A NOTE ON RANDOMLY SCALED SCALE-DECORATED POISSON POINT PROCESSES

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ABSTRACT. In this article, we shall study the properties of randomly scaled scale-decorated Poisson point processes and obtain a characterization based on its Laplace functional. In the way of deriving the characterization, we shall show that the clusters are independently and identically distributed. A connection with randomly shifted decorated Poisson point process is also obtained.

1. INTRODUCTION

Poisson point processes are turned out to be very important in the context of extreme value theory for independently and identically distributed random variables (see Chapter 3 in Resnick (1987), Chapter 7 in Resnick (2007) and references therein). Recently, it has been observed that the dependence among the random variables affects the asymptotic behavior of extremes resulting in slower rate of growth of extremes, clusters around the extreme points etc. So the class of all Poisson point processes is not large enough to accommodate all possible extremal processes and it motivates the necessity of different generalizations of Poisson point processes. One of many possible generalizations is randomly scaled scale-decorated Poisson point processes (SScDPPP) which is contained in the class of all Cox cluster point processes. We shall see that as a particular class of Cox cluster processes, SScDPPP enjoys many nice properties which will be investigated in this article.

Loosely speaking, a scale-decorated Poisson point process (ScDPPP) is a cluster Poisson point process where cluster around each Poisson point is an independent copy of another point process. If we multiply each point in ScDPPP by a positive random variable which independent of the ScDPPP, then the resulting point process will be called SScDPPP. This generalization is large enough to accommodate all possible extremal processes for the random variables known in literature till now for the random variables with regularly varying tails. The notion SScDPPP is introduced in Bhattacharya et al. (2017a) in the context of extremes of branching random walk (BRW). Note that ScDPPP was appeared for the first time in Davydov et al. (2008) where an equivalence between strictly α-stable (StαS) point process and ScDPPP is obtained. In another article Bhattacharya et al. (2017b), it turned out to be very difficult to verify that the limiting extremal process is an SScDPPP (predicted in Brunet and Derrida (2011)) and appropriately scaled superposition of regularly varying point processes converges weakly to a StαS point process without the characterization of SScDPPP based on its Laplace functional. This characterization will be derived in this article. The tools and the approach used in this article to obtain the characterization and study the properties of SScDPPP is borrowed from Subag and Zeitouni (2015a). However it does not follow from the aforementioned article as a randomly shifted decorated Poisson point process (SSDPPP) can be transformed into an SScDPPP on the positive half of the real line by considering exponential of each point in SSDPPP but it is not always possible to transform an SScDPP to an SSDPPP.

The extreme positions of the particles at the nth generation in BRW turned out to be very important to study due to its connection to many important models including random energy model, Gaussian free field, Gaussian multiplicative chaos, tree polymers etc. (see Subag and Zeitouni (2015a) and references therein). It was predicted in Brunet and Derrida (2011) that the limiting extremal point process in BRW satisfies some distributional stability property and should admit

some representation in terms of Poisson point process. In case of displacements having exponentially decaying tail, the stability property is termed as exponentially stable point process and the representation is termed as randomly shifted decorated Poisson point process (SSDPPP). It has been shown in Maillard (2013) that decorated Poisson point process (DPPP) and exponentially stable point process are two sides of the same coin. This was implicitly observed in Davydov et al. (2008) (see Example 8.6). This result is generalized in Subag and Zeitouni (2015a) where equivalence between exponentially stable point process with random shift and SSDPPP is established using a novel technique based on shift-Laplace functional. In Madaule (2015), the extremal process is derived in case of displacements with exponentially decaying tail. It has also been shown that the extremal process satisfies the predictions in Brunet and Derrida (2011) relying upon the work Maillard (2013). Although, it has been shown that the extremal process admits a representation as SSDPPP , the law of decoration was missing. The distribution of the decoration can be obtained from the shift-Laplace functional of the extremal process using the construction given in Subag and Zeitouni (2015a). The equivalence turned out to be very useful in studying extremal point process in p-spin spherical spin glass model (see Subag and Zeitouni (2015b)), genealogy of extremal process of BRW (see Mallein (2016)), extremal process of two-dimensional Gaussian free field (see Biskup and Louidor (2016)) etc.

The investigation for BRW with independently and identically distributed displacements having regularly varying tail was initiated in Durrett (1979) and Durrett (1983). Then the extremal process is studied in Bhattacharya et al. (2017a) where the limit is shown to be SScDPPP. Then the independence among the displacements coming from the same parent is relaxed in Bhattacharya et al. (2017b) by assuming that the displacements are jointly regularly varying. Under this assumption, it turned out to be very difficult to verify that the limit process is SScDPPP. Relying on the results proved in this article, it has been shown that the limit admits an SScDPPP representation which supports the predictions given in Brunet and Derrida (2011). In the same article, a characterization of StαS point process based on its Laplace functional turned out to be very important which we derive in this article as a consequence of the main result of this article. These results are also used in Bhattacharya et al. (2016) to study multitype BRW with displacements having regularly varying tails.

The article is organized as follows. In Section 2, we have introduced some notations and stated main results of this article. The expression for scaled Laplace functional of SScDPPP has been derived in Section 3. In Section 4, the properties of the scaled Laplace functional has been studied. The construction of the scale-decoration is given in Section 5. In Section 6, proofs of main results and the connection between SSDPPP and SScDPPP are given.

2. Notation and main results

2.1. **Notation.** Suppose $\bar{\mathbb{R}}_0$ denotes the punctured space $[-\infty, \infty] \setminus \{0\}$ and $\mathscr{M}(\bar{\mathbb{R}}_0)$ denotes the space of all Radon measures on $\bar{\mathbb{R}}_0$ which does not put any mass on $\{\pm\infty\}$. The scalar multiplication on $\mathscr{M}(\bar{\mathbb{R}}_0)$ by a positive real number b is denoted by $\mathbf{S}_b$ and is defined as follows: if $\mathcal{P} = \sum_i \boldsymbol{\delta}_{u_i} \in \mathscr{M}(\bar{\mathbb{R}}_0)$, then

$$\mathbf{S}_b\,\mathcal{P} = \sum_i \boldsymbol{\delta}_{bu_i}.$$

In other words, a scalar multiple of a point measure is obtained by multiplying each point of the measure by a positive real number. A point process on $\bar{\mathbb{R}}_0$ is an $\mathscr{M}(\bar{\mathbb{R}}_0)$-valued random variable defined on $(\Omega, \mathcal{F}, \mathbf{P})$ that does not charge any mass at $\pm\infty$. The following definition of strictly α-stable point process was introduced in Davydov et al. (2008).

Definition 2.1 (**StαS point process**; Davydov et al. (2008))**.** A point process N (on $\bar{\mathbb{R}}_0$) is called a strictly α-stable (StαS) point process ($\alpha > 0$) if for every $b_1, b_2 > 0$,

$$(2.1) \qquad\qquad \mathbf{S}_{b_1}\,N_1 + \mathbf{S}_{b_2}\,N_2 \overset{d}{=} \mathbf{S}_{(b_1^\alpha + b_2^\alpha)^{1/\alpha}}\,N,$$

where N_1, N_2 are independent copies of N, $+$ denotes superposition of point processes and $\overset{d}{=}$ denotes equality in distribution.

