

# Computability and Representations of the Zero Set<sup>1</sup>

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## Abstract

In this note we give a new representation for closed sets under which the robust zero set of a function is computable. We call this representation the component cover representation. The computation of the zero set is based on topological index theory, the most powerful tool for finding robust solutions of equations.

*Keywords:* Robust zero set, index theory, computability, component cover representation

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## 1 Introduction

In this paper we study the computability of the set of zeroes of a function  $f$  on a subset of Euclidean space. It is well-known that the set of zeroes is upper-semicomputable i.e. given a description of  $f$ , we can effectively enumerate all compact rational balls or boxes  $\bar{I}$  which are disjoint from the zero set  $Z(f) := \{x \mid f(x) = 0\}$ . It is also known that the zero set is not lower-semicomputable using the lower Fell topology on closed sets. i.e. we cannot effectively enumerate all compact rational balls or boxes which intersect  $Z(f)$ . There are two main obstructions to the lower computation, namely a lack of *robustness* and non-isolation of zeroes, sometimes known as *hovering*. However, existential information on the zero set can be obtained using topological index theory.

Topological index theory is the most powerful tool for computing zeroes (or fixed-points) of a continuous function. The index is an effectively-computable integer-valued function defined on pairs  $(f, U)$  such that  $Z(f) \cap \partial U = \emptyset$ , and the main result is that if  $\text{ind}(f, U) \neq 0$ , then  $f$  has a zero in  $U$ . In this article, we address the

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question of how to use the information provided by the index to give an intrinsic description of the zero set. We show that we can compute a *component covering* of the *robust zero set* of  $f$ . In the language of the computable analysis of Weihrauch [8], we give a representation of the set of closed sets such that the robust zero set is computable from a name of  $f$ .

One important difficulty is to handle sets  $U$  for which  $\text{ind}(f, U) = 0$ , since in this case, the index gives no information about the existence of a zero in  $U$ . Using the generalised Hopf degree theorem (see Milnor [4]) we can prove that we can perturb  $f$  in a neighbourhood  $U$  of an isolated component of the zero set with  $\text{ind}(f, U) = 0$  to remove all zeros in  $U$ .

The most celebrated result on locating zeros or fixed-points is Brouwer’s fixed-point theorem, that any self-map of the unit ball in  $\mathbb{R}^n$  has a fixed point. Curiously for one so concerned with constructive issues in mathematics, the result itself is non-constructive. Indeed the theorem cannot be constructivised, for there is no continuous selection of the fixed-point function. The results of this paper can be seen as an attempt to effectivise the content of Brouwer’s theorem.

The computability of zeroes in one-dimension has been considered by a number of authors; see Taylor [7] and references therein. Here, the intermediate value theorem is sufficient to find *straddling intervals* of the zero set. If the function does not *hover* i.e. there are no intervals of zeroes, then the zero set is lower-semicomputable in the Fell topology. The *robust zeroes* of this paper correspond to the “stable zeroes” of Taylor.

## 2 Preliminaries

We now give some technical preliminaries on computability theory, algebraic topology and fixed-point theory which we need later. We will write  $C(X, Y)$  for the set of continuous functions from  $X$  to  $Y$ ,  $\mathcal{O}(X)$  for the set of open subsets of  $X$ , and  $\mathcal{A}(X)$  for the set of closed subsets of  $X$ .

### 2.1 Computability theory

Let  $\mathcal{B}$  be the base of  $\mathbb{R}^n$  consisting of coordinate aligned boxes with rational vertices i.e. sets of the form  $(a_1, b_1) \times \cdots \times (a_n, b_n)$  with  $a_i, b_i \in \mathbb{Q}$  and  $a_i < b_i$  for  $i = 1, \dots, n$ .

We use the following descriptions of points, sets and maps. A *representation* of a class of objects is an encoding of that class in terms of some machine-readable set, such as binary or integer sequences. A *name* of an object is the encoding of that object. We will need the following representations.

- (i) A  $\theta_{<}$ -name of an open set  $U$  is an encoding of all  $I \in \mathcal{B}$  such that  $\bar{I} \subset U$ .
- (ii) A  $\psi_{>}$ -name of a closed set  $A$  is an encoding of all  $I \in \mathcal{B}$  such that  $\bar{I} \cap A = \emptyset$ .
- (iii) A  $\psi_{<}$ -name of a closed set  $B$  is an encoding of all  $J \in \mathcal{B}$  such that  $J \cap B \neq \emptyset$ .
- (iv) A  $\delta$ -name of a continuous function  $f : X \rightarrow Y$  is an encoding of all  $(I, J) \in \mathcal{B}_X \times \mathcal{B}_Y$  such that  $f(\bar{I}) \subset J$ .

