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IDENTIFIABILITY OF NONLINEAR HOMOGENEOUS POLYNOMIAL SYSTEMS

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Abstract: New results are presented concerning the state isomorphism approach to global identifiability analysis of parameterized classes of nonlinear homogeneous systems with specified initial states. For such systems, the local state isomorphism for a pair of indistinguishable parameter vectors is homogeneous of degree one. Under certain conditions, which may only be satisfied for homogeneous polynomial systems, the local state isomorphism is linear. Here, the key issue is whether or not the observability rank condition holds at the origin. The controllability rank condition is shown to play a truly secondary role. The results are generalized to the multivariable case and a worked example demonstrates how identifiability analysis may be simplified along these lines. *Copyright © 2003 IFAC*

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1. INTRODUCTION

Consider a parameterized class of nonlinear dynamical state-space systems with initial conditions, described by:

$$\dot{x}(t,p) = f(x(t,p),p) + g(x(t,p),p)u(t), \quad (1)$$

$$y(t,p) = h(x(t,p),p), \quad (2)$$

$$x(0,p) = x_0(p). \quad (3)$$

The vector fields $f(.,p) : M_p \rightarrow \mathbb{R}^n$ and $g(.,p) : M_p \rightarrow \mathbb{R}^n$ are assumed to be real analytic on some state space domain M_p, which is an open connected subset of $\mathbb{R}^n$ containing the initial state $x_0(p)$. The scalar output function $h(.,p) : M_p \rightarrow \mathbb{R}$ is also assumed to be real analytic and the parameter vector p takes its values in some parameter set $\Omega \subseteq \mathbb{R}^q$. The scalar input function $u(\cdot) : [0,\tau] \rightarrow \mathbb{R}$ is chosen from some set $U[0,\tau]$ of bounded and measurable controls. Here $\tau > 0$ is a fixed constant, and it is assumed that the system of equations (1)-(3) has a unique solution on M_p over the entire time interval $[0,\tau]$. (Some subtleties of a technical nature are involved in this set-up, but we will not go into this; see also (Sussmann, 1977).)

For this class of systems, parameter indistinguishability and (global) identifiability can be studied by various methods. One particular approach of interest is based on the *local state isomorphism theorem*, see also (Sussmann, 1977; Hermann and Krener, 1977; Tunali and Tarn, 1987). This theorem can be viewed as the nonlinear counterpart of a well-known result from realization theory, which states that if two minimal linear state-space systems are equivalent from an input-output point of view then there exists a nonsingular linear state-space transformation which takes one system into the other.

To make these notions precise, let Σ_p denote the input-output mapping which maps the input space $U[0,\tau]$ to a suitable space $Y[0,\tau]$ of output functions, according to the rule $u(.) \mapsto y(.,p)$ as generated by the equations

(1)-(3). (The dependence of the input-output mapping on τ will be suppressed in the notation for the sake of readability.) Then two parameter vectors p and $\widetilde{p}$ are said to be *indistinguishable* if $\Sigma_p = \Sigma_{\widetilde{p}}$. Indistinguishability is an equivalence relation. The system (1)-(3) is called *globally identifiable* if the mapping $p \mapsto \Sigma_p$ is injective on Ω, i.e., no pair of distinct indistinguishable parameter vectors exists.

Reformulated for the purpose of identifiability analysis, the following form of the local state isomorphism theorem can essentially be found in (Vajda *et al.*, 1989).