A point process M will be called **randomly scaled strictly α-stable point process** if $M \stackrel{d}{=} \mathbf{S}_U N$ where N is a StαS point process and U is a positive random variable independent of N.

It has been established in Davydov et al. (2008), that a point process (on $\mathbb{R}$) is StαS if and only if it admits a series representation of a special kind (analogous to the LePage series representation for stable processes). To be more precise, we need the following definition which is introduced in Bhattacharya et al. (2017a).

Definition 2.2 (Scale-decorated Poisson point process)**.** A point process N is called a scale decorated Poisson point process with intensity measure m and scale-decoration $\mathcal{P}$ (denoted by $N \sim ScDPPP(m, \mathcal{P})$) if there exists a Poisson random measure $\Lambda = \sum_{i=1}^{\infty} \boldsymbol{\delta}_{\lambda_i}$ on $(0, \infty)$ with intensity measure m and a point process $\mathcal{P}$ such that

$$N \stackrel{d}{=} \sum_{i=1}^{\infty} \mathbf{S}_{\lambda_i} \mathcal{P}$$

where $(\mathcal{P}_i : i \geq 1)$ is a collection of independent copies of $\mathcal{P}$.

As mentioned above, it has been observed in Davydov et al. (2008) (see Example 8.6 therein) that a point process N is StαS if and only if $N \sim ScDPPP(m_\alpha, \mathcal{P})$ where $\mathcal{P}$ is a point process on $\bar{\mathbb{R}}_0$ and m_α is measure on $(0, \infty)$ with $m_\alpha\big((x, \infty)\big) = \alpha x^{-\alpha-1}$ for every $x > 0$. The light-tailed analogue of this result has been proved in a novel approach by Maillard (2013).

A point process M is called a **randomly scaled scale-decorated Poisson point process** with mean measure m, scale-decoration $\mathcal{P}$ and random scale U (denoted by $M \sim SScDPPP(m, \mathcal{P}, U)$) if $M \stackrel{d}{=} \mathbf{S}_U N$ where $N \sim ScDPPP(m, \mathcal{P})$ and U is a positive random variable independent of N.

Let $C_c^+(\bar{\mathbb{R}}_0)$ denote the space of all nonnegative continuous functions defined on $\bar{\mathbb{R}}_0$ with compact support (and hence vanishing in a neighbourhood of 0). By an abuse of notation, for a measurable function $f : \bar{\mathbb{R}}_0 \to [0, \infty)$, we denote by $\mathbf{S}_y f(\cdot)$ the function $f(y\cdot)$. For a point process N on $\bar{\mathbb{R}}_0$ and any $y > 0$, one has $\int f \, d\, \mathbf{S}_y N = \int \mathbf{S}_y f \, d\, N$. The Laplace functional of a point process N will be denoted by

$$\Psi_N(f) = \mathbf{E}\left(\exp\left\{ -N(f) \right\} \right), \tag{2.2}$$

where $N(f) = \int f \, d\, N$. In parallel to the notion of shift-Laplace functional from Subag and Zeitouni (2015a), we define the **scaled Laplace functional** as

$$\Psi_N(f \| y) := \Psi_N(\mathbf{S}_{y^{-1}} f) \tag{2.3}$$

for some $y > 0$. Let $g : (0, \infty) \to (0, \infty)$ be a measurable function. We define $[g]_{sc}$ as the class of all positive measurable functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that for some $y > 0$, $f(x) = g(yx)$ for all $x > 0$. Let us define by $\Phi_\alpha(x)$ the Frechét distribution function, i.e., for each $\alpha > 0$,

$$\Phi_\alpha(x) = \exp(-x^{-\alpha}), \qquad x > 0. \tag{2.4}$$

Definition 2.3 (Scale-uniquely supported)**.** The scaled Laplace functional of the point process N is uniquely supported on $[g]_{sc}$ if for any $f \in C_c^+(\bar{\mathbb{R}}_0)$, there exists a constant c_f (depending on f only) such that $\Psi_N(f \| y) = g(yc_f)$ for all $y > 0$.

2.2. **Main Result.** The notion of scale-uniquely supported is intimately tied to the behaviour of SScDPPP. Sometimes, it is not possible to write down the SScDPPP representation, but it is easy to study properties of its scaled Laplace functional. In those cases, the following theorem turns out to be useful; see e.g., Bhattacharya et al. (2017a) and Bhattacharya et al. (2016). The theorem can be very useful to study the weak limit of a sequence of point processes as existence of the weak limit can be guaranteed by studying properties of scaled Laplace functional of the limit.

Theorem 2.4. *Let N be a locally finite point process on $\bar{\mathbb{R}}_0$ satisfying the following assumptions:*

$$\mathbf{P}(N(\bar{\mathbb{R}}_0) > 0) > 0 \text{ and } \mathbf{E}\left(N(\bar{\mathbb{R}}_0 \setminus (-a, a)) \right) < \infty \tag{2.5}$$

for some $a > 0$. Then the following statements are equivalent:

(Prop1) $\Psi_N(f\|\cdot)$ *is scale-uniquely supported on* $[g]_{sc}$ *for all* $f \in C_c^+(\bar{\mathbb{R}}_0)$ *for some function* $g :$
$\mathbb{R}_+ \to \mathbb{R}_+$.

(Prop2) $\Psi_N(f\|\cdot)$ *is scale-uniquely supported on* $[g]_{sc}$ *for all* $f \in C_c^+(\bar{\mathbb{R}}_0)$, *where*

$$(2.6) \qquad g(y) = \mathbf{E}\left(\Phi_\alpha(ycW^{-1})\right),$$

for some $\alpha > 0$, *for some* $c > 0$ *and some positive random variable* W.

(Prop3) $N \sim SScDPPP(m_\alpha, \mathcal{P}, W)$ *for some point process* $\mathcal{P}$, *some positive random variable* W
and some positive scalar $\alpha > 0$ *(same as in Prop2) where* $m_\alpha(\cdot)$ *is a measure on* $(0, \infty)$,
such that $m_\alpha((x, \infty)) = x^{-\alpha}$ *for every* $x > 0$.

The next result is an immediate corollary of the above proposition. In Davydov et al. (2008),
it has been established that B1 and B2 are equivalent.

Corollary 2.5. *Assume that* $\mathbf{P}(N(\bar{\mathbb{R}}_0) > 0) > 0$. *Fix* $\alpha > 0$. *Then the following statements are*
equivalent:

(B1) *N is a scale-decorated Poisson point process on* $\bar{\mathbb{R}}_0$ *with Poisson intensity* $m_\alpha(dx)$, *where*
m_α *is as defined in Prop3.*

(B2) *N is a strictly α-stable point process.*

(B3) *The scaled Laplace functional of N is scale-uniquely supported on the class* $[\Phi_\alpha]_{sc}$.

