\ &= \int_{-\infty}^\infty e^{-\lambda(t-s)}\chi_{[s,\infty)}(t)f(t)\, dt.\end{aligned}$$

Define $g : \mathbb{R} \to L^1(\mathbb{R})$ by $(g(t))(s) = e^{-\lambda(t-s)}\chi_{[s,\infty)}(t)$. We have

$$||g(t)||_{L^1(\mathbb{R})} = \int_{-\infty}^\infty e^{-\lambda(t-s)}\chi_{[s,\infty)}(t)\, ds = \int_{-\infty}^t e^{-\lambda(t-s)}\, ds = \frac{1}{\lambda}.$$

Since also g is continuous as a map $\mathbb{R} \to L^1(\mathbb{R})$, hence in particular strongly measurable, this shows that $g \in L^\infty(\mathbb{R}; L^1(\mathbb{R}))$ and our claim is proved. From Lemma 7.3.10 we deduce that $L^1(\mathbb{R}; Y)^\odot \subset L^\infty(\mathbb{R}; Y^*)$. Let $h \in L^1(\mathbb{R}; Y)^\odot$. We claim that h is continuous. Let ϕ_n be any continuous function with compact support such that $\phi_n(t) = 1$ for all $t \in [-n, n]$. Clearly it suffices to prove that $h\phi_n$ is continuous for all n. Since each $h\phi_n$ is compactly supported and since obviously $h \in L^1(\mathbb{R}; Y)^\odot$ implies $h\phi_n \in L^1(\mathbb{R}; Y)^\odot$, we may consider $h\phi_n$ as an element of $L^1([-N_n, N_n]; Y)^\odot$ for some N_n large enough. Since $L^1([-N_n, N_n])$ is $\odot$-reflexive with respect to translation modulo $[-N_n, N_n]$ (cf. Example 1.3.9) we have by Theorem 7.3.8 that

$$\begin{aligned}L^1([-N_n, N_n]; Y)^\odot &= L^1([-N_n, N_n])^\odot \tilde{\otimes}_\epsilon Y^* \subset C([-N_n, N_n])\tilde{\otimes}_\epsilon X^* \\ &= C([-N_n, N_n]; Y^*).\end{aligned}$$

Hence $h\phi_n \in C([-N_n, N_n]; Y^*)$. This proves that $L^1(\mathbb{R}; Y)^\odot \subset C(\mathbb{R}; Y^*)$. But then we must have that actually $h \in BUC(\mathbb{R}; Y^*)$: h is bounded as an element of $L^\infty(\mathbb{R}; Y^*)$, and uniformly continuous since otherwise the map $t \mapsto T^*(t)h$ is easily seen not to be norm-continuous. This shows $L^1(\mathbb{R}; Y)^\odot \subset BUC(\mathbb{R}; Y^*)$. The reverse inclusion holds trivially. ////

This theorem is the L^1-analogue of Theorem 7.1.2. Now in general it is not true that

$$BUC(\mathbb{R};Y) = BUC(\mathbb{R})\tilde{\otimes}_\epsilon Y$$

holds. In fact, any function in $BUC(\mathbb{R})\tilde{\otimes}_\epsilon Y$ must have relatively compact range whereas it is easy to construct functions $F \in BUC(\mathbb{R}; C_0(\mathbb{R}))$ not having relatively compact range as follows. Let $f \in C_0(\mathbb{R})$ be any non-zero function. Then the set of translates $\{T(t)f : t \in \mathbb{R}\}$ is not relatively compact, so we can take $F(t) = T(t)f$.

Remark 7.3.12. (i) The above examples show that for translation on $L^p(\mathbb{R})$, $1 \le p < \infty$ the conclusion of Theorem 7.3.7 does not hold for every Banach space.
In fact, let X be *any* fixed Banach space and let $T_0(t)$ be a C_0-semigroup on X with generator A_0. We claim that if for every Y the formula $(X\tilde{\otimes}_\pi Y)^\odot = X^\odot\tilde{\otimes}_\epsilon Y^*$ holds, then $R(\lambda, A_0)$ must be compact. Let $Y = X^*$ and assume $(X\tilde{\otimes}_\pi Y)^\odot = X^\odot\tilde{\otimes}_\epsilon Y^*$. Then $R(\lambda, A)^*(T) = TR(\lambda, A_0)$ is a compact operator for every $T \in (X\tilde{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*) = \mathcal{L}(X, X^{**})$. In particular, letting $T : X \to X^{**}$ be the natural embedding, it follows that $R(\lambda, A_0)$ itself is compact. See also [Pa1], where $Y = l^\infty$ is taken.

(ii) Concerning 7.3.4, the situation is different and weak compactness of $R(\lambda, A_0)$ is not necessary in order that $(X\tilde{\otimes}_\epsilon Y)^\odot = \overline{X^\odot \otimes Y^*}^{X^*\tilde{\otimes}_\pi Y^*}$ holds for every Banach space Y. In fact, inspection of the proof of Theorem 7.3.4 shows that a necessary and sufficient condition for this is that $TR(\lambda, A_0)$ is nuclear for every operator $T \in \mathcal{L}^i(X, Y^*)$. An example of a semigroup without weakly compact resolvent but satisfying this condition (by Theorem 7.1.2 !) is translation in $C_0(\mathbb{R})$.

By combining 7.3.4 and 7.3.7 one can under suitable assumptions describe the second semigroup dual of the ϵ- and the π-tensor product. In order to apply 7.3.4 and 7.3.7 we formally need the assumption that $X^{\odot *}$ has the a.p. The proof below however shows that it suffices to have that X^* has the a.p.

For $L^1(\mu)\tilde{\otimes}_\pi Y$ the following result was first proved by de Pagter (unpublished).

Proposition 7.3.13. *Suppose $R(\lambda, A_0)$ is compact. Then:*

(i) $(X\tilde{\otimes}_\pi Y)^{\odot\odot}$ is the closure in $X^{\odot *}\tilde{\otimes}_\pi Y^{**}$ of $X \otimes Y^{**}$. If either X is complemented in $X^{\odot *}$ or Y is an $L^1(\mu)$-space then $(X\tilde{\otimes}_\pi Y)^{\odot\odot} = X\tilde{\otimes}_\pi Y^{**}$;

(ii) If either $X^\odot$ is complemented in X^* or $Y = C_0(\Omega)$, Ω locally compact Hausdorff, then $(X\tilde{\otimes}_\epsilon Y)^{\odot\odot} = X\tilde{\otimes}_\epsilon Y^{**}$.

Proof: First we prove (ii). By Corollary 7.3.5 we have $(X\tilde{\otimes}_\epsilon Y)^\odot = X^\odot\tilde{\otimes}_\pi Y^*$. The conclusion now follows from Theorem 7.3.7 in case $X^{\odot *}$ has the a.p. However, inspection of the proof of Theorem 7.3.7 shows that the a.p. was needed for showing that $R(\lambda, A_0)$ could be approximated by finite rank operators in the uniform operator topology. Hence what we must show in the present case is that $R(\lambda, A_0^\odot)$ *can be approximated by finite rank operators.* That this is true when X^* has the a.p., i.e. under Assumption 7.3.3 (regardless whether $X^{\odot *}$ has the a.p.), is shown by the following argument. Fix $\lambda \in \varrho(A_0)$. Since X^* has the a.p., $R(\lambda, A_0)$ is the uniform limit of finite rank operators $\Phi_n \in X^* \otimes X$. Then for $\mu \in \varrho(A_0)$, $R(\lambda, A_0)R(\mu, A_0)$ is the uniform limit of $\Phi_n R(\mu, A_0)$. Since $R(\mu, A_0^*)X^* \subset X^\odot$ it follows that $\Phi_n R(\mu, A_0) \in X^\odot \otimes X$. Moreover,

$$||R(\lambda, A_0^*)R(\mu, A_0^*) - (\Phi_n R(\mu, A_0))^*|| = ||R(\mu, A_0)R(\lambda, A_0) - \Phi_n R(\mu, A_0)||,$$

hence $\mu R(\lambda, A_0^\odot)R(\mu, A_0^\odot) = \mu R(\lambda, A_0^*)R(\mu, A_0^*)|_{X^\odot}$ is the uniform limit of $\mu\Phi_n R(\mu, A_0^*)|_{X^\odot} \in X \otimes X^\odot \subset X^{\odot *} \otimes X^\odot$. Since

$$R(\lambda, A_0^\odot) = \lim_{\mu\to\infty} \mu R(\lambda, A_0^\odot)R(\mu, A_0^\odot)$$

in the uniform operator topology (this follows from the resolvent equation for $A_0^\odot$), we can conclude that $R(\lambda, A_0^\odot)$ can be approximated by finite rank operators. As we noted above, from these considerations we can conclude that

$$(X^\odot \tilde{\otimes}_\pi Y^*)^\odot = X^{\odot\odot} \tilde{\otimes}_\epsilon Y^{**},$$

and since $R(\lambda, A_0)$ is compact we have $X^{\odot\odot} = X$, and (ii) is proved.

The first assertion of (i) is proved by a similar argument. Now suppose that X is complemented in $X^{\odot *}$. Then trivially every $T \in \mathcal{L}(X, Y^*)$ admits an extension to an operator in $\mathcal{L}(X^{\odot *}, Y^*)$. Also, if X is an $L^1(\mu)$-space, then Y^* is injective [LT, Section I.2.f] and this again implies that every $T \in \mathcal{L}(X, Y^*)$ admits an extension to an operator in $\mathcal{L}(X^{\odot *}, Y^*)$. In other words, in either case the natural map (induced by restriction $\pi : X^{\odot *} \to X$)

$$\pi : \mathcal{L}(X^{\odot *}, Y^*) \to \mathcal{L}(X, Y^*)$$

is *surjective*. But since $\mathcal{L}(Z, Y^*) = (Z\tilde{\otimes}_\pi Y)^*$ this shows that the canonical inclusion map

$$j : X\tilde{\otimes}_\pi Y \to X^{\odot *}\tilde{\otimes}_\pi Y$$

is an embedding. Applying this to Y^{**} instead of Y (and noting that Y^{***} is an $L^1(\mu)$-space if Y^* is) we obtain that $X\tilde{\otimes}_\pi Y^{**}$ can be regarded as a closed subspace of $X^{\odot *}\tilde{\otimes}_\pi Y^{**}$ and this proves the second assertion. ////

Notes. The fact that $C_0(\mathbb{R})^\odot = L^1(\mathbb{R})$ with respect to translation was first proved by Plessner [Pl].

The results of Sections 7.1 and 7.3 are joint work with Günther Greiner and are taken from [GNe]. They answer a question of Odo Diekmann, who asked (private communication) for a characterization of $C_0(\mathbb{R} \times K)^\odot$. This space is relevant in the context of the theory of structured populations.

Section 7.2 is based on [DU].

We have not dealt with $L^p(\mu; Y)$, $1 < p < \infty$. Most results from Section 7.3 have an analogue for these spaces too. We only sketch the ideas; for more complete information see [GNe]. For $L^p(\mu)$ one needs yet another tensor product, then so-called *l-tensor product*. This is a tensor product for Banach lattices and $L^p(\mu)\tilde{\otimes}_l Y$ is naturally isomorphic to $L^p(\mu; Y)$. Since the l-tensor product $X\tilde{\otimes}_l Y$ is asymmetric, one has to deal separately with the cases (i): $T_0(t)\tilde{\otimes}_l I$ and (ii): $I\tilde{\otimes}_l T_1(t)$. The respective results are:

(i) Suppose $T_0(t)$ is a positive C_0-semigroup on a Banach lattice X whose resolvent $R(\lambda, A_0)$ is r-compact. Then $(X\tilde{\otimes}_l Y)^{\odot}$ is the closure in $X^*\tilde{\otimes}_l Y^*$ of $X^{\odot}\tilde{\otimes}_l Y^*$. If $X^{\odot}$ is a sublattice of X^*, then $(X\tilde{\otimes}_l Y)^{\odot} = X^{\odot}\tilde{\otimes}_l Y^*$. An operator T on a Banach lattice X is called *r-compact* if the modulus $|T|$ exists and there is a sequence of finite rank operators $\Phi_n \in X^* \otimes X$ such that

$$\lim_{n\to\infty} \| \, |T - \Phi_n| \, \| = 0.$$

(ii) If either $R(\lambda, A_1)$ is weakly compact and X does not contain a sublattice isomorphic to l^1, or $R(\lambda, A_1)$ is compact, then $(X\tilde{\otimes}_l Y)^{\odot} = X^*\tilde{\otimes}_l Y^{\odot}$.

