Two More Linear Programming Models for the Nonogram

Ko Anthonisse

Email: info@anthonisse.biz

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[1] An earlier version was distributed on a small scale in 2013

Introduction

The challenge of the nonogram puzzle is to mark cells in a rectangular grid in such a way that the constraints, as specified for each row and column, are satisfied. The specification has the form of a segment-size-sequence of numbers. Each number specifies the size of a segment, a segment is a contiguous set of marked cells. The segments should occur in the order as specified and subsequent segments should be separated by at least one blank (unmarked) cell. The example in Figure 1 was copied from [1].

	X		X	
		X		
X				X
	X	X	X	

Figure 1a. Example of a simple nonogram, solved

	1	1 1	1 1	1 1	1
1 1					
1					
1 1					
3					

Figure 1b. Example of a simple nonogram, unsolved

	1	1 1	1 1	1 1	1
1 1		X		X	
1			X		
1 1	X				X
3		X	X	X	

Figure 1c. Example of a simple nonogram, solved

The nonogram can be solved by search techniques. Another approach is to use integer linear programming. The two ILP models to be developed in this paper are alternatives for the model as presented by Bosch in [2]. The models will be generalized to solve colored nonograms, those are alternatives for the model by Mingote and Azevedo in [3]. They also can be extended to 3-dimensional puzzles.

Focus on a line

A *line* is either a row or a column of the grid, together with its segment-size-sequence of numbers. In this section the ILP constraints for an arbitrary line will be developed. The following notation and terminology will be used:

w = # cells in the line,
t = # segments in the specification,
v_k = size of segment k ($k=1,\dots,t$).

Segment k will be said to contain v_k *items*, the specification contains a total of $d = \Sigma_k\ v_k$ items which are to be *assigned* to the cells.

Some more notation is introduced with the help of an example. Consider the line

1, 3, 2										

Figure 2. Example of a line

where, w=10, t=3 and there are 1+3+2=6 items to be assigned to the 9 cells.

Let s_i denote the segment which contains item i.

i	1	2	3	4	5	6
s_i	1	2	2	2	3	3

Figure 3. Membership of items to segments

The next figures show the assignments with all items in their minimum and maximum position respectively.

1, 3, 2	X		X	X	X		X	X		

Figure 4. Minimum positions of items

1, 3, 2			X		X	X	X		X	X

Figure 5. Maximum position of items

Let a_i and b_i denote the minimum and maximum position of item i respectively.

i	1	2	3	4	5	6
a_i	1	3	4	5	7	8
b_i	3	5	6	7	9	10

Figure 6. Minimum and maximum positions of items

In general,

$$a_1 = 1$$
$$a_i = a_{i-1} + 1 + \delta(s_{i-1}, s_i) \qquad (i=2,...,d)$$

$$b_d = w$$
$$b_i = b_{i+1} - 1 - \delta(s_i, s_{i+1}) \qquad (i=d-1,...,1)$$

where

$$\delta(x,y) = \begin{cases} 0 \text{ if } x=y \\ 1 \text{ otherwise.} \end{cases}$$

For each cell c there is a set of items I_c which could be assigned to that cell. If the cell is marked, then it is by one of those items. It is easily seen that $I_c = \{ f_c , \dots , g_c \}$ where f_c and g_c are the first and last item respectively that could be assigned to cell c.

c	1	2	3	4	5	6	7	8	9	10
I_c	1	1	1, 2	2, 3	2, 3, 4	3, 4	4, 5	5, 6	5, 6	6
f_c	1	1	1	2	2	3	4	5	5	6
g_c	1	1	2	3	4	4	5	6	6	6

Figure 7. Possible assignments to cells

The contents of Figure 7 are readily derived from the contents of Figure 6.