Suppose that the point process N in Theorem 2.4 is supported on the positive part of the real
line. Then these results are equivalent to the results in Subag and Zeitouni (2015a) via one to one
correspondence between the spaces $\mathscr{M}((0, \infty])$ and $\mathscr{M}((-\infty, \infty])$ given by $\sum \delta_{a_i} \leftrightarrow \sum \delta_{\log a_i}$. In
particular, the assumption of monotonicity of g can be dropped from Corollary 3 of the aforemen-
tioned reference.

2.3. **Connection to SDPPP.** Here we shall establish a connection between the notions SScDPPP
and SDPPP (introduced in Subag and Zeitouni (2015a)). We shall show that if we consider an
SScDPPP on $(0, \infty]$ and take logarithm transform of each atom, then the transformed point process
is an SDPPP on $\bar{\mathbb{R}} \setminus \{-\infty\}$. Conversely if we consider an SDPPP on $\bar{\mathbb{R}} \setminus \{-\infty\}$ and take exponential
transform of its atoms, then the transformed point process is a SScDPPP point process on $\bar{\mathbb{R}} \setminus$
$[-\infty, 0]$. Based on this connection, we shall derive a slightly extended version (relaxing the property
"increasing" of h) of Corollary 3 in Subag and Zeitouni (2015a), see Theorem 2.9 below.

Here, we recall the basic notations and definitions from Subag and Zeitouni (2015a). Let $\mathcal{P} =$
$\sum \delta_{p_i}$ be a point process, then by $\theta_x \mathcal{P}$ we denote the shifted point process $\sum \delta_{x+p_i}$ for every $x \in \mathbb{R}$.
By $\bar{\mathbb{R}}_{-\infty}$, we denote the space $[-\infty, \infty] \setminus \{-\infty\}$.

Definition 2.6 (Decorated Poisson point process, Brunet and Derrida (2011), Maillard (2013)).
A point process Q is called a decorated Poisson point process of Poisson intensity m and decoration
$\mathcal{P}$ (denoted by $Q \sim DPPP(m, \mathcal{P})$) if $Q \stackrel{d}{=} \sum_{i=1}^{\infty} \theta_{\lambda_i} \mathcal{P}_i$ where $\Lambda = \sum_{i=1}^{\infty} \delta_{\lambda_i}$ is a Poisson random
measure with intensity m and $\mathcal{P}$ is some point process on $\bar{\mathbb{R}}_{-\infty}$ and $\mathcal{P}_i$'s are independent copies of
the point process $\mathcal{P}$.

Definition 2.7 (Randomly shifted decorated Poisson point process, Subag and Zeitouni (2015a)).
A point process T is called a *randomly shifted decorated Poisson point process* of Poisson intensity
m and decoration $\mathcal{P}$ and shift U (denoted by $T \sim SDPPP(m, \mathcal{P}, U)$) if for $Q \sim DPPP(m, \mathcal{P})$
and some independent random variable U, $T \stackrel{d}{=} \theta_U Q$.

Subag and Zeitouni (2015a) introduced shift-Laplace functional as

$$(2.7) \qquad L_T(f|y) = \mathbf{E}\left(\exp\left\{-\int \theta_{-y} f \, dT\right\}\right)$$

where $\theta_{-y} f(x) = f(x - y)$ for every non-negative measurable function $f : \mathbb{R} \to \mathbb{R}$. By $f \approx g$, we
mean the two functions f and g are equal up to translation and let $[g]$ denotes the equivalence
class of g under the relation.

Definition 2.8 (Uniquely supported). A shift-Laplace functional is uniquely supported on $[h]$ if
$L_T(f|\cdot) \approx h(\cdot)$ for every $f \in C_c^+(\bar{\mathbb{R}}_{-\infty})$.

Theorem 2.9 (Slight imporevement of Corollary 3 in Subag and Zeitouni (2015a))**.** *Let T be a point process such that $\mathbf{P}(T(\bar{\mathbb{R}}_{-\infty}) > 0) = 1$. Then the following are equivalent.*

(a) *$L_T(f|\cdot)$ is uniquely supported on $[h]$ for some function $h : \mathbb{R} \to \mathbb{R}$.*

(b) *$L_T(f|\cdot)$ is uniquely supported on $[h]$, where*

$$
(2.8) \qquad h(y) = \mathbf{E}\left(\exp\left\{ - e^{-c(y-U)} \right\} \right)
$$

for some random variable U and $c > 0$.

(c) *$T \sim SDPPP(e^{-cx}dx, \mathcal{P}, U)$ for some point process $\mathcal{P}$, random variable U and $c > 0$ (same as in (b)).*

3. Proof of "(Prop3) implies (Prop1)"

The proof is a heavy-tailed analogue of the proof of Theorem 9 (Converse Part) in Subag and Zeitouni (2015a). Let $N \sim SScDPPP(m_\alpha, \mathcal{P}, W)$ where W is a positive random variable. We shall compute

$$
(3.1) \qquad \Psi_N(f\|y) = \mathbf{E}\left[\mathbf{E}\left(\exp\left\{ - \sum_{i=1}^{\infty} \mathbf{S}_W \, \mathbb{S}_{\lambda_i} \mathcal{P}_i(\mathbf{S}_{y^{-1}} f) \right\} \Big| W \right) \right]
$$

for any $f \in C_c^+(\bar{\mathbb{R}}_0)$ where $\{\lambda_i\}_{i=1}^{\infty}$ be the atoms of the Poisson point process Λ with mean measure m_α on $(0, \infty]$. Fix $\eta > 0$. Define $I(\eta) = \{i : \lambda_i > \eta\}$ to be the collection of all indices of the atoms in the interval $(\eta, \infty]$. We introduce another point process

$$
N_\eta = \mathbf{S}_W \sum_{i \in I(\eta)} \mathbf{S}_{\lambda_i} \mathcal{P}_i.
$$

It is clear that by monotone convergence theorem, the right hand side of (3.1) equals

$$
(3.2) \qquad \lim_{\eta \to 0} \mathbf{E}\left[\mathbf{E}\left(\exp\left\{ - \sum_{i \in I(\eta)} \mathbf{S}_{W\lambda_i} \mathcal{P}_i(\mathbf{S}_{y^{-1}} f) \right\} \Big| W \right) \right].
$$

Note that $|I(\eta)|$ (cardinality of the random set $I(\eta)$) is a Poisson random variable with mean $\eta^{-\alpha}$ and it is independent of W and $\{\mathcal{P}_i\}_{i \geq 1}$. The conditional expectation in (3.2) can then be written as

$$
(3.3) \qquad \mathbf{E}\left[\mathbf{E}\left(\exp\left\{ - \sum_{i \in I(\eta)} \mathbf{S}_{W\lambda_i} \mathcal{P}_i(\mathbf{S}_{y^{-1}} f) \right\} \Big| I(\eta), W \right) \Big| W \right].
$$

