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Semiparametric Models: Progress and Problems[1]

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Semiparametric models, models which incorporate both parametric (finite-dimensional) and nonparametric (infinite-dimensional) components, have received increasing use and attention in statistics in recent years. This paper reviews developments in this very large and rich class of models which spans the middle ground between parametric and nonparametric models. Attention is devoted to a preliminary classification of such models with comments on recent work, to lower bounds for estimation, to two potentially useful methods for construction of efficient estimates, and to open problems.

1. Introduction

Models for phenomena involving randomness play a key role in statistics. If $\mathbf{P}_{all}$ denotes the collection of all probability distributions on a sample space $\mathbf{X}$ of the observations X, a model $\mathbf{P}$ is a subset of $\mathbf{P}_{all}$: thus we assume in constructing a model $\mathbf{P}$ that X has a distribution P in $\mathbf{P}$, and we write $X \cong P \in \mathbf{P}$. The sample space $\mathbf{X}$ is the set of all possible observations.

A statistician uses the observations X to make inferences about the 'true' probability distribution P, and hence about real-world phenomena in question. A common form of inference is *point estimation.* For example, if X represents the life expectancy or survival time of an individual who has been given a new medical treatment, the statistician may be interested in using a sample of such individuals to estimate $\nu(P) \equiv P(X \geqslant t)$, the probability of survival beyond t time units. The choice of a model $\mathbf{P}$ can have a major effect on inferences about $\nu(P)$: If the model $\mathbf{P}$ is too small, the statistician runs the risk that the model will not contain the 'true' P, and the consequent price is bias in estimation of $\nu(P)$. In this case the model is not sufficiently large to be realistic and may fail to capture the essential features of the phenomena in question. On the other hand, if the model $\mathbf{P}$ is too large, the statistician may find himself in the position of estimating too many parameters from too little data. This tradeoff

1. This is a revised version of a paper presented at the Centenary Session of the International Statistical Institute, Amsterdam 1985, and which has appeared in the proceedings of that conference (*Bull. Int. Stat. Inst. 51 (4),* 23.1.1-23.1.20). It is reproduced here by kind permission of the International Statistical Institute.

between realism and parsimony is an ever-present theme in statistics; for interesting discussions of some aspects of model-building see Chapters 2 and 4 of COX and SNELL [23] and STONE [74].

Parametric models $\mathbf{P}_0 \equiv \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset \mathbb{R}^d$ for some d play a dominant role in classical statistical theory. Such models, with a finite-dimensional parameter space Θ, form the basis of much of classical statistics. A difficulty with such parametric models is that typically a parametric model $\mathbf{P}_0$ is a relatively small subset of $\mathbf{P}_{all}$, and hence the 'true' distribution P of X may not be contained in $\mathbf{P}_0$.

One approach to this difficulty is the completely nonparametric approach: assume only that $P \in \mathbf{P}_{all}$ or a slight restriction of $\mathbf{P}_{all}$ requiring only some smoothness or monotonicity assumptions. While this approach seems to be feasible when the dimensionality of the sample space is small, it fails to take advantage of structure in the phenomena being modeled and begins to run into difficulty when the dimensionality of the sample space (and hence of the parameter space, $\mathbf{P}_{all}$ itself) is large.

A compromise strategy which gains in model realism and the flexibility needed to make use of the larger data sets which are increasingly available is the semiparametric approach: assume that some aspects or components of the model are parametric or finite-dimensional, while other aspects or components are allowed to be nonparametric or infinite-dimensional. Then the resulting *semiparametric model* $\mathbf{P}$ is typically of the form

$$\mathbf{P} = \{P_{\theta,G} : \theta \in \Theta,\ G \in \mathbf{G}\}$$

where $\Theta \subset \mathbb{R}^d$ for some d and $\mathbf{G}$ is some (large) collection of functions. We also write

$$\mathbf{P} = \{P_\theta : \theta = (\theta_1, \theta_2) \text{ with } \theta_1 \in \Theta_1 \subset \mathbb{R}^d,\ \theta_2 \in \Theta_2\},$$

where Θ_2 is a collection of functions.

This semiparametric approach has proved to be very useful in a wide range of problems, and promises to play an increasingly important role in statistics. Our object here is to survey this extremely rich and flexible class of models (Section 2), and to briefly review the developing inference methods with emphasis on lower bounds for estimation and construction of efficient estimates of the parametric component of such models (Section 3 and 4). The survey of models and review of inference methods may be read independently of one another. The final section discusses open problems.

The notion of a semiparametric model is very general, and is already being used, at least implicitly, in situations involving observations which are not independent and identically distributed (iid). For simplicity, however, we restrict attention here to the iid case: throughout this paper $X_1, \ldots, X_n$ are iid $P \in \mathbf{P}$ where $\mathbf{P}$ is a parametric or semiparametric model.

2. Classes of semiparametric models

Little effort has been made to classify or categorize semiparametric models. While such an effort may be premature, it may also help to identify related models and aid in developing methods to apply to new problems. The following scheme should be regarded as provisional and temporary.

The classification of models given here has two fundamental categories: *basic models*, and *derived models*. The basic models consist of exponential family models, group models, and transformation models. The derived models include regression models, convolution models, mixing models, censoring models, and biased sampling models. Although this scheme is both redundant and possibly incomplete, it includes all the semiparametric models with which I am now familiar. The rest of this section elaborates on these categories, and provides examples of the models of the various types with some brief comments on recent work.

2.1. Basic Models

The following basic models serve as building blocks in the construction of semiparametric models.

2.1.1. Exponential family models. (A). These are familiar parametric models with density (with respect to some measure m)

$$p(x,\theta) = c(\theta)\exp(\sum_{i=1}^{k} Q_i(\theta)T_i(x))h(x)$$

for $\theta \in \Theta \subset \mathbb{R}^k$, $x \in \mathbf{X} \subset \mathbb{R}^d$. While these are themselves completely parametric (finitely dimensional) models, they serve as building blocks for many interesting semiparametric models.

2.1.2. Group models. (B).

(1). The classical parametric model of this type is obtained as follows: suppose that $Y \simeq G \equiv P_0$, a fixed distribution on $\mathbf{X}$, and let $\mathbf{V}$ denote a group of (one to one) transformations on $\mathbf{X}$ parametrized by $\theta \in \Theta \subset \mathbb{R}^k$. If $v_\theta \in \mathbf{V}$, let $X \equiv v_\theta(Y) \simeq P_\theta$ for $\theta \in \Theta$.

Examples:

(a) Location. $\mathbf{X} = \mathbb{R}^d$, $v_\theta(x) = x + \theta$ with $\theta \in \mathbb{R}^d$, and $P_\theta = P_0(\cdot - \theta)$.

(b) Elliptic distributions. $\mathbf{X} = \mathbb{R}^d$, $v_\theta(x) = \theta^{-1/2}x$ where θ is positive definite and symmetric; $G \equiv P_0$ is spherically symmetric on $\mathbb{R}^d$. Then $\mathbf{P} = \{P_\theta : \theta \in \Theta\}$ is the P_0-family of elliptic distributions.

(c) Two-sample models. $\mathbf{X} = \mathbf{X}_0 \times \mathbf{X}_0$, $\mathbf{V} = \mathbf{V}_0 \times \mathbf{V}_0$ where $\mathbf{V}_0$ is a group of transformations on $\mathbf{X}_0$, $\theta = (\mu, \nu) \in \Theta_0 \times \Theta_0 \equiv \Theta$, $Y = (W, Z)$ with $W, Z \simeq P_0$ independent, and $X = (v_\mu(W), v_\nu \circ v_\mu(Z))$.

(2). By letting the distribution P_0 in (1) range over some large class of probability distributions $\mathbf{G}$ small enough to still allow identification of θ, or at least some important functions of θ, yields a semiparametric model

$$\mathbf{P}=\{P_{\theta,G}:\theta\in\Theta, G\in\mathbf{G}\}.$$

Examples:
(a) If $\mathbf{X}=\mathbb{R}^1$ in 1(a) above and $\mathbf{G}$ is the family of distributions symmetric about 0, $\mathbf{P}$ is the classical symmetric location family.
(b) If $\mathbf{X}$ and Θ are as in 1(b) above and $\mathbf{G}$ is the family of all spherical symmetric distributions, then $\mathbf{P}$ is the family of all elliptic distributions; see e.g. BICKEL [6].
(c) If $\mathbf{X}$ and Θ are as in 1(c) and $\mathbf{G}$ is arbitrary, then ν is still identifiable; see STEIN [71] or PFANZAGL [64].
(3). Classical nonparametric statistical theory uses transformation groups which are not parametrizable by a Euclidean space; for example, all continuous monotone transformations from $\mathbb{R}$ to $\mathbb{R}$. See LEHMANN [51] page 24 and 25 for 'semiparametric subgroups' of the large group and note that examples 2(a) and 2(b) are of this type. A wealth of other 'semiparametric group' families are undoubtedly possible.

2.1.3 Transformation models. (C). These models typically map $(\theta,P)\rightarrow P_\theta$ where $\theta\in\Theta\subset\mathbb{R}^k$ and $P\in\mathbf{G}$, a collection of probability distributions on $\mathbf{X}$. The key feature is that the map $P_\theta=\psi(\theta,P)$ acts on P, or some function that is one-to-one with P, rather than on X as in the case of a group model.

The classical example of this type of model is that of a family of 'Lehmann alternatives' defined as follows (see LEHMANN [50]): Let $\mathbf{X}=\mathbb{R}^1$, suppose that $Y\cong G$ and let $\{B(\cdot,\theta):\theta\in\Theta\subset\mathbb{R}^k\}$ be a family of monotone transformations from [0,1] to [0,1] with $B(0,\theta)=0$, $B(1,\theta)=1$ for all $\theta\in\Theta$. Then $X\cong P_{\theta,G}$ has df (distribution function) $F_{\theta,G}(x)=B(G(x),\theta)$. Here are some particular cases.

Examples:
(a) $B_a(\mu,\theta)=1-(1-\mu)^\theta$ with $0<\theta<\infty$. This yields the *proportional hazards model*: $\Lambda_F(x)=\theta\Lambda_G(x)$ where Λ_F is the cumulative hazard function corresponding to F; see LEHMANN [50] and COX [22].
(b) $B_b(\mu,\theta)=\dfrac{\theta\mu}{\theta\mu+(1-\mu)}=\dfrac{\theta\mu(1-\mu)^{-1}}{1+\theta\mu(1-\mu)^{-1}}$ with $0<\theta<\infty$. This yields the *proportional odds model*

$$\frac{F(x)}{1-F(x)}=\theta\frac{G(x)}{1-G(x)};$$

see BENNETT [2].
(c) $B_c(\mu,\theta,\nu)=1-[1-\nu\theta\log(1-\mu)]^{-1/\nu}$, $0<\nu<\infty$, $\theta>0$. This yields the *semiparametric Pareto model* suggested by CLAYTON and CUZICK [19]. Note

that $B_c(\mu,\theta,\nu)\to B_a(\mu,\theta)$ as $\nu\to 0$ while Bennett's B_b is related to Clayton and Cuzick's B_c by

$$B_c(1-\exp(-\frac{\mu}{1-\mu}),\theta,1) = B_b(\mu,\theta).$$

These three models can all be written in the form

$$h(X) = -\log(\theta) + \epsilon \tag{1}$$

where $h(x)\equiv\log\Lambda_G(x)=\log[-\log(1-G(x))]$ and ϵ has the distribution:
(a) $F(x)=1-\exp(-e^x)$ (extreme value);
(b) $F(x)=1/(1+e^{-x})$ (logistic);
(c) $F(x)=1-1/(1+\nu x)^{1/\nu}$ (Pareto).

Because of the generality allowed for the transformations h, rank methods and partial likelihoods play an important role in analyzing these models. Note that (1) yields a transformation family linear model if $\theta=\exp(\gamma z)$, and shows that these models can be viewed as special cases of a type of model involving smooth transformations of both X and z considered by Breiman and Friedman [11]; see 2.2.1 below and Doksum [24].

2.2 Derived models
The following classes of models are all derived from the basic models given above.

2.2.1 Regression models. (D). Given a basic model of one of the three types described above, there is a straightforward recipe for constructing related regression models:
1. Start with an exponential family, group or transformation model $\mathbf{P}=\{P_{\theta,G}:\theta\in\Theta, G\in\mathbf{G}\}$ where θ is the finite-dimensional Euclidean component of the model and G is the nonparametric or infinite-dimensional component of the basic model.
2. Suppose that $Z\cong H$ on $\mathbb{R}^d$.
3. Given $Z=z$, replace θ (or a component thereof) in the basic model by a semiparametric regression function $r(\gamma,z)$ taking values in Θ where $\gamma\in\Gamma\subset$ some $\mathbb{R}^k$. Different forms for r ranging from parametric to nonparametric regression models, with many interesting intermediate semiparametric forms, are possible. For example:
   (a) Linear model: $r(\gamma,z)=\gamma z$;
   (a′) Exponential linear model: $r(\gamma,z)=\exp(\gamma z)$;
   (b) Nonlinear: $r(\gamma,z)=r_0(\gamma,z)$ for a fixed known nonlinear function r_0;
   (c) Nonparametric: $r(\gamma,z)=r(z)$, with r smooth;
   (d) Semiparametric: $r(\gamma,z)=\gamma z_1+r(z_2)$, where $z=(z_1,z_2)$, and r is smooth;

(e) Projection pursuit: $r(\gamma,z)=r(\gamma z)$ where $|\gamma| = 1$ and $r:\mathbb{R}^1 \rightarrow \mathbb{R}^1$ is smooth;
(f) Signal-noise: $r(\gamma z)$ where $r:\mathbb{R}^1 \rightarrow \mathbb{R}^1$ is periodic with period 1 so that γ is a frequency parameter.

Combining various types of regression functions illustrated by (a) - (f) with the basic models A, B or C yields a rich collection of regression models, including parametric, semiparametric, and nonparametric models. STONE [74] gives an interesting survey and further references. A few selected examples with brief comments concerning recent work follow.
Examples:

(a) Combining basic model A with the regression model D(a) yields linear exponential family regression models; see e.g. LEHMANN [51] Chapter 3, pages 196 - 207.
(b) Combining the basic model B1(a) where P_0 is normal with D(a) yields classical parametric normal theory regression models; the extension to B2(a) yields semiparametric linear regression models with arbitrary (symmetric) error distributions.
(c) The basic model B1(a) (with P_0 a fixed distribution on $\mathbb{R}^1$; e.g. normal) combined with the semiparametric regression model D(d) leads to a very interesting class of regression models introduced by ENGLE, GRANGER, RICE and WEISS [26] to study effects of weather on electricity demand, and by WAHBA [79]. This model has one nonparametric component, the smooth regression function r. Generalizations with two nonparametric components by allowing the error distribution to be arbitrary are also of interest. A special case has been studied by SCHICK [70], while STONE [74] discusses a spectrum of related regression models.
(d) Combining B2(a) with D(e) leads to a model related to projection-pursuit regression; see FRIEDMAN and STUETZLE [27], STONE [74], and HUBER [37].
(e) Combining C1(a) with D(a′) yields Cox's (1972) proportional hazards model. Many variants on this model are possible and deserve further exploration. Replacement of the exponential with some other (fixed) non-negative function has been considered by PRENTICE and SELF [67], while C1(c) combined with D(a′) has been explored by CLAYTON and CUZICK [19]. TIBSHIRANI [76] considers a version of Cox's model with the linear function in $\exp(\gamma z)$ replaced by a sum of smooth but otherwise arbitrary functions $\sum_{i=1}^{k} r_i(z_i)$. See 2.2.2 below for related mixture models involving unobserved covariates.
(f) Combination of B1(a) or B2(a) with D(f) yields a semiparametric 'signal plus noise' model which extends classical parametric signal plus noise models. For the latter, see IBRAGIMOV and HAS'MINSKII [38]. MCDONALD [56] has some interesting preliminary work on semiparametric extensions. These models are of interest in astrophysical applications; see e.g. LAFLER and KINMAN [44] or STELLINGWERF [72].

