The optimality of (s,S) inventory policies in the infinite period model *

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Summary The infinite period stationary inventory model is considered. There is a constant lead time, a nonnegative set-up cost, a linear purchase cost, a holding and shortage cost function, a fixed discount factor β, $0 < \beta \leqslant 1$, and total backlogging of unfilled demand. Both the total discounted cost ($\beta < 1$) and the average cost ($\beta = 1$) criteria are considered. Under the assumption that the negatives of the one period holding and shortage costs are unimodal, a unified proof of the existence of an optimal (s,S) policy is given. As a by-product of the proof upper and lower bounds on the optimal values of s and S are found. New results simplify the algorithm of VEINOTT and WAGNER for finding an optimal (s,S) policy for the case $\beta < 1$. Further it is shown that the conditions imposed on the one period holding and shortage costs can be weakened slightly.

1. Introduction

We consider the infinite period stationary inventory model in which demands for a single item in periods 1, 2, ... are independent, identically distributed random variables. At the beginning of each period an order may be placed for any positive quantity of stock. There is a constant lead time, a fixed set up cost, a linear purchase cost, a holding an shortage cost function, a fixed discount factor β, $0 < \beta \leqq 1$, and total backlogging of unfilled demand.

In the finite period nonstationary model the existence of an (s,S) policy minimizing the total expected cost is shown under different conditions by SCARF [9, 10] and VEINOTT [12]. Under SCARF's assumption that the one period expected holding and shortage costs are convex IGLEHART has examined the infinite period stationary model [4, 5]. In [4] it is proved that an (s,S) policy exists which minimizes the total expected discounted cost and in [5] the existence of an (s,S) policy minimizing the average expected cost per period is shown (see also [11], pp. 530–531). VEINOTT has replaced SCARF's assumption that the one period holding and shortage costs are convex by the weaker assumption that negatives of these costs are unimodal. Under a slight weakening of VEINOTT's assumption JOHNSON [6] has proved that an optimal (s,S) policy exists under the total discounted cost criterion in the infinite period stationary model. Further a proof of the existence of an optimal (s,S) policy under the average cost criterion is indicated. However the approach of JOHNSON, based on HOWARD's policy improvement method [3], is typically for the discrete demand case.

In this paper the infinite period model is considered for both the total expected discounted cost and the average cost criteria. Under the assumption that the negatives of the one period expected holding and shortage costs are unimodal ** a unified proof

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** This assumption will be weakened slightly in remark 5.2.

of the existence of an optimal (s, S) policy is given. For the case $\beta = 1$ the proof generalizes a proof of IGLEHART [5] and for the case $\beta < 1$ the proof is new. As a by-product of the proof upper and lower bounds on the optimal s and S are found, which are similar to those in [6, 11, 12]. New results of this paper simplify the algorithm of VEINOTT and WAGNER [11] for finding an optimal (s, S) policy for the case $\beta < 1$ (see theorem 5.4). In this paper we treat the discrete demand case. However, by making obvious modifications the proofs and the results carry over to the continuous demand case.

2. Model formulation

We consider the infinite period stationary inventory model in which the demands $\underline{\xi}_1, \underline{\xi}_2, \ldots$ for a single item in periods 1, 2, ... are independent, nonnegative, discrete random variables* with the common probability distribution $p_j = P\{\underline{\xi}_t = j\}$, $(j \geqq 0; t \geqq 1)$. Assume $\mu = \mathscr{E}\underline{\xi}_t$ is finite and $p_0 < 1$. At the beginning of each period the stock on hand plus on order is reviewed. An order may then be placed for any nonnegative, integral quantity of stock. An order placed in period t is delivered at the beginning of period $t+\lambda$, where λ is a known nonnegative integer. The demand is assumed to take place at the end of each period. All unfilled demand is backlogged and there is no obsolescence of stock.

There is specified a fixed discount factor β, $0 < \beta \leqq 1$, so that a unit cost incurred n periods in future has a present value β^n.

The following costs are considered. In any period the cost of ordering z units is $K\delta(z)+cz$, where $K \geqq 0$, $\delta(0) = 0$, and $\delta(z) = 1$ for $z > 0$. Assume that the ordering cost is incurred on the time of delivery of the order. We can always take care that this assumption is satisfied by an appropriate discounting of the ordering cost. Let $g(i)$ be the holding and shortage cost in a period when i is the amount of stock on hand at the beginning of that period just after any additions to stock.

Let $\underline{T}_0 = 0$ and $\underline{T}_n = \underline{\xi}_1 + \ldots + \underline{\xi}_n$, $n \geqq 1$. Define $p_j^{(n)} = P\{\underline{T}_n = j\}$, $(j \geqq 0; n \geqq 0)$. Assume for any integer k that

$$L(k) = \sum_{j=0}^{\infty} g(k-j) p_j^{(\lambda)} \tag{2.1}$$

exists and is finite. If at the beginning of the present period t the stock on hand plus on order, just after ordering in that period, is k, then at the beginning of period $t+\lambda$, just after delivery for period $t+\lambda$, the stock on hand is $k - \underline{T}_\lambda$. Hence $L(k)$ is the expected holding and shortage cost in period $t+\lambda$ when k is the stock on hand plus on order just after ordering in period t. Define for any integer k

$$G_\beta(k) = L(k) + (1-\beta)ck \tag{2.2}$$

* Random variables are underlined.

The function $G_\beta(k)$ will appear to be important and it is referred to as the one period expected holding and shortage cost. Observe that $(1-\beta)cz$, $z > 0$, can be interpreted as the saving in ordering cost when the placing of an order for z units is delayed one period.

The following conditions are imposed on $G_\beta(k)$:

i. There exists a finite integer S_0 such that $G_\beta(i) \leqq G_\beta(j)$ for $j \leqq i \leqq S_0$ and $G_\beta(i) \geqq G_\beta(j)$ for $i \geqq j \geqq S_0$.

ii. $\lim_{|k|\to\infty} G_\beta(k) > G_\beta(S_0)+K$.