Theorem 1. Let p and $\widetilde{p} \in \Omega$ be two parameter vectors for each of which the system (1)-(3) is locally reduced at the initial state, i.e., both the controllability rank condition (CRC) and the observability rank condition (ORC) are satisfied. Then p and $\widetilde{p}$ are indistinguishable if and only if there exists an open neighborhood $\widetilde{V}$ of $x_0(\widetilde{p})$, an open neighborhood V of $x_0(p)$ and a real analytic diffeomorphism $\lambda : \widetilde{V} \to V$ with the following properties:
(i) $\text{rank}\,(\nabla\lambda(\widetilde{x})) = n$, for all $\widetilde{x} \in \widetilde{V}$,
(ii) $\lambda(x_0(\widetilde{p})) = x_0(p)$,
(iii) $f(\lambda(\widetilde{x}), p) = \nabla\lambda(\widetilde{x}) \cdot f(\widetilde{x}, \widetilde{p})$, for all $\widetilde{x} \in \widetilde{V}$,
(iv) $g(\lambda(\widetilde{x}), p) = \nabla\lambda(\widetilde{x}) \cdot g(\widetilde{x}, \widetilde{p})$, for all $\widetilde{x} \in \widetilde{V}$,
(v) $h(\lambda(\widetilde{x}), p) = h(\widetilde{x}, \widetilde{p})$, for all $\widetilde{x} \in \widetilde{V}$,
where $\nabla\lambda(\widetilde{x})$ denotes the Jacobian matrix of the mapping λ evaluated at the point $\widetilde{x}$, and the symbol $\cdot$ denotes matrix-vector multiplication. For a given choice of $\widetilde{V}$, this real analytic diffeomorphism λ is unique.

See (Isidori, 1989; Hermann and Krener, 1977; Sussmann, 1977) for more details on the CRC and ORC.

The state isomorphism approach to identifiability analysis is centered around the following procedure. Assume that p and $\widetilde{p} \in \Omega$ are indistinguishable and let the systems in the model class be locally reduced at the initial states. Then Thm. 1 guarantees the existence of a unique local state isomorphism λ. From the literature, an explicit representation of λ can often be computed, see for instance (Isidori, 1989), which may be useful. The five conditions on λ stated in Thm. 1 give rise to a set of equations which involve p and $\widetilde{p}$, and which hold on an open neighborhood V of the initial state $x_0(\widetilde{p})$. If these equations imply that λ is the identity mapping and $p = \widetilde{p}$ yields the only feasible solution, global identifiability holds. Otherwise, unidentifiability follows, and the construction provides the various conditions for indistinguishable pairs of parameters and their associated state isomorphisms.

Now, if λ is known in advance to have a certain structure, such as being linear or affine, one may exploit this to substantially reduce the complexity of the computations in the various stages of the procedure sketched above. Thus, it becomes of interest to study particular subclasses of systems for which such properties of λ can be established. One such class is that of *homogeneous systems*. For a definition and some properties of homogeneous functions used in this paper, see App. A. A worked example is included in Sect. 6. Because of space limitations, all the proofs are given elsewhere; see (Peeters and Hanzon, 2002).

2. PRELIMINARY RESULTS FOR HOMOGENEOUS SYSTEMS

In a recent paper, see (Hanzon and Peeters, 2001), it has been established for homogeneous vector fields $f(.,p)$ and $g(.,p)$ and linear $h(.,p)$, that if the systems are all locally reduced at the initial states $x_0(p)$, then any local state isomorphism λ for a pair of indistinguishable parameters is homogeneous of degree 1.

Now, if a real analytic function is homogeneous of degree 1 on an open domain containing the origin, then it has to be linear. In (Hanzon and Peeters, 2001) this fact is exploited, yielding that λ is linear under the additional condition that the ORC always holds at the origin. Note that to facilitate the ORC to be properly evaluated at the origin, smoothness of $f(.,p)$, $g(.,p)$ and $h(.,p)$ at the origin is required. Together with homogeneity and real analyticity this implies that $f(.,p)$, $g(.,p)$ and $h(.,p)$ are *homogeneous polynomials* and that their degrees of homogeneity k, ℓ and m are nonnegative integers.

The following two theorems summarize and slightly extend these results by also allowing the output function $h(.,p)$ to be homogeneous on M_p, which notably may or may not contain the origin.

Theorem 2. Consider the parameterized class of systems given by (1)-(3). For all $p \in \Omega$, let the CRC and the ORC be satisfied at $x_0(p)$ and let $f(.,p)$, $g(.,p)$ and $h(.,p)$ be real analytic and homogeneous on M_p, of degrees k, ℓ and $m \in \mathbb{R}$, respectively. Let p and $\widetilde{p} \in \Omega$ be indistinguishable and let λ be the associated real analytic local state isomorphism as in Thm. 1. Then λ is homogeneous of degree 1.