2.2.2 Mixture models. (E). Mixture models can usually be viewed as the result of unobserved heterogeneity as follows: suppose that $X=(Y,Z)$ has a distribution of the form

$$P_{\theta,G,H}(Y\in A,Z\in B)=\int_B P_{\theta,G}(Y\in A\mid Z=z)dH(z).$$

Then if we can only observe Y, the observations have the *mixture distribution*

$$P_{\theta,G,H}(Y\in A)=\int P_{\theta,G}(Y\in A\mid Z=z)dH(z).$$

Examples:

(a) Paired exponentials. Suppose that $(Y\equiv(Y_1,Y_2)\mid Z=z)\cong$ (exponential (z), exponential(θz)):

$$f(y\mid z)=\theta z^2\exp(-(zy_1+\theta zy_2))1_{[0,\infty)}(y_1)1_{[0,\infty)}(y_2)$$

and suppose $Z\cong H$ on $\mathbb{R}^+$. Then

$$f(y)\equiv f_{\theta,H}(y)=\int_0^\infty \theta z^2\exp(-z(y_1+\theta y_2))dH(z);$$

see e.g. LINDSAY [53]. Here θ is a parametric component and H a nonparametric component of the model, and the mixed distribution is parametric while the mixing distribution is nonparametric. Generalizations of this model, including regression type models, have been studied and advocated for use in modeling micro-economic data by HECKMAN and SINGER [35].

(b) Dependent proportional hazards or fraility models. Suppose that $(Y\equiv(Y_1,Y_2)\mid Z=z)$ has joint survival function

$$P_G(Y_1\geqslant y_1,Y_2\geqslant y_2\mid Z=z)=[1-G_1(y_1)]^z[1-G_2(y_2)]^z$$

with $G=(G_1,G_2)$ and suppose that $Z\cong$Gamma(ν,λ). Then with $\theta=(\nu,\lambda)$,

$$P_{\theta,G}(Y_1\geqslant y_1,Y_2\geqslant y_2)=\frac{\lambda^\nu}{[\lambda+\Lambda_1(y_1)+\Lambda_2(y_2)]^\nu}$$

where $\Lambda_i\equiv-\log(1-G_i)$, $i=1,2$. In this case the mixed distribution is nonparametric while the mixing distribution is a parametric family. This model, which serves as an alternative to (a), has been studied by CLAYTON [16] and OAKES [63], and has been generalized by GILL [28]. Related regression models are discussed by RIDDER and VERBAKEL [68] and ELBERS and RIDDER [25].

(c) Errors in variables models. Suppose that $X=(Y,Z)$ with

$$Y_1=Z+\epsilon_1$$

$$Y_2=\alpha+\beta Z+\epsilon_2$$

where $Z \cong H$ (non-Gaussian) and $\epsilon \equiv (\epsilon_1, \epsilon_2) \cong N(0, \Sigma)$. The resulting mixture model is an *errors in variables regression* model. Consistent maximum likelihood estimates were obtained by KIEFER and WOLFOWITZ [42], but lower bounds for estimation of (α, β) together with asymptotically efficient estimates attaining the bounds were first obtained by BICKEL and RITOV [9].

(d) If $(Y \mid Z = z) \cong$ exponential (z) and $Z \cong H$, then

$$P_H(Y \geqslant y) = \int_0^\infty \exp(-yz) dH(z).$$

Estimation of H via nonparametric maximum likelihood methods in this and more general situations has been considered by LAIRD [45] and JEWELL [39]. While the estimates are known to be consistent, little is known about the efficiency of the estimates or their rate of convergence.

Other results concerning mixing models and efficient estimation have also been obtained by LAMBERT and TIERNEY [46], [47], and by HAS'MINSKII and IBRAGIMOV [34].

2.2.3 Censoring models. (F). These models are derived from other models of one of the above types as follows: Suppose that $X \cong P_{\theta,G} \in \mathbf{P}$, and suppose that T is a many-to-one function on the sample space $\mathbf{X}$ of X. Then we can observe only $X^* \equiv T(X) \cong P^*_{\theta,G}$.

Examples:

(a) Mixing. The mixing models of E are censoring models with $X^* \equiv T(Y, Z) = Y$.

(b) Random right censorship. In this type of censoring, which has received much use in survival analysis, $X^* \equiv (X_1^*, X_2^*) \equiv T(X_1, X_2) \equiv (X_1 \wedge X_2, 1_{[X_1 \leqslant X_2]})$. Random right censoring meshes extremely well with Cox's proportional hazards regression model as discussed in D(e). On the other hand, however, this type of censoring can make estimation quite difficult. For example, estimation for the linear regression model D(b) with arbitrary right censoring of the dependent variable has been considered by MILLER [61] and by BUCKLEY and JAMES [13]; see HALPERN and MILLER [60]. RITOV [69] has, in spite of the difficulties, computed information lower bounds and produced asymptotically efficient estimators achieving the bounds. TIBSHIRANI [75] considered a version of this censored regression model with the linear (parametric) regression function replaced by a smooth regression function.

(c) Convolution. Here $X^* \equiv T(X_1, X_2) \equiv X_1 + X_2$ where X_1 and X_2 are independent. The traffic model of BRANSTON [10] is a model which results from this convolution type of censoring combined with a simple mixture model.

2.2.4 Biased sampling models. (G). Suppose that $X \cong P_{\theta,G} \in \mathbf{P}$, a semiparametric model. Then suppose that $K_i(x)$, $i=1,\ldots,s$ is a collection of known non-negative *biasing kernels* and that λ_i, $i=1,\ldots,s$ is a probability distribution on $\{1,\ldots,s\}$. Then the *biased sampling distribution* corresponding to $P_{\theta,G}$, $\underline{K}=(K_1,\ldots,K_s)$, and $\underline{\lambda}=(\lambda_1,\ldots,\lambda_s)$ is

$$P_{\theta,G,\lambda}(X \in A, I=i) = \frac{\int_A K_i(x) P_{\theta,G}(dx)}{\int_{\mathbf{X}} K_i(x) P_{\theta,G}(dx)} \lambda_i \tag{2}$$

for $i=1,\ldots,s$. Here are some examples of this type of model.

Examples:

(a) Vardi's selection bias model. Suppose that $P_{\theta,G}=G$ and $K_1,\ldots,K_s$ are biasing functions with $\int K_i dG < \infty$ for $i=1,\ldots,s$, and $\lambda_i \geqslant 0$ satisfy $\sum_{i=1}^{s} \lambda_i = 1$. Then

$$P_{G,\lambda}(X \in A, I=i) = \frac{\int_A K_i dG}{\int_{\mathbf{X}} K_i dG} \lambda_i, \quad i=1,\ldots,s.$$

VARDI [78] gives a condition which guarantees existence of the nonparametric maximum likelihood estimate of G. The particular case with $\mathbf{X}=\mathbb{R}^1$, $K_1(x)=1, K_2(x)=x$, which involves the length-biased distribution $\int_0^x y dG(y)/\mu$ corresponding to G was studied by VARDI [77], and the further special case with $\lambda_1=0=1-\lambda_2$ was considered earlier by COX [21]. Consistency, asymptotic normality, and efficiency of Vardi's nonparametric maximum likelihood estimator are addressed in a forthcoming paper by GILL and WELLNER [29].

(b) Choice-based sampling models. Suppose that $X \equiv (\underline{Y}, Z)$, where $Z \cong H$ is a vector of covariates, and $(\underline{Y} \mid Z=z) \cong \text{Multinomial}_k(1, \underline{p}(\theta,z))$ (where k denotes the number of cells and the number of trials is 1); we will write $[Y=y]$ for the event that outcome y occurs, $y=1,\ldots,k$. A frequently used model for the p's is the multinomial - logit model with

$$P_\theta(Y=y \mid Z=z) = p_y(\theta,z) = \frac{\exp(\theta_y z)}{\sum_{y'=1}^{k} \exp(\theta_{y'} z)},$$

but in any case this part of the model is parametric; the nonparametric part of the model is G. To get a 'choice-based sampling model', let $K_i(x) \equiv K_i(y,z) = 1_{D_i}(y)$, $i=1,\ldots,s$ where $D_1,\ldots,D_s$ are known subsets of $\{1,\ldots,k\}$. Then the biased sampling model (2) becomes

$$P_{\theta,G}(Y=y, Z \in B, I=i) = \frac{\int_B 1_{D_i}(y) P_\theta(Y=y \mid Z=z) dG(z)}{\int \sum_{y=1}^{k} 1_{D_i}(y) P_\theta(Y=y \mid Z=z) dG(z)} \lambda_{i.}$$

This type of model has received considerable use in econometrics; see COSSLETT [20] for some history and further references. Estimation for this model was considered by MANSKI and LERMAN (*Econometrika 45* (1977), 1977-1988). The efficiency of their estimators of θ and generalizations were treated by COSSLETT [20]. In general the 'choice functions' or biasing kernels may depend on both y and z; see MANSKI and MCFADDEN (*Structural Analysis of Discrete Data* (1981), MIT Press).

(c) Truncated regression models. Suppose that $X=(Y,Z)$ with $Y=\theta Z+\epsilon$ where $\epsilon \cong G$ with density g and $Z \cong H$ are independent. Thus the basic semiparametric model is a linear regression model with unknown error distribution G. If $s=1$ and $K(x)=K(y,z)=1_{(-\infty,y_0]}(y)$ where y_0 is a fixed constant, then

$$P_{\theta,G}(Y \in A, Z \in B)=\frac{\int_B \int_{(-\infty,y_0]\cap A} g(y-\theta z)dydH(z)}{\int \int_{(-\infty,y_0]} g(y-\theta z)dydH(z)}.$$

This truncated regression model has been investigated by BHATTACHARYA, CHERNOFF, and YANG [5]. Motivated by a controversy in astronomy concerning Hubble's law, they constructed $\sqrt{n}$-consistent estimators of the regression parameters θ. Further results for this model have been obtained by JEWELL [41], who also gives additional examples. JEWELL [40] has also considered estimation for generalizations of this model with $s \geqslant 2$ corresponding to stratified sampling on the dependent variable Y.

3. BOUNDS FOR ESTIMATION

Lower bounds for the variances of estimators play an important role in statistical theory, setting a baseline or standard against which estimators can be compared. In their classical form such bounds assert that *any* unbiased estimator θ_n of θ has variance no smaller than $(nI(\theta))^{-1} \equiv b(\theta)/n$:

$$\mathrm{Var}_\theta[\hat{\theta}_n] \geqslant \frac{b(\theta)}{n}$$

In other words $b(\theta)/n$ is the smallest variance we can hope for in an unbiased estimator $\hat{\theta}_n$ of θ. If $\hat{\theta}_n^b$ is an estimator which asymptotically *achieves* the bound (in the sense that $\sqrt{n}(\hat{\theta}_n^b-\theta) \rightarrow_d N(0,b(\theta))$), then we say that $\hat{\theta}_n^b$ is *asymptotically efficient*. If the statistician uses an estimator $\hat{\theta}_n^a$ which is inefficient, then he has not used the data to best advantage and is essentially wasting observations. Hence if $\hat{\theta}_n^a$ is another estimator with $\sqrt{n}(\hat{\theta}_n^a-\theta) \rightarrow_d N(0,a(\theta))$ where $a(\theta) \geqslant b(\theta)$ necessarily, then the limiting ratio of sample sizes which yields equal standard deviations (and hence also equal variances) of $\hat{\theta}_n^b$ and $\hat{\theta}_n^a$ is called the *asymptotic relative efficiency* $e_{a,b}$ of $\hat{\theta}_n^a$ with respect to $\hat{\theta}_n^b$; evidently $e_{a,b}=b(\theta)/a(\theta) \leqslant 1$. If the estimator $\hat{\theta}_n^a$ has asymptotic relative efficiency 1/2 relative to an (efficient) estimator $\hat{\theta}_n^b$ and the estimator $\hat{\theta}_n^b$ requires $n_b=100$

observations to yield a given variance, then $n_a = 200$ observations will be needed to achieve the same variance using the inefficient estimator $\hat{\theta}_n^a$; half the data are 'wasted' by the use of $\hat{\theta}_n^a$. Thus in the search for 'good' estimators and other inference procedures, statisticians are interested in answers to the questions: A. How well can we do? What are the lower bounds for estimation in the model at hand? B. How can we construct efficient estimates, i.e. estimates which achieve the bounds?

Our aim in this section is to briefly survey classical (Cramér - Rao) and modern (Hâjek - Le Cam) bounds for estimation in 'regular' parametric models. The Hâjek - Le Cam approach has led to the development of lower bounds for estimation in nonparametric and semiparametric models. Bounds of this type have been established by BERAN [3], KOSHEVNIK and LEVIT [43], LEVIT [52], MILLAR [57], [58], [59], PFANZAGL [64], and BEGUN et al. [1]. We give a brief introduction to these bounds for semiparametric models at the end of this section. A thorough treatment will be given in the forthcoming monograph by BICKEL, KLAASSEN, RITOV, and WELLNER [7].

3.1. Cramér - Rao lower bounds

First consider the case of a 'regular' parametric model: suppose that $X_1, \ldots, X_n$ are iid $P_\theta \in \mathbf{P} \equiv \{P_\theta : \theta \in \Theta\}$ where $\Theta \subset \mathbb{R}^d$ is open, that $\mathbf{P}$ is dominated by a (sigma-finite) measure μ on $\mathbf{X}$, and let $p(\cdot, \theta) \equiv \frac{dP_\theta}{d\mu}$ for $\theta \in \Theta$. Then the classical *log-likelihood* of an observation X is

$$\mathbf{l}(\theta, X) \equiv \log p(X, \theta),$$

the *scores vector* $\dot{\mathbf{l}}$ is

$$\dot{\mathbf{l}}(\theta, X) \equiv \nabla \mathbf{l}(\theta, X) = \frac{1}{p(X,\theta)}\left(\frac{\partial}{\partial \theta_1} p(X,\theta), \ldots, \frac{\partial}{\partial \theta_d} p(X,\theta)\right)^{\mathsf{T}},$$

and the *Fisher information matrix* for θ is

$$I(\theta) = E_\theta[\dot{\mathbf{l}}(\theta, X)\dot{\mathbf{l}}(\theta, X)^{\mathsf{T}}].$$

Assume that $I(\theta)$ is positive definite so that $I(\theta)^{-1}$ exists.

