FAILURE OF THE TRILINEAR OPERATOR SPACE GROTHENDIECK THEOREM

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ABSTRACT. We give a counterexample to a trilinear version of the operator space Grothendieck theorem. In particular, we show that for trilinear forms on ℓ_∞, the ratio of the symmetrized completely bounded norm and the jointly completely bounded norm is in general unbounded. The proof is based on a non-commutative version of the generalized von Neumann inequality from additive combinatorics.

1. INTRODUCTION

In the following, let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and let $\Phi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ be a bilinear form. The fundamental theorem in the metric theory of tensor products, better known as Grothendieck's theorem or GT [Gro53] implies that if $\mathcal{A}, \mathcal{B}$ are commutative, then the following holds. There exists a universal constant K such that $\|\Phi\| \leq \|\Phi\|_{\gamma_2} \leq K\|\Phi\|$, where $\|\Phi\|$ is the operator norm and $\|\Phi\|_{\gamma_2}$ is the factorization norm, which quantifies how well the bilinear form factorizes through the inner product of Hilbert spaces:

$$\|\Phi\|_{\gamma_2} = \inf\{\|\Psi_1\|\|\Psi_2\|\},$$

where the infimum is taken over Hilbert spaces $\mathcal{H}$ and linear maps $\Psi_1 : \mathcal{A} \to \mathcal{H}, \Psi_2 : \mathcal{B} \to \mathcal{H}$ such that for any $a \in \mathcal{A}, b \in \mathcal{B}$, we have $\Phi(a, b) = \langle \Psi_1(a), \Psi_2(b) \rangle$. In the same work, Grothendieck conjectured that the assumption that $\mathcal{A}, \mathcal{B}$ are commutative is unnecessary. This was first proved by Pisier in [Pis78] under some approximation assumptions and later in full generality by Haagerup [Haa85]. These results are not only important to Banach space theory, but also found applications

in quantum information theory [Tsi87, CJPPG15, RV15, AAI$^+$16], computer science [KN12, NRV14, BRS17] and combinatorics [CZ17].

1.1. The operator space GT.

The fact that C^*-algebras have a natural operator space structure [Pis03] invites the study of Grothendieck's theorem in this context. In this setting, the relevant norms are the so-called completely bounded norms, which we introduce below; we refer to [Pis12] for much more detailed information. We will identify $M_d(\mathcal{A})$, the space of $\mathcal{A}$-valued $d \times d$ matrices, with $\mathcal{A} \otimes M_d$ (and similarly for $\mathcal{B}$).

The *completely bounded norm* of Φ is defined by

$$\|\Phi\|_{\mathrm{cb}} = \sup_{d \in \mathbb{N}} \|\Phi_d\|,$$

where $\Phi_d : M_d(\mathcal{A}) \times M_d(\mathcal{B}) \to M_d$ is the "lift" given by

$$\left(\sum_i a_i \otimes X_i, \sum_j b_j \otimes Y_j \right) \mapsto \sum_{i,j} \Phi(a_i, b_j) X_i Y_j.$$

The *jointly completely bounded norm* of Φ is defined by

$$\|\Phi\|_{\mathrm{jcb}} = \sup_{d \in \mathbb{N}} \|\widetilde{\Phi}_d\|,$$

where $\widetilde{\Phi}_d : M_d(\mathcal{A}) \times M_d(\mathcal{B}) \to M_{d^2}$ is given by

$$\left(\sum_i a_i \otimes X_i, \sum_j b_j \otimes Y_j \right) \mapsto \sum_{i,j} \Phi(a_i, b_j) X_i \otimes Y_j.$$

It is easy to see that $\|\Phi\|_{\mathrm{jcb}} \leq \|\Phi\|_{\mathrm{cb}}$ (just consider the operators $X_i \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes Y_j$ when computing $\|\Phi_d\|$). It follows from Grothendieck's theorem that if $\mathcal{A}, \mathcal{B}$ are commutative C^*-algebras, then these norms are equivalent. However, their ratio is unbounded in general. An important difference between these two norms is that only the second is commutative, by which we mean the following. Define $\Phi^{\mathsf{T}} : \mathcal{B} \times \mathcal{A} \to \mathbb{C}$ by $\Phi^{\mathsf{T}}(b, a) = \Phi(a, b)$. Then, the jointly completely bounded norm is invariant with respect to this operation, but the completely bounded norm generally is not. The following "symmetrized" version of the completely bounded norm, introduced in [OP99], is again commutative in this sense:

$$\|\Phi\|_{\mathrm{sym}} = \inf \left\{ \|\Psi_1\|_{\mathrm{cb}} + \|\Psi_2^{\mathsf{T}}\|_{\mathrm{cb}} : \Phi = \Psi_1 + \Psi_2 \right\},$$

where the infimum is over bilinear forms $\Psi_1, \Psi_2 : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$. It turns out that this norm is equal to an operator space version of the factorization norm mentioned above, provided Hilbert spaces are endowed with the right operator space structure [Pis12, Section 18]. It

still holds that $\|\Phi\|_{\mathrm{jcb}} \leq \|\Phi\|_{\mathrm{sym}}$. Indeed, it follows from the above that for any decomposition $\Phi = \Psi_1 + \Psi_2$, we have

$$\|\Phi\|_{\mathrm{jcb}} \leq \|\Psi_1\|_{\mathrm{jcb}} + \|\Psi_2\|_{\mathrm{jcb}} = \|\Psi_1\|_{\mathrm{jcb}} + \|\Psi_2^{\mathsf{T}}\|_{\mathrm{jcb}} \leq \|\Psi_1\|_{\mathrm{cb}} + \|\Psi_2^{\mathsf{T}}\|_{\mathrm{cb}}.$$

Pisier and Shlyakhtenko [PS02] proved that under certain conditions on the C^*-algebras, the jointly completely bounded norm and the symmetrized completely bounded norm are equivalent, giving an operator space version of Grothendieck's theorem. This result was refined by Haagerup and Musat [HM08] showing the following result.