Theorem 3. Consider the parameterized class of systems given by (1)-(3). For all $p \in \Omega$, let $f(.,p)$, $g(.,p)$ and $h(.,p)$ be homogeneous polynomials on M_p containing the origin, of nonnegative integer degrees k, ℓ and m, respectively. Let the CRC and the ORC be satisfied at $x_0(p)$ and let the ORC be satisfied at the origin too. Let p and $\widetilde{p} \in \Omega$ be indistinguishable and let λ be the associated real analytic local state isomorphism as in Thm. 1. Then λ is linear.

At this point it is not required that the homogeneous polynomial systems also satisfy the CRC at the origin. In fact, if the degrees k and ℓ are both strictly positive, then the CRC cannot hold at the origin. In the next section it will be made clear to what extent the roles played by CRC and the ORC at the origin are different, and how they are interrelated.

Remark 4. In (Joly-Blanchard and Denis-Vidal, 1998) and (Hanzon and Peeters, 2001) it has recently been demonstrated that a number of results reported in the literature on the local isomorphism approach to identifiability analysis are invalid. Identifiability of a class of polynomial systems was previously studied in (Chappell *et al.*, 1990) but Thm. 2 in that paper is invalid: it is based on an invalid result of (Vajda and Rabitz, 1989) and a counter example is provided by Example 1 in (Peeters and Hanzon, 2002).

3. ON THE ROLES PLAYED BY THE CRC AND THE ORC AT THE ORIGIN

As mentioned above, if $f(.,p)$, $g(.,p)$ and $h(.,p)$ are homogeneous and real analytic and if the CRC and the ORC can be properly evaluated at the origin, then $f(.,p)$, $g(.,p)$ and $h(.,p)$ are actually *polynomial*.

The following theorem indicates that for a homogeneous polynomial system which is locally reduced at the initial state, requiring the CRC to hold at the origin is a stronger condition than requiring the ORC to hold at the origin.

Theorem 5. For a homogeneous polynomial system, if the ORC holds at some state x_0 and the CRC holds at the origin, then the CRC and the ORC hold everywhere.

Thm. 3 and Thm. 5 jointly lead to the following result.

Corollary 6. Consider the parameterized class of systems given by (1)-(3). For all $p \in \Omega$, let $f(.,p)$, $g(.,p)$ and $h(.,p)$ be homogeneous polynomials on M_p containing the origin, of nonnegative integer degrees k, ℓ and m, respectively. Let the ORC and the CRC be satisfied at $x_0(p)$ and let the CRC be satisfied at the origin too. Let p and $\widetilde{p} \in \Omega$ be indistinguishable and let λ be the associated real analytic local state isomorphism as in Thm. 1. Then λ is linear.

This shows that, for homogeneous polynomial systems, evaluation of the CRC at the origin does not bring any additional prior information on the possible linearity of λ compared to the information already obtainable by evaluation of the ORC at the origin. However, if it is easier to verify the CRC at the origin than the ORC (and if the CRC happens to hold) Cor. 6 may still prove to be of practical value.

Conversely, situations may occur where the ORC holds at the origin and Thm. 3 applies, while the CRC does not hold at the origin. An obvious class of systems for which this happens is that of the *bilinear systems*, where f, g and h are all linear. In that case the CRC cannot hold at the origin, but it is well possible for the ORC to hold at the origin (and also for the CRC to hold outside the origin). Another example is constituted by the following SISO system, given by

$$\begin{aligned}\dot{x}_1 &= x_1 x_3,\\ \dot{x}_2 &= x_3^2,\\ \dot{x}_3 &= x_2^2 + u,\\ y &= x_1 + x_2,\end{aligned}$$

with the non-zero initial state $x_0(p) = (1, 0, 0)^T$. For this system the ORC holds everywhere and the CRC holds at all states with $x_1 \neq 0$. To see this, first consider the functions:

$$\begin{aligned}\omega_1(x) &:= h(x) = x_1 + x_2,\\ \omega_2(x) &:= L_g L_f h(x) = x_1 + 2x_3,\\ \omega_3(x) &:= L_g L_f L_g L_f h(x) = x_1.\end{aligned}$$

Together, their gradients constitute a constant invertible 3×3 matrix, which makes clear that the ORC holds everywhere.