One form of the classical Crámer-Rao inequality for unbiased estimates $a^{\mathsf{T}}\hat{\theta}_n$ of $a^{\mathsf{T}}\theta$, where a is a fixed vector in $\mathbb{R}^d$, is:

$$n \mathrm{Var}_\theta[a^{\mathsf{T}}\hat{\theta}_n] \geqslant a^{\mathsf{T}} I(\theta)^{-1} a = \sup_{b \in \mathbb{R}^d} \frac{(a^{\mathsf{T}} b)^2}{b^{\mathsf{T}} I(\theta) b}. \tag{1}$$

If we focus on estimation of the first component $\theta_1 \in \mathbb{R}^1$ of θ, it follows immediately from (1), the definition of $I(\theta)$, and standard L_2-projection or regression theory that

$$n\mathrm{Var}_\theta[\hat{\theta}_1] \geqslant \sup_{b\in\mathbb{R}^d} \frac{b_1^2}{b^\mathsf{T} I(\theta) b} \equiv I^{11}(\theta) \tag{2}$$

$$= \frac{1}{\inf\limits_{c\in\mathbb{R}^d, c_1=1} E_\theta[\dot{\mathbf{l}}_1 - c_2\dot{\mathbf{l}}_2 - \ldots - c_d\dot{\mathbf{l}}_d]^2}$$

$$= \frac{1}{I_{11}(\theta) - I_{12}(\theta)I_{22}^{-1}(\theta)I_{21}(\theta)} \equiv \frac{1}{I_{11}^*(\theta)}$$

where

$$I(\theta) \equiv \begin{bmatrix} I_{11}(\theta) & I_{12}(\theta) \\ I_{21}(\theta) & I_{22}(\theta) \end{bmatrix}, \quad I(\theta)^{-1} = \begin{bmatrix} I^{11}(\theta) & I^{12}(\theta) \\ I^{21}(\theta) & I^{22}(\theta) \end{bmatrix}$$

denote the partitions of $I(\theta)$ and $I(\theta)^{-1}$ corresponding to the partition of $\theta = (\theta_1, \underline{\theta}_2^\mathsf{T})^\mathsf{T}$ with $\underline{\theta}_2 = (\theta_2, \ldots, \theta_d)^\mathsf{T}$. Thus when θ_1 is the parameter of interest and $\underline{\theta}_2 = (\theta_2, \ldots, \theta_d)^\mathsf{T}$ are nuisance parameters, the *effective information* $I_{11}^*(\theta)$ for θ_1 is

$$I_{11}^*(\theta) = I_{11} - I_{12}I_{22}^{-1}I_{21} = E_\theta(\dot{\mathbf{l}}_1^{*2}), \tag{3}$$

where the *efficient score function* $\dot{\mathbf{l}}^*$ for θ_1 is

$$\dot{\mathbf{l}}_1^* \equiv \dot{\mathbf{l}}_1 - I_{12}I_{22}^{-1}\dot{\mathbf{l}}_2 = \dot{\mathbf{l}}_1 - \Pi(\dot{\mathbf{l}}_1 \mid [\underline{\dot{\mathbf{l}}}_2]) \tag{4}$$

and the *efficient influence curve* $\tilde{\mathbf{l}}_1$ for estimation of θ_1 is

$$\tilde{\mathbf{l}}_1 = I_{11}^*(\theta)^{-1}\dot{\mathbf{l}}_1^*, \tag{5}$$

so that

$$E_\theta(\tilde{\mathbf{l}}_1^2) = I_{11}^*(\theta)^{-1} = I^{11}(\theta).$$

It is easily seen that the effective information I_{11}^* for θ_1 is just the squared length of the component $\dot{\mathbf{l}}_1^*$ of $\dot{\mathbf{l}}_1$ which is orthogonal to $\dot{\mathbf{l}}_2, \ldots, \dot{\mathbf{l}}_d$ in $L_2(P_\theta)$: in other words, the efficient score function is obtained by subtracting from $\dot{\mathbf{l}}_1$ its projection $\Pi(\dot{\mathbf{l}}_1 \mid [\underline{\dot{\mathbf{l}}}_2]) = I_{12}I_{22}^{-1}\underline{\dot{\mathbf{l}}}_2$ on the space $[\underline{\dot{\mathbf{l}}}_2]$ spanned by $\dot{\mathbf{l}}_2, \ldots, \dot{\mathbf{l}}_d$ in $L_2(P_\theta)$.

If the nuisance parameters $\underline{\theta}_2 = (\theta_2, \ldots, \theta_d)^\mathsf{T}$ are known, the bound (2) may be replaced by

$$n\mathrm{Var}_\theta[\hat{\theta}_1] \geqslant \frac{1}{I_{11}(\theta)}, \tag{6}$$

and, of course,

$$I_{11}(\theta) \geqslant I_{11}^*(\theta) = I_{11} - I_{12}I_{22}^{-1}I_{21}$$

where equality holds if and only if

$$I_{12} = I_{21}^{\mathsf{T}} = 0 \text{ or iff } \dot{\mathbf{l}}_1 \perp \dot{\mathbf{l}}_2, \ldots, \dot{\mathbf{l}}_d \text{ in } L_2(P_\theta). \tag{7}$$

Thus lack of knowledge of $\underline{\theta}_2 \equiv (\theta_2, \ldots, \theta_d)^{\mathsf{T}}$ decreases the information for θ_1 unless (7) holds; in this case the lower bounds (2) and (6) agree, suggesting that θ_1 can be estimated as well when θ_2 is unknown as when θ_2 is known. This possibility was recognized by STEIN [71] in a paper which initiated the theory of *adaptive estimation*.

3.2. Hájek - Le Cam lower bounds

Two different but closely related asymptotic formulations of the classical Cramér - Rao lower bounds have proved useful: One is the convolution-type representation theorem of HÁJEK [32] and LE CAM [48] which has been further developed and applied by BERAN [3], [4] and MILLAR [59]. The other is the local asymptotic minimax approach; see HÁJEK [33] for a nice exposition and history, MILLAR [58], and LE CAM [49] for additional remarks.

Both types of lower bounds are formulated in terms of *locally asymptotically normal families*: Suppose that $\underline{X} = (X_1, \ldots, X_n) \cong P_{n,\theta}$ has density $p_n(\cdot,\theta)$, $\theta \in \Theta \subset \mathbb{R}^d$, and set

$$\mathbf{l}_n(\theta) = \log p_n(\underline{X},\theta).$$

If $\theta_n \equiv \theta + hn^{-1/2}$, so that

$$\mathbf{l}_n(\theta_n) - \mathbf{l}_n(\theta) = \log[p_n(\underline{X},\theta_n) / p_n(\underline{X},\theta)],$$

then $\mathbf{P} \equiv \{P_{n,\theta} : \theta \in \Theta\}$ is *locally asymptotically normal* (LAN) at θ if there is a vector of $L_2(P_\theta)$ functions $\underline{\dot{\mathbf{l}}}_n(\theta)$ and a nonsingular matrix $I(\theta)$ such that, with

$$\mathbf{l}_n(\theta_n) - \mathbf{l}_n(\theta) = \underline{\dot{\mathbf{l}}}_n(\theta)^{\mathsf{T}} h - \frac{1}{2} h^{\mathsf{T}} I(\theta) h + \mathbb{R}_n(\theta,h), \tag{8}$$

it follows that, in $P_{n,\theta}$-probability,

(i) $R_n(\theta,h) \to_p 0$ uniformly on bounded h-sets, and

(ii) $\underline{\dot{\mathbf{l}}}_n(\theta) \to_d N(0, I(\theta))$.

Thus $\mathbf{l}_n(\theta_n) - \mathbf{l}_n(\theta) \to_d N(-\frac{1}{2}\sigma^2, \sigma^2)$ with $\sigma^2 = h^{\mathsf{T}} I(\theta) h$. In 'regular families' $\mathbf{P}$ (with iid observations) $\underline{\dot{\mathbf{l}}}_n(\theta) = n^{-1/2} \Sigma_{i=1}^n \underline{\dot{\mathbf{l}}}(\theta, X_i)$ where $\underline{\dot{\mathbf{l}}}$ is the scores vector (for $n = 1$) and $I(\theta)$ is the information matrix.

Because of our interest here in the parametric component θ of a semi-parametric model $\mathbf{P} = \{P_{\theta,G}\}$, we formulate versions of the convolution and asymptotic minimax bounds for the first component θ_1 of θ.

A sequence of estimators T_{1n} of θ_1 is *regular* at θ if, under P_{θ_n}

$$\sqrt{n}(T_{1n} - \theta_{1n}) \to_d T_1$$

for every $\theta_n = \theta + n^{-1/2} h$ where the distribution $\mathbf{L}(T_1)$ of T_1 does not depend on h.

THEOREM 1 (HÁJEK, 1970). *Suppose that* $\mathbf{P}$ *is LAN at* θ *and that* T_{1n} *is a regular estimator with limit distribution* $\mathbf{L}(T_1)$. *Then*

$$T_1 \cong Z_1 + W_1 \tag{9}$$

where $Z_1 \cong N(0,1/I_{11}^*(\theta))$, $I_{11}^*(\theta)$ *is as in (3), and* W_1 *is independent of* Z_1.

Thus any regular estimator T_{1n} of θ_1 must have a limit distribution which is at least as dispersed as $N(0,1/I_{11}^*(\theta))$, and it makes sense to call a regular estimator T_{1n} asymptotically efficient if it converges in distribution to Z_1; i.e. if $W_1=0$ in (9).
Now suppose that $w:\mathbb{R}^1 \to \mathbb{R}^+$ satisfies:
(i) $w(x)=w(-x)$ for all $x \in \mathbb{R}^1$;
(ii) $w(0)=0$, $w(x)$ increases in $x \geqslant 0$;
(iii) $Ew(\sigma Z) < \infty$ for all $\sigma > 0$ where $Z \cong N(0,1)$.

THEOREM 2 (HÁJEK, 1972). *Suppose that* $\mathbf{P}$ *is LAN at* θ *and that* w *satisfies (i) - (iii). Then, for any estimator* T_{1n} *of* θ_1,

$$\lim_{M\to\infty} \liminf_{n\to\infty} \sup_{\sqrt{n}|\theta_n - \theta| \leqslant M} E_{\theta_n} w(\sqrt{n}(T_{1n} - \theta_{1n})) \geqslant Ew(Z_1) \tag{10}$$

where $Z_1 \cong N(0,1/I_{11}^*(\theta))$ *as in theorem 1.*

If the uniformity in h in (i) of the definition of a LAN family is relaxed to just pointwise convergence, then theorems 1 and 2 continue to hold, but the bounds may not be attainable. Furthermore, if attention is restricted to regular estimates, then (10) holds without the supremum on the lefthand side.

3.3. Bounds for semiparametric models

The Hájek-Le Cam convolution and asymptotic minimax bounds stated above for a parametric model $\mathbf{P}_0$ continue to hold in a wide range of regular nonparametric and semiparametric models. All of the extensions make use, in some form, of the *tangent space* $\dot{\mathbf{P}}$ (at (θ,G)) for the model $\mathbf{P}$. For a parametric model $\mathbf{P}_0$ the tangent space $\dot{\mathbf{P}}_0$ (at $\theta \in \Theta$) is just the linear subspace $[\dot{\mathbf{l}}_1, \ldots, \dot{\mathbf{l}}_d]$ of $L_2(P_\theta)$ spanned by $\dot{\mathbf{l}}_1, \ldots, \dot{\mathbf{l}}_d$. For a semiparametric model $\mathbf{P}=\{P_{\theta,G}:\theta \in \Theta \subset \mathbb{R}^d, G \in \mathbf{G}\}$, the tangent space $\dot{\mathbf{P}} \subset L_2(P_{\theta,G})$ is simply the set of all possible score functions of one-dimensional regular parametric submodels (at (θ,G)).

For $\theta_0 \in \Theta, G_0 \in \mathbf{G}$, let $\mathbf{P}_\theta$ and $\mathbf{P}_G$ denote the submodels of $\mathbf{P}$ with $G=G_0$ and $\theta=\theta_0$ respectively:

$$\mathbf{P}_\theta \equiv \{P_{\theta,G_0} \in \mathbf{P}:\theta \in \Theta\}, \quad \mathbf{P}_G \equiv \{P_{\theta_0,G} \in \mathbf{P}:G \in \mathbf{G}\}.$$

If $\dot{\mathbf{P}}_\theta$ and $\dot{\mathbf{P}}_G$ denote the corresponding tangent spaces, then $\dot{\mathbf{P}}_\theta \oplus \dot{\mathbf{P}}_G \subset \dot{\mathbf{P}}$ and typically equality holds. Here $\dot{\mathbf{P}}_G$ plays the role that $[\dot{\mathbf{l}}_2, \ldots, \dot{\mathbf{l}}_d]$ played for the parametric model $\mathbf{P}_0$, and the *efficient score function* for θ extending (4) is:

$$\dot{\mathbf{l}}_\theta^* = \dot{\mathbf{l}}_\theta - \Pi(\dot{\mathbf{l}}_\theta \,|\, \dot{\mathbf{P}}_G) \qquad (11)$$

so that $\dot{\mathbf{l}}_\theta^* \perp \dot{\mathbf{P}}_G$ in $L_2(P_{\theta,G})$, and the *effective information* for θ in the model $\mathbf{P}$ is

$$I^*(\theta) = E_{\theta,G}(\dot{\mathbf{l}}_\theta^* \dot{\mathbf{l}}_\theta^{*\mathsf{T}}). \qquad (12)$$

In the special case when $\dot{\mathbf{l}}_\theta^* = \dot{\mathbf{l}}_\theta \perp \dot{\mathbf{P}}_G$, then $I^*(\theta) = I(\theta) \equiv E_{\theta,G}(\dot{\mathbf{l}}_\theta \dot{\mathbf{l}}_\theta^{\mathsf{T}})$ and *adaptation to* G is possible; this is the situation emphasized by STEIN [71] and BICKEL [6].

Now versions of theorems 1 and 2 for the parametric component θ of the semiparametric model $\mathbf{P}$ continue to hold with θ_1 replaced by θ and $1/I_{11}^*(\theta)$ replaced by $I^*(\theta)^{-1}$ where $I^*(\theta)$ is given in (12); see KOSHEVNIK and LEVIT [43], LEVIT [52], BEGUN et al. [1], and PFANZAGL [64], [65]. A complete treatment will be given in BICKEL, KLAASSEN, RITOV, and WELLNER [7].

4. Construction of asymptotically efficient estimates: two approaches

Suppose that $\mathbf{P} = \{P_{\theta,G} : (\theta, G) \in \Theta \times \mathbf{G}\} \equiv \{P_\theta : \theta = (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2\}$ with $\Theta_2 = G$ is a 'regular' semiparametric model. A first stage in analyzing the model is to calculate scores for θ and information lower bounds as outlined in Section 2 above if possible. A second step is to construct estimators $(\bar{\theta}_n, \bar{G}_n)$ which are $\sqrt{n}$-consistent. A third stage is to find estimators $(\hat{\theta}_n, \hat{G}_n)$ of (θ, G) which are efficient in the sense that they *achieve* the information lower bounds (perhaps in the weakened sense of convergence in distribution for fixed (θ, G) rather than locally uniformly as required by the definition of regular estimates given in Section 3).

Two classical methods of constructing asymptotically efficient estimators $\hat{\theta}_n$ in regular parametric models are the methods of maximum likelihood estimation and Bayes estimation; see LEHMANN [51] and IBRAGIMOV and HAS'MINSKII [38], though, as LEHMANN makes clear, the emphasis in likelihood estimation, even in parametric models, should be on the scores and score equations rather than on maximizing likelihoods per se since the scores often lead to efficient estimates even when likelihoods themselves are unbounded.

Our aim here is to outline two useful approaches to the construction of asymptotically efficient estimates of the parametric part θ of a semiparametric model $\mathbf{P}$.

4.1. Method 1: Efficient score equation

Suppose that it is possible to calculate the *efficient score function* $\dot{\mathbf{l}}_1^*$ for θ_1,

$$\dot{\mathbf{l}}_1^* = \dot{\mathbf{l}}_1 - I_{12} I_{22}^{-1} \dot{\mathbf{l}}_2 = \dot{\mathbf{l}}_1 - \Pi(\dot{\mathbf{l}}_1 \,|\, \dot{\mathbf{P}}_{\theta_2})$$

and the *effective information*

$$I_{11}^*(\theta) = E_\theta(\dot{\mathbf{l}}_1^{*2}).$$

Furthermore, suppose that $\bar{\theta}_n$ is a $\sqrt{n}$-consistent estimator of θ, $\sqrt{n}(\bar{\theta}_n - \theta) = O_P(1)$. Then define $\hat{\theta}_{1n}$ to be either a solution of the *efficient score equation*

$$\sum_{i=1}^{n} \dot{\mathbf{l}}_1^*(\hat{\theta}_{1n}, \bar{\theta}_{2n}, X_i) = 0,$$

or a one-step approximation thereof:

$$\hat{\theta}_{1n} = \bar{\theta}_{1n} + \frac{\frac{1}{n}\sum_{i=1}^{n} \dot{\mathbf{l}}_1^*(\bar{\theta}_n, X_i)}{I_{11}^*(\bar{\theta}_n)} \tag{1}$$

$$= \bar{\theta}_{1n} + \frac{1}{n}\sum_{i=1}^{n} \tilde{\mathbf{l}}_1(\bar{\theta}_n, X_i)$$

where $\tilde{\mathbf{l}}_1$ is the efficient influence curve for θ_1, see (3.5). Additional smoothing may also be required in forming the sums in (1), but we have omitted it here for simplicity. Once an efficient estimator $\hat{\theta}_{1n}$ of θ_1 is found, method 2 can often be used to construct an efficient estimator of θ_2.