Now the decision variables for the line are

$$x_{ic} = \begin{cases} 1 \text{ if item i is assigned to cell c} \\ 0 \text{ otherwise.} \end{cases} \quad (i=1,\dots d;\ c=a_i, \dots, b_i) \qquad (1)$$

The first group of constraints ensures that each item is assigned to exactly one cell.

$$\sum_{c=a_i}^{b_i} x_{ic} = 1 \quad (i=1, \dots, d) \qquad (2)$$

The second group of constraints is about two subsequent items i-1 and i. If these two items belong to the same segment, then they must be assigned to adjacent cells. If item i-1 is assigned to cell c then item i must be assigned to cell c+1, if item i-1 is not assigned to cell c then item i cannot be assigned to cell c+1. If the two subsequent items do not belong to the same segment then the first one is the last item of a segment and the other one is the first item of the next segment. If item i-1 is assigned to cell c then item i must be assigned to cell c+2 or beyond. Thus the constraints are:

$$\left.\begin{aligned} & x_{i-1,c} = x_{i,c+1} \quad \text{if } s_{i-1} = s_i \\ & x_{i-1,c} \le \sum_{j=c+2}^{b_i} x_{ij} \quad \text{if } s_{i-1} \ne s_i \end{aligned}\right\} \quad (i=2, ..., d;\ c=a_{i-1}, \dots, b_{i-1}) \qquad (3)$$

The set of d constraints (2) can be replaced by just 2 constraints,

$$\sum_{c=a_1}^{b_1} x_{1c} = 1 \qquad (2a)$$

and

$$\sum_{i=1}^{d} \sum_{c=a_i}^{b_i} x_{ic} = d \qquad (2b)$$

From (2a) and (3) it is clear that $\Sigma_c\, x_{ic} \geq 1$ for all i. Then (2) follows from (2b).

Application to the grid, model I

The grid has m rows and n columns, so there are m+n lines. The lines ℓ =1, …, m correspond with the rows of the grid, the lines ℓ =m+1, …, m+n correspond with the columns. Each line has its own set of data, variables and constraints as described above.

Some additional notation is required. For line ℓ, d_ℓ is its number of items, $a_{\ell i}$ and $b_{\ell i}$ are the minimum and maximum positions of these items, $I_{\ell c}$ denotes the set I_c of line ℓ, and, finally, $x_{\ell ic}$ denotes its set of variables x_{ic}.

The cell (u,v) of the grid is the intersection of the lines u and n+v. If that cell is marked then in both lines an item should be assigned to that cell. If that cell is not marked then in neither of the lines an item should be assigned to that cell. This is described by the constraints:

$$\sum_{i \in I_{uv}} x_{uiv} \quad = \quad \sum_{i \in I_{m+v,u}} x_{m+v,iu} \qquad (u=1, \ldots, m;\ v=1, \ldots, n) \qquad (4)$$

The variables (1) for each line and the constraints (2a), (2b) and (3) for each line, together with the constraints (4) for each cell constitute a complete model for solving the nonogram.

The m+n constraints (2b) can be replaced by a single constraint

$$\Sigma_{\ell ic}\, x_{\ell ic} \;=\; \Sigma_\ell\, d_\ell \qquad (2c)$$

As the problem is a satisficing problem there is no need for an objective function. If the ILP solver requires an objective function then any function can be used. The constraint (2c) can be deleted if the left hand side is chosen as the objective function, which then must be minimized. If the value of the right hand side is not attained in the solution of the model then the nonogram has no solution.

Figure 8 summarizes the contents of model I.

variables	(1)
constraints	(2a), (2c), (3), (4)

Figure 8. Model I summary

The trivial nonogram

	1	1
1		
1		

Figure 9. A trivial nonogram

has the following model.

$x_{111} + x_{112} = 1$
$x_{211} + x_{212} = 1$
$x_{311} + x_{312} = 1$
$x_{411} + x_{412} = 1$

$x_{111} - x_{311} = 0$
$x_{112} - x_{411} = 0$
$x_{211} - x_{312} = 0$
$x_{212} - x_{412} = 0$

$x_{111} + x_{112} + x_{211} + x_{212} + x_{311} + x_{312} + x_{411} + x_{412} = 4$

Application to the grid, model II
In the next model the decision variables are defined at the level of the grid.

$$X_{uv\alpha\beta} = \begin{cases} 1 & \text{if in row u item } \alpha \text{ is assigned to cell v} \\ & \text{and in column v item } \beta \text{ is assigned to cell u} \\ 0 & \text{otherwise.} \end{cases} \qquad (u=1, \ldots, m;\ v=1, \ldots, n;\ \alpha \in I_{uv};\ \beta \in I_{m+v,u}) \qquad (5)$$

The following relations hold between these $X_{uv\alpha\beta}$ and the $x_{\ell ic}$ in the previous model.