It easily follows that conditioned on the event $I(\eta) = k$,

$$
\sum_{i \in I(\eta)} \delta_{\lambda_i} \overset{d}{=} \sum_{i=1}^{k} \delta_{\eta X_i}
$$

for every $\eta > 0$ and $\{X_i\}_{i \geq 1}$ be an i.i.d. collection of $Pareto(\alpha)$ random variables with probability density function $f_X(x) = \alpha x^{-\alpha-1}$ for $x > 1$. Using this fact, the conditional expectation in (3.3) can be written as,

$$
\mathbf{E}\left[\prod_{i \in I(\eta)} \mathbf{E}\left(\exp\left\{ - \mathcal{P}_i(\mathbf{S}_{\eta X_i W y^{-1}} f) \right\} \Big| I(\eta), W \right) \Big| W \right]
$$

$$
(3.4) \qquad = \mathbf{E}\left[\left(\mathbf{E}\left(\Psi_{\mathcal{P}_1}(f\|y\eta^{-1}W^{-1}X_1^{-1}) \right) \Big| W \right)^{|I(\eta)|} \Big| W \right].
$$

As $I(\eta)$ is a Poisson random variable with mean $\eta^{-\alpha}$, using the expression for probability generating function for the Poisson random variable we obtain the following expression for the right hand side of (3.4)

$$
(3.5) \qquad \exp\left\{ \eta^{-\alpha} \mathbf{E}\left(\psi_{\mathcal{P}_1}(f\|y\eta^{-1}X_1^{-1}W^{-1}) \Big| W \right) - 1 \right\}.
$$

It can easily be computed using probability density function of of X_1, that

$$(3.6) \qquad \mathbf{E}\left(\Psi_{\mathcal{P}_1}(f\|y\eta^{-1}X_1^{-1}W^{-1})\Big|W\right) = y^{-\alpha}\eta^{\alpha}W^{\alpha}\int_{x>y^{-1}\eta W}\Psi(f\|x^{-1})m_{\alpha}(\mathrm{d}\,x).$$

Combining expressions in (3.6), (3.5) and (3.2) and using monotone convergence theorem, we get

$$\Psi_N(f\|y) = \mathbf{E}\left(\exp\left\{-y^{-\alpha}W^{-\alpha}\int(1-\Psi_{\mathcal{P}}(f\|x^{-1}))m_{\alpha}(\mathrm{d}\,x)\right\}\right)$$

$$= \mathbf{E}\left(\Phi_{\alpha}\left(yW^{-1}c_f\right)\right)$$

where

$$c_f = \int_{x>0}(1-\Psi_{\mathcal{P}}(f\|x^{-1}))m_{\alpha}(\mathrm{d}\,x).$$

4. Properties of the scaled Laplace functional

Suppose that N is point process with scaled Laplace functional which is uniquely supported on g. In this section our aim is to study the properties of the function g. This study is in parallel to Subag and Zeitouni (2015a). We shall show that g is continuous and using continuity of g, we derive that g is monotone. Then, we shall show that g is a distribution function which implies that g is increasing. Then using the fact that N is locally finite, we shall show that $(1-g)$ is regularly varying at ∞.

Let N be a point process satisfying the assumptions stated in Theorem 2.4. The first step will be to show that g is a continuous and increasing function. To be more specific, we shall show that g is a distribution function. Then we shall determine the form of g.

Continuity of g follows from dominated convergence theorem. Note that

$$g(yc_f) = \mathbf{P}(N(\bar{\mathbb{R}}_0) = 0) + \mathbf{P}(N(\bar{\mathbb{R}}_0) > 0)\,\mathbf{E}\left(\exp\{-N(\mathbf{S}_{y^{-1}}f)\}|N(\bar{\mathbb{R}}_0) > 0\right).$$

So in is enough to consider the case $\mathbf{P}(N(\bar{\mathbb{R}}_0) = 0) = 0$. Following the same same arguments in Lemma 12 of Subag and Zeitouni (2015a), it is easy to see that

$$(4.1) \qquad 0 = \inf_{y\in\mathbb{R}_+} g(y) < g(x) < \sup_{y\in\mathbb{R}_+} g(y) = 1$$

and g does not attain its lower bound.

In the next step, we shall show that g is monotone. Let the super-level set of g is denoted by

$$\mathrm{SL}_x = \{y\in\mathbb{R}_+ : g(x) > y\}.$$

To show that g is monotone, it is enough to show that either SL_x has unbounded component or SL_x has component with left end point 0. Suppose that SL_x has a bounded component which is denoted (y_x, y_x') such that $y_x > 0$. Following the argument in Subag and Zeitouni (2015a), we can show that SL_x has uncountable components. This is contradiction to the fact that SL_x can have at most countable components as SL_x is an open set (g is continuous). Note that we can obtain a relation between components of SL_x and components of super-level set of $\Psi_N(af\|y)$ using scaling relation instead of shift for all $a > 0$ and we use the ratio of end points to establish disjointness of the intervals instead of differences.

We shall again use the method of contradiction to show that g does not attain its maximum. Suppose that g attains its maximum and g is increasing. Fix $f\in C_c^+(\bar{\mathbb{R}}_0)$ and define

$$y_0 = \min\{y\in\mathbb{R}_+ : \Psi_N(f\|y) = 1\}.$$

Note that $\Psi_N(f\|y) < \Psi_N(2^{-1}f\|y)$ for all $y\in\mathbb{R}_+$ i.e. $\Psi_n(2^{-1}f\|y) = 1$ for all $y \geq y_0$. Using the fact that f and $2^{-1}f$ has the same support, we get that

$$y_0 = \min\{y\in\mathbb{R}_+ : \Psi_N(2^{-1}f\|y) = 1\}$$

Note that $\Psi_N(f\|y)$ is uniquely supported on g for every $f \in C_c^+(\bar{\mathbb{R}}_0)$. This implies that $c_f = c_{2^{-1}f}$. This means $\Psi_N(f\|y) = \Psi_N(2^{-1}\|y)$ for all $y \in \mathbb{R}_+$. This contradicts the fact that $\mathbf{P}(N(\bar{\mathbb{R}}_0) = 0) = 0$.

The following two properties of g can easily be derived in parallel to Corollary 13 and 14 in Subag and Zeitouni (2015a). For future reference, we are stating the properties as facts.

Fact 4.1. Suppose that the conditions in Theorem 2.4 holds and $\mathbf{P}(N(\bar{\mathbb{R}}_0) = 0) = 0$. Let $\{c_n\}_{n \geq 1}$ be a sequence of positive real numbers, such that $g(c_n \cdot) \to h(\cdot)$ pointwise, then either $c_n \to 0$ or ∞ and h is a constant function with value in $\{0, 1\}$, or $c_n \to c$ and $h(y) = g(cy)$.

Fact 4.2. Let $f \geq 0$ be a measurable function on $\bar{\mathbb{R}}_0$ and there exists a sequence of functions $\{f_n\}_{n \geq 1}$ in $C_c^+(\bar{\mathbb{R}}_0)$ such that f_n converges to f monotonically and pointwise. Under the assumptions of Fact 4.1, if $\Psi_N(f\|y) \in (0, 1)$ for some $y \in \mathbb{R}_+$ then $\Psi_N(f\|\cdot)$ is uniquely supported on the class $[g]_{sc}$.