While no general theorem yet exists, the estimator $\hat{\theta}_{1n}$ defined above (or variations thereon involving suitable smoothing and truncation) has been shown to be asymptotically efficient in several important problems, a notable example being the errors in variables models studied by Bickel and Ritov [9]. Roughly speaking, the fact that $\dot{\mathbf{l}}_1^*$ is orthogonal to $\dot{\mathbf{l}}_2, \ldots, \dot{\mathbf{l}}_d$, the scores for θ_2, permits the use of an inefficient estimator $\bar{\theta}_{2n}$ to estimate out the 'nuisance parameter' θ_2. This should be contrasted with solving (or approximating by a one-step solution)

$$\sum_{i=1}^{n} \dot{\mathbf{l}}_1(\theta_1, \bar{\theta}_{2n}) = 0$$

for θ_1, a method which is known to produce inefficient estimates of θ_1 in general; see e.g. Gong and Samaniego [30].

The main drawback of the method is that it requires calculation of the efficient score function $\dot{\mathbf{l}}_1^*$. Thus the method depends heavily on being able to calculate projections onto $[\underline{\dot{\mathbf{l}}_2}] = \dot{\mathbf{P}}_{\theta_2} = \dot{\mathbf{P}}_G$, which often necessitates calculation of the inverse of the information operator $\underline{\dot{\mathbf{l}}_2}^T \underline{\dot{\mathbf{l}}_2} = I_{22}$. When $\dot{\mathbf{l}}_1^* = \dot{\mathbf{l}}_1$ so $\dot{\mathbf{l}}_1$ is orthogonal to $[\dot{\mathbf{l}}_2] = \dot{\mathbf{P}}_{\theta_2}$, then 'adaptation' with respect to $\theta_2 = G$ is possible, and method 1 becomes essentially the method used to construct efficient estimates in this case; see e.g. Stone [73] and Bickel [6].

4.2. Method 2: Efficient estimation of θ_2 for known θ_1

Now suppose that an *efficient* estimate $\tilde{\theta}_{2n}$ of θ_2 is available if θ_1 is *known*. We denote this estimator by $\tilde{\theta}_{2n}(\theta_1)$ because it depends on the 'known' value of θ_1. Substitution of this estimate of θ_2 into the ordinary score for θ_1 (as if θ_2 were known and equal to $\tilde{\theta}_{2n}$ yields the 'condensed' or 'concentrated' score equation

$$\sum_{i=1}^{n} \dot{\mathbf{l}}_1(\theta_1, \tilde{\theta}_{2n}(\theta_1), X_i) = 0$$

which we can solve for $\theta_1 \equiv \hat{\theta}_1$. Or, if $\overline{\theta}_{1n}$ is a $\sqrt{n}$-consistent estimate of θ_1, a one-step approximation thereof:

$$\hat{\theta}_{1n} = \overline{\theta}_{1n} + \frac{\frac{1}{n}\sum_{i=1}^{n} \dot{\mathbf{l}}_1(\overline{\theta}_{1n}, \tilde{\theta}_{2n}(\overline{\theta}_{1n}), X_i)}{\frac{1}{n}\sum_{i=1}^{n} \dot{\mathbf{l}}_1^{2}(\overline{\theta}_{1n}, \tilde{\theta}_{2n}(\overline{\theta}_{1n}))}; \tag{2}$$

as in the case of (1), more smoothing may be needed in forming the sums in (2), we have omitted it here for simplicity. This is a frequently used device in parametric models, but the method is equally useful for semiparametric models. While no general results concerning the estimator (2) seem to be known, this method has been used by RITOV [69] to construct efficient estimates for censored regression models.

5. PROBLEMS

Statisticians have a large, well-stocked tool-box for dealing with classical parametric models, and a growing companion set of tools for handling completely nonparametric models. The choice of tools for dealing with the rich middle ground of semiparametric models is, however, still relatively limited, and the few available tools are not all well suited for the job. Many important problems remain. Here is a partial list:

(a). *Calculation of lower bounds.* If the projection $\Pi(\dot{\mathbf{l}}_\theta \,|\, \dot{\mathbf{P}}_G)$ in Section 3 can be calculated, then so can the efficient score function $\dot{\mathbf{l}}_\theta^*$, the effective information $I_{11}^*(\theta)$, and the efficient influence curve $\tilde{\mathbf{l}}_1$. In many models this projection is simply a conditional expectation, and hence can be calculated easily; but in other models such as the dependent proportional hazards model of 2.E(b) the projection calculation is apparently intractable. More systematic methods, possibly involving iterative, numerical techniques, are needed.

(b). *Construction of efficient estimates.* HUANG [36] has made a preliminary study of method 1 outlined in Section 4, but general results concerning the asymptotic efficiency of methods 1 and 2, or variations thereof involving more smoothing, are still needed. Other methods including minimum Hellinger distance estimates, minimum Kullback-Leibler discrepancy estimators, and maximum-likelihood estimators obtained via EM-algorithms all need further development and sharpening in the context of semiparametric models. Efficient estimates are still unknown for many of the models given in Section 2.

(c). *Identifiability and regularity criteria.* For many semiparametric models, further work on identifiability and conditions for regularity of submodels is still needed before work on estimation can get underway. For examples of such studies, see the papers by HECKMAN and SINGER [35] and ELBERS and RIDDER [25] concerning identifiability issues for the models of 2.E(b) and 2.E(c). Classical regularity investigations of translation and parametric models, which carry over to many group models are given by HÁJEK [31], [33].

(d). *Hypothesis testing.* As yet no adequate theory of hypothesis testing exists for semiparametric models. One type of testing problem concerns testing hypotheses within a nested family of semiparametric models: for example, consider testing $\Lambda_2 = \gamma\Lambda_1$ for some $0<\gamma<\infty$ in the Clayton-Oakes model of example E(b). Or, of interest in survival analysis, test the assumption of a proportional hazards regression model against some general family of alternatives. Another rather different testing problem would involve testing non-nested semiparametric models against one another, e.g. a Cox-type regression model against a more classical linear regression model or perhaps a semiparametric mixed regression model.

(e). *Asymptotics for estimates based on smoothing.* Construction of efficient estimates for many of the models discussed above require smoothing techniques involving density or conditional expectation estimators. While the asymptotics for such smoothing processes are available, they need further development, study, and refinement to ease their systematic application to the construction of efficient estimates in a wide range of semiparametric models.

(f). *Robustness; connections and problems.* Efficient estimation in semiparametric models has many interesting connections with questions of robustness. Just as classical robustness theory has focused on neighborhoods of parametric models (often a one - sample location model), a generalization suggested by BICKEL and LEHMANN [8] concerns neighborhoods of semiparametric models, which they called 'nonparametric models with natural parameters'. For example, are the partial likelihood estimtors for the Cox proportional hazards model robust in some appropriate sense (with respect to the assumption of proportional hazards)? As more experience is gained with efficient estimates for semiparametric models, this more general type of robustness outlined by BICKEL and LEHMANN [8] can begin to be considered. Many challenging problems remain.

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Numerical Time-Stepping in Partial Differential Equations[1]

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1. Introduction

The numerical solution of initial-value or initial boundary-value problems in partial differential equations (PDEs) has been studied for a considerable time already. Thanks to widely available computer facilities many important and interesting PDE problems from the engineering and physical sciences are nowadays solved by numerical methods. An outstanding field of applications is that of fluid dynamics, for example. In fact, computational fluid dynamics is still growing and seems to develop itself as an almost independent and self-supporting branch of science lying between mathematics and physics.

Due to the wide diversity in PDEs, there are many features which play a role in the construction and analysis of numerical methods: hyperbolic or parabolic character, number of space dimensions, nonlinearities, large gradients in the solution and discontinuities (shocks), shape of region, etc. In this note we will present a brief introduction to time-stepping schemes for time-dependent problems. Our aim is a presentation accessible to the non-specialist in numerical methods. We shall discuss the construction of some simple numerical time-stepping procedures and their stability. It should be emphasized that no part of the material presented here is new. In fact, the schemes and their properties we discuss have been known in the literature for a considerable time already. However, we also stress that these schemes, in spite of their simplicity, remain of continuing practical interest.

As a concrete example we shall use the linear *convection-diffusion equation*

1. Report of a lecture presented at the *General CWI-colloquium*

$$u_t+(v\cdot\nabla)u=\epsilon\Delta u, \quad t>0, \ x\in\Omega\subset\mathbb{R}^d, \tag{1.1}$$

where (scalar) $u(x,t)$ is the convected and diffused variable, $v=(v_1, \ldots, v_d)^T$ is the convecting velocity vector (here constant), and $\epsilon>0$ is a diffusivity parameter. We recall that ∇ is the gradient operator and Δ the Laplacian.

2. Explicit and implicit schemes

For the time being we consider a general, linear time-dependent PDE of the form

$$u_t=Lu, \quad t>0, \ x\in\Omega\subset\mathbb{R}^d. \tag{2.1}$$

If we define the space-operator L by $Lu\equiv-(v\cdot\nabla)u+\epsilon\Delta u$, the convection-diffusion equation (1.1) is obtained. We will not specify boundary conditions for (2.1) as we do not discuss their influence here. The development of the time-stepping schemes will be carried through as if we were studying the initial-value problem.

Time-stepping schemes for the numerical integration of the evolution equation (2.1) are *step-by-step methods.* A step-by-step method proceeds from an approximation at $t=t_n$ one step of size τ to an approximation at $t=t_{n+1}$ where $t_{n+1}=t_n+\tau$. The choice of a scheme for (2.1) depends on various problem features as we mentioned in the introduction. A fundamental property of any scheme is that of stability. In this connection it makes sense to classify time-stepping schemes under three headings:

a) explicit schemes;
b) implicit schemes;
c) explicit-implicit schemes;

which we shall use for the purpose of this presentation.

In the remainder of this section we shall discuss a simple example of an explicit and implicit scheme. Subsequently, in section 3 we shall present an interesting example of type c.

2.1. The explicit and implicit Euler scheme

To begin with, we shall discuss the *construction* of the forerunners of all integration schemes, viz., the *explicit* and *implicit Euler rule.* We note that the notions employed are of a much wider applicability. Let u be a solution of (2.1). Suppose that u is sufficiently differentiable and consider the Taylor expansion of $u(x,t_n+\tau)$ about t_n:

$$u(x,t_n+\tau)=u(x,t_n)+\tau u_t(x,t_n)+\frac{1}{2}\tau^2 u_{tt}(x,t_n)+\cdots. \tag{2.2}$$

When stepping from t_n to t_{n+1} any scheme tries to approximate, in some way or another, a truncated part of this series. Let us truncate after two terms:

$$u(x,t_n+\tau)\simeq u(x,t_n) + \tau u_t(x,t_n). \tag{2.3}$$

This is an approximate relation between exact solution values. By replacing this approximate relation by an exact one, but now between approximate solution values, a numerical scheme is obtained. This scheme is then said to be *consistent of order one* (in time) since (2.2) is approximated up to $O(\tau^2)$. The replacement itself determines the actual type of scheme.

Using the differential equation (2.1), (2.3) can be rewritten to

$$u(x,t_n+\tau)\simeq u(x,t_n) + \tau Lu(x,t_n). \tag{2.4}$$

We now define the approximation $U^n(x)$ for $u(x,t_n)$ by the exact relation

$$U^{n+1}(x) = U^n(x) + \tau LU^n(x), \tag{2.5}$$

which is the simplest of all integration schemes, viz., the *explicit Euler rule.* Note, however, that U^n is still space-continuous and L a space differential operator. To get a fully discrete approximation we next replace L by an appropriate *finite difference operator* L_h [1,11,13,14,15] so that (2.5) is replaced by

$$U_j^{n+1} = U_j^n + \tau L_h U_j^n. \tag{2.6}$$

The precise form of L_h is not of interest at the moment. Here we only note that U_j^n is a grid variable and approximates u at the space-time point (x_j,t_n) where x_j is a grid point from the finite difference grid Ω_h covering the space domain Ω. We observe that $L_h U_j^n$ is always a linear combination of grid values defined on a stencil around x_j. If the space dimension d is greater than 1, then j is a multi-index.

At each step with (2.6) an approximation error is made, the so-called *local discretization* or *truncation error,* which is accumulated during the stepping forward in time to the so-called *global discretization error*

$$\epsilon_j^n = u(x_j,t_n) - U_j^n. \tag{2.7}$$

The local error is found by recovering the truncated Taylor series (2.3) or (2.4) from (2.6). For this purpose we write down a perturbed version of (2.6), viz.,

$$\begin{aligned} u(x_j,t_{n+1}) &= u(x_j,t_n) + \tau L_h u(x_j,t_n) + \\ &\quad \tau(L-L_h)u(x_j,t_n) + \tau\rho(x_j,t_n) \\ &= u(x_j,t_n) + \tau u_t(x_j,t_n) + \tau\rho(x_j,t_n). \end{aligned} \tag{2.8}$$

Comparison with (2.2) shows that $\rho(x_j,t_n)$, which is the local error due to Euler's formula, satisfies

$$\rho(x_j,t_n)=\frac{1}{2}\tau u_{tt}(x_j,t_n) + O(\tau^2), \tag{2.9}$$

which once more reveals the first order consistency of Euler's rule. The quantity

$$\alpha(x_j,t_n) = (L-L_h)u(x_j,t_n) \tag{2.10}$$

is called the *space truncation error.* This error has nothing to do with the time-stepping scheme and originates from the replacement of L by L_h. If the grid Ω_h is refined, then $\alpha(x_j,t_n)$ should diminish accordingly. The (total) *local error* of the discretization (2.6) of (2.1) thus is given by

$$\beta(x_j,t_n) = \alpha(x_j,t_n) + \rho(x_j,t_n). \tag{2.11}$$

Finally, if we subtract (2.6) from its perturbed version (2.8), we get by definition of ϵ_j^n (2.7)

$$\epsilon_j^{n+1} = (1+\tau L_h)\epsilon_j^n + \tau\beta(x_j,t_n). \tag{2.12}$$

The accumulation of the local errors $\beta(x_j,t_n)$, $0 \leqslant n \leqslant N-1$, to the global error ϵ_j^n at a fixed time $T=N\tau$, is described by this recursion. The *convergence question,* i.e., the question under which conditions on τ and h the global error ϵ^N at $t=T$ will decrease and how fast, obviously turns out to be a *stability question.* Loosely speaking, recursion (2.12) may be called *stable* if at each step the amplification of ϵ_j^n is not larger than by a factor $1+O(\tau)$. Of course this depends on the metric used on the operator $1+\tau L_h$, and thus on L_h and L. The interested reader is referred to [15] where the convergence question is extensively discussed. The notion of stability, in the sense of VON NEUMANN, will be taken up again in section 2.2.