If, in row u item i is assigned to cell v then in column v an item $\beta \in I_{m+v,u}$ must be assigned to cell u, thus $X_{uvi\beta} = 1$ for one β.
If item i in row u is not assigned to cell v, then $X_{uvi\beta} = 0$ should hold for all $\beta \in I_{m+v,u}$. This leads to the relations

$$x_{uiv} = \sum_{\beta \in I_{m+v,u}} X_{uvi\beta} \qquad (u=1, \ldots, m;\ i=1, \ldots, d_u;\ v=a_{ui}, \ldots, b_{ui}) \qquad (6)$$

Similarly for column v:

$$x_{m+v,iu} = \sum_{\alpha \in I_{uv}} X_{uv\alpha i} \qquad (v=1, \ldots, n;\ i=1, \ldots, d_{m+v};\ v=a_{m+v,i}, \ldots, b_{m+v,i}) \qquad (7)$$

For each row, (6) can be substituted into its constraints (2a) and (3). For each column, (7) can be substituted into its constraints (2a) and (3). Both (6) and (7) can be substituted into (2c).

Substitution of (6) into the left hand side of (4) and of (7) into the right hand side of (4) yields:

$$\sum_{i \in I_{uv}} \sum_{\beta \in I_{m+v,u}} X_{uvi\beta} = \sum_{i \in I_{m+v,u}} \sum_{\alpha \in I_{uv}} X_{uv\alpha i} \qquad (u=1, \ldots, m;\ v=1, \ldots, n) \qquad (8)$$

Replacing i by α in the left hand side, i by β in the right hand side and interchanging the positions of the two ∑ in the right hand side leads to:

$$\sum_{A \in I_{uv}} \sum_{\beta \in I_{m+v,u}} X_{uv\alpha\beta} = \sum_{\alpha \in I_{uv}} \sum_{\beta \in I_{m+v,u}} X_{uv\alpha\beta} \qquad (u=1, \ldots, m;\ v=1, \ldots, n) \qquad (9)$$

The conclusion is that the constraints (9) and thereby (4) are redundant now.

Figure 10 summarizes the contents of model II.

variables	(5)
constraints	(2a), (2c), (3) all with substitutions (6), (7)

Figure 10. Model II summary

Model II for the trivial nonogram (Figure 9) is

$X_{1111} + X_{1211} = 1$
$X_{2111} + X_{2211} = 1$
$X_{1111} + X_{2111} = 1$
$X_{1211} + X_{2211} = 1$

$X_{1111} + X_{1211} + X_{2111} + X_{2211} = 2$

Problem sizes
In the next three tables N denotes the number of variables and M denotes the number of constraints in a model.

Table 1 presents the model sizes for a 10x10 grid with a single segment in each line.

d=#items in segment	Model I		Model II		Model Bosch	
	N	M	N	M	N	M
1	200	121	100	21	300	400
2	360	301	324	201	280	560
3	480	441	576	341	260	680
4	560	541	784	441	240	760
5	600	601	900	501	220	800
6	600	621	900	521	200	800
7	560	601	784	501	180	760
8	480	541	576	441	160	680
9	360	441	324	341	140	560
10	200	301	100	201	120	400

Table 1. 10x10 grid, 1 segment in each line

It should be noted that $N_I = d x (N_B - 100)$, where N_I and N_B are the number of variables in model I end model Bosch respectively. In model Bosch there is a variable for each position for the first item of each segment, plus a variable for each cell of the grid. In model I there is a variable for each position for each item of each segment. Items of the same segment have the same number of such variables.