Because of (ii) we may assume that S_0 is the smallest integer for which (i) holds. Let s_1 be the smallest integer for which

$$G_\beta(s_1) \leqq G_\beta(S_0)+(1-\beta p_0)K \tag{2.3}$$

and let S_1 be the largest integer for which

$$G_\beta(S_1) \leqq G_\beta(S_0)+\beta K \tag{2.4}$$

Let S^0 be the largest integer at which $G_\beta(k)$ attains its absolute minimum. Observe that $s_1 \leqq S_0 \leqq S^0 \leqq S_1$ and that $G_\beta(k) = G_\beta(S_0)$ for $S_0 \leqq k \leqq S^0$.

Let us define the state of the system in a period as the stock on hand plus on order just before ordering in that period. We take the set I of all integers as the set of all possible states. Let us say that in state i decision k, $k \geqq i$, is made when $k-i$ units are ordered. We impose the following mild restrictions on the choice of an ordering decision. There are given finite integers $m \leqq s_1$ and $M \geqq S_1$, such that nothing is ordered if the stock on hand plus on order $i \geqq M$, at most $M-i$ units are ordered if $i < M$, and at least $m-i$ units are ordered if $i < m$. Let $A(i)$ denote the set of feasible decisions in state i. Then $A(i) = \{i\}$ for $i \geqq M$ and $A(i) = \{k \mid \max(i,m) \leqq \leqq k \leqq M\}$ for $i < M$.

A policy R for controlling the inventory system is a set of functions $D_k(h_{t-1}, i_t)$, $k \in A(i_t)$; $t \geqq 1$, satisfying

$$D_k(h_{t-1}, i_t) \geqq 0, k \in A(i_t) \quad \text{and} \quad \sum_{k\in A(i_t)} D_k(h_{t-1}, i_t) = 1$$

for every "history" $h_{t-1} = (i_1, k_1, \ldots, i_{t-1}, k_{t-1})$ and all $i_t \in I$, $t = 1, 2, \ldots$, where i_n and k_n is the observed state and the observed decision in period n.

The interpretation being: if at the beginning of period t the history h_{t-1} has been observed and the system is in state i_t, then $k-i_t$ units are ordered with probability $D_k(h_{t-1}, i_t)$.

Let $C(m, M)$ denote the class of all possible policies. A policy R is said to be stationary deterministic if $D_k(h_{t-1}, i_t = i) = D_k(i)$, independent of h_{t-1} and t, and if $D_k(i) = 1$, or 0.

Given a policy $R \in C(m, M)$ and an initial state $i \in I$, define $\underline{i}_t$ and $\underline{k}_t$ as the state and the decision in period t ($t \geqq 1$).

For the case $\beta < 1$ we take as optimality criterion

$$V_\beta(i;R) = \sum_{t=1}^{\infty} \beta^{t-1} \mathscr{E}_R\{K\delta(\underline{k}_t - \underline{i}_t) + (\underline{k}_t - \underline{i}_t)c + L(\underline{k}_t)|\underline{i}_1 = i\},$$

where $\mathscr{E}_R$ denotes the expectation under policy R. We note that the expectations exist and that $V_\beta(i;R)$ is finite (this is proved by $m \leqq \underline{k}_t \leqq \max(i, M)$, given $\underline{i}_1 = i$, and $\underline{i}_{t+1} = \underline{k}_t - \underline{\xi}_t$ for $t \geqq 1$). The quantity $V_\beta(i;R)$ represents the total expected discounted cost over the periods $\lambda+1, \lambda+2, \ldots$, all discounted to the beginning of period $\lambda+1$, when i is the state in period 1 and the policy R is followed. Observe that the cost over the first λ periods cannot be influenced by any policy.

For the case $\beta = 1$ we take as optimality criterion

$$g(i;R) = \liminf_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \mathscr{E}_R\{K\delta(\underline{k}_t - \underline{i}_t) + (\underline{k}_t - \underline{i}_t)c + L(\underline{k}_t)|\underline{i}_1 = i\}.$$

We note that $g(i;R)$ is finite. When the limit exists $g(i;R)$ represents the average expected cost per period when the initial state is i and policy R is followed.

Using the relation $\underline{i}_{t+1} = \underline{k}_t - \underline{\xi}_t$, $t \geqq 1$, and the boundness of the sequence $\{\mathscr{E}_R(\underline{i}_{n+1}|\underline{i}_1 = i)\}$, $n \geqq 0$, it is easy to verify that (see also [11])

$$V_\beta(i;R) = \sum_{t=1}^{\infty} \beta^{t-1} \mathscr{E}_R\{K\delta(\underline{k}_t - \underline{i}_t) + G_\beta(\underline{k}_t)|\underline{i}_1 = i\} - ci + \beta\mu c/(1-\beta), \quad i \in I,$$

and

$$g(i;R) = \liminf_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \mathscr{E}_R\{K\delta(\underline{k}_t - \underline{i}_t) + G_1(\underline{k}_t)|\underline{i}_1 = i\} + \mu c, \qquad i \in I.$$

Since the terms $-ci + \beta\mu c/(1-\beta)$ and μc are not affected by the choice of R, it is convenient to redefine $V_\beta(i;R)$ by setting

$$V_\beta(i;R) = \sum_{t=1}^{\infty} \beta^{t-1} \mathscr{E}_R\{K\delta(\underline{k}_t - \underline{i}_t) + G_\beta(\underline{k}_t)|\underline{i}_1 = i\}, \quad i \in I, \tag{2.5}$$

and to redefine $g(i;R)$ by setting

$$g(i;R) = \liminf_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \mathscr{E}_R\{K\delta(\underline{k}_t - \underline{i}_t) + G_1(\underline{k}_t)|\underline{i}_1 = i\}, \quad i \in I \tag{2.6}$$