Fact 4.1 easily follows from the continuity of g and Fact 4.2 follows from a combination of monotone convergence theorem and Fact 4.1.

In this step, we shall show that g is a distribution and hence an increasing function. Define

$$(4.2) \qquad \mathrm{maxmod}(N) = \inf\{y \in \mathbb{R}_+ : N(\{x : |x| > y\}) = 0\}$$

i.e. maxima of the absolute values of the points associated to the point process N. Define $A = \{x \in \bar{\mathbb{R}} : |x| > 1\} = [-\infty, 1) \cup (1, \infty]$ and

$$\infty \mathbb{1}_A(x) = \begin{cases} \infty & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Also note that $\Psi_N(\infty \mathbb{1}_A \| y) = \mathbf{P}(\mathrm{maxmod}(N) \leq y)$. It is clear that $\infty \mathbb{1}_A \notin C_c^+(\bar{\mathbb{R}}_0)$. Consider the sequence of functions $(f_n : n \geq 1)$ defined below

$$f_n(x) = \begin{cases} n & \text{if } |x| \geq 1 + \frac{1}{n}, \\ n^2(x-1) & \text{if } 1 \leq x < 1 + \frac{1}{n}, \\ n^2(x+1) & \text{if } -1 - \frac{1}{n} \leq x \leq -1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $f_n \in C_c^+(\bar{\mathbb{R}}_0)$ for all $n \geq 1$ and $f_n \in (0, 1)$. Note that f_n converges monotonically and pointwise to $\infty \mathbb{1}_A$. So using Fact 4.2, we get that there exists a positive real number c_{max} which satisfies

$$(4.3) \qquad \mathbf{P}\Big(\mathrm{maxmod}(N) \leq y\Big) = \Psi_N(\infty \mathbb{1}_A \| y) = g(c_{max} y)$$

for all $y \in \mathbb{R}_+$ (in parallel to equation (6.1) in Subag and Zeitouni (2015a)). Hence g is a distribution function and as a consequence, we show that g is increasing.

In the final step, we study the tail behavior of the distribution function g and show that $1 - g$ is regularly varying at ∞. Let ν denotes the mean measure of the point process N, that is, $\nu(B) = \mathbf{E}(N(B))$ for every Borel subset of $\mathbb{R}$. We shall first show that $\nu((b, \infty)) < \infty$ for all $b > 0$. It is clear from (2.5) in Theorem 2.4 that $\nu((ay, \infty])$ is finite for $y \geq 1$. Consider a collection of positive real numbers $(c_a(t) : t > 0)$ such that $g(c_a(t)y) = \Psi_N(t\mathbb{1}_{(a,\infty)}\|y)$. Note that $t\mathbb{1}_{(a,\infty)} \notin C_c^+(\bar{\mathbb{R}}_0)$. However, we can construct a sequence of functions $f_n^{(t)} \in C_c^+(\bar{\mathbb{R}}_0)$ such that $f_n^{(t)}$ converges to monotonically and pointwise to the function $t\mathbb{1}_{(a,\infty)}$ for every $t > 0$. Using Fact 4.2, we can show that there exists a constant $c_a(t)$ such that $g(c_a(t)y) = \Psi_N(t\mathbb{1}_{(a,\infty)}\|y)$ holds for every $t > 0$. In parallel to the steps in Proposition 16 in Subag and Zeitouni (2015a), it is easy to see that

$$(4.4) \qquad \nu((ay, \infty)) = \lim_{t \downarrow 0} \frac{1}{t}\Big(1 - g(yc_a(t))\Big) < \infty$$

for all $y \geq 1$. So for all pairs $y_1, y_2 \geq 1$, we have

$$(4.5) \qquad \frac{\nu((ay_1, \infty))}{\nu((ay_2, \infty))} = \lim_{t \downarrow 0} \frac{1 - g\Big(c_a(t)y_1\Big)}{1 - g\Big(c_a(t)y_2\Big)}.$$

It is clear that the right hand side of (4.5) is finite as the ratio in left hand side is finite. Using the fact that g is increasing, we get $c_a(t) \uparrow \infty$ as $t \downarrow 0$. So, the right hand side of (4.5) becomes

$$\lim_{x \to \infty} \frac{1 - g(y_1 y_2^{-1} x)}{1 - g(x)}$$

and depends only on the ratio $y_1 y_2^{-1}$. This fact implies that

$$(4.6) \qquad \lim_{x \to \infty} \frac{1 - g(yx)}{1 - g(x)}$$

exists and finite for every $y \in \mathbb{R}_+$. Now it is important to note that

$$(4.7) \qquad \nu((ay, \infty)) = \lim_{x \to \infty} \frac{1 - g(yx)}{1 - g(x)} \lim_{t \to 0} \frac{1 - g\Big(c_a(t)\Big)}{t}.$$

It is clear that the first term in (4.7) is finite as the ratio (4.6) is finite for every $y \in \mathbb{R}_+$ and the second term is finite from (4.4). Hence we have established the fact that $\nu((b, \infty)) < \infty$ for every $b > 0$. Similarly, one can show that $\nu((-\infty, -b)) < \infty$. Finally, these results imply that $\nu(B) < \infty$ for all Borel subsets of $\mathbb{R}$ which are bounded away from 0.

Let $\theta(y)$ denotes the expression in (4.6). Then it is easy to verify that $\theta(\cdot)$ satisfies famous Hamel equation $\theta(yz) = \theta(y)\theta(z)$. So we can write $\theta(y) = y^{-\alpha}$ for all $y \geq 1$ and some $\alpha \in \mathbb{R}$. Using the fact that g is increasing, we get that $\theta(y) \leq 1$ for $y \geq 1$. So we obtain $\alpha \geq 0$. Proof of the fact that $\alpha \neq 0$ is very similar to that in Proposition 16 in Subag and Zeitouni (2015a). One needs to replace the interval $(0, 1)$ by A and shift by scale to prove it. So, we get that there exists $\alpha > 0$, such that

$$\lim_{x \to \infty} \frac{1 - g(xy)}{1 - g(x)} = y^{-\alpha}$$

for all $y > 0$ (in parallel to equation (6.2) in Subag and Zeitouni (2015a)). As g is the distribution function of $\mathrm{maxmod}(N)$, it is clear that $\mathrm{maxmod}(N)$ has regularly varying tail at ∞.

5. DISTRIBUTION OF THE SCALE-DECORATION

In this section, we shall construct the scale-decoration $SD(N)$ of N from its scaled Laplace functional. The crucial step will be to compute the distribution of the weak limit $\hat{N}$ of $\mathbf{S}_{y^{-1}} N$ conditioned on the event $\{\mathrm{maxmod}(N) > y\}$. Then we shall show that the decoration $SD(N)$ corresponding to the point process N, has the same distribution as $\mathbf{S}_{(\mathrm{maxmod}(\hat{N}))^{-1}} \hat{N}$.