Theorem 1.1 (Operator space GT). *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and let $\Phi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ be a bilinear form. Then, $\|\Phi\|_{\mathrm{jcb}} \leq \|\Phi\|_{\mathrm{sym}} \leq 2\|\Phi\|_{\mathrm{jcb}}$.*

1.2. **Trilinear operator space GT.** A natural question is whether Theorem 1.1 generalizes to trilinear forms. In particular, Pisier [Pis12, Problem 21.3] asked the following : Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ be C^*-algebras and let S_3 denote the permutation group on $\{1, 2, 3\}$. For a trilinear form $\Phi : \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \to \mathbb{C}$ and $\pi \in S_3$, define the trilinear form $\Phi \circ \pi : \mathcal{A}_{\pi(1)} \times \mathcal{A}_{\pi(2)} \times \mathcal{A}_{\pi(3)} \to \mathbb{C}$ by

$$\text{(1)} \qquad \Phi \circ \pi(a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)}) = \Phi(a_1, a_2, a_3).$$

Define

$$\|\Phi\|_{\mathrm{sym}} = \inf \Big\{ \sum_{\pi \in S_3} \|\Psi_\pi \circ \pi\|_{\mathrm{cb}} : \Phi = \sum_{\pi \in S_3} \Psi_\pi \Big\},$$

where the infimum is over trilinear forms $\Psi_\pi : \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \to \mathbb{C}$ indexed by S_3. Define $\|\Phi\|_{\mathrm{jcb}}$ in the obvious way, using three-fold tensor products. Then, is it true that $\|\Phi\|_{\mathrm{jcb}} \leq \|\Phi\|_{\mathrm{sym}} \leq K\|\Phi\|_{\mathrm{jcb}}$ for some absolute constant $K \in (0, \infty)$?

This question was originally formulated by asking if any trilinear form $\Phi : \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \to \mathbb{C}$ that is *jointly completely bounded*, which is to say that $\|\Phi\|_{\mathrm{jcb}} < \infty$, is always *completely bounded*, which is to say that $\|\Phi\|_{\mathrm{sym}} < \infty$. However, this formulation is equivalent by the Open Mapping Theorem.

Here we answer this question in the negative. In particular, we show that such an inequality can fail already in the simplest possible scenario; that is for commutative C^*-algebras.

Theorem 1.2. *There exist absolute constants $C > 0$ and $c > 0$ such that the following holds. For every $n \in \mathbb{N}$, there exists a trilinear form $\Phi : \ell_\infty^n \times \ell_\infty^n \times \ell_\infty^n \to \mathbb{C}$ such that $\|\Phi\|_{\mathrm{sym}} \geq Cn^c\|\Phi\|_{\mathrm{jcb}}$.*

Other trilinear versions of Grothendieck's theorem have already been shown to fail in the past. Smith [Smi88] gave counterexamples to hoped-for trilinear versions of a Grothendieck-type theorem for completely bounded bilinear forms on C^*-algebras due to Pisier [Pis78] and Haagerup [Haa85]. Blecher [Ble89] introduced the notion of tracially completely bounded multilinear forms. These maps form a subspace strictly contained in the space of completely bounded multilinear forms. It was shown there that bounded bilinear forms on C^*-algebras are always tracially completely bounded, which may be interpreted as another Grothendieck-type theorem, but that this is false for trilinear forms in general. However, these works did not concern the jointly completely bounded norm, which is the appropriate norm in the context of operator spaces. In [PGWP$^+$08] it was shown that the operator norm and the jointly completely bounded norm are not equivalent for trilinear forms on commutative C^*-algebras (proving the existence of bounded trilinear forms which are not jointly completely bounded). This can be understood as a failure of yet another version of Grothendieck's theorem. The main result in [PGWP$^+$08] was later quantitatively improved in [BV13], but the optimal ratio between these norms as a function of the dimension is still an open problem.

Remarkably, both the jointly and symmetrized completely bounded norms again turn out to play an important role in quantum information theory. While the first appears naturally in the context of tripartite entanglement, in particular as the quantum bias of three-player XOR games (or equivalently, the quantum value of a tripartite correlation Bell inequality) [PV16], the second norm was recently used in the context of quantum algorithms, to give a characterization of quantum query complexity [ABP18]. In a sense, Theorem 1.2 can be also read as an absence of a direct connection between these topics.

The proof of Theorem 1.2 uses a non-commutative version of the generalized von Neumann inequality from additive combinatorics. This inequality allows us to upper bound the jointly completely bounded norm of certain structured trilinear forms, given by a function f on a finite Abelian group Γ, by the Gowers 3-uniformity norm of f. An argument of Varopoulos can be used to show that the symmetrized completely bounded norm of such trilinear forms is always at least $|\Gamma|\|f\|_{\ell_2}^2$. A simple explicit choice of a function f from the group $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ for prime $p \geq 5$ to the complex unit circle gives the result with $c = 1/8$. This follows from an elementary Weyl-type exponential sum estimate used to upper bound the Gowers 3-uniformity norm of f by $(2/p)^{1/8}p^2$,

while the Varopoulos bound shows that the symmetrized completely bounded norm is at least p^2. In the last section we comment on possible modifications of our construction.

