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Slower variation of the generation sizes induced by heavy-tailed environment for geometric branching

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Motivated by seminal paper of Kozlov et al. Kesten et al. (1975) we consider in this paper a branching process with a geometric offspring distribution parametrized by random success probability A and immigration equals 1 in each generation. In contrast to above mentioned article, we assume that environment is heavy-tailed, that is $\log A^{-1}(1-A)$ is regularly varying with a parameter $\alpha > 1$, that is that $\mathbb{P}\Big(\log A^{-1}(1-A) > x\Big) = x^{-\alpha}L(x)$ for a slowly varying function L. We will prove that although the offspring distribution is light-tailed, the environment itself can produce extremely heavy tails of distribution of the population at nth generation which gets even heavier with n increasing. Precisely, in this work, we prove that asymptotic tail $\mathbb{P}(Z_l \geq m)$ of lth population Z_l is of order $\Big(\log^{(l)} m\Big)^{-\alpha} L\Big(\log^{(l)} m\Big)$ for large m, where $\log^{(l)} m = \log \ldots \log m$. The proof is mainly based on Tauberian theorem. Using this result we also analyze the asymptotic behavior of the first passage time T_n of the state $n \in \mathbb{Z}$ by the walker in a neighborhood random walk in random environment created by independent copies $(A_i : i \in \mathbb{Z})$ of $(0, 1)$-valued random variable A.

1. Introduction and main results

We consider branching process appeared in Kesten et al. (1975) to study limit theorems for hitting times associated to the Random Walk in Random Environment (RWRE). We describe briefly the model RWRE and the associated geometric Branching Process in Random Environment (BPRE). Consider a collection $(A_i : i \in \mathbb{Z})$ of i.i.d. (independently and identically distributed) $(0, 1)$-valued random variables. Let $\mathcal{A}$ be the natural σ-field associated to the collection $(A_i : i \in \mathbb{Z})$. Let $(X_k : k \in \mathbb{N})$ be a collection of $\mathbb{Z}$-valued random variables such that, $X_0 = 0$

$$\mathbb{P}\left(X_{k+1} = X_k + 1 \middle| \mathcal{A}, X_0 = i_0, \ldots, X_k = i_k \right) = A_{i_k} = 1 - \mathbb{P}\left(X_{k+1} = X_k - 1 \middle| \mathcal{A}, X_0 = i_0, \ldots, X_k = i_k \right)$$

for all $i_j \in \mathbb{Z}$, $1 \leq j \leq k$ and $k \geq 1$. The collection $(A_i : i \in \mathbb{Z})$ is called the random environment. For this random walk Kesten et al. (1975) studied asymptotic distribution (after appropriate normalization) of a sequence of hitting times $T_n = \inf\{k > 0 : X_k = n\}$ of the state $n \in \mathbb{Z}$ by the walker in the random environment. Following the arguments (see after Remark 3 in page 148 of Kesten et al. (1975)) given in the aforementioned work, we have

$$T_n = n + 2 \sum_{i=-\infty}^{\infty} U_i^{(n)} \tag{1.1}$$

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where $U_i^{(n)} := Card\{k < T_n : X_k = i, X_{k+1} = i-1\}$ denotes the number of times moved left being at state $\{i\}$ with $Card(K)$ is the cardinality of the set K. Under the following assumptions (see assumption (1.2) in Kesten et al. (1975)) on the environment

$$\mathbb{E}\left(\log\frac{1-A}{A}\right) < 0 \quad \text{but} \quad \mathbb{E}\left(\frac{1-A}{A}\right) \geq 1, \tag{1.2}$$

it follows that $X_k \to \infty$ almost surely as $k \to \infty$. So $\sum_{i=-\infty}^{0} U_i^{(n)}$ is finite almost surely and can be ignored in asymptotic analysis of T_n. It is also easy to observe that $\sum_{i=n+1}^{\infty} U_i^{(n)} = 0$ almost surely as the walker cannot reach i before hitting n for all $i \geq n+1$. Thus the asymptotic behavior of T_n is solely determined by the asymptotic behavior of $\sum_{i=1}^{n} U_i^{(n)}$. The following observation

$$\sum_{i=1}^{n} U_i^{(n)} \overset{d}{=} \sum_{l=0}^{n-1} Z_l. \tag{1.3}$$

has been used in Kesten et al. (1975) to derive the asymptotics of $\sum_{i=1}^{n-1} U_i^{(n)}$, where Z_n denotes the size of the nth generation of a BPRE with one immigrant in each generation. The BPRE is constructed in such a way that

$$Z_n = \sum_{i=1}^{Z_{n-1}+1} B_{n,i}, \tag{1.4}$$

where $(B_{n,i} : i \geq 1)$ are independent copies of the geometric random variable B_n such that

$$\mathbb{P}(B_n = k) = A_{n-1}\left(1 - A_{n-1}\right)^k \text{ for all } k \geq 0,\ n \geq 1 \tag{1.5}$$

conditioned on $\mathcal{A}$. Kesten et al. (1975) derived central limit theorem for $n^{-1/\kappa}T_n$ if there exists a $\kappa > 0$ such that

$$\mathbb{E}\left(\exp\left\{\kappa\log\frac{1-A}{A}\right\}\right) = 1. \tag{1.6}$$

