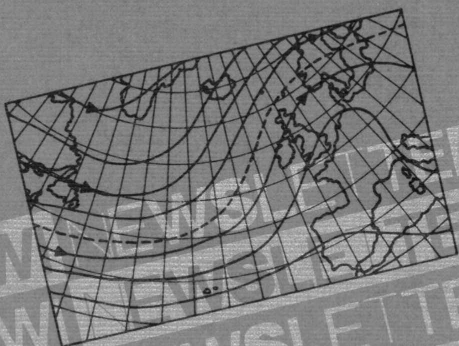
CWI NEWSLETTER

Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

Quarterly, Issue no. 16
September 1987



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The CWI Newsletter is published quarterly by the Centre for Mathematics and Computer Science (Centrum voor Wiskunde en Informatica), Kruislaan 413, 1098 SJ Amsterdam, The Netherlands. The Newsletter will report on activities being conducted at the Centre and will also contain articles of general interest in the fields of Mathematics and Computer Science, including book reviews and mathematical entertainment. The editors encourage persons outside and in the Centre to contribute to the Newsletter. Normal referee procedures will apply to all articles submitted.

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ISSN 0168-826x

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The Centre for Mathematics and Computer Science (CWI) is the research institute of the Stichting Mathematisch Centrum (SMC), which was founded on 11 February 1946.

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- Applied Mathematics;
- Mathematical Statistics;
- Operations Research and System Theory;
- Numerical Mathematics;
- Software Technology;
- Algorithms and Architecture;
- Interactive Systems

There are also a number of supporting sectors, in particular the Computer Systems and Telematics Sector, and an extensive Library.

The subdivision of the research into eight departments is less rigid than it appears, for there exists considerable collaboration between the departments. This is a matter of deliberate policy, not only in the selection of research topics, but also in the selection of the permanent scientific staff.

Group Induced Orderings with some Applications in Statistics

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We discuss sufficient conditions on a compact group G for a function be decreasing with respect to certain group induced orderings, and present a class of composition theorems. We give an application of group induced orderings to linear statistical models, in particular a new proof of the Gauss-Markov Theorem. Furthermore, we indicate a possible application of such orderings to general experimental design problems.

1. Introduction

The origins of group induced orderings date back at least to the work of ADO [33]. In a paper concerned with majorization and variations thereof, Rado observed that classical majorization (see MARSHALL and OLKIN [24], Chapter 1 for an historical sketch concerning majorization) is equivalent to a pre-ordering defined by the group of permutation matrices.ecall that for two column vectors x, y in $\mathbb{R}^n$, x is *majorized* by y (often written $x \leqslant y$) if the conditions

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \ , \ k = 1,...,n-1 \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]} \end{cases} \tag{1.1}$$

are satisfied where $x_{[1]} \geqslant ... \geqslant x_{[n]}$ and $y_{[1]} \geqslant ... \geqslant y_{[n]}$ are the ordered coordinates of x and y. An important characterization of majorization due to HARDY, LITTLEWOOD and POLYA [19] is that

$$x \leqslant y \quad \text{iff} \quad x = Py \tag{1.2}$$

where P is an $n \times n$ doubly stochastic matrix.

Now, let $\mathcal{P}_n$ denote the group of $n \times n$ permutation matrices. BIRKHOFF [3] proved that $\mathcal{P}_n$ is exactly the set of extreme points of the convex set of doubly stochastic matrices. Thus each doubly stochastic matrix has the representation

* This work was supported in part by National Science Foundation Grant No. DMS 83-19924.

$$P = \sum_g \alpha_g g \tag{1.3}$$

where the sum runs over $\mathcal{P}_n$ and the non-negative weights α_g satisfy $\Sigma\alpha_g = 1$. Combining (1.2) and (1.3) shows that

$$x \leqslant y \quad \text{iff} \quad x = \sum_g \alpha_g g y \tag{1.4}$$

for some set of non-negative weights α_g adding up to 1. The set $O_y = \{gy \mid g \in \mathcal{P}_n\}$ is the *orbit* of y under the action of the group $\mathcal{P}_n$ on $\mathbb{R}^n$. Further, the convex hull of O_y consists of points of the form

$$x = \Sigma\alpha_g g y$$

and is denoted by $C(y)$. We are thus led to Rado's observation that

$$x \leqslant y \quad \text{iff} \quad x \in C(y). \tag{1.5}$$

Equation (1.5) was then used by ADO [33] as a definition to study relatives of majorization defined by subgroups of $\mathcal{P}_n$. More precisely, if G is any subgroup of $\mathcal{P}_n$, define $x \leqslant (G) y$ to mean $x \in C_G(y)$ where $C_G(y)$ denotes the convex hull of the set $\{gy \mid g \in G\}$.

The idea of group induced orderings on $\mathbb{R}^n$ arose in quite a different context in MUDHOLKAR [27]. Given a compact subgroup G of the orthogonal group O_n, write

$$x \leqslant y \quad \text{iff} \quad x \in C(y) \tag{1.6}$$

where again $C(y)$ denotes the convex hull of the orbit $O_y = \{gy \mid g \in G\}$. The dependence of $\leqslant$, $C(y)$ and O_y on G is suppressed notationally. A real valued function f defined on $\mathbb{R}^n$ is *decreasing* if

$$x \leqslant y \text{ implies } f(x) \geqslant f(y). \tag{1.7}$$

Mudholkar's result gives a sufficient condition that the convolution of two functions be decreasing.

THEOREM 1 (MUDHOLKAR [27]). *Suppose f_1 and f_2 are non-negative measurable functions defined on $\mathbb{R}^n$ which satisfy*
(i) $f_i(x) = f_i(gx)$, $x \in \mathbb{R}^n$, $g \in G$, $i = 1,2$;
(ii) *for each* $c > 0$ *and* $i = 1,2$, $\{x \mid f_i(x) \geqslant c\}$ *is a convex set.*
If

$$h(y) = \int f_1(y-x) f_2(x) dx$$

is finite for each $y \in \mathbb{R}^n$, then h is decreasing in the sense of (1.7).

The impetus for Mudholkar's work as well as some more recent work on group induced orderings has come from problems in multivariate probability inequalities. Such problems often involve obtaining tight upper and/or lower bounds on a function defined on $\mathbb{R}^n$ or some subset of $\mathbb{R}^n$. To see how group induced orderings are applied to such problems, again let G be a compact subgroup of

O_n and let $\leqslant$ denote the pre-ordering defined by G. Thus, $x \leqslant y$ iff $x \in C(y)$. Consider a real valued function f defined on $\mathbb{R}^n$ which satisfies

$$\begin{cases} \text{(i)} \ f(x)=f(gx); \quad x \in \mathbb{R}^n, \ g \in G \\ \text{(ii)} \ f \text{ is concave.} \end{cases} \tag{1.8}$$

First observe that f satisfies (1.7). To see this consider $x \leqslant y$, so

$$x = \sum_g \alpha_g gy. \tag{1.9}$$

From (1.8), we have

$$f(x) = f(\Sigma \alpha_g gy) \geqslant \Sigma \alpha_g f(gy) = \Sigma \alpha_g f(y) = f(y).$$

Thus, concave invariant functions are necessarily decreasing in the sense of (1.7) and lower bounds on $f(x)$ are obtained when $x \in C(y)$. Upper bounds on f satisfying (1.8) are obtained via the following observation. Given any y, let

$$\mathbf{y} = \int gy \nu(dg)$$

where ν is the unique invariant probability measure on the compact group G. Obviously $\mathbf{y} \leqslant y$ since $\mathbf{y}$ is a 'convex combination' of points in the orbit of y. In fact, $\mathbf{y}$ is the smallest element in $C(y)$ in the sense that $x \in C(y)$ implies $\mathbf{y} \leqslant x$. To see this, observe that $\mathbf{x} \leqslant x$ and for $x \in C(y)$ we have

$$x = \sum_g \alpha_g gy.$$

Therefore the invariance of ν yields

$$\begin{aligned} \mathbf{x} &= \int hx \nu(dh) = \int h(\Sigma \alpha_g gy) \nu(dh) = \Sigma \alpha_g \int hgy \nu(dh) \\ &= \Sigma \alpha_g \int hy \, \nu(dh) = \Sigma \alpha_g \mathbf{y} = \mathbf{y}. \end{aligned}$$

Thus, for f satisfying (1.8), the double inequality

$$f(\mathbf{y}) \geqslant f(x) \geqslant f(y) \tag{1.10}$$

is valid for all $x \in C(y)$. Further (1.10) is sharp in the sense that there are points in $C(y)$ so that both of the inequalities are equalities.

It is inequality (1.10) which has proved to be so useful in many applications. When $G = \mathcal{P}_n$, the book by MARSHALL and OLKIN [24] provides a host of examples. The main focus of this paper is a discussion of conditions on a compact group G so that usable sufficient conditions can be given which imply that a function is decreasing, and thus that (1.10) holds. In the case that $G = \mathcal{P}_n$, there are three general sets of conditions on a function f which imply that f is decreasing. A differential condition due to OSTROWSKI [30] is discussed in MARSHALL and OLKIN ([24], p. 57). A second type of condition, established by MARSHALL and OLKIN [23], shows that the convolution of two decreasing functions is again decreasing. Both sets of conditions were shown to have complete analogues when the group G is a reflection group (see EATON and PERLMAN [13]). A third set of conditions involves the so-called composition theorem and convolution families of probability densities (see PROSCHAN and SETHURAMAN [31], HOLLANDER, PROSCHAN and SETHURAMAN [20], and NEVIUS, PROSCHAN and SETHURAMAN [29]). These are the types of conditions on which our discussion centers.

General group induced orderings are introduced in Section 2. The line of development described here comes from EATON [6, 9, 10]. This development provides a description of what is currently known concerning differential conditions which imply that a function is decreasing (as defined in (1.7)). After presenting two standard examples, we apply the theory to give a group induced ordering on real skew symmetric matrices.

In Section 3, we discuss a class of composition theorems which yield sufficient conditions for certain functions to be decreasing. These theorems have applications in probability and statistics via multivariate probability inequalities - for example, see INOTT [34], MARSHALL and OLKIN [23], EATON and PERLMAN [13], PROSCHAN and SETHURAMAN [31], MARSHALL and OLKIN [24], TONG [37], EATON [7], EATON [9], and EATON [10].

An application of group induced orderings to linear statistical models is presented in Section 4. A new proof of the classical Gauss-Markov Theorem is given. Under slightly strengthened assumptions, this classical result is then extended to a more general class of loss functions.

In Section 5, we discuss some open problems connected with group induced orderings. In addition, we indicate a possible application of such orderings to experimental design problems.

Before beginning a general discussion of group induced orderings, it is useful to consider an example which is prototypical of many statistical applications of such orderings. This example concerns what might be called the k-sample Behrens-Fisher problem and its solution dates back to HSU [21] and HAJEK [17].

EXAMPLE 1. Consider random samples from k normal populations, say X_{ij}, $j=1,...,n_i+1$ and $i=1,...,k$ where the distribution of X_{ij} is

$$\mathcal{L}(X_{ij}) = N(\mu_i, \sigma_i^2).$$

Here the mean μ_i and the variance σ_i^2 are both unknown. The problem is to construct a confidence interval (perhaps approximate) for a known linear combination of the means - say

$$\theta = \sum_i c_i \mu_i$$

with $c_1,...,c_k$ known constants. The sample means

$$\overline{X}_i = (n_i+1)^{-1} \sum_j X_{ij}$$

and the sample variances

$$s_i^2 = n_i^{-1} \sum_j (X_{ij} - \overline{X}_i)^2$$

are the MVUE (Minimum variance unbiased estimators) for the population means and variances respectively. Thus

$$\hat{\theta} = \sum_i c_i \overline{X}_i$$

is the MVUE for θ and

$$\mathfrak{L}(\hat{\theta}) = N(\theta,\tau^2)$$

where

$$\tau^2 = \sum_i c_i^2 (n_i+1)^{-1} \sigma_i^2.$$

Further,

$$\hat{\tau}^2 = \sum_i c_i^2 (n_i+1)^{-1} s_i^2$$

is the MVUE for τ^2 so it seems reasonable to try to construct a confidence interval for θ based on the approximate pivotal quantity

$$W = \frac{\hat{\theta}-\theta}{\hat{\tau}}.$$

For a fixed constant d, the interval $(\hat{\theta}-d\hat{\tau},\ \hat{\theta}+d\hat{\tau})$ has confidence coefficient

$$\psi = P\left\{\frac{(\hat{\theta}-\theta)^2}{\hat{\tau}^2} \leqslant d^2\right\}$$

where ψ is a function of $\sigma_1^2,...,\sigma_k^2$. Thus, the assessment of the above interval as an inferential procedure depends on finding upper and more importantly, lower bounds on ψ. To this end, set

$$Z = \left\{\frac{\hat{\theta}-\theta}{\tau}\right\}^2$$

so Z has the χ_1^2 distribution (chi-square with one degree of freedom distribution). Now, define w_{ij} by

$$w_{ij} = \frac{c_i^2 (n_i+1)^{-1} n_i^{-1} \sigma_i^2}{\tau^2} \quad , j=1,...,n_i$$

for $i=1,...,k$. Obviously $0 \leqslant w_{ij}$ and

$$\sum_i \sum_j w_{ij} = 1.$$

Because $(n_i s_i^2)/\sigma_i^2$ has a $\chi_{n_i}^2$ distribution, it follows easily that $\hat{\tau}^2/\tau^2$ has the same distribution as

$$V = \sum_i \sum_j w_{ij} U_{ij}$$

where $\{U_{ij} \mid j=1,...,n_i;\ i=1,...,k\}$ is a collection of $n=\Sigma n_i$ i.i.d. (independent and identically distributed) χ_1^2 random variables.

The analysis above and the independence of $\hat{\theta}$ and $\hat{\tau}$ show that

$$\psi = \psi(w) = P\{Z \leqslant d^2(\sum_i \sum_j w_{ij} U_{ij})\}$$

where w is the n-dimensional vector with coordinate w_{ij}, and Z is independent of the U_{ij}. Therefore bounding ψ involves studying $\psi(w)$. For notational convenience, the double subscript notation is now dropped and we consider vectors w in $\mathbb{R}^n$ which satisfy

(i) $0 \leqslant w_i$, $i = 1,...,n$;

(ii) $\sum_1^n w_i = 1$;

(iii) n_1 coordinates of w are the same, n_2 coordinates of w are the same,...,n_k coordinates of w are the same where $n = \Sigma n_i$.

Let $A \subseteq \mathbb{R}^n$ be the set of w's satisfying these conditions. The function which needs to be bounded is

$$\psi(w) = P\{Z \leqslant d^2 w'U\}$$

where U is an n-vector of i.i.d. χ_1^2 random variables and w' is the transposed of w. Because Z and U are independent, $\psi(w)$ can be written

$$\psi(w) = \mathcal{E}(F(d^2 w'U))$$

where F is the distribution function of Z. Since Z is χ_1^2, F is a concave function so that ψ is a concave function.

Now, let $\mathcal{P}_n$ be the group of $n \times n$ permutation matrices. Since the coordinates of U are i.i.d., it follows that

$$\mathcal{L}(U) = \mathcal{L}(gU), \quad g \in \mathcal{P}_n.$$

In other words, U is exchangeable and so $\psi(w) = \psi(gw)$ for $g \in \mathcal{P}_n$. Thus ψ satisfies (1.8) and hence the analysis leading to (1.10) is valid. In particular, for any $w \in A$, the vector

$$\mathbf{w} = \frac{1}{n!} \sum_g gw$$

satisfies $g\mathbf{w} = \mathbf{w}$ for all $g \in P_n$. This implies that

$$\mathbf{w} = \frac{1}{n}[1,1,...,1]'$$

and hence $\psi(w) \leqslant \psi(\mathbf{w})$ for all $w \in A$. A moment's reflection shows that

$$\psi(\mathbf{w}) = P\{F_{1,n} \leqslant d^2\}$$

where $F_{1,n}$ has the F-distribution with 1 and n degrees of freedom.

A lower bound for ψ on the set A is obtained as follows. Recall that n_1 is the smallest sample size. Define $\overline{w}$ by

$$\overline{w} = \frac{1}{n_1}[1,1,...,1,0,0,...,0]' \in A$$

where $\bar{w}$ has n_1 coordinates equal to one and the remainder are zero. The classical definition (1.1) of majorization yields $w \leqslant \bar{w}$ for all $w \in A$ so that $w \in C(\bar{w})$. Hence

$$\psi(\bar{w}) \leqslant \psi(w), \quad w \in A.$$

Again, it is easy to show

$$\psi(\bar{w}) = P\{F_{1,n_1} \leqslant d^2\}$$

so that computable tight upper and lower bounds on $\psi(w)$ have been found.

2. Group induced orderings

Our formal treatment of group induced orderings is restricted to the finite dimensional case and to the case that the group is a compact group of linear transformations. More precisely, let $(V, (\cdot,\cdot))$ be a finite dimensional inner product space. As usual $GL(V)$ denotes the group of non-singular linear transformations on V. The orthogonal group of $(V, (\cdot,\cdot))$ is

$$O(V) = \{g \mid g \in GL(V), (gx, gx) = (x, x) \text{ for } x \in V\}.$$

In what follows, G is a closed subgroup of $O(V)$ so G is compact. Given $x \in V$, $O_x = \{gx \mid g \in G\}$ is the *orbit* of x and $C(x)$ denotes the convex hull of O_x. Because G is compact, both O_x and $C(x)$ are compact subsets of V.

Definition 2.1. For x, $z \in V$, write $z \leqslant x$ iff $z \in C(x)$. The dependence of $\leqslant$ on G is suppressed notationally. Here are some easily verifiable facts about the relation $\leqslant$.

Proposition 2.1. *For* $x \in V$
(i) $gC(x) = C(gx) = C(x), \quad g \in G$;
(ii) $z \leqslant x$ *iff* $g_1 z \leqslant g_2 x$ *for some* $g_1, g_2 \in G$;
(iii) $z \in C(x)$ *iff* $C(z) \subseteq C(x)$;
(iv) $z \leqslant y$ *and* $y \leqslant x$ *implies* $z \leqslant x$;
(v) $z \leqslant x$ *and* $x \leqslant z$ *iff* $z \in O_x$.

Proof. Property (i) follows from the invariance of the orbit O_x and the fact that

$$O_x = O_{gx}, \quad g \in G.$$

(ii) follows directly from (i). For (iii), $C(z) \subseteq C(x)$ obviously implies $z \in C(x)$. Conversely, $z \in C(x)$ implies $gz \in C(x)$ for all $g \in G$ by (ii). Thus $C(z) \subseteq C(x)$ since $C(x)$ is convex. If $z \leqslant y$ and $y \leqslant x$, then by (iii) $C(z) \subseteq C(y) \subseteq C(x)$ so $z \leqslant x$ and (iv) holds. To prove (v), if $z \in O_x$, then $z = gx$ for some $g \in G$ so by (ii) $z \leqslant x$ and $x \leqslant z$. Conversely, assume $z \leqslant x$ and $x \leqslant z$. Then for some integer r,

$$z = \sum_{i=1}^{r} \alpha_i g_i x$$

where $g_1x,...,g_rx$ are distinct vectors, $0 \leqslant \alpha_i$ and $\Sigma\alpha_i = 1$. Thus,

$$||z|| = ||\Sigma\alpha_i g_i x|| \leqslant \Sigma\alpha_i||g_i x|| = \Sigma\alpha_i||x|| = ||x||. \qquad (2.1)$$

Similarly $||x|| \leqslant ||z||$ so $||x|| = ||z||$. But there is equality in the inequality (2.1) iff all the α_i except one are zero because the norm $||\cdot||$ derived from an inner product is strictly convex. Thus, $z \in O_x$. □

The relation $\leqslant$ is called a *pre-ordering* in what follows. (The term 'ordering' is usually reserved for relations which are reflexive, transitive and $x \leqslant y \leqslant x$ implies $x = y$.) A real valued function f on V is *decreasing* if $x \leqslant y$ implies that $f(x) \geqslant f(y)$. If $-f$ is decreasing, then f is *increasing.* Observe that any decreasing function f must satisfy

$$f(x) = f(gx), \quad x \in V, \ g \in G$$

because $x \leqslant gx \leqslant x$ for all x,g.

In order to decide whether or not $z \leqslant x$, it is necessary to have a verifiable criterion to decide whether or not $z \in C(x)$. The use of support functions for this purpose was developed in EATON [6, 9] and in GIOVAGNOLI and WYNN [16]. Given x, $u \in V$, define m on $V \times V$ by

$$m[u,x] = \sup_{g \in G}(u,gx). \qquad (2.2)$$

The use of the square brackets in the definition of m is to distinguish $m[\cdot,\cdot]$ from the inner product $(\cdot,\cdot)$ on the right hand side of (2.2).

PROPOSITION 2.2. *The function m satisfies*
(i) $m[u,x] = m[x,u]$;
(ii) $m[g_1u,g_2x] = m[u,x]$ *for* $g_1,g_2 \in G$;
(iii) $z \leqslant x$ *iff* $m[u,z] \leqslant m[u,x]$ *for all* $u \in V$.

PROOF. Properties (i) and (ii) follow from the fact that G is a subgroup of $O(V)$. For (iii), if $z \leqslant x$, then

$$z = \Sigma\alpha_i g_i x$$

as in (1.1). Thus

$$m[u,z] = \sup_g(u,gz) = \sup_g(u,g(\Sigma\alpha_i g_i x)) = \sup_g \Sigma_{\alpha_i}(u,gg_i x) \leqslant$$
$$\Sigma\alpha_i \sup_g(u,gg_i x) = \Sigma\alpha_i \sup_g(u,gx) = \Sigma\alpha_i m[u,x] = m[u,x].$$

That the right-hand side of (iii) implies $z \leqslant x$ can be proved directly from the Separating Hyperplane Theorem (see EATON [10], Proposition A.3). Alternatively, the fact that $u \mapsto m[u,x]$ is the support function of $C(x)$ (see ROCKAFELLER [35], Chapter 13) can be used to give a proof. □

Part (ii) of Proposition 2.2 shows that m is an invariant function of each of its arguments. Thus m is determined by its values on the quotient space V/G. In all of the applications that I know, it is possible to 'represent' V/G by a convex cone contained in V. Further, this representation turns out to be important in characterizing the pre-ordering $\leq$.

At this point in our discussion, we restrict our attention to the *group induced cone orderings.* In essence these are the pre-orderings where we know a differential characterization of the decreasing functions.

DEFINITION 2.2. The pre-ordering $\leq$ defined on $(V,(\cdot,\cdot))$ by G is a *group induced cone ordering* if there exists a closed (non-empty) convex cone $F \subseteq V$ such that

(i) for each $x \in V$, $O_x \cap F$ is not empty;

(ii) for $u, x \in F$, $m[u,x] = (u,x)$.