When $\beta < 1$ a policy $R^* \in C(m, M)$ is called optimal if

$$V_\beta(i;R^*) \leqq V_\beta(i;R) \qquad \text{for all } i \in I, \text{ all } R \in C(m,M) \tag{2.7}$$

When $\beta = 1$ a policy $R^* \in C(m, M)$ is called optimal if

$$g(i;R^*) \leqq g(i;R) \qquad \text{for all } i \in I, \text{ all } R \in C(m,M) \tag{2.8}$$

We shall need the following two basic theorems:

Theorem 2.1 (BLACKWELL)

Let $\beta < 1$ and $R^* \in C(m, M)$. If

$$V_\beta(i; R^*) = \min_{k \in A(i)} \{K\delta(k-i) + G_\beta(k) + \beta \sum_{j=0}^{\infty} V_\beta(k-j; R^*) p_j\}, \quad i \in I, \tag{2.9}$$

then the policy R^* is optimal.

Proof

Fix some integer $i_0 \in I$. Let $M_0 = \max(i_0, M)$. Since the equation (2.9) holds for all $i \leqq M_0$ and $K\delta(k-i) + G_\beta(k)$, $k \in A(i)$ and $i \leqq M_0$, is bounded, a direct application of theorem 6(f) in [1] shows that $V_\beta(i; R^*) \leqq V_\beta(i; R)$ for all $i \leqq M_0$ and all $R \in C(m, M)$. Hence in particular $V_\beta(i_0; R^*) \leqq V_\beta(i_0; R)$ for all $R \in C(m, M)$. This proofs the theorem.

Theorem 2.2 (ROSS)

Let $\beta = 1$ and suppose there exists a set of numbers $\{g, v(i)\}$, $i \in I$, such that

$$v(i) = \min_{k \in A(i)} \{K\delta(k-i) + G_1(k) - g + \sum_{j=0}^{\infty} v(k-j) p_j\}, \quad i \in I, \tag{2.10}$$

and $\mathscr{E}_R(v(\underline{i}_n) | \underline{i}_1 = i)/n \to 0$ as $n \to \infty$ for all $i \in I$ and all $R \in C(m, M)$.

Let R^* be any policy which, for each i, prescribes a decision which minimizes the right side of (2.10), then R^* is optimal. Further $g(i; R^*) = g$ for all $i \in I$ and the limit in (2.6) exists for R^*.

Proof

This theorem is a direct consequence of the proof of theorem 1 in [8] (see also [2]).

3. Some results from renewal theory

We have defined $p_j^{(n)} = P\{\underline{T}_n = j\}$, where $\underline{T}_0 = 0$ and $\underline{T}_n = \underline{\xi}_1 + \ldots + \underline{\xi}_n$, $n \geqq 1$. We note that $p_j^{(1)} = p_j$. It is assumed that $p_0 < 1$. The formula

$$p_j^{(n)} = \sum_{k=0}^{j} p_{j-k} p_k^{(n-1)}, \qquad j \geqq 0;\ n \geqq 1, \tag{3.1}$$

is well-known. Define

$$m_\beta(j) = \sum_{n=1}^{\infty} \beta^n p_j^{(n)} \quad \text{and} \quad M_\beta(j) = \sum_{k=0}^{j} m_\beta(k), \qquad j \geqq 0, \tag{3.2}$$

where β fixed and $0 < \beta \leqq 1$. From (3.1) and (3.2) it follows that the numbers $m_\beta(j)$ can be computed from

$$m_\beta(j) = \beta p_j + \beta \sum_{k=0}^{j} p_{j-k} m_\beta(k), \qquad j \geqq 0 \tag{3.3}$$

When $\beta < 1$, we have clearly $M_\beta(j) \leqq \beta/(1-\beta)$, $j \geqq 0$, and consequently $m_\beta(j) \to 0$ as $j \to \infty$. It is known [7] that the renewal quantity $m_1(j)$, $j \geqq 0$, is bounded and that

$$\lim_{j \to \infty} M_1(j)/j = 1/\mu \tag{3.4}$$

For any integer $k \geqq 0$, define $\underline{N}(k) = \max\{n | \underline{T}_n \leqq k\}$. It is known that $\mathscr{E}\underline{N}(k) = M_1(k)$ [7]. Hence $M_1(k)$ is the expected number of periods before the cumulative demand exceeds k. The excess random variable $\underline{\gamma}(k)$ is defined by $\underline{\gamma}(k) = \underline{T}_{\underline{N}(k)+1} - k$. Using a standard probabilistic argument it follows [7]

$$P\{\underline{\gamma}(k) \leqq j\} = F(k+j) - F(k) + \sum_{h=0}^{k} \{F(k+j-h) - F(k-h)\} m_1(h), \quad j \geqq 1, \tag{3.5}$$

where $F(n) = p_0 + \ldots + p_n$, $n \geqq 0$.

4. The (s,S) policy

An (s, S) policy, $s, S \in I$ and $s \leqq S$, has the following simple form: when the stock on hand plus on order $i < s$, order $S-i$ units; for $i \geqq s$, order nothing.

It is known [5, 11] (see [2] for a complete proof), that

$$a_1(s, S) \stackrel{\text{def}}{=} \{G_1(S) + \sum_{j=0}^{S-s} G_1(S-j) m_1(j) + K\} / \{1 + M_1(S-s)\} \tag{4.1}$$

represents for each initial state the average expected cost per period when the (s, S) policy is followed.

For the case $\beta < 1$ it is shown in [11] that for the (s, S) policy,

$$V_\beta(i; (s, S)) = \begin{cases} a_\beta(s, S)/(1-\beta), & i < s, \\ G_\beta(i) + \sum_{j=0}^{i-s} G_\beta(i-j) m_\beta(j) + \\ \quad + \{a_\beta(s, S)/(1-\beta)\}\{\beta - (1-\beta) M_\beta(i-s)\}, & i \geqq s, \end{cases} \tag{4.2}$$

where

$$a_\beta(s, S) = \{G_\beta(S) + \sum_{j=0}^{S-s} G_\beta(S-j) m_\beta(j) + K\} / \{1 + M_\beta(S-s)\} \tag{4.3}$$

Lemma 4.1

Let $0 < \beta \leqq 1$. There exist finite integers s^* and S^* such that $a_\beta(s^*, S^*) \leqq a_\beta(s, S)$ for all $s, S \in I$, $s \leqq S$.