Note that the assumption in (1.6) implies that the random variable $\log A^{-1}(1-A)$ has an exponentially decaying right tail. Under above assumptions, after appropriate scaling, T_n has the same asymptotical tail like scaled $\sum_{l=0}^{n-1} Z_l$ and converges to a κ-stable random variable if $\kappa \in (0, 2)$ and Gaussian random variable if $\kappa \geq 2$ (see main result in Kesten et al. (1975)).

The aim of this article is to study the asymptotic behavior of branching process Z_n under the assumption that $\log A^{-1}(1-A)$ has a regularly varying (instead of exponentially decaying) tail. We assume then that

$$\mathbb{P}\left(\log A^{-1}(1-A) < x\right) = \begin{cases} 1 - x^{-\alpha}L(x) & \text{if } x > \eta \\ G(x) & \text{if } x < \eta \end{cases} \tag{1.7}$$

for some $\eta > 0$ and $\alpha > 1$ where $L(\cdot)$ is a slowly varying function i.e. $\lim_{x\to\infty} L(tx)/L(x) = 1$. We assume that G is chosen in such a way that (1.2) holds. Note that $\log A^{-1}(1-A)$ is a real-valued random variable. In (1.7), we only put restrictions on the right-tail of the distribution of $\log A^{-1}(1-A)$ and we do not assume anything about the left-tail. It is clear that the probability of the walker moving to right is small if the value of A is close to 0 which causes large values of T_n. Further, $\log(a^{-1}(1-a))$ is a decreasing function of $a \in (0, 1)$. Hence the tail behavior of A near 0 is same as the tail behavior of $\log A^{-1}(1-A)$ near infinity. Thus large values of $\log A^{-1}(1-A)$ cause large values of T_n. As we are interested in the probability of large values of T_n, right-tail of the random variable $\log A^{-1}(1-A)$ only matters.

Note that (1.2) implies that the BPRE under consideration is subcritical without immigrant and hence becomes extinct eventually for almost all environments. The formula for T_n involves the first n generations of the subcritical BPRE. Thus there is a positive probability that the extinction of BPRE may happen before generation n. The immigration is important for survival of the tree till generation n. But it does not contribute too much to the large values of T_n as it is constant through out all the generations. There is another interpretation of the immigrant. Note that we are considering here nearest-neighbor random walk on $\mathbb{Z}$ and so the walker has to spend at least one unit of time at each state i before hitting n for all $i = 0, 1, 2, \dots, n-1$. Hence one immigrant in each generation appears in the description (see (1.4)) of the BPRE.

Following (Vatutin et al., 2013), if there exists $\beta > 0$, given by the following equation

$$\mathbb{E}\left[\exp\left\{\beta\log\mathbb{E}(Z_1|\mathcal{A})\right\}\log\mathbb{E}(Z_1|\mathcal{A})\right] = 0$$

which would become

$$\mathbb{E}\left[\exp\left\{\beta\log\frac{1-A}{A}\right\}\log\frac{1-A}{A}\right] = 0$$

in our case, then the asymptotic behavior of BPRE crucially depends on the parameter β. The parameter β may not exist always and it does not exist in our case. Indeed, since the right-tail of $\log A^{-1}(1-A)$ is regularly varying as stated in (1.7), thus does not exist any $\beta > 0$ such that

$$\mathbb{E}\left[\left|\exp\left\{\beta \log\frac{1-A}{A}\right\}\log\frac{1-A}{A}\right|\right] < \infty.$$