Condition (i) says that each orbit intersects F. Since the relation $x \leq y$ is invariant in both x and y, it is sufficient to characterize $\leq$ for $x, y \in F$. Condition (ii) simply says that the support function m is just the inner product when restricted to $F \times F$. Let M be the linear span of F so that F has a non-empty interior as a subset of the linear space M. Further, let

$$F_M^\star = \{w \in M \mid (w,x) \geq 0 \text{ for all } x \in F\}.$$

Thus, $F_M^\star$ is the dual cone of F relative to the subspace M.

PROPOSITION 2.3. *Assume $\leq$ is a group induced cone ordering. For $x, y \in F$, the following are equivalent:*

(i) $x \leq y$;

(ii) $y - x \in F_M^\star$.

PROOF. When $x \leq y$, Proposition 2.2 (iii) together with Definition 2.2 (ii) shows that for $u \in F$

$$(u,x) = m[u,x] \leq m[u,y] = (u,y).$$

so $y - x \in F_M^\star$. For the converse, just read the above argument backwards. □

Proposition 2.3 shows that $\leq$ is a cone ordering on F as defined in MARSHALL, WALKUP and WETS [25]. The convex cone which defines the cone ordering is $F_M^\star$ while the domain of definition of the ordering is F. Recall that a subset $T^\star \subseteq F_M^\star$ is a *positive spanning set* for $F_M^\star$ if every element u of $F_M^\star$ has the form

$$u = \sum_1^r a_i t_i$$

where $t_i \in T^\star$, $a_i \geq 0$ for $i = 1,...,r$ and r is some positive integer. A positive spanning set $T^\star \subseteq F_M^\star$ is a *frame* for $F_M^\star$ if no proper subset of $T^\star$ is a positive spanning set. A direct application of the results in MARSHALL, WALKUP and WETS [25] yields the following necessary and sufficient condition that an invariant function with a differential be decreasing when $\leq$ is a group induced cone ordering.

THEOREM 2.1. *Suppose $\leqslant$ is a group induced cone ordering on $(V,(\cdot,\cdot))$ with F and $F_M^{\star}$ as above. Let f be a real valued function which is invariant (i.e. $f(x)=f(gx)$ for $x \in V$ and $g \in G$), and suppose f has a differential df. Let $T^{\star}$ be a positive spanning set for $F_M^{\star}$. The following are equivalent:*
(i) *$x \leqslant y$ implies $f(x) \geqslant f(y)$ for all $x,y \in V$;*
(ii) *$(t, df(x)) \leqslant 0$ for all $x \in F$ and $t \in T^{\star}$.*

In applications of Theorem 2.1, one tries to find a frame $T^{\star}$ for $F_M^{\star}$ when attempting to verify (ii). In the following example, we show that the above theory applies and yields the classical results concerning majorization.

EXAMPLE 2.1. (Majorization). Let $V=\mathbb{R}^n$ with the usual inner product and consider the pre-ordering $\leqslant$ induced by the group of permutation matrices $\mathcal{P}_n$. The usual choice for the convex cone F is

$$F = \{x \mid x_1 \geqslant ... \geqslant x_n\}$$

where $x_1,...,x_n$ are the coordinates of x. Obviously, every orbit intersects F. Since F has non-empty interior, $M=\mathbb{R}^n$ for this example. The fact that

$$m[u,x] = \sup_g u'gx = u'x$$

for $x,u \in F$ is the famous rearrangement inequality of HARDY, LITTLEWOOD and POLYA ([19], p.261). Thus, we see that $\leqslant$ is a group induced cone ordering (as in Definition 2.2).

The dual cone of F is easily shown to be

$$F^{\star} = \{u \mid \sum_1^k u_i \geqslant 0,\ k=1,...,n-1,\ \sum_1^n u_i = 0\}.$$

A frame for $F^{\star}$ is

$$T^{\star} = \{t_1,...,t_{n-1}\}$$

where $t_1 \in \mathbb{R}^n$ has its ith coordinate equal to one, its $(i+1)$st coordinate equal to minus one, and all other coordinates equal to zero. Proofs of these assertions can be found in EATON [10].

For $x, y \in F$, Proposition 2.2 shows that $x \leqslant y$ iff $y-x \in F^{\star}$ iff

$$\sum_1^k y_i \geqslant \sum_1^k x_i, \qquad k=1,...,n-1$$

$$\sum_1^n y_i = \sum_1^n x_i.$$

These are just the classical conditions for majorization for elements of F. For elements not in F, one simply permutes the coordinates so the permuted vector is in F, and then applies the above conditions.

Now, let f be a $\mathcal{P}_n$ invariant real valued function defined on $\mathbb{R}^n$ and assume f has a differential df. Theorem 2.1 shows that f is decreasing iff

$$t'_i(df(x)) \leqslant 0, \quad i = 1, \ldots, n, \quad x \in F$$

which is easily seen to be equivalent to the conditions

$$\frac{\partial f}{\partial x_1}(x) \leqslant \ldots \leqslant \frac{\partial f}{\partial x_n}(x), \quad x \in F.$$

These are exactly the OSTROWSKI [30] conditions for f to be decreasing (Schur concave). This completes Example 2.1.

EXAMPLE 2.2. For this example, take V to be the real vector space of $n \times n$ real symmetric matrices with inner product

$$(x,y) = \operatorname{tr} xy$$

where tr denotes the trace. Let O_n act on V by

$$x \rightarrow gxg'$$

for $x \in V$ and $g \in O_n$. The Spectral Theorem for real symmetric matrices implies that for each x, there is a $g \in O_n$ such that

$$z = gxg'$$

is an $n \times n$ diagonal matrix with diagonal elements z_{ii} which satisfy $z_{11} \geqslant \ldots \geqslant z_{nn}$. Thus, the convex cone

$$F = \{z \mid z \in V, \ z \text{ is diagonal}, \ z_{11} \geqslant \ldots \geqslant z_{nn}\}$$

intersects every orbit under the action of O_n on V. For u, $x \in F$,

$$m[u,x] = \sup_g \operatorname{tr} ugxg' = \sum_{i=1}^{n} u_{ii} x_{ii} = \operatorname{tr} ux = (u,x).$$

The second equality is a consequence of results of VON NEUMANN [28] and FAN [14] (see also Example 6.4 in EATON [10]). Hence the pre-ordering $\leqslant$ induced on V by O_n is a group induced cone ordering.

It is clear that the subspace M generated by F is just the space of all $n \times n$ real diagonal matrices. Using the results of Example 2.1, it is routine to show that the dual cone F^*_M (of F in M) is

$$F^*_M = \{z \mid z \in M, \ \sum_1^k z_{ii} \geqslant 0, \ k = 1, \ldots, n-1, \ \sum_1^n z_{ii} = 0\}.$$

As in Example 2.1, a frame for F^*_M is

$$T^* = \{t_1, \ldots, t_n\}$$

where $t_i \in F^*_M$ has its (i,i) element equal to one, its $(i+1, \ i+1)$ element equal to minus one, and all other elements are zero.

Given $x \in V$, when $gxg' = z$ is in F, then the diagonal elements of z are just the ordered eigenvalues of x. To interpret what the pre-ordering $\leqslant$ means in terms of eigenvalues, consider $x,y \in V$ and write

$$z = g_1 x g_1', \quad w = g_2 y g_2'$$

with z and w in F. Then $x \leqslant y$ iff $z \leqslant w$ iff $w - z \in F_M^*$ iff

$$\sum_1^k w_{ii} \geqslant \sum_1^k z_{ii}, \; k = 1,...,n-1; \; \sum_1^n w_{ii} = \sum_1^n z_{ii}.$$

In other words, $x \leqslant y$ iff the eigenvalues of y majorize the eigenvalues of x. This was proved by KARLIN and RINOTT [22] from first principles, by ALBERTI and UHLMANN [1] in a book related to mathematical physics, and by EATON [6, 9] using the general theory of group induced cone orderings described above.

To describe the decreasing functions, first note that if f is decreasing, then $f(x)$ is only a function of the eigenvalues of x. Because of the above characterization of $\leqslant$ in terms of majorization, f is decreasing on V iff as a function of the eigenvalues of x, it is decreasing in the sense of majorization (as in Example 2.1).

Here is a new example of a group induced cone ordering.

EXAMPLE 2.3. Let V be the real vector space of $n \times n$ real skew symmetric matrices, with inner product $(x,y) = \operatorname{tr} xy'$. The case of n even, say $n = 2r$, is treated below. When n is odd, the details are slightly different, but the same general argument applies. The group O_n acts on V via

$$x \rightarrow gxg'; \; x \in V, \; g \in O_n.$$

This group action produces a canonical form for x which can be described as follows. Let $E_1,...,E_r$ be defined by

$$E_i = \begin{bmatrix} 0 & & \cdots & & 0 \\ & \ddots & & & \\ \vdots & & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & & \\ & & & \ddots & \\ 0 & & \cdots & & 0 \end{bmatrix}$$

where the 2×2 block

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is located on the diagonal in rows and columns $2i-1$ and $2i$, $i = 1,...,r$. Given $x \in V$, there exists a $g \in O_n$ such that

$$gxg' = \sum_{i=1}^r \theta_i E_i$$

where the real numbers $\theta_1,\ldots,\theta_r$ satisfy

$$\theta_1 \geqslant \theta_2 \geqslant \ldots \geqslant \theta_r \geqslant 0.$$

For a proof of this standard result, see MEHTA ([26], p. 221). Thus the convex cone

$$F = \{x \mid x = \sum_1^r \theta_i E_i \ \text{ with } \theta_1 \geqslant \ldots \geqslant \theta_r \geqslant 0\}$$

intersects every orbit under the action of O_n on V. When $x \in F$, say

$$x = \sum_1^r \theta_i E_i,$$

then the singular values of x (by definition, the singular values are the ordered non-negative square roots of the ordered eigenvalues of xx') are easily shown to be

$$\theta_1, \theta_1, \theta_2, \theta_2, \ldots, \theta_r, \theta_r.$$

The results of VON NEUMANN [28] and FAN [14] show that for

$$x = \sum_1^r \theta_i E_i \text{ and } u = \sum_1^r \alpha_i E_i \text{ in } F,$$

we have

$$m[u,x] = \sup_g \operatorname{tr} u(gxg')' = 2 \sum_1^r \alpha_i \theta_i = \operatorname{tr} ux' = (u,x).$$

Therefore O_n induces a cone ordering $\leqslant$ on V as in Definition 2.2.

To describe the pre-ordering $\leqslant$ more completely, let

$$M = \{x \mid x = \sum_1^r \alpha_i E_i,\ a_i \in \mathbb{R},\ i = 1,\ldots,r\}.$$

Clearly M is the linear subspace of V generated by F. It is not too hard to show that the dual cone of F in M is

$$F_M^* = \{x \mid x = \sum_1^r a_i E_i,\ \sum_1^k a_i \geqslant 0,\ k = 1,\ldots,r\}.$$

Therefore, for $x,y \in F$, say

$$x = \sum_1^r \theta_i E_i \text{ and } y = \sum_1^r \eta_i E_i,$$

we see that $x \leqslant y$ iff

$$\sum_1^k \eta_i \geqslant \sum_1^k \theta_i, \quad k = 1,\ldots,r. \tag{2.3}$$

This relationship among $\theta_1 \geqslant \ldots \geqslant \theta_r \geqslant 0$ and $\eta_1 \geqslant \ldots \geqslant \eta_r \geqslant 0$ is sometimes called *submajorization* - that is, the vector of θ's is submajorized by the vector of η's (see the discussion in MARSHALL and OLKIN ([24], p. 10) and in EATON ([10], Example 6.2, p. 157)).

For x and y in V, the relation $x \leqslant y$ can be described as follows. Let $\theta_1,\theta_1,...,\theta_r,\theta_r$ be the singular values of x and let $\eta_1,\eta_1,...,\eta_r,\eta_r$ be the singular values of y. Then $x \leqslant y$ iff the singular values of y submajorize the singular values x - that is, iff the inequalities

$$\sum_1^k \eta_i \geqslant \sum_1^k \theta_i,\ k = 1,...,r$$

hold. These inequalities are related to the group induced cone ordering given in Example 6.2 in EATON [10].

Finally, suppose f is an O_n-invariant function defined on V. Then f is determined by its values on F so we write

$$h(\theta) = f(\sum_1^r \theta_i E_i) \text{ for } \sum_1^r \theta_i E_i \text{ in } F.$$

Assume h has a differential. It follows from MARSHALL, WALKUP and WETS [25] that the conditions

$$\frac{\partial h}{\partial \theta_1}(\theta) \leqslant ... \leqslant \frac{\partial h}{\partial \theta_r}(\theta) \leqslant 0 \tag{2.4}$$

imply that

$$h(\theta) \geqslant h(\eta)$$

whenever (2.3) holds. Thus the conditions (2.4) imply that an invariant function f is decreasing.

Other examples of group induced cone orderings can be found in EATON and PERLMAN [13], ALBERTI and UHLMANN [1], EATON [6,9] and EATON [10].

3. Composition theorems

For group induced cone orderings, the results of Theorem 2.1 provide necessary and sufficient conditions for a differentiable invariant function to be decreasing. These conditions are certainly the most widely used for proving that functions are decreasing. However, in special situations there are other sufficient conditions which are sometimes easier to verify than the differential condition. In this section, we review a few of the main results.

Here is a common situation in probability and statistics to which group induced orderings and the double inequality (1.10) can sometimes be applied. Let $\mathcal{X} \subseteq \mathbb{R}^k$ be the sample space of a random vector. Also, let $\Theta \subset \mathbb{R}^k$ be a parameter space for a class of probability models for X. Assume that λ is a σ-finite measure on the Borel sets of $\mathcal{X}$ and assume that X has a density (with respect to λ) $f(\cdot \mid \theta)$ where $\theta \in \Theta$. For any integrable function h, consider

$$\psi(\theta) = \mathcal{E}_\theta h(X) = \int_{\mathcal{X}} h(x) f(x \mid \theta) \lambda(dx). \tag{3.1}$$

The question is: Under what conditions on h, $f(\cdot|\cdot)$, and λ can we hope to apply the ideas of group induced orderings in order to conclude that ψ is decreasing (or increasing)? Notice that Mudholkar's result mentioned in Section 1 provides one set of sufficient conditions that ψ be decreasing when θ is a translation parameter.

To give another example, let $\mathcal{X} \subseteq \mathbb{R}^k$ be the set of vectors x whose coordinates $x_1, \ldots, x_k$ are non-negative integers which satisfy

$$\sum_{1}^{k} x_i = n.$$

Here n is a fixed positive integer. Take λ to be counting measure on $\mathcal{X}$. Let $\Theta \subseteq yk$ be the set of θ's with coordinates $\theta_1, \ldots, \theta_k$ which satisfy

$$\theta_i \geq 0, \ \sum_{1}^{k} \theta_i = 1.$$

The density of the multinomial distribution, $\mathfrak{M}(k, \theta, n)$ is

$$f(x \mid \theta) = \frac{n!}{x_1! \ldots x_k!} \prod_{i=1}^{k} \theta_i^{x_i}, \ x \in \mathcal{X}.$$

The group $\mathcal{P}_k$ of permutation matrices acts on $\mathcal{X}$ and Θ. Thus we have the group induced pre-ordering $\leqslant$ on both $\mathcal{X}$ and Θ.

THEOREM 3.1 (RINOTT [34]). *Suppose h is a real valued function defined on $\mathcal{X}$ which is decreasing. Then*

$$\psi(\theta) = \mathcal{E}_\theta h(X) = \int_{\mathcal{X}} h(x) f(x \mid \theta) \lambda(dx)$$

is a decreasing function defined on Θ.

Rinott's proof consists of showing that ψ satisfies the differential conditions of Example 2.1. NEVIUS, PROSCHAN and SETHURAMAN [29] developed another method for establishing this result which is discussed later in this section.

MARSHALL and OLKIN [23] established a convolution theorem which strengthens Mudholkar's Theorem in the case that the group is $\mathcal{P}_k$ is acting on yk.

THEOREM 3.2 (MARSHALL and OLKIN [23]). *Suppose f_1 and f_2 are non-negative functions defined on $\mathbb{R}^k$ which are decreasing (in the pre-ordering of majorization). If*

$$f_3(\theta) = \int_{\mathbb{R}^k} f_1(x) f_2(x - \theta) dx$$

exists for $\theta \in yk$, then f_3 is decreasing.

These two theorems turn out to be closely connected with the fact that $\mathcal{P}_k$ is a reflection group. To explain the connection, we now turn to a discussion of such groups. In the inner product space $(V,(\cdot,\cdot))$, Let u be a vector of length one. Define the linear transformation R_u by

$$R_u x = x - 2(u,x)u, \; x \in V.$$

Clearly $R_u u = -u$, $R_u x = x$ if $(u,x)=0$ and $R_u = R_u^{-1}$. Thus, $R_u \in O(V)$ reflects vectors across the hyperplane $\{x \mid (u,x)=0\}$. Any such transformation is a *reflection.*

DEFINITION 3.1. A closed group $G \subseteq O(V)$ is a *reflection group* if there is some set of reflections $\mathcal{R} = \{R_u \mid u \in \Delta\}$ such that G is the closure of the group generated algebraically by $\mathcal{R}$.

The structure of reflection groups is completely known, see EATON and PERLMAN ([13], Section 3) for a discussion. In particular, the pre-orderings induced by reflection groups are all group induced cone orderings (i.e. Definition 2.2). However, the groups in Examples 2.2 and 2.3 are not reflection groups. Perhaps the most relevant example here is $\mathcal{P}_k$ acting on $\mathbb{R}^k$. To see that $\mathcal{P}_k$ is a reflection group, just take

$$\Delta = \{u \mid u = t_i/\sqrt{2}, \; i = 1,...,k-1\}$$

where $t_1,...,t_{k-1}$ are given in Example 2.1.

In what follows, we focus on a given set

$$\mathcal{R} = \{R_u \mid u \in \Delta\} \subseteq O(V)$$

of reflections rather than on the reflection group G generated by $\mathcal{R}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{R}$-invariant Borel subsets of V.

DEFINITION 3.2. A real valued function f defined on $\mathcal{X} \times \mathcal{Y}$ is a *decreasing reflection* (DR) function if
(i) $f(x,y) = f(R_u x, R_u y)$ for $R_u \in \mathcal{R}$;
(ii) for $u \in \Delta$, if $(u,x)(u,y) \geq 0$, then $f(x,y) \geq f(x, R_u y)$.

Condition (ii) which is the essence of the definition, means that when x and y are on the same side of the hyperplane $\{x \mid (u,x)=0\}$, then f does not increase when one of the arguments is reflected across the hyperplane. For a statistical interpretation of DR functions when $G = \mathcal{P}_n$, see EATON ([10], Chapter 3). When $G = \mathcal{P}_n$, properties of DR functions have been used in a variety of applications. For example, SAVAGE [36] applied the ideas to some non-parametric problems while EATON [4] isolated properties (i) and (ii) in a paper on ranking problems. In the context of majorization PROSCHAN and SETHURAMAN [29] proved the important Composition Theorem for DR functions when $G = \mathcal{P}_n$.

To describe the Composition Theorem in the case of general reflection groups, let

$$\mathcal{R} = \{R_u \mid u \in \Delta\}$$

be a given set of reflections.

Suppose $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Borel subsets of $(V,(\cdot,\cdot))$ which are invariant under each reflection in $\mathcal{R}$. Further, let λ be a σ-finite measure defined on the Borel subsets of $\mathcal{Y}$ and assume λ is invariant under each reflection in $\mathcal{R}$.

THEOREM 3.3 (COMPOSITION THEOREM). *Suppose f_1 and f_2 are DR functions defined on $\mathcal{X}\times\mathcal{Y}$ and $\mathcal{Y}\times\mathcal{Z}$ respectively and suppose*

$$f_3(x,z) = \int f_1(x,y)f_2(y,z)\lambda(dy).$$

exists for each x and z. Then f_3 is a DR function on $\mathcal{X}\times\mathcal{Y}$.

PROOF. That f_3 satisfies (i) of Definition 3.2 is an easy consequence of the invariance assumption on λ and the fact that f_1 and f_2 are DR functions. Now, consider $R_u \in \mathcal{R}$ and $x\in\mathcal{X}$, $z\in\mathcal{Z}$ which satisfy $(u,x)(u,z)\geq 0$. It must be shown that

$$\begin{aligned}\delta &= f_3(x,z)-f_3(x,R_u z) \qquad (3.2)\\ &= \int f_1(x_1,y)[f_2(y,z) - f_2(y,R_u z)]\lambda(dy)\geq 0.\end{aligned}$$

Decompose the region of integration $\mathcal{Y}$ into

$$\mathcal{Y}_+ = \{y\,|\,(u,y)>0\},\ \mathcal{Y}_0 = \{y\,|\,(u,y)=0\},\ \mathcal{Y}_- = \{y\,|\,(u,y)<0\}.$$

In (3.2), the integral over the set $\mathcal{Y}_0$ is zero because $f(y,R_u z)=f(y,z)$ for $y\in\mathcal{Y}_0$. Using the change of variable $y\mapsto R_u y$, the integral over $\mathcal{Y}_-$ is transformed into an integral over $\mathcal{Y}_+$. Then the invariance assumptions on f_1, f_2 and λ show that δ can be written

$$\delta = \int_{\mathcal{Y}_+} [f_1(x,y) - f_1(x,R_u y)][f_2(y,z) - f_2(y,R_u z)]\lambda(dy).$$

Because f_1 and f_2 are DR functions, the integrand is non-negative over $\mathcal{Y}_+$ since $(u,x)(u,z)\geq 0$. Thus $\delta\geq 0$ and the proof is complete. □

Now, we turn to a connection between DR functions and the decreasing (or increasing) functions. This connection was first established in HOLLANDER, PROSCHAN and SETHURAMAN [20] for the case $G=\mathcal{P}_n$.

THEOREM 3.4. *Let G be the reflection group generated by the set of reflections $\mathcal{R}=\{R_u\,|\,u\in\Delta\}$. For a function f_0 defined on V, the following are equivalent:*

(i) *f_0 is decreasing (increasing);*

(ii) *the function $f(x,y)=f_0(x-y)$ ($f(x,y)=f_0(x+y)$) is a DR function and satisfies $f(x,y)=f(gx,gy)$, $g\in G$.*

PROOF. The proof of this result depends on the structure theory for reflection groups and is not given here. A proof in the case of $G=\mathcal{P}_n$ can be found in HOLLANDER, PROSCHAN and SETHURAMAN [20]. A discussion of the general case can be found in EATON ([10], Chapter 6). □

In some cases, the conclusion of Theorem 3.4 is true for f_0 defined only on a G-invariant subset, say $\mathcal{X}$, of V. The G-induced ordering on $\mathcal{X}$ is the restriction of the G-induced ordering on V. For example, if $G=\mathcal{P}_n$ and $\mathcal{X}$ is the set of all vectors in $\mathbb{R}^n$ with integer coordinates then Theorem 3.4 is valid. Also if $\mathcal{X}$ is the set of vectors all of whose coordinates are non-negative, then Theorem 3.4 is valid. These two cases are used in the Poisson example at the end of this section.