With respect to the CRC note that the first component of the vector field f has a factor x_1, while the first component of g is zero. Now suppose that v is a homogeneous polynomial vector field of which the first component has a factor x_1, so that it can be written as $v(x) = (x_1 \widetilde{v}_1(x), v_2(x), v_3(x))^T$, with $\widetilde{v}_1$, v_2 and v_3 polynomial. Then it is straightforward to show that the first components of the Lie brackets $[f, v]$ and $[g, v]$ both also contain a factor x_1. As a consequence, the first component of any repeated Lie bracket of f and g is always zero when evaluated at a point for which $x_1 = 0$ and the CRC does not hold. On the other hand the CRC does hold if $x_1 \neq 0$ since $[f, g](x) = (-x_1, -2x_3, 0)^T$ and $[[f, g], g](x) = (0, 2, 0)^T$.

If $f(.,p)$ and $g(.,p)$ are polynomial vector fields, one may also investigate the situation where the CRC holds at the origin directly, without explicit reference to the output function $h(.,p)$ or the ORC at the origin. For such systems, the local state isomorphism λ is merely affine.

Theorem 7. Consider the parameterized class of systems given by (1)-(3). For all $p \in \Omega$, let $f(.,p)$ and $g(.,p)$ be homogeneous polynomials on M_p containing the origin, of nonnegative integer degrees k and ℓ, respectively. Let the ORC and the CRC be satisfied at $x_0(p)$ and let the CRC be satisfied at the origin too. Let p and $\widetilde{p} \in \Omega$ be indistinguishable and let λ be the associated real analytic local state isomorphism as in Thm. 1. Then λ is affine.

Note that nothing special other than real analyticity is required from the output function $h(.,p)$. In case $h(.,p)$ is also homogeneous on M_p (and therefore polynomial, since M_p contains the origin) it follows from Thm. 2 that λ is also homogeneous of degree 1. Hence, λ is actually linear. This constitutes an alternative proof of Cor. 6.

4. ON THE SCOPE OF THE RESULTS

Starting from the assumptions of real analyticity and homogeneity of the system, for the CRC and the ORC to be properly defined at the origin the functions f, g and h are required to be homogeneous polynomials and their degrees of homogeneity k, ℓ and m to be nonnegative integers.

Then the repeated Lie brackets of f and g, as well as the repeated Lie derivatives of h along f and g, are all homogeneous polynomials too. Their exact degrees of homogeneity depend on how many times f and g occur in the repeated operations. Note that the roles of f and g are entirely interchangeable, both with respect to the CRC and the ORC. Each time a Lie bracket is formed with f (resp. g) and each time a Lie derivative is taken along f (resp. g) the degree of homogeneity increases by $k-1$ (resp. $\ell-1$). Clearly, for $k=0$ (or $\ell=0$) the degree decreases by 1. For $k=1$ (or $\ell=1$) the degree remains the same, and for $k \geq 2$ (or $\ell \geq 2$) the degree (strictly) increases.

At the origin, a homogeneous polynomial vector field evaluates to zero unless its degree of homogeneity is zero, i.e., unless it is constant. Therefore, for the CRC to hold at the origin a necessary and sufficient condition is that n independent constant vector fields are contained in the Control Lie Algebra generated by f and g. With respect to the ORC one needs to consider the observability co-distribution $d\mathcal{O}$ spanned by the gradient fields associated with the repeated Lie derivatives of h along f and g. At the origin these gradients evaluate to zero, except for the repeated Lie derivatives which are are linear. Thus, for the ORC to hold at the origin, a necessary and sufficient condition is that n independent linear functions are contained in the Observability Space. With respect to the degrees k, ℓ and m the following observations can now be made.