Also in this case, (i) can be extended to the translation group on $L^p(\mathbb{R})$ and we obtain:

(iii) With respect to translation on $L^p(\mathbb{R})$, $1 < p < \infty$, we have $L^p(\mathbb{R}; Y)^{\odot} = L^q(\mathbb{R}; Y^*)$, $p^{-1} + q^{-1} = 1$.

Since $L^p(\mathbb{R}; Y)^* = L^q(\mathbb{R}; Y^*)$ if and only if Y^* has the RNP, it follows that the adjoint of translation on $L^p(\mathbb{R}; Y)$ is strongly continuous if and only if Y^* has the RNP.

Chapter 8

The adjoint of a positive semigroup

In this chapter the adjoint of a positive semigroup is studied. If $T(t)$ is positive, then so is $T^*(t)$, but there is a problem with $T^{\odot}(t)$: in order to give meaning to the sentence '$T^{\odot}(t)$ is positive', the semigroup dual must have the structure of a Banach lattice. We study in Section 8.1 under what conditions this is the case. In Section 8.2 we look at the adjoints of positive semigroups on AM-spaces and generalize several classical results on the translation semigroup. In Section 8.3 we study the adjoint of a multiplication semigroup on an arbitrary Banach lattice. It is shown that essentially the only two types of $\odot$-reflexive multiplication semigroups are those on reflexive Banach lattices and on Banach lattices with unconditional basis. In Section 8.4 we apply some of the results to Banach function spaces.

8.1. When is $E^{\odot}$ a sublattice?

We start this section with fixing some terminology and recalling some facts on Banach lattices. For the proofs we refer to [AB], [S4] and [M].

A partially ordered real vector space $(E, \leq)$ is called a *Riesz space* if the following axioms are satisfied:

(R1) $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in E$;

(R2) $ax \geq 0$ for all $a \geq 0$ and $x \geq 0$;

(R3) for all $x, y \in E$ the least upper bound $x \vee y$ and the greatest lower bound $x \wedge y$ exist.

In (R3) it suffices to assume that only $x \vee y$ (or $x \wedge y$) exist. A Riesz space is also called a *vector lattice*. A real Banach space E which is also a Riesz space is called a *Banach lattice* if the norm has the following lattice property: $|x| \leq |y|$ implies $||x|| \leq ||y||$. Here $|x| = x \vee (-x)$ is the *modulus* of x; the vectors $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$ are the *positive part* and the *negative part* of x respectively. We have $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

A linear subspace F of a Riesz space E is called a *Riesz subspace* of E if for all $y_1, y_2 \in F$ the least upper bound $y_1 \vee y_2$, taken in E, belongs to F. In that case also $y_1 \wedge y_2$ and $|y_1|$ and $|y_2|$ belong to F. Conversely, if F is a linear subspace with the property that $|y| \in F$ for every $y \in F$, then F is already a Riesz subspace. A closed Riesz subspace of a Banach lattice will be called a *sublattice*. The closure of a Riesz subspace is always a sublattice. An *ideal* of a Riesz space E is a linear subspace F with the property that $x \in F$ whenever $|x| \leq |y|$ for some $y \in F$. Clearly every ideal is a Riesz subspace. An ideal F is called a *band* if for any subset of F its least upper bound, if it exists, belongs to F. Every band of a Banach lattice is closed. A band B is called a *projection band* if there is a band $B_\perp$ such that there is a direct sum decomposition $E = B \oplus B_\perp$. We say that x and y are *disjoint*, notation $x \perp y$, if $|x| \wedge |y| = 0$. If $x \in B$ and $y \in B_\perp$, then $|x| \wedge |y| \in B \cap B_\perp = \{0\}$, so any such decomposition is automatically disjoint. A projection associated with a projection band is called a *band projection*. A linear operator P is a band projection if and only if $0 \leq P \leq I$. Here we use the following notation: An operator T is *positive*, $T \geq 0$, if $Tx \geq 0$ for all $x \geq 0$. We say $T \geq S$ if $T - S \geq 0$. Every positive operator on a Banach lattice is bounded.

The dual space E^* of a Banach lattice is again a Banach lattice if for $u \geq 0$ we define $\langle x^* \vee y^*, u\rangle := \sup\{x^*(u_1) + y^*(u_2) : u_i \geq 0, u_1 + u_2 = u\}$. Moreover, $x^* \geq 0$ if and only if $\langle x^*, x\rangle \geq 0$ for all $x \geq 0$, and $T \geq 0$ if and only if $T^* \geq 0$.

A semigroup $T(t)$ on a Banach lattice E is *positive*, notation $T(t) \geq 0$, if for all $t \geq 0$ the operator $T(t)$ is positive. The basic question we address in this section is the following: If $T(t)$ is a positive C_0-semigroup on E, under what conditions is $E^\odot$ a Banach lattice? That $E^\odot$ be a Banach lattice is desirable, since then $T^\odot(t)$ is a *positive* semigroup again. Whether $E^\odot$ is always a Banach lattice, or even a sublattice of E^*, was an open problem for some time. Unfortunately, the following counterexample, due to A. Grabosch and R. Nagel [GNa], solves the problem in the negative.

Example 8.1.1. Let $E := L^1[0,1] \times L^1[0,1]$ with norm $||(f,g)|| := ||f|| + ||g||$. Consider the operator

$$A = \begin{pmatrix} d/dx & 0 \\ 0 & d/dx \end{pmatrix}$$

with domain

$$D(A) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in E : \; f, g \in AC[0,1], \; \begin{pmatrix} f(1) \\ g(1) \end{pmatrix} = B \begin{pmatrix} f(0) \\ g(0) \end{pmatrix} \right\}.$$

Here $AC[0,1]$ denotes the linear space of all absolutely continuous functions on $[0,1]$, and B is a real 2×2 matrix. The operator A generates a positive C_0-semigroup on E.

Identify the dual E^* with $L^\infty[0,1] \times L^\infty[0,1]$, where for $\phi, \psi \in L^\infty[0,1]$ we put $\langle(\phi,\psi),(f,g)\rangle := \langle\phi, f\rangle + \langle\psi, g\rangle$ and $||(\phi,\psi)|| = \sup(||\phi||_\infty, ||\psi||_\infty)$. Since for all $\begin{pmatrix} f \\ g \end{pmatrix} \in D(A)$,

$$\langle A^* \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \rangle = \langle \phi, f' \rangle + \langle \psi, g' \rangle$$

$$= \int_0^1 \langle \phi(s) f'(s)\, ds \rangle + \int_0^1 \psi(s) g'(s)\, ds \rangle$$

one sees easily from the fundamental theorem of calculus [Ru2, p. 167] that $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in D(A^*)$ if and only if $\phi, \psi \in Lip[0,1]$ and

$$\phi(1)f(1) - \phi(0)f(0) + \psi(1)g(1) - \psi(0)g(0) = 0.$$

Using the definition of $D(A)$, the last condition is equivalent to

$$\begin{pmatrix} \phi(0) \\ \psi(0) \end{pmatrix} = B^t \begin{pmatrix} \phi(1) \\ \psi(1) \end{pmatrix}.$$

Here $Lip[0,1]$ is the linear space of all Lipschitz continuous functions on $[0,1]$ and B^t is the adjoint matrix of B. Since $E^{\odot} = \overline{D(A^*)}$, one obtains

$$E^{\odot} = \left\{ \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in C[0,1] \times C[0,1] : \begin{pmatrix} \phi(0) \\ \psi(0) \end{pmatrix} = B^t \begin{pmatrix} \phi(1) \\ \psi(1) \end{pmatrix} \right\}.$$

It follows that $E^{\odot}$ is a sublattice of $C[0,1] \times C[0,1]$, and hence of E^*, if and only if B is a lattice homomorphism on $\mathbb{R}^2$. This is the case if and only if B is a positive diagonal- or off-diagonal matrix.

The situation is even worse: $E^{\odot}$ can even fail to be a Banach lattice with respect to its own ordering. Indeed, suppose that for some $x^{\odot} := \begin{pmatrix} y_0^* \\ z_0^* \end{pmatrix} \in E^{\odot}$ its modulus taken with respect to $X^{\odot}$,

$$|x^{\odot}|_{\odot} := \inf \left\{ \begin{pmatrix} \psi \\ z^* \end{pmatrix} \in X_+^{\odot} : \pm \begin{pmatrix} y_0^* \\ z_0^* \end{pmatrix} \leq \begin{pmatrix} \psi \\ z^* \end{pmatrix} \right\},$$

exists in $X^{\odot}$. Since the defining condition of $X^{\odot}$ refers only to the boundary points of $[0,1]$, it follows by a simple extension argument for continuous functions that $|x^{\odot}|_{\odot} = |x^{\odot}| = \begin{pmatrix} |y_0^*| \\ |z_0^*| \end{pmatrix}$. Therefore $E^{\odot}$ is a Banach lattice if and only it is a sublattice of E^*.

This example shows that the positivity of $T(t)$ is too weak a condition in order to obtain a satisfactory duality theory for positive semigroups. There are two possible remedies: either impose stronger conditions on $T(t)$ or restrict the attention to more well-behaved Banach lattices. We will pursue both strategies.

An operator T on a Riesz space E is called a *Riesz space homomorphism* if $T|x| = |Tx|$ for all $x \in E$. If a positive operator T is invertible with positive inverse, then T and its inverse are Riesz space homomorphisms. In this case T is called a *Riesz space isomorphism*. Two Riesz spaces are said to be *Riesz space isomorphic* if there is a Riesz space isomorphism between them. In the context of Banach lattices, a Riesz space homo(iso)morphism is automatically also an homo(iso)morphism of the underlying Banach spaces (since positive maps between Banach lattices are bounded), i.e. it is a *Banach lattice homo(iso)morphism*.

A semigroup $S(t)$ on E is called a *lattice semigroup* if for all $t \geq 0$ the operator $S(t)$ is a lattice homomorphism.

From now on $T(t)$ is a positive C_0-semigroup on E.

Theorem 8.1.2. *If $T^*(t)$ is a lattice semigroup on E, then $E^{\odot}$ is a sublattice of E^*.*

Proof: Let $x^{\odot} \in E^{\odot}$. We must prove that $|x^{\odot}| \in E^{\odot}$. But

$$\big|T^*(t)|x^{\odot}| - |x^{\odot}|\big| = \big||T^*(t)x^{\odot}| - |x^{\odot}|\big| \leq |T^*(t)x^{\odot} - x^{\odot}|,$$

where we used the fact that $\big||x| - |y|\big| \leq |x - y|$ holds in an arbitrary Riesz space. By the lattice property of the norm it follows that

$$\big\|T^*(t)|x^{\odot}| - |x^{\odot}|\big\| \leq \|T^*(t)x^{\odot} - x^{\odot}\|.$$

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Corollary 8.1.3. *If $T(t)$ extends to a positive group, then $E^{\odot}$ is a sublattice.*

These two sufficient conditions for $E^{\odot}$ to be a sublattice seem to be the only ones known.

We will now study when $E^{\odot}$ is an *ideal* in E^*. For this we need some more terminology. A subset G of a Riesz space E is said to be *order bounded* if there exists an $x \in E$ such that $|y| \leq x$ holds for all $y \in G$. A Riesz space E is said to be *Dedekind complete* (resp. *σ-Dedekind complete*) if every order bounded set (resp. every countable order bounded set) has a least upper bound. The dual of every Banach lattice is Dedekind complete. Every band in a Dedekind complete Riesz space is a projection band. A net $(x_\alpha)_{\alpha \in I}$ is said to be *directed upward*, notation $x_\alpha \uparrow$, if for any two indices $\alpha, \beta \in I$ such that $\alpha \leq \beta$ we have $x_\alpha \leq x_\beta$. If a net is directed upward and bounded from above by x, then we write $x_\alpha \uparrow\leq x$. If x is the least upper bound of $(x_\alpha)_{\alpha \in I}$, then we write $x_\alpha \uparrow x$. A Banach lattice has *order continuous norm* if $0 \leq x_\alpha \uparrow x$ implies that $x = \lim_\alpha x_\alpha$ in norm. Every Banach lattice with order continuous norm is Dedekind complete. Moreover, we have:

(i) Every $L^1(\mu)$-space has order continuous norm;

(ii) A σ-Dedekind complete Banach lattice has order continuous norm if and only if E does not contain a closed subspace isomorphic to l^∞.