Taken together, Theorems 3.3 and 3.4 provide a very easy proof of the so-called Convolution Theorem for the case of a reflection group (EATON and PERLMAN [13]). Again, let G be a reflection group acting on $(V,(\cdot,\cdot))$.

THEOREM 3.5 (CONVOLUTION THEOREM). *Suppose f_1 and f_2 are non-negative decreasing (in the pre-ordering defined by G) functions defined on V. Let dx denote Lebesgue measure on V and assume*

$$f_3(y) = \int_V f_1(y-x)f_2(x)dx$$

exists for each $y \in V$. Then f_3 is decreasing.

PROOF. From Theorem 3.4, it suffices to show that

$$f(y,z) = f_3(y-z) = \int_V f_1(y-z-x)f_2(x)dx$$

is a DR function. The invariance of f_3 follows from the G-invariance of f_1, f_2 and dx. Using the translation invariance of Lebesgue measure, we have

$$f(y,z) = \int_V f_1(y-x)f_2(x-z)dx.$$

Theorem 3.4 shows $f_1(y-x)$ and $f_2(x-z)$ are both DR functions on $V \times V$. The Composition Theorem then yields that f is a DR function and hence that f_3 is decreasing. □

Applications of the Convolution Theorem can be found in MARSHALL and OLKIN [23,24], EATON and PERLMAN [13] and EATON [7]. The validity of this result for non-reflection groups is discussed in Section 5.

The main applications of the Convolution Theorem in statistics is to problems involving a translation parameter. For non-translation parameter problems there is one special case where arguments similar to that used in the proof of Theorem 3.5 can be used to show functions are decreasing or increasing. An example will illustrate the main idea. Again consider the reflection group $\mathcal{P}_n$ acting on $\mathbb{R}^n$ and let $\mathcal{X}$ be those vectors in $\mathbb{R}^n$ which have integer coordinates. Counting measure on $\mathcal{X}$ is denoted by λ. Further let Θ be those vectors in $\mathbb{R}^n$ with all coordinates positive. Given $\theta \in \Theta$, consider the density (on $\mathcal{X}$, with respect to λ) given by

$$f(x \mid \theta) = \begin{cases} \prod_{j=1}^{n} \dfrac{e^{-\theta_i}\theta_i^{x_i}}{x_i!} & \text{if } x_i \geqslant 0,\ i=1,\dots,n \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(\cdot|\theta)$ is the density function of a random vector X with independent coordinates $X_1,...,X_n$ and X_i has a Poisson distribution with parameter θ_i, $i=1,...,n$. Let h be an increasing function defined on $\mathcal{X}$. (Functions which are defined only on $\{x \mid x \in \mathcal{X}, x_i \geqslant 0 \;\; i=1,...,n\} = \mathcal{X}^+$ and are increasing have increasing extensions defined on $\mathcal{X}$.) Here is the argument used by HOLLANDER, PROSCHAN and SETHURAMAN [20] to show that

$$\psi(\theta) = \int_{\mathcal{X}} h(x) f(x \mid \theta) \lambda(dx) \qquad (3.3)$$

is increasing. First, ψ is increasing iff $\psi(\theta+\eta)=k(\theta,\eta)$ is a DR function on $\Theta \times \Theta$ (by Theorem 3.4 applied to the convex $\mathcal{P}_n$- invariant set Θ rather than $\mathbb{R}^n$). But

$$\psi(\theta+\eta) = \int h(x) f(x \mid \theta+\eta) \lambda(dx). \qquad (3.4)$$

Now, the density $f(\cdot|\cdot)$ has the *convolution property* - that is, for all $x \in \mathcal{X}$,

$$f(x \mid \theta+\eta) = \int f(x-y \mid \theta) f(y \mid \eta) \lambda(dy). \qquad (3.5)$$

Such parametric families are called *convolution families.* Substituting (3.5) into (3.4) and interchanging integrations yields

$$\psi(\theta+\eta) = \int f(y \mid \eta) [\int h(x) f(x-y \mid \theta) \lambda(dx)] \lambda(dy).$$

Changing variables in the inside integral, the translation invariance of λ gives

$$\psi(\theta+\eta) = \int f(y \mid \eta) [\int h(x+y) f(x \mid \theta) \lambda(dx)] \lambda(dy).$$

But, a routine argument shows that $f(\cdot|\cdot)$ is a DR function. Since h is increasing, $(x,y) \mapsto h(x+y)$ is a DR function, so

$$(y,\theta) \mapsto \int h(x+y) f(x \mid \theta) \lambda(dx)$$

is a DR function by the Composition Theorem. A second application of the Composition Theorem then shows that $(\theta,\eta) \mapsto \psi(\theta+\eta)$ is a DR function so ψ is increasing.

The essence of the above argument is two applications of Theorem 3.5 together with the observation that $f(\cdot|\theta)$ is a convolution family. Other applications of this argument can be found in HOLLANDER, PROSCHAN and SETHURAMAN [20] and MARSHALL and OLKIN [24]. The result of RINOTT [34] given in Theorem 3.1 above follows from the above result for the Poisson distribution via an easy conditioning argument (see NEVIUS, PROSCHAN and SETHURAMAN [29]).

4. THE GAUSS-MARKOV THEOREM

In this section, we use group induced orderings to provide a new proof of the classical Gauss-Markov Theorem. This new proof suggests some strengthened versions of this classical result under some slightly stronger assumptions.

In an inner product space $(V,(\cdot,\cdot))$, a *linear statistical model* for a random vector Y consists of the specification of

(i) a known linear subspace M in which the mean vector μ of Y is assumed to lie;

(ii) a known set γ of possible positive definite covariances for the random vector Y.

Throughout this discussion, it is assumed that the identity is an element of γ.

The linear unbiased estimators of μ have the form AY where A is a linear transformation on V which satisfies

$$Ax = x \text{ for } x \in M. \tag{4.1}$$

Let $\mathcal{L}$ be the set of all linear transformations satisfying (4.1). Typically, one tries to choose $A \in \mathcal{L}$ to minimize some measure of average loss of the form

$$\psi(A) = \mathcal{E}K(AY-\mu). \tag{4.2}$$

A classical choice for the function K, in the context of the Gauss-Markov Theorem, is the quadratic form

$$K(x) = (x, Bx),\ x \in V \tag{4.3}$$

where B is some fixed self adjoint positive definite linear transformation on V.

In the present context, the Gauss-Markov Theorem takes the following form. Let $A_0 \in \mathcal{L}$ be the orthogonal projection onto M.

THEOREM 4.1. *Assume that $\Sigma(M) \subseteq M$ for each $\Sigma \in \gamma$ (so the regression subspace is an invariant subspace of each of the covariances in the model for Y). Then for each non-negative definite B and each $\Sigma \in \gamma$, the function*

$$\psi(A) = \mathcal{E}(AY-\mu,\ B(AY-\mu))$$

is minimized at $A = A_o$. Conversely, if B is positive definite and if ψ is minimized at $A = A_o$ for each $\Sigma \in \gamma$, then $\Sigma(M) \subseteq M$ for each $\Sigma \in \gamma$.

This form of the Gauss-Markov Theorem is discussed in EATON [8] where a proof can be found. In the present generality, the result applies to both univariate and multivariate analysis of variance models as well as some types of linear models with patterned covariances.

To formulate things in terms of subgroups of $O(V)$, first let $Q = (I - A_o)$ be the orthogonal projection onto $M^{\perp}$ - the orthogonal complement of M. Then set

$$g_o = I - 2Q. \tag{4.4}$$

Clearly $g_o = g_o' = g_o^{-1} \in O(V)$, so $G_o = \{I, g_o\}$ is a two element subgroup of $O(V)$. The following result connects G_o to a basic condition in Theorem 4.1.

LEMMA 4.1. *The following are equivalent*

(i) *$\Sigma(M) \subseteq M$ for all $\Sigma \in \gamma$;*

(ii) *$g_o \Sigma = \Sigma g_o$ for all $\Sigma \in \gamma$.*

PROOF. Condition (ii) is clearly equivalent to

(iii) $A_o \Sigma = \Sigma A_o$ for all $\Sigma \in \gamma$.

That (iii) and (i) are equivalent is well known (for example, see HALMOS [18]). □

LEMMA 4.2. *For each* $A \in \mathcal{L}$,

$$\frac{A + Ag_o}{2} = A_o. \tag{4.5}$$

PROOF. A bit of algebra shows that

$$\frac{A + Ag_o}{2} = AA_o.$$

Because $A \in \mathcal{L}$,

$$\begin{cases} AA_o x = x & \text{for } x \in M \\ AA_o x = 0 & \text{for } x \in M^{\perp}. \end{cases}$$

Since AA_o is a linear transformation, and agrees with A_o on M and $M^{\perp}$, obviously $AA_o = A_o$. □

Note that

$$\frac{A + Ag_o}{2}$$

is just the average (with respect to the invariant probability measure on G_o) of $\{Ag \mid g \in G_o\}$. Thus A_o is in the convex hull of the orbit $\{Ag \mid g \in G_o\}$ for every $A \in \mathcal{L}$.

Here is Theorem 4.1 expressed in terms of G_o.

THEOREM 4.2. *Given the linear model for Y, assume that*

$$\Sigma g_o = g_o \Sigma, \ \Sigma \in \gamma. \tag{4.6}$$

Then for each positive semi-definite B and for each $\Sigma \in \gamma$,

$$\psi(A) = \mathcal{E}(AY - \mu, B(AY - \mu))$$

is minimized at $A = A_o$.

PROOF. A standard result in the calculus of random vectors (see EATON [8], Chapter 2) shows that when Cov $(Y) = \Sigma$,

$$\psi(A) = \mathcal{E}(AY - \mu, B(AY - \mu)) = \operatorname{tr} A \Sigma A' B$$

where tr denotes the trace. Because of assumption (4.6),

$$\psi(Ag_o) = \psi(A), \ A \in \mathcal{L}, \tag{4.7}$$

so ψ is a G_o invariant function. Because Σ and B are non-negative definite, it is easy to verify that ψ is a convex function defined on the convex set $\mathcal{L}$. Using Lemma 4.2 and (4.7), we have for any $A \in \mathcal{L}$,

$$\psi(A_o) = \psi(\tfrac{1}{2}(A+Ag_o)) \leq \tfrac{1}{2}\psi(A) + \tfrac{1}{2}\psi(Ag_o) = \psi(A)$$

and the proof is complete. □

The above argument is just a special case of the argument given in Section 1 to derive inequality (1.10) (for concave rather than convex functions). In our previous terminology, G_o acts on $\mathcal{L}$ and ψ is an invariant convex function. Thus, for $A \in \mathcal{L}$, ψ must be minimized at 'the center of the orbit of A.'

We now turn to a generalization of Theorem 4.2. As before the linear model for Y in $(V,(\cdot,\cdot))$ consists of the regression subspace M and the set of covariances γ for Y. Elements A of $\mathcal{L}$ yield linear unbiased estimators AY for $\mu \in M$. Let G be a subgroup of $O(V)$ such that
(i) $G_o \subseteq G$;
(ii) $gx = x$ for $x \in M$, $g \in G$.
The group G acts on the left of $\mathcal{L}$ via the group action

$$A \mapsto Ag^{-1}.$$

Thus, G defines an induced group ordering on $\mathcal{L}$ that is, write $A_1 \leq A_2$ iff A_1 is an element of the convex hull of the orbit

$$\{Ag^{-1} \mid g \in G\}.$$

LEMMA 4.3. *Given $A \in \mathcal{L}$, $A_o \leq A$ where A_o is the orthogonal projection onto M.*

PROOF. Let ν denote the invariant probability measure on G and set

$$A_1 = \int Ag^{-1}\nu(dg).$$

Then $A_1 \in \mathcal{L}$ and $A_1 \leq A$. With g_o as in (4.4), the invariance of ν yields

$$A_1 g_o = \int Ag^{-1}g_o\nu(dg) = \int A(g_o g)^{-1}\nu(dg) = \int Ag^{-1}\nu(dg) = A_1.$$

Thus,

$$A_1 = \tfrac{1}{2}(A_1 + A_1 g_o)$$

and so by lemma 4.2, $A_1 = A_o$. Hence $A_o \leq A$. □

The above lemma shows that

$$h(A_o) \leq h(A)$$

for any convex G-invariant function defined on $\mathcal{L}$. Here is a generalization of Theorem 4.2.

THEOREM 4.3. *In the linear model for Y, assume that $g\Sigma = \Sigma g$ for $g \in G$, $\Sigma \in \gamma$. For each positive semi-definite B and for each $\Sigma \in \gamma$, the function*

$$\psi(A) = (AY-\mu, B(AY-\mu))$$

is increasing in the pre-ordering defined by G and

$$\psi(A_o) \leq \psi(A),\ A \in \mathcal{L}.$$

PROOF. As in Theorem 4.2,

$$\psi(A) = \operatorname{tr} A \Sigma A' B,$$

and so ψ is convex. The invariance of ψ follows from the assumption. This completes the proof. □

Somewhat stronger conclusions can be obtained with invariance assumptions on the distribution of the error vector

$$E = Y - \mu.$$

The group G is as above. However, we now consider more general loss functions (rather than only quadratic forms) to measure the performance of linear unbiased estimators. First, consider

$$\psi(A) = \mathcal{E} H(AY - \mu),\ A \in \mathcal{L} \tag{4.8}$$

as a measure of loss for using AY to estimate μ. Of course, H is assumed to be measurable and such that

$$\mathcal{E}\,|H(AY-\mu)| < +\infty$$

for all $A \in \mathcal{L}$ and $\Sigma \in \gamma$.

THEOREM 4.4. *Assume the distribution of E is the same as the distribution of gE for each g∈G. Then ψ in (4.8) is an invariant function - that is,*

$$\psi(Ag^{-1}) = \psi(A),\ \ A \in \mathcal{L},\ g \in G.$$

Further, if H is a convex function, then ψ is a convex function so ψ is increasing in the pre-ordering defined by G, and in particular,

$$\psi(A_o) \leq \psi(A),\ \ A \in \mathcal{L}.$$

PROOF. Because $A \in \mathcal{L}$, $A\mu = \mu$ for all $\mu \in M$. The assumption on the distribution of E yields,

$$\begin{aligned}\psi(A) &= \mathcal{E}H(AY-\mu) = \mathcal{E}H(A(Y-\mu)) = \mathcal{E}H(AE)\\ &= \mathcal{E}H(Ag^{-1}E) = \psi(AG^{-1}).\end{aligned}$$

The first assertion follows.

When H is convex, obviously ψ is convex and hence increasing. □

As an example of the previous result, consider the standard univariate linear regression model with homoscedastic normal errors. Then, Y has a normal distribution on $\mathbb{R}^n$, say $N_n(\mu, \sigma^2 I_n)$, where μ lies in a known linear subspace M.

In this case, the error vector $E = Y - \mu$ is $N_n(0, \sigma^2 I_n)$ and hence the distribution of E is invariant under all orthogonal transformations. Thus, the appropriate group for this problem is

$$G = \{g \mid g \in O_n,\ gx = x \text{ for } x \in M\}.$$

Theorem 4.4 shows that when H is convex,

$$\psi(A) = \mathcal{E} H(AY - \mu)$$

is minimized at $A = A_o$. Thus the usual least squares estimator minimizes the expected loss (among linear unbiased estimators) for all convex H. In the normal case, this result has been strengthened even further. Let C be a convex symmetric subset of M - that is, C is convex, $C \subseteq M$ and $C = -C$. As a measure of loss, consider

$$\psi_1(A) = P\{AY - \mu \in C\}.$$

Berk and Hwang [2] proved that

$$\psi_1(A_o) \leqslant \psi_1(A)$$

for all $A \in \mathfrak{L}$. This result has been extended in a variety of directions in Eaton [10] where group induced orderings also play a role.

5. Discussion

There are a variety of open questions related to the results discussed in the previous sections. First, we discuss differential characterizations of the decreasing functions when the compact group $G \subseteq O(V)$ acts on $(V, (\cdot,\cdot))$ as in Section 2. A necessary condition for a real valued function f, with a differential df, to be decreasing is

Proposition 5.1 (Eaton [5]). *If f is decreasing, then*

$$(gx - x, df(x)) \geqslant 0 \quad g \in G,\ x \in V. \tag{5.1}$$

Proof. For $\alpha \in [0,1]$, $x \in V$ and $g \in G$,

$$\phi(\alpha) = f((1-\alpha)x + \alpha gx) \geqslant f(x)$$

because f is decreasing. Expanding ϕ in a Taylor series about $\alpha = 0$ yields

$$\phi(\alpha) = \phi(0) + \phi'(0)\alpha + o(\alpha).$$

Since $\phi(\alpha) \geqslant \phi(0)$ and

$$\phi'(0) = (gx - x, df(x)),$$

we have

$$\alpha(gx - x, df(x)) + o(\alpha) \geqslant 0.$$

Dividing by α and letting $\alpha \to 0$ gives (5.1). □

It is known (see EATON and PERLMAN [13]) that (5.1) is necessary and sufficient for f to be decreasing when G is a reflection group. For Examples 2.2 and 2.3, it can be shown that (5.1) is necessary and sufficient for f with a differential to be decreasing. However, there are instances of interest where the question is open. For example, take $V=\mathbb{R}^n$ and let $G=\{\pm g \mid g\in\mathcal{P}_n\}$. This group is not a reflection group and the pre-ordering induced by G is not a group induced cone ordering (condition (ii) of Definition 2.2 fails, see EATON ([11], Example 6.6)). A differential characterization of the decreasing functions is not known for this example.

Condition (5.1) can be rewritten in a form similar to that in Theorem 2.2 (ii). Let $H(x)$ be the convex cone generated by

$$\{x-gx \mid g\in G\}.$$

Then (5.1) is equivalent to

$$(t,df(x))\leqslant 0 \text{ for all } t\in H(x). \qquad (5.2)$$

An important question is whether or not (5.2) implies that f is decreasing. Counterexamples are not known.

Next, we turn to Composition and Convolution Theorems. In statistical applications, the Convolution Theorem (CT) deals mainly with translation parameter problems. The *only* cases for which CT is known to be valid are when G is a product of reflection groups (see EATON [9] for a discussion) or when G acts transitively on $\{x \mid x\in V, (x,x)=1\}$. Further, CT is known to be false for finite rotation groups acting on $\mathbb{R}^2$ (see EATON [9], Example 4.1). However, there are important cases which arise in practice where the question has not been settled. For example, take $G=\{\pm g \mid g\in\mathcal{P}_n\}$ acting on $\mathbb{R}^n$, $n\geqslant 3$. A necessary condition for CT to hold is described in EATON ([9], Proposition 10). The only known counterexamples to CT violate this necessary condition.

The Composition Theorem (CoT) was used in Section 3 to show that the function ψ in (3.3) is increasing. The argument employed there was rather special because the parametric family in question was a convolution family. In fact, the only applications of CoT to settle questions relating to the monotonicity of functions ψ of the form (3.1) involve convolution families (see HOLLANDER, PROSCHAN and SETHURAMAN [20]). Conditions which yield monotonicity of ψ in (3.1) for non-convolution families would be most useful.

Finally, we offer a few comments on possible applications of group induced orderings to experimental design. These comments are prompted, at least in part, by the recent article of PUKELSHEIM [32]. In essence an experimental design problem consists of a measurable space $\mathcal{X}$ (the design space) and a class $\mathcal{M}$ of probability measures defined on the σ-algebra of $\mathcal{X}$. Elements of $\mathcal{M}$ are interpreted as 'designs.' Symmetry properties of designs are most naturally defined in terms of a group G of bimeasurable transformations defined on $\mathcal{X}$. Given $g\in G$ and a design $\xi\in\mathcal{M}$, define the new design $g\xi$ by

$$(g\xi)(B) = \xi(g^{-1}B) \qquad (5.3)$$

for each measurable set B. Now, assume that

$$\left.\begin{array}{l}\text{(i) } \mathfrak{M} \text{ is a convex set} \\ \text{(ii) } \xi\in\mathfrak{M} \text{ implies that } g\xi\in\mathfrak{M} \text{ for all } g\in G.\end{array}\right\} \tag{5.4}$$

Under the assumptions (5.4), the group G acts on $\mathfrak{M}$ and it is clear that

$$g(\alpha\xi_1+(1-\alpha)\xi_2) = \alpha g\xi_1 + (1-\alpha)g\xi_2 \tag{5.5}$$

for real numbers $\alpha\in[0,1], g\in G$ and $\xi_1,\xi_2\in M$. In other words, elements of G act affinely on $\mathfrak{M}$. This suggests that we define the group induced pre-ordering on $\mathfrak{M}$ as follows:

$$\xi_1 \leqslant \xi_2 \text{ iff } \xi_1\in C(\xi_2) \tag{5.6}$$

where $C(\xi_2)$ is the convex hull of $\{g\xi_2 \,|\, g\in G\}$. This is precisely the type of situation considered in Section 2, except that in most cases, $\mathfrak{M}$ is a convex subset of an infinite dimensional linear space. A design $\xi\in\mathfrak{M}$ is *invariant* if

$$g\xi = \xi \quad \text{for } g\in G.$$

In order to select a 'good' design from M, one ordinarily specifies a real valued *criterion function* Φ defined on $\mathfrak{M}$. Many common criterion functions satisfy

$$\left.\begin{array}{l}\text{(i) } \Phi(\alpha\xi_1 + (1-\alpha)\xi_2) \geqslant \alpha\Phi(\xi_1) + (1-\alpha)\Phi(\xi_2) \\ \text{(ii) } \Phi(\xi) = \Phi(g\xi), \quad g\in G.\end{array}\right\} \tag{5.7}$$

That is, attention is focused on criterion functions which are concave and G-invariant (see PUKELSHEIM [32] for a discussion of these two conditions in the context of experimental design problems in linear models).

A design ξ_o is called *Φ-optimal* if ξ_o maximizes Φ over $\mathfrak{M}$. To see how the pre-ordering plays a role, consider

$$\xi_1 = \Sigma\alpha_g g\xi_2$$

where the finite sum ranges over some subset of G and the non-negative weights α_g sum to 1. Then the conditions (5.7) on Φ yield

$$\Phi(\xi_1) = \Phi(\Sigma\alpha_g g\xi_2) \geqslant \Sigma\alpha_g\Phi(g\xi_2) = \Sigma\alpha_g\Phi(\xi_2) = \Phi(\xi_2).$$

In other words, $\xi_1\leqslant\xi_2$ implies that $\Phi(\xi_1)\geqslant\Phi(\xi_2)$ so Φ is decreasing.