Proof

We shall show that there exist finite integers A and B, such that $a_\beta(s, S) > G_\beta(S_0)+K$ for $S > A$ and regardless of s, and $a_\beta(s, S) > G_\beta(S_0)+K$ for $s < B$ and regardless of S. These inequalities together with $a_\beta(S_0, S_0) = G_\beta(S_0)+K/\{1+M_\beta(0)\} \leqq G_\beta(S_0)+ +K$ imply the lemma. Only the existence of A will be proved. The existence of B can be shown in a quite similar way.

By the properties of $G_\beta(j)$ there exist a number $\delta > 0$ and integers u_1 and u_2, $u_1 < u_2$, such that both for $j \geqq u_2$ and $j \leqq u_1$ we have that $G_\beta(j) \geqq G_\beta(S_0)+K+\delta$. We distinguish three cases.

a. $S \geqq s > u_2$. Since $K \geqq 0$, we have $a_\beta(s, S) \geqq G_\beta(S_0)+K+\delta > G_\beta(S_0)+K$.

b. $S > u_2$, $u_1 \leqq s \leqq u_2$. Separating the summation range of $\sum_{j=0}^{S-s} G_\beta(S-j)m_\beta(j)$ into $[0, S-u_2]$ and $[S-u_2+1, S-s]$ and using $G_\beta(S-j) \geqq G_\beta(S_0)+K+\delta$ for $0 \leqq j \leqq S-u_2$, $G_\beta(k) \geqq G_\beta(S_0)$ for $k \in I$, yields after regrouping terms

$$a_\beta(s, S) \geqq G_\beta(S_0)+K+\delta+\{K+\delta\}\,\{M_\beta(S-u_2)-M_\beta(S-s)\}/\{1+M_\beta(S-s)\}.$$

Since $M_\beta(j)$ is nondecreasing and $u_1 \leqq s \leqq u_2$, we have that

$$|M_\beta(S-u_2)-M_\beta(S-s)|/\{1+M_\beta(S-s)\} \leqq \{M_\beta(S-u_1)-M_\beta(S-u_2)\}/\{1+M_\beta(S-u_2)\}$$

There exists a finite integer $A_1 > u_2$, such that the right side of this inequality is $\leqq \delta/2(\delta+K)$ for $S > A_1$. For the case $\beta < 1$ this follows from the fact $m_\beta(j) \to 0$ as $j \to \infty$, and for the case $\beta = 1$ this follows from (3.4) and the boundness of $m_1(j)$, $j \geqq 0$. Hence for each s, $u_1 \leqq s \leqq u_2$,we have $a_\beta(s, S) > G_\beta(S_0)+K$ for $S > A_1$.

c. $S > u_2$, $s < u_1$. After separating the summation range of $\sum_{j=0}^{S-s} G_\beta(S-j)m_\beta(j)$ into $[0, S-u_2]$, $[S-u_2+1, S-u_1]$ and $[S-u_1+1, S-s]$ it is straightforward to prove that a finite integer $A_2 > u_2$ exists, such that for each s, $s < u_1$, we have $a_\beta(s, S) > G_\beta(S_0)+K$ for $S > A_2$. Taking $A = \max(A_1, A_2)$ ends the proof.

For any β, $0 < \beta \leqq 1$, let a_β^* be the absolute minimum of the function $a_\beta(s, S)$, $s, S \in I$, $s \leqq S$.

Lemma 4.2

Let $0 < \beta \leqq 1$ and let s^* and S^* be any integers such that $a_\beta(s^*, S^*) = a_\beta^*$.

a. If $m_\beta(S^*-s^*+1) > 0$, then $G_\beta(s^*-1) \geqq a_\beta^*$.
b. If $s^* = S^*$, then $G_\beta(s^*) \leqq a_\beta^*$.
c. If $s^* < S^*$ and if $m_\beta(S^*-s^*) > 0$, then $G_\beta(s^*) \leqq a_\beta^*$.
d. If $p_1 > 0$, then $G_\beta(s^*-1) \geqq a_\beta^* \geqq G_\beta(s^*)$.
e. If $G_\beta(s^*-1) \geqq a_\beta^* \geqq G_\beta(s^*)$, then $s_1 \leqq s^*$; $s^* \leqq S^0$ when $K = 0$; and $s^* \leqq S_0$ when $K > 0$.

Proof

a. Consider $a_\beta(s,S)$ as function of $\Delta = S-s$ and S. Put $b_\beta(\Delta,S) = a_\beta(S-\Delta,S)$. The function $b_\beta(\Delta,S)$ is minimal for $\Delta = \Delta^* = S^*-s^*$ and $S = S^*$. Hence $b_\beta(\Delta^*+1,S^*)-b_\beta(\Delta^*,S^*) \geqq 0$. This inequality leads after some straightforward calculations to

$$m_\beta(\Delta^*+1)\{G_\beta(s^*-1)(1+M_\beta(\Delta^*))-(G_\beta(S^*)+\sum_{j=0}^{\Delta^*} G_\beta(S^*-j)m_\beta(j)+K)\} \geqq 0.$$