2. Proof of Theorem 1.2

2.1. **Preliminaries.** We use the following notational conventions and basic facts. Denote $[n] = \{1, \ldots, n\}$ and $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$. For a set S let $(e_s)_{s \in S}$ be the standard basis for $\mathbb{C}^S$. Below, the set S will vary but will be clear from the context. Let $B_{M_d} = \{X \in M_d : \|X\| \le 1\}$, where $\|X\|$ denotes the usual operator norm on M_d. Recall that the commutator of $X, Y \in M_d$ is defined by $[X, Y] = XY - YX$ and that X, Y are said to commute if their commutator is zero. We will use the standard notation ℓ_∞^n for the n-dimensional commutative C^*-algebra given by $\mathbb{C}^n$ endowed with the sup norm and coordinate-wise multiplication. We refer to a trilinear form $\Phi : \ell_\infty^n \times \ell_\infty^n \times \ell_\infty^n \to \mathbb{C}$ as a trilinear form on ℓ_∞^n.

Note that ℓ_∞^n can be identified with the space of $n \times n$ diagonal matrices endowed with the operator norm. In turn, this implies that $M_d(\ell_\infty^n)$ can be identified with the space of maps $X : [n] \to M_d$, where $X(i)$ corresponds to the ith diagonal block of an element in $M_d(\ell_\infty^n)$. As such, the unit ball of $M_d(\ell_\infty^n)$ consists of the maps X such that $X(i) \in B_{M_d}$ for all $i \in [n]$. Then, it is not hard to see that

$$(2) \quad \|\Phi\|_{\mathrm{cb}} = \sup\left\{\|\Phi_d(X, Y, Z)\|_{M_d} : d \in \mathbb{N},\ X, Y, Z : [n] \to B_{M_d}\right\},$$

where

$$(3) \qquad \Phi_d(X, Y, Z) = \sum_{i,j,k=1}^{n} \Phi(e_i, e_j, e_k) X(i) Y(j) Z(k).$$

Note that if $\Phi = \sum_{\pi \in S_3} \Psi_\pi$ for some trilinear forms Ψ_π, then this decomposition holds also for the "lifts": $\Phi_d = \sum_{\pi \in S_3} (\Psi_\pi)_d$. Similarly,

$$(4) \quad \|\Phi\|_{\mathrm{jcb}} = \sup\left\{\|\widetilde{\Phi}_d(X, Y, Z)\|_{M_{d^3}} : d \in \mathbb{N},\ X, Y, Z : [n] \to B_{M_d}\right\},$$

where

$$(5) \qquad \widetilde{\Phi}_d(X, Y, Z) = \sum_{i,j,k=1}^{n} \Phi(e_i, e_j, e_k) X(i) \otimes Y(j) \otimes Z(k).$$

2.2. **The example.** Let Γ be a finite Abelian group and $f_0 : \Gamma \to \mathbb{C}$ be some function. Identify ℓ_∞^Γ with the function space $L_\infty(\Gamma)$. Define the trilinear form $\Phi : L_\infty(\Gamma) \times L_\infty(\Gamma) \times L_\infty(\Gamma) \to \mathbb{C}$ by

$$(6) \qquad \Phi(f_1, f_2, f_3) = \sum_{x,y \in \Gamma} f_0(y) f_1(x) f_2(x+y) f_3(x+2y).$$

Theorem 1.2 is based on a form as above, for the group $\mathbb{Z}_p$ with prime $p \geq 5$. To get an example for arbitrary integer $n \geq 4$, one can choose an odd prime between $n/2$ and n (which exists by Bertrand's postulate) and embed Φ as in (6) based on this group into a trilinear form on ℓ_∞^n in the obvious way. In the following two subsections we upper and lower bound the jointly completely bounded norm and the symmetrized completely bounded norm, respectively.

2.3. **Bounding the jointly completely bounded norm.** Let Φ be a trilinear form as in (6). We bound its jointly completely bounded norm using a non-commutative version of the generalized von Neumann inequality. The scalar version of this inequality, a basic tool in additive combinatorics, shows that the operator norm of Φ can be bounded from above in terms of the Gowers uniformity norm of f_0. It was observed already in [BBB+19] that this inequality holds also for the jointly completely bounded norm; in fact, they prove a more general version than what we use here. Here, we give an alternative short proof—a straightforward adaptation of the standard proof of the scalar case [TV06, Chapter 11]—for the version that is sufficient for our purpose. To state the inequality, we first define the Gowers uniformity norms (we refer to [TV06] for more information on these norms).

For a finite set S, denote by

$$\mathbb{E}_{s \in S}[f(s)] = \frac{1}{|S|} \sum_{s \in S} f(s).$$

Definition 2.1 (Gowers uniformity norms)**.** Let $t \geq 2$ be an integer, let Γ be a finite Abelian group and $f : \Gamma \to \mathbb{C}$ be some function. Then, the Gowers U^t-norm of f is given by

$$\|f\|_{U^t(\Gamma)}^{2^t} = \left| \mathbb{E}_{x,h_1,\ldots,h_t} \left[(\Delta_{h_1} \cdots \Delta_{h_t} f)(x) \right] \right|,$$

where $\Delta_h f(x) = \overline{f(x)} f(x+h)$.

For example, the 8th power of the Gowers U^3-norm is given by

$$(7) \quad \left|\mathbb{E}\left[\overline{f(x)}f(x+h_1)f(x+h_2)f(x+h_3)\overline{f(x+h_1+h_2)}\times \right.\right.$$
$$\left.\left.\overline{f(x+h_1+h_3)f(x+h_2+h_3)}f(x+h_1+h_2+h_3)\right]\right|,$$

where the expectation is over independent uniform $x, h_1, h_2, h_3 \in \Gamma$.

Our upper bound on $\|\Phi\|_{\text{jcb}}$ is based on the following inequality.

Proposition 2.2. *Let Γ be a finite Abelian group and let $f_0 : \Gamma \to \mathbb{C}$ be some function. Then, for Φ as in (6), we have*

$$\|\Phi\|_{\text{jcb}} \leq |\Gamma|^2 \|f_0\|_{U^3(\Gamma)}.$$

To prove this result, let us introduce the following non-commutative version of the Gowers uniformity norms. It is unknown if these also define norms, but here we do not need them to be.