(iii) A Banach lattice has order continuous norm if and only if every ideal is a band (and hence a projection band).

(iv) If E has order continuous norm and $G \subset E_+$ is relatively weakly compact, then also sol$G$ is relatively weakly compact.

Here the *solid hull* sol$G$ of a subset $G \subset E$ is the set

$$\mathrm{sol}G := \{x \in E : \exists y \in G \text{ with } |x| \leq |y|\}.$$

Note that for positive semigroups we have $R(\lambda, A) \geq 0$ for sufficiently large $\lambda > 0$; this is an easy consequence of the Laplace transform (0.5).

Lemma 8.1.4. *If $E^\odot$ is contained in a sublattice of E^* with order continuous norm, then $E^\odot$ is an ideal.*

Proof: Let F be a sublattice of E^* with order continuous norm, containing $E^\odot$.

Step 1. First let $0 \leq x^* \leq y^*$ with $y^* \in E^\odot$. We will show that $x^* \in E^\odot$. Choose $\lambda_0 > 0$ be such that $R(\lambda, A) \geq 0$ for $\lambda \geq \lambda_0$. Put

$$G := \{\lambda R(\lambda, A^*)y^* : \lambda \geq \lambda_0\}.$$

Since $y^* \in E^\odot$, this set is relatively compact subset of F, hence certainly relatively weakly compact in F. Let

$$\mathrm{sol}_F G := \{f \in F : \exists g \in G \text{ with } |f| \leq |g|\}$$

be the solid hull of G in F. By (iv) above, $\mathrm{sol}_F G$ is relatively weakly compact in F. Since $E^\odot \subset F$ and $0 \leq \lambda R(\lambda, A^*)x^* \leq \lambda R(\lambda, A^*)y^*$ for all $\lambda \geq \lambda_0$, it is clear that

$$H := \{\lambda R(\lambda, A^*)x^* : \lambda \geq \lambda_0\} \subset \mathrm{sol}_F G.$$

In particular, H is relatively weakly compact in F. Let z^* be any weak accumulation point of H as $\lambda \to \infty$. Then z^* is also a weak*-accumulation point of H. But by weak*-continuity we know that $\lambda R(\lambda, A^*)x^*$ weak*-converges to x^*, and therefore necessarily $z^* = x^*$. Since each $\lambda R(\lambda, A^*)x^*$ belongs to $E^\odot$, it follows that x^* belongs to the weak closure of $E^\odot$. Hence $x^* \in E^\odot$.

Step 2. Suppose $|x^*| \leq |y^*|$ with $y^* \in E^\odot$. We will show that $x^* \in E^\odot$. Since $0 \leq (x^*)^+ \leq |y^*|$ and similarly for $(x^*)^-$, we may assume that $x^* \geq 0$. For $\lambda \geq \lambda_0$ put

$$z^*_\lambda := |\lambda R(\lambda, A^*)y^*| \wedge x^*.$$

Then, since $x^* \geq 0$ and $\lambda R(\lambda, A^*) \geq 0$,

$$0 \leq z^*_\lambda \leq |\lambda R(\lambda, A^*)y^*| \leq \lambda R(\lambda, A^*)|y^*|,$$

and since $\lambda R(\lambda, A^*)|y^*|$ is a positive element in $E^\odot$, it follows from Step 1 that $z^*_\lambda \in E^\odot$. But since $y^* \in E^\odot$ we have $\lim_{\lambda\to\infty} |\lambda R(\lambda, A^*)y^*| = |y^*|$, and therefore

$$\lim_{\lambda\to\infty} z^*_\lambda = \lim_{\lambda\to\infty} |\lambda R(\lambda, A^*)y^*| \wedge x^* = |y^*| \wedge x^* = x^*.$$

Since $E^\odot$ is closed it follows that $x^* \in E^\odot$. ////

In the $\odot$-reflexive case one can prove a converse. For this we need the following lemma, which is an immediate consequence of Corollary 3.2.9 and property (ii) preceding Lemma 8.1.4.

Lemma 8.1.5. *Suppose E is σ-Dedekind complete. If E is $\odot$-reflexive with respect to a C_0-semigroup, then E has order continuous norm.*

Theorem 8.1.6. *Suppose E is $\odot$-reflexive with respect to a positive C_0-semigroup. Then the following assertions are equivalent:*
(i) $E^\odot$ is an ideal;
(ii) $E^\odot$ is contained in a sublattice with order continuous norm;
(iii) $E^\odot$ is a σ-Dedekind complete sublattice.

Proof: The implication (iii)$\Rightarrow$(ii) follows from Lemma 8.1.5 and Corollary 3.2.8, (ii)$\Rightarrow$(i) follows from Lemma 8.1.4 and (i)$\Rightarrow$(iii) follows from the fact that the dual of a Banach lattice is always Dedekind complete. ////

Theorem 8.1.7. *Suppose E^* has order continuous norm. If $T(t)$ is a positive C_0-semigroup on E, then $E^\odot$ is a projection band in E^*.*

This follows from Lemma 8.1.4 and property (iii) preceding it. Examples of Banach lattices whose duals have order continuous norm are the $C(K)$ spaces and more generally, AM-spaces (see Section 8.2).

In the following lemma we use the fact that every band in the dual of a σ-Dedekind complete Banach lattice is sequentially weak*-closed [S2].

Lemma 8.1.8. *Suppose $T(t)$ is a C_0-semigroup on a σ-Dedekind complete Banach lattice E. Then the band generated by $E^\odot$ is E^*.*

Proof: The band F generated by $E^\odot$ is weak*-sequentially closed. Take $x^* \in E^*$ arbitrary. Since $\lambda_n R(\lambda_n, A)^* x^* \to x^*$ weak* for some sequence $\lambda_n \to \infty$, and since $\lambda_n R(\lambda_n, A)^* x^* \in E^\odot$, it follows that $x^* \in F$ and hence $F = E^*$.

Corollary 8.1.9. *Let $T(t)$ be a positive C_0-semigroup on c_0. Then $c_0^\odot = c_0^*$.*

This follows from Theorem 8.1.7, Lemma 8.1.8 and the fact that both c_0 and $l^1 = c_0^*$ have order continuous norm. More generally, Corollary 8.1.9 is valid for every σ-Dedekind complete Banach lattice whose dual has order continuous norm.

8.2. Positive semigroups on $C(K)$

In this section we study the adjoints of positive C_0-semigroups on $E = C(K)$ with K compact Hausdorff. For these semigroups there is very detailed information. We encountered already one result in Section 8.1, viz. $E^{\odot}$ is always a projection band.

The main result of this section is Theorem 8.2.6, which asserts that $T^*(t)$ is $C_{>0}$ as soon as it is weakly Borel measurable. This is quite surprising, since it is a priori not clear that weak measurability already implies separability of the orbits $t \mapsto T^*(t)x^*$.

Example 8.2.1. Let $E = l^2(\mathrm{IR})$ and define $T(t)$ by $T(t)f(s) := f(s+t)$. Since each $f \in E$ is non-zero for at most countably many values of s, and since the same holds for each $f^* \in E^* = l^2(\mathrm{IR})$, it follows that $\langle f^*, T(t)f\rangle = 0$ except for at most countably many values of $T(t)$. Hence $T(t)$ is weakly Borel measurable, but evidently $t \mapsto T(t)f$ is $C_{>0}$ if and only if $f = 0$.

Admittedly, this example is slightly misleading since $T(t) = T^{**}(t)$ is not the adjoint of a *strongly continuous* semigroup. Therefore let us give a somewhat more sophisticated example.

Example 8.2.2. Let JF be the *James function space*, which is defined as the completion of the linear span of the characteristic functions of subintervals of $[0,1]$ with respect to the norm

$$||f|| = \sup\Big(\sum_{i=0}^{k-1}\Big|\int_{t_i}^{t_{i+1}} f(t)dt\Big|^2\Big)^{\frac{1}{2}},$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < .... < t_k = 1$ of $[0,1]$. It is easy to show that $L^1[0,1] \subset JF$ as sets and $||f|| \leq ||f||_{L^1}$ for $f \in L^1[0,1]$. JF is separable and is spanned by the characteristic functions of intervals with rational endpoints. For more information on JF we refer to [LS]. Define the translation group $T(t)$ by

$$T(t)f(x) = f(x + t \bmod 1).$$

Then is it obvious that $||T(t)|| = 1$. From $||T(t)\chi_{[p,q]} - \chi_{[p,q]}|| \leq ||T(t)\chi_{[p,q]} - \chi_{[p,q]}||_{L^1}$ it follows that $T(t)$ acts in a strongly continuous way on characteristic functions and consequently $T(t)$ is a C_0-group on JF.

It is easy to see that any interval $[p,q] \subset [0,1]$ belongs to JF^* by putting

$$\langle [p,q], f\rangle = \int_{[p,q]} f(t)dt,$$

and $||[p,q]|| = 1$. Let $0 < s < 1$, let $0 \leq p < q < 1-s$ and take $t > 0$ sufficiently small. Then $T^*(t+s)[p,q] - T^*(s)[p,q] = [q+s, q+t+s] - [p+s, p+t+s]$, so $||T^*(t+s)[p,q] - T^*(s)[p,q]|| = 2$ and therefore $T^*(t)$ is not $C_{>0}$.

Since JF is separable and by [LS] contains no copy of l^1, by the Odell-Rosenthal theorem (cf. Section 5.2) each $x^{**} \in JF^{**}$ is the weak*-limit of a sequence in JF. Therefore, for all $x^* \in JF^*$ and $x^{**} \in JF^{**}$ the map $t \mapsto \langle x^{**}, T^*(t)x^*\rangle$ is Borel measurable, being the pointwise limit of a sequence of continuous functions.

Since we are going to apply the Riddle-Saab-Uhl Theorem 5.2.11, we start with a reduction to the separable case.

Lemma 8.2.3. *Let $T(t)$ be a positive C_0-semigroup on a Banach lattice E. If $T^*(t)$ is not $C_{>0}$, then there is a separable closed $T(t)$-invariant sublattice $F \subset E$ such that the adjoint $T_F^*(t)$ of the restrictions of $T(t)$ to F is not $C_{>0}$.*

Proof: Fix any $y^* \in B_{E^*}$ and $t_0 > 0$ such that $T^*(t_0)y^* \notin E^{\odot}$. There is a sequence $t_n \downarrow 0$, a number $\epsilon > 0$ and a sequence (x_n) of norm-1 vectors in E such that

$$|\langle T^*(t_0+t_n)y^* - T^*(t_0)\, y^*, x_n\rangle| > \epsilon.$$

Let F_1 be the sublattice of E generated by the countable set of vectors (x_n). It is not hard to see that F_1 is separable. Let G_1 be the linear span of the set $\{T(t)x : t \geq 0, x \in F_1\}$. By the strong continuity of $T(t)$ also G_1 is separable. Suppose the separable subspaces F_i and G_i have been chosen for $i = 1, ..., N$. Let F_{N+1} be the sublattice generated by G_N and let G_{N+1} be the linear span of the set $\{T(t)x : t \geq 0, x \in F_{N+1}\}$. Note that $F_n \subset F_{n+1}$ for all n. Let F_0 be the linear subspace $\bigcup_{n\geq 1} F_n$. Then F_0 is separable and F_0 is a sublattice of E: for if $x \in F_0$, say $x \in F_N$, then $|x| \in F_N \subset F_0$ since F_N is a sublattice. Consequently F_0 is a sublattice. By a similar argument it is shown that $T(t)F_0 \subset F_0$. Let $F := \overline{F_0}$. Then F is a closed separable $T(t)$-invariant sublattice. Denote the restriction of $T(t)$ to F by $T_F(t)$. Let $i : F \to E$ denote the inclusion map and put $y_F^* := i^*y^*$. We will show that $T_F^*(t_0)y_F^* \notin F^{\odot}$. Indeed, for each n we have

$$\begin{aligned} ||T_F^*(t_0+t_n)y_F^* - T_F^*(t_0)y_F^*|| &\geq |\langle y_F^*, T_F(t_0+t_n)x_n - T_F(t_0)x_n\rangle| \\ &= |\langle y^*, T(t_0+t_n)x_n - T(t_0)x_n\rangle| > \epsilon. \end{aligned}$$

////

If $||x+y|| = ||x|| + ||y||$ holds for every disjoint pair of positive vectors in E, then E is called an *abstract L^1-space* or simply an *AL-space*. Abstract L^p-spaces are defined similarly. Every AL-space has order continuous norm. In fact, every AL-space is lattice isometric to some $L^1(\mu)$-space (Kakutani representation theorem).