The time-stepping scheme (2.6) is called the explicit Euler rule. A scheme is called explicit if the approximation at the new step point t_{n+1} is based only on previous approximations. The appellation *implicit* becomes clear if we slightly change (2.6) to the first order implicit Euler rule

$$U_j^{n+1} = U_j^n + \tau L_h U_j^{n+1}. \tag{2.13}$$

Here, U_j^{n+1} also appears in the right-hand side of the approximating equation which essentially requires the inversion of the operator $1-\tau L_h$ at each time-step. In practice, this inversion implies the inversion of an associated, well-structured finite difference or finite element matrix [11, 12], which is carried out either by some form of Gaussian elimination or in an iterative fashion. For this reason one time step with (2.13) is more costly than one step with the explicit scheme (2.6), especially if $d>1$. However, it still may be attractive to use (2.13), viz., if stability restricts the step size value τ in the explicit scheme. In virtually all applications the implicit Euler rule is stable for all $\tau>0$ (unconditional stability [3]). In section 2.2 we shall illustrate this for the convection-diffusion equation (1.1). Finally, despite their low order of consistency, both Euler rules are frequently employed in numerical practice. This is particularly the case if problems are highly complicated as in computational fluid dynamics (see, e.g., [2] and [9] for an application of the implicit and explicit scheme, respectively).

2.2. Von Neumann stability

A central theme in the development and analysis of time-stepping schemes is that of *stability*. Consider again the explicit Euler rule and its error scheme (2.12). Stability there was taken to mean that, in some metric the amplification of ϵ_j^n to ϵ_j^{n+1} is not larger than by a factor $1+O(\tau)$. In practice it is usual to insist on a stability condition which guarantees an *amplification factor not larger than 1,* since this best mimics the behaviour of the true solution of the problem, at least for the convection-diffusion equation (1.1). For constant coefficient problems, such as (1.1), most common is a *Fourier analysis* as proposed by VON NEUMANN (see [15]). Here we illustrate this analysis for equation (1.1).

Unless otherwise stated we let $d=1$, so we consider the one-dimensional problem

$$u_t + vu_x = \epsilon u_{xx}, \quad t>0, \tag{2.14}$$

whereby we suppose that L_h is defined by second order *central finite differences:* $x_j = jh$ and [1, 11, 13, 15]

$$L_h U_j = \frac{\epsilon}{h^2}(U_{j+1} - 2U_j + U_{j-1}) - \frac{v}{2h}(U_{j+1} - U_{j-1}). \tag{2.15}$$

For the sake of brevity it is convenient to introduce the finite difference operators

$$\delta^2 U_j = U_{j+1} - 2U_j + U_{j-1}, \quad HU_j = (U_{j+1} - U_{j-1})/2. \tag{2.16}$$

Then the explicit Euler scheme (2.6) applied to (2.14) reads

$$U_j^{n+1} = (1 + \frac{\tau\epsilon}{h^2}\delta^2 - \frac{\tau v}{h}H)U_j^n. \tag{2.17}$$

Consider the convection-diffusion equation (2.14). Corresponding to the initial data $u(x,0) = \exp(i\omega x), i^2 = -1, \omega \in \mathbb{R}$, the solution of (2.14) satisfies

$$u(x_j, t_n) = e^{-n(\tau\epsilon\omega^2 + i\tau v\omega)} e^{i\omega x_j}. \tag{2.18}$$

We see that the absolute value of the exponential is less than or equal to one for all $\epsilon \geq 0, v \in \mathbb{R}$, and any Fourier mode $\exp(i\omega x)$. The *Fourier (or von Neumann) stability method* applied to (2.17) now consists of examining Fourier modes

$$U_j^n = \xi^n e^{i\omega x_j}, \tag{2.19}$$

and deriving conditions on τ and h such that, in agreement with the behaviour of the exponential in (2.18) the (complex valued) amplification factor ξ satisfies

$$|\xi| \leq 1. \tag{2.20}$$

If (2.20) is true, the time-stepping scheme is called *stable in the (strict) sense of von Neumann.* The adjective strict refers to the fact that $|\xi| \leq 1$ rather than

$|\xi| \leqslant 1+O(\tau)$ which is the original condition [15]. Hereafter we will omit this adjective. Obviously, for any method ξ approximates the exponential in (2.18). Here we do not discuss this further but concentrate on the stability question.

Some remarks are in order. Fourier analysis is based on the hypothesis that the problem and its approximating scheme admit solutions such as (2.18) and (2.19), respectively. Strictly speaking, Fourier analysis applies only to the problem with periodic boundary conditions or to the pure initial-value problem on the infinite x-axis. Also, it is required that the initial function possesses a Fourier series. Further, we take the wave number ω continuous in $\mathbb{R}$ while the given mesh actually allows us to consider only a discrete set. Loosely speaking, the error made here is $O(h^2)$ and therefore we will follow precedent and let ω be continuous [15]. Despite these constraints, 'the von Neumann method is generally the best single technique for analysing the stability of difference schemes. It should always be part of a stability analysis, even if other techniques are also employed' (quotation from [9], p.890).

On substituting the Fourier mode (2.19) into the difference scheme (2.17), the value for $\xi = U_j^{n+1}/U_j^n$ is obtained. As a function of the phase angle $\theta = \omega h$ it is given by

$$\xi = 1-\alpha(1-\cos\theta)-ic\sin\theta, \tag{2.21}$$

where

$$\begin{cases} \alpha = \dfrac{2\epsilon\tau}{h^2} \text{ (is called the diffusion parameter)} \\ c = \dfrac{\nu\tau}{h} \text{ (is called the Courant number)} \end{cases} \tag{2.22}$$

Here we have used the simple properties

$$\delta^2 e^{i\omega x_j} = 2(\cos\theta - 1)e^{i\omega x_j}, \quad He^{i\omega x_j} = i(\sin\theta)e^{i\omega x_j} \tag{2.23}$$

We are now ready to establish conditions on the diffusion parameter α and the Courant number c in order that the explicit Euler - central difference scheme (2.17) is stable in the sense of von Neumann. It can be shown that $|\xi| \leqslant 1$ for all θ if and only if

$$c^2 \leqslant \alpha \leqslant 1. \tag{2.24}$$

Hence this pair of inequalities is necessary and sufficient for stability in the sense of VON NEUMANN (see [9] and the reference therein) The diffusion barrier $\alpha \leqslant 1$ implies that

$$\tau \leqslant \frac{1}{2\epsilon}h^2. \tag{2.25}$$

which is a severe restriction on the time-step τ if ϵ is not small. Such a restriction on τ is found for virtually all explicit methods and unacceptable for numerical practice. Inequality (2.25) is acceptable only if ϵ is small, hence if $|v| >> \epsilon$ (dominating convection term). Unfortunately then the *convection-diffusion barrier* $c^2 \leqslant \alpha$, or

$$\tau \leqslant \frac{2\epsilon}{v^2}, \tag{2.26}$$

spoils the game. In fact, in the absense of diffusion ($\epsilon = 0$) we always have instability illustrating that for the purely hyperbolic problem the explicit Euler - central difference scheme is of no use at all.

Similar negative results are found for the multi-dimensional problem (1.1). For full details we refer to the interesting and previously mentioned paper [9] by Hindmarsh, Gresho and Griffiths. They present a comprehensive study regarding the numerical stability of the simple explicit Euler formula combined with various spatial discretizations, including finite elements. In the next section we shall discuss an alternative scheme, nearly as simple as explicit Euler, which can be used with standard central differences and does not suffer from the convection diffusion barrier $c^2 \leqslant \alpha$.

To contrast the explicit Euler-finite difference scheme with its *implicit* counterpart (see (2.13))

$$U_j^{n+1} = U_j^n + \left(\frac{\tau\epsilon}{h^2}\delta^2 - \frac{\tau v}{h}H\right)U_j^{n+1}, \tag{2.27}$$

let us compute the amplification factor ξ for this implicit scheme. Using relations (2.22 and 2.23) we get

$$\xi = (1 + \alpha(1 - \cos\theta) + i\, c \sin\theta)^{-1}, \tag{2.28}$$

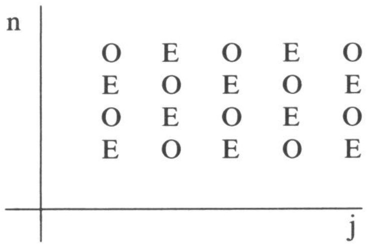
and an elementary calculation shows that $|\xi| \leqslant 1$, for all θ, for any nonnegative value of the diffusion parameter α and any value of the Courant number c. Consequently, stability limits such as (2.25) and (2.26) do not exist for the implicit Euler - finite difference scheme. The scheme is said to be *unconditionally stable.*

For numerical time-stepping purposes the property of unconditional stability is ideal in the sense that we then have the freedom to adjust the stepsize τ completely to the accuracy (in time) desired. Unfortunately, as we mentioned before, implicit steps may be rather expensive when compared with explicit steps, particularly so in two-dimensional and three-dimensional spaces. In practice, the choice of using explicit or implicit time-stepping is generally influenced by various factors, e.g., ease of programming (explicit schemes are invariably easier to apply than implicit ones) and computer facilities (available memory storage and central processor time). No doubt problems exist which must be treated implicitly just for the sake of numerical stability. In other cases, however, implicit time-stepping may be a bit superfluous and then appropriate explicit or explicit-implicit schemes can be quite useful.

3. AN EXPLICIT-IMPLICIT SCHEME

The appellation *explicit-implicit* refers to the fact that such schemes are based on a combination of explicit and implicit calculations. The objective of such a combination is always to reduce the computational effort of a fully implicit step to an acceptable level and in such a way that the resulting combination still offers attractive stability properties (in the literature one often uses the phrase *splitting* instead of explicit-implicit). Here we shall present an interesting example, viz., the *odd-even hopscotch* scheme which combines the explicit and implicit Euler rules (2.6) and (2.13). This combination was introduced by GORDON [4]. GOURLAY [5,6] has made a thorough study of Gordon's combination and has suggested various generalizations (see also [7], p.777 for more references).

Consider the explicit Euler - finite difference scheme (2.6) and its implicit counterpart (2.13). Let us suppose that the problem is one-dimensional (just for simplicity of presentation) and that L_h is a 3-point operator, i.e., $L_h U_j$ is a linear combination of U_{j-1}, U_j, U_{j+1}, for example the convection diffusion operator (2.15). Next consider the time-space mesh in the figure below



where the mesh points have been divided into two sets of points, viz., the points E where $(n+j)$ is even and the points O where $(n+j)$ is odd. GORDON'S idea was now to use the explicit scheme at the odd points, for some fixed value of n, and then, for the same value of n, the implicit scheme at the even points:

$$U_j^{n+1} = U_j^n + \tau L_h U_j^n \quad , \quad (n+j) \text{ odd}, \tag{3.1}$$

$$U_j^{n+1} = U_j^n + \tau L_h U_j^{n+1} \quad , \quad (n+j) \text{ even}. \tag{3.2}$$

For example, for (2.15) this can be written as

$$U_j^{n+1} = U_j^n + \frac{\tau\epsilon}{h^2}(U_{j+1}^n - 2U_j^{n+1} + U_{j-1}^n) - \frac{\tau\nu}{2h}(U_{j+1}^n - U_{j-1}^n) \tag{3.1a}$$

$$U_j^{n+1} = U_j^n + \frac{\tau\epsilon}{h^2}(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) - \frac{\tau\nu}{2h}(U_{j+1}^n - U_{j-1}^n) \tag{3.2a}$$

Because first all U_j^{n+1}, where $(n+1+j)$ is even, are computed, for a fixed n, it immediately follows that Gordon's scheme (3.1a) - (3.1b) is essentially explicit and therefore easy to program. Note that at the following time step the roles of odd and even formulas are interchanged. GOURLAY has introduced the name hopscotch for this type of explicit-implicit scheme because of its progress through the time-space mesh.

As observed by GOURLAY the scheme (3.1) can be reformulated in terms of a single equation as follows:

$$U_j^{n+1} = U_j^n + \tau\theta_j^n L_h U_j^n + \tau\theta_j^{n+1} L_h U_j^{n+1}, \tag{3.3}$$

where $\theta_j^n = 1$ for $(n+j)$ odd and $\theta_j^n = 0$ for $(n+j)$ even. This formulation is useful as a starting point for the von Neumann stability analysis which we shall discuss now for the central difference operator (2.15).

Before we can apply the von Neumann stability analysis some preparatory work has to be done. First we write down the step from t_{n+1} to t_{n+2}

$$U_j^{n+2} = U_j^{n+1} + \tau\theta_j^{n+1} L_h U_j^{n+1} + \tau\theta_j^{n+2} L_h U_j^{n+2}, \tag{3.4}$$

and subtract (3.3) from (3.4):

$$U_j^{n+2} = 2U_j^{n+1} - U_j^n + \tau\theta_j^{n+2} L_h U_j^{n+2} - \tau\theta_j^n L_h U_j^n. \tag{3.5}$$

This equation takes a particularly simple form at the *even points* since there $\theta_j^n = \theta_j^{n+2} = 0$, viz.,

$$U_j^{n+2} = 2U_j^{n+1} - U_j^n \quad , \ (n+j) \text{ even.} \tag{3.6}$$

Note that this is a simple *three-level extrapolation* formula. To get a workable three-level formula for the odd points the expression for U_j^{n+1} from (3.3) is substituted into (3.4). Then, taking into account that for the odd points $\theta_j^{n+1} = 0$ and $\theta_j^n = \theta_j^{n+2} = 1$, we get

$$U_j^{n+2} = U_j^n + \tau L_h U_j^n + \tau L_h U_j^{n+2} \quad , \ (n+j) \ \text{odd.} \tag{3.7}$$

Finally, by inserting (2.15) and the relation (3.6) for the even points, we arrive at the following *three-level scheme at the odd points:*

$$U_j^{n+2} = U_j^n + \frac{2\tau\epsilon}{h^2}(U_{j+1}^{n+1} - (U_j^{n+2} + U_j^n) + U_{j-1}^{n+1}) \tag{3.8}$$

$$- \frac{\tau\nu}{h}(U_{j+1}^{n+1} - U_{j-1}^{n+1}) \quad , \ (n+j) \ \text{odd.}$$

If we ignore the start and the completion of the odd-even hopscotch process, we see that this process produces numerical solutions at the complete set of uncoupled odd points which satisfy the three-level scheme (3.8). This three-level scheme is known as the *leapfrog-Du Fort-Frankel scheme.* This equivalence is employed for the von Neumann stability analysis of the hopscotch scheme. Let us substitute the Fourier mode (2.19) into (3.8). We then find that the associated amplification factor ξ must be a root of the quadratic equation

$$(1+\alpha)\xi^2 - 2(\alpha\cos\theta - i\,c\sin\theta)\xi - (1-\alpha) = 0. \tag{3.9}$$

For von Neumann stability we thus want both roots of (3.9) on the unit disk for all θ. Using well-known results [8,10] this is the case if and only if the complex number

$$\lambda = 1 - (1 - \cos\theta) - i\,c\sin\theta \tag{3.10}$$

satisfies $|\lambda| \leqslant 1$ for all θ. We see that λ is of type (2.21) (with $\alpha = 1$) and we immediately conclude that scheme (3.8), and thus also the odd-even hopscotch - finite difference scheme (3.1) - (3.2) with L_h given by (2.15), is stable in the sense of von Neumann iff $\tau|v| \leqslant h$ or

$$c^2 \leqslant 1. \tag{3.11}$$

This is in marked contrast to the result (2.24) for the explicit Euler - central difference scheme. We see that by alternating the explicit and implicit Euler rules in the odd-even hopscotch way — a simple and essentially explicit process — we are rid of the convection-diffusion barrier $c^2 \leqslant \alpha$. Moreover, the diffusion parameter α is no longer present in the inequality (3.11).