A function is *computable* if it is possible to compute a name of the output from a name of the input. For more details on computability theory in analysis and topology, see [8].

## 2.2 Algebraic topology

Algebraic topologic is a well-developed and technical branch of mathematics, and to give more than a cursory treatment is beyond the scope of this paper. The interested reader is invited to read one of the many textbooks on the subject; Munkres [5] is an accessible introduction, and Spanier [6] a detailed reference.

Recall that a homology theory is a function from the category of topological spaces and continuous functions to the category of graded abelian groups and graded homomorphisms. This means that for any natural number  $n$  and any space  $X$ , we have an abelian group  $H_n(X)$ , and for any continuous function  $f : X \rightarrow Y$  we have a group homomorphism  $H_n(f) : H_n(X) \rightarrow H_n(Y)$ . We will sometimes write  $f_*$  as a shorthand for the graded homomorphism  $H_*(f)$ . We write  $H_*(X)$  and  $H_*(f)$  to denote the sequences of homology groups and homomorphisms of, respectively,  $X$  and  $f$ .

In the case that  $X$  is a *compact absolute neighbourhood retract*, which includes the case that  $X$  is a compact topological manifold, the homology is independent of the homology theory chosen. If  $X$  is a connected orientable  $n$ -dimensional topological manifold, it can be shown that  $H_n(X) \cong \mathbb{Z}$ , and that choosing an orientation of  $X$  corresponds to choosing an isomorphism  $H_n(X) \equiv \mathbb{Z}$ . Algorithms for computing the homology of simplicial and cubical maps are detailed in the book [3].

## 2.3 Degree theory

The index theory which we use is most easily developed using the *degree* of a continuous function between closed manifolds of the same dimension.

**Definition 2.1** [Degree] Let  $M$  and  $N$  be connected oriented closed  $n$ -dimensional manifolds. Then the *degree*  $\deg(f)$  of a continuous function  $f : M \rightarrow N$  is the integer  $d$  such that the homology  $H_n(f) : H_n(M) \rightarrow H_n(N)$  with respect to the standard generator  $\omega_M$  of  $H_n(M)$  and  $\omega_N$  of  $H_n(N)$  is given by  $H_n(f)(\omega_M) = d \cdot \omega_N$ . In one dimension, we need to subtract 1 from the homology of  $H_0(f)$ .

If  $M$  is not connected, then we can define the degree of  $f$  to be the sum of the degrees of  $f$  restricted to the components of  $M$ .

Recall that a regular value of a differentiable function  $f : X \rightarrow Y$  is a point  $y \in Y$  such that for every  $x \in f^{-1}(y)$ , we have  $\text{rank } Df(x) = \dim(Y)$ . By Sard's theorem, the set of regular values of any differentiable function  $f$  has full measure in  $Y$ .

**Definition 2.2** [Brouwer degree] Let  $M$  and  $N$  be differentiable manifolds of the

same dimension, and  $f$  be a differentiable function. Define

$$\deg(f) := \sum_{x \in f^{-1}(y)} \operatorname{sgn}(\det Df(x))$$

where  $y$  is any regular value of  $f$ . It can be shown that the sum is independent of the value of  $y$  chosen.

#### 2.4 Index theory

The index is an integer-valued function defined on pairs  $(f, U)$ , where  $f : X \rightarrow \mathbb{R}^n$  is a function on Euclidean space, and  $U$  is a bounded open set such that  $Z(f) \cap \partial U = \emptyset$ . The fundamental observation used in the definition of the index is that if  $f$  has no zeroes in a set  $M$ , then we can define a continuous function  $h : M \rightarrow S^{n-1}$  by  $h(x) = f(x)/\|f(x)\|$ . The main property is that  $\operatorname{ind}(f, U) \neq 0$  implies that  $Z(f) \cap U \neq \emptyset$ . Many of the statements here can be derived from equivalent statements for the Lefschetz fixed-point index; see Brown [2], or using differentiable degree theory; see Milnor [4].

**Definition 2.3** [Index] Let  $f : X \rightarrow \mathbb{R}^n$  and  $U \subset X$  be a connected bounded open set such that  $\partial U$  is a closed topological manifold and  $Z(f) \cap \partial U = \emptyset$ . Let  $h : \partial U \rightarrow S^{n-1}$  be the continuous function given by  $h(x) = f(x)/\|x\|$ . We define  $\operatorname{ind}(f, U) = \deg(h)$ .