Definition 2.3. Let Γ be a finite Abelian group and let $F : \Gamma \to M_d$. Define $\|F\|_{U^1(\Gamma)} = \|\mathbb{E}_{x,h\in\Gamma}F(x)^*F(x+h)\|^{\frac{1}{2}} = \|\mathbb{E}_{x\in\Gamma}F(x)\|$ and for $k \geq 2$ define

$$\|F\|_{U^k(\Gamma)} = \left\|\mathbb{E}_{x,h_1,\ldots,h_k}(\Delta_{h_1}\cdots\Delta_{h_k}F)(x)\right\|^{\frac{1}{2^k}},$$

where (with abuse of notation) $(\Delta_h F)(x) = F(x)^*F(x+h)$.

Proposition 2.2 follows from the following key lemma, which is a non-commutative version of the generalized von Neumann theorem.

Lemma 2.4. *Let $d \in \mathbb{N}$ and let Γ be a finite Abelian group. Let $A_0, A_1, A_2, A_3 : \Gamma \to B_{M_d}$ be maps such that for all $x, y \in \Gamma$ and distinct $i, j = 0, 1, 2, 3$, we have $[A_i(x), A_j(y)] = [A_i(x)^*, A_j(y)] = 0$. Then,*

$$\left\|\mathbb{E}_{x,y\in\Gamma}A_0(y)A_1(x)A_2(x+y)A_3(x+2y)\right\| \leq \|A_0\|_{U^3(\Gamma)}.$$

A version of Lemma 2.4 with k-term arithmetic progressions instead of 3-term arithmetic progressions also holds with the right-hand side replaced with $\|A_0\|_{U^k(\Gamma)}$. More generally, other known variants of the scalar case hold also in this non-commutative setting. The proof of Lemma 2.4 uses the following "matrix van der Corput lemma".

Lemma 2.5. *Let Γ be a finite Abelian group, let S be a finite set and for each $s \in S$ let $F_s : \Gamma \to M_d$. Then, for any map $\mathbf{B} : S \to B_{M_d}$,*

$$\left\|\mathbb{E}_{s\in S}\mathbb{E}_{x\in\Gamma}\mathbf{B}(s)F_s(x)\right\| \leq \left\|\mathbb{E}_{s\in S}\mathbb{E}_{x,h\in\Gamma}(\Delta_h F_s)(x)\right\|^{\frac{1}{2}}.$$

Proof: Let $F(s) = \mathbb{E}_{x \in \Gamma} F_s(x)$. The Cauchy–Schwarz inequality and boundedness of $\mathbf{B}$ give

$$\left\|\mathbb{E}_{s \in S}\mathbb{E}_{x \in \Gamma}\mathbf{B}(s)F_s(x)\right\| = \left\|\mathbb{E}_{s \in S}\mathbf{B}(s)F(s)\right\|$$

$$\leq \left\|\mathbb{E}_{s \in S}\mathbf{B}(s)\mathbf{B}(s)^*\right\|^{\frac{1}{2}}\left\|\mathbb{E}_{s \in S}F(s)^*F(s)\right\|^{\frac{1}{2}}$$

$$\leq \left\|\mathbb{E}_{s \in S}\mathbb{E}_{x,y \in \Gamma}F_s(x)^*F_s(y)\right\|^{\frac{1}{2}}.$$

The claim now follows by substituting $y = x + h$.   □

Proof of Lemma 2.4: We will repeatedly use the fact that the map $(x, y) \mapsto (x - y, y)$ on $\Gamma \times \Gamma$ is bijective. The proof uses Lemma 2.5 three times, with different choices of $S, \mathbf{B}$ and F_s.

First, let $S = \Gamma$, let $\mathbf{B} = A_1$ and let $F_x(y) = A_2(x+y)A_3(x+2y)A_0(y)$. Then Lemma 2.5 and commutativity of the A_i give

$$\left\|\mathbb{E}_{x,y}A_0(y)A_1(x)A_2(x+y)A_3(x+2y)\right\|^8 = \left\|\mathbb{E}_x\mathbf{B}(x)\mathbb{E}_yF_x(y)\right\|^8$$

$$(8) \qquad\qquad\qquad\qquad \leq \left\|\mathbb{E}_x\mathbb{E}_{y,h_1}(\Delta_{h_1}F_x)(y)\right\|^4.$$

Using the above-mentioned change of variables, the right-hand side of (8) equals

$$(9) \qquad\qquad \left\|\mathbb{E}_{x,h_1}\mathbb{E}_yF_{x-y}(y)^*F_{x-y}(y+h_1)\right\|^4.$$

Second, using the properties of the maps A_2, A_3, A_0, it follows that

$$F_{x-y}(y)^*F_{x-y}(y+h_1) =$$
$$A_2(x)^*A_2(x+h_1)A_3(x+y)^*A_3(x+y+2h_1)\,(\Delta_{h_1}A_0)(y).$$

Now let $S = \Gamma \times \Gamma$ and factor the above as

$$\mathbf{B}(x, h_1) = A_2(x)^*A_2(x+h_1)$$
$$F_{x,h_1}(y) = A_3(x+y)^*A_3(x+y+2h_1)\,(\Delta_{h_1}A_0)(y).$$

From Lemma 2.5 and another change of variables, it follows that the right-hand side of (9) is at most

$$(10) \quad \left\|\mathbb{E}_{x,h_1}\mathbb{E}_{y,h_2}(\Delta_{h_2}F_{x,h_1})(y)\right\|^2 =$$

$$\left\|\mathbb{E}_{x,h_1,h_2}\mathbb{E}_yF_{x-y,h_1}(y)^*F_{x-y,h_1}(y+h_2)\right\|^2.$$