From $m_\beta(\Delta^*+1) > 0$ and the definition of $a_\beta(s^*,S^*)$ follows now (a).
b. Since $s^* = S^*$, we have $a_\beta^* = a_\beta(s^*,s^*) = G_\beta(s^*)+K/\{1+M_\beta(0)\} \geqq G_\beta(s^*)$.
c. Assertion (c) follows after some straightforward calculations from $b_\beta(\Delta^*-1,S^*)-b_\beta(\Delta^*,S^*) \geqq 0$ and $m_\beta(\Delta^*) > 0$.
d. If $p_1 > 0$, then $m_\beta(k) > 0$ for $k \geqq 1$. From (a), (b) and (c) follows now (d).
e. Since $G_\beta(s^*) \leqq a_\beta^* \leqq a_\beta(S_0,S_0) = G_\beta(S_0)+K/\{1+M_\beta(0)\} = G_\beta(S_0)+(1-\beta p_0)K$ the definition (2.3) implies $s_1 \leqq s^*$. Next we distinguish between $K = 0$ and $K > 0$. Consider first the case $K = 0$. We have then $a_\beta(s,S) \geqq G_\beta(S_0) = a_\beta(S_0,S_0)$ for all s and S. Hence $a_\beta^* = G_\beta(S_0)$. From $a_\beta^* \geqq G_\beta(s^*)$ it follows $G_\beta(s^*) \leqq G_\beta(S_0)$. Thus $s^* \leqq S^0$, since S^0 is the largest integer which minimizes $G_\beta(k)$. Consider next the case $K > 0$. Assume to the contrary $s^* > S_0$. Since $G_\beta(k)$ is nondecreasing on $[S_0,\infty)$, we have then $a_\beta^* = a_\beta(s^*,S^*) \geqq G_\beta(s^*-1)+K/\{1+M_\beta(S^*-s^*)\} >$ $> G_\beta(s^*-1)$. This contradicts $G_\beta(s^*-1) \geqq a_\beta^*$. Thus $s^* \leqq S_0$.

Lemma 4.3

Let $0 < \beta \leqq 1$. There exist integers s^* and S^* such that $a_\beta(s^*,S^*) = a_\beta^*$ and $G_\beta(s^*-1) \geqq$ $\geqq a_\beta^* \geqq G_\beta(s^*)$. If $K = 0$, then $s^* = S_0$ and $S^* = S_0$ satisfy these conditions.

Proof

By lemma 4.1 there exist integers s' and S' such that $a_\beta(s',S') = a_\beta^*$. When $m_\beta(S'-s'+1) = 0$, we have by the definition of $a_\beta(s,S)$ that also $a_\beta(s'-1,S') = a_\beta^*$. However by $\beta > 0$ we have that $m_\beta(S'-s) > 0$ for infinite many values of s. This proves now that there exist integers s and S such that $a_\beta(s,S) = a_\beta^*$ and $m_\beta(S-s+1)$ > 0. By lemma 4.2(a) we have now proved that the set $T = \{(s,S)|a_\beta(s,S) =$ $= a_\beta^* \leqq G_\beta(s-1)\}$ is non-empty. Let (s^*,S^*) be a policy from T such that $S^*-s^* \leqq S-s$ for all $(s,S) \in T$. We shall show that $a_\beta^* \geqq G_\beta(s^*)$. When $s^* = S^*$ this follows from lemma 4.2(b). Consider now the case $s^* < S^*$. Suppose to the contrary that $G_\beta(s^*) > a_\beta^*$. By lemma 4.2(c) we have then $m_\beta(S^*-s^*) = 0$. Next it follows from the definition of $a_\beta(s,S)$ that $a_\beta(s^*+1,S^*) = a_\beta^*$. By $G_\beta(s^*) > a_\beta^*$ we have now the contradiction $(s^*+1,S^*) \in T$.

If $K = 0$, then $a_\beta^* = a_\beta(S_0,S_0) = G_\beta(S_0) < G_\beta(S_0-1)$. This ends the proof.

5. The optimality of an (s,S) policy

In this section we shall give a unified proof of the existence of an optimal (s,S) policy. As a by-product of the proof we find for $K > 0$ the important new result that any (s,S) policy such that $a_\beta(s,S) = a_\beta^*$ and $G_\beta(s-1) \geqq a_\beta^* \geqq G_\beta(s)$, is optimal and has the property $s_1 \leqq s \leqq S_0 \leqq S \leqq S_1$. To give the existence proof, we shall define a function $v_\beta^*(i)$ which will be shown to satisfy a functional equation, which is closely related to (2.9) and (2.10).

From now on s^* and S^* are two fixed integers for which $a_\beta(s^*,S^*) = a_\beta^*$ and $G_\beta(s^*-1) \geqq a_\beta^* \geqq G_\beta(s^*)$, where β fixed and $0 < \beta \leqq 1$. For the case $K = 0$ we take $s^* = S^* = S_0$ (see lemma 4.3). The function $v_\beta^*(i)$, $i \in I$, is defined as follows

$$v_\beta^*(i) = \begin{cases} 0, & i < s^*, \\ G_\beta(i) - a_\beta^* + \beta \sum_{j=0}^{i-s^*} v_\beta^*(i-j)p_j, & i \geqq s^* \end{cases} \tag{5.1}$$

Remark 5.1

In this remark we motivate this definition for the case $\beta = 1$. Suppose that $\{g, v(i)\}$, $i \in I$, is a set of numbers satisfying (2.10) and suppose further that the right side of (2.10) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geqq s^*$. Then

$$v(i) = G_1(i) - g + \sum_{j=0}^{\infty} v(i-j)p_j, \quad i \geqq s^*, \text{ and } v(i) = K + v(S^*), \quad i < s^*.$$

When c is a constant, then $\{g, v(i)+c\}$, $i \in I$, satisfies also (2.10). Normalizing $v(i)$ to be zero at $i = s^*-1$, explains now definition (5.1) for $\beta = 1$.