It is not easy to derive the distribution of $\hat{N}$ directly. We shall construct a collection of point processes $\{N^{(y)} : y \geq 1\}$ from N. It will be shown that $\{N^{(y)} : y \geq 1\}$ is a tight family of point processes and we shall obtain the weak limit N_*. Then we shall construct another collection of point processes $\{N_{(y)} : y \geq 1\}$. We shall show that the family of the point processes is tight and weak limit $\widetilde{N}$ has the same distribution as $\hat{N}$. From the weak limit $\widetilde{N}$, we shall derive the distribution of $SD(N)$. This construction of scale-decoration is motivated from Subag and Zeitouni (2015a). The following property of g is in parallel to equation (6.3) in Subag and Zeitouni (2015a)

$$\lim_{m \to 0} c_{mf} = \infty \quad \text{for all } f \in C_c^+(\bar{\mathbb{R}}_0).$$

Let $N^{(y)}$ denotes the point process such that for every Borel subset $B \subset \bar{\mathbb{R}}_0$, we have

$$N^{(y)}(B) = N\Big(yB \cap \{(-\infty, -y) \cup (y, \infty)\}\Big)$$

conditioned on the event that $\{\mathrm{maxmod}(N) > y\}$. Following the same steps of Lemma 22 in Subag and Zeitouni (2015a), it can be shown that $N^{(y)}$ is tight family of point processes. As a

consequence, existence of the limit of the scaled Laplace functionals of $N^{(y)}$ will imply existence of the weak limit N^* of the collection of the point processes $\{N^{(y)} : y \geq 1\}$ as $y \to \infty$. The limit of the scaled Laplace functionals will correspond to the point process N^*. Now, we shall derive the scaled Laplace functional of N_* for a function $f \in C_c^+(\bar{\mathbb{R}}_0)$ with support contained in A. Exactly the same steps in display (6.4) of Subag and Zeitouni (2015a), lead to

$$\Psi_{N^*}(f\|x) = \lim_{y \to \infty} \Psi_{N^{(y)}}(f\|x)$$

$$= 1 - \lim_{y \to \infty} \frac{1 - g(yxc_f c_{max}^{-1})}{1 - g(y)}$$

(5.1)
$$= 1 - x^{-\alpha}(c_f c_{max}^{-1})^{-\alpha}.$$

If we consider $f = \infty \mathbb{1}_A$ in the left hand side of (5.1), then in the right hand side, we get $c_f = c_{max}$. This implies that $\max \mathrm{mod}\, N^*$ is a $Pareto(\alpha)$ random variable.

For every $y > 0$, we define a point process $N_{(y)}$ which has same distribution as that of $\mathbf{S}_{y^{-1}} N^*$ conditioned on the event $\{\mathrm{maxmod}(N^*) > y\}$. For $y \geq 1$ and any $f \in C_c^+(\bar{\mathbb{R}}_0)$ with support contained in A, we have

(5.2)
$$\Psi_{N_{(y)}}(f) = 1 - (c_f c_{max}^{-1})^{-\alpha}.$$

which is independent of y. Define $N_{(y)}|_A(B) = N_{(y)}(A \cap B)$. From (5.2), it is clear that distribution of $N_{(y)}|_A$ does not depend on y. Following the steps of Lemma 24 in Subag and Zeitouni (2015a), we obtain

(5.3)
$$N_{(y)} \overset{d}{=} N_{(ym)}|_{y^{-1}A}$$

for all $y \geq 1$ and $m \geq 1$. Our next step is to show that weak limit of $N_{(y)}$ exists as $y \to \infty$. In order to show existence, we shall show that the family of point processes $\{N_{(y)} : y \geq 1\}$ is tight and finite dimensional distribution converges. Consider a sequence of real numbers $\{y_n\}$ such that $y_n \to \infty$ as $n \to \infty$. Our aim is to show that for any $f \in C_c^+(\bar{\mathbb{R}}_0)$,

$$\lim_{t \to \infty} \limsup_{n \to \infty} \mathbf{P}\left(N_{y_n}(f) > t\right) = 0.$$

Fix $f \in C_c^+(\bar{\mathbb{R}}_0)$ and choose a large enough $x > 1$, such that the support of f is contained in $x^{-1}A$ (choice of x depends on f). As $y_n \to \infty$, we can find a large enough n_0, such that $y_n > x$ for all $n \geq n_0$. For all $n \geq n_0$, we get $y_n = xz_n$ such that $z_n > 1$ and hence we have

(5.4)
$$N_{(xz_n)}|_{x^{-1}A} \overset{d}{=} N_{(x)}.$$

So (5.4) follows immediately. The same argument implies convergence of finite dimensional distributions.

Recall that our aim is to find the distribution of $\widehat{N}$ where $\widehat{N}$ is the weak limit of $\mathbf{S}_{y^{-1}} N$ conditioned on the event $\{\mathrm{maxmod}(N) > y\}$. We shall show that weak limit $\widetilde{N}$ of $N_{(y)}$ has the same distribution as that of $\hat{N}$. Consider a collection of sets $\{A_i\}_{i=1}^l$ which are bounded away from 0. Then, we get the following expression for distribution of $\widetilde{N}$

$$\mathbf{P}\left(\widetilde{N}(A_i) \geq k_i, 1 \leq i \leq l\right)$$

$$= \lim_{y \to \infty} \frac{\mathbf{P}\left(N^*(yA_i) \geq k_i, 1 \leq i \leq l, N^*(yA) > 0\right)}{\mathbf{P}(N^*(yA) > 0)}$$

(5.5)
$$= \lim_{y \to \infty} \lim_{t \to \infty} \frac{\mathbf{P}(N(tA) > 0)}{\mathbf{P}(N(tyA) > 0)} \frac{\mathbf{P}\left(N(tyA_i) \geq k_i, i \leq i \leq l, N(tyA) > 0\right)}{\mathbf{P}(N(tA) > 0)}.$$

Note that after cancellation, each of the terms in the right hand side of (5.5) does not involve t and y separately, but involves the product ty. So the separate limits $\lim_{t \to \infty} \lim_{y \to \infty}$ can be replaced by $\lim_{ty \to \infty}$ and we get expression for the distribution of $\widehat{N}$.

Here we shall study some properties of $\widetilde{N}$. As $N_{(y)}$ converges weakly to $\widetilde{N}$, then $N_{(ym)}|_{y^{-1}A}$ converges weakly to $\widetilde{N}|_{y^{-1}A}$ as $m \to \infty$. Now use (5.3) to observe that $\widetilde{N}|_{y^{-1}A} \stackrel{d}{=} N_{(y)}$. In parallel to Corollary 26 in Subag and Zeitouni (2015a), it is easy to see that for any $f \in C_c^+(\overline{\mathbb{R}}_0)$,

$$(5.6) \qquad \mathbf{E}\left[\exp\left\{ -\int f \, \mathrm{d}\, \mathbf{S}_{y^{-1}} \, \widetilde{N} \right\} \middle| \mathrm{maxmod}(\widetilde{N}) > y \right]$$

is independent of $y \geq 1$. It is important to observe that $\mathrm{maxmod}(\widetilde{N}) > 1$. Using the fact that $\widetilde{N}|_A \stackrel{d}{=} N^*$, we get that $\mathrm{maxmod}(\widetilde{N}) \stackrel{d}{=} \mathrm{maxmod}(N^*)$ and so $\mathrm{maxmod}(\widetilde{N})$ follows $Pareto(\alpha)$ distribution.