Third, it follows from the properties of A_3, A_0 that

$$F_{x-y,h_1}(y)^*F_{x-y,h_1}(y+h_2) = A_3(x+2h_1)^*A_3(x)A_3(x+h_2)^* \times$$
$$A_3(x+2h_1+h_2)(\Delta_{h_2}\Delta_{h_1}A_0)(y).$$

Finally set $S = \Gamma \times \Gamma \times \Gamma$ and factor the above as

$$\mathbf{B}(x, h_1, h_2) = A_3(x + 2h_1)^* A_3(x) A_3(x + h_2)^* A_3(x + 2h_1 + h_2)$$
$$F_{x,h_1,h_2}(y) = (\Delta_{h_2} \Delta_{h_1} A_0)(y).$$

Again by Lemma 2.5, the right-hand side of (10) is at most

$$\left\| \mathbb{E}_{x,h_1,h_2} \mathbb{E}_{y,h_3} (\Delta_{h_3} F_{x,h_1,h_2})(y) \right\| = \left\| \mathbb{E}_{h_1,h_2,h_3,y} (\Delta_{h_3} \Delta_{h_2} \Delta_{h_1} A_0)(y) \right\|,$$

giving the claim. □

Proof of Proposition 2.2: For any $d \in \mathbb{N}$ and $X, Y, Z : \Gamma \to B_{M_d}$, define $A_0, A_1, A_2, A_3 : \Gamma \to B_{M_{d^3}}$ by

$$A_0(x) = f_0(x) \operatorname{Id} \otimes \operatorname{Id} \otimes \operatorname{Id}$$
$$A_1(x) = X(x) \otimes \operatorname{Id} \otimes \operatorname{Id}$$
$$A_2(x) = \operatorname{Id} \otimes Y(x) \otimes \operatorname{Id}$$
$$A_3(x) = \operatorname{Id} \otimes \operatorname{Id} \otimes Z(x).$$

Then, the statement follows trivially from Lemma 2.4 and noting that the factor $|\Gamma|^2$ comes from replacing sums with expectations. □

Remark 2.6. Note that Lemma 2.4 also applies if M_d is replaced by the the space $B(\mathcal{H})$ of bounded operators on some (possibly infinite-dimensional) Hilbert space $\mathcal{H}$. Moreover, the upper bound stated in Proposition 2.2 even applies if one replaces the jointly completely bounded norm by

$$\|\Phi\|_{\text{com}} := \sup \left\{ \left\| \sum_{i,j,k=1}^{n} \Phi(e_i, e_j, e_k) X_1(i) X_2(j) X_3(k) \right\|_{\mathcal{B}(\mathcal{H})} \right\},$$

where the supremum is over maps $X_1, X_2, X_3 : [n] \to B_{\mathcal{B}(\mathcal{H})}$ such that $[X_i(k), X_j(l)] = 0$ and $[X_i(k)^*, X_j(l)] = 0$ for all $k, l \in [n]$ and $i \neq j$.

Proposition 2.7. *Let $p \geq 5$ be a prime, $\omega = e^{2\pi i/p}$ and $f_0 : \mathbb{Z}_p \to \mathbb{C}$ be the function given by $f_0(x) = \omega^{x^3}$. Then,*

$$\|f_0\|_{U^3(\mathbb{Z}_p)} \leq (2/p)^{1/8}.$$

Proof: A straightforward calculation shows that for $x, h_1, h_2, h_3 \in \mathbb{Z}_p$, we have

$$(\Delta_{h_1} \Delta_{h_2} \Delta_{h_3} f_0)(x) = \omega^{6h_1 h_2 h_3}.$$

It follows that

$$\|f_0\|_{U^3(\mathbb{Z}_p)}^8 = \mathbb{E}_{h_1,h_2 \in \mathbb{Z}_p} \left[\mathbb{E}_{h_3 \in \mathbb{Z}_p} [\omega^{6h_1 h_2 h_3}] \right].$$

The inner expectation over h_3 is 1 if $h_1 = 0$ or $h_2 = 0$ and, since 6 is coprime relative to p and $\mathbb{Z}_p$ is a field, it is 0 otherwise. Hence, the right-hand side equals $(2p - 1)/p^2$, which gives the claim. $\qquad\square$

Corollary 2.8. *Let $p \geq 5$ be a prime, let $\Gamma = \mathbb{Z}_p$ and let f_0 be as in Proposition 2.7. Then, for Φ as in (6), we have*

$$\|\Phi\|_{\mathrm{jcb}} \leq p^2 (2/p)^{1/8}.$$

2.4. **Bounding the symmetrized completely bounded norm.** To lower bound the symmetrized completely bounded norm, we first prove the following result.

Lemma 2.9. *Let Ψ be a trilinear form on ℓ_∞^n. Then,*

$$\|\Psi\|_{\mathrm{sym}} \geq \sup \big\{ \|\Psi_d(X_1, X_2, X_3)\|_{M_d} : d \in \mathbb{N}, \ X_1, X_2, X_3 : [n] \to B_{M_d} \big\},$$

where the supremum is over maps X_i such that $[X_i(k), X_j(l)] = 0$ for all $k, l \in [n]$ and $i \neq j$.[1]

This result was already proved in [OP99] in much greater generality and the authors showed that the quantities appearing in Lemma 2.9 are equivalent. Since the proof of the inequality we need is straightforward, we add it for completeness.