We are interested in the annealed behavior of the generation sizes $(Z_n : n \geq 1)$ of the BPRE in this paper. Our first result Theorem 1.1 shows that $\mathbb{P}(Z_1 \geq m) \sim (\log m)^{-\alpha} L(\log m)$ and hence has slowly varying tail. It is clear that $\mathbb{E}(Z_1) = \infty$. We would also like to stress the fact that this behavior is not totally unexpected. Note that the tail behavior of $n^{-1/\kappa} T_n$ is regularly varying if $\kappa \in (0, 2)$ as the limit is stable random variable under the assumption stated in (1.6). So it is natural to guess that Z_1 has slowly varying tail under the assumption (1.7) though the form of the slowly varying function is far from being obvious. We derive exact form of the slowly varying function in Theorem 1.1 for Z_1 and Theorem 1.2 for Z_l with $l \geq 2$. These results are used finally to derive the asymptotics for T_n in Theorem 1.4. To the best of our knowledge, this kind of example in BPRE is missing in the literature where generation sizes have exponentially decaying tail given the environment but have slowly varying tail after averaging out the effect of random environment. As a consequence of slowly varying tail of Z_1, it is easy to guess that the annealed behavior of generation sizes is very similar to a GW tree with infinite mean. Branching process with infinite mean is well-studied in literature and a brief review indicating contribution of this article in that literature is given after stating main results of this paper.

Theorem 1.1. *Under the assumptions* (1.5), (1.2) and (1.7),

$$\lim_{m\to\infty} \frac{\mathbb{P}(Z_1 > m)}{(\log m)^{-\alpha} L(\log m)} = 1.$$

Theorem 1.2. *Under the assumptions* (1.5), (1.2) and (1.7),

$$\lim_{m\to\infty} \frac{\mathbb{P}(Z_l > m)}{(\log^{(l)} m)^{-\alpha} L(\log^{(l)} m)} = \alpha^{-\alpha} \tag{1.8}$$

for $l \geq 2$ *where* $\log^{(l)} m = \underbrace{\log \ldots \log}_{l \text{ many}} m$.

Corollary 1.3. *Under the assumptions in* Theorem 1.2, *we have*

$$\lim_{l\to\infty}\lim_{m\to\infty} \frac{\mathbb{P}\left(\log^{(l)} Z_l > m\right)}{m^{-\alpha} L(m)} = \alpha^{-\alpha}.$$

Theorem 1.1 shows that the tail of Z_1 is surprisingly heavy and it is slowly varying. What is more surprising, with each new generation is getting even more heavy and the tail is slowly varying. What should be underlined, this type of behavior is a consequence of an environment only, not branching mechanism which is of geometric type. In our opinion it is first time that such unusual behavior has been observed in the context of branching processes. As a consequence of slowly varying tail of Z_1, annealed behavior of the considered branching process seems to be similar to the branching processes with infinite mean (see Seneta (1973), Hudson and Seneta (1977), Davies (1978), Grey (1977), Cohn (1977), Schuh and Barbour (1977) for example). The asymptotic study in this paper is different as we are studying the asymptotics by looking at the tail behavior of the generation sizes rather than their probability generating functions.

As a corollary we can get another very important result concerning the first passage time T_n of the state $n \in \mathbb{Z}$ by the walker in a nearest neighbor random walk in random environment created by i.i.d. $(0, 1)$-valued random variables $(A_i : i \in \mathbb{Z})$ with generic A.

Theorem 1.4. *Under the assumptions* (1.2), (1.5) and (1.7),

$$\lim_{m\to\infty} \frac{\mathbb{P}\left(\log^{(n-1)}\left[2^{-1}\left(T_n - n - 2\sum_{i\leq 0} U_i^{(n)}\right)\right] > m\right)}{m^{-\alpha} L(m)} = \alpha^{-\alpha} \tag{1.9}$$

for all $n \geq 2$.

Remark 1.5. In Theorem 1.4 we also identify the asymptotic distribution of the first passage time T_n of the state $n \in \mathbb{Z}$ by the walker in a neighborhood random walk in random environment. The counterpart of so-called 'scaling' in the central limit theorem takes the surprising form of taking n-times logarithm. This is also a consequence of heavy-tailed environment. In this case roughly one needs $\exp_{(n)}$ trials to cross the barrier created by heavy-tailed environment where $\exp_{(n)}$ is the inverse function of $\log^{(n)}$. Indeed, the large values of $\log A^{-1}(1 - A)$ by (1.7) correspond to values of A close to 0. This is related, by single one jump principle, with the phenomenon that there is a place on a lattice line that blocks move to the right and hence one has to wait long time to get to the state n.