If $||x+y|| = \sup(||x||, ||y||)$ holds for every pair of disjoint positive vectors in E, then E is called an *abstract M-space* or *AM-space*. Every AM-space E is order isometric to a sublattice of a $C(K)$-space for some compact Hausdorff K. Moreover, if E has a order unit $u > 0$, then E is lattice isometric to a $C(K)$-space and the isomorphism can be chosen in such a way that u corresponds to the constant one function on K (Kakutani-Krein representation theorem). An *order unit* is a vector u with the property that the ideal generated by u is E. Every Banach lattice with an order unit can be given an equivalent lattice norm such that it becomes an AM-space whose closed unit ball is the order interval $[-u, u]$.

The dual of an AM-space is an AL-space. The dual of an AL-space is an AM-space with order unit $u > 0$ satisfying $\langle u, x\rangle = ||x||$ for all $x \geq 0$.

Theorem 8.2.4. *Suppose $T(t)$ is a positive contraction semigroup on an AM-space E. Then the following assertions are equivalent:*

(i) $T^*(t)$ is $C_{>0}$;

(ii) $T^*(t)$ is weakly Borel measurable;

(iii) $E_{\odot\odot} = kE^{\odot\odot}$.

Proof: (i)⇒(iii) follows from Corollary 5.2.8.

(iii)⇒(i): By Theorem 8.1.7, $E^{\odot}$ is a projection band in E^*. Let $\pi : E^* \to E^{\odot}_{\perp}$ be the band projection onto the orthogonal complement of $E^{\odot}$. Let u be the order unit of the AM-space E^{**} satisfying $\langle u, x^*\rangle = ||x^*||$ for all $x^* \geq 0$. Let $x^* \in E^*$ be arbitrary. Since $||T(t)|| \leq 1$ for all $t \geq 0$, if $t \geq s$ we have

$$\begin{aligned} ||\pi T^*(t)x^*|| &= ||\pi T^*(t-s)\big(\pi T^*(s)x^* + (1-\pi)T^*(s)x^*\big)|| \\ &= ||\pi T^*(t-s)(\pi T^*(s)x^*)|| \leq ||\pi T^*(s)x^*||, \end{aligned}$$

using that band projections have norm ≤ 1. Therefore, for $x^* \geq 0$ the map

$$t \mapsto \langle T^{**}(t)\pi^* u, x^*\rangle = \langle u, \pi T^*(t)x^*\rangle = ||\pi T^*(t)x^*||$$

is a non-increasing function, hence measurable. Hence, $t \mapsto T^{**}(t)\pi^* u$ is weak*-measurable and for each $t > 0$ the weak*-integral

$$v(t) := weak^* \int_0^t T^{**}(\sigma)\pi^* u \, d\sigma$$

exists. A calculation as in the proof of Theorem 0.1.1 shows that it defines an element of $E_{\odot\odot}$, so by assumption (iii) we have $v(t) \in kE^{\odot\odot}$. But on the other hand, we have $\pi^* u \in E^{\odot\perp}$, the annihilator of $E^{\odot}$, hence also $T^{**}(t)\pi^* u \in E^{\odot\perp}$ for all $t \geq 0$, which implies that $v(t) \in E^{\odot\perp}$. Therefore, necessarily $v(t) = 0$. Now let $0 \leq y^* \in E^{\odot}_{\perp}$. Then

$$0 = \langle v(t), y^*\rangle = \int_0^t \langle u, \pi T^*(\sigma)y^*\rangle \, d\sigma = \int_0^t ||\pi T^*(\sigma)y^*|| \, d\sigma.$$

The latter integrand is non-increasing, which forces that $\pi T^*(t)y^* = 0$ for all $t > 0$, i.e. $T^*(t)y^* \in E^{\odot}$ for all $t > 0$.

(i)⇒(ii) is trivial.

(ii)⇒(i): If F is a closed $T(t)$-invariant subspace of E, then by identifying F^{**} with the double annihilator $F^{\perp\perp} \subset E^{**}$, it is easily checked that the adjoint of the restricted semigroup $T_F(t)$ is weakly Borel measurable again. Since a sublattice of an AM-space is an AM-space, by Lemma 8.2.3 we may assume without loss of generality that E is separable. But then by Corollary 5.2.12 we have $E_{\odot\odot} = kE^{\odot\odot}$ and by what we have already proved $T^*(t)$ is $C_{>0}$. ////

It follows from this theorem that the adjoints of the translation group on $C_0(\mathbb{R})$ and the rotation group on $C(T)$ are not weakly Borel measurable, cf. Example 5.2.13. This is well-known; a simple proof can be found in [Fe].

If E is a $C(K)$-space, then essentially every positive semigroup is already a contraction semigroup:

Lemma 8.2.5. *Let $T(t)$ be a positive C_0-semigroup on an AM-space with order unit u. Then there is a $\lambda > 0$ and an equivalent AM-norm relative to which $e^{-\lambda t}T(t)$ is a contraction semigroup.*

Proof: By replacing u by $|u|$, we may assume $u > 0$. By representing E as a $C(K)$-space with u corresponding to the function 1_K, from the denseness of $D(A)$ it follows that there is an order unit $v \in D(A)$. Choose an equivalent lattice norm $||\cdot||$ in such a way that the closed unit ball B_E is precisely the order interval $[-v, v]$. Since $D(A) \subset \text{Fav}(T(t))$ there is a constant K such that for $0 \le t \le 1$,

$$||T(t)|| = ||T(t)v|| \le ||v|| + ||T(t)v - v|| \le (1 + tK)||v|| = 1 + tK.$$

Hence

$$||e^{-Kt}T(t)|| \le 1, \qquad \forall\, 0 \le t \le 1.$$

Choose $M > 0$ so large that $||e^{-Mt}T(t)|| \le 1$ for all $t \ge 1$. Then $\lambda := \max(K, M)$ will do. ////

Theorem 8.2.6. *Let $T(t)$ be a positive C_0-semigroup on a $C(K)$-space. Then the following assertions are equivalent:*

(i) $T^*(t)$ is $C_{>0}$;

(ii) $T^*(t)$ is weakly Borel measurable;

(iii) $E_{\odot\odot} = kE^{\odot\odot}$.

The next result is due to B. de Pagter. It generalizes a classical result of Wiener and Young [WY], which asserts that if $\mu \in C_0(\mathbb{R})^*$ is singular with respect to the Lebesgue measure, then μ is singular with respect to almost all of its own translates: $\mu_t \perp \mu$, a.a-t.

Theorem 8.2.7. *Suppose either E has a quasi-interior point or E^* has order continuous norm. Let $T(t)$ be a positive C_0-semigroup on E. If $x^* \perp E^{\odot}$, then $T^*(t)x^* \perp x^*$ for almost all $t \ge 0$.*

8.3. Multiplication semigroups

In Section 8.2 we considered positive C_0-semigroups on a special class of Banach lattices. In the present section we let E be arbitrary but instead confine ourselves to a special class of semigroups, viz. multiplication semigroups. Recall that we take Banach lattices to be *real*. This will be important for some of the results in this section.

A C_0-semigroup $T(t)$ on a Banach lattice E is called a *multiplication semigroup* if for all $t \geq 0$ the operator $T(t)$ is *band preserving*, i.e. $T(t)B \subset B$ holds for every band B of E. The name 'multiplication semigroup' is justified by the well-known facts that every band preserving operator on an $L^p(\mu)$-space can be represented as multiplication with some bounded measurable function and every band preserving operator on $C(K)$ can be represented as multiplication with a function in $C(K)$.

In order to be able to deal with band preserving operators in a convenient way we start by showing that a large class of band preserving operators can be represented as multiplication operators on a suitably chosen representation space (Lemma 8.3.2).

A vector u is called a *weak order unit* if the band generated by u is norm dense, and a *quasi-interior point* if the ideal generated by u is norm-dense. Note that every separable Banach space has quasi-interior point $u > 0$: take a dense sequence (x_n) in $B_E \cap E_+$ and let $u = \sum_n 2^{-n} x_n$. Our main tool is an analogue of the Kakutani-Krein representation theorem valid for Banach lattices with quasi-interior points, due to Davies [Da1], Lotz [Lo1] and Schaefer [S3]; see also [Na1].

In the following $\overline{\mathbb{R}}$ denotes the two-point compactification of $\mathbb{R}$.

Theorem 8.3.1. *Suppose E is a Banach lattice with quasi-interior point $u > 0$. There exists a compact Hausdorff space K_u such that E is Riesz space isomorphic to a Riesz space $\hat{E}$ of continuous $\overline{\mathbb{R}}$-valued functions on K_u. Moreover, each $\hat{x} \in \hat{E}$ is finitely-valued on an open dense subset of K_u. The space $\hat{E}$ contains $C(K_u)$ as a dense ideal and u can be identified with the constant one function.*

Here $\hat{E}$ is given the norm of E. The space K_u is called the *representation space* of E. Often we will identify $\hat{E}$ with E.

E is in fact an 'ideal' in the space of all continuous functions $K_u \to \overline{\mathbb{R}}$ in the following sense: if $x \in E$ and $f : K_u \to \overline{\mathbb{R}}$ satisfies $0 \leq |f(s)| \leq |x(s)|$ for all $s \in K_u$, then f represents an element of E.

A compact Hausdorff space is *Stonean* if the closure of every open set is open. If E is Dedekind complete, then K_u is Stonean.

Following [Na1], call a subset $N \subset K_u$ *E-null* if the ideal $\{f \in E : f(s) = 0, \ \forall s \in N\}$ is norm-dense in E. A set N is E-null if and only if there exists a $g \geq 0$ in E such that $N \subset \{g = \infty\}$. If N is E-null, then $K_u = \beta(K_u \backslash N)$, the Stone-Čech compactification of $K_u \backslash N$. One can use this to define for any two $f, g \in E$ a continuous function $fg : K_u \to \overline{\mathbb{R}}$ in the following way. Set $fg(s) := f(s)g(s)$ for $s \notin \{f = \pm\infty\} \cup \{g = \pm\infty\}$ and extend this function in a unique way to a continuous $\overline{\mathbb{R}}$-valued function on K_u.

A linear operator A with domain $D(A)$ on E is *band preserving* if $A(B \cap D(A)) \subset B$ for every band B in E. Similarly, A is called *positive* if $Ax \geq 0$ for all $0 \leq x \in D(A)$. Of course this makes only sense if $D(A)$ contains 'many' positive elements, e.g. if $D(A)$ is a Riesz subspace.

Lemma 8.3.2. *Suppose A is a positive band preserving operator on a Banach lattice E with $D(A)$ a Riesz subspace. Suppose further that $D(A)$ contains a quasi-interior point $u > 0$. Then there is a continuous function $0 \leq g : K_u \rightarrow \overline{\mathbb{R}}$ such that $Af = gf$ holds for all $f \in D(A)$.*

Proof: Define $g : K_u \rightarrow \overline{\mathbb{R}}$ to be the function representing Au. The set $N := \{s \in K_u : g(s) = \infty\}$ is E-null. Fix $s \notin N$. If $0 \leq h \in D(A)$ is any element satisfying $0 < h(s) < \infty$ then we claim that $Ah(s) = g(s)h(s)$ holds. By multiplying h with a scalar we may assume that $h(s) = 1 = u(s)$. Since $D(A)$ is a Riesz subspace, $h \wedge (1+\epsilon)u \in D(A)$ for all $\epsilon > 0$. Note that $(h \wedge (1+\epsilon)u)(\tau) = h(\tau)$ holds on some open neighbourhood of s. Hence, since A is band preserving, it follows that $Ah(s) = A(h \wedge (1+\epsilon)u)(s)$. By the positivity of A we obtain

$$Ah(s) = A(h \wedge (1+\epsilon)u)(s) \leq A((1+\epsilon)u)(s) = (1+\epsilon)g(s)u(s) = (1+\epsilon)g(s)h(s).$$

Since ϵ is arbitrary it follows that $Ah(s) \leq g(s)h(s)$. Arguing similarly with $h \vee (1-\epsilon)u$ we obtain $Ah(s) \geq g(s)h(s)$. This proves the claim. If $0 \leq f \in D(A)$ is arbitrary, let $N_f := \{f = \infty\}$. We will show that $Af = gf$ holds outside the E-null set $N_f \cup N$. If $f(s) > 0$ this follows from the claim. If $f(s) = 0$, then note that

$$Af(s) = A(f+u)(s) - Au(s) = g(s)(f+u)(s) - g(s)u(s) = g(s)f(s).$$

The proof is completed by the observation that $K_u = \beta(K_u \backslash F)$ for any E-null set $F \subset K_u$. ////

For bounded operators there is an improvement of Lemma 8.3.2 for which we need the following two well-known results [AB, Ch. 8 and 15], [Wi].