The function $v_\beta^*(i)$, $i \in I$, is uniquely determined by the renewal equation (5.1). Iterating (5.1) and using relations (3.1) and (3.2) together with the fact $p_j^{(n)} \to 0$ as $n \to \infty$ for each j, yields

$$v_\beta^*(i) = \begin{cases} 0, & i < s^*, \\ G_\beta(i) + \sum_{j=0}^{i-s^*} G_\beta(i-j)m_\beta(j) - a_\beta^*\{1 + M_\beta(i-s^*)\}, & i \geqq s^* \end{cases} \tag{5.2}$$

For convenience we define the function

$$J_\beta(k) = G_\beta(k) - a_\beta^* + \beta \sum_{j=0}^{\infty} v_\beta^*(k-j)p_j, \qquad k \in I \tag{5.3}$$

From (5.1) and (5.3) it follows

$$J_\beta(k) = G_\beta(k) - a_\beta^*, \qquad k < s^*, \tag{5.4}$$

and

$$J_\beta(k) = v_\beta^*(k), \qquad k \geqq s^* \tag{5.5}$$

Theorem 5.1

a. $J_\beta(k)$ is nonincreasing on $(-\infty, s^*-1]$,
b. $K + J_\beta(S^*) = 0$, $J_\beta(s^*-1) \geqq 0$,
c. $J_\beta(k) \geqq J_\beta(S^*)$ for all $k \in I$,
d. $J_\beta(k) \leqq 0$ for $s^* \leqq k \leqq S_0$,
e. $J_\beta(k)$ is nonincreasing on $[s^*, S_0]$,
f. $J_\beta(k) - J_\beta(i) \geqq G_\beta(k) - G_\beta(i) - \beta K$ for $k \geqq i \geqq S_0$.

Proof

a. Since $G_\beta(j)$ is nonincreasing on $(-\infty, S_0]$ and by lemma 4.2(e) we have $s^* \leqq S_0$, it follows directly from (5.4) that (a) holds.
b. From (5.2), (5.5), $a_\beta^* = a_\beta(s^*, S^*)$ and the definition of $a_\beta(s, S)$ it follows that $J_\beta(S^*) = -K$. By (5.4) we have that $J_\beta(s^*-1) = G_\beta(s^*-1) - a_\beta^* \geqq 0$.
c. Since $K \geqq 0$, we have by (a) and (b) that $J_\beta(k) \geqq J_\beta(s^*-1) \geqq J_\beta(S^*)$ for $k < s^*$. Hence it remains to show $J_\beta(k) \geqq J_\beta(S^*)$ for $k \geqq s^*$. Suppose there exists an integer $k \geqq s^*$, say $k = r$, such that $J_\beta(r) < J_\beta(S^*)$. From $J_\beta(S^*) = -K$ and the formulas (5.2) and (5.5) it follows then

   $$a_\beta^* > \{G_\beta(r) + \sum_{j=0}^{r-s^*} G_\beta(r-j) m_\beta(j) + K\} / \{1 + M_\beta(r - s^*)\}.$$

   Since by the definition of $a_\beta(s, S)$ the right side of this inequality is $a_\beta(r, s^*)$, we have obtained a contradiction. Thus (c) holds.
d. Since $G_\beta(k)$ is nonincreasing on $[s^*, S_0]$, by (5.5) and (5.2) we have that $J_\beta(k) \leqq \{G_\beta(s^*) - a_\beta^*\}\{1 + M_\beta(k - s^*)\} \leqq 0$ for $s^* \leqq k \leqq S_0$.
e. From (5.1) and (5.5) it follows that

   $$J_\beta(k) = G_\beta(k) - a_\beta^* + \beta \sum_{j=0}^{k-s^*} J_\beta(k-j) p_j, \qquad k \geqq s^* \tag{5.6}$$

   By (d) and (5.6) we have for $s^* \leqq i \leqq k \leqq S_0$ that

   $$J_\beta(i) - J_\beta(k) \geqq G_\beta(i) - G_\beta(k) + \beta \sum_{j=0}^{i-s^*} \{J_\beta(i-j) - J_\beta(k-j)\} p_j.$$

   Iterating this inequality, yields for $s^* \leqq i \leqq k \leqq S_0$,

   $$J_\beta(i) - J_\beta(k) \geqq G_\beta(i) - G_\beta(k) + \sum_{j=0}^{i-s^*} \{G_\beta(i-j) - G_\beta(k-j)\} m_\beta(j).$$

The assertion (e) follows from this inequality and the fact that $G_\beta(k)$ is nonincreasing on $[s^*, S_0]$.

f. By (b) and (c) we have that $J_\beta(k) \geqq -K$, $k \in I$. Further we have by (d) that $J_\beta(k) \leqq 0$ for $s^* \leqq k \leqq S_0$. Using (5.6) it follows now that for $k \geqq i \geqq S_0$

$$J_\beta(k) - J_\beta(i) \geqq G_\beta(k) - G_\beta(i) + \beta \sum_{j=0}^{i-S_0} \{J_\beta(k-j) - J_\beta(i-j)\} p_j + \\ -\beta K\{F(k-s^*) - F(i-S_0)\}.$$

Iterating this inequality, yields for $k \geqq i \geqq S_0$,

$$J_\beta(k) - J_\beta(i) \geqq G_\beta(k) - G_\beta(i) + \sum_{j=0}^{i-S_0} \{G_\beta(k-j) - G_\beta(i-j)\} m_\beta(j) + \\ -\beta K\left[F(k-s^*) - F(i-S_0) + \sum_{j=0}^{i-S_0} \{F(k-s^*-j) - F(i-S_0-j)\} m_\beta(j)\right].$$

Since $m_\beta(j) \leqq m_1(j)$, $j \geqq 0$, we have by (3.5) that the coefficient of $-\beta K$ is less than or equal to $P\{\gamma(i-S_0) \leqq k-i+S_0-s^*\}$ and hence is less than or equal to 1. This observation and the fact that $G_\beta(k)$ is nondecreasing on $[S_0, \infty)$ imply now assertion (f).