Define the scale-decoration point process,

$$SD(N) = \mathbf{S}_{\mathrm{maxmod}(\widetilde{N})^{-1}} \, \widetilde{N}.$$

Our first step will be to observe that $\mathrm{maxmod}(\widetilde{N})$ and $SD(N)$ are independent. In parallel to the steps of Lemma 27 in Subag and Zeitouni (2015a), we can write down the conditional Laplace functional of $SD(N)$ given $\{\mathrm{maxmod}(\widetilde{N}) > y\}$ as

$$\mathbf{E}\left[\exp\left\{ -\int f \, \mathrm{d}\, \mathbf{S}_{(\mathrm{maxmod}(\mathbf{S}_{y^{-1}} \, \widetilde{N}))^{-1}} \, \mathbf{S}_{y^{-1}} \, \widetilde{N} \right\} \middle| \mathrm{maxmod}(N) > y \right]$$

which does not involve y using (5.6). Hence $SD(N)$ and $\mathrm{maxmod}(\widetilde{N})$ are independent. So we can write $\widetilde{N} = \mathbf{S}_X \, SD(N)$ where X is a $Pareto(\alpha)$ random variable independent of $SD(N)$.

Let $L(N) \sim ScDPPP(m_\alpha(\mathrm{d}\,x), \mathbf{S}_{c_{max}^{-1}} \, SD(N))$. Then we observe that $L(N)$ has the scaled Laplace functional

$$(5.7) \qquad \Psi_{L(N)}(f\|y) = \exp\{-y^{-\alpha} c_f^{-\alpha}\}$$

which is uniquely supported on Φ_α. The derivation is similar to the computation done in Section 3.

6. Rest of The Proofs

6.1. **Proof of Theorem 2.4.** Suppose that (Prop2) holds for some positive random variable W and positive scalar α. Consider the point process $L'(N) \sim SScDPPP(m_\alpha(\mathrm{d}\,x), \mathbf{S}_{c_{max}^{-1}} \, SD(N), W)$. then in the light of scaled Laplace functional computed in , it follows that the scaled Laplace functional of $L'(N)$ is same as that given in (Prop2).

Under (Prop1), we have shown that g is a distribution function. Now we consider a random variable W_g such that W_g follows the distribution g and independent of $L(N) \sim ScDPPP(m_\alpha(\mathrm{d}\,x), \mathbf{S}_{c_{max}^{-1}} \, SD(N))$. Consider also a Fréchet-α random variable W_F such that W_F is independent of N. It is easy to see that

$$\Psi_{\mathbf{S}_{W_g} \, L(N)}(f) = \Psi_{\mathbf{S}_{W_F} \, N}(f)$$

using multiplicative convolution. Here it is clear that $\mathbf{S}_{W_F} \, N \stackrel{d}{=} \mathbf{S}_{W_g} \, L(N)$. Now we shall use the transfer principle to establish that there exists some random variable $\widehat{W}_g$ and a point process $\widehat{L(N)}$ which are independent of each other, such that

$$\mathbf{S}_{W_F} \, N = \mathbf{S}_{\widehat{W}_g} \, \widehat{L(N)}$$

almost surely. Hence we get that $N = \mathbf{S}_{W_F^{-1} \widehat{W}_g} \, \widehat{L(N)}$ establishing (Prop3). This completes the proof.

6.2. **Proof of Corollary 2.5.** It is easy to verify that (B1) implies (B3) and (B2) by computing scaled Laplace functional. Suppose that (B3) holds, then we can construct $SD(N)$ and obtain c_{max} so that $L(N) \sim ScDPPP(m_\alpha(\mathrm{d}\,x), \mathbf{S}_{c_{max}^{-1}} \, SD(N))$ has the same scaled Laplace functional as N. So (B1) follows. To show the equivalence, we only have to show that B3 implies (B2).

Using the fact that N is a Radon measure and $\mathbf{P}(N(\bar{\mathbb{R}}_0) > 0)$, it is clear that $\Psi_N(f\|y) \in (0,1)$ for every $f \in C_c^+(\bar{\mathbb{R}}_0)$ and $y \in \mathbb{R}_+$. Define $\phi(a) = \log \Psi_N(f\|a^{-1})$. Then using the standard way of approximating real numbers by rational ones and then continuity of $\phi(\cdot)$, it is easy to see that

$$\phi(x) = x^{-\alpha}\phi(1).$$

Hence it is clear that $\Psi_N(f\|y)$ is uniquely supported on $[\Phi_\alpha]_{sc}$.

6.3. **Proof of Theorem 2.9.** We shall prove this theorem as a corollary to Theorem 2.4. Let $\mathscr{M}_{-\infty}$ denotes the space of all counting measures on $\bar{\mathbb{R}}_{-\infty} = \bar{\mathbb{R}} \setminus \{\infty\}$. We denote the space of all counting measures on $\mathbb{R}_+$ by $\mathscr{M}_+$. Let $\mathbf{Exp} : \mathscr{M}_{-\infty} \to \mathscr{M}_+$ be a bijection such that $\mathbf{Exp}(\sum \delta_{a_i}) = \sum \delta_{\exp(a_i)}$ where $a_i \in \bar{\mathbb{R}}_{-\infty}$ for all $i = 1, 2, \ldots$ and similarly $\mathbf{Log} : \mathscr{M}_+ \to \mathscr{M}_{-\infty}$ such that $\mathbf{Log}(\sum \delta_{a_i}) = \sum \delta_{\log a_i}$ where $a_i \in \mathbb{R}_+$ for all $i = 1, 2, \ldots$. It is very easy to see that $\mathbf{Exp}^{-1} = \mathbf{Log}$. We also define an operator $\mathscr{T} : C_c^+(\bar{\mathbb{R}}_{-\infty}) \to C_c^+(\mathbb{R}_+)$ such that $\mathscr{T}(f)(x) = f(\log x)$ for all $x \in \mathbb{R}_+$ and for every $f \in C_c^+(\bar{\mathbb{R}}_{-\infty})$. It is clear that $\mathscr{T}^{-1} : C_c^+(\mathbb{R}_+) \to C_c^+(\bar{\mathbb{R}}_{-\infty})$ such that $\mathscr{T}^{-1}(u)(x) = u(e^x)$ for all $x \in \bar{\mathbb{R}}_{-\infty}$ and for every $u \in C_c^+(\mathbb{R}_+)$. Now we would like to state some easy consequences of change of variable formula. Suppose that $T \in \mathscr{M}_{-\infty}$ and $u \in C_c^+(\mathbb{R}_+)$ and $t > 0$, then

$$(6.1) \qquad \int \mathbf{S}_t\, u\, \mathrm{d}\, \mathbf{Exp}(T) = \int \theta_{\log t} \mathscr{T}^{-1}(u)\, \mathrm{d}\, T$$

similarly, we get

$$(6.2) \qquad \int \mathbf{S}_t\, u\, \mathrm{d}\, N = \int \theta_{\log t} \mathscr{T}^{-1}(u)\, \mathrm{d}\, \mathbf{Log}\, N$$