Lemma 8.3.3. *For an operator T on a Banach lattice E the following properties are equivalent:*
(i) T is band preserving;
(ii) T is ideal preserving;
(iii) there exists a number $\lambda > 0$ such that $T \in [-\lambda I, \lambda I]$.

By (iii) an operator T satisfying one of these equivalent conditions is bounded. Also note that by (iii) the adjoint of T is again band preserving. The lemma fails for unbounded operators. Take for instance multiplication with the function $g(s) = s^{-1}$ on $C[0,1]$ with maximal domain. This operator is band preserving but not ideal preserving.

There is a natural partial ordering on $\mathcal{L}(E)$ by declaring $S \leq T$ if $T - S$ is positive. For an arbitrary bounded operator T on E the least upper bound of T and $-T$, taken in $\mathcal{L}(E)$, need not exist. In case it exists we denote it by $|T|$. The operator $|T|$ is called the *modulus* of T.

Lemma 8.3.4. *T is a band preserving operator on E, then $|T|$ exists and satisfies $|Tx| = |T||x| = \big||T|x\big|$.*

Lemma 8.3.5. *Let E be a Banach lattice with quasi-interior point $u > 0$. If T is a band preserving operator on E, then there exists a $g_u \in C(K_u)$ such that $Tx = g_u x$ holds for all $x \in E$.*

Proof: Since $|T|$ exists by Lemma 8.3.4, also T^+ and T^- exist. By Lemmas 8.3.2 and 8.3.3(ii), both T^+ and T^- can be represented by functions in $C(K_u)$. ////

We will now state the two main generation theorems for multiplication semigroups. We omit the somewhat lengthy proofs, since here we are mainly interested in duality of multiplication semigroups and we will not need these results.

Theorem 8.3.6. *Suppose A is a densely defined positive operator on a Banach lattice E. If $D(A)$ is an ideal and A preserves closed ideals, then A is closable and $-\overline{A}$ generates a multiplication semigroup. Conversely, if A generates a multiplication semigroup, then $D(A)$ is an ideal, A preserves closed ideals, and there is a $\lambda \in \mathbb{R}$ such that $\lambda - A$ is positive.*

Theorem 8.3.7. *Suppose A is a positive band preserving operator with $D(A)$ a Riesz subspace which generates a dense ideal. Then $-A$ admits a unique extension to a multiplication semigroup.*

Now we will start with the duality theory of multiplication semigroups.

Proposition 8.3.8. *If $T(t)$ is a multiplication semigroup on a Banach lattice E, then $E^{\odot}$ is an order dense ideal in E^* and $T^{\odot}(t)$ is a multiplication semigroup on $E^{\odot}$.*

Proof: Suppose $0 \le |x^*| \le |y^*|$ holds with $y^* \in E^{\odot}$. Since each $T^*(t)$ is a band preserving operator by Lemma 8.3.3, by Lemma 8.3.4 we have

$$|T^*(t)x^* - x^*| = |T^*(t) - I||x^*| \le |T^*(t) - I||y^*| = |T^*(t)y^* - y^*|.$$

By the lattice property of the norm, $x^* \in E^{\odot}$ and therefore $E^{\odot}$ is an ideal. From 8.3.3(iii) and the Laplace transorm formula it follows that for $\lambda \in \varrho(A)$ large enough there is a $\mu = \mu(\lambda) > 0$ such that $0 \le R(\lambda, A^*) \le \mu I$. Therefore, for arbitrary x^* we have $0 \le \mu^{-1} R(\lambda, A^*)x^* \le x^*$. Since $R(\lambda, A^*)x^* \in E^{\odot}$, this proves order denseness. Finally $T^{\odot}(t)$ is band preserving by Lemma 8.3.3(iii) and the fact that $T^*(t)$ is band preserving. ////

Theorem 8.3.9. *Suppose $T(t)$ is a multiplication semigroup on a Banach lattice E. If E^* has order continuous norm, then $E^{\odot} = E^*$.*

Proof: First we claim that A^* is band preserving. Fix $x^* \in D(A^*)$ and let π be the band projection onto the band generated by x^*. It follows from Lemma

8.3.3 that $R(\lambda, A^*) = R(\lambda, A)^*$ is band preserving, so

$$\begin{aligned} R(\lambda, A^*)(\lambda - A^*)x^* &= \pi R(\lambda, A^*)(\lambda - A^*)x^* \\ &= \pi R(\lambda, A^*)\big(\pi(\lambda - A^*)x^* + (1-\pi)(\lambda - A^*)x^*\big) \\ &= \pi R(\lambda, A^*)\pi(\lambda - A^*)x^* = R(\lambda, A^*)\pi(\lambda - A^*)x^*. \end{aligned}$$

The injectivity of $R(\lambda, A^*)$ implies that $(\lambda - A^*)x^* = \pi(\lambda - A^*)x^* \in B_{x^*}$, where B_{x^*} is the band generated by x^*. This proves the claim. Since E^* has order continuous norm, $E^\odot$ is band (since it is a closed ideal by Proposition 8.3.8). Hence for $x^* \in D(A^*)$ it follows that $A^*x^* \in B_{x^*} \subset E^\odot$. Since $A^\odot$ is the part of A^* in $E^\odot$ it follows that $A^\odot = A^*$ and hence $E^\odot = E^*$. ////

This theorem is an improvement of Corollary 1.3.2 for multiplication semigroups. If E^* does not have order continuous norm it can happen that $E^\odot$ is a proper subspace of E^*, as is shown by the example $E = L^1[0,1]$, $Af(s) = -s^{-1}f(s)$ with $D(A)$ maximal. However for arbitrary E one can show that the adjoint is always strongly continuous for $t > 0$. We turn now to the proof of this. The *growth bound* λ of a C_0-semigroup is the number

$$\lambda := \inf\{\omega : \exists M \text{ such that } \|T(t)\| \leq Me^{\omega t}\}.$$

The ideal generated by an element $u \in E$ will be denoted by E_u.

Lemma 8.3.10. *Let $T(t)$ be a multiplication semigroup on a Banach lattice E. Then there exists a $\lambda \in \mathbb{R}$ such that $0 \leq T(t) \leq e^{\lambda t}$ for all $t \geq 0$.*

Proof: First we show that $T(t)$ is positive. Let $u \geq 0$. By Lemma 8.3.5 on $\overline{E_u}$ the operator $T(t/2)$ is represented as multiplication with some bounded function g_u. Then $T(t)x = T(t/2)T(t/2)x = g_u^2 x \geq 0$ since g_u^2 is a positive function (recall that we deal with real Banach lattices only). In order to obtain the other estimate, let $T_{\mathbb{C}}$ denote the complexification of T. If T is a bounded multiplication operator, then for the spectral radius $r(T_{\mathbb{C}})$ of $T_{\mathbb{C}}$ we have $r(T_{\mathbb{C}}) = \|T_{\mathbb{C}}\| = \|T\|$, cf. [Ar1, Satz 1.8]. For the growth bound λ of $T_{\mathbb{C}}(t)$ we have by [Na2, Prop. AIII.1.1]

$$\|T(t)\| = r(T_{\mathbb{C}}(t)) = e^{\lambda t}.$$

The desired estimate now follows from the inequality $T(t) \leq \|T(t)\|I$ [Ar1, Satz 1.8]. ////

We remark that spectral theory and complexification is not essential here. Instead one could work with representation spaces and show that the semigroup property implies that the functions $g_u(t)$ representing $T(t)$ on closed principal ideals must be exponentials. This approach is more laborious however.

Let E be a Banach lattice with quasi-interior point $u > 0$ and let K_u be its representation space. Every element $x^* \in E^*$ can be identified with a Borel measure $\mu \in (C(K_u))^*$. Indeed, the restrictions $(x^*)^+$ and $(x^*)^-$ are *positive* linear forms on $C(K_u)$, hence bounded, and the Riesz representation theorem applies.

Theorem 8.3.11. *Suppose $T(t)$ is a multiplication semigroup on a Banach lattice E. Then $T^*(t)$ is $C_{>0}$.*

Proof: First we observe the following. If $f : \mathbb{R}_+ \to \mathbb{R}$ is a function satisfying $0 \leq f \leq 1$ and $f(s+t) = f(s)f(t)$ for all $s, t > 0$ then there is a $0 \leq \lambda \leq \infty$ such that $f(t) = e^{-\lambda t}$ for all $t > 0$. Since this is a variation on a well-known theme, we only sketch the (elementary) proof. We first claim that f is continuous. For this we may assume $f(t) > 0$ for all $t > 0$, for if $f(t) = 0$ for some t it follows easily that $f = 0$ for all $t > 0$ and we may take $\lambda = \infty$. If f has a discontinuity at $t_0 > 0$ it is easy to produce a $t_1 \in (0, t_0)$ with $f(t_1) > 1$, a contradiction, so f is continuous. That f now must be an exponential is standard.

Proof of the theorem: By Lemma 8.3.10, after replacing $T(t)$ by $e^{-\lambda t}T(t)$ without loss of generality we may assume that $0 \leq T(t) \leq I$. Fix $x^* \in E^*$. Since $0 \leq T^*(t) \leq I$, by Lemma 8.3.5 there is a continuous function $0 \leq g_t \leq 1$ on the representation space $K = K_{x^*}$ such that $T^*(t)y^* = g_t y^*$ for all y^* in the closed ideal generated by x^* in E^*. We claim that there is a continuous positive $h : K \to \overline{\mathbb{R}}$ such that $g_t = e^{-ht}$. To see this, note that $T^*(t+s)x^* = T^*(t)T^*(s)x^*$ shows that $g_{t+s}(\xi) = g_t(\xi)g_s(\xi)$ holds for all $\xi \in K$. Hence by the above observation we find numbers $0 \leq h(\xi) \leq \infty$ such that $g_t(\xi) = e^{h(\xi)t}$ for all $t > 0$. Since the g_t are continuous, also h must be. This proves the claim. Now let $t_0 > 0$ be fixed. Then as $t \downarrow t_0$ we have $g_t(\xi) \uparrow g_{t_0}(\xi)$ pointwise. Identifying bounded functionals with measures as above, by Lebesgue's dominated convergence it follows that $T^*(t)x^* = g_t \to g_{t_0} = T^*(t_0)x^*$ weakly. By Theorem 0.2.1, applied to the closed linear span of $\{T^*(t)x^* : t \geq t_0\}$, we find that $T^*(t)x^* \to T^*(t_0)x^*$ strongly. Since x^* and t_0 are arbitrary the proof is complete. ////

This theorem fails in the complex case. A counterexample is the semigroup on l^1 defined by $T(t)x_n := e^{int}x_n$, where $\{x_n\}_{n=1}^\infty$ is the unit vector basis of l^1.

The final result of this section is a 'classification' of all $\odot$-reflexive multiplication semigroups. Trivially, if X is reflexive, then X is $\odot$-reflexive with respect to any C_0-semigroup on X. If $E = c_0$ or l^1 then E is $\odot$-reflexive with respect to the multiplication semigroup $T(t)$ defined by $T(t)x_n = e^{-nt}x_n$, where x_n is the nth unit vector. More generally, if $\{x_n\}_{n=1}^\infty$ is an unconditional basis of E, then (up to an equivalent norm) E is a Banach lattice and $T(t)x_n = e^{-nt}$ is a $\odot$-reflexive multiplication semigroup on E.

Banach lattices with unconditional basis are atomic. Recall that $x \in E$ is called an *atom* if the ideal generated by x is one-dimensional and that a Banach lattice is *atomic* if there exists a maximal orthogonal system $\{x_\alpha\}_\alpha$ with each x_α an atom. The band E_a generated by all atoms of E is called the *atomic part* of E and is an atomic Banach lattice. Finite-dimensional Banach lattices are atomic. See [S4] for more information.