When the group G is compact (as in some applications), a repetition of the argument leading to (1.10) shows the Φ is maximized over the set of invariant designs in $\mathfrak{M}$. More precisely, let ν be the invariant probability measure on the compact group G. For $\xi\in\mathfrak{M}$, let

$$\bar{\xi} = \int g\xi\nu(dg). \tag{5.8}$$

This is shorthand notation for $\bar{\xi}$ defined by

$$\bar{\xi}(B) = \int (g\xi)(B)\nu(dg) = \int \xi(g^{-1}B)\nu(dg). \tag{5.9}$$

Obviously $\bar{\xi}$ is invariant and because $\bar{\xi}$ is in $C(\xi)$,

$$\Phi(\bar{\xi}) \geqslant \Phi(\xi). \tag{5.10}$$

Therefore, given any design ξ, there is an invariant design $\bar{\xi}$ with $\Phi(\bar{\xi}) \geqslant \Phi(\xi)$. Hence Φ is maximized on the set of invariant designs.

The purpose of the above discussion is to show that group orderings can be applied to *general* design problems rather than just linear model design problems as discussed in PUKELSHEIM [32]. The important observation is that the group G acts in a very natural way on the designs ξ. The idea of inducing a group action on one space when the group acts on a second space is very well known and is widely used in invariance applications in statistics (for example, see EATON ([8], Chapter 7) for as systematic discussion). Recent work on group induced orderings in experimental design can be found in GIOVAGNOLI, PUKELSHEIM and WYNN [15].

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TECHNICAL CONTRIBUTIONS

I. Castellani (INRIA, Sophia-Antipolis)
Permutation of transitions: an event structure semantics for CCS and SCCS

E.M. Clarke (Carnegie-Mellon University, Pittsburgh)
Compositional model checking

R. de Nicola (Istituto di Elaborazione dell'Informazione, Pisa)
Temporal, spatial and causal observations of concurrent systems

H. Gaifman (Hebrew University, Jerusalem)
Analysis of concurrency with the help of partial order

S. Katz (Technion Haifa)
Exploiting interleaving set temporal logic to simplify correctness proofs

M. Nielsen (Aarhus University)
Partial order semantics

E.-R. Olderog (Christian-Albrechts-University, Kiel)
Nets, terms and formulas: three views of concurrent processes

W. Reisig (GMD, St. Augustin)
Linear time and branching time logic for partial order semantics

J.J.M.M. Rutten (Centre for Mathematics and Computer Science)
Correctness and full abstraction of metric semantics for concurrency

M.W. Shields (University of Kent at Canterbury)
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C. Stirling (University of Edinburgh)
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P.S. Thiagarajan (The Institute of Mathematical Sciences, Madras)
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Interleaving and partial order in data flow

Research and Education in Concurrent Systems

Regime Behaviour and Predictability Properties of the Atmospheric Circulation Studied with Limited-Resolution Spectral Models

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A brief review is presented of a project carried out at the CWI during the years 1983-1987. These investigations were supported by the Netherlands Foundation for the Technical Sciences (STW), future Technical Science Branch of the Netherlands Organization for Scientific Research (NWO). The aim was to obtain a better understanding of the chaotic properties of the atmosphere, which may contribute to the development of long-range weather prediction models. It is argued that a method for investigating this problem is to analyse highly simplified atmospheric spectral models, since the results may provide clues on how to analyse more complicated models as well as real data. It appears that low-order models possess multiple equilibria, with the corresponding flow patterns resembling large-scale preferent states of the atmospheric circulation. Vacillatory behaviour, in which the system alternately visits different flow regimes, is obtained either by adding stochastic perturbations to the equations or by including a sufficient number of modes in the spectral expansions. The predictability properties of these systems are discussed and particular attention is given to the forcing terms which are added to the spectral equations in order to account for the effect of the neglected modes and physical processes not included in the model.

1. Introduction

During 1983-1987 research was done at the CWI in the STW project 'Mathematical methods for the analysis of atmospheric spectral models'. The aim of this study was to obtain a better understanding of the dynamics of the atmospheric circulation in the midlatitudes (roughly between the 30 and 60 degree latitude), especially in relation to the problem of long-range weather predictions. Modern weather forecasts are based on the results of detailed and complicated numerical models, such as that of the European Centre for Medium Range Weather Forecasting (ECMWF) in Reading, England. It has long been assumed that the period over which the weather is predicted could be increased forever if the numerical models would be further improved and the initial state (determined by means of observations) better prescribed. However, nowadays it is known that there is a fundamental limit to the period over which the weather can be predicted, i.e., it cannot be enlarged by carrying out more and better observations. In most cases this predictability horizon of the atmosphere is encountered after a period of 5 to 14 days (OPSTEEGH [21]). As a consequence, it is not possible to give forecasts for periods longer than about two weeks. Unfortunately, there is precisely a strong need for accurate long-range weather predictions. This is because on time scales between a week and three months frequently climatic anomalies occur which have large social consequences, for example excessive droughts, heat-waves, etc.

In order to understand why forecasts fail on the long term we first present some qualitative arguments. The weather as we (in midlatitudes) experience it is the result of day to day variations in the geographical distribution of high- and low-pressure belts. These so-called synoptic-scale eddies have typical horizontal dimensions of 1000 km, a life span of about a week and are embedded in a belt of predominantly westerly winds. The latter result from an approximate balance between thermal forcing, due to the equator-pole temperature gradient, and the Coriolis force (induced by the rotation of the earth) acting on a moving fluid. Furthermore, due to the presence of a large-scale topography (to which in particular the Himalaya, Rocky Mountains and the oceans contribute) and thermal differences between land and ocean, ultra-long quasi-stationary waves are generated which give the flow a meandering structure. These planetary waves have much larger dimensions (about 10000 km) and longer lives (of the order of several months) than the synoptic-scale eddies. Thus, the atmospheric circulation is characterized by two distinct scales of motion: a planetary scale and a synoptic scale. Little is known about the subtle interplay between these scales of motion. It appears that synoptic-scale eddies develop spontaneously as initially small perturbations of the locally unstable planetary-scale circulation. Moreover, the planetary-scale flow tends to steer and organize the eddies along preferent paths, which are the stormtracks. On the other hand the eddies themselves influence the evolution of the planetary waves. The consequences for the predictability of the atmospheric circulation were systematically studied by LORENZ [18]. He demonstrated that interactions between different scales of motion are the principal cause for the limited predictability of the atmosphere.

As a result of the feedback between the planetary waves and the synoptic-scale eddies quasi-stable flow configurations occur which cause short-range climatic anomalies. The existence of such large-scale preferent states of the atmospheric circulation (sometimes called weather regimes) has been known for a long time. They can be divided into three major types: zonal (high-index) states with strong western winds and small wave amplitudes, meridional (low-index) states with large waves embedded in a weak zonal flow and transitional states which have characteristics of both the high- and low-index states. Typical flow configurations of these regimes in the European region are shown in Figure 1. The situation in Figure 1c is that of a persistent anticyclone near Scandinavia which blocks the standard passage of depressions over Western Europe, in this way causing persistent weather conditions in this region.

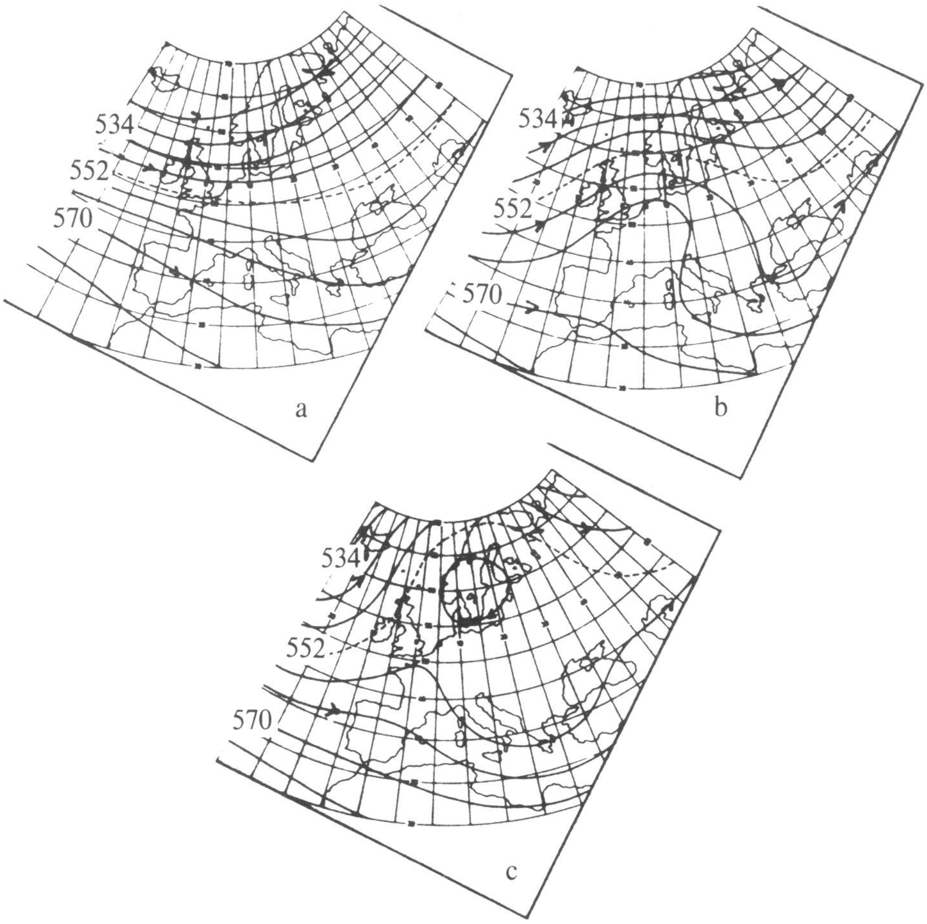

FIGURE 1. Geographical distribution of the height (in geopotential decameter) of the 500 mb level for the Wz (a), HM (b) and HFa (c) winter Grosswetterlagen, which are of zonal, mixed and meridional type, respectively. The isohyps are approximate streamlines of the flow; arrows indicate flow direction. From VAN DIJK ET AL. [29].

Obviously, the atmosphere can be considered as a chaotic system which shows vacillatory behaviour, i.e., it irregularly visits different preferent states. As discussed in DOLE [9] and REINHOLD [23], quasi-stable flow patterns suddenly develop and disappear without any clear indication why. Furthermore, the life span of the weather regimes is highly variable without having a preferent time scale. Within the framework of long-term weather predictions it is important to obtain a better understanding of the dynamics responsible for this vacillatory behaviour. A method for studying this problem consists of analysing highly simplified models which represent qualitative features of the atmospheric circulation. The motivation for doing this is that from the results indications may be found how to consider more complicated models as well as real data. In this way we hope to enhance our understanding in the atmospheric dynamics. This method has been adapted in the STW project mentioned previously and a review of the results will be presented in this paper.

2. QUASI-GEOSTROPHIC DYNAMICS

In order to study the variability of the atmospheric circulation we should start from the full equations of motion. However, they are too complicated to deal with analytically and therefore they are simplified by the application of scale analysis, being a standard technique in geophysical fluid dynamics, see PEDLOSKY [22]. The method requires an a priori specification of the type of motion to be studied. Next it yields, by means of physical arguments, characteristic scales for the flow with which the equations of motion are written in a dimensionless form. The resulting system will contain several dimensionless parameters. The aim of the method is to find small parameters. Then, by means of standard perturbation techniques, simplified equations are derived which describe the type of motion under consideration.

Here we consider a flow near some central latitude $\phi=\phi_0$ on the Northern Hemisphere distant from equator and pole. Let it have a horizontal (parallel to the earth's surface) length scale k^{-1}, a vertical length scale H (which is the depth of the fluid) and a time scale σ^{-1}, such that

$$H << k^{-1} << r_0, \quad \sigma << f_0 \equiv 2\Omega \sin \phi_0, \tag{2.1}$$

where f_0 is the Coriolis parameter at $\phi=\phi_0$, Ω the angular speed of rotation of the earth and r_0 the radius of the earth. The first condition implies that the flow is nearly horizontal and 2-dimensional. The latter means that to a first approximation the momentum equations reduce to a balance between the Coriolis force and pressure gradient force, which is the geostrophic balance. Clearly, (2.1) is satisfied for large-scale atmospheric motions near $\phi_0=45°$ N where $k^{-1} \sim 10^6$ m, $H \sim 10^4$ m, $\sigma^{-1} \sim 10^5$ s, $f_0=10^{-4}$ s^{-1} and $r_0 \sim 6.4 \cdot 10^6$ m. Under these conditions it is shown by PEDLOSKY [22] that the equations of motion reduce to one nonlinear partial differential equation. With the additional assumption that the flow is barotropic (i.e., density is a function of pressure only) the result reads (in a dimensionless form)

$$\underset{(1)}{\frac{\partial}{\partial t}\nabla^2\psi} + \underset{(2)}{J(\psi, \nabla^2\psi)} + \underset{(3)}{\gamma J(\psi, h)} + \underset{(4)}{\beta\frac{\partial\psi}{\partial x}} + \underset{(5)}{C\nabla^2(\psi-\psi^*)} = 0. \tag{2.2}$$

This is the barotropic vorticity equation. Here t is time and $\psi(x,y,t)$ a streamfunction to which all state variables (velocities, density, pressure and temperature) are related. At a fixed time the flow is along the streamlines ψ = constant. Furthermore

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}), \quad J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}, \tag{2.3}$$

$$dx = kr_0 \cos\phi_0 d\lambda, \quad dy = kr_0 d\phi,$$

where λ is longitude and ϕ is latitude. In Eq. (2.2) term (1) represents the local change of the relative vorticity $\nabla^2\psi$ (which is the vertical component of the curl of the velocity vector), (2) the advection of vorticity by the flow itself, (3) the production of vorticity due to the presence of a large-scale topography $h(x,y)$ (in particular the high mountains and the oceans) with characteristic amplitude h_0, (4) the planetary vorticity advection due to the variation of the Coriolis parameter with latitude and (5) represents both a damping of vorticity and an external vorticity forcing (modelling the equator-pole temperature gradient) indicated by the function $\psi^*(x,y)$. The dimensionless parameters are

$$\gamma = \frac{f_0 h_0}{\sigma H}, \quad \beta = \frac{\beta_0}{\sigma k} = \frac{2\Omega\cos\phi_0}{\sigma k r_0}, \quad C = \frac{f_0\delta_E}{2\sigma H}, \tag{2.4}$$

where δ_E is the depth of the boundary layer near the earth's surface in which frictional effects are important. The flow described by Eq. (2.2) is called quasi-geostrophic because the small departures from the geostrophic balance determine the evolution of the flow (PEDLOSKY [22]).

3. Derivation of spectral models by Galerkin projection techniques

The barotropic vorticity equation (2.2) is still difficult to handle, mainly because of its nonlinear structure. A way to obtain approximate solutions is to apply Galerkin projection techniques where explicit use is made of the boundary conditions to the equation (VOIGT ET AL. [30]). This spectral method will be discussed for a specific example. Its application to models used for numerical weather prediction is described by JARRAUD and BAEDE [15]. Consider Eq. (2.2) in a rectangular channel of length L and width $B=(bL/2)$. The dimensionless length and width are 2π and πb, respectively. We investigate the existence of travelling wave solutions in the zonal x-direction. At the boundaries $y=0$ and $y=\pi b$ the meridional velocity component is assumed to be zero and it follows that the mean zonal velocity component over these boundaries should be constant. Consequently, the boundary conditions read

$$\psi(x+2\pi, y, t) = \psi(x,y,t), \tag{3.1}$$

$$\frac{\partial\psi}{\partial x} = 0 \text{ and } \frac{\partial}{\partial t}\int_0^{2\pi}\frac{\partial\psi}{\partial y}dx = 0 \text{ at } y=0, y=\pi b.$$

Applying the spectral method, we expand the streamfunction $\psi(x,y,t)$ in a series of eigenfunctions $\{\phi_j\}_j$ of the Laplace operator ∇^2 with corresponding eigenvalues λ_j, thus

$$\psi(x,y,t) = \sum_j \psi_j(t)\phi_j(x,y), \quad j=(j_1,j_2). \tag{3.2}$$

Each mode $\psi_j\phi_j$ satisfies the boundary conditions and the eigenfunctions are orthonormalized with respect to the domain average. In this case

$$\{\phi_j\} = \sqrt{2}\cos(j_2 y/b) \tag{3.3a}$$

$$\{\phi_j\} = \sqrt{2}\exp(ij_1 x)\sin(j_2 y/b) \tag{3.3b}$$

$$\lambda_j = j_1^2 + \frac{j_2^2}{b^2}, \quad |j_1|, j_2 = 1,2,\ldots\ldots \tag{3.3c}$$

The functions (3.3a) describe $(0,j_2)$ zonal flow modes (because they are independent of x) and the functions in (3.3b) describe $(|j_1|,j_2)$ wave modes. The topography and forcing streamfunction are represented by

$$h(x,y) = \cos(x)\sin(y/b), \tag{3.4}$$

$$\psi^*(x,y) = \sqrt{2}\{\psi_{01}^*\cos(y/b)+\psi_{02}^*\cos(2y/b)\}.$$

Projecting Eq. (2.2) on the eigenfunctions (3.3), which is called a Galerkin projection, we obtain the spectral model

$$\lambda_j\dot{\psi}_j = \frac{1}{2}\sum_l \sum_m c_{jlm}(\lambda_l-\lambda_m)\psi_l\psi_m + \gamma\sum_l \sum_m c_{jlm}\psi_l h_m +$$

$$+ \sum_l b_{jl}\psi_l - c\lambda_j(\psi_j-\psi_j^*), \tag{3.5}$$

consisting of an infinite number of coupled ordinary differential equations. Here a dot denotes differentiation with respect to time,

$$c_{jlm} = <\phi_j, J(\phi_l,\phi_m)>, \quad b_{jl} = \beta<\phi_j, \frac{\partial\phi_l}{\partial x}> \tag{3.6}$$

are the interaction coefficients and $<,>$ denotes an inner product on the domain considered. It appears that nonlinear contributions always occur as triads in which two modes interact and affect the evolution of a third mode. Developing (3.6) using (3.3) we find that there are two types of nonlinear triads: one involving a zonal flow mode and two wave modes and one involving three wave modes. The underlying physical mechanism is discussed in PEDLOSKY [22].

4. THE TRUNCATION PROBLEM

In practice the expansion (3.2) is truncated after a finite number of eigenfunctions. Only the large-scale modes are resolved since it is observed that most energy of quasi-geostrophic flow is contained in the long waves. The result is a dynamical system of the type

$$\dot{x} = f_\mu(x)+F(t) \quad \text{in } \mathbb{R}^N. \tag{4.1}$$

Here N is the truncation number, $\mathbb{R}^N$ the phase space, $x=(x_1,x_2,...,x_N)$ real-valued velocity amplitudes of the modes (to be specified in the next sections) and $f_\mu(x)$ an N-dimensional vector field depending on x and parameters $\mu=(\mu_1,\mu_2,...,\mu_m)$. Finally, the $F(t)$ represent the effect of the neglected modes on the dynamics of the retained modes. We remark that if (4.1) is used as a forecast model $F(t)$ should also account for the effect of physical processes and boundary conditions not (correctly) incorporated in the model. These forcing terms are unknowns by definition.

A convenient approach in theoretical studies concerning (4.1) is to neglect the effect of the forcing terms a priori and consider the properties of spectral models with increasing truncation numbers. The underlying motivation is that they will at least represent properties of the original vorticity equation. Some formal indications that this idea is correct have been found by CONSTANTIN ET AL. [4]. They showed that for spectral models of the Navier-Stokes equations a minimum number N_s of eigenfunctions could be selected such that solutions of truncated spectral models with $N \geqslant N_s$ and $F(t)=0$ have equal attractor properties as the solutions of the original system. Although it is not clear whether these results are applicable to the barotropic vorticity equation, they at least suggest that it is useful to consider truncated spectral models.

In principle we would like to investigate the properties of (4.1) for arbitrary values of N. However, we remark that it is not possible to carry out such an analysis systematically since the systems have a complicated dynamics due to the large number of nonlinear terms in the equations. Therefore, as a first step, it becomes worthwhile to study low-order spectral models, in which only a few modes are retained, and investigate in what sense they reflect features like transitions between weather regimes and a flow with a limited predictability. An important advantage is that they can be analysed with techniques originating from the theory of dynamical systems, see GUCKENHEIMER and HOLMES [14] and THOMPSON and STEWART [28], whereas from the results indications may be found how to study more complicated models as well as real data.

The structure of the vector fields of the spectral models discussed in this paper is such that nontransient solutions are found in bounded subsets of the phase space. These can be either regular sets, including stationary points (equilibrium flow patterns), limit cycles (oscillating flow) and invariant tori (quasi-periodically oscillating flow), as well as irregular sets which are in fact strange attractors (chaotic flow). These sets of limit points are determined from a numerical bifurcation analysis of the spectral model, using adapted routines of the software package AUTO of DOEDEL [8].

A spectral model is assumed to give at least a qualitative description of the atmospheric circulation if trajectories irregularly visit different preferent regions in phase space. In this way the index cycle mentioned in the introduction is simulated. If this behaviour does not occur the truncation is apparently too severe and more modes should be included in the spectral expansions. Another possibility is to take account of the effect of the synoptic-scale transient eddies on the dynamics of planetary-specific scale flow by adding specific forcing terms to the spectral equations. However, this requires a thorough understanding of the interactions between different scales of motion, this being one of the major problems in modern dynamic meteorology. We will return to this point in the Sections 6 and 7.

5. A THREE- AND SIX-COMPONENT MODEL

The fact that the Galerkin projection technique, discussed in Section 3, can be applied to the partial differential equations describing the dynamics of large-scale atmospheric flow was first realized by SILBERMAN [26]. Later on a number of other spectral models have been developed, see the review in DE SWART [5]. It appears that already extremely low-order spectral models show qualitative features of the circulation. The simplest example is the three-component model of CHARNEY and DEVORE [2] in which only the (0,1) zonal flow mode and the (1,1) wave mode are retained. This implies that we assume ψ^*_{02} in (3.4) to be zero. The stationary points of this model can be computed analytically. There are either one or three of them depending on the model parameters. As a characteristic situation we will consider a channel of length 5000 km ($= 2\pi/k$) and width 4000 km centered at latitude $\phi_0 = 45°$ where $f_0 = 10^{-4}$ s^{-1} and $\beta_0 = 1.6 \cdot 10^{-11}$ m^{-1} s^{-1}. The vertical length scale is taken $H = 10^4$ m, the time scale $\sigma^{-1} = 10^5$ s (about one day), the mountain amplitude $h_0 = 10^3$ m and the dissipation time scale about ten days. This yields the parameter values $b = 1.6$, $\beta = 1.25$, $\gamma = 1$ and $c = 0.1$. In Figure 2 the x_1-component of the stationary points (where $x_1 = \psi_{01}/b$) is presented as a function of the external forcing $x_1^* = \psi^*_{01}/b = U/U_0$, where U is a velocity scale for the forcing and $U_0 = \sigma/k = 7.8$ ms $^{-1}$.