Proof: Let $\Psi = \sum_{\pi \in S_3} \Psi_\pi$ be some decomposition. Let d be a positive integer and let $X_1, X_2, X_3 : [n] \to M_d$ be maps with commuting ranges as in the lemma. By the triangle inequality,

$$(11) \qquad \|\Psi_d(X_1, X_2, X_3)\|_{M_d} \leq \sum_{\pi \in S_3} \|(\Psi_\pi)_d(X_1, X_2, X_3)\|_{M_d}.$$

We claim that each term on the right-hand side equals

$$\|(\Psi_\pi \circ \pi)_d(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)})\|_{M_d}.$$

This implies the lemma because the above is clearly at most $\|\Psi_\pi \circ \pi\|_{\mathrm{cb}}$. To see the claim, first observe that by commutativity, it holds that for every $i_1, i_2, i_3 \in [n]$ and $\pi \in S_3$, we have

$$(12) \qquad X_1(i_1) X_2(i_2) X_3(i_3) = X_{\pi(1)}(i_{\pi(1)}) X_{\pi(2)}(i_{\pi(2)}) X_{\pi(3)}(i_{\pi(3)}).$$

Let χ be some trilinear form on ℓ_∞^n. Recall from (1) that

$$(13) \qquad \chi\big(e_{i_{\pi^{-1}(1)}}, e_{i_{\pi^{-1}(2)}}, e_{i_{\pi^{-1}(3)}}\big) = (\chi \circ \pi)(e_{i_1}, e_{i_2}, e_{i_3}).$$

---

[1] Note that in contrast with the norm $\|\Psi\|_{\mathrm{com}}$ defined above, here we do not require that $[X_i(k)^*, X_j(l)] = 0$.

Then,

$$\chi_d(X_1, X_2, X_3) = \sum_{i_1,i_2,i_3=1}^{n} \chi(e_{i_1}, e_{i_2}, e_{i_3}) X_1(i_1) X_2(i_2) X_3(i_3)$$

$$\overset{(12)}{=} \sum_{i_1,i_2,i_3=1}^{n} \chi(e_{i_1}, e_{i_2}, e_{i_3}) X_{\pi(1)}(i_{\pi(1)}) X_{\pi(2)}(i_{\pi(2)}) X_{\pi(3)}(i_{\pi(3)})$$

$$\overset{(13)}{=} \sum_{i_1,i_2,i_3=1}^{n} (\chi \circ \pi)(e_{i_1}, e_{i_2}, e_{i_3}) X_{\pi(1)}(i_1) X_{\pi(2)}(i_2) X_{\pi(3)}(i_3)$$

$$= (\chi \circ \pi)_d(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(2)}).$$

Applying this to $\chi = \Psi_\pi$ for each π gives the claim. $\qquad\square$

A trilinear form Ψ on $\mathbb{C}^n$ is *symmetric* if $\Psi \circ \pi = \Psi$ holds for every $\pi \in S_3$. A *slice* of a (not necessarily symmetric) trilinear form Ψ is an $n \times n$ matrix obtained by fixing one of the three coordinates (so there are $3n$ slices), for example

$$M_i = (\Psi(e_i, e_j, e_k))_{j,k=1}^{n}.$$

We will denote

$$\Delta(\Psi) = \max\{\|M\| : M \text{ is a slice of } \Psi\}.$$

Also define

$$\|\Psi\|_{\ell_2} = \Big(\sum_{i,j,k=1}^{n} |\Psi(e_i, e_j, e_k)|^2 \Big)^{\frac{1}{2}}.$$

The following lemma, due to Varopoulos [Var74], is the key to our lower bound on $\|\Psi\|_{\text{sym}}$. Again, the proof is simple, so we add it for completeness.

Lemma 2.10. *Let Ψ be a symmetric trilinear form on ℓ_∞^n. Then,*

$$\|\Psi\|_{\text{sym}} \geq \frac{\|\Psi\|_{\ell_2}^2}{\Delta(\Psi)}.$$

Proof: For each $i \in [n]$, let $M_i = (\Psi(e_i, e_j, e_k))_{j,k=1}^{n}$ be the slice obtained by fixing the first coordinate to i. Define $W_i = \Delta(\Psi)^{-1} M_i$ and note that this has operator norm at most 1. For each $i \in [n]$, define the $(2n+2) \times (2n+2)$ block matrix

$$X_i = \begin{bmatrix} \begin{array}{c|c|c|c} e_i & & & \\ \hline & W_i^* & & \\ \hline & & & \\ \hline & & e_i^* & \end{array} \end{bmatrix},$$

where the row and column blocks have size $1, n, n, 1$, respectively, and where the empty blocks are filled with zeros. Then, for all $i, j \in [n]$,

$$X_i^* X_i = \begin{bmatrix} \boxed{\begin{smallmatrix} 1 \\ W_i W_i^* \\ e_i e_i^* \end{smallmatrix}} \end{bmatrix} \quad \text{and} \quad X_i X_j = \begin{bmatrix} \boxed{\begin{smallmatrix} W_i^* e_j \\ e_i^* W_j^* \end{smallmatrix}} \end{bmatrix}.$$

The first identity shows that $\|X_i\| = \max\{1, \|W_i\|\} \leq 1$. Since Ψ is symmetric, we have $M_j e_i = M_i e_j$ for all i, j. Therefore, the second identity shows that these matrices commute with each other. Moreover,

$$X_i X_j X_k = \begin{bmatrix} \boxed{\begin{smallmatrix} e_i^* W_j^* e_k \end{smallmatrix}} \end{bmatrix} = \frac{1}{\Delta(\Psi)} \begin{bmatrix} \boxed{\begin{smallmatrix} \Psi(e_i, e_j, e_k) \end{smallmatrix}} \end{bmatrix}.$$

Hence, by Lemma 2.9, we get that

$$\|\Psi\|_{\mathrm{sym}} \geq \left\| \sum_{i,j,k=1}^{n} \Psi(e_i, e_j, e_k) X_i X_j X_k \right\|_{M_d} \geq \frac{1}{\Delta(\Psi)} \sum_{i,j,k=1}^{n} |\Psi(e_i, e_j, e_k)|^2.$$