We will prove that reflexive Banach lattices and atomic Banach lattices on countably many atoms (i.e., Banach lattices with an unconditional basis) are essentially the only ones which can be $\odot$-reflexive with respect to a multiplication semigroup. This was conjectured by Ben de Pagter (private communication).

Lemma 8.3.12. *If E is $\odot$-reflexive with respect to a multiplication semigroup $T(t)$, then E has order continuous norm.*

Proof: From Proposition 8.3.8 we know that $E^{\odot\odot}$ is a Dedekind complete Banach lattice. Also $E^{\odot\odot}$ is $\odot$-reflexive with respect to $T^{\odot\odot}(t)$, so $E^{\odot\odot}$ has order continuous norm by Lemma 8.1.5. Let $0 \leq x_\alpha \uparrow x$ in E. Since j is positive we have $0 \leq jx_\alpha \uparrow\leq jx$ and hence (jx_α) is norm convergent to $\sup jx_\alpha$. Thus (x_α) is norm convergent as well and its limit must be x, which shows that E has order continuous norm. ////

Let K be a subset of a Banach space X. In the next lemma we use the standard fact [AB, Thm. 10.17] that if for each $\epsilon > 0$ there exists a weakly compact subset $K_\epsilon \subset X$ such that $K \subset K_\epsilon + \epsilon B_X$, then K is weakly compact.

Lemma 8.3.13. *If a Banach lattice E is $\odot$-reflexive with respect to a multiplication semigroup $T(t)$, then $T(t)$ is weakly compact.*

Proof: Let (x_n) be a bounded sequence in E and let $t > 0$ be fixed. By the Eberlein-Shmulyan theorem it suffices to show that the sequence $(T(t)x_n)$ has a weakly convergent subsequence. The closed linear span of (x_n) is contained in the closed ideal of E generated by $\sum_n 2^{-n}|x_n|$. By Lemma 8.3.3(ii) this ideal is invariant under $T(t)$. Therefore without loss of generality we may assume that E has a quasi-interior point $u > 0$. Also by Lemma 8.3.10 we may assume $0 \leq T(t) \leq I$. Let the generator A be represented on K_u as multiplication with a continuous $\overline{\mathbb{R}}$-valued function $g \leq 0$. Define the open sets F_n by

$$F_n := \{s \in K_u : -n < g(s) \leq 0\}$$

and let G_n be its closure. Since E is Dedekind complete, K_u is Stonean. This implies that G_n is clopen and $\chi_{G_n} \in C(K_u)$. Define band projections π_n on E by $\pi_n x := \chi_{G_n} x$ and denote the corresponding bands by B_n. The restriction of the semigroup $T(t)$ to each B_n is uniformly continuous by construction. Hence $B_n^\odot = B_n^*$ and $B_n = B_n^{\odot\odot} = B_n^{**}$ so each B_n is reflexive. For notational clarity we denote the closed unit ball of E by U_E. For $t > 0$ we have

$$T(t)U_E \subset U_{B_n} + e^{-nt}U_E$$

since $0 \leq T(t) \leq e^{-nt}$ holds on the orthogonal complement of B_n. By the above remark the weak compactness of $T(t)$ follows. ////

Theorem 8.3.14. *Let E be $\odot$-reflexive with respect to a multiplication semigroup. Then E has order continuous norm, and either E contains an infinite-dimensional reflexive band or E has an unconditional basis.*

Proof: Suppose there are no infinite-dimensional reflexive bands in E. Let E_a denote the atomic part of E. We will show that $E = E_a$. If not, then $E = E_a \oplus B$ for some non-empty band B. The proof of the previous lemma shows how to find a reflexive band in B which by assumption must be finite-dimensional. But finite-dimensional Banach lattices are atomic, a contradiction to the definition of E_a. So E is atomic. On each atom x_α we have $T(t)x_\alpha = e^{-\lambda_\alpha} x_\alpha$ for some number λ_α. If there were uncountably many atoms, then there would be an $n \in \mathbb{N}$ such that uncountably many of the λ_α satisfy $|\lambda_\alpha| < n$. Consider the band B generated by the corresponding atoms. This band is infinite-dimensional and clearly $T(t)$ is uniformly continuous on it, so $B^{\odot\odot} = B^{**}$. But B is $\odot$-reflexive with respect to the restriction of $T(t)$, so B is reflexive, a contradiction. ////

A Dunford-Pettis space cannot contain a complemented infinite-dimensional reflexive subspace, for then the associated projection $\pi = \pi^2$ would be compact. Recall that every AM-space and every AL-space is Dunford-Pettis [S4].

Corollary 8.3.15. *If E is a Dunford-Pettis lattice which is $\odot$-reflexive with respect to a multiplication semigroup $T(t)$, then E is has an unconditional basis and $T(t)$ is compact.*

8.4. Applications to Banach function spaces

In this section we apply some of the results of the previous section to the setting of Banach function spaces.

Let (Ω, Σ, μ) be a σ-finite measure space and let $L^0(\mu)$ denote the linear space of real-valued μ-measurable functions on Ω which are finite a.e. As usual μ-a.e. equal functions are identified. A linear subspace E of $L^0(\mu)$, equipped with a norm $||\cdot||$, is called a *Banach function space* (over (Ω, Σ, μ)) if E is a Banach space with respect to $||\cdot||$ and $f \in L^0(\mu)$, $g \in E$ with $|f| \leq |g|$ a.e. implies that $f \in E$ and $||f|| \leq ||g||$. Note that every Banach function space is a σ-Dedekind complete Banach lattice.

We say that *E is carried by Ω* if there is no subset E of Ω of positive measure with the property that $f = 0$ a.e. on E for all $f \in E$, or equivalently if for every $E \subset \Omega$ of positive measure there is a subset $F \subset E$ of positive measure such that the characteristic function χ_F belongs to E. Ω always contains a subset Ω_0 such that E is carried by $\Omega \backslash \Omega_0$. Therefore we will assume henceforth without loss of generality that E is carried by Ω.

The *associate space* (sometimes called the Köthe dual) of E is defined by

$$E' = \{g \in L^0(\mu) : \int_\Omega |fg| \, d\mu < \infty, \forall f \in E\}.$$

E' is a Banach function space with respect to the norm given by

$$||g|| = \sup_{||f||\le 1} \left| \int_\Omega fg \, d\mu \right|.$$

Every $g \in E'$ defines a bounded linear functional $\phi_g \in E^*$ via the formula

$$\langle \phi_g, f \rangle = \int_\Omega fg \, d\mu, \qquad \forall f \in E.$$

We have $||g||_{E'} = ||\phi_g||_{E^*}$. Therefore E' can be identified with a closed subspace of E^*. In fact E' is even a band in E^*.

If $T(t)$ is a C_0-semigroup on a Banach function space E, then one may ask under what conditions we have $E^{\odot} \subset E'$. Trivially, this is true when E has order continuous norm, since then $E' = E^*$.

Proposition 8.4.1. *Suppose E is a C_0-semigroup on a Banach function space E. Then $E^{\odot} \subset E'$ if and only if E has order continuous norm.*

Proof: If E has order continuous norm, then $E' = E^*$, so trivially $E^{\odot} \subset E'$ holds. Conversely, suppose $E^{\odot} \subset E'$. Since E' is a band in E^*, by Lemma 8.1.8 we have $E^* \subset E'$, forcing $E' = E^*$. ////

By Lemma 8.1.5 this applies in particular to $\odot$-reflexive semigroups. In the case of *positive* semigroups one can show that $E^{\odot} \subset E'$ if and only if $f_n \downarrow 0$ implies $||R(\lambda, A)f_n|| \to 0$, see [NP].

Let $h \in L^0(\mu)$ be a real-valued measurable function and define the operator A_h by

$$D(A_h) = \{f \in E : hf \in E\};$$
$$A_h f = hf, \quad f \in D(A_h).$$

Note that A_h is a closed operator. Put

$$\Omega_n = \{s \in \Omega : |h(s)| \le n\}, \tag{8.1}$$

let χ_{Ω_n} be its characteristic function and define the band projections

$$P_n : E \to E, \quad P_n f = \chi_{\Omega_n} f. \tag{8.2}$$

Since $|P_n f| \le |f|$ for all f, P_n indeed maps E into E. In fact, from the lattice property of the norm we see immediately that P_n is a contraction mapping.

In general $D(A_h)$ need not be dense, as the example $E = L^\infty(0,1)$, $h(s) = s^{-1}$ shows.

Lemma 8.4.2. *$D(A_h)$ is an ideal. Moreover, $D(A_h)$ is dense if and only if $\lim_n ||P_n f - f|| = 0$ for all $f \in E$.*

Proof: Suppose $g \in D(A_h)$ and let $f \in E$ be a function satisfying $|f| \leq |g|$. By assumption we have $hg \in E$, hence also $|hg| \in E$. But $|hf| \leq |hg|$, so $hf \in E$ which implies that $f \in D(A_h)$. This proves the first assertion.

Suppose $||P_n f - f|| \to 0$ for all $f \in E$. To prove that $D(A_h)$ is dense it suffices to show that $P_n f \in D(A_h)$ for all $f \in E$. But on Ω_n we have $|h(s)| \leq n$, so

$$|hP_n f| \leq |nP_n f| \leq n|f|$$

showing that $hP_n f \in E$ and hence $P_n f \in D(A_h)$. Conversely, suppose $D(A_h)$ is dense. First let $f \in D(A_h)$. Then

$$|P_n f - f| = |\chi_{(\Omega\setminus\Omega_n)} f| \leq \frac{1}{n}|hf| = \frac{1}{n}|A_h f|.$$

Hence by the lattice property of the norm,

$$||P_n f - f|| \leq \frac{1}{n}||A_h f|| \to 0, \quad n \to \infty.$$

Since $D(A_h)$ is dense and $||P_n|| \leq 1$ for all n, the general case follows from a density argument. ////

The following theorem characterizes those $h \in L^0(\mu)$ which give rise to a generator of a C_0-semigroup. It follows readily from the previous lemma (cf. [NP]), but since it is a special case of Theorem 8.3.6 we omit the proof.

Theorem 8.4.3. *A_h generates a C_0-semigroup on E if and only if $D(A_h)$ is dense and $h \leq K$ for some constant K.*

It is an easy consequence of the definition [Za1] that E has order continuous norm if and only if for all $f \in E$ and decreasing sets $F_1 \supset F_2 \supset \ldots \downarrow \emptyset$ we have $||f\chi_{F_n}|| \to 0$. Using this equivalent formulation together with Lemma 8.4.2 and Theorem 8.4.3 we obtain:

Theorem 8.4.4. *E has order continuous norm if and only if A_h generates a C_0-semigroup on E for every h which is bounded from above.*

Proof: Suppose E has order continuous norm. Take h with $h \leq K$ and define the sets Ω_n and maps P_n according to formulas (8.1) and (8.2). Since

$$\Omega_1 \subset \Omega_2 \subset \ldots \uparrow \Omega,$$

for all $f \in E$ we get

$$||P_n f - f|| = ||f\chi_{\Omega\setminus\Omega_n}|| \to 0.$$

Hence by Lemma 8.4.2 $D(A_h)$ is dense. Then Theorem 8.4.3 shows that A_h is a generator on E.

Conversely, let $\Omega = F_0 \supset F_1 \supset F_2 \supset \ldots \downarrow \emptyset$. Define $h \in L^0(\mu)$ by

$$h(s) = -n, \quad s \in F_n \setminus F_{n+1}.$$

Then

$$\Omega_n = \{s \in \Omega : |h(s)| \leq n\} = \Omega \backslash F_{n+1}.$$

Since by assumption A_h is a generator on E, hence in particular $D(A_h)$ is dense, we get by Lemma 8.4.2

$$||f\chi_{F_{n+1}}|| = ||f\chi_{\Omega\backslash F_{n+1}} - f|| = ||P_n f - f|| \to 0.$$

////

From now on we assume h to be fixed and bounded from above. Then A_h is the generator of a semigroup $T(t)$ on E. In the following theorem we will give a representation for the semigroup dual $E^{\odot}$. Let $[P_n^* E^*]_{n=1}^{\infty}$ denote the closed linear span in E^* of the subspaces $P_n^* E^*$, $n = 1, 2, ...$

Theorem 8.4.5. $E^{\odot} = [P_n^* E^*]_{n=1}^{\infty}$.