As we noted above, the scheme (3.8) is unconditionally stable for the parabolic part of the problem. Unfortunately, concerning this part the scheme shows a deficiency with respect to accuracy. If we substitute u and let $\tau, h \rightarrow 0$, the scheme approaches

$$u_t + vu_x = \epsilon u_{xx} - \epsilon\frac{\tau^2}{h^2}u_{tt} + O(\tau^2) + O(h^2) + O(\epsilon\tau^4/h^2). \tag{3.12}$$

This implies that for $\tau \rightarrow 0$, τ/h fixed and $\tau|v| \leqslant h$, the numerical solution approaches the solution of a wrong equation. It is plausible to expect that in practise this deficiency is not very serious as long as ϵu_{tt} does not take too large values. If it should lead to inaccuracies the most simple remedy is to reduce the time step a little. One could also conceive of eliminating the term $\epsilon\tau^2 h^{-2} u_{tt}$ by an extrapolation device [16]. Finally we want to remark that the hopscotch process can be equally well applied for all θ in two-dimensional and three-dimensional spaces. The only restriction is that L_h allows the odd-even uncoupling [5,6,16].

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The Seventh MTNS Symposium

Stockholm, June 10-14, 1985

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1. THE CONFERENCE AND THE CITY

The Seventh International Symposium on the Mathematical Theory of Networks and Systems (MTNS) was held in Stockholm, from Monday the 10th to Friday the 14th of June, 1985. The conference sessions took place in the Royal Institute of Technology in the Swedish capital. The renowned institute is located somewhat north of the center of the city, forming a large complex of red brick buildings which offered the MTNS symposium a variety of lecture rooms for the parallel sessions (of which there were often five or six), as well as a simple but effective auditorium for the plenary lectures.

The MTNS conference takes place once every two years. It started out as an informal meeting (now counted as the zeroth MTNS) organized by R. W. NEWCOMB in 1972 at the University of Maryland, College Park. The first two symposia were in fact held under the heading OTNS (Operator Theory of Networks and Systems), but the name was changed to reflect the importance of various other mathematical disciplines in network and system theory. The MTNS symposium is now the leading conference in a field that might be described as 'Mathematical System Theory' to distinguish it from its larger neighbor, the more applications-oriented area known as 'Systems and Control'. The systems and control field itself is covered by many conferences, such as the ones organized by the IEEE, INRIA, and IFAC.

One of the interesting features of MTNS is that it is a truly international undertaking. There is no particular organization which is responsible for the conference; rather, its continuity is ensured by an international committee of scientists, chaired by PAUL A. FUHRMANN of the Ben Gurion University of the Negev in Israel. The site of the conference is different each time, and the preceding years have seen meetings at various locations in the United States, Canada, the Netherlands, and Israel. For 1985, ANDERS LINDQUIST of the

Division of Optimization and Systems Theory of the Royal Institute of Technology volunteered to organize the symposium. LINDQUIST did the job together with CHRISTOPHER I. BYRNES, whose home base is Arizona State University at Tempe, Arizona. The Swedish-Arizonan cooperation resulted in a smoothly-running conference.

For myself, it was the first visit to the Swedish capital. Stockholm is a remarkable city, built on a conglomerate of islands, some larger, some smaller. The situation called for a comparison with my home town, Amsterdam, which is also a pretty watery place. I would formulate as a conjecture that the number of bridges is smaller in Stockholm than it is in Amsterdam, but that their total length is larger. On the other hand, I don't hesitate to formulate as a theorem (rigorously proven) that there are more places in Amsterdam than there are in Stockholm where a person can have a beer after a hard day's work.

The seventh MTNS was the largest ever held. In the List of Participants, I counted 285 attendants from 26 countries. The researchers from the United States were most numerous (74), followed by those from host nation Sweden. The third largest party came, and this is no surprise for those who know the MTNS conference, from the Netherlands (25 participants). Groups of ten or more researchers also came from France (18), Italy (18), the United Kingdom (14), Canada (12), Poland (11), Israel (11), and the Federal Republic of Germany (10). The List of Participants also mentions six Belgians, four Chinese, four Japanese, and four Soviet citizens. The large size of the Dutch delegation is an indication of the high level of activity that the small country by the sea maintains in the field of system theory — or, at least, in those aspects of the field that are traditionally emphasized at the MTNS symposia.

2. MAIN LECTURES

There were nine plenary speakers at the symposium, and the organization had put the lecturers simply in alphabetical order. As a result, it was J. ACKERMANN from the Institut für Dynamik der Flugsysteme in Oberpfaffenhofen, West-Germany, who gave the opening talk. He discussed the problem of 'simultaneous stabilization': how to construct a single controller which will stabilize several different systems (say, an airplane under different flight conditions). Ackermann's solution essentially came down to defining a parametrized class of controllers, and using a heuristic scheme to look for a feasible solution. There has also been some theoretical work in this area using heavy mathematical equipment. However, results of BYRNES and GHOSH giving a criterion for a 'generic' class of systems were put aside by ACKERMANN, who noted: 'practical cases are not generic'. What we have to learn from this remarkable statement is, I presume, that one has to be very careful in assuming that sets looking 'fat' or 'thin' from a certain mathematical point of view correspond to situations that 'almost always' or 'almost never' occur in practice.

One of the aspects I like about the system theory field is the fact that there is often a philosophical touch which helps to bear the weight of the technicalities. Several of the plenary lectures exhibited this feature very clearly, and among those the talk given by JAN C. WILLEMS of the University of Groningen. WILLEMS presented some of his recent work under the title 'Modelling, complexity and approximation of linear systems'. This work addresses the field of system identification, which, roughly speaking, deals with the problem of how to obtain a (dynamic) system model from observed data. There are some basic issues here which, according to the speaker, still need a fair amount of clearing up; 'causality structure', 'complexity', and 'approximation' are a few of the keywords in this context. The lecture was followed by a lively discussion, and I think this is welcome in a field where perhaps too often the approach is: select a model class, pick an error criterion, and compute compute compute. This is not to say that one has to stop doing computations, but certainly a complement is needed in terms of a discussion on the basics of the field.

Notwithstanding the alphabetic order of the speakers, WILLEMS was followed by two more main lecturers. W. MURRAY WONHAM of the University of Toronto informed the audience about recent progress in the field of 'discrete event systems', an area in which he himself was the first to initiate a major research program, about five years ago. Discrete event systems, as defined by WONHAM, are intended to describe certain types of decision situations, characterized by a finite number of states which go 'on' or 'off' at irregularly spaced points in time, influenced by internal dynamics as well as by so-called 'supervisory control'. These are rather untraditional objects to look at in a discipline where people are used to work with rather neat differential or difference equations. Nevertheless, it seems that some analogies can be drawn, so that, perhaps, new applications (such as production planning and control) come into reach. It has taken some time before a firm structure developed in this new field, but now there are at least some algorithms which can solve certain types of problems, be it that the amount of computation time needed seems to easily become a hurdle. Recently, there has also been a group at INRIA in France which considers discrete event systems in a setting that is somewhat different from Wonham's. It may well be that here we have a research area of which we will hear a lot more in the future. One can, at least, recognize a trend away from the vector space or manifold structures which have pervaded the system theory field for so long.

As was to be expected, the final plenary lecture was given by GEORGE ZAMES of McGill University. ZAMES shocked his audience slightly by noting, near to the end of his lecture, that he had given almost the same talk already once before, at an IEEE conference in 1976. Indeed, what ZAMES said had a very philosophical touch, and was, therefore, not tied to a specific time and place. The lecture was followed by a heated debate about the relation of π and 22/7, or, what one has to watch out for when using rational approximations for things that are not rational. ZAMES had stated that any useful system representation should be able to incorporate the uncertainty about the 'real', 'physical' system, and had claimed that transfer matrix representations are the only ones that fulfill this criterion; obviously, there were some in the audience who disagreed.

3. Adaptive control

Among the very many subjects that were discussed in the parallel sessions, perhaps most attention was drawn by adaptive control. This is a field which has been very actively explored over quite a few years now, and by a large group of researchers. Roughly, what one tries to do in adaptive control is to define controller structures that include an 'intelligent' (if one may use this word) reaction to a perceived suboptimal behavior of the controlled system, due, perhaps, to incorrect modelling or to drift of the parameters of the system. A precise definition of the problem would be difficult to give, and, in fact, the term 'adaptive control' should rather be understood as denoting a collection of disciplines, all working with the above idea in mind, but in several directions.

One branch tries to attack the problem by splitting it in two: first try to identify the parameters of the system to be controlled, then apply the control action that would be optimal for what you think that the system is. The 'adaptiveness' then comes from the combination of both aspects in one ongoing procedure. This is sometimes called a 'certainty equivalence' approach, because at each step one acts as if one is certain about the controlled system. One of the best-known results in system theory is that this approach is justified for linear systems if the uncertainty appears as additive Gaussian noise and if the cost criterion is of the quadratic integral type (the 'separation principle'). Of course, such a strong result is not expected for other types of uncertainty and other cost criteria, but, under suitable circumstances, the approach may still be feasible. Another direction is given by 'model reference adaptive control', which updates the controller parameters on the basis of the difference between the outputs of the actual system and of a reference model that one is trying to 'follow'. In these schemes, the control structure is still understood as basically consisting of a linear controller which contains, however, parameters that will vary along with the system dynamics. Looking at the control structure as a whole, one sees that this set-up actually defines a nonlinear controller (which, of course, explains much of the difficulty of the field).

So, from a certain point of view, adaptive control simply means that one is trying to find a nonlinear controller which will reach some specified design goal for a linear system containing some unknown parameters, that is, for a class of linear systems. It then becomes a natural question to ask, how big such a class of linear systems can be, in order that there exists a nonlinear controller which will reach a minimal design goal, say, asymptotic stability. Several sufficient conditions for this to happen have been around for a long time, but it was not clear how close these were to being necessary. In 1983,

A. S. MORSE explicitly stated a conjecture in which he formulated the problem in its simplest instance: does there exist a universal controller of the form $\dot{z}(t) = f(z(t), x(t))$, $u(t) = g(z(t), x(t))$, where $z(t)$ is in $\mathbb{R}^m$ (an 'm^{th} order controller') and f and g are differentiable functions, which will stabilize the class of systems of the form $\dot{x}(t) = x(t) + \lambda u(t)$ ($x(t) \in \mathbb{R}, \lambda \in \mathbb{R} \setminus \{0\}$ fixed but unknown)? A solution to this problem had been known for a long time for the case where λ is restricted to either the positive or the negative reals. MORSE conjectured that this condition — the sign of λ must be known — is also necessary, so that the answer to the above question would be negative for λ arbitrarily ranging over $\mathbb{R}$. The conjecture was explained by EDUARDO SONTAG to his colleague ROGER D. NUSSBAUM at Rutgers University. NUSSBAUM, not burdened by many years of study in adaptive control, was quick to show that Morse's conjecture is false, by producing a universal controller of the desired type. He even showed that it is sufficient to use a first-order controller ($m = 1$) and to let f and g be real-analytic functions. By way of consolation, NUSSBAUM also proved that Morse's conjecture is true when f and g are restricted to be polynomials. (The solution was published in *Systems and Control Letters*, *vol. 3* (1983), pp. 243-246.)

Nussbaum's result indicated that it should be possible to design universal controllers for much larger classes of systems than had been considered before. Research in this direction culminated recently in work by BENGT MÅRTENSSON, who is a Ph. D. student at Lund University in Sweden. The MTNS meeting in Stockholm was the first occasion for MÅRTENSSON to present his results at a major conference. He spoke in one of the parallel sessions. MÅRTENSSON showed how to construct a universal controller for any class of linear systems having the property that there is a uniform bound on the orders of the *linear* controllers that can be used to stabilize any *particular* element from the class. The order of the universal controller can be taken equal to this bound. This result encompasses all previous results on sufficient conditions for stabilizing adaptive control, including, of course, the one by NUSSBAUM.

In addition to the sufficiency result by MÅRTENSSON, CHRIS BYRNES, in his plenary talk, announced a proof of the necessity of the same condition. So, it seems that the problem has been completely solved. However, there are a few points that still call for discussion. In his presentation, MÅRTENSSON emphasized that his universal controller is useless from the applied point of view. The excursions that will take place in the controlled system before stabilization sets in are so large that any practical use is precluded. MÅRTENSSON even succeeded to obtain computer simulation results in the simplest cases, due to overflow problems. So it may be that the algorithm converges only on a cosmological time scale, which, although the requirements of the mathematical problem are still met, does not really represent a solution to the engineering problems that supposedly motivate the study of adaptive control.

Therefore, it remains to be seen what the real conclusion will be from the development that apparently reached its summit at the MTNS meeting. As a corollary to his result (which has now been published in *Systems and Control Letters, vol. 6* (1985), pp. 87-91), MÅRTENSSON showed that if one allows the order of the nonlinear controller to vary with time, then one can construct a universal controller for the whole class of linear systems that can be stabilized at all by a finite-order controller. In a sense, this is a disappointing result since it shows that no interesting conditions come out from the problem of constructing a universal stabilizing controller. Apparently, the requirement of asymptotic stability in itself is too weak to distinguish between systems that are 'easy' or 'difficult' to control; a distinction which, of course, is felt very clearly in practice. So, it appears that something stronger should be looked for, which brings us back (a little wiser, though) to the problem that has bothered the mathematical study of adaptive control all along: how to obtain a sharply defined and not too intractable mathematical question that properly reflects at least some of the aspects that one has to reckon with in actual control applications. Time will tell whether Mårtensson's result marks the beginning or the end of a development.

Abstracts

of Recent CWI Publications

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CWI Tract 19. T.M.V. Janssen. *Foundations and Applications of Montague Grammar. Part 1, Philosophy, Framework, Computer Science.*
AMS 08A99, 03G15, 68F05, 68F20, 03B15; CR F.3.2, I.2.7, F.3.1, F.3.0, F.4.3; 206 pp.
Abstract: The present volume is one of the two tracts which are based on my dissertation 'Foundations and applications of Montague grammar'. The two volumes present an interdisciplinary study in mathematics, philosophy, computer science, logic, and linguistics. No knowledge of specific results in these fields is presupposed, although occasionally terminology or results from them are mentioned. Throughout the text it is assumed that the reader is acquainted with fundamental principles of logic, in particular of model theory, and that he is used to a mathematical kind of argumentation. The contents of the volumes have a linear structure: first the approach is motivated, next the theory is developed, and finally it is applied. Volume 1 contains an application to programming languages, whereas volume 2 is devoted completely to the consequences of the approach for natural languages. The volumes deal with many facets of syntax and semantics, discussing rather different kind of subjects from this interdisciplinary field. They range from abstract universal algebra to linguistic observations, from the history of philosophy to formal language theory, and from idealized computers to human psychology.

CS-R8512. J.W. de Bakker, J.-J.Ch. Meyer & E.-R. Olderog. *Infinite streams and finite observations in the semantics of uniform concurrency.*
AMS 68B10, 68C01; CR D.3.1, F.3.2, F.3.3; 23 pp.; **key words:** concurrency, denotational semantics, streams, uniform languages, observations, Smyth ordering, parallel composition, topological closedness.

Abstract: Two ways of assigning meaning to a language with uniform concurrency are presented and compared. The language has uninterpreted elementary actions from which statements are composed using sequential composition, nondeterministic choice, parallel composition with communication, and recursion. The first semantics uses infinite streams in the sense which is a refinement of the linear time semantics of De Bakker et al. The second semantics uses the finite observations of Hoare et al., situated 'in between' the divergence and readiness semantics of Olderog & Hoare. It is shown that the two models are isomorphic and that this isomorphism induces an equivalence result between the two semantics. Furthermore, a definition of the hiding operation which is inspired by the infinite streams approach is presented. Finally, the continuity of this operation is proved in the framework of finite observations.

CS-R8517. J.C.M. Baeten, J.A. Bergstra & J.W. Klop. *Ready trace semantics for concrete process algebra with priority operator.*
AMS 68B10, 68C01, 68D25, 68F20; CR F.1.1, F.1.2, F.3.2, F.4.3; 21 pp.; **key words:** process algebra, concurrency, readiness semantics, failure semantics, ready trace semantics, priority operator.