(1) The CRC may hold at the origin only if $k=0$ or $\ell=0$. If $k=\ell=0$, then all the repeated Lie brackets of f and g are identically zero, so one should also have that $n \leq 2$.

(2) The ORC may hold at the origin in each of the following three situations:
(a) $k=0$ or $\ell=0$ (and m arbitrary);
(b) $k=m=1$ and $\ell \geq 1$, or $k \geq 1$ and $\ell=m=1$;
(c) $k \geq 2$ and $\ell \geq 2$ and $n=m=1$.
Case (a) contains the linear systems (for $k=1$, $\ell=0$ and $m=1$), which are well-known to involve linear state space transformations. Case (b) includes the bilinear systems (for $k=\ell=m=1$) which are also well-known to involve linear state space transformations (see (D'Alessandro *et al.*, 1974)). Case (c) is not of much interest with respect to identifiability analysis, since $n=m=1$. One then deals with a scalar first-order differential equation in x, which is directly observable from the output y being a linearly scaled version of x.

5. GENERALIZATION TO THE MULTIVARIABLE CASE

To generalize the results of the previous section to the multi-input multi-output case is straightforward. The nonlinear systems under consideration are required to have the form

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{r} g_i(x(t))u_i(t), \tag{4}$$

$$y_j(t) = h_j(x(t)), \quad j = 1, \ldots, s, \tag{5}$$

$$x(0) = x_0. \tag{6}$$

Here one is dealing with r input signals u_i, $i = 1, \ldots, r$, entering the system via the dynamical equation in a linear way, each having its own associated input vector field g_i. Also, one now has s output signals y_j, $j = 1, \ldots, s$, each obtained from the state x by means of an associated output function h_j. This constitutes a common class of nonlinear systems often studied in the literature. Note that the concepts of the CRC and the ORC are defined in the literature to apply to these multivariable systems too (see, e.g., (Isidori, 1989)).

For the theorems of the previous sections to hold, the input vector fields $g_1, \ldots, g_r$ are required to be real analytic and homogeneous of degree $\ell_1, \ldots, \ell_r$, respectively. The output functions $h_1, \ldots, h_s$ are required to be real analytic and homogeneous of degree $m_1, \ldots, m_s$, respectively. The Control Lie Algebra $\mathcal{C}$ and the Observation Space $\mathcal{O}$ have to be constructed in the usual way, by taking repeated Lie brackets of $f, g_1, \ldots, g_r$ and by taking repeated Lie derivatives of all the output functions $h_1, \ldots, h_s$ with respect to f, $g_1, \ldots, g_r$. Obviously, more complicated sequences of Lie brackets and Lie derivatives can now be formed, but their total number remains countable and systematic procedures are easily developed to enumerate them. All the theorems of the previous sections are then seen to allow for immediate generalization to this multivariable set-up.

With respect to the issue of the limitations imposed on the degrees of homogeneity $k, \ell_1, \ldots, \ell_r, m_1, \ldots, m_s$ in case the CRC or the ORC holds at the origin, the following holds.

(1) For the CRC to hold at the origin, at least one of the degrees $k, \ell_1, \ldots, \ell_r$ must be equal to zero. If all of them are equal to zero, then n should not exceed $r+1$.

(2) For the ORC to hold at the origin, one of the following three situations must hold:
(a) at least one of the degrees k, $\ell_1, \ldots, \ell_r$ is equal to 0;
(b) the degrees $k, \ell_1, \ldots, \ell_r$ are all ≥ 1, at least one of them is equal to 1, and moreover at least one of the degrees $m_1, \ldots, m_s$ is equal to 1;
(c) the degrees $k, \ell_1, \ldots, \ell_r$ are all ≥ 2, and at least n of the degrees $m_1, \ldots, m_s$ are equal to 1.