This concludes the proof. $\qquad\qquad\square$

Proposition 2.11. *Let $p \geq 3$ be a prime, let $\Gamma = \mathbb{Z}_p$, let $f_0 : \Gamma \to \mathbb{T}$ and let Φ be a trilinear form as in (6). Then, $\|\Phi\|_{\ell_2}^2 = p^2$ and $\Delta(\Phi) = 1$.*

Proof: The first assertion is straightforward to check. Let $\{e_x : x \in \Gamma\}$ denote the standard basis for $\mathbb{C}^\Gamma$. Fix a $x \in \Gamma$ and consider the slice obtained by fixing the first coordinate of the tensor corresponding to Φ to x:

$$\begin{aligned} M_x &= \sum_{y,z \in \Gamma} \Phi(e_x, e_y, e_z) e_y \otimes e_z \\ &= \sum_{y,z \in \Gamma} \sum_{u,v \in \Gamma} f_0(v) e_x(u) e_y(u+v) e_z(u+2v) e_y \otimes e_z \\ &= \sum_{y,z \in \Gamma} \sum_{v \in \Gamma} f_0(v) e_y(x+v) e_z(x+2v) e_y \otimes e_z \\ &= \sum_{y,z \in \Gamma} f_0(y-x) e_z(2y-x) e_y \otimes e_z \\ &= \sum_{y \in \Gamma} f_0(y-x) e_y \otimes e_{2y-x}. \end{aligned}$$

Since for our group $\mathbb{Z}_p$, the map $y \mapsto 2y$ is injective, it follows that M_x is a unitary matrix and therefore has norm 1. The other slices can similarly be seen to have norm 1. $\qquad\qquad\square$

2.5. **Putting everything together.** To apply Lemma 2.10 we need to symmetrize our form. We do this so as to approximately preserve $\Delta(\Phi)$, $\|\Phi\|_{\ell_2}$ and $\|\Phi\|_{\text{jcb}}$.[2] To this end, we first consider the trilinear form $E : \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}$ given by

$$E(u, v, w) = u_1 v_2 w_3.$$

For a trilinear form Ψ on $\mathbb{C}^n$, the trilinear form $\Psi \otimes E$ on $\mathbb{C}^{3n}$ is given by

$$(\Psi \otimes E)(x \otimes u, y \otimes v, z \otimes w) = \Psi(x, y, z)E(u, v, w),$$

for $x, y, z \in \mathbb{C}^n$ and $u, v, w \in \mathbb{C}^3$. If Ψ is a trilinear form on ℓ_∞^n, then we define the symmetrized version of Ψ to be the trilinear form $\overline{\Psi}$ on ℓ_∞^{3n} by

$$(14) \qquad \overline{\Psi} = \sum_{\pi \in S_3} (\Psi \otimes E) \circ \pi.$$

It is easy to see that $\overline{\Psi}$ is symmetric. Moreover, as per (1), for any $x_i \in \mathbb{C}^n$ and $u_i \in \mathbb{C}^3$ for $i = 1, 2, 3$, we have

$$(15) \quad \overline{\Psi}(x_1 \otimes u_1, x_2 \otimes u_2, x_3 \otimes u_3) =$$

$$\sum_{\pi \in S_3} \Psi\big(x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, x_{\pi^{-1}(3)}\big) E\big(u_{\pi^{-1}(1)}, u_{\pi^{-1}(2)}, u_{\pi^{-1}(3)}\big) =$$

$$\sum_{\pi \in S_3} \Psi\big(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}\big) E\big(u_{\pi(1)}, u_{\pi(2)}, u_{\pi(3)}\big).$$

Proposition 2.12. *Let Ψ be a trilinear form on ℓ_∞^n. Then, its symmetrization $\overline{\Psi}$ as in (14) satisfies:*

- $\Delta(\overline{\Psi}) = \Delta(\Psi)$

- $\|\overline{\Psi}\|_{\ell_2}^2 = 6\|\Psi\|_{\ell_2}^2$

- $\|\overline{\Psi}\|_{\text{jcb}} \leq 6\|\Psi\|_{\text{jcb}}.$

Proof: We begin with the first item. The lower bound $\Delta(\overline{\Psi}) \geq \Delta(\Psi)$ follows easily from the fact that

$$\overline{\Psi}(x \otimes e_1, y \otimes e_2, z \otimes e_3) = \Psi(x, y, z).$$

By symmetry of $\overline{\Psi}$, for the upper bound $\Delta(\overline{\Psi}) \leq \Delta(\Psi)$, it suffices to show that for any $i \in [n]$ and $a \in [3]$, the slice corresponding to the

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[2]Perhaps a more natural symmetrization to consider is $\Phi_s = \sum_\pi \Phi \circ \pi$. However, the relevant values can be dramatically affected by this procedure, since we could get a zero tensor from a non-zero one.

bilinear form $B : \mathbb{C}^{3n} \times \mathbb{C}^{3n} \to \mathbb{C}$ given by

$$B(x, y) = \overline{\Psi}(e_i \otimes e_a, x, y)$$

has operator norm at most $\Delta(\Psi)$. Let $x, y \in \mathbb{C}^{3n}$ be unit vectors. Write $x = x_1 \otimes e_1 + x_2 \otimes e_2 + x_3 \otimes e_3$ for $x_1, x_2, x_3 \in \mathbb{C}^n$ and similarly for y. Then,

$$
\begin{aligned}
|B(x, y)| &= \Big| \sum_{b,c=1}^{3} \overline{\Psi}(e_i \otimes e_a, x_b \otimes e_b, y_c \otimes e_c) \Big| \\
&= \Big| \sum_{b,c=1}^{3} \sum_{\pi \in S_3} \big((\Psi \otimes E) \circ \pi \big)(e_i \otimes e_a, x_b \otimes e_b, y_c \otimes e_c) \Big| \\
&= \Big| \sum_{\pi \in S_3} \sum_{b,c=1}^{3} (\Psi \circ \pi)(e_i, x_b, y_c) \cdot (E \circ \pi)(e_a, e_b, e_c) \Big|.
\end{aligned}
$$