Proof of Theorem 1.4. It follows from (1.1) and (1.3) that it is enough to prove

$$\lim_{m\to\infty} \frac{\mathbb{P}\left(\log^{(n-1)} \left(\sum_{i=1}^{n-1} Z_i \right) > m \right)}{\left(m^{-\alpha} L(m) \right)} = \alpha^{-\alpha}. \tag{1.10}$$

We shall prove it using upper and lower bounds of the probability in (1.10). Note that

$$\mathbb{P}\left[\log^{(n-1)} \left(\sum_{i=1}^{n-1} Z_i \right) > m \right] \geq \mathbb{P}\left(\log^{(n-1)} Z_{n-1} \geq m \right) \sim \alpha^{-\alpha} m^{-\alpha} L(m) \tag{1.11}$$

for all $n \geq 2$ as $m \to \infty$ using Theorem 1.2. So we are done with the lower bound. We have to prove now the upper bound. We shall first observe that $\{\sum_{i=1}^{n-1} Z_i > m\} \subset \cup_{i=1}^{n-1} \{Z_i > (n-1)^{-1} m\}$. Thus we have

$$\mathbb{P}\left(\sum_{l=1}^{n-1} Z_l > m \right) \leq \sum_{l=1}^{n-1} \mathbb{P}(Z_l \geq (n-1)^{-1} m)$$

for all $n \geq 2$. For large enough m, we have

$$\begin{aligned}
&\sum_{l=1}^{n-1} \mathbb{P}\left(Z_l > (n-1)^{-1} m \right) \\
&\sim \left[\log m - \log(n-1) \right]^{-\alpha} L(\log m - \log(n-1)) \\
&\qquad + \sum_{l=2}^{n-1} \alpha^{-\alpha} \left[\log^{(l)} m - \log^{(l)}(n-1) \right]^{-\alpha} L\left[\log^{(l)} m - \log^{(l)}(n-1) \right] \\
&\sim \alpha^{-\alpha} \left(\log^{(n-1)}(m) \right)^{-\alpha} L(\log^{(n-1)}(m))
\end{aligned} \tag{1.12}$$

for all $n \geq 2$. This implies that

$$\limsup_{m\to\infty} \frac{\mathbb{P}\left[\log^{(n-1)} \left(\sum_{l=1}^{n-1} Z_l \right) > m \right]}{m^{-\alpha} L(m)} \leq \alpha^{-\alpha}.$$

for every $n \geq 2$. This completes the proof. □

2. Proofs

Proof of Theorem 1.1. Note that $\mathbb{P}(Z_1 \geq m) = \mathbb{E}\left((1-A)^m \right)$. To study the asymptotics of above expectation as $m \to \infty$, we have to understand the tail behavior of A near 0. It follows from the assumption (1.7) that

$$\mathbb{P}(A > a) = \begin{cases} G\left(\log \frac{1-a}{a} \right) & \text{if } a > \left(1 + e^{\eta} \right)^{-1} \\ 1 - R_\alpha \left(\log \frac{1-a}{a} \right) & \text{if } 0 < a < (1+e^{\eta})^{-1}, \end{cases} \tag{2.1}$$

where $R_\alpha(a) = a^{-\alpha} L(a)$ for all $a > 0$. Hence we obtain the following equation:

$$\mathbb{P}(Z_1 \geq m) = \int_0^{(1+e^{\eta})^{-1}} (1-a)^m dR_\alpha \left(\log \frac{1-a}{a} \right) + \int_{(1+e^{\eta})^{-1}}^{\infty} (1-a)^m dG \left(\log \frac{1-a}{a} \right). \tag{2.2}$$

Using the fact that $(1-a) < e^{\eta}(1+e^{\eta})^{-1}$ if $a > (1+e^{\eta})^{-1}$, we can see that the second integral in (2.2) can be bounded by $e^{m\eta}(1+e^{\eta})^{-m}$ which decays exponentially with m. It is then enough to consider the first integral. Substituting $y = \log \left(\frac{1-a}{a} \right)$, we obtain the following expression for the first integral in (2.2) as

$$\int_\eta^\infty \left(1 - (1+e^y)^{-1} \right)^m dR_\alpha(y) = \int_\eta^\infty \left(1 + e^{-y} \right)^{-m} dR_\alpha(y). \tag{2.3}$$

Again substituting $e^u = (1+e^{-y})$, (2.3) can be transformed into

$$\int_0^{\log(1+e^{-\eta})} e^{-mu} dR_\alpha \left(-\log(e^u - 1) \right). \tag{2.4}$$

This expression helps to understand the behavior of the integral as $m \to \infty$ since this is the Laplace transform of the measure $R_\alpha(-\log(e^u - 1))$ and we can use Tauberian Theorem 1.7.1′ in Bingham et al. (1987). Note that

$$\lim_{u\to 0} \frac{\Big(-\log(e^u-1)\Big)^{-\alpha} L\Big(-\log(e^u-1)\Big)}{\Big(-\log u\Big)^{-\alpha} L\Big(-\log u\Big)} = 1 \tag{2.5}$$

and thus

$$\lim_{u\to 0} \frac{R_\alpha\Big(-\log(e^u-1)\Big)}{\big(-\log u\big)^{-\alpha} L(-\log u)} = 1.$$

This gives

$$\lim_{m\to\infty} \frac{\int_0^{\log(1+e^{-\eta})} e^{-mu} dR_\alpha\Big(-\log(e^u-1)\Big)}{\Big(\log m\Big)^{\alpha} L\Big(\log m\Big)} = 1 \tag{2.6}$$

which completes the proof. □

We shall prove Theorem 1.2 based on the following result.