6. A WORKED EXAMPLE

Consider the following class of homogeneous polynomial systems, given by

$$\begin{aligned}
\dot{x}_1 &= -px_1^2x_3 + x_2x_3^2, \\
\dot{x}_2 &= -2p^2x_1^3 + 2px_1x_2x_3 + x_1x_3^2 + x_3^3, \\
\dot{x}_3 &= u, \\
y_1 &= x_1, \\
y_2 &= x_3,
\end{aligned}$$

with the initial state $x_0(p) = (0, 0, 1)^T$.

To verify the ORC, the following functions ω_1^p, ω_2^p and ω_3^p are determined, involving Lie derivatives of the components of h along the vector fields f and g:

$$\begin{aligned}
\omega_1^p(x) &:= h_1(x) = x_1, \\
\omega_2^p(x) &:= L_f h_1(x) = -px_1^2x_3 + x_2x_3^2, \\
\omega_3^p(x) &:= h_2(x) = x_3.
\end{aligned}$$

The corresponding matrix of gradient vectors takes the form

$$\begin{pmatrix} 1 & 0 & 0 \\ -2px_1x_3 & x_3^2 & -px_1^2 + 2x_2x_3 \\ 0 & 0 & 1 \end{pmatrix},$$

having full rank 3 whenever $x_3 \neq 0$, which holds for instance for the initial state vector $x_0(p)$.

To verify the CRC, one may compute $[[[f, g], g], g](x) = \begin{pmatrix} 0 \\ -6 \\ 0 \end{pmatrix}$ and $[[[[[f, g], g], g], f], g], g](x) = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}$. Together with $g = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ this shows that the CRC is satisfied everywhere.

An explicit representation of λ can be computed from ω_1^p, ω_2^p and ω_3^p, since according to Thm. 1 it holds on an open neighborhood of the initial state that $\omega_i^p \circ \lambda = \omega_i^{\tilde{p}}$ for all i. One has:

$$\begin{aligned}
\tilde{x}_1 &= x_1, \\
-\tilde{p}\tilde{x}_1^2\tilde{x}_3 + \tilde{x}_2\tilde{x}_3^2 &= -px_1^2x_3 + x_2x_3^2, \\
\tilde{x}_3 &= x_3.
\end{aligned}$$

This gives the mapping $x = \lambda(\tilde{x})$ described by

$$\begin{aligned}
x_1 &= \tilde{x}_1, \\
x_2 &= \tilde{x}_2 + (p - \tilde{p})\frac{\tilde{x}_1^2}{\tilde{x}_3}, \\
x_3 &= \tilde{x}_3,
\end{aligned}$$

which is indeed homogeneous of degree 1 but nonlinear for $p \neq \tilde{p}$. But since the CRC holds everywhere, and in particular at the origin, it follows from Cor. 6 that λ is linear. Hence one can conclude that $p = \tilde{p}$, whence identifiability holds.

It is illustrative to note that without the prior knowledge of linearity of λ provided by Cor. 6, one would normally proceed by imposing the properties (i)–(v) of λ as given by Thm. 1. It then happens that the conditions (i) and (ii) and (v) are obviously satisfied.

However, condition (iv) imposes the restriction

$$-(p - \tilde{p})\frac{\tilde{x}_1^2}{\tilde{x}_3^2} = 0.$$

This again leads to the conclusion $p = \tilde{p}$, since this identity should hold on an open neighborhood of the initial state. As before, only the trivial solution with λ equal to the identity mapping is obtained and once again identifiability is concluded to hold.

From Thm. 5 it actually follows that the ORC apparently holds everywhere too, despite the partial analysis above which so far has only shown the ORC to hold at points for which $x_3 \neq 0$. To verify this, one can compute the function ω_4^p defined by

$$\omega_4^p(x) := L_g L_g L_f h_1(x) = 2x_2.$$

Then the matrix of gradient vectors of ω_1^p, ω_3^p and ω_4^p becomes equal to the constant nonsingular matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}.$$

Indeed, the ORC is satisfied everywhere, and in particular at the origin so that Thm. 3 applies. If one now chooses the functions ω_1^p, ω_3^p and ω_4^p to establish again an explicit representation of λ (instead of the functions ω_1^p, ω_2^p and ω_3^p that were used above) then it is found directly that the identity mapping is the unique candidate for the mapping λ in this example.