In Table 2 the model sizes for another 10x10 grid are presented, now each line has the same number of segments, each segment consist of a single item.

#segments in line	Model I		Model II		Model Bosch	
	N	M	N	M	N	M
1	200	121	100	21	300	400
2	320	281	256	181	420	660
3	360	361	324	261	460	760
4	320	361	256	261	420	700
5	200	281	100	181	300	480

Table 2. 10x10 grid, 1 item in each segment

Table 3 shows the model sizes for another 10x10 grid, now each line has 5 items, distributed over an increasing number of segments.

#segments in line	Model I		Model II		Model Bosch	
	N	M	N	M	N	M
1	600	601	900	501	220	800
2	500	521	625	421	300	780
3	400	441	400	341	340	720
4	300	361	225	261	340	620
5	200	281	100	181	300	480

Table 3. 10x10 grid, 5 items in each line

Table 4, finally, shows the average sizes of randomly generated nonograms. In this table, W denotes the size of the WxW grid. The treshold T was used to mark the cells in the grid (a cell is marked iff RANDOM <= T). For each line, the segments were calculated from the marking of the cells. From these, the model sizes were calculated. For each nonogram the ratios N_I/N_B, M_I/M_B, N_{II}/N_B and M_{II}/M_B were calculated, where subscript I and II refer to model I en II respectively and subscript B refers to model Bosch. In the table, the columns for Model I and Model II report the averages of these ratios, the columns for Model Bosch report the average N and M respectively. Each row of the table is based upon 128 nonograms.

		Model I		Model II		Model Bosch	
W	T	average N_I/N_B	average M_I/ M_B	average N_{II}/N_B	average M_{II}/ M_B	average N_B	average M_B
10	0.2	0.4	0.3	1.0	0.1	334.7	595.6
15	0.2	0.4	0.3	0.9	0.2	983.4	1822.6
20	0.2	0.5	0.3	1.0	0.2	2168.9	4177.4
25	0.2	0.5	0.3	1.0	0.2	4003.0	7916.9
30	0.2	0.5	0.3	1.0	0.2	6711.0	13571.5
35	0.2	0.5	0.3	1.1	0.2	10405.8	21328.2
40	0.2	0.5	0.3	1.1	0.2	15234.1	31555.7
10	0.3	0.5	0.3	1.0	0.2	363.0	706.8
15	0.3	0.5	0.3	1.0	0.2	1040.7	2123.9
20	0.3	0.6	0.3	1.1	0.2	2272.7	4816.3
25	0.3	0.6	0.3	1.2	0.2	4199.5	9105.8