Proposition 2.1 (*Lemma 3.8 in Jessen and Mikosch (2006)*). *Consider an i.i.d. sequence $(X_i : i \geq 1)$ of non-negative random variables independent of the integer-valued non-negative random variable K. Define $S_K = \sum_{i=1}^{K} X_i$. If $K, X_1 > 0$ are regularly varying with indices $\gamma_1 \in [0, 1)$ and $\gamma_2 \in [0, 1)$ respectively. Then*

$$\mathbb{P}(S_K > x) \sim \mathbb{P}\Big[K > \big(\mathbb{P}(X > x)\big)^{-1}\Big] \sim x^{-\gamma_1\gamma_2}(L_X(x))^{\gamma_2} L_K(x^{\gamma_1}(L_X(x))^{-1}).$$

Proof of Theorem 1.2. We shall prove the result using induction. Note that $Z_2 \stackrel{d}{=} \sum_{i=1}^{Z_1+1} B_{2,i}$ where $(B_{2,i} : i \geq 1)$ is a collection of independent copies of Z_1. Then we can use Proposition 2.1 with $K = Z_1$, $X_i = B_{2,i}$, $\gamma_1 = 0$, $\gamma_2 = 0$, $L_K(x) \sim (\log x)^{-\alpha} L(\log x)$ and $L_X(x) \sim (\log x)^{-\alpha} L(\log x)$ to obtain

$$\begin{aligned}
\mathbb{P}(Z_2 > m) &\sim \mathbb{P}\Big(Z_1 + 1 > \big(\mathbb{P}(Z_1 > m)\big)^{-1}\Big) \\
&\sim \mathbb{P}\big(Z_1 > (\log m)^\alpha (L(\log m))^{-1}\big) \\
&\sim \Big[\log\Big((\log m)^\alpha \big(L(\log m)\big)^{-1}\Big)\Big]^{-\alpha} L\Big(\log\big[(\log m)^\alpha\big(L(\log m)\big)^{-1}\big]\Big) \\
&= \big(\alpha(\log^{(2)} m)\big)^{-\alpha}\Big[1 - \frac{\log L(\log x)}{\alpha \log^{(2)} m}\Big]^{-\alpha} L\Big[\alpha(\log^{(2)} m)\Big(1 - \frac{\log L(\log m)}{\alpha \log^{(2)} m}\Big)\Big] \\
&\sim \alpha^{-\alpha}(\log^{(2)} m)^{-\alpha} L(\alpha \log^{(2)} m).
\end{aligned} \tag{2.7}$$

We have used the fact that $\lim_{m\to\infty} \log L(\log m)/\log^{(2)} m = 0$ which can be proved using Potter's bound given in Feller (1971, Lemma 2, page 277). Hence the result is proved for $l = 2$. Note that $Z_{l+1} \stackrel{d}{=} \sum_{l=1}^{Z_l+1} B_{l,i}$ where $(B_{l,i} : i \geq 1)$ are independent copies of Z_1. We shall assume that (1.8) holds for Z_l with $l \geq 3$. Then we obtain similarly to the previous asymptotics

$$\begin{aligned}
\mathbb{P}\Big(Z_{l+1} > m\Big) &\sim \mathbb{P}\Big(Z_l + 1 > \big(\mathbb{P}(Z_1 > m)\big)^{-1}\Big) \\
&\sim \alpha^{-\alpha}\Big(\log^{(k+1)} m + \log^{(k-1)} \alpha - \log^{(k)} L(\log m)\Big)^{-\alpha} \\
&\qquad L\big(\alpha \log^{(k+1)} m + \alpha \log^{(k-1)} \alpha - \alpha \log^{(k)} L(\log m)\big) \\
&\sim \alpha^{-\alpha}\big(\log^{(k+1)} m\big) L(\log^{(k+1)} m).
\end{aligned}$$

We have again used Potter's bound to show that $\lim_{m\to\infty} \log^{(k)} L(\log m)/\log^{(k+1)} m = 0$. Hence we conclude the proof. □

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