Abstract: We consider a process semantics intermediate between bisimulation semantics and readiness semantics, called here ready trace semantics. The advantage of this semantics is that, while retaining the simplicity of readiness semantics, it is still possible to augment this process model with the mechanism of atomic actions with priority (the θ operator). It is shown that in readiness semantics and a fortiori in failure semantics such an extension with θ is impossible. Ready trace semantics is considered here in the simple setting of concrete process algebra, that is: without abstraction (no silent moves), moreover for finite processes only. For such finite processes without silent moves a complete axiomatisation of ready trace semantics is given via the method of process graph transformations.

CS-R8518. M.L. Kersten, H. Weigand, F. Dignum & J. Boom. *A conceptual modeling expert system.*
CR H.2.1, I.2.1, I.2.7; 14 pp.; **key words:** logical design, expert systems, natural language parsing.

Abstract: This paper describes the architecture of an interactive conceptual modeling expert system, called ACME. The input to ACME is a natural language description of the application domain, which is decomposed by a parser into so-called predications. From these predications a preliminary EAR model can be extracted readily, which is subsequently improved by the EAR modeling expert using structural and semantic rules stored in a knowledge base. In an iterative process the user resolves the inconsistency and ambiguity problems discovered by ACME and enhances the conceptual model.

CS-R8519. Ming Li & P.M.B. Vitányi. *Tape versus queue and stacks: the lower bounds.*
AMS 68C40, 68C25, 68C05, 94B60, 10-00; CR F.1.1, F.1.3, F.2.3; 24 pp.; **key words:** multitape Turing machine, stack, queue, pushdown stores, determinism, nondeterminism, on-line, off-line, time complexity, lower bound, simulation by one tape, algorithmic information theory, Kolmogorov complexity.

Abstract: Several optimal or nearly optimal lower bounds are derived on the time needed to simulate queue, stacks (stack = pushdown store) and tapes by one off-line single-head tape-unit with one-way input, both deterministic and nondeterministic. The techniques rely on algorithmic information theory (Kolmogorov complexity).

CS-R8520. N.W.P. van Diepen & W.P. de Roever. *Program derivation through transformations: the evolution of list-copying algorithms.*
AMS 68B10, 68C05, 68E10; CR D.2, D.2.2, D.2.4, E.1, F.3, F.3.1, I.2.2; 60 pp.; **key words:** program verification, Hoare logic, program transformation, list traversal, Deutsch-Schorr-Waite algorithm, list-copying, Robson's algorithm, Clark's algorithm, graph algorithm.

Abstract: The introduction of Hoare Logic made it feasible to supply correctness proofs of small sequential programs. While correctness proofs of larger programs could be given in principle, the increased size of such a proof warranted additional organization. The present paper puts emphasis on the technique of program transformation to show the derivability and to prove the correctness of some fast list-copying algorithms developed by Robson, Fisher and Clark. This subject was motivated by an earlier paper on the same topic by Lee, De Roever, and Gerhart. Some transformation rules necessary for the correctness proofs are given. Other proof techniques used include data refinement and the use of auxiliary variables and structures.

CS-R8521. J.C.M. Baeten & J.A. Bergstra. *Global renaming operators in concrete process algebra.*
AMS 68B10, 68C01, 68D25, 68F20; CR F.1.1, F.1.2, F.3.2, F.4.3; 30 pp.; **key words:** process algebra, concurrency, renaming operator, trace set.

Abstract: Renaming operators are introduced in concrete process algebra (concrete means that abstraction and silent moves are not considered). Examples of renaming operators are given: encapsulation, pre-abstraction and localization. We show that renamings enhance the defining power of concrete process algebra by using the example of a queue. We give a definition of the trace set of a process, see when equality of trace sets implies equality of processes, and use trace sets to define the restriction of a process. Finally, we describe processes with actions that have a side effect on a state space and show how to use this for a translation of computer programs into process algebra.

CS-R8522. J.C.M. Baeten, J.A. Bergstra & J.W. Klop. *An operational semantics for process algebra.*
AMS 68B10, 68C01, 68D25, 68F20; CR F.1.1, F.1.2, F.3.2, F.4.3; 30 pp.; **key words:** process algebra, concurrency, operational semantics, true concurrency, real-time behaviour, rewrite rules, object-oriented, Petri net theory.

Abstract: We consider operational rewrite rules, expressing the dynamic behaviour of configurations of objects. This gives an algebraic way to generalize Petri net theory. We give a detailed description of operational rewrite rules for various process algebras, and show that, in each case, the operational semantics thus obtained is equivalent to the regular denotational semantics of processes. This theory can be used to describe features like true concurrency and real-time behaviour for concurrent, communicating processes.

CS-R8523. J.A. Bergstra, J.W. Klop & E.-R. Olderog. *Readies and failures in the algebra of communicating processes.*
AMS 68B10, 68C01, 68D25, 68F20; CR F.1.1, F.1.2, F.3.2, F.4.3; 38 pp.; **key words:** process algebra, concurrency, readiness semantics, failure semantics, bisimulation semantics.

Abstract: Readiness and failure semantics are studied in the setting of ACP (Algebra of Communicating Processes). A model of process graphs modulo readiness and failure equivalence respectively is constructed, and an equational axiom system is presented which is complete for this graph

model. An explicit representation of the graph model is given, the failure model, whose elements are failure sets. Furthermore, a characterization of failure equivalence is obtained as the maximal congruence which is consistent with trace semantics. By suitably restricting the communication format in ACP, this result is shown to carry over to Milner's CCS and Hoare's CSP. In the above we restrict ourselves to finite processes without τ-steps. At the end of the paper a comment is made on the situation for infinite processes with τ-steps: notably we obtain that failure semantics is incompatible with Koomen's fair abstraction rule, a proof principle based on the notion of bisimulation.

CS-N8507. J.C. van Vliet. *STARS and stripes.*
CR D.2; 7 pp.; **key words:** STARS, Software Engineering Institute, Ada, software engineering environments, software engineering methodology, reusable software, software tools, software transition, measurements, defense applications, business practices, education.
Abstract: The overall goal of DoD's Software Initiative is to meet DoD's future software needs by an order of magnitude improvement in the state of the practice, and to hasten the transition to new technology. The STARS Program (Software Technology for Adaptable, Reliable Systems) is one of the components of the Software Initiative. Other components are the Ada Program and the Software Engineering Institute. The STARS Program is to result in a fully integrated environment which captures all phases of the software life cycle. From April 30 to May 2, 1985, the first DoD/Industry STARS Program Conference was organized, which brought together representatives of government, industry, and the academic community to review and discuss the STARS Program and other components of the Software Initiative.

CS-N8508. S. van Veen & E. de Vink. *Semantics of logic programming.*
AMS 68B05, 68C01; 41 pp.; **key words:** Concurrent Prolog.
Abstract: In this note several semantics of logic programming are given and equivalence of these semantics is proven. The technique of transition systems is used to describe the semantics of a subset of Prolog and Concurrent Prolog. A notion of fairness is introduced in order to model infinite computations of logic programs.

CS-N8509. M.J.A.C. Andreoli. *Language primitives in B for graphic editing.* (In Dutch.)
AMS 69D26, 69D43, 69K34; 26 pp.; **key words:** programming language *B*.
Abstract: This report investigates the addition of commands to a language such as *B* to make 'graphic editing' (interactive input via a graphics interface) possible and easy.

OS-R8506. J.M. Schumacher. *Residue formulas for meromorphic matrices.*
AMS 15A54, 47A56, 70J10, 70J25, 93D05; 13 pp.; **key words:** vibrating structure, residue matrix, partial fraction expansion, local ring, Lyapunov stability.
Abstract: In the analysis of the vibrations of mechanical systems, it is not only important to compute the resonance frequencies, but also to find the so-called 'participation matrices' which govern the distribution of the energy over the various resonance modes. These matrices appear as residue matrices for certain meromorphic matrix-valued functions (transfer matrices from forces to displacements), the poles of which correspond to the resonance frequencies. Also, these poles are simple as a consequence of the law of conservation of energy. So the problem comes down to the computation of the residue at a simple pole of a meromorphic matrix. This matrix is in general not given through its entries, but rather as the inverse of another matrix or as a fraction of holomorphic matrices. Extending earlier results of Lancaster and of Gohberg and Sigal, we work out a convenient residue formula for matrices in fractional form. Several variants will be discussed as

well. In all versions, one constructs a 'normalizing matrix' which is invertible if and only if the pole one considers is simple, and one writes down a formula for the residue which features the inverse of the normalizing matrix. Proofs are based on the 'local Smith form' for meromorphic matrices. The normalizing matrix can also be used in stability tests, and we show an application of this.

OS-R8507. J.H. van Schuppen. *Stochastic realization problems motivated by econometric modeling.*
AMS 93E03, 93E12, 93B15, 90A15, 90A16, 90A20; 17 pp.; **key words:** stochastic realization, factor analysis, latent structure analysis, conditional independence, causality.

Abstract: The econometric modeling of time series by linear stochastic models has been criticized by R.E. Kalman. Instead he proposes formulating this modeling problem as a stochastic realization problem. In this note Kalman's approach is followed and in a non-dynamic framework generalized to multivariate stochastic realization problems. The special case of the three-variate Gaussian stochastic realization problem is investigated in some detail. In a dynamic context the stochastic realization problem is posed of representing an observed process such that the inherent causality or dependency relation between the components is made explicit.

OS-R8508. R.K. Boel & J.H. van Schuppen. *Overload control for SPC telephone exchanges refined models and stochastic control.*
AMS 93E20, 90B22, 60K25; 11 pp.; **key words:** overload control, stochastic control, queueing theory, communication systems.

Abstract: In telephone networks switching and connecting operations are performed by the exchanges. The Stored Program Control (SPC) exchanges which are nowadays installed are computer controlled. One of the problems with these exchanges is the severe performance degradation during periods in which the demand for service exceeds the design capacity. The problem of overload control is then to maximize the number of successfully completed calls. In this paper two models for overload control of an SPC exchange are proposed that are refinements of an earlier model. A stochastic control problem for one of these models is shown to have a bang-bang type of optimal solution.

OS-R8509. J.W. Polderman. *A note on the structure of two subsets of the parameter space in adaptive control problems.*
AMS 93C40; 10 pp.; **key words:** adaptive LQ control, C^{ω}-manifold.

Abstract: In this note we study the geometric structure of two subsets of the parameter space that are of interest in the context of adaptive LQ-control. The first set can be considered as the set of possible limit points of an adaptive control algorithm, whereas the second can be seen as the set of desirable limit points. Our main result is that these sets are C^{ω}-manifolds.

OS-R8510. O.J. Boxma & B. Meister. *Waiting-time approximations for cyclic-service systems with switch-over times.*
AMS 60K25, 60K30, 68M20; 16 pp.; **key words:** queueing system, cyclic service, switch-over times, mean waiting time.

Abstract: Mean waiting-time approximations are derived for a single-server multi-queue system with nonexhaustive cyclic service. Non-zero switch-over times of the server between consecutive queues are assumed. The main tool used in the derivation is a pseudo-conservation law recently found by Watson. The approximation is simpler and, as extensive simulations show, more accurate than existing approximations. Moreover, it gives very good insight into the qualitative behavior of cyclic-service queueing systems.

OS-R8511. O.J. Boxma. *A queueing model of finite and infinite source interaction.*
AMS 60K25, 68M20; 11 pp.; **key words:** $M/M/1$ queue, finite source, interaction.
Abstract: An $M/M/1$ service system is considered, which also serves one finite source. The joint distribution of queue length at the $M/M/1$ queue and position of the finite source customer in the system is determined. This leads to exact expressions for various performance measures; such expressions yield insight into the interaction of finite and infinite sources.

OS-R8512. G.A.P. Kindervater & H.W.J.M. Trienekens. *Experiments with parallel algorithms for combinatorial problems.*
AMS 90C27, 68Q10, 68R05; 19 pp.; **key words:** parallel computer, SIMD, MIMD, pipelining, dataflow, branch and bound, dynamic programming, divide and conquer, knapsack, shortest paths, change-making.
Abstract: In the last decade many models for parallel computation have been proposed and many parallel algorithms have been developed. However, few of these models have been realized and most of these algorithms are supposed to run on idealized, unrealistic parallel machines. The parallel machines constructed so far all use a simple model of parallel computation. Therefore, not every existing parallel machine is equally well suited for each type of algorithm. The adaptation of a certain algorithm to a specific parallel architecture may severely increase the complexity of the algorithm or severely obscure its essence. Little is known about the performance of some standard combinatorial algorithms on existing parallel machines. In this paper we present computational results concerning the solution of knapsack, shortest paths and change-making problems by branch and bound, dynamic programming, and divide and conquer algorithms on the ICL-DAP (an SIMD computer), the Manchester dataflow machine and the CDC-CYBER-205 (a pipeline computer).

OS-R8513. P.J.C. Spreij. *Recursive parameter estimation for counting processes with linear intensity.*
AMS 62F12, 93E12; 23 pp.; **key words:** recursive estimation, counting process, martingale, Lyapunov function, central limit theory.
Abstract: Recursive estimation algorithms are presented for counting processes that have an intensity process which is linear in the parameter. Strong consistency and asymptotic normality of the estimators generated by the algorithms are proved.

NM-R8519. P.J. van der Houwen & B.P. Sommeijer. *Reduction of dispersion in hyperbolic difference schemes by adapting the space discretization.*
AMS 65M20, 76B15; 12 pp.; **key words:** hyperbolic equations, difference schemes, spatial discretization, dispersion, shallow water equations.
Abstract: A fourth-order accurate difference scheme for systems of hyperbolic equations is presented. The dispersion in this scheme can be reduced if it is known in advance in which region the frequencies of the dominant Fourier components are located. All method parameters are explicitly expressed in terms of the bounds on the dominating frequencies. The performance of the method is illustrated by an application to the shallow water equations.

NM-R8520. S.P. Spekreijse. *Second order accurate upwind solutions of the 2D steady Euler equations by the use of a defect correction method.*
AMS 65N05, 76G15, 76H05; 11 pp.; **key words:** steady Euler equations, defect correction method.

Abstract: In this paper a description is given of first and second order finite volume upwind schemes for 2D steady Euler equations in generalized coordinates. These discretizations are obtained by projection-evolution stages, as suggested by Van Leer. The first order schemes can be solved efficiently by multigrid methods. Second order approximations are obtained by a defect correction method. In order to maintain monotone solutions, a limiter is introduced for the defect correction method.

NM-R8521. F.W. Wubs. *Stabilization of explicit methods for hyperbolic initial-value problems.*
AMS 65M10, 65M20; 12 pp.; **key words:** stabilization, hyperbolic equations, method of lines, residual averaging.

Abstract: It is well known that explicit methods are subject to a restriction on the time step. This restriction is a drawback if the variation in time is so small that accuracy considerations would allow a larger time step. In this case, implicit methods are more appropriate because they do allow large time steps. However, in general, they require more storage and are more difficult to implement than explicit methods. In this paper, we propose a technique by which it is possible to stabilize explicit methods for quasi-linear hyperbolic equations. The stabilization turns out to be so effective that explicit methods become a good alternative to unconditionally stable implicit methods.

NM-R8522. J.G. Blom & H. Brunner. *The numerical solution of nonlinear Volterra integral equations of the second kind by collocation and iterated collocation methods.*
AMS 65R20, 45D05, 45L10; 28 pp.; **key words:** nonlinear Volterra integral equation, polynomial splines, collocation, iterated collocation, superconvergence, error estimates, variable stepsize.