Proof: $D(A_h^*)$ is an ideal. This follows from the abstract theory, but let us give an elementary proof. Suppose $|\phi| \leq |\psi|$ with $\psi \in D(A_h^*)$. Clearly,

$$\langle h\phi, f\rangle := \langle \phi, hf\rangle = \langle \phi, A_h f\rangle, \qquad \forall f \in D(A_h)$$

defines a linear functional $h\phi$ on $D(A_h)$ and for $f \in D(A_h)$,

$$\langle h\phi, f\rangle = \langle \phi, hf\rangle \leq \langle |\phi|, |hf|\rangle \leq \langle |\psi|, |hf|\rangle = \langle |h\psi|, |f|\rangle \leq ||A_h^*\psi||\ ||f||.$$

Therefore, $h\phi$ is bounded on $D(A_h)$. Since $D(A_h)$ is dense, $h\phi$ extends to a bounded linear functional on E. This proves that $\phi \in D(A_h^*)$.

We will now prove the inclusion $[P_n^* E^*]_{n=1}^{\infty} \subset E^{\odot}$. Let $\phi \in P_n^* E^*$, say $\phi = P_n^*\psi$. We have to show that $\phi \in E^{\odot}$. Since $D(A_h^*)$ is an ideal, so is its closure $E^{\odot}$. Therefore it suffices to show that $|\phi| \in E^{\odot}$. Fix $\epsilon > 0$ and choose $t_0 > 0$ so small that for any $0 \leq t \leq t_0$ and $|\alpha| \leq n$ we have $|e^{\alpha t} - 1| < \epsilon$. Since we have $|\phi| = |P_n^*\psi| = P_n^*|\psi|$, and hence for $t \leq t_0$,

$$\begin{aligned} |\langle T^*(t)|\phi| - |\phi|, f\rangle| &= |\langle |\psi|, P_n(e^{th}f - f)\rangle| = |\langle |\psi|, \chi_{\Omega_n}(e^{th} - 1)f\rangle| \\ &\leq \epsilon\langle |\phi|, |f|\rangle| \leq \epsilon||\phi||\ ||f||. \end{aligned}$$

Hence

$$\|\ T^*(t)|\phi| - |\phi|\ \| \leq \epsilon||\phi||$$

showing that $|\phi| \in E^{\odot}$ and therefore also $\phi \in E^{\odot}$. Since $E^{\odot}$ is a closed linear subspace this implies that $[P_n^* E^*]_{n=1}^{\infty} \subset E^{\odot}$.

To conclude the proof we show the reverse inclusion. Since $\overline{D(A_h^*)} = E^{\odot}$ it suffices to prove that $D(A_h^*) \subset [P_n^* E^*]_{n=1}^{\infty}$. Let $\phi \in D(A_h^*)$. Since $D(A_h^*)$ is an ideal, we may without loss of generality assume that $\phi \geq 0$. It suffices to prove that $||P_n^*\phi - \phi|| \to 0$ as $n \to \infty$. For any $f \in D(A_h)$ we have

$$|\langle P_n^*\phi - \phi, f\rangle| = |\langle \phi, \chi_{(\Omega\backslash\Omega_n)}f\rangle| \leq \frac{1}{n}|\langle \phi, |hf|\rangle| = \frac{1}{n}\langle |h\phi|, |f|\rangle \leq \frac{1}{n}||A^*\phi||\ ||f||.$$

This shows that $||P_n^*\phi - \phi|| \leq n^{-1}||A_h^*\phi|| \to 0.$ ////

Remark 8.4.6. A similar result can be proved for the abstract multiplication semigroups considered in Section 8.3.

Finally, we will deal with the case where Ω is compact Hausdorff space and μ is a regular Borel measure. In this case it is natural to see what improvements can be obtained when we require $h \in L^0(\mu)$ to be *continuous*. In fact we will ask something slightly different, viz. that $|h|$ is a continuous function $\Omega \to \overline{\mathbb{R}}$. For such functions we put $\Omega_\infty = \{s \in \Omega : h(s) = -\infty\}$. Since $h \in L^0(\mu)$, necessarily $\mu(\Omega_\infty) = 0$. We will say that $f \in E$ is *compactly supported* if there is a compact $K \subset \Omega \backslash \Omega_\infty$ such that $f = \chi_K f$ a.e. and we define E_c to be the linear subspace of E consisting of all compactly supported functions. Of course E_c depends on h. A functional $\phi \in E^*$ is said to be compactly supported if there is a compact $K \subset \Omega \backslash \Omega_\infty$ such that $\langle \phi, f \rangle = \langle \phi, \chi_K f \rangle$ for all $f \in E$.

Theorem 8.4.7. *Suppose $|h| : \Omega \to \overline{\mathbb{R}}$ is continuous and h is bounded from above. A_h generates a C_0-semigroup if and only if E_c is dense in E. In this case $E^{\odot}$ is the closure of the compactly supported functionals.*

Proof: Suppose A_h generates a C_0-semigroup. Since $|h|$ is continuous, we see that the sets $\Omega_n \subset \Omega \backslash \Omega_\infty$ defined by (8.1) are closed in Ω, hence compact. Now take $f \in E$ arbitrary. By assumption $D(A_h)$ is dense, so by Lemma 8.4.2 we have $||P_n f - f|| \to 0$. Since $P_n f$ is supported in the compact set Ω_n, this proves that E_c is dense in E.

For the converse, assume E_c to be dense. In view of Theorem 8.4.3 we must show that $D(A_h)$ is dense. In fact we will show that $E_c \subset D(A_h)$. Indeed, let $f \in E_c$ be supported in the compact set $K \subset \Omega \backslash \Omega_\infty$. Since $|h|$ is continuous as a function $K \to \mathbb{R}$, we see that h is bounded on K. This implies that $h \in D(A_h)$.

The assertion on $E^{\odot}$ is proved in the same way, using the characterization from Theorem 8.4.5. ////

Example 8.4.8. (i) Let $E = L^1(\mathbb{R})$, $h(t) = -|t|$. Letting $\Omega = \overline{\mathbb{R}}$ we conclude from Theorem 8.4.5 that $E^{\odot}$ is the closed ideal in L^∞ generated by $C_0(\mathbb{R})$.

(ii) Let $E = L^1(D)$ with D the closed unit disc in $\mathbb{C}$. Suppose h is continuous in D with $\lim_{s \to t} h(s) = -\infty$ for all $t \in \partial D$. Then $E^{\odot}$ is the closed ideal in $L^\infty(D)$ generated by the subpace of continuous functions which are zero on ∂D.

Notes. The problem of determining when $E^{\odot}$ is a sublattice is difficult and up to now there is no characterization of the positive semigroups with this property. Example 8.1.1 is taken from [GNa]. Theorem 8.1.2 and its corollary can be found in [Cea4]. Theorem 8.1.7 is due to de Pagter [Pa3], whose proof is essentially that of Lemma 8.1.4, the only difference being that, in order to obtain a slightly more general result, we use the resolvent rather than $T(t)$. Lemmas 8.1.5 and 8.1.8 are taken from [NP] and Theorem 8.1.6 is new.

Example 8.2.1 is due to Phillips [Ph1]; see also [Fe], who studies semigroups under very weak measurability conditions. Example 8.2.2 up to Theorem 8.2.6 are from [Ne8]. Theorem 8.2.7 due to de Pagter [Pa3].

The results of Section 8.3 are taken from [Ne6]. The proofs of Theorems 8.3.6 and 8.3.7 are rather long and technical, and rely on the representation theory of Theorem 8.3.1. Note the following special case of Theorem 8.3.7: *Let $u > 0$ be a quasi-interior point of E and let $0 \geq v \in E$. Then there exists a unique multiplication semigroup on E, with generator A_v, such that $u \in D(A_v)$ and $A_v u = v$.* This result can be used to prove the following converse of Lotz's theorem: *Suppose E is a Banach lattice with quasi-interior point. If every C_0-semigroup on E is uniformly continuous, then there is a compact Hausdorff space K such that E is Banach lattice isomorphic to $C(K)$. Moreover, E has the Grothendieck property*; see [Ne7]. Of course E, being a $C(K)$-space, also has the Dunford-Pettis property. It seems that this result is quite optimal: in [Le] an example is constructed of a Banach lattice E with weak order unit which does not have the Dunford-Pettis property, such that every C_0-semigroup on E is uniformly continuous.

Theorem 8.3.11 was stated without proof in [Ne6]. Theorem 8.3.14 is a minor improvement of the corresponding result in [Ne6].

For the basic theory concerning Banach function spaces we refer to the books [KPS], [Za1], [Za2]. The results of Section 8.4 are taken from [NP].

Open problems

In this short section we discuss some open problems.

The first five problems are about positive semigroups $T(t)$ on a Banach lattice E.

(1) *Characterise the positive C_0-semigroups for which $E^\odot$ is a sublattice of E^*.* Some information can be found in Section 8.1, but in general this problem seems to be very difficult.

(2) *Is $kE^{\odot\odot}$ always a sublattice of E^{**}?*

(3) *Is $E^*/E^\odot$ always either zero or nonseparable?* We ask this simply because of the lack of a counterexample. None of the semigroups with $X^*/X^\odot$ separable discussed so far is positive.

(4) *Discuss the problem $E_{\odot\odot} = kE^{\odot\odot}$ for positive semigroups on AL-spaces E.* In Section 8.2 we did this for AM-spaces. The only thing that seems to be known for AL-spaces is that $E_{\odot\odot} \neq kE^{\odot\odot}$ if $T(t)$ is rotation on $E = L^1(T)$; see Example 5.2.13.

(5) *If $T^*(t)$ is weakly Borel measurable, does it follow that $T^*(t)$ is strongly continuous for $t > 0$?* This is true for $E = C(K)$; see Theorem 8.2.6.

The last two problems are about arbitrary C_0-semigroups on a Banach space X.

(6) *Describe the Mackey topology of $(X, \sigma(X, X^\odot))$.* In particular it would be interesting to know whether this topology is quasi-complete, for this would explain why the Eberlein-Shmulyan theorem holds for $\sigma(X, X^\odot)$. See also the notes of Chapter 2.

Finally we observe that a C_0-group on X can be thought of as a strongly continuous representation of $\mathbb{R}$ in $\mathcal{L}(X)$.

(7) *Extend the theory of adjoint groups to the more general setting of strongly continuous representations of locally compact groups in $\mathcal{L}(X)$.*

Appendix

Integration in Banach spaces

Let (Ω, Σ, μ) be a finite measure space and let X be a Banach space. A function $f : \Omega \to X$ is called a *step function* if it can be written in the form

$$f = \sum_{n=1}^{N} \chi_{E_n} x_n,$$

where $x_n \in X$, $E_n \in \Sigma$ are disjoint, and χ_{E_n} is the characteristic function of E_n. A function $f : \Omega \to X$ is said to be *strongly μ-measurable* if there is a sequence (f_n) of step functions converging to f μ-almost everywhere.

A strongly μ-measurable function is called *Bochner integrable* with respect to μ if there exists a sequence (f_n) of step functions such that

$$\lim_{n\to\infty} \int_\Omega \|f - f_n\| \, d\mu = 0.$$

Note that the scalar function $\|f(\cdot) - f_n(\cdot)\|$ is indeed measurable, so this definition makes sense. If f is Bochner integrable, for $E \in \Sigma$ we define

$$\int_E f \, d\mu := \lim_{n\to\infty} \int_E f_n \, d\mu,$$

where the integral of the step function $f_n = \sum_{k=1}^{N} \chi_{E_k} x_k$ is defined in the obvious way:

$$\int_E f_n \, d\mu := \sum_{k=1}^{N} \mu(E_k \cap E) x_k.$$

It is easily checked that $\int_E f \, d\mu$ does not depend on the particular choice of (f_n).

There is the following characterization of Bochner integrable functions (*Bochner's theorem*).

Theorem A.1. *A strongly μ-measurable function f is Bochner integrable with respect to μ if and only if $\int_\Omega \|f\| \, d\mu < \infty$.*

One has $\| \int_E f \, d\mu \| \leq \int_E \|f\| \, d\mu$. Many of the classical theorems on the Lebesgue integral, e.g. Lebesgue's dominated convergence theorem, Egoroff's theorem, Fubini's theorem and Lebesgue's differentiation theorem, extend to the Bochner integral. For reference we state the latter explicitly.