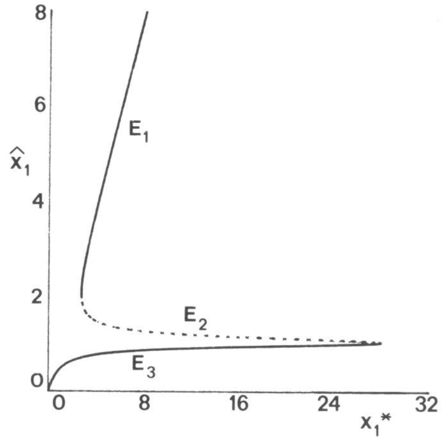

FIGURE 2. The x_1-component of the stationary points $\hat{x}$ of the three-component model for the parameter values discussed in the text. A solid line denotes that the solution is stable whereas a dashed line refers to an unstable solution.

In Figure 3 the streamfunction patterns associated with the equilibria E_1, E_2 and E_3 occurring for $x_1^* = 4$ are shown. Note their strong resemblance to the circulation patterns shown in Figure 1. Based on this agreement CHARNEY and DEVORE [2] suggest that equilibria of spectral models indicate large-scale preferent states of the atmospheric circulation. The existence of multiple equilibria is a consequence of the presence of topography, forcing and dissipation.

However, no flow with a limited predictability and no vacillatory behaviour (i.e., an index cycle) is found: the nontransient solutions are always stationary. The reason for this behaviour is a lack of nonlinear interactions in the model.

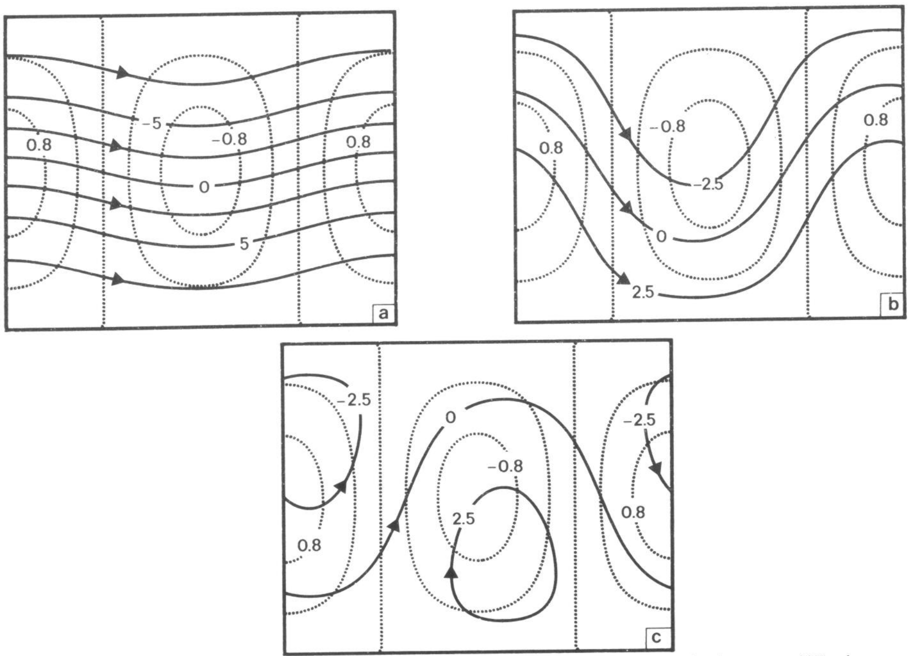

FIGURE 3. Nondimensional streamfunction patterns of the equilibria E_1 (a), E_2 (b) and E_3 (c) for the parameter values discussed in the text. The arrows indicate the flow direction which is along the streamlines ψ=constant. Here a difference $\Delta\psi = 1$ corresponds to a zonal transport of $2.6 \cdot 10^7$ m^2 s^{-1}.The dashed lines represent contours of the topography (10^3 m).

Therefore, we extend the model by including also the (0,2) zonal flow mode and the (1,2) wave mode in the spectral expansions, resulting in a six-component model. In Figure 4 the x_1- and $x_4(=\psi_{02}/b)$-component of its stationary points are shown as a function of x_1^* in case $x_4^*(=\psi_{02}^*/b)=0$ and all other parameter values similar as before. Clearly, equilibria of the three-component model are also equilibria of the six-component model but stability properties can be different because of the increased number of degrees of freedom. Furthermore, the model contains a new type of nonlinear interactions involving a zonal flow component and two different wave modes. As a result additional equilibria are found. However, the nontransient behaviour can be more complicated. In DE SWART [5,6] it is shown that also stable periodic orbits exist, indicated by the presence of Hopf bifurcation points in Figure 4, as well as strange attractors. However, the latter have only a limited domain of attraction in phase space and the chaotic solutions remain in the low-index regime forever. Thus no simulation of an index cycle is obtained. This conclusion remains unchanged if x_4^* is given nonzero values. It is due to the presence of only one triad of nonlinear interactions in the model. Thus in order to obtain vacillating solutions either forcing terms must be added to the equations or more modes should be included in the spectral expansions. Both possibilities will be subsequently considered.

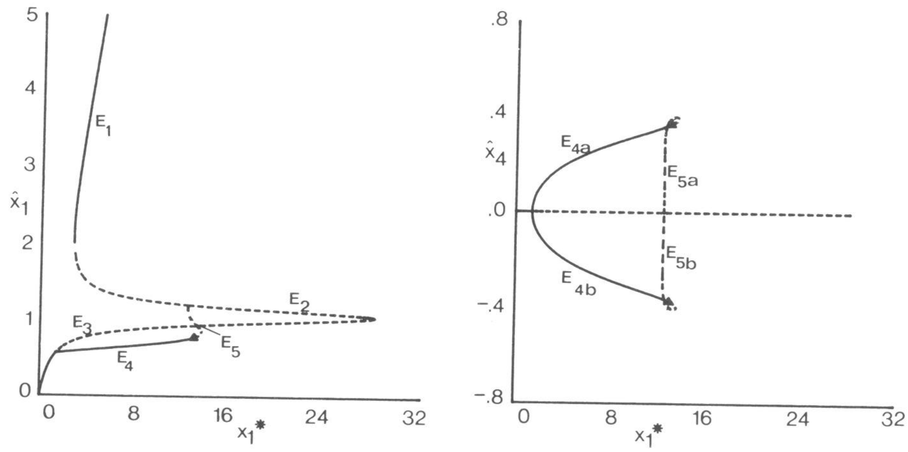

FIGURE 4. As Figure 2, but the x_1-component (left) and x_4-component (right) of the six-component model. A triangle denotes a Hopf bifurcation point.

6. EFFECT OF STOCHASTIC PERTURBATIONS ON LOW-ORDER SPECTRAL MODELS

A first attempt to model the effect of the unresolved modes on the three component model was carried out by EGGER [10]. He added stochastic forcing terms of Gaussian white noise with a fixed small intensity to the equations. The noise forces the system to visit alternately the attraction domains of the two stable equilibria, thus in this way an index cycle is simulated. A justification for choosing this type of forcing was given by EGGER and SCHILLING [11] who showed, using atmospheric data, that the forcing terms $F(t)$ in (4.1) can be modelled by coloured-noise processes. These are stationary and Gaussian Markov processes and contain white noise as a limit for the correlation time tending to zero. In DE SWART and GRASMAN [7] the effect of coloured-noise forcing on the three-component-model of Section 5 having three different stationary points is discussed. For simplicity we only consider the effect of white-noise forcing. Then Eq. (4.1) become

$$dx = f_u(x)dt + \epsilon dW \quad \text{in } \mathbb{R}^3, \tag{6.1}$$

where the three components of $W(t)$ are mutually independent Wiener processes and ϵ is the noise intensity which is assumed to be small ($\epsilon << 1$). Let the stable equilibria E_1 and E_3 have the attraction domains Ω_1 and Ω_3 with boundaries $\partial\Omega_1$ and $\partial\Omega_3$, respectively. We investigate the distribution of residence times $\tau(x)$ starting from a state $x \in \Omega_i$ $(i=1,3)$ of the system in these attraction domains. The expected value $<\tau(x)> = T(x)$ gives a measure of the persistence of a large-scale preferent state of the atmospheric circulation. In GORDINER [13] it is shown that $T(x)$ obeys

$$\frac{1}{2}\epsilon^2 \nabla^2 T(x) + f_\mu(x) \cdot \nabla T(x) = -1 \text{ in } \Omega_i, \qquad (6.2)$$

$$T=0 \text{ at } \partial\Omega_i \ (i=1 \text{ or } 3).$$

An asymptotic solution of this elliptic differential equation, valid for low-intensity noise ($\epsilon << 1$), is derived in MATKOWSKY ET AL. [20] by application of singular perturbation techniques. Outside a boundary layer near $\partial\Omega_i$ it reads

$$T_i \sim C_i e^{K_i/\epsilon^2}, \quad K_i = \lim_{x \to E_2} Q(x). \qquad (6.3)$$

Here $Q(x)$ is the solution of the eikonal equation

$$\frac{1}{2}(\nabla Q(x))^2 + f_\mu(x) \cdot \nabla Q(x) = 0, \quad Q(E_i) = 0, \qquad (6.4)$$

which can be solved by means of the ray method, see LUDWIG [19]. However, the computed residence times are of the order of months whereas from observations we expect life spans of weather regimes in the order of two weeks. We will discuss this indiscrepancy in Section 8. Furthermore, it is found that the most probable region of exit from the attraction domains is an ϵ-neighbourhood of the unstable equilibrium E_2. Here the system remains for a characteristic time

$$T_2 \sim \frac{1}{\lambda} \log\left(\frac{1}{\epsilon}\right), \qquad (6.5)$$

where λ is the largest positive real part of the deterministic system linearized at E_2.

Once the stochastic dynamical system is in its statistical equilibrium it is characterized by the expected residence times in the different regimes. However, in this way no information is obtained about the time scale over which the effect of initial conditions is important. This can be investigated with a discrete-state Markov process model. For the randomly forced spectral models discussed here we can derive such a model with three states: a zonal state (1), a transitional state (2) and a meridional state (3). Let Q_{ij} denote the transition probability per unit of time from state i to j and let $p_i(t)$ denote the probability for the system to be in state i at time t. Then the $p_i(t)$ satisfy

$$\begin{aligned} \dot{p}_1 &= -(Q_{12}+Q_{21})p_1 - Q_{21}p_3 + Q_{21}, \\ \dot{p}_3 &= -Q_{23}p_1 - (Q_{32}+Q_{23})p_3 + Q_{23}, \\ p_2 &= 1 - p_1 - p_3, \end{aligned} \qquad (6.6)$$

where

$$Q_{21} = Q_{23} = \frac{1}{2T_2}, \quad Q_{12} = \frac{1}{T_1}, \quad Q_{32} = \frac{1}{T_3}. \qquad (6.7)$$

In Figure 5 the probability functions $p_i(t)$ are given for a process with $T_1 = 9$, $T_2 = 1$ and $T_3 = 31$ that starts in state 1, 2 and 3, respectively.

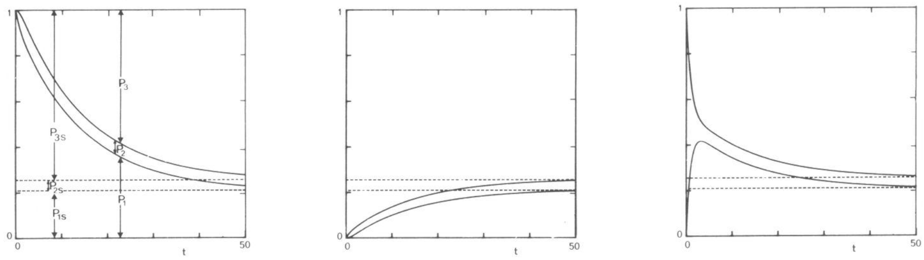

FIGURE 5. Evolution of the probability distribution of the Markov process starting in state 1 (left), 2 (middle) and 3 (right) respectively. The dotted lines represent the stationary distribution.

From these figures it is seen that once an initial state is given, the Markov model contains more information about the system than the stationary probability distribution for a period of about fifty days.

7. TEN COMPONENTS: DETERMINISTIC CHAOS AND VACILLATION

As discussed in Section 5, a second possibility for simulating an index cycle with spectral models of the quasi-geostrophic barotropic potential vorticity equation is to include more modes in the spectral expansions. LEGRAS and GHIL [16] have studied a 25-component model and found that solutions could visit different preferent regions in phase space. In DE SWART [6] a method is discussed to derive a 'minimum-order' spectral model which has, for fixed parameter values, multiple unstable regular solutions and a strange attractor. It is expected that trajectories starting from arbitrary initial conditions converge to this attractor. After that they must vacillate between different preferent regions in phase space which are close to the (weakly) unstable regular solutions. It is claimed that, by using a rectangular truncation of the eigenfunction expansions in wave number space, the minimum number of components is ten. The model describes the evolution of two zonal flow profiles (a (0,1) and (0,2) mode) and four waves (the (1,1), (1,2), (2,1) and (2,2) modes) in a barotropic atmosphere. Compared to the six-component model of Section 5 it contains a new type of nonlinear interaction involving three waves: the (1,1), (1,2) and (2,1) modes. In Figure 6 the x_1- and x_4-component of the stationary points of this model are shown for the same parameter values as discussed in Section 5. Due to the presence of the wave triad, isolated branches of equilibria occur. By letting x_4^* become nonzero all regular solutions may be turned unstable. For $x_4^* = -8$ the model represents a flow vacillating between three preferent regimes where the latter are actually unstable periodic solutions of the model, see Figure 7. The wave triad provides for interaction between clearly distinct scales of motion: a planetary scale and synoptic scale. This behaviour is similar to what is observed in the atmosphere.

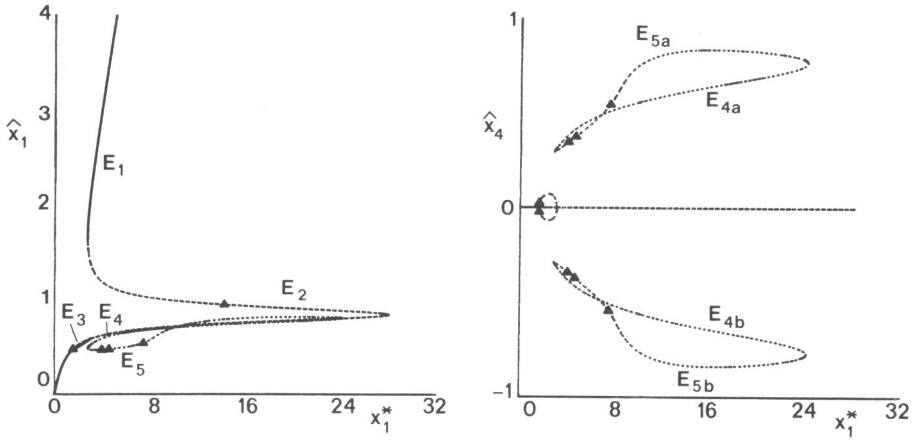

FIGURE 6. As Figure 4, but for the ten-component model.

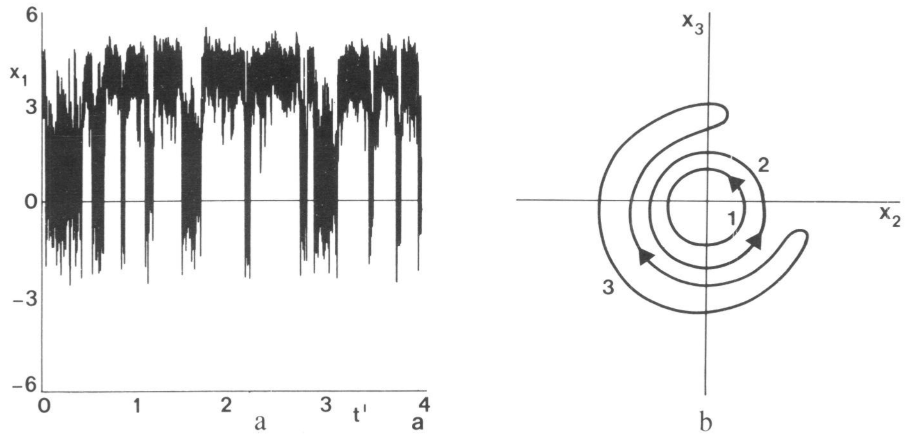

FIGURE 7a. The x_1-component of a chaotic solution of the ten-component model as a function of time. Here $t' = (t - 1000)/500$ and the dimensional period is approximately six years.

b. Sketch of the unstable periodic orbits projected onto the $x_2 - x_3$ plane. The preferent regions of the strange attractor are small tubes around these orbits.

By computing the spectrum of Lyapunov exponents, using the method of WOLF ET AL. [31] the existence of a global strange attractor is shown.

Lyapunov exponents measure the average divergence between nearby orbits in phase space whereas chaos is defined by at least one positive exponent. As discussed in FARMER ET AL. [12] from the spectrum of Lyapunov exponents we can estimate the fractal dimension of the attractor which also yields an upper bound to the number of degrees of freedom of the chaotic flow.

In practice initial conditions are never known with infinite precision. Thus small errors are introduced in the system which will grow during its evolution because of the chaotic dynamics. Consequently, the predictability of the flow is limited: a time scale of average prediction is given by the reciprocal of the sum of all positive Lyapunov exponents (SCHUSTER [25]). However, of more interest to meteorologists is the dependence of predictability on the state of the system (TENNEKES ET AL. [27]). In DE SWART [6] it is argued that the local eigenvalues at each point of an orbit may determine the time evolution of small errors on this orbit. In that case the eigenvectors corresponding to the eigenvalues with positive real part determine the geographical distribution of the error growth. However, this is on the condition that the time scale of error growth is small compared to the time scale on which the flow itself evolves. This method can be applied to spectral models showing long periods of quasi-stationary behaviour.

The impact of neglected short-scale modes on a planetary-scale model was studied by considering the chaotic ten-component model to represent the real atmosphere and the six-component model of Section 5 (for identical parameter values) to be a forecast model. For obtaining equivalence between solutions of the two systems forcing terms must be added to the equations of the forecast model. It appears that these forcing terms have an unpredictable nature and that they cannot be modelled by the simple stochastic processes used in Section 6. We will discuss these results in the next section.

8. Concluding remarks

In this final section we briefly discuss the relevance of our investigations to a better understanding of the atmospheric circulation. It was remarked in the introduction that an accurate modelling of the feedback between quasi-stationary planetary-scale motion and transient synoptic-scale eddies is important for the development of long-range weather forecast models. Here we have argued that this problem may be studied by considering simplified models which still represent the chaotic properties and vacillatory behaviour of the atmosphere. Next we investigated whether they provide clues on how to analyse more complicated models as well as real data.

As discussed in Section 5, already extremely low-order spectral models show qualitative features of the atmospheric circulation. They possess multiple equilibria for a range of parameter values and the corresponding flow patterns resemble large-scale preferent states of the atmospheric circulation. However, we remark that the existence of weather regimes has never been convincingly demonstrated by a systematic data analysis; only recently some indications have been found (BENZI ET AL. [1]).

It has been found that the three- and six-component models cannot simulate a flow vacillating between different weather regimes. In order to meet this imperfection, in Section 6 stochastic forcing terms were added to the spectral equations. They are assumed to account for the effect of neglected modes and physical processes not incorporated in the model. A justification for choosing white noise and coloured noise parametrizations is found from the data study of EGGER and SCHILLING [11]. The noise forces the system to visit alternately attraction domains of the stable equilibria. During a transition the system remains a characteristic time near an unstable equilibrium. This suggests that stable and unstable regular solutions of a spectral model may have some relevance for the dynamics of the atmospheric circulation. A method was discussed for computing expected residence times near the equilibria of the unperturbed system. Comparing the results with observational data it appears that the computed life spans of the weather regimes are a factor of 10 larger than those in the atmosphere.

A systematic way for investigating the effect of neglected modes on a truncated spectral model has been discussed in Section 7. Here a ten-component model is considered which is a 'minimum-order' model representing a finitely predictable flow having two distinct scales of motion (a planetary and synoptic scale) and vacillating between different preferent regimes. We assumed this model to represent the real atmosphere and considered a six-component subsystem as a forecast model. To the subsystem forcing terms were added such that its solutions are equivalent to those of the full model projected onto the modes which also belong to the subsystem. It was found that these forcing terms have an unpredictable nature and that they cannot be modelled by coloured-noise processes. This result is in agreement with that of LINDENBERG and WEST [17], who analysed explicit expressions for the forcing terms representing the effect of the neglected modes on truncated spectral models of the barotropic vorticity equation. It does not contradict the result of EGGER and SCHILLING [11] since the latter authors also include the effect of neglected physical processes in their definition of the forcing terms.

We remark that effects of topography are over-estimated in barotropic models since they act directly on the entire fluid column. Baroclinic multi-level models of the quasi-geostrophic potential vorticity equation give better results at this point. Again multiple equilibria are found (CHARNEY and STRAUS [3]) and due to the presence of baroclinic instability mechanisms vacillatory behaviour is even more easily produced. The two-level twenty-component model of REINHOLD and PIERREHUMBERT [24] is probably the simplest model containing all basic physical mechanisms: topographic, barotropic and baroclinic instability as well as the occurrence of wave triad interactions.

The problems with low-order spectral models in general is that unrealistically large external forcing values (corresponding to an equator-pole temperature difference of more than 150° C) are required in order to produce vacillatory behaviour. Moreover, the characteristic lives of the regimes in the models are much larger than those obtained from atmospheric data. These imperfections are probably due to the severe truncation in both the horizontal and vertical direction. A better description of the atmospheric circulation is expected from multi-level high resolution models. Since their structure is extremely complicated they are difficult to analyse. Alternatively, we can study lower-dimensional spectral models which include an appropriate parametrization of the synoptic forcing terms. This problem remains to be investigated in more detail.

ACKNOWLEDGEMENTS
I would like to thank Dr. J. Grasman and Dr. J.D. Opsteegh (both at the State University of Utrecht) for their various and valuable contributions to the research carried out in this project.

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Symbolic Computing Day

Spring WTI Meeting

A Symposium 'Symbolic Computing' organized by the Werkgemeenschap Theoretische Informatica (WTI) will take place on *Tuesday, April 19, 1988,* at the Centre for Mathematics and Computer Science (CWI), Kruislaan 413, in Amsterdam. Time: from *9.45 (sharp)* to *16.30.* Further information can be obtained from K.R. Apt (CWI, 020-5924135).