Abstract: The subject of this paper is a variable-stepsize one-step method of collocation type for solving general nonlinear second kind Volterra integral equations. We extend the iterated collocation method corresponding to polynomial spline collocation to nonlinear Volterra integral equations of the second kind. The resulting superconvergence properties of either the collocation approximation or the iterated collocation approximation are used to obtain (local and global) error estimates which in turn form the basis of a variable stepsize code. The performance of this code is illustrated by means of numerous test problems.

NM-R8523. P.W. Hemker. *Defect correction and higher order schemes for the multigrid solution of the steady Euler equations.*
AMS 65N05, 65N30, 76G13; 17 pp.; **key words:** steady Euler equations, multigrid methods, higher order schemes, defect correction.

Abstract: In this paper we describe first and second order finite volume schemes for the solution of steady Euler equations for inviscid flow. The solution for the first order scheme can be efficiently computed by an FAS multigrid procedure. Second order accurate approximations are obtained by linear interpolation in the flux- or the state space. The corresponding discrete system is solved (up to truncation error) by defect correction iteration. An initial estimate for the second order solution is computed by Richardson extrapolation. Examples of computed approximations are given, with emphasis on the effect for the different possible discontinuities in the solution.

NM-N8501. D.T. Winter. *Information about CWI Ada facilities #1.*
AMS 69D49; 8 pp.; **key words:** Ada.

Abstract: This report describes the technical characteristics of the hardware and software for Ada program development available at CWI, based on a Data General MV4000 mini computer.

NM-N8502. H.J.J. te Riele. *Applications of supercomputers in mathematics.*
AMS 65V05, 65N20; 31 pp.; **key words:** vector computers, parallel processing, computational mathematics.

Abstract: These course notes provide a concise survey of the role played by vector and parallel processors in the solution of problems in computational mathematics. Some vectorization and parallelization techniques are discussed. Many examples illuminate the discussion.

MS-R8506. H.J. Ader, D.J. Kuik, E. Opperdoes & B.F. Schriever. *The use of conversational packages in statistical computing.*
AMS 62-04; 33 pp.; **key words:** conversational computing, statistical computing, evaluation of software, conversational statistical packages.

Abstract: It is not generally recognized that conversational computing demands a different way of using statistical analysis. However, special problems arise while applying statistical techniques repeatedly on the same dataset. The question is put whether unexperienced users should be protected against this kind of improper use of a conversational statistical package (CSP). We list some formal aspects of conversational communication. Thereafter user-objectives in the use of the package are considered and consequences for the user-interface formulated. Finally some technical questions are treated.

MS-R8509. H.C.P. Berbee. *Chains with infinite connections: uniqueness and Markov representation.*
AMS 60G10, 60K35, 60J20; 9 pp.; **key words:** g-measure, chain with infinite connection, Markov representation.

Abstract: If for a process $(\xi_n)_{n=-\infty}^{\infty}$ the conditional distribution of ξ_n given the past does not depend on n for e.g. $n \geqslant 0$, then the process may be called a chain with infinite connections. Under a well-known continuity condition on this conditional distribution the process is shown to be distributed as an instantaneous function of a countable state Markov chain. Also under a certain weaker continuity condition uniqueness of the distributions of the stationary chains is obtained.

AM-R8512. N.M. Temme. *Uniform asymptotic expansion for a class of polynomials biorthogonal on the unit circle.*
AMS 33A65, 41A60, 33A30, 30E15; 6 pp.; **key words:** biorthogonal polynomials, uniform asymptotic expansion, hypergeometric function.

Abstract: An asymptotic expansion including error bounds is given for polynomials $\{P_n, Q_n\}$ that are biorthogonal on the unit circle with respect to the weight function $(1-e^{i\theta})^{\alpha+\beta}(1-e^{-i\theta})^{\alpha-\beta}$. The asymptotic parameter is n; the expansion is uniform with respect to z in compact subsets of $\mathbb{C} \setminus \{0\}$. The point $z=1$ is an interesting point, where the asymptotic behaviour of the polynomials changes markedly. The approximants in the expansions are confluent hypergeometric functions. The polynomials are special cases of Gauss hypergeometric functions. The results of the paper apply in fact to these functions for the case that in the function ${}_2F_1(-a,b;c;\zeta)$ a is positive and large, b and c are fixed and ζ is the uniformity parameter with $\zeta=0$ as 'transition' point.

AM-R8513. N.M. Temme. *On the computation of the incomplete gamma functions for large values of the parameters.*
AMS 33A15, 65D20, 41A60; 7 pp.; **key words:** incomplete gamma function, uniform asymptotic expansion, computation of special functions.
Abstract: A method for computing incomplete gamma functions is given for the case that the parameters are positive and both large. The method is based on earlier results of the author on uniform asymptotic expansions of these functions. It is concluded that the method may be considered as an addition to Gautschi's algorithm, which becomes inefficient precisely in the case that the method described here is best applicable.

AM-R8514. N.M. Temme. *Incomplete Laplace integrals: uniform asymptotic expansions with application to the incomplete beta function.*
AMS 41A60, 30E15, 33A15, 44A10; 26 pp.; **key words:** uniform asymptotic expansion of integrals, incomplete gamma function, incomplete beta function, incomplete Laplace integral, construction of error bounds.
Abstract: The incomplete Laplace integral

$$\frac{1}{\Gamma(\lambda)}\int_{\alpha}^{\infty} t^{\lambda-1}e^{-zt}f(t)dt$$

is considered for large values of z. Both λ and α are uniformity parameters in $[0,\infty)$. The basic approximant is an incomplete gamma function, that is, the above integral with $f=1$. Also, a loop integral in the complex plane is considered with the same asymptotic features. The asymptotic expansions are furnished with error bounds for the remainders in the expansions. The results of the paper combine four kinds of asymptotic problems considered earlier. An application is given for the incomplete beta function. The present investigations are a continuation of earlier work of the author for the above integral with $\alpha=0$, and build significantly on this special case.

AM-R8515. H.E. de Swart & J. Grasman. *Effect of stochastic perturbations on a spectral model of the atmospheric circulation.*
AMS 76C15, 60J70, 34A50; 18 pp.; **key words:** stochasticly forced spectral model, Fokker-Planck equation, characteristic residence time, eikonal equation, discrete state Markov process.
Abstract: The dynamics of a low order spectral model of the barotropic potential vorticity equation, forced by random perturbations, is studied as a function of the memory and intensity of the noise. The unperturbed deterministic system has three equilibria, and for arbitrary initial conditions trajectories in phase space always tend to one of the two stable equilibria representing preferent circulation patterns of the atmosphere. The noise forces the system to visit alternately the two attraction domains of the stable equilibria. During the transition the system will remain for some time in a neighbourhood of the unstable equilibrium. Characteristic residence times in the attraction domains and in the domain near the unstable equilibrium are calculated by combined analytical and numerical methods. Furthermore the alternation of preferent states is studied with a discrete state Markov process model. It consists of three states, which are related to the equilibria of the low order spectral model. Transition probabilities are derived from the characteristic residence times of the stochasticly forced dynamical system. The eigenvalues of the Markov model yield information about the time scale over which the effect of the initial state is present in the system.

AM-R8516. H.A. Lauwerier. *Hopf bifurcation in host-parasitoid models.*
AMS 58F14, 58F08, 92A15, 39A10; 18 pp.; **key words:** Hopf bifurcation, host-parasitoid model, Arnold normal form, discrete dynamical systems, iterated planar maps.

Abstract: For a wide class of host-parasitoid models a reduction to Arnold's normal form can be carried out in an explicit way. In the case of Hopf bifurcation the shape and size of the elliptic limit curve can be derived in terms of the parameters of the model. Some models have a rich bifurcation behaviour with both forward and backward Hopf bifurcation, and with a transition zone in the parameter plane for which there exists a pair of limit curves, one stable and one unstable. The theory is confirmed and illustrated by numerical experiments.

PM-R8507. M. Hazewinkel. *Parametrization problems for spaces of linear input-output systems.*
AMS 93B30, 93E12; 8 pp.; **key words:** identification, parametrization problems, linear dynamical input-output systems.

Abstract: This note introduces and discusses the general problem of finding good parametrizations of sets of possible models, mainly in the context of finite dimensional dynamic input-output models. The general problem is addressed in particular in the case where it is impossible to find one global parametrization.

PM-N8501. A.E. Brouwer. *Recursive constructions of mutually orthogonal Latin squares.*
AMS 05B15; 15 pp.; **key words:** Latin squares, transversal designs.

Abstract: Two $n \times n$ matrices (a_{ij}) and (b_{ij}) with entries in an alphabet S of size n are called orthogonal if for all $(s,t) \in S \times S$ there is a unique position (i,j) such that $a_{ij} = s$ and $b_{ij} = t$. In this report, large sets of pairwise orthogonal matrices are constructed, and the connection with Latin squares is explained.

CWI Activities

Winter 1985

With each activity we mention its frequency and (between parentheses) a contact person at CWI. Sometimes some additional information is supplied, such as the location if the activity will not take place at CWI.

Study group on Analysis on Lie groups. Jointly with University of Leiden. Biweekly. (T.H. Koornwinder)

Seminar on Algebra and Geometry, on Multilinear Forms. 20 January, 17 February, 17 March, 21 April. (A.M. Cohen)

NATO - ARW Workshop on Geometries and Groups, Finite and Algebraic. 24-28 March at Leeuwenhorst, Noordwijkerhout.
The objective is to provide an overview of current activities in: diagram geometries, chamber systems with transitive automorphism groups, finite quotients of affine buildings, properties of finite (twisted and nontwisted) Chevalley groups, and geometries related to finite simple groups, and to explore the use of new results in Algebraic Groups for Finite Group Theory, and Geometry and vice versa. Invited speakers:
M. Aschbacher, F. Buekenhout, B.N. Cooperstein, R.L. Griess, W.M. Kantor, G. Seitz, E.E. Shult, S. Smith, T.A. Springer, B. Stellmacher, G. Stroth, F.G. Timmesfeld, J. Tits. (A.M. Cohen)

Cryptography working group. Monthly. (J.H. Evertse)

Colloquium 'STZ' on System Theory, Applied and Pure Mathematics. Twice a month. (J. de Vries)

Seminar on Integrable Systems. 27 January, 24 February, 24 March. (M. Hazewinkel)

Orthogonal Polynomials Day. 31 January. Invited speakers:
E. Hendriksen (Amsterdam). P. Nevai (Columbus, USA), D. Stanton (Minneapolis, USA), E.A. van Doorn (Twente, The Netherlands). (T.H. Koornwinder)

Study group 'Biomathematics'. Lectures by visitors or members of the group. Jointly with University of Leiden. Bimonthly. (J. Grasman)

Study group on Nonlinear Analysis. Lectures by visitors or members of the group. Jointly with University of Leiden. (O. Diekmann)

Progress meetings of the Applied Mathematics Department. New results and open problems on the research topics of the department: biomathematics, mathematical physics, asymptotic and applied analysis, image analysis. Weekly. (N.M. Temme)

Study group on Statistical Image Analysis. Biweekly. (R.D. Gill)

Colloquium 'Models for Discrete Variables'. Biweekly.

The emphasis will be on generalized linear models and variants thereof for discrete variables. After a short introduction to log-linear models, based on Fienberg's book 'The Analysis of Cross-Classified Categorical Data' and the standard package GLIM, we will turn to a number of modern developments in the data-analysis of models for discrete variables, such as: Goodman's bilinear models, comparison with other models and models for continuous data, comparison of cross-products in different tables, Lauritzen and Speed's graphical models, computation of asymptotic standard errors, also for more complex sample setups, quasi-likelihood, formalisation of model-selection and testing with the same data. (A. Verbeek)

Seminar on Asymptotic Statistics. 15,21,29 January.

E. Mammen will give three lectures about recent developments in asymptotic statistics. Especially Le Cam's theory of the deficiency distance between statistical experiments and strong approximation of experiments will be treated. (R.D. Gill)

Study group on Combinatorial Optimization. Biweekly. (J.K. Lenstra)

System Theory Days. Irregular. (J.H. van Schuppen & J.M. Schumacher)

Study group on System Theory. Biweekly. (J.M. Schumacher)

Eleventh Conference on the Mathematics of Operations Research and System Theory. 15,16,17 January at Lunteren. Invited speakers:

A.R. Conn (Waterloo, Canada), G. Cornuéjols (Carnegie-Mellon, USA), R.P. Guidorzi (Bologna, Italy), T.J. Ott (Holmdel, USA), J.P. Quadrat (Paris, France), P. Whittle (Cambridge, UK). (J.M. Schumacher)

Colloquium on Queueing Theory and Performance Evaluation. Irregular. (O.J. Boxma)

International Seminar on Teletraffic Analysis and Computer Performance Evaluation. 2-6 June. (O.J. Boxma)

Progress meetings on Numerical Mathematics. Weekly. (H.J.J. te Riele)

International Colloquium on Numerical Aspects of Vector and Parallel Processors. Monthly, every last Friday. (H.J.J. te Riele)

Study group on Numerical Software for Vector Computers. Monthly. (H.J.J. te Riele)

Study group on Differential and Integral Equations. Lectures by visitors or group members. Irregular. (H.J.J. te Riele)

Study group on Graphics Standards. Monthly. (M. Bakker)

Study group on Dialogue Programming. (P.J.W. ten Hagen)

Post-academic Course on Modern Techniques in Software Engineering. 13,14,27,28 February. (J.C. van Vliet)

Seminar National Concurrency Project. Jointly with Universities of Leiden & Eindhoven and several industrial research establishments. 7 February, 7 March, 9 May. (J.W. de Bakker)

National Study Group on Concurrency. Jointly with Universities of Leiden & Eindhoven and several industrial research establishments. 21 February, 21 March, 23 May. (J.W. de Bakker)

Computer Science Colloquium. 24 January, 20 February, 20 March, 17 April, 22 May. (A. Janssen)

Colloquium Knowledge Based Systems. Biweekly. (M.L. Kersten & P.J.F. Lucas)

Process Algebra Meeting. Weekly. (J.W. Klop)

Visitors to CWI from Abroad

P.J. Courtois (Philips Research Laboratory, Brussels, Belgium) 18 November. S. Csörgo (Bolyai Mathematical Institute, Szeged University, Hungary) 16 November. Dao Huu Ho (University of Hanoi, North Vietnam) 20 November. A.K. Dave (CRAY Research Ltd., London, UK) 29 November. H. Fujii (Kyoto Sangyo University, Japan) 15-21 December. M. Guevara (McGill University, Montreal, Canada) 10 January. M. Harris (University of Minnesota, Minneapolis, USA) 18-19 November. F. Herrlich (University of Bochum, West Germany) 18 November. E. Koenigsberg (University of California, Berkeley, USA) 19 December. T. Koski (Türkü, Finland) 9 December. H.R. Lerche (University of Heidelberg, West Germany) 30 September - 4 October. A. Ligtenberg (AT&T Bell Laboratories, Murray Hill, USA) 8 November. E. Mammen (University of Heidelberg, West Germany) December 1985 - January 1986. J.F. Paris (University of California, La Jolla, USA) 20 December. I. Phillips (Imperial College London, UK) 9-13 December. G. Picci (University of Padua, Italy) 7-17 October. P. Rabinowitz (Weizmann Institute of Science, Rehovot, Israel) 23 October. S.M.N. Ruijsenaars (University of Tübingen, West Germany) 28 October. R. Scherer (University of Karlsruhe, West Germany) 1-3 November. J. Smith (John Hopkins University, Baltimore, USA) September 1985 - June 1986. P. Tran-Gia (University of Stuttgart, West Germany) 22 October. P.J. Weinberger (AT&T Bell Laboratories, Murray Hill, USA) 28-30 0ctober, 1 November. L.A. Wolsey (CORE, Louvain-la-Neuve, Belgium) 13 December. H. Yoshikawa (University of Tokyo, Japan) 5 November.

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