Theorem A.2. *Let $f : [0,1] \to X$ be Bochner integrable with respect to the Lebesgue measure. Then for almost all t we have*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} f(\sigma) \, d\sigma = f(t).$$

One of the nice features of the Bochner integral is the following result, due to Hille (see [DU]).

Theorem A.3. *Let $A : D(A) \subset X \to Y$ be a closed linear operator on X. Suppose that both $f : \Omega \to D(A)$ and $Af : \Omega \to Y$ are Bochner integrable with respect to μ. Then for all $E \in \Sigma$ we have*

$$A\Big(\int_E f \, d\mu \Big) = \int_E Af \, d\mu.$$

In particular this shows that the Bochner integral commutes with bounded linear operators and functionals.

We now define the concept of weak measurability. If $\Gamma \subset X^*$ is a linear subspace, then a function $f : \Omega \to X$ is *Γ-measurable* with respect to μ if the function $\langle x^*, f(\cdot) \rangle$ is μ-measurable for each $x^* \in \Gamma$. If $\Gamma = X^*$, then f is called *weakly μ-measurable.* If $X = Y^*$ is a dual space and if $\Gamma = Y$, then f is called *weak*-μ-measurable.* The subspace Γ is said to be *norming* for X if

$$\|x\| = \sup_{x^* \in B_\Gamma} |\langle x^*, x \rangle|, \qquad \forall x \in X,$$

where B_Γ is the unit ball of Γ. A function $f : \Omega \to X$ is *μ-essentially separably valued* if there is a μ-null set $E \in \Sigma$ such that $f(\Omega \backslash E)$ is contained in a separable subspace of X.

The following result is known as the *Pettis measurability theorem.*

Theorem A.4. *Suppose f is μ-essentially separably valued and Γ-measurable with respect to μ, with Γ norming. Then f is strongly μ-measurable.*

Note that all our measurability concepts so far refer to some measure μ. In Chapter 5 we need one more measurability concept which does not refer to a particular measure but only to the underlying σ-algebra. Suppose Ω is a topological space and Σ is its Borel σ-algebra. Then a function $f : \Omega \to X$ is called *weakly Borel measurable* if $\langle x^*, f(\cdot) \rangle$ is Borel measurable for all $x^* \in X^*$. If f is weakly Borel measurable, then clearly f is μ-measurable with respect to every positive regular Borel measure on (Ω, Σ).

The drawback of the Bochner integral is that one has to impose rather strong measurability assumptions on functions to be integrated. In order to deal with a more general class of functions, e.g. weak*- or weakly measurable functions, we will now discuss the weak*-integral and the Pettis integral.

Suppose $g : \Omega \to X^*$ is weak*-μ-measurable and suppose further that for each $x \in X$ the function $\langle g(\cdot), x\rangle$ belongs to $L^1(\mu)$ (briefly we say that g is weak* L^1). For each $E \in \Sigma$ define a map $T_E : X \to L^1(\mu)$, $T_E x = \langle g\chi_E(\cdot), x\rangle$. We claim that T_E is closed. Indeed, if $x_n \to x$ in X and $T_E x \to y$ in $L^1(\mu)$, then there is a subsequence (x_{n_i}) such that $T_E x_{n_i}$ converges to y μ-almost everywhere. But $\langle g\chi_E(\omega), x_{n_i}\rangle \to \langle g\chi_E(\omega), x\rangle$ for all $\omega \in \Omega$, and therefore $y = T_E x$. This proves the claim. By the closed graph theorem T_E is bounded. This implies that the linear map x_E^* defined by

$$\langle x_E^*, x\rangle := \int_E \langle g(\cdot), x\rangle \, d\mu$$

is bounded. The element $x_E^* \in X^*$ is called the *weak*-integral* (or *Gelfand integral* of g over E with respect to μ, notation $x_E^* = weak^* \int_E g \, d\mu$. By definition the weak*-integral satisfies

$$\langle weak^* \int_E g \, d\mu, x\rangle = \int_E \langle g(\cdot), x\rangle \, d\mu, \quad \forall\ E \in \Sigma \text{ and } x \in X.$$

From this one immediately derives that the weak*-integral commutes with adjoints T^* of bounded operators T on X:

$$\begin{aligned}\langle T^*\Big(weak^* \int_E g \, d\mu\Big), x\rangle &= \langle weak^* \int_E g \, d\mu, Tx\rangle = \int_E \langle g(\cdot), Tx\rangle \, d\mu \\ &= \int_E \langle T^* g(\cdot), x\rangle \, d\mu = \langle weak^* \int_E T^* g \, d\mu, x\rangle.\end{aligned}$$

For arbitrary bounded operators on X^* this is in general not true.

Although the existence of the weak*-integral in its most general form depends on the closed graph theorem, in most applications its use can be avoided. If for instance $\sup_{\omega \in E} \|f(\omega)\| < \infty$, then the boundedness of x_E^* is immediately clear. If $f : [0,1] \to X^*$ is a bounded weak*-continuous function, then one can even avoid the Lebesgue integral and define $\int_E f \, d\mu$ as a weak*-Riemann integral in the obvious way. We note this here because the majority of the weak*-integrals used in this thesis (but not all) belong to this class.

We finally turn to the Pettis integral. If $f : \Omega \to X$ is weakly μ-measurable and weakly L^1, then by the same argument as above, each $E \in \Sigma$ defines an element $x_E^{**} \in X^{**}$ such that

$$\langle x_E^{**}, x^*\rangle = \int_E \langle x^*, f(\cdot)\rangle \, d\mu$$

for all $x^* \in X^*$. If for all $E \in \Sigma$ the element x_E^{**} actually belongs to X, then f is said to be *Pettis integrable* with respect to μ. It follows from the definition that the Pettis integral commutes with elements of X^* and with bounded linear operators on X.

If f is Bochner integrable, then f is also Pettis integrable and the integrals are the same. If f takes its values in a dual space, then f is also weak*-integrable, again with the same integral. Similarly, if a function f with values in a dual space is Pettis integrable, then it is also weak*-integrable and the integrals are the same.

Notes. This material is standard. A very nice exposition with complete proofs can be found in [DU, Ch. 2].

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Italic numbers refer to the sections where the item is defined.

Symbols

DE GEADJUNGEERDE
VAN EEN HALFGROEP VAN LINEAIRE OPERATOREN

Nederlandse samenvatting

De dualiteitstheorie van halfgroepen van operatoren is voor het eerst systematisch bestudeerd door Phillips in de 50er jaren. De resultaten van Phillips werden verder uitgewerkt in het standaardwerk over halfgroepen van Hille en Phillips. Pas in de 80er jaren kwam deze theorie weer in de belangstelling door een reeks van toepassingen, met name in de perturbatietheorie, die lieten zien dat geadjungeerde halfgroepen in veel gevallen een goed abstrakt kader leveren voor allerlei uiteenlopende problemen.

Dit proefschrift behandelt de abstrakte dualiteitstheorie van halfgroepen van lineaire operatoren op Banach ruimten. Met name in de latere hoofstukken ligt de nadruk op de vraag onder welke omstandigheden informatie omtrent de sterke continuiteit van de geadjungeerde halfgroep kan worden gewonnen. In de eerste vier hoofdstukken wordt de algemene theorie behandeld, zoals de elementaire eigenschappen van de ruimte $X^{\odot}$, de $\sigma(X, X^{\odot})$-topologie, $\odot$-reflexiviteit en de Favard klasse van een halfgroep. De overige vier hoofdstukken handelen over onderwerpen van struktuurtheoretische aard. In de eerste twee daarvan wordt bestudeerd in welke mate de continuiteit van de geadjungeerde halfgroep samenhangt met de struktuur van de onderliggende Banachruimte en in de laatste twee worden speciale klassen van halfgroepen bestudeerd, namelijk halfgroepen van de vorm $T(t)\otimes I$ op het tensorprodukt van twee Banachruimten en positieve halfgroepen of Banachroosters.

Curriculum Vitae

Jan van Neerven werd geboren op 23 december 1964 in Heerlen. Daar bezocht hij het Grotius College en vervolgens in Amsterdam het Vossiusgymnasium. In 1983 begon hij zijn studie Biologie aan de Universiteit van Amsterdam. Een jaar later begon hij aan dezelfde universiteit met de studie Wiskunde. In 1987 behaalde hij het doctoraalexamen Biologie bij professor F. Lopes da Silva met als afstudeerrichting Neurobiologie en in 1988 dat van Wiskunde bij professor H. Lauwerier met als afstudeerrichting Mathematische Fysica, beide cum laude. Voor zijn afstudeerscriptie bij Biologie ontving hij de Unilever Prijs 1987.

In 1988 begon hij zijn promotieonderzoek aan het Centrum voor Wiskunde en Informatica te Amsterdam bij prof. O. Diekmann. In het kader hiervan bezocht hij van september 1990 tot april 1991 het Mathematisch Instituut van de Eberhard-Karls-Universität Tübingen.

STELLINGEN

1. De terminologie 'adjoint semigroup' verdient de voorkeur boven 'dual semigroup'.

2. Zij $T(t)$ een positieve C_0-halfgroep op een Banachrooster E. Als $x^* \perp E^{\odot}$, dan geldt $\limsup_{t\downarrow 0} \|T^*(t)x^* - x^*\| \geq 2\|x^*\|$.

3. Zij $Q(t)$ de C_0-halfgroep op c_0 gegeven door $(Q(t)x)_n := \sum_{k=1}^{\infty} \binom{t}{k} x_{k+n}$ en zijn e_n^* de standaard eenheidsvectoren van l^1. *Dan is $c_0^{\odot}$ het gesloten lineaire opspansel van de vectoren $e_n^* + e_{n+1}^*$.* In het bijzonder heeft $c_0^{\odot}$ co-dimensie één in l^1. Dit is een oplossing van Probleem 4.13.5 in het proefschrift van A. di Bucchianico.

4. Alle 'natuurlijke' afbeeldingen in dit proefschrift zijn inderdaad natuurlijk met betrekking tot geschikt gekozen categorieën.

5. De volgende omkering van de stelling van Lotz over uniforme continuiteit van C_0-halfgroepen op Banachruimten met de Grothendieck- en de Dunford-Pettis eigenschap is waar. *Zij E een Banach rooster met quasi-inwendig punt. Als iedere C_0-halfgroep op E uniform continu is, dan heeft E de Grothendieck- en de Dunford-Pettis eigenschap.* Er volgt zelfs dat E als Banach rooster isomorf is met een $C(K)$-ruimte. Deze beweringen zijn onjuist voor Banachroosters met een zwakke orde-eenheid.

6. Zij E een reëel Banach rooster en zij $E_{\mathbb{C}}$ diens complexificatie. *Onder de natuurlijke identificatie van (reële) vectorruimten $E_{\mathbb{C}} \simeq E \otimes \mathbb{R}^2$, gegeven door $x + iy \leftrightarrow x \otimes (1,0) + y \otimes (0,1)$, is de norm die $E_{\mathbb{C}}$ tot een complex Banach rooster maakt precies de l-tensor norm van $E \otimes \mathbb{R}^2$.*

7. De beweringen in 1 Koningen 7,23 en 2 Kronieken 4,2 van het Oude Testament zijn onjuist.

8. De beste wijze meer scholieren te interesseren voor de studie wiskunde is hun fantasie te prikkelen met de vele tegen-intuitieve resultaten die zonder speciale wiskundige kennis geformuleerd kunnen worden.

9. De universiteiten schieten tekort in hun morele taak wiskundigen en natuurwetenschappers er op te wijzen dat, indien zij in de wapenindustrie werken, zij, en niet politici of militairen, de hoofdverantwoordelijkheid dragen voor de gevolgen van hun produkten.

10. De uitleg die de Rijksuniversiteit te Leiden geeft aan lid 4 van artikel 35 van het promotiereglement ("Het proefschrift en met name het voor- en nawoord blijven gespeend van dankbetuigingen met dien verstande dat 'acknowledgements' zoals gangbaar in de internationale wetenschappelijke literatuur zijn toegestaan.") is een onjuiste weergave van wat gangbaar is in de internationale wetenschappelijke literatuur.

Behorende bij het proefschrift *The adjoint of a semigroup of linear operators.*

Amsterdam, Jan van Neerven