By 'Symbolic Computing' we mean programming on non-numeric domains using non-imperative methods. The lectures will attempt to delineate the area. They will be all of an introductory character.

PROGRAM

Relational databases
P.M.G. Apers (U. Twente)

Deductive databases
K.R. Apt (CWI, Amsterdam and UT Austin)

Functional programming
H.P. Barendregt (KU Nijmegen)

Logic programming and PROLOG
M. Bezem (CWI, Amsterdam)

Mathematical formula manipulation from a user's point of view
A.M. Cohen (CWI, Amsterdam and RU Utrecht)

Complexity of symbolic computing
P. van Emde Boas (UvA and CWI, Amsterdam)

Term rewriting systems
J.W. Klop (CWI, Amsterdam and VU Amsterdam)

Abstracts of Recent CWI Publications

When ordering any of the publications listed below please use the order form at the back of this issue.

CWI Tract 43. H.L. Bodlaender. *Distributed Computing, Structure and Complexity.*
AMS 68B20, 68C05; CR C.2.1; 294 pp.
Abstract: By connecting several processors or computer systems, for instance in a network, one obtains a distributed system. Distributed systems have important advantages over conventional (mainly) sequential computer systems. In this book some fundamental problems in the area of distributed computing are analyzed. An extensive analysis is made of the concept of uniform emulation: a method for obtaining structure-preserving, efficient simulations of (large) processor-networks on smaller processor-networks. Several new lower bounds and upper bounds are obtained for extrema-finding (or election) on rings of processors. An analysis is made of a fundamental load-balancing problem on rings of processors. Deadlock is an undesirable property of store-and-forward packet switching networks. A large class of controllers is introduced, that avoid store-and-forward deadlock and use only 'local information'.

CWI Syllabus 15. *Vacantiecursus 1983: Complexe getallen.*
Abstract: This syllabus (in Dutch) contains reprints of lectures on complex numbers, presented at the Vacation Course for high school mathematics teachers in 1983. They cover interesting aspects of complex numbers with applications in a variety of mathematical disciplines.

CS-R8735. K.R. Apt, L. Bougé & Ph. Clermont. *Two normal form theorems for CSP programs.*
AMS 68Q55, 68Q2; 12 pp.
Abstract: We define two normal forms for *CSP* programs. In the First Normal Form, each process

contains only one I/O repetitive command and all its I/O commands appear as guards of this command. In the Second Normal Form, it is moreover required that all guards of this I/O repetitive command are in fact I/O guards. We describe an inductive method which transforms any *CSP* program into an equivalent program in first or second normal form. The involved equivalence notion is discussed. It is shown in particular that no transformation into second normal form can preserve deadlock-freedom.

CS-R8736. M. Bezem. *Consistency of rule-based expert systems.*
AMS 68T30, 68T15; CR I.2.3, I.2.4, F.4.1; 10 pp.; **key words:** knowledge-based systems, rule-based expert systems, knowledge representation, consistency.
Abstract: Consistency of a knowledge-based system has become a topic of growing concern. Every notion of consistency presupposes a notion of semantics. We present a theoretical framework in which both the semantics and the consistency of a knowledge base can be studied. This framework is based on first order many-sorted predicate logic and is sufficiently rich to capture an interesting class of rule-based expert systems and deductive databases. We also provide criteria which allow us to isolate cases in which the consistency test is feasible.

CS-R8737. P.R.H. Hendriks. *Type-checking mini-ML: an algebraic specification with user-defined syntax.*
AMS 68BXX; CR D.2.1, D.3.1, F.3.1, F.3.2, F.4.2; 32 pp.; **key words:** software engineering, algebraic specifications, formal definition of programming languages, specification languages, executable specifications, user-definable syntax, syntax definition formalism, type-checker, polymorphism, type inference.
Abstract: An algebraic specification of a type-checker for Mini-ML, a sublanguage of ML, is given. As specification formalism a combination of the algebraic specification formalism ASF and the syntax definition formalism SDF is used.

CS-R8738. R. van Liere & P.J.W. ten Hagen. *Logical input devices and interaction.*
CR I.3.4, I.3.6; 15 pp.; **key words:** logical input devices, dialogue systems, dialogue programming.
Abstract: The *logical input device* model, as adopted in the standardized graphics packages GKS and PHIGS, has been an accepted basis for producing device-independent graphics systems. However, when used in highly *interactive* graphical applications, the logical input device model does not provide sufficient support for a number of fundamental issues inherent to interaction. This paper reopens a discussion which questions the functionality provided by the logical input device model when brought in conjunction with interaction. In particular, the logical input device model does not support the notion of input/output symmetry.

CS-R8739. J.C.M. Baeten & W.P. Weijland. *Semantics for prolog via term rewrite systems.* AMS 68Q45, 68Q50, 68N15; CR F.4.1, F.4.2, D.3.1, I.2.3, F.3.2; 12 pp.; **key words:** prolog, logic programming, term rewrite system, priority rewrite system, depth-first search.
Abstract: We present semantics for logic programs using term rewrite systems. Reading program lines from left to right (so reversing the arrows), considering the result as a rewrite system, immediately gives the usual declarative semantics (the least Herbrand model). Then, we add a priority ordering on the rewrite rules, and obtain a procedural semantics for Prolog with depth-first search rule. This gives us different semantics in the same setting.

CS-R8740. W.P. Weijland. *Semantics for logic programs without occur check.*
AMS 68Q40, 68Q55; CR F.3.1, F.3.2, F.4.1; 25 pp.; **key words:** logic programming, occur-check, infinite trees, fixed point semantics.

Abstract: For reasons of efficiency, in almost all implementations of Prolog the *occur check* is left out. This mechanism should protect the program against introducing *circular bindings of variables*. In practice the occur check is very expensive, however, and it is left to the skills of the user to avoid these circular bindings in the program. In this paper a semantics of Prolog without occur check is introduced, by considering circular bindings $\{x/f(x)\}$ as *recursive equations* $\{x=f(x)\}$. The new kind of resolution, i.e.: SLD-resolution without occur check, is referred to as *CSLD*-resolution. Important theorems such as soundness and completeness of CSLD-resolution are established. Moreover, the *finite failure set* turns out to be precisely the complement of the greatest fixed point of a monotonic mapping T'_P on the complete Herbrand base. Soundness and completeness of the negation as failure rule can be obtained in this new setting.

CS-R8741. K.R. Apt. *Introduction to logic programming.*
CR F.4.1, F.3.2; 55 pp.

Abstract: We provide a systematic and self-contained introduction to the theory of logic programming.

CS-R8742. P. Bernus & Z. Létray. *Intelligent systems interconnection: what should come after open systems interconnection?*
AMS 68T30; CR I.2.4, I.2.6, C.1.3, C.2.4; 12 pp.; **key words:** representation languages, agents, CAD, conversation theory, knowledge representation architectures.

Abstract: The battle for the best knowledge representation language seems to be terminated. Not because anyone had found one, but because fairly obviously there is none. We propose to turn our attention to general architectures of knowledge representation. If theories to be represented become specified, we can use such general constructs in the development of individual architectures. Theories are generally used by some agent for specific purposes. Agents need to communicate with each other in order to make use of their theories. We claim that a theory of computer-agenthood should describe communication and conversation among agents and the way they interact with the environment. A functional architecture for intelligent systems interconnection (ISI) is proposed. Subsuming functional layers of representations and theories are identified, without making commitments for particular choices in any one layer. We attempt to show the road to bring together represented conscious - and unconscious - as well as not represented inherent knowledge. We do it in order to combine intelligence with effectiveness.

CS-R8743. P.M.B. Vitányi. *Locality, communication, and interconnect length in multicomputers.*
AMS 68C05, 68C25, 68A05, 68B20, 94C99; CR B.7.0, C.2, D.4, F.2.7, G.2.4; 15 pp.; **key words:** multicomputers, complexity of computation, locality, communication, wire length, general communication network, edge-symmetric graph, n-cube, cube-connected cycles, tree, Euclidean embedding, scalability, optical computing.

Abstract: We derive a lower bound on the average interconnect (edge) length in d-dimensional embeddings of arbitrary graphs, expressed in terms of diameter and symmetry. It is optimal for all graph topologies we have examined, including complete graph, star, binary n-cube, cube-connected cycles, complete binary tree, and mesh with wrap-around (e.g., torus, ring). The lower bound is technology independent, and shows that many interconnection topologies of today's

multicomputers do not scale well in the physical world ($d=3$). The new proof technique is simple, geometrical and works for wires with zero volume, e.g., for optical (fibre) or photonic (fibreless, laser) communication networks. Apparently, while getting rid of the 'von Neumann' bottleneck in the shift from sequential to non-sequential computation, a new communication bottleneck arises because of the interplay between locality of computation, communication, and the number of dimensions of physical space. As a consequence, realistic models for non-sequential computation should charge extra for communication, in terms of time and space.

CS-R8744. V. Akman, P.J.W. ten Hagen & T. Tomiyama. *Design as a formal, knowledge engineered activity.*
CR D.2.m, I.2.1, I.2.3, I.2.4, I.2.5, J.6; 22 pp.; **key words:** intelligent CAD, design theory, logic, theory of knowledge, qualitative physics, object oriented programming, logic programming, knowledge engineering, prototyping.

Abstract: We summarize the framework and the preliminary results of the project *Intelligent Integrated Interactive CAD (IIICAD)* conducted at CWI. In e.g. mechanical computer aided engineering, currently much research aspires at using knowledge engineering but truly unifying approaches (i.e. 'more' than expert systems) are lacking. IIICAD aims at filling this gap using the theory of CAD. Here, we show the concepts and the architecture of IIICAD. We then formulate how IIICAD can help in integrating existing CAD tools and establishing a workbench for design. We demonstrate the relevance and use of qualitative physics in machine design. As for the IIICAD software development methodology, we consider several aspects of software engineering as well as knowledge engineering, for CAD systems tend to be very large. IIICAD has a kernel language based on logic programming and object oriented programming paradigms. We explain the combination of these paradigms and give a taste of our IIICAD prototype which is presently under construction.

CS-R8745. V. Akman, P.J.W. ten Hagen, J.L.H. Rogier & P. Veerkamp. *Knowledge engineering in design.*
CR I.2.3, I.2.4, J.6; 18 pp.; **key words:** nonstandard logics, object-oriented programming, naive physics, extensional/intensional descriptions.

Abstract: We present in a unifying framework the principles of the IIICAD (Intelligent, Integrated, and Interactive Computer-Aided Design) system. IIICAD is a generic design apprentice currently under development at CWI. IIICAD incorporates three kinds of design knowledge. First, it has general knowledge about the stepwise nature of design based on a set-theoretic design theory. Second, it has domain-dependent knowledge belonging to the specific design areas where it may actually be used. Finally, it maintains knowledge about the previously designed objects; this is somewhat similar to software reuse. Furthermore, IIICAD uses AI techniques in the following areas: (i) formalisation of design processes; extensional vs. intensional descriptions; modal and other nonstandard logics as knowledge representation tools, (ii) common sense reasoning about the physical world (naive physics); coupling symbolic and numerical computation, (iii) integration of object-oriented and logic programming paradigms; development of a common base language for design.

CS-R8746. R. van Liere & P.J.W. ten Hagen. *Resource management in DICE.*
CR I.3.4, I.3.5, I.3.6; 28 pp.; **key words:** user interface management systems, dialogue languages, resource management, window management.

Abstract: A framework is presented for integrating a general resource management facility into the dialogue cell language. It is shown that, by using resources as the basis for input and output, the coupling of input and output at the physical device level can be achieved. By integrating the resource manager in the dialogue cell language, correlations between higher level input and output

can be defined and maintained. Context sensitive resource rules are defined as extensions to the corresponding activation rules of dialogue cells themselves. By applying various inheritance mechanisms a resource specification can be done virtually. The dialogue run-time system dynamically binds these virtual specifications with physical devices. The resource manager is implemented by augmenting the dialogue grammar with specific resource information. In this way a potentially ambiguous dialogue can be unambiguously parsed. An $O(n\log n)$-time algorithm, with n indicating the number of overlapping windows, is given which detects ambiguous resource configurations.

CS-R8747. W.P.Weijland. *Correctness proofs for systolic algorithms: a palindrome recognizer.*

AMS 68Q35, 68Q60, 68Q10, 68Q55; CR B.7.1, D.2.4, F.3.2; 19 pp.; **key words:** concurrency, process algebra, systolic array, VLSI, correctness proof.

Abstract: In designing VLSI-circuits it is very useful, if not necessary, to construct the specific circuit by placing simple components in regular configurations. Systolic systems are circuits built up from arrays of cells and therefore very suitable for formal analysis and induction methods. In the case of a palindrome recognizer a correctness proof is given using bisimulation semantics with asynchronous cooperation. The proof is carried out in the formal setting of the *Algebra of Communicating Processes,* which provides us with an algebraic theory and a convenient proof system. An extensive introduction to this theory is included in this paper. The palindrome recognizer has also been studied by Hennessy in a setting of failure semantics with synchronous cooperation.

CS-R8748. J.A. Bergstra, J.W. Klop & E.-R. Olderog. *Readies and failures in the algebra of communicating processes.*

AMS 68Q05, 68Q10, 68Q55, 68Q45; CR F.1.2, F.3.1, F.3.2; 54 pp.; **key words:** process algebra, concurrency, readiness semantics, failure semantics, bisimulation semantics.

Abstract: Readiness and failure semantics are studied in the setting of ACP (Algebra of Communicating Processes). A model of process graphs modulo readiness equivalence, respectively failure equivalence, is constructed, and an equational axiom system is presented which is complete for this graph model. An explicit representation of the graph model is given, the failure model, whose elements are failure sets. Furthermore, a characterisation of failure equivalence is obtained as the maximal congruence which is consistent with trace semantics. By suitably restricting the communication format in ACP, this result is shown to carry over to subsets of Hoare's CSP and Milner's CCS. Also, the characterisation implies a full abstraction result for the failure model. In the above we restrict ourselves to finite processes without τ-steps. At the end of the paper a comment is made on the situation for infinite processes with τ-steps: notably we obtain that failure semantics is incompatible with Koomen's fair abstraction rule, a proof principle based on the notion of bisimulation. This is remarkable because a weaker version of Koomen's fair abstraction rule is consistent with (finite) failure semantics.

CS-R8749. J. Heering, P. Klint & J. Rekers. *Principles of lazy and incremental program generation.*

AMS 68N20; CR D.1.2, D.3.4; 8 pp.; **key words:** program generator, fourth generation language, greedy, lazy, and incremental program generation, lazy and incremental generation of lexical scanners, lazy and incremental generation of parsers, lazy and incremental compilation, dynamic compilation.

Abstract: Current program generators usually operate in a *greedy* manner in the sense that a program must be generated in its entirety before it can be used. If generation time is scarce, or if the

input to the generator is subject to modification, it may be better to be more cautious and to generate only those parts of the program that are indispensable for processing the particular data at hand. We call this *lazy program generation.* Another, closely related, strategy is *incremental program generation.* When its input is modified, an incremental generator will try to make a corresponding modification in its output rather than generate a completely new program. It may be advantageous to use a combination of both strategies in program generators that have to operate in a highly dynamic and/or interactive environment.

CS-R8750. S.J. Mullender & P.M.B. Vitányi. *Distributed match-making.*
AMS 68C05, 68C25; CR C.2.1, F.2.2, G.2.2; 25 pp.; **key words:** locating objects, locating services, mutual exclusion, replicated data management, distributed algorithms, computational complexity, store-and-forward computer networks, network topology.

Abstract: In many distributed computing environments, processes are concurrently executed by nodes in a store-and-forward communication network. Distributed control issues as diverse as name server, mutual exclusion, and replicated data management involve making matches between such processes. We propose a formal problem called 'distributed match-making' as the generic paradigm. Algorithms for distributed match-making are developed and the complexity is investigated in terms of messages and in terms of storage needed. Lower bounds on the complexity of distributed match-making are established. Optimal algorithms, or nearly optimal algorithms, are given for particular network topologies.

CS-R8751. Ming Li & P.M.B. Vitányi. *A very simple construction for atomic multiwriter register.*
AMS 68C05, 68C25, 68A05, 68B20; CR B.3.2, B.4.3, D.4.1, D.4.4; 7 pp.; **key words:** shared variable (register), concurrent reading and writing, atomicity, multiwriter variable.

Abstract: This paper introduces a new and conceptually very simple algorithm to implement an atomic n-reader n-writer variable directly from atomic 1-reader 1-writer variables, using bounded tags. The algorithm is developed top-down from the Vitányi-Awerbuch unbounded tag method. This is the first direct such construction, and considerably improves the complexity of all known compound constructions. The algorithm uses new techniques, but its main virtue is that it is *conceptually very simple and easily proved correct.*

CS-R8752. Ming Li &, P.M.B. Vitányi. *Tape versus queue and stacks: The lower bounds.*
AMS 68C40, 68C25, 68C05, 94B60, 10-00; CR F.1.1, F.1.3, F.2.3; 24 pp.; **key words:** multitape Turing machine, stack, queue, pushdown stores, determinism, nondeterminism, on-line, off-line, time complexity, lower bounds, simulation by one tape, Kolmogorov complexity.

Abstract: Several new optimal or nearly optimal lower bounds are derived on the time needed to simulate queues, stacks (stack = pushdown store) and tapes by one off-line single-head tape-unit with one-way input, for both the deterministic case and the nondeterministic case. The techniques rely on algorithmic information theory (Kolmogorov complexity).

CS-R8753. M.M. de Ruiter. *C-GKS, a C implementation of GKS, the graphical kernel system.*
AMS 69K32, 69K34, 69K30; 83 pp.; **key words:** computer graphics, graphics systems, standardization.

Abstract: GKS, the ISO international standard for 2D graphics software, is specified in a language independent form. This document contains the GKS language binding to C of the GKS datatypes and functions. Furthermore, it contains remarks about the implemented GKS functions. The user of this document is supposed to be familiar with GKS.

CS-R8754. E. Kranakis & K.N. Oikonomou. *Fixpoint representations of characteristic sets of linear-time temporal formulas.*
AMS 68N05, 68N15; CR F.3.1, F.4.1; 33 pp.; **key words:** temporal logic, fixpoint, set-transformer, temporal functional equation, walk, universal and existential characteristic set.

Abstract: We present an algebraic-axiomatic method for computing existential and universal characteristic sets of linear-time temporal logic formulas on directed graphs. The set of all nodes ν of a given graph (model) such that all (respectively, some) infinite walks starting from ν satisfy a formula ϕ is called the universal (respectively, existential) characteristic set of ϕ. We reduce the computation of the characteristic set to finding the least or greatest fixpoint of a system of set equations. Our method is sufficient to handle the following subsets of the logic $L(\Box,\Diamond,\bigcirc,\wedge,\vee,\sim)$: formulas in which the temporal connective $\Diamond$ applies only to boolean subformulas, formulas in which $\Box$ does not occur, and formulas that express general fairness properties of concurrent systems, such as impartiality, justice, and fairness. The representation of the characteristic sets obtained are model-independent, in the sense that the same representation holds for all graphs, and regardless of whether or not they are finite or infinite.

CS-R8755. J.N. Kok & J.J.M.M. Rutten. *Contractions in comparing concurrency semantics.*
AMS 68B10, 68C01; CR D.3.1, F.3.2, F.3.3; 40 pp.; **key words:** concurrency, imperative languages, denotational semantics, operational semantics, metric spaces, contractions, semantic equivalence.

Abstract: We define for a number of concurrent imperative languages both operational and denotational semantic models as fixed points of contractions on complete metric spaces. Next, we develop a general method for comparing different semantic models by relating their defining contractions and exploiting the fact that contractions have a unique fixed point.

CS-R8756. Y. Yamaguchi, F. Kimura & P.J.W. ten Hagen. *Interaction management in CAD systems with a history mechanism.*
CR H.1.2, I.3.5, I.3.6, J.6, D.4.8; 12 pp.

Abstract: User friendliness is one of the unresolved problems in CAD systems. There are many possible directions for improving user friendliness. Understanding of the modeling process is one of the most important directions. It is natural for a user to describe the model in terms of its evolution. We call this concept *model derivation.* To construct and use model derivation, we propose a *history mechanism* which keeps and manipulates the history of the modeling process. The history mechanism manages high level interactions by introducing powerful symbolic computation to manipulate the history. Since the history representation is based on the operation's syntax and separated from the internal model representation, it is easy to apply the history mechanism to any modeling system which uses established techniques. Thus the system designer can easily introduce model derivation without reducing efficiency of the implementation.

CS-R8757. L.C. van der Gaag. *A network approach to the certainty factor model.*
AMS 68TXX; CR I.2.3, I.2.5; 17 pp.; **key words:** expert systems, plausible

reasoning, certainty factor model.

Abstract: Most expert knowledge is of an ill-defined and heuristic nature. Therefore, many present-day rule-based expert systems include a mechanism for modelling and manipulating imprecise knowledge. For a long time, probability theory has been the primary quantitative approach for handling uncertainty. Other mathematical models of uncertainty have been proposed during the last decade, several of which depart from probability theory. The certainty factor model proposed by the authors of the MYCIN system is an example of an ad hoc model. The aim in developing the model was primarily to develop a method that was of practical use. The certainty factor model is computationally simple, a property that has led to its considerable success. In this paper, we use so-called inference networks to demonstrate the application of the model in a rule-based top-down reasoning expert system. This approach enables us to show some inadequacies of the notational convention used by the creators of the model. We propose a syntactically correct formalism and use this formalism to discuss several properties of the model.

CS-R8758. A. Israeli, M. Li & P.M.B. Vitányi. *Simple multireader registers using time-stamp schemes.*
AMS 68C05, 68C25, 68A05, 68B20; CR B.3.2, B.4.3, D.4.1, D.4.4; 8 pp.; **key words:** shared variable (register), concurrent reading and writing, atomicity, multiwriter variable.

Abstract: We use the theory of time-stamp schemes to implement an atomic 1-writer n-reader variable (register) from n^2 atomic 1-writer 1-reader variables, using bounded time-stamps. The number of time-stamps needed is $(2n+2)^2$, so this scheme uses $O(n^2 \log n)$ control bits altogether. The construction is simple, transparent, and optimal in worst-case number of control bits per subvariable. A similar scheme is given that uses only 2-bit variables written by readers, and $2n$-bit variables written by the writer. This uses altogether $O(n^2)$ control bits altogether. This scheme is optimal in worst-case overall number of control bits. Apart from being optimal in several ways, our constructions add an intuitive dimension which lacks in previous algorithms for this problem.