Observe that $(E \circ \pi)(e_a, e_b, e_c)$ equals 1 if $a = \pi(1), b = \pi(2), c = \pi(3)$ and 0 otherwise. Hence, the above is at most

$$\sum_{\pi \in S_3 : \pi(1)=a} \big| (\Psi \circ \pi)(e_i, x_{\pi(2)}, y_{\pi(3)}) \big| \leq \Delta(\Psi) \sum_{\pi \in S_3 : \pi(1)=a} \|x_{\pi(2)}\| \, \|y_{\pi(3)}\|.$$

By the Cauchy-Schwarz inequality, the last sum is at most

$$\Big(\sum_{\pi \in S_3 : \pi(1)=a} \|x_{\pi(2)}\|^2 \Big)^{\frac{1}{2}} \Big(\sum_{\pi \in S_3 : \pi(1)=a} \|y_{\pi(3)}\|^2 \Big)^{\frac{1}{2}} \leq 1.$$

This proves the first item.

The second item is a straightforward calculation. It follows from (15) that

$$
\begin{aligned}
\|\overline{\Psi}\|_{\ell_2}^2 &= \sum_{i_1,i_2,i_3=1}^{n} \sum_{j_1,j_2,j_3=1}^{3} \Big| \overline{\Psi}(e_{i_1} \otimes e_{j_1}, e_{i_2} \otimes e_{j_2}, e_{i_3} \otimes e_{j_3}) \Big|^2 \\
&= \sum_{i_1,i_2,i_3=1}^{n} \sum_{j_1,j_2,j_3=1}^{3} \Big| \sum_{\pi \in S_3} \Psi(e_{i_{\pi(1)}}, e_{i_{\pi(2)}}, e_{i_{\pi(3)}}) E(e_{j_{\pi(1)}}, e_{j_{\pi(2)}}, e_{j_{\pi(3)}}) \Big|^2.
\end{aligned}
$$

Observe that $E(e_{j_{\pi(1)}}, e_{j_{\pi(2)}}, e_{j_{\pi(3)}}) = 1$ only if $j_1, j_2, j_3 \in [3]$ are distinct and that in that case there is a unique $\pi \in S_3$ for which this holds. Since for any fixed $\pi \in S_3$ we have

$$\sum_{i_1,i_2,i_3=1}^{n} |\Psi(e_{i_{\pi(1)}}, e_{i_{\pi(2)}}, e_{i_{\pi(3)}})|^2 = \|\Psi\|_{\ell_2}^2,$$

and there are 6 ways to choose j_1, j_2, j_3 distinct, the second item follows.

For the third item, observe that the jointly completely bounded norm is commutative, which is to say that $\|\Psi \circ \pi\|_{\text{jcb}} = \|\Psi\|_{\text{jcb}}$ for every $\pi \in S_3$. The claim then follows from the identity $\|\Psi \otimes E\|_{\text{jcb}} = \|\Psi\|_{\text{jcb}}$ and triangle inequality. To see the identity, recall the expressions (4) and (5) for the jointly completely bounded norm. Let d be a positive integer and let $X, Y, Z : [n] \times [3] \to B_{M_d}$. Then,

$$\sum_{i,j,k=1}^{n} \sum_{a,b,c=1}^{3} \Psi(e_i, e_j, e_j) E(e_a, e_b, e_c) X(i,a) \otimes Y(j,b) \otimes Z(k,c)$$

$$= \sum_{i,j,k=1}^{n} \Psi(e_i, e_j, e_j) X(i,1) \otimes Y(j,2) \otimes Z(k,3).$$

Taking norms and suprema over X, Y, Z and $d \in \mathbb{N}$ gives identity. $\quad\square$

With this, the proof of Theorem 1.2 is straightforward.

Proof of Theorem 1.2: Let $p \geq 5$ be a prime number and let $\Gamma = \mathbb{Z}_p$. Let Φ be a trilinear form as in (6) and let $f_0 : \Gamma \to \mathbb{C}$ be as in Proposition 2.7. Let $\overline{\Phi}$ be the symmetrization of Φ as in (14). Then, it follows from Corollary 2.8 and Proposition 2.12 that $\|\overline{\Phi}\|_{\text{jcb}} \leq 6p^2(2/p)^{1/8}$. On the other hand, it follows from Lemma 2.10, Proposition 2.11 and Proposition 2.12 that $\|\overline{\Phi}\|_{\text{sym}} \geq 6p^2$. $\quad\square$

3. Alternative tensors

A straightforward argument based on splitting the tensor associated to the trilinear form Ψ as in (6) into real and complex parts shows that Theorem 1.2 holds also for a trilinear form $\Phi : \ell_\infty^n \times \ell_\infty^n \times \ell_\infty^n \to \mathbb{C}$ whose associated tensor is real, which is to say that $\Phi(e_i, e_j, e_k) \in \mathbb{R}$ for every $i, j, k \in [n]$. Alternatively, one can directly get such a form by replacing Proposition 2.7 in our construction with the following statement [TV06, Exercise 11.1.17], giving a random example.

Proposition 3.1. *Let Γ be a finite Abelian group and $f : \Gamma \to \{-1, 1\}$ be a uniformly random mapping. Then, $\|f\|_{U^t(\Gamma)} \leq O_t(|\Gamma|^{-1/2^t})$ with probability $1 - o_t(1)$.*

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