The main result of index theory is the following:

**Theorem 2.4** If  $\operatorname{ind}(f, U)$  is defined and nonzero, then  $f$  has a zero in  $U$ .

The index satisfies a number of properties which are useful in computations, stated below. Recall that a homotopy is a continuous function  $F : [0, 1] \times X \rightarrow Y$ . We typically set  $f_t(x) = F(t, x)$ , and think of  $f_t$  as a continuously-varying parameterised family of functions  $f_t : X \rightarrow Y$  connecting  $f_0$  and  $f_1$ .

**Restriction** If  $Z(f) \cap U = Z(f) \cap V$ , then  $\operatorname{ind}(f, U) = \operatorname{ind}(f, V)$ .

**Additivity** If  $U_1 \cap U_2 = \emptyset$ , then  $\operatorname{ind}(f, U_1 \cup U_2) = \operatorname{ind}(f, U_1) + \operatorname{ind}(f, U_2)$ .

**Localisation** If  $f|_U = g|_U$ , then  $\operatorname{ind}(f, U) = \operatorname{ind}(g, U)$ .

**Homotopy** If  $f_0$  is homotopic to  $f_1$  via a homotopy  $f_t$  such that  $Z(f_t) \cap \partial U = \emptyset$  for all  $t \in [0, 1]$ , then  $\operatorname{ind}(f_0, U) = \operatorname{ind}(f_1, U)$ .

We can also think of the index as applying to components of the zero set. Let  $C$  be a set of zeroes of  $f$ , and suppose there exists an open set  $U$  such that  $Z(f) \cap U = C$  and  $Z(f) \cap \partial U = \emptyset$ . Then we can define  $\operatorname{ind}(f, C) := \operatorname{ind}(f, U)$  since by the restriction property, this is independent of  $U$ .

In order to consider the case  $\operatorname{ind}(f, U) = 0$ , we will need to use the following generalised version of the Hopf theorem. A proof can be found in Milnor [4].

**Theorem 2.5** Let  $M$  be a manifold of dimension  $n$ , and  $f, g : M \rightarrow S^n$ . Then  $f$  and  $g$  are homotopic if, and only if,  $\deg(f) = \deg(g)$ .

### 2.5 Computing the index

We now consider computability of the index. The following lemma, which we state without proof, shows that we can effectively approximate a set  $U$  such that  $Z(f) \cap \partial U = \emptyset$  by a set containing the same zeroes, but with a more regular boundary.

**Lemma 2.6** *Let  $C \subset X$  be a compact set, and let  $U$  be a bounded connected open set such that  $C \cap \partial U = \emptyset$ . Then there exists an open set  $V$  such that  $C \cap V = C \cap U$ , that  $C \cap \partial V = \emptyset$ , and that  $\partial V$  is an  $n - 1$  dimensional manifold. Further, given a  $\theta_{<}$  name of  $U$  and a  $\psi_{>}$ -name of  $\bar{U}$ , we can compute a name of  $\partial V$ .*

The following result shows that the degree of a continuous function is computable. The proof relies on computational homology theory; see [3] for details.

**Lemma 2.7** *Let  $M$  be an  $n - 1$ -dimensional manifold and  $f : M \rightarrow S^{n-1}$ . Then the degree of  $f$  is computable.*

The following result shows that the index function is effectively computable.

**Theorem 2.8** *The index  $\text{ind} : C(X, \mathbb{R}^n) \times \mathcal{O}(X) \rightarrow \mathbb{Z}$  has a computably open domain, and the index is computable on its domain.*

## 3 Computability of the zero set

The following theorem is well known; a proof can be found in Weihrauch [8][Theorem 6.2.9(ii)]

**Theorem 3.1** *The zero-set operator  $Z : C(X, \mathbb{R}^n) \rightarrow \mathcal{A}(X)$  is  $(\delta; \psi_{>})$ -computable.*

If  $x$  is contained in an isolated component  $C$  of the zero set, then  $x$  is robust if  $\text{ind}(f, C) \neq 0$ . However, if  $C$  is not isolated, then there are other components of the zero set approaching  $C$ .

**Definition 3.2** Let  $C$  be a component of  $Z(f)$ . We say  $C$  is a *robust* component of  $Z(f)$  if for all open  $U$  with  $C \subset U$ , there exists a neighbourhood  $W$  of  $f$  in  $C(X, \mathbb{R}^n)$  such that for all  $\tilde{f} \in W$ ,  $\text{fix}(\tilde{f}) \cap U \neq \emptyset$ . The set of *robust zeroes* of  $f$ , denoted  $\text{RZ}(f)$ , is defined by

$$\text{RZ}(f) := \bigcup \{C \mid C \text{ is a robust component of } Z(f)\} \tag{1}$$

We now define the *component cover representation* of a set.