CS-R8759. J.J.M.M. Rutten & J.I. Zucker. *A semantic approach to fairness.*
AMS 68B10, 68C01, 68C05; CR D.1.3, F.1.2; 25 pp.; **key words:** fairness, semantic domains of metric processes, fair infinite iteration, alternation of random choices.

Abstract: In the *semantic* framework of metric process theory, we undertake a general investigation of *fairness* of processes from two points of view: (1) *intrinsic fairness* of processes, and (2) *fair operations* on processes. Regarding (1), we shall define a 'fairification' operation on processes called **Fair** such that for every (generally unfair) process p the process **Fair**(p) is fair, and contains precisely those paths of p that are fair. Its definition uses systematic alternation of random choices. The second part of this paper treats the notion of fair operations on processes: suppose given an operator on processes (like merge, or infinite iteration), we want to define a fair version of it. For the operation of infinite iteration we define a fair version, again by a 'fair scheduling' technique.

CS-R8761. J. Heering, P. Klint & J. Rekers. *Incremental generation of lexical scanners.*
AMS 68N20; CR D.1.2, D.3.4; 24 pp.; **key words:** program generator, lazy and incremental generation of lexical scanners, finite automata, subset construction.

Abstract: It is common practice to specify textual patterns by means of a set of regular expressions and to transform this set into a finite automaton to be used for the scanning of input strings. In many applications, the costs of this preprocessing phase can be amortized over many uses of the constructed automaton. In this paper new techniques for lazy and incremental scanner generation are presented. The lazy technique postpones the construction of parts of the automaton until they

are really needed during the scanning of input. The incremental technique allows modifications to the original set of regular expressions to be made and reuses as many parts of the previous automaton as possible. This is interesting in situations where modifications to the definition of lexical syntax and the use of the generated scanner alternate frequently, for instance, in environments for the interactive development of language definitions.

CS-R8762. H.A. Lauwerier & J.A. Kaandorp. *Fractals (mathematics, programming and applications).*
AMS 58F13, 69K30, 69K34; 33 pp.; **key words:** fractals, self-similar sets, non-linear complex mappings, fractal growth.
Abstract: This tutorial consists of three subjects. The first subject is the mathematics of fractals, several classes of fractals (Cantor sets, Koch's curve, Levy's Curve Mandelbrot and Julia sets) are discussed together with self-similarity and fractal dimension. The second subject is the generation of fractal objects, in which several methods are discussed for creating fractals (formal languages, Iterated Function Systems, geometric construction of fractals, non-linear complex mappings). The last subject is the application of fractals for modelling natural objects natural. In this subject the estimation of the fractal dimension and simulation of natural objects from nature are discussed.

OS-R8712. O.J. Boxma & W.P. Groenendijk. *Two queues with alternating service and switching times.*
AMS 30E25, 60K25, 68M20; 20 pp.; **key words:** queueing system, alternating service, switching time, Riemann boundary value problem.
Abstract: This paper is concerned with a system of two queues, attended by a single server who alternately serves one customer of each queue (if not empty). The server experiences switching times in his transition from one queue to the other. It is shown that the joint stationary queue-length distribution, at the instants at which the server becomes available to a queue, can be determined via transformation to a Riemann boundary value problem. The latter problem can be completely solved for general service- and switching-time distributions. The stationary distributions of the waiting times at both queues, and of the cycle times of the server, are also derived. The obtained results, and in particular the extensive numerical data for moments of waiting times and cycle times, yield insight into the behaviour of more general cyclic-service models. Such models are frequently used to analyse polling systems.

OS-R8713. P.R. de Waal. *Performance analysis and optimal control of an $M/M/1/k$ queueing system with impatient customers.*
AMS 60K25, 68M20 90B22, 93E20; 16 pp.; **key words:** communication systems, queues, impatient customers, stochastic control.
Abstract: A simple $M/M/1/k$ queue with impatient customers is presented as a model for communication systems operating under overload conditions. The performance analysis and optimal control problem for this model are discussed. An efficient algorithm for computing the optimal control is presented along with numerical results.

OS-R8714. J.K. Lenstra, D.B. Shmoys & E. Tardos. *Approximation algorithms for scheduling unrelated parallel machines.*
AMS 90B35, 90C27, 68Q25, 68R05; 10 pp.; **key words:** scheduling, parallel machines, approximation algorithm, worst case analysis, linear programming, integer programming, rounding.
Abstract: We consider the following scheduling problem. There are m parallel machines and n independent jobs. Each job is to be assigned to one of the machines. The processing of job j on

machine i requires time p_{ij}. The objective is to find a schedule that minimizes the makespan. Our main result is a polynomial algorithm which constructs a schedule that is guaranteed to be no longer than twice the optimum. We also present a polynomial approximation scheme for the case that the number of machines is fixed. Both approximation results are corollaries of a theorem about the relationship of a class of integer programming problems and their linear programming relaxations. In particular, we give a polynomial method to round the fractional extreme points of the linear program to integral points that nearly satisfy the constraints. In contrast to our main result, we prove that no polynomial algorithm can achieve a worst-case ratio less then 3/2 unless $P = NP$. We finally obtain a complexity classification for all special cases with a fixed number of processing times.

OS-R8715. M. Desrochers, J.K. Lenstra, M.W.P. Savelsbergh & F. Soumis. *Vehicle routing with time windows: optimization and approximation.*
AMS 90B05, 90B35, 90C27; 15 pp.; **key words:** traveling salesman problem, vehicle routing problem, pickup and delivery problem, dial-a-ride problem, time window constraints, branch and bound, dynamic programming, state space relaxation, set partitioning, column generation, construction, iterative improvement, incomplete optimization.
Abstract: This is a survey of solution methods for routing problems with time constraints. Among the problems considered are the traveling salesman problem, the vehicle routing problem, the pickup and delivery problem, and the dial-a-ride problem. We present optimization algorithms that use branch and bound, dynamic programming and set partitioning, and approximation algorithms based on construction, iterative improvement and incomplete optimization.

OS-R8716. J.M. Anthonisse, J.K. Lenstra & M.W.P. Savelsbergh. *Functional description of CAR, an interactive system for computer aided routing.*
AMS 90B05, 90C27, 90C50, 68U05; 15 pp.; **key words:** physical distribution, clustering, routing, man-machine interaction, color graphics.
Abstract: CAR is an interactive software package which can be used to support operational distribution management. It has been developed at the Centre for Mathematics and Computer Science (CWI) in the period 1983-1986. This document contains a general description of CAR, a detailed description of the interface between CAR and software in its environment (data entry and report generation), and a user manual in the form of a functional description of all available commands.

OS-R8717. O.J. Boxma & G.A.P. Kindervater. *A queueing network model for analyzing a class of branch and bound algorithms on a master-slave architecture.*
AMS 60K25, 68M20, 68Q10, 90C27; 16 pp.; **key words:** parallel computing, branch and bound, queueing network, fluid flow approximation.
Abstract: Partitioning methods lend themselves very well to implementation on parallel computers. In recent years, branch and bound algorithms have been tested on various types of architectures. In this paper, we develop a queueing network model for the analysis of a class of branch and bound algorithms on a master-slave architecture. The analysis is based on a fluid flow approximation. Numerical examples illustrate the concepts developed. Finally, related branch and bound algorithms are studied using a machine repair queueing model.

OS-R8718. A. Schrijver. *Edge-disjoint homotopic paths in straight-line planar graphs.*
AMS 05C10, 68R10, 90C27, 05C38, 57M15; 15 pp.; **key words:** planar, graph, homotopic, edge-disjoint, VLSI, straight-line.

Abstract: Let G be a planar graph, embedded without crossings in the euclidean plane $\mathbb{R}^2$, and let $I_1, \ldots, I_p$ be some of its faces (including the unbounded face), considered as open sets. Suppose there exist (straight) line segments $L_1, \ldots, L_t$ in $\mathbb{R}^2$ such that $G \cup I_1 \cdots \cup I_p = L_1 \cup \cdots \cup L_t \cup I_1 \cup \cdots \cup I_p$ and such that each L_i has its end points in $L_1 \cup \cdots \cup I_p$. Let $C_1, \ldots, C_k$ be curves in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$ with end points in vertices of G. We describe conditions under which there exist pairwise edge-disjoint paths $P_1, \ldots, P_k$ in G so that P_i is homotopic to C_i in $\mathbb{R}^2 \setminus (I_1 \cup \cdots \cup I_p)$, for $i = 1, \ldots, k$. This extends results of Kaufmann and Mehlhorn for graphs derived from the rectangular grid.

OS-R8719. A. Schrijver. *Decomposition of graphs on surfaces and a homotopic circulation theorem.*

AMS 05CXX, 57MXX, 90C27, 05C10, 05C38, 57N06; 49 pp.; **key words:** surface, homotopic, graph, curve, circulation, VLSI, crossing.

Abstract: We prove the following theorem. Let G be an Eulerian graph embedded (without crossings) on a compact orientable surface S. Then the edges of G can be decomposed into cycles $C_1, \ldots, C_t$ in such a way that for a closed curve D on S:

$$\text{mincr}\,(G,D) = \sum_{i=1}^{t} \text{mincr}\,(C_i,D).$$

Here $\text{mincr}(G,D)$ denotes the minimum number of crossings of G and $\tilde{D}$, among all closed curves $\tilde{D}$ homotopic to D (so that $\tilde{D}$ does not intersect vertices of G). Similarly, $\text{mincr}(C,D)$ denotes the minimum number of crossings of $\tilde{C}$ and $\tilde{D}$, among all closed curves $\tilde{C}$ and $\tilde{D}$ homotopic to C and D, respectively. As a corollary we derive the following 'homotopic circulation theorem'. Let G be a graph embedded on a compact orientable surface S, let $c: E \rightarrow \mathbb{Q}_+$ be a 'capacity' function, let $C_1, \ldots, C_k$ be cycles in G, and let $d_1, \ldots, d_k \in \mathbb{Q}_+$ be 'demands'. Then there exist circulations $x_1, \ldots, x_k$ in G so that each x_i decomposes fractionally into d_i cycles homotopic to C_i $(i = 1, \ldots, k)$ and so that the total flow through any edge does not exceed its capacity, if and only if for each closed curve D on S not intersecting vertices of G we have that the sum of the capacities of the edges intersected by D (counting multiplicities) is not smaller than $\Sigma_{i=1}^{k} d_i \cdot \text{mincr}(C_i, D)$. This applies to a problem posed by K. Mehlhorn in relation to the automatic design of integrated circuits.

OS-R8720. G.A.P. Kindervater & J.K. Lenstra. *Parallel computing in combinatorial optimization.*

AMS 90C27, 68Q15, 68Q25, 68RXX; 33 pp.; **key words:** parallel computer, computational complexity, polylog parallel algorithm, $\mathcal{P}$-completeness, sorting, shortest paths, minimum spanning tree, matching, maximum flow, linear programming, knapsack, scheduling, traveling salesman, dynamic programming, branch and bound.

Abstract: This is a review of the literature on parallel computers and algorithms that is relevant for combinatorial optimization. We start by describing theoretical as well as realistic machine models for parallel computations. Next, we deal with complexity theory for parallel computations and illustrate the resulting concepts by presenting a number of polylog parallel algorithms and $\mathcal{P}$-completeness results. Finally, we discuss the use of parallelism in enumerative methods.

OS-R8721. M. Desrochers, J.K. Lenstra & M.W.P. Savelsbergh. *A classification scheme for vehicle routing and scheduling problems.*

AMS 90B05, 90B35, 68T30; 12 pp.; **key words:** classification, routing, scheduling, model, algorithm.

Abstract: We propose a classification scheme for a class of models that arise in the area of vehicle

routing and scheduling and illustrate it on a number of problems that have been considered in the literature. The classification scheme may serve as a first step towards the development of a model and algorithm management system in this area.

OS-N8702. J.M. Anthonisse, K.M. van Hee & J.K. Lenstra. *Resource-constrained project scheduling: an international exercise in DSS development.*
AMS 90B35, 90C50; 11 pp.; **key words:** decision support system, resource-constrained project scheduling, comparative evaluation.

Abstract: The International Institute for Applied Systems Analysis in Laxenburg, Austria, coordinates an international exercise in the development of decision support systems. The participants will independently develop a number of interactive planning systems for resource-constrained project scheduling, in the hope of generating knowledge and experience in the design, analysis and implementation of decision support systems. This report specifies the rules of the exercise.

NM-R8715. W.M. Lioen, M. Louter-Nool & H.J.J. te Riele. *Optimization of the real level 2 BLAS on the Cyber 205.*
AMS 65V05, 65F05, 65F30; CR 5.14, 4.6; 13 pp.; **key words:** matrix-vector operations, Level 2 BLAS, Cyber 205 optimization.

Abstract: The results of the implementation and optimizaton of the real Level 2 BLAS routines on the Cyber 205 vectorcomputer are presented. The Level 2 BLAS routines perform three types of matrix-vector operations, viz., matrix-vector multiplication, rank-1 and rank-2 updates, and solution of triangular systems of equations. The performance of the routines varies between 60% and 80% of the maximum Cyber 205 performance, for general matrices of order 500, and for band matrices of order 30000 with 6 non-zero diagonals.

NM-R8716. C. Nebbeling & B. Koren. *An experimental-computational investigation of transonic shock wave-turbulent boundary layer interaction in a curved test section.*
AMS 35B30, 65N50, 76G15, 76H05; 17 pp.; **key words:** steady Euler equations, transonic flows, grid generation and adaptation, boundary conditions.

Abstract: This paper describes an experimental investigation of a transonic shock wave-turbulent boundary layer interaction in a curved test section in which the flow has been computed by a 2-D Euler flow method. The test section has been designed such that the flow field near the shock wave at the convex wall corresponds to that near the shock wave at the upper surface of a transonic airfoil. The ratio between the radius of curvature of the wind tunnel wall and the thickness of the undisturbed boundary layer is about 80, being a mean value for modern transonic wings at cruising flight conditions. The Mach number distributions from the Euler flow computations are compared to those obtained from holographic interferometry, at flow Mach numbers upstream of the shock wave of 1.15 and 1.37. For these two Mach numbers boundary layer measurements in the interaction region have been performed by means of static pressure and pitot pressure probe traverses. Moreover, extended surface pressure measurements have been made at several upstream Mach numbers M_u. In particular attention is paid to the effects of flow curvature and static pressure increase downstream of the shock wave, in relation to changing boundary layer parameters and separation phenomena.

NM-R8717. W.H. Hundsdorfer. *Convergence of Runge-Kutta methods on classes of stiff initial value problems.*
AMS 65L05, 65M20; 9 pp.; **key words:** numerical analysis, stiff initial value problems, implicit Runge-Kutta methods, *B*-convergence.

Abstract: For certain stiff initial value problems the order of convergence of implicit Runge-Kutta methods can be much lower than for nonstiff problems. In this paper we consider for some classes of stiff initial value problems convergence results which are independent of the stiffness, such as the B-convergence results for nonlinear dissipative problems.

NM-R8718. J. Kok. *Proposal for standard mathematical packages in Ada.*
AMS 69D49, 65-04; 23 pp.; **key words:** Ada, high level language, elementary functions, mathematical packages, standard functions, scientific libraries.

Abstract: On behalf of the Ada-Europe Numerics Working Group we propose Ada packages of mathematical types, constants, operators and subprograms to be added to the standard Ada program library. These include packages of elementary mathematical functions, of mathematical constants, of random number generators, and of (cartesian and polar) complex types and related arithmetic operations.

NM-R8719. P.P.M. de Rijk. *NUMVEC FORTRAN library manual. Chapter: Simultaneous linear equations. Routine: SVDTJP and LSQMNS.*
AMS 65F15, 65F20, 65F25, 15A18; CR 5.14; 9 pp.; **key words:** least squares problems, one-sided Jacobi algorithm, rank deficiency, Singular Value Decomposition, vector computing.

Abstract: This document describes two NUMVEC FORTRAN Library routines. SVDTJP computes the Singular Value Decomposition of a real rectangular matrix A, using a one-sided Jacobi algorithm. LSQMNS finds the minimum norm least-squares solution of a system $Ax \approx b$,' where the real rectangular matrix A has been decomposed into its Singular Value Decomposition using SVDTJP.

NM-R8720. E.D. de Goede & J.H.M. ten Thije Boonkkamp. *Vectorization of the odd-even hopscotch scheme and the alternating direction implicit scheme for the two-dimensional Burgers' equations.*
AMS 65V05, 65M05, 76DXX; 15 pp.; **key words:** vector computers, Burgers' equations, odd-even hopscotch scheme, alternating direction implicit scheme, vectorization.

Abstract: A vectorized version of the odd-even hopscotch (OEH) scheme and the alternation direction implicit (ADI) scheme have been implemented on vector computers for solving two-dimensional Burgers' equations on a rectangular domain. This paper examines the efficiency of both schemes on vector computers. Data structures and techniques employed in vectorizing both schemes are discussed, accompanied by performance details.

NM-R8721. J.H.M. ten Thije Boonkkamp. *Residual smoothing for accelerating the ADI iteration method for elliptic difference equations.*
AMS 65F10, 65N20; 14 pp.; **key words:** elliptic difference equation, ADI iteration, residual smoothing, smoothed ADI iteration.

Abstract: Residual smoothing is a simple technique for accelerating the rate of convergence of iterative methods for elliptic difference equations. In this paper, we combine residual smoothing with the ADI iteration method, which can be done in several ways. When applied in the proper way, residual smoothing can considerably reduce the number of iterations and thus the computing time of the ADI scheme. The parameter values of the smoothed ADI scheme are chosen such that the high- and low frequency components in the iteration error are damped very well. Due to the residual smoothing, the other components in the error are also properly damped. Numerical examples demonstrate the performance results of the ADI scheme and the smoothed ADI scheme.

NM-R8722. W.M. Lioen. *Multigrid methods for elliptic PDEs.*
AMS 65V05, 65N20, 65F10; CR 5.17; 15 pp.; **key words:** elliptic PDEs, Galerkin approximation, multigrid methods, software, sparse linear systems, zebra relaxation.

Abstract: After a brief introduction in multigrid methods we discuss some of the algorithmic choices in the NUMVEC[1] Library routine MGZEB (which is a highly vectorised multigrid code for the solution of linear systems resulting from the 7-point discretisation of general linear 2^{nd} order elliptic PDEs in two dimensions). Since the relaxation process is the most expensive part of a multigrid iteration cycle, we adapted the datastructure to avoid Cyber 205 stride-problems when executing zebra relaxation. After discussing the effects of vectorisation and of choosing another datastructure, we will also have a glance at large problems on the Cyber 205.

NM-R8723. P.J. van der Houwen, B.P. Sommeijer & G. Pontrelli. *A comparative study of Chebyshev acceleration and residue smoothing in the solution of nonlinear elliptic difference equations.*
AMS 65N10; 23 pp.; **key words:** numerical analysis, elliptic boundary value problems, smoothing matrices.

Abstract: We compare the traditional and widely-used Chebyshev acceleration method with an acceleration technique based on residue smoothing. Both acceleration methods can be applied to a variety of function iteration methods and allow therefore a fair comparison. The effect of residue smoothing is that the spectral radius of the Jacobian matrix associated with the system of equations can be reduced substantially, so that the eigenvalues of the iteration matrix of the iteration method used are considerably decreased. Comparative experiments clearly indicate that residue smoothing is superior to Chebyshev acceleration. For a model problem we show that the rate of convergence of the smoothed Jacobi process is comparable with that of ADI methods. The smoothing matrices by which the residue smoothing is achieved, allow for a very efficient implementation, thus hardly increasing the computational effort of the iteration process. Another feature of residue smoothing is that it is directly applicable to nonlinear problems without affecting the algorithmic complexity. Moreover, the simplicity of the method offers excellent prospects for execution on vector and parallel computers.

NM-R8724. W.H. Hundsdorfer & J.G. Verwer. *Stability and convergence of the Peaceman-Rachford ADI method for initial boundary value problems.*
AMS 65M10, 65M15, 65M20; CR 5.17; 28 pp.; **key words:** numerical analysis, time dependent PDEs, alternating direction implicit methods, Peaceman-Rachford method, method of lines, stability, error bounds.

Abstract: In this paper an analysis will be presented for the ADI (alternating direction implicit) method of Peaceman and Rachford applied to initial boundary value problems for partial differential equations in two space dimensions. We shall use the method of lines approach. Motivated by developments in the field of stiff nonlinear ordinary differential equations, our analysis will focus on problems where the semi-discrete system, obtained after discretization in space, satisfies a one sided Lipschitz condition with a constant independent of the grid spacing. For such problems unconditional stability and convergence results will be derived.

NM-R8725. J.J.F.M. Schlichting & H.A. van der Vorst. *Solving bidiagonal systems of linear equations on the CDC Cyber 205.*

1. NUMVEC is a CWI library of NUMerical software for VECtor computers in FORTRAN.

AMS 65F05, 65V05, 69C12; CR 5.14, 4.6; 46 pp.; **key words:** bidiagonal linear system, Cyber 205, scalar optimization, cyclic reduction, recursive doubling, vectorization.

Abstract: This paper examines the efficiency of some different techniques for the solution of bidiagonal systems of linear equations on a CDC Cyber 205. Special attention is paid to exploiting the capabilities of the scalar processor and the estimation of execution times. Three categories of algorithms for the solution of bidiagonal systems of linear equations are described. The first category consists of straightforward scalar algorithms written in standard FORTRAN, optimized by means of commonly known techniques like loop unrolling. The second category consists of vector algorithms on recursive doubling, cyclic reduction and a partitioning technique. The third category consists of scalar codes written in assembly code designed to fully exploit the parallelism in the scalar processor; a method is developed to predict the execution time of optimal code for recursive problems. The predicted and measured performances of the routines described are compared and analysed.

MS-R8707. R.D. Gill & S. Johansen. *Product-integrals and counting processes.*
AMS 60J27, 62G05, 60H20, 45D05; 36 pp.; **key words:** Markov process, multiplicative integral, Volterra integral equation, intensity measure, exponential semimartingale, compact (Hadamard) differentiability, survival analysis, product-limit (Kaplan-Meier) estimator.

Abstract: The basic theory of the product-integral $\Pi(\mathbf{1}+dX)$ is summarized and applications in probability and statistics are discussed, in particular to non-homogeneous Markov processes, counting process likelihoods and the product-limit estimator.

MS-R8708. R. Helmers. *On the Edgeworth expansion and the bootstrap approximation for a studentized U-statistic.*
AMS 62E20, 62G05, 60F05; 15 pp.; **key words:** Edgeworth expansions, bootstrap approximations, studentized U-statistics, bootstrap confidence intervals, Edgeworth based confidence intervals, studentized L-statistics, studentized M-estimators.