where $N \in \mathscr{M}_+$ and $u \in C_c^+(\mathbb{R}_+)$. Suppose that $T \in \mathscr{M}_{-\infty}$ and shift-Laplace functional of T is shift-uniquely supported. Then our first step is to show that scaled Laplace functional of $\mathbf{Exp}\, T$ is scale-uniquely supported. Consider $h \in C_c^+(\mathbb{R}_+)$ and $t > 0$ to see that

$$\Psi_{\mathbf{Exp}(T)}(\mathbf{S}_{t^{-1}}\, u) = \mathbf{E}\left(\exp\left\{ -\int \mathbf{S}_{t^{-1}}\, u\, \mathrm{d}\, \mathbf{Exp}(T) \right\} \right)$$
$$= h\left(\log t - \tau_{\mathscr{T}^{-1}(u)} \right)$$
$$= g\left(t e^{-\tau_{\mathscr{T}^{-1}(u)}} \right)$$

where $g(x) = h(\log x)$. Hence we are done with the first step.

Our next step will be to show that if $N \sim SScDPPP(m_\alpha, \mathcal{P}, W)$ then $\mathbf{Log}(N) \sim SDPPP(e^{-\alpha x}\, \mathrm{d}\, x, \mathbf{Log}(\mathcal{P}), \log W)$. Suppose that our claim is true. Then using the first step and Theorem 2.4, we obtain a point process N admitting SScDPPP representation. Now we use the claim to observe that $\mathbf{Log}\, N$ is an SSDPPP and shift-uniquely supported.

So we are only remained with proof of the claim. We shall prove the claim by computing Laplace functional of $\mathbf{Log}\, N$. Fix $f \in C_c^+(\bar{\mathbb{R}}_{-\infty})$. We get

$$\Psi_{\mathbf{Log}(N)}(f) = \mathbf{E}\left(\exp\left\{ -\int f\, \mathrm{d}\, \mathbf{Log}(N) \right\} \right)$$
$$= \mathbf{E}\left(\exp\left\{ -\int \mathscr{T}(f)\, \mathrm{d}\, N \right\} \right)$$
$$= \mathbf{E}\left(\exp\left\{ -W^\alpha c_{\mathscr{T}(f)}^{-\alpha} \right\} \right)$$
$$= \mathbf{E}\left(\exp\left\{ -e^{\alpha(\log W - \log c_{\mathscr{T}(f)})} \right\} \right)$$

Now to get the complete description of the decoration we need to compute

$$c_{\mathscr{T}(f)} = \int_0^\infty \left(1 - \Psi_{\mathcal{P}}(\mathbf{S}_x\, \mathscr{T}(f)) \right) \alpha x^{-\alpha-1}\, \mathrm{d}\, x$$
$$= \int_{-\infty}^\infty e^{-\alpha x}\left(1 - L_{\mathbf{Log}(\mathcal{P})}(f|-x) \right) \mathrm{d}\, x$$

Finally we get,

$$\Psi_{\mathbf{Log}(N)}(f) = \mathbf{E} \left(\exp \left\{ -e^{\alpha \left[\log W - \log \left(\int_{-\infty}^{\infty} e^{-\alpha x} \left(1 - L_{\mathbf{Log}(\mathcal{P})}(f|-x) \right) \mathrm{d}\, x \right) \right]} \right\} \right)$$

which implies that $\mathbf{Log}\, N \sim SSDPPP(e^{-\alpha x}\, \mathrm{d}\, x, \mathbf{Log}(\mathcal{P}), \log W)$.

Acknowledgement

References

A. Bhattacharya, K. Maulik, Z. Palmowski, and P. Roy. Extremes of multi-type branching random walks: Heaviest tail wins. *arXiv preprint arXiv:1612.00692*, 2016.

A. Bhattacharya, R. S. Hazra, and P. Roy. Point process convergence for branching random walks with regularly varying steps. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 53(2):802–818, 2017a.

A. Bhattacharya, R. S. Hazra, and P. Roy. Branching random walks, stable point processes and regular variation. *Stochastic Processes and their Applications,* to appear. arXiv:1601.01656, 2017b.

M. Biskup and O. Louidor. Full extremal process, cluster law and freezing for two-dimensional discrete gaussian free field. *arXiv preprint arXiv:1606.00510*, 2016.

É. Brunet and B. Derrida. A branching random walk seen from the tip. *J. Stat. Phys.*, 143(3):420–446, 2011. ISSN 0022-4715. doi: 10.1007/s10955-011-0185-z. URL http://dx.doi.org/10.1007/s10955-011-0185-z.

Y. Davydov, I. Molchanov, and S. Zuyev. Strictly stable distributions on convex cones. *Electron. J. Probab.*, 13:no. 11, 259–321, 2008. ISSN 1083-6489. doi: 10.1214/EJP.v13-487. URL http://dx.doi.org/10.1214/EJP.v13-487.

R. Durrett. Maxima of branching random walks vs. independent random walks. *Stochastic Process. Appl.*, 9(2):117–135, 1979. ISSN 0304-4149. doi: 10.1016/0304-4149(79)90024-3. URL http://dx.doi.org/10.1016/0304-4149(79)90024-3.

R. Durrett. Maxima of branching random walks. *Z. Wahrsch. Verw. Gebiete*, 62(2):165–170, 1983. ISSN 0044-3719. doi: 10.1007/BF00538794. URL http://dx.doi.org/10.1007/BF00538794.

T. Madaule. Convergence in law for the branching random walk seen from its tip. *Journal of Theoretical Probability*, pages 1–37, 2015. ISSN 0894-9840. doi: 10.1007/s10959-015-0636-6. URL http://dx.doi.org/10.1007/s10959-015-0636-6.

P. Maillard. A note on stable point processes occurring in branching brownian motion. *Electronic Communications in Probability*, 18(5):1–9, 2013.

B. Mallein. Genealogy of the extremal process of the branching random walk. *arXiv preprint arXiv:1606.01748*, 2016.

S. I. Resnick. *Extreme values, regular variation, and point processes*, volume 4 of *Applied Probability. A Series of the Applied Probability Trust*. Springer-Verlag, New York, 1987. ISBN 0-387-96481-9. doi: 10.1007/978-0-387-75953-1. URL http://dx.doi.org/10.1007/978-0-387-75953-1.

S. I. Resnick. *Heavy-tail phenomena*. Springer Series in Operations Research and Financial Engineering. Springer, New York, 2007. ISBN 978-0-387-24272-9; 0-387-24272-4. Probabilistic and statistical modeling.

E. Subag and O. Zeitouni. Freezing and decorated Poisson point processes. *Comm. Math. Phys.*, 337(1):55–92, 2015a. ISSN 0010-3616. doi: 10.1007/s00220-015-2303-2. URL http://dx.doi.org/10.1007/s00220-015-2303-2.

E. Subag and O. Zeitouni. The extremal process of critical points of the pure p-spin spherical spin glass model. *Probability Theory and Related Fields*, pages 1–48, 2015b.

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