As the example demonstrates, some care should be taken in computing a candidate function λ. Note how the interplay between the various results of this paper can sometimes avoid certain computations, such as the verification of the conditions of Thm. 1 on λ, or the computation of the function ω_4^p (for which there are no a priori guidelines which sequence of Lie derivatives of which output components to consider).

7. CONCLUSIONS

In this paper several new theorems are presented concerning the class of homogeneous systems. The first result shows that the state isomorphism for a pair of indistinguishable systems is, under certain conditions, homogeneous of degree one. The second result shows that, under certain conditions and for homogeneous polynomial systems, this state isomorphism is linear in case the ORC holds at the origin. They generalize preliminary results reported in (Hanzon and Peeters, 2001) to the multivariable case, and with homogeneous rather than linear output functions.

The third result shows that, again for homogeneous polynomial systems, linearity of the state isomorphism also holds in case the CRC holds at the origin, but this is due to the fact that the ORC then also holds at the origin. Fourth, if the CRC holds at the origin and the vector fields making up the dynamical equation of the system are homogeneous polynomials, then the state isomorphism is affine, for any real analytic output function.

A worked example is presented, demonstrating the construction of the state isomorphism λ and illustrating how the results may be used in system identifiability analysis to reduce the complexity of the computations. In the given examples calculations can be done by hand, but for more complex cases the calculations can be done (at least in principle) by constructive algebra methods and exact and symbolic computation.

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Appendix A. HOMOGENEOUS FUNCTIONS

The following definition of a homogeneous function of n real variables is employed in this paper.

Definition 8. Let $f : D \to \mathbb{R}$ be a function on some open non-empty domain $D \subseteq \mathbb{R}^n$. Let $k \in \mathbb{R}$. Then f is called homogeneous of degree k on the domain D if for all $x \in D$ there exists an open neighborhood I_x of 1 in the interval $[0, \infty)$, on which it holds that $\phi x \in D$ for all $\phi \in I_x$ and for which

$$f(\phi x) = \phi^k f(x), \quad \forall \phi \in I_x. \qquad \text{(A.1)}$$

If f is homogeneous and $x \in D$, let $I_x^{\max}$ be the largest subinterval of $[0, \infty)$ containing 1 on which $\phi x \in D$ for all $\phi \in I_x^{\max}$. Then the validity of property (A.1) extends to all $\phi \in I_x^{\max}$ and not just to $\phi \in I_x$.

The set of all *differentiable* homogeneous functions f of degree k on some open domain D can be characterized as the solution set of a first order partial differential equation. This is the content of Euler's Theorem.

Theorem 9. (Euler's Theorem.) Let $f : D \to \mathbb{R}$ be differentiable on some open non-empty domain $D \subseteq \mathbb{R}^n$. Let $k \in \mathbb{R}$. Then f is homogeneous of degree k if and only if

$$\nabla f(x) \cdot x = k f(x), \quad \forall x \in D. \qquad \text{(A.2)}$$

In the literature one sometimes finds that the degree of homogeneity k is restricted to be a nonnegative integer. While the nonnegative integers are the only relevant values in case of real analyticity of f at the origin, it may be useful to allow k to attain other real values too for functions which are defined on a domain outside the origin.

Proposition 10. Let f be homogeneous of degree $k \in \mathbb{R}$ on an open connected domain $D \subseteq \mathbb{R}^n$ containing the origin. Moreover, let f be real analytic on the punctured domain $D - \{0\}$ and at least $[k] + 1$ times continuously differentiable at the origin, where $[k]$ denotes the largest integer smaller than or equal to k. Then f is a homogeneous polynomial on D. In particular, f is real analytic at the origin too, and if f is not identically zero, then k is a nonnegative integer which is equal to the total degree of f.

If f is real analytic on an open connected domain D which includes the origin, then it is homogeneous of degree 0 if and only if it is constant, and it is homogeneous of degree 1 if and only if it is linear.