30	0.3	0.6	0.3	1.2	0.2	6952.1	15357.3
35	0.3	0.6	0.3	1.2	0.2	10752.3	23974.8
40	0.3	0.6	0.3	1.3	0.3	15664.8	35352.8
10	0.4	0.5	0.3	1.1	0.2	347.9	736.5
15	0.4	0.6	0.3	1.2	0.2	981.4	2184.9
20	0.4	0.7	0.3	1.3	0.2	2090.7	4825.7
25	0.4	0.7	0.3	1.4	0.3	3799.0	8988.6
30	0.4	0.7	0.3	1.4	0.3	6244.5	15020.1
35	0.4	0.7	0.3	1.4	0.3	9578.7	23338.1
40	0.4	0.7	0.3	1.5	0.3	13843.6	34093.8
10	0.5	0.6	0.4	1.3	0.2	313.9	717.5
15	0.5	0.7	0.4	1.5	0.3	839.8	2043.9
20	0.5	0.8	0.4	1.5	0.3	1781.0	4459.5
25	0.5	0.8	0.4	1.6	0.3	3118.4	8065.7
30	0.5	0.8	0.4	1.7	0.3	5017.9	13252.8
35	0.5	0.9	0.4	1.7	0.3	7667.9	20447.4
40	0.5	0.9	0.4	1.7	0.3	11014.3	29771.5
10	0.6	0.8	0.4	1.5	0.3	267.9	665.9
15	0.6	0.9	0.4	1.8	0.3	691.3	1828.0
20	0.6	1.0	0.4	1.9	0.3	1379.2	3797.3
25	0.6	1.0	0.4	2.0	0.3	2399.3	6781.2
30	0.6	1.1	0.4	2.1	0.3	3764.0	10906.8
35	0.6	1.1	0.4	2.2	0.4	5574.6	16482.2
40	0.6	1.1	0.4	2.2	0.4	7912.3	23756.8
10	0.7	0.9	0.4	1.9	0.3	221.2	600.1
15	0.7	1.1	0.5	2.3	0.4	535.9	1542.3
20	0.7	1.3	0.5	2.6	0.4	1021.9	3078.8
25	0.7	1.4	0.5	2.8	0.4	1702.4	5311.2
30	0.7	1.4	0.5	2.9	0.4	2606.3	8359.3
35	0.7	1.5	0.5	3.0	0.4	3748.6	12312.3
40	0.7	1.5	0.5	3.0	0.4	5233.6	17522.6
10	0.8	1.0	0.5	2.4	0.3	177.4	524.2
15	0.8	1.5	0.6	3.1	0.4	405.0	1273.9
20	0.8	1.8	0.7	3.6	0.5	742.6	2427.8
25	0.8	2.0	0.7	4.0	0.6	1195.6	4046.6
30	0.8	2.2	0.7	4.4	0.6	1746.7	6093.1
35	0.8	2.3	0.8	4.6	0.6	2456.8	8793.9
40	0.8	2.4	0.8	4.7	0.6	3295.2	12055.8

Table 4. Problem sizes for random nonograms

Apparently, the number of variables in model II is close to twice that number in model I. As T increases from 0.2 to 0.8 , the number of variables in model I (as a fraction of that number in Bosch) increases from 0.5 to 2.0 (at the average). Similarly, the number of variables in model II increases from 1.0 to 4.0 (at the average). Similarly, the number of constraints in model I increases from 0.3 to 0.7.

Computational results

A few examples were solved by CPLEX (many thanks to Lex Schrijver). The solution times are presented in the next table:

	Model I	Model II	Model Bosch
Picross 35	0.00	0.01	0.01
Picross 36	0.02	0.04	0.02
Picross 37	0.01	0.04	0.00
Picross 38	0.03	0.09	0.07
Picross 39	0.03	0.08	0.03
Picross 40	0.03	0.07	0.01
Bosch figure 5	0.38	53.88	2.78

Table 5. Solution times (seconds) for a few nonograms

The Picross examples were borrowed from the website www.picross.co.uk/puzzleimages/Hard on 10-03-2013. The Bosch example is from [2].

It is worth noting that in nearly all cases, CPLEX found the solution in the Presolve stage. The exceptions are Picross 38 with model Bosch and Bosch figure 5 with all models.

Colored nonograms

In the case of colored nonograms the specifications of the lines are extended with a color for each segment, where "blank" is not considered as a color. Let $p_k = p(k)$ denote the color of segment k and let $q_i = q(i) = p(s_i)$ denote the color of item i in the line. If two subsequent segments in a line have different colors, then there is no need for one or more blank cells between the two segments, as they are distinguished by their colors. This leads to a modification of constraints (3):

The summation does not start at $j = c + 2$ but at $j = c + 2 - \delta(q_{i-1}, q_i)$. (3a)

The set I_c has been introduced as the set of eligible items for cell c. Now let C_c denote the set of colors which occur in I_c.
I_c can be partitioned (according to the colors of the items) into one or more sets P_{cp}, each containing the items of color p which are available for cell c.

For the sequel, $C_{\ell c}$ denotes the set C_c of line ℓ and $P_{\ell cp}$ denotes the set P_{cp} of line ℓ.
Finally, $K_{uv} = C_{uc} \cap C_{m+v,c}$, that is the set of available colors for cell (u,v) of the grid.