**Definition 3.3** Let  $A$  be a closed set. A name of  $A$  in the component cover representation encodes all tuples  $(J_1, \dots, J_k) \in \mathcal{B}^*$  such that  $\bigcup_{i=0}^k J_i$  is a superset of a component  $C$  of  $A$ .

The following result shows that isolated components of the zero set with index equal to zero can be removed by perturbation.

**Proposition 3.4** *Suppose  $X$  is an open subset of  $\mathbb{R}^n$  for  $n \neq 2$ , that  $f : X \rightarrow \mathbb{R}^n$  and  $Z(f)$  is compact. Then if  $C$  is an isolated component of  $\text{fix}(f)$ , and  $U$  is a neighbourhood of  $C$  such that  $Z(f, \text{cl}(U)) = C$ , and  $\text{ind}(f, U) = 0$ , then there exists arbitrarily small perturbations  $f_\epsilon$  of  $f$  such that  $Z(f_\epsilon) \cap U = \emptyset$ .*

**Proof.** [Sketch] By making a small perturbation of  $U$ , if necessary, we can assume that  $\partial U$  is a manifold  $M$ . By making  $U$  smaller if necessary, we can assume that  $|f(x)| < \epsilon/2$  for all  $x \in U$ . Let  $V$  be an open set such that  $C \subset V$ ,  $\bar{V} \subset U$ , and  $\bar{U} \setminus V$  is homeomorphic to  $M \times I$ . We define our perturbation  $f_\epsilon$  of  $f$  as follows. Choose a constant  $c \in \mathbb{R}^n \setminus \{0\}$  with  $|c| < \epsilon/2$  and set  $f_\epsilon(x) = c$  for  $x \in V$ . For  $x \in U \setminus \bar{V}$ , we define coordinates  $x = (t, y)$  with  $t \in [0, 1]$  and  $y \in M$ ,  $f_\epsilon(t, y)$  via a homotopy from  $f$  to  $c$ . The resulting function  $f_\epsilon$  is an  $\epsilon$ -perturbation of  $f$  with no zeroes in  $U$ .  $\square$

The following theorem is the main result of the paper. It states that a name of robust zero set in the component cover representation can be computed from a name of  $f$ .

**Theorem 3.5** *The operator  $\text{RZ} : C(X, \mathbb{R}^n) \rightarrow \mathcal{A}(X)$  is computable using the component cover representation of  $\mathcal{A}(X)$ .*

**Proof.** [Sketch] Suppose  $x$  is a point in the robust zero set of  $f$  contained in a component  $C$ , and let  $U$  be an open neighbourhood of  $C$ . Then there exists open  $V \subset U$  such that  $\text{ind}(f, V) \neq 0$ , which proves that  $U$  contains a component of  $\text{RZ}(f)$ . Hence we can compute a list of all open  $U$  containing components of  $\text{RZ}(f)$ .  $\square$

## 4 Examples

We now give some examples in one dimension illustrating the representation of the robust zero set. Note that if  $I = (a, b)$  is an open bounded interval in  $\mathbb{R}$  and  $a, b \notin Z(f)$ , then

$$\text{ind}(f, I) = \begin{cases} 1, & \text{if } f(a) < 0 < f(b); \\ -1, & \text{if } f(b) < 0 < f(a); \\ 0, & \text{otherwise.} \end{cases}$$

**Example 4.1** Suppose  $f(x) = x(1 - \cos(1/x))$ . Then  $f$  has a zero for  $x = 0$  and  $x = 1/2\pi n$  for  $n \in \mathbb{Z} \setminus \{0\}$ . However, since  $f(x) \geq 0$  for all  $x$ , we have  $\text{ind}(f, z) = 0$  for all isolated zeroes of  $f$ . By definition, even the non-isolated zero at  $x = 0$  is non-robust. Indeed the perturbation  $f_\epsilon(x) := f(x) + \epsilon = x(1 - \cos(1/x)) + \epsilon$  has no zeroes.

**Example 4.2** Let  $f(x) = x(x - 1)^2$ . Then  $f$  has a robust zero at  $x = 0$  and a non-robust zero at  $x = 1$ . The index of  $f$  on the set  $U = (-1, 1)$  is not defined due to the zero at  $x = 1$ , but we can still prove that  $U$  contains a zero by computing the index of  $f$  on  $V = (-1/2, 1/2)$ .

## 5 Conclusion

In this paper, we have given a new “lower” representation for the set of closed subsets of Euclidean space, and shown that the sets of robust fixed-points and robust zeroes are computable for this representation. It remains to investigate the computability of the robust fixed-point set in two dimensions. It would also be interesting to investigate the computability of the robust fixed-point set in terms of the Borel hierarchy of sets in the sense of Brattka [1].

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