Theorem 5.2

a. The set of numbers $\{a_\beta^*, v_\beta^*(i)\}$, $i \in I$, satisfies

$$v_\beta^*(i) = \min_{k \geqq i} \{K\delta(k-i) + G_\beta(k) - a_\beta^* + \beta \sum_{j=0}^{\infty} v_\beta^*(k-j) p_j\}, \qquad i \in I \tag{5.7}$$

The right side of (5.7) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geqq s^*$.

b. $s_1 \leqq s^* \leqq S_0 \leqq S^* \leqq S_1$.

Proof

a. By (5.3) we have for each $i \in I$ that

$$K\delta(k-i) + G_\beta(k) - a_\beta^* + \beta \sum_{j=0}^{\infty} v_\beta^*(k-j) p_j = K\delta(k-i) + J_\beta(k), \qquad k \geqq i.$$

Recall $\delta(0) = 0$, and $\delta(j) = 1$ for $j > 0$. Let us consider $K\delta(k-i) + J_\beta(k)$ for i fixed and $k \geqq i$. We distinguish three cases.

Case 1. $i < s^*$. By theorem 5.1(a), 5.1(b) and 5.1(c) we have that

$$J_\beta(i) \geqq J_\beta(s^*-1) \geqq K + J_\beta(S^*) = \min_{k > i} \{K + J_\beta(k)\}.$$

Hence the right side of (5.7) is minimized by $k = S^*$ for $i < s^*$. By theorem 5.1(b) and (5.1) we have that $K+J_\beta(S^*) = 0 = v_\beta^*(i)$, $i < s^*$. This proves assertion (a) for $i < s^*$.

Case 2. $s^* \leqq i \leqq S_0$. By theorem 5.1(b), 5.1(c), 5.1(d) and (5.5) we have that $K+J_\beta(k) \geqq K+J_\beta(S^*) = 0 \geqq J_\beta(i) = v_\beta^*(i)$ for $k > i$. This proves (a) for $s^* \leqq i \leqq S_0$.

Case 3. $i > S_0$. Since $G_\beta(k)$ is nondecreasing on $[S_0, \infty)$ it follows from theorem 5.1(f) and (5.5) that $K+J_\beta(k) \geqq J_\beta(i) = v_\beta^*(i)$ for $k > i$. This proves (a) for $i > S_0$.

b. By lemma 4.2(e) and the choice $s^* = S_0$ when $K = 0$, we have that $s_1 \leqq s^* \leqq S_0$. Assume to the contrary that $S^* < S_0$. Then $G_\beta(S_0) < G_\beta(S^*)$. By using theorem 5.1(d), 5.1(e) and (5.6) it is now easy to verify that $S^* < S_0$ implies $J_\beta(S_0) < J_\beta(S^*)$. This contradicts theorem 5.1(c). Thus $S^* \geqq S_0$. By theorem 5.1(f) and 5.1(c) we have next that $G_\beta(S^*) - G_\beta(S_0) - \beta K \leqq 0$. From (2.4) it follows now that $S^* \leqq S_1$.

Consider first the case $\beta = 1$. By theorem 5.2 we have that

$$v_1^*(i) = \min_{k \geqq i} \{K\delta(k-i) + G_1(k) - a_1^* + \sum_{j=0}^{\infty} v_1^*(k-j)p_j\}, \qquad i \in I, \tag{5.8}$$

where the right side of (5.8) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geqq s^*$. Since $m \leqq s_1 \leqq s^*$ and $S^* \leqq S_1 \leqq M$ we see that the set of numbers $\{a_1^*, v_1^*(i)\}$, $i \in I$, satisfies (2.10), where the right side of (2.10) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geqq s^*$. Since $v_1^*(j) = 0$ for $j < s^*$, and under the condition $\underline{i}_1 = i$ we have $\underline{i}_t \leqq \max(i, M)$ for $t \geqq 1$, the sequence $\{\mathscr{E}_R(v(\underline{i}_n) | \underline{i}_1 = i)\}$, $n \geqq 1$, is bounded. Thus $\mathscr{E}_R(v(\underline{i}_n) | \underline{i}_1 = i)/n \to 0$ as $n \to \infty$ for all $i \in I$ and all $R \in C(m, M)$. The optimality of the (s^*, S^*) policy follows now from theorem 2.2.

Summarizing, we have proved (see also the lemmas 4.2 and 4.3):

Theorem 5.3 (*average cost criterion*)

Let $\beta = 1$. If $K = 0$, then the (S_0, S_0) policy is optimal. If $K > 0$, then any (s, S) policy such that $a_1(s, S) = a_1^*$ and $G_1(s-1) \geqq a_1^* \geqq G_1(s)$, is optimal and has the property that $s_1 \leqq s \leqq S_0 \leqq S \leqq S_1$. If $p_1 > 0$, then $a_1(s, S) = a_1^*$ implies $G_1(s-1) \geqq a_1^* \geqq G_1(s)$.

Consider next the case $\beta < 1$. From (4.2) and (5.2) it follows that

$$v_\beta^*(i) = V_\beta(i; (s^*, S^*)) - a_\beta^*/(1-\beta), \qquad i \in I \tag{5.9}$$

Substituting (5.9) in (5.7) yields

$$V_\beta(i; (s^*, S^*)) = \min_{k \geqq i} \{K\delta(k-i) + G_\beta(k) + \beta \sum_{j=0}^{\infty} V_\beta(k-j; (s^*, S^*))p_j\}, \qquad i \in I, \tag{5.10}$$

where the right side of (5.10) is minimized by $k = S^*$ for $i < s^*$ and by $k = i$ for $i \geqq s^*$. Since $m \leqq s_1 \leqq s^*$ and $S^* \leqq S_1 \leqq M$ the function $V_\beta(i;(s^*,S^*))$, $i \in I$, satisfies also (2.9). This proves the optimality of the (s^*,S^*) policy.