Now the constraints (4) can be replaced by

$$\sum_{i \in P_{uvp}} x_{uiv} = \sum_{i \in P_{m+v,up}} x_{m+v,iu} \qquad (u=1, \ldots, m;\ v=1, \ldots, n;\ p \in K_{uv}) \qquad (4a)$$

Figure 11 summarizes the contents of colored model I.

variables	(1)
constraints	(2a), (2c), (3a), (4a)

Figure 11. Colored model I summary

The colored model I has the same number of variables as the non-colored model. Due to (4a) the number of constraints will increase with the number of colors.

The model by Mingote and Azevedo in [3] is an extension of the model by Bosch in [2]. As mentioned above, in model Bosch there is a variable for each position for the first item of each segment, plus a variable for each cell of the grid. For each cell, Mingote and Azevedo replace the single variable by as many variables as the nonogram has colors.

The extension of model II to the colored case amounts to redefining the variables (5):

$$X_{uv\alpha\beta} = \begin{cases} 1 & \text{if in row u item } \alpha \text{ is assigned to cell v} \\ & \text{and in column v item } \beta \text{ is assigned to cell u} \\ 0 & \text{otherwise.} \end{cases} \quad (u=1, \dots, m;\ v=1, \dots, n;\ p \in K_{uv};\ \alpha \in P_{uvp};\ \beta \in P_{m+v,up}) \qquad (5a)$$

The assignment of α and β to the same cell of the grid is only possible if α and β have the same color. The number of variables decreases when the number of colors increases and all other things remain unchanged.

Like before, there are relations between these $X_{uv\alpha\beta}$ and the $x_{\ell ic}$.

In row u

$$x_{uiv} = \sum_{\beta \in P_{m+v,u,q(i)}} X_{uvi\beta} \qquad (u=1, \dots, m;\ i=1, \dots, d_u;\ v=a_{ui}, \dots, b_{ui}) \qquad (6a)$$

In column v

$$x_{m+v,iu} = \sum_{\alpha \in P_{uvq(i)}} X_{uv\alpha i} \qquad (v=1, \dots, n;\ i=1, \dots, d_{m+v};\ v=a_{m+v,i}, \dots, b_{m+v,i}) \qquad (7a)$$

In (6a) and (7a) respectively, β and α should have the color q(i) of item i.

For each row, (6a) can be substituted into its constraints (2a) and into its (3a). For each column, (7a) can be substituted into its constraints (2a) and into its (3a). Both (6a) and (7a) can be substituted into (2c).

Substitution of (6a) into the left hand side of (4a) and of (7a) into the right hand side of (4a) yields:

$$\sum_{i \in P_{uvp}} \sum_{\beta \in P_{m+v,u,p}} X_{uvi\beta} = \sum_{i \in P_{m+v,up}} \sum_{\alpha \in P_{uvp}} X_{uv\alpha i} \qquad (u=1, \ldots, m;\ v=1, \ldots, n;\ p \in K_{uv}) \qquad (8a)$$

Replacing i by α in the left hand side, i by β in the right hand side and interchanging the positions of the two ∑ in the right hand side leads to:

$$\sum_{\alpha \in P_{uvp}} \sum_{\beta \in P_{m+v,u,p}} X_{uv\alpha\beta} = \sum_{\alpha \in P_{uvp}} \sum_{\beta \in P_{m+v,up}} X_{uv\alpha\beta} \qquad (u=1, \ldots, m;\ v=1, \ldots, n;\ p \in K_{uv}) \qquad (9a)$$

The conclusion is that the constraints (9a) and thereby (4a) are redundant.

Figure 12 summarizes the contents of colored model II.

variables	(5a)
constraints	(2a), (2c), (3a) all with substitutions (6a), (7a)

Figure 10. Colored model II summary

This completes the extension of both models for the colored case.

3-dimensional nonograms

The 3-dimensional nonogram is a block of cells. Each cell is the intersection of 3 lines. These lines correspond with a cell on each of three faces of the block, these faces having one common corner-point. The constraints on the lines are as above. A cell of the block is marked if and only if an item is assigned to that cell in each of the three lines. This can be assured with the help of additional constraints like (4).

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