Abstract: The asymptotic accuracy of the estimated one-term Edgeworth expansion and the bootstrap approximation for a studentized U-statistic is investigated. It is shown that both the Edgeworth expansion estimate and the bootstrap approximation are asymptotically closer to the exact distribution of a studentized U-statistic than the normal approximation. The conditions needed to obtain these results are weak moment assumptions on the kernel h of the U-statistic and a non-lattice condition for the distribution of $g(X_1)=E[h(X_1,X_2)|X_1]$. As an application improved Edgeworth and bootstrap based confidence intervals for the mean of a U-statistic are obtained. Extensions to studentized statistical functions admitting a second order von Mises expansion, such as studentized L-statistics with smooth weights and studentized M-estimators of maximum likelihood type, are also briefly discussed.

MS-R8709. R.D. Gill. *Non- and semi-parametric maximum likelihood estimators and the von Mises method (part I).*
AMS 62G05, 62G20, 60B12, 60F17, 46A05; 26 pp.; **key words:** non-parametric maximum likelihood, von Mises method, compact differentiation, Hadamard differentiation, asymptotically efficient estimation.

Abstract: After introducing the approach to von Mises derivatives based on compact differentiation due to Reeds, we show how non-parametric maximum likelihood estimators can often be defined by solving infinite dimensional score equations. Each component of the score

equation corresponds to the derivative of the log likelihood for a one-dimensional parametric submodel. By means of examples we show that it usually is not possible to base consistency and asymptotic normality theorems on the implicit function theorem. However (in part II) we show for a particular class of models, that once consistency (in a rather strong sense) has been established by other means, asymptotic normality and efficiency of the non-parametric maximum likelihood estimator can be established by the von Mises method. This revised version of an earlier report contains a new section on applications to the bootstrap resampling scheme.

MS-R8710. A.L.M. Dekkers & L. de Haan. *On a consistent estimate of the index of an extreme-value distribution.*
AMS 62F12, 62G30; 17 pp.; **key words:** extreme-value theory, order statistics, strong consistency, asymptotic normality.
Abstract: An easy proof is given for the weak consistency of Pickands' estimate of the main parameter of an extreme-value distribution. Moreover, further natural conditions are given for strong consistency and for asymptotic normality of the estimate.

MS-R8711. A.L.M. Dekkers & L. de Haan. *Large quantile estimation under extreme-value conditions.*
AMS 62F25, 62G30; 13 pp.; **key words:** quantile estimation, extreme-value theory.
Abstract: A large quantile is estimated by a combination of extreme or intermediate order statistics. This leads to an asymptotic confidence interval.

MS-R8712. E.V. Khmaladze. *An innovation approach to goodness of fit tests in* $\mathbb{R}^m$.
AMS 62G10, 62F03; 12 pp.; **key words:** empirical processes, multivariate innovation process, Doob transformation.
Abstract: We present a solution to the goodness-of-fit problem for multivariate observations, using the innovation process for the (sequential) empirical distribution function with respect to a conveniently chosen linear ordering or scanning system in $\mathbb{R}^m$.

MS-R8713. C.C. Heesterman. *A central limit theorem for M-estimators by the von Mises method.*
AMS 62F12, 62G05; 13 pp.; **key words:** asymptotic normality of M-estimators, compact differentiation, Hadamard differentiation, δ-method, M-estimator, von Mises functional.
Abstract: Asymptotic normality of M- or maximum likelihood type estimators has long since been a result by Huber (1967). Reeds (1976) argued that this could also have been established as an application of the δ-method using the tool of compactly differentiating von Mises functionals with respect to the (empirical) distribution function F_n. If slightly adapted, this alternative approach is shown to be quite fruitful, hopefully maybe even in the non-parametric case. A corrected version of the proof by REEDS is given.

MS-R8714. A.J. de Koning. *On single lane roads.*
AMS 60K30, 60G35; 14 pp.; **key words:** traffic flow, stochastic processes.
Abstract: A road which narrows at a bottleneck from an ∞-lane road to a one-lane road is studied with the aid of two stochastic processes. Special attention is given to headways and gaps. At the bottleneck an equilibrium headway can be viewed as the maximum of a shifted exponential random variable and a minimum headway. After the bottleneck the situation becomes far more

complicated. However, limiting results are obtained for headways and gaps at a large distance from the bottleneck. The asymptotic behaviour of headway and gaps is largely determined by the behaviour of the desired speed distribution at the lower extreme of its support.

AM-R8706. Ph. Clément, O. Diekmann, M. Gyllenberg, H.J.A.M. Heijmans & H.R. Thieme. *Perturbation theory for dual semigroups III. Nonlinear Lipschitz continuous perturbations in the sun-reflexive case.*
AMS 47D05, 47H20; 15 pp.; **key words:** strongly continuous semigroup, dual semigroup, weakly * continuous semigroup, Favard class, weak * Riemann integral, variation-of-constants formula, inhomogeneous initial value problem, nonlinear Lipschitz continuous perturbation, semilinear equation, principle of linearized stability, Volterra integral equation.
Abstract: We consider nonlinear Lipschitz perturbations of the infinitesimal generator of a linear C_0-semigroup on a non-reflexive Banach space. It is allowed that the perturbation maps the space into a bigger space which arises in a natural way when considering dual semigroups. Using a generalized variation-of-constants formula we show that the perturbed operator generates a strongly continuous nonlinear semigroup. We study regularity properties of this semigroup and prove the principle of linearized stability.

AM-R8707. H.J.A.M. Heijmans. *Mathematical morphology: an algebraic approach.*
AMS 69K40, 06A23; 17 pp.; **key words:** image processing, mathematical morphology, morphological transformation, complete lattice, automorphism group.
Abstract: Mathematical morphology is a theory on morphological transformations which form the basic components for a number of algorithms in quantitative image analysis. In this paper we present an overview of the basic principles of mathematical morphology, and initiate a generalization of the theory by taking the object space to be an arbitrary complete lattice.

AM-R8708. H.R. Thieme & J.A.P. Heesterbeek. *How to estimate the efficacy of periodic control of an infectious plant disease.*
AMS 92A15, 15A18; 11 pp.; **key words:** epidemiology, deterministic model, spectral radius, positive symmetric matrix.
Abstract: Certain infectious plant diseases are controlled by inspection and subsequent hand removal of diseased parts. In this paper we give two sets of criteria from which one can conclude whether this control effort is adequate or not. These criteria do not require knowledge of the infection- or detection rate of the disease but only use the structure of the contact matrix. Computer experiments give a feeling of how many inspections are needed in order to draw a conclusion.

AM-R8709. K. Soni & N.M. Temme. *On a biorthogonal system associated with uniform asymptotic expansions.*
AMS 41A10, 41A60, 30E15, 33A20, 33A70; 22 pp.; **key words:** approximation by polynomials, biorthogonal functions, uniform asymptotic expansions.
Abstract: In 1987 Soni and Sleeman introduced a family of polynomials which are related to the coefficients in a uniform asymptotic expansion of a class of integrals. In this expansion parabolic cylinder functions (Weber functions) occur as basic approximants and the resulting series is of Bleistein type. In the present paper a family of rational functions is introduced, and the two families form a biorthogonal system, on a contour in the complex plane. The system can be viewed as

a generalization of the families $\{z^n\}$ and $\{z^{-n-1}\}$, which occur in Taylor expansions and the Cauchy integrals of analytic functions. Explicit representations of the rational functions are given together with rigorous estimates. These results are used to establish convergence of expansions of certain functions in terms of the polynomials and the rational functions. The main motivation to study this system stems from the above mentioned problem on the asymptotic expansion of a class of integrals. It is shown how to use the system in order to construct bounds for the remainders in the asymptotic expansion. An instructive example is worked out in detail.

AM-R8710. H.E. de Swart. *Analysis of a six-component barotropic spectral model: chaotic motion, predictability and vacillation.*
AMS 86A10, 76E20, 34C35; 25 pp.; **key words:** low-order model, vacillation between weather regimes, bifurcation analysis, homoclinic orbits and chaos.
Abstract: A low-order spectral model of the barotropic potential vorticity equation in a β-plane channel is considered. Its physical and mathematical properties are investigated by a numerical bifurcation analysis of the steady states and periodic solutions. The two parameters varied are the external forcing and width-length ratio of the channel with which the topographic and barotropic instability mechanisms respectively can be controlled. Particular interest is paid to the existence of solutions describing a flow with a limited predictability and which can vacillate between different preferent regimes. It appears that, depending on the parameter values and initial conditions, the long-term behaviour of the flow can be either stationary, periodic, quasi-periodic or chaotic. An important scenario is found which leads to the generation of strange attractors. It includes the occurrence of homoclinic orbits for specific parameter values, which connect an unstable stationary point with itself. For nearby parameter values chaotic orbits exist which move in small tubes around the homoclinic orbits, in agreement with Silnikov's theory. The chaotic motion, characterized by a positive Lyapunov exponent, describes irregular flow predictable on a time scale given by the reciprocal of this exponent. However, despite its interesting properties the model cannot describe transitions between different preferent regimes. It is argued that this is due to the structure of the equations as well as to the severe truncation of the spectral expansions.

PM-R8705. D.J. Smit. *String theory and algebraic geometry of moduli spaces.*
35 pp.
Abstract: It is shown how the algebraic geometry of the moduli space of Riemann surfaces entirely determines the partition function of Polyakov's string theory. This is done by using elements of Arakelov's intersection theory applied to determinants of families of differential operators parametrized by moduli space. As a result we write the partition function in terms of an exponential of Arakelov's Green functions and Faltings' invariant on Riemann surfaces. Generalizing to arithmetic surfaces, i.e. surfaces which are associated to an algebraic number field K, we establish a connection between string theory and the infinite primes of K. As a result we conjecture that the usual partition function is a special case of a new partition function on the moduli space defined over K.

PM-R8706. A.I. Zayed. *Jacobi polynomials as generalized Faber polynomials.*
AMS 33A65, 30C10; 14 pp.; **key words:** Jacobi polynomials, Faber polynomials.
Abstract: Let $\mathbb{B}$ be an open bounded subset of the complex z-plane with closure $\overline{\mathbb{B}}$ whose complement $\overline{\mathbb{B}}^c$ is a simply connected domain on the Riemann sphere. Let $z=\psi(w)$ map the domain $|w|>\rho$ $(\rho>0)$ one-to-one conformally onto the domain $\overline{\mathbb{B}}^c$ such that $\psi(\infty)=\infty$. Let $R(w)=\Sigma_{n=0}^{\infty}c_n w^{-n}$, $c_0\neq 0$ be analytic in the domain $|w|>\rho$ with $R(w)\neq 0$. Let $F(z)=\Sigma_{n=0}^{\infty}b_n z^n$, $b_n\neq 0$, $F^*(z)=\sum_{n=0}^{\infty}\frac{1}{b_n}z^n$ be analytic in $|z|<1$ and analytically continuable to any point outside

$|z|<1$ along any path not passing through the points $z=0,1,\infty$. The generalized Faber polynomials $\{P_n(z)\}_{n=0}^{\infty}$ of $\mathbb{B}$ are defined by

$$\frac{t\psi'(t)}{\psi(t)}R(t)F(\frac{z}{\psi(t)}) = \sum_{n=0}^{\infty} P_n(z)\frac{1}{t^n},\ |t|>\rho$$

The aim of this paper is to show that
1) if the Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ are generalized Faber polynomials of any region $\mathbb{B}$, then it must be the elliptic region $\{z:|z+1|+|z-1|<\rho+\frac{1}{\rho},\rho>1\}$.
2) the only Jacobi polynomials that can be classified as generalized Faber polynomials are the Chebyshev polynomials of the first kind, some normalized Gegenbauer polynomials, some normalized Jacobi polynomials of type $\{P_n^{(\alpha,\alpha+1)}\}_{n=0}^{\infty}$, $\{P_n^{(\beta+1,\beta)}\}_{n=0}^{\infty}$ and there are no others, no matter how one normalizes them.
3) the Hermite and Laguerre polynomials cannot be generalized Faber polynomials of any region.

PM-R8707. J. van Bon, A.M. Cohen & H. Cuypers. *Graphs related to Held's simple group.*
AMS 20B25, 05C25, 20D08; 15 pp.; **key words:** Held's simple group, multiplicity free permutation representation.

Abstract: We analyse the permutation representations of low degree of Held's simple group *He*. We also determine its primitive multiplicity free permutation representations and show that there is no graph on which it or its automorphism group acts as a distance transitive group of automorphisms. In doing so, we supply a computerfree construction of *He*.

PM-R8708. D.J. Smit. *Algebraic and arithmetic geometry in string theory.*
AMS 81E13, 81C35, 14H01, 14H25; 13 pp.; **key words:** Polyakov path integral, Grothendieck-Riemann-Roch theorem, Belavin-Kniznik theorem, algebraic number fields, Riemann-Roch on Spec($\mathcal{O}_K$), rational points.

Abstract: In the first part we review how elements of algebraic geometry can be used to give an algebraic formula for the string partition function. In the second part we generalize these ideas to include arithmetic surfaces, i.e. surfaces defined over an algebraic number field *K*. We will calculate explicitly the volume of the lattice formed by *K*-rational tangent vectors at a *K*-rational point in moduli space, with respect to the Polyakov measure.

PM-R8709. M. Hazewinkel. *Introduction to nilpotent approximation filtering.*
AMS 93E11, 93B30, 93E10, 93D25, 60H15, 93B15, 17B65, 17B99, 57R25, 35H05; 6 pp.; **key words:** nonlinear filtering, estimation Lie algebra, topologically nilpotent Lie algebra, Duncan-Mortenson-Zakai equation, unnormalized conditional probability density, robustness, nilpotent approximation, Wei-Norman theory, stochastic differential equations.

Abstract: The so-called reference probability or unnormalized probability method for nonlinear filtering problems leads to a (robust) infinite dimensional filter of bilinear type. If the associated Lie algebra is topologically solvable or nilpotent an infinite dimensional version of Wei-Norman theory applies. If not then ideas of nilpotent approximation lead to (potential) approximation filters. This note is not so much a definite report on results as an outline of a research program.

PM-R8710. G. Brassard, D. Chaum & C. Crépeau. *Minimum disclosure proofs of knowledge.*
CR C.2.0, E.3; 45 pp.; **key words:** bit commitment, blob, cryptographic

protocol, cryptography, discrete logarithm, interactive proof, minimum disclosure, quantum cryptography, satisfiability, unconditional, security, zero-knowledge.

Abstract: Protocols are given for allowing a 'prover' to convince a 'verifier' that the prover knows some verifiable secret information, without allowing the verifier to learn anything about the secret. The secret can be probabilistically or deterministically verifiable, and only one of the prover or verifier need have constrained resources. This paper unifies and extends models and techniques previously put forward by the authors, and compares some independent related work.

PM-N8701. J.T.M. van Bon. *Distance regular antipodal covers of Johnson and Hamming graphs.*
AMS 05C75; 4 pp.; **key words:** distance regular antipodal graph, Hamming graph, Johnson graphs.

Abstract: We determine distance regular antipodal covers of Johnson and Hamming graphs and some graphs related to them.

CWI Activities

With each activity we mention its frequency and (between parentheses) a contact person at CWI. Sometimes some additional information is supplied, such as the location if the activity will not take place at CWI.

Study group on Analysis on Lie groups. Jointly with University of Leiden. Once a month. (T.H. Koornwinder)

Seminar on Algebra and Geometry. Jointly with the Universities of Eindhoven and Utrecht. Biweekly. (A.M. Cohen)
Finite geometry and distance transitive graphs.

Cryptography working group. Monthly. (H. den Boer)

Study group Biomathematics. Lectures by visitors or members of the group. Jointly with University of Leiden. Irregular. (O. Diekmann)

Progress meetings of the Applied Mathematics Department. Weekly. (N.M. Temme)
New results and open problems on the research topics of the department: biomathematics, mathematical physics, asymptotics and applied analysis, image analysis.

Study group on Statistical and Mathematical Image Analysis. Every three weeks. (R.D. Gill)

Progress meetings of the Mathematical Statistics Department. Biweekly. (K.O. Dzhaparidze)
Talks by members of the department on recent developments in research and consultation.

System Theory Days. Irregular. (J.H. van Schuppen, J.M. Schumacher)

Study group on System Theory. Biweekly. (J.H. van Schuppen)

Colloquium on Queueing Theory and Performance Evaluation. Irregular. (O.J. Boxma)

Thirteenth Conference on the Mathematics of Operations Research. 13,14,15 January 1988 at Lunteren.
Invited speakers: A. Federgruen (Tel Aviv, Israel), D. Shmoys (Cambridge, USA), L. Trotter (Augsburg, West Germany), J. Walrand (Berkeley, USA), L. van Wassenhove (Leuven, Belgium). (B.J. Lageweg)

Seventh Benelux Meeting on Systems and Control. 2,3,4 March 1988 at Heijen, Limburg. (J.M. Schumacher)

Progress meetings on Numerical Mathematics. Including lectures by visitors. Weekly. (H.J.J. te Riele)

Study group on Graphics Standards. Monthly. (M. Bakker)

National Study Group on Concurrency. Jointly with Universities of Leiden & Eindhoven and several industrial research establishments. 29 January, 26 February, 25 March, 20 May 1988. (J.W. de Bakker)

REX Workshop: 'Linear Time, Branching Time and Partial Order in Logics and Models for Concurrency'. Jointly with Universities of Leiden & Eindhoven. 30 May - 3 June 1988 at Noordwijkerhout.
Invited speakers: J.F.A.K. van Benthem (Amsterdam), E.A. Emerson (Austin, USA), A. Pnueli (Rehovot, Israel), M. Hennessy (Sussex, UK), J.W. Klop (Amsterdam), A. Mazurkiewicz (Warsaw, Poland), G. Winskel (Cambridge, UK). (J.W. de Bakker)

Post-academic Course on Modern Techniques in Software Engineering. (N. van Diepen)
Various lectures present modern techniques and methods for the construction of complex software systems. The course is meant for persons with a background in computer science, who are or will be actively involved in the construction of those systems.

Study group on Logical Aspects of Artificial Intelligence. (P.J.F. Lucas)

Study group on Dialogue Programming. (P.J.W. ten Hagen)

Process Algebra Meeting. Weekly. (J.W. Klop)

Course on DICE. 14,15,21,22,28,29 January 1988. (R. van Liere)

Course on C++ voor C-programmers. 1,2 March 1988. (G. van Rossum)

Study group on User Interface. 15 January, 19 February, 18 March, 15 April, 20 May, 17 June 1988. (P.J.W. ten Hagen)

Working group on Knowledge Representation. (V. Akman)
The Interactive Systems Department is planning to start a working group on knowledge representation beginning in January 1988. The aim of this working group is two-fold: (i) to study past achievements in Knowledge Representation and (ii) to discuss new developments. We expect that these studies and discussions will help us understand current problems more clearly and hopefully lead to new, enlightening, and useful theoretical developments. We emphasize that this working group is supposed to conduct theoretical studies and we assume some maturity on the part of the participants.

Second Eurographics Workshop on Intelligent CAD Systems - Implementational Issues. 12,13,14,15 April 1988 at Veldhoven. (V. Akman)

Visitors to CWI from Abroad

C.A. Addison (Chr. Michelsen Institute, Bergen, Norway) 16 September. F. Arbab (University of Southern California, USA) 3-28 August. B. Awerbuch (MIT, USA) 13-17 July. A.J. Baddely (CSIRO, Sidney, Australia) 20 September - 3 October. O.E. Barndorff-Nielsen (University of Aarhus, Denmark) 28-29 September. B. Braams (Princeton University, New Jersey, USA) 3 July. S. Burys (University Jagiellonski, Krakòw, Poland) 7-18 September. J. Carroll (NIHE, Dublin, Ireland) 16 September. A.K. Chandra (IBM Th.J. Watson Research, Yorktown Heights, USA) 9-10 July. G. Cousineau (Ecole Normale Supérieure, Paris, France) 28 August. J.H. Davenport (University of Bath, UK) 14-15 September. L.M. Delves (University of Liverpool, UK) 16 September. R.K. Dewar (Courant Institute, New York, USA) 19-21 August. P. De Vijver (Philips Research Labs., Brussels, Belgium) 28-29 September. P.P.B. Eggermont (University of Delaware, USA) September 1987 - July 1988. I. Elshoff (University of Arizona, Tucson, USA) 1 June - 14 November. M. Erl (NAG, Oxford, UK) 14-15 September. Fengsu Chen (Shanghai Institute of Electric Power, Shanghai, China) 22-25 July. T. Flannagan (University of Cambridge, UK) 31 July. B. Ford (NAG, Oxford, UK) 16 September. M.R. Gomez (INESC, Lisbon, Portugal) 31 August. S. Grumbach (INRIA, Rocquencourt, France) 22-24 July. H.J.G. Gunderson (University of Aarhus, Denmark) 28-29 September. R. Haggenmüller (SIEMENS, München, West Germany) 14-16 September. M. Hennessy (University of Sussex, Brighton, UK) 24-26 September. G.S. Hodgson (NAG, Oxford, UK) 14-16 September. J. Jenssen (University of Aarhus, Denmark) 28-29 September. C.B. Jones (University of Manchester, UK) 24 August. Ms. K. Kanchanasut (Asian Institute of Technology, Bangkok, Thailand) 24-28 August. W. Klein (University of Karlsruhe, West Germany) 14-15 September. F. Krückeberg (GMD, Sankt

Augustin, Bonn, West Germany) 15 September. M-H. Lallemand (INRIA, Sophia-Antipolis, France) 21 September - 17 October. J.J. Lauture (European Commission, Brussels, Belgium) 15 September. E.L. Lawler (University of Berkeley, USA) 1-7 September. D. Leivant (Carnegie Mellon University, Pittsburgh, USA) 13 August. M. Martelli (University of Pisa, Italy) 28 August. J. Moller (University of Aarhus, Denmark) 28-29 September. S. Morse (Yale University, New Haven, USA) 27 July. R. Nisbet (University of Strathclyde, Glasgow, UK) 21 September. J. Pintér (Research Center for Water Resources Development, Budapest, Hungary) 18 September. G. Pontrelli (CNR, Rome, Italy) 1 March - 3 October. S. Scholtz (TU-Dresden, East Germany) one week in autumn. D. Scott (Concordia University and University of Montréal, Canada) 14 August. K.E. Shuler (University of California, San Diego, USA) 1-25 September. F. Sommen (RU Gent, Belgium) 14 August. G.T. Symm (NPL, Teddington, UK) 16 September. Y. Tomiyama (NTT, Tokyo, Japan) 6 July. C. Ullrich (University of Karlsruhe, West Germany) 14-16 September. A. Vanderbauwhede (RU Gent, Belgium) 17-20 August. J. Vitter (Brown University, USA and INRIA, Paris, France) 7-8 July. J. Wolff von Gudenberg (University of Karlsruhe, West Germany) 14-16 September. A. Zöllner (SIEMENS, München, West Germany) 14-16 September. J.I. Zucker (SUNY at Buffalo, USA) 19-31 August.

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