Summarizing, we have proved (see also the lemmas 4.2 and 4.3):

Theorem 5.4 (*the total discounted cost criterion*)

Let $\beta < 1$. If $K = 0$, then the (S_0,S_0) policy is optimal. If $K > 0$, then any (s,S) policy such that $a_\beta(s,S) = a_\beta^*$ and $G_\beta(s-1) \geqq a_\beta^* \geqq G_\beta(s)$, is optimal and has the property that $s_1 \leqq s \leqq S_0 \leqq S \leqq S_1$. If $p_1 > 0$, then $a_\beta(s,S) = a_\beta^*$ implies $G_\beta(s-1) \geqq a_\beta^* \geqq G_\beta(s)$.

Remark 5.2

In this remark we shall show that for the case $\beta = 1$ an optimal (s,S) policy also exists under the following weaker assumptions about $G_1(k)$: (i) there exists a finite integer S_0 at which $G_1(k)$ takes on its absolute minimum; (ii) there exist finite integers $v < S_0$ and $V > S_0$ such that $G_1(k)$ is nonincreasing on $[v,S_0]$, $G_1(k)$ is nondecreasing on $[S_0,V]$ and both $w = \inf_{k<v} G_1(k)$ and $W = \inf_{k>V} G_1(k)$ are larger than $G_1(S_0)+K$. We may assume that S_0 is the smallest integer for which (*i*) holds. Clearly, $s_1 \geqq v$ and $S_1 \leqq V$, where s_1 and S_1 are defined by (2.3) and (2.4) Define $\hat{G}_1(k)$ as follows: $\hat{G}_1(k) = \min(w,G_1(k))$ for $k \leqq S_0$ and $\hat{G}_1(k) = \min(W,G_1(k))$ for $k \geqq S_0$. Let $\hat{g}(i;R)$ and $\hat{a}_1(s,S)$ correspond to $\hat{G}_1(k)$. Since $G_1(k) \geqq \hat{G}_1(k)$, $k \in I$, we have that $g(i;R) \geqq \hat{g}(i;R)$ for all $i \in I$ and all $R \in C(m,M)$. The function $\hat{G}_1(k)$ satisfies the conditions (*i*) and (ii) from section 2 and $\hat{G}_1(k) = G_1(k)$ on $\{s_1,S_1\}$. Let s^* and S^* be any integers such that $\hat{a}_1(s^*,S^*) = \min \hat{a}_1(s,S)$ and $\hat{G}_1(s^*-1) \geqq \hat{a}_1(s^*,S^*) \geqq \hat{G}_1(s^*)$. We take $s^* = S^* = S_0$ when $K = 0$. By lemma 4.3 such integers exists. By theorem 5.3 we have $s_1 \leqq s^* \leqq S_0 \leqq S^* \leqq S_1$ and $\hat{g}(i;R) \geqq \hat{a}_1(s^*,S^*)$ for all i and all R. Since $G_1(k) = \hat{G}_1(k)$ on $[s_1,S_1]$, we have by (4.1) that $a_1(s^*,S^*) = \hat{a}_1(s^*,S^*)$. Hence $g(i;R) \geqq a_1(s^*,S^*)$ for all i,R and $\min a_1(s,S) = \min \hat{a}_1(s,S)$. Further a modification of the proof of lemma 4.2(e) shows that lemma 4.2(e) remains valid (by using that $a_1^* \leqq G_1(s^*-1)$ and $a_1^* \leqq G_1(S_0)+K$, it can now be proved that $s^* \leqq S_0$ when $K > 0$, since both $S_0 < s^* \leqq S_1$ and $s^* > S_1$ lead to a contradiction). It is now not difficult to see that under the weakened assumptions about $G_1(k)$ theorem 5.3 remains valid.

For the case $\beta < 1$ it can be proved in an analogous way that theorem 5.4 remains valid when the weaker assumption $\inf_{k<s_1} G_\beta(k) > G_\beta(S_0)+K$ is substituted for $G_\beta(k)$ is nonincreasing for $k < s_1$. We note that in general the condition $G_\beta(k)$ is nondecreasing for $k > S_1$ cannot be weakened for the case $\beta < 1$.

Remark 5.3

Consider now the continuous demand case, in which the distribution function $F(\xi)$ of the demand variables ξ_t, $t \geqq 1$, has a probability density $f(\xi)$.

Define $F^{(n)}(\xi) = P\{\xi_1 + \ldots + \xi_n \leqq \xi\}$, $n \geqq 1$, and let

$$G_\beta(y) = \begin{cases} g(y)+(1-\beta)cy & \text{if } \lambda = 0, \\ \int\limits_0^\infty g(y-\xi)f^{(\lambda)}(\xi)\mathrm{d}\xi+(1-\beta)cy & \text{if } \lambda \geqq 1, \end{cases}$$

where $f^{(\lambda)}(\xi)$ is the density of $F^{(\lambda)}(\xi)$ and $0 < \beta \leqq 1$. It is assumed that: (i) there exists a finite number S_0 such that $G_\beta(y)$ is nonincreasing for $y \leqq S_0$ and nondecreasing for $y \geqq S_0$; (ii) $G_\beta(y) > G_\beta(S_0)+K$ for $|y|$ sufficient large; (iii) $G_\beta(y)$ is differentiable. We may assume that S_0 is the smallest number for which (i) holds. Let s_1 be the smallest number for which $G_\beta(s_1) = G_\beta(S_0)+K$ and let S_1 be the largest number for which $G_\beta(S_1) = G_\beta(S_0)+\beta K$.

Let

$$M_\beta(\xi) = \sum_{n=1}^\infty \beta^n F^{(n)}(\xi).$$

Its derivative $m_\beta(\xi)$ satisfies

$$m_\beta(\xi) = \beta f(\xi)+\beta \int\limits_0^\xi f(\xi-\eta)m(\eta)\mathrm{d}\eta, \qquad \xi \geqq 0.$$

Let for any β, $0 < \beta \leqq 1$,

$$a_\beta(s, S) = \{G_\beta(S)+ \int\limits_0^{S-s} G_\beta(S-\xi)m_\beta(\xi)\mathrm{d}\xi+K\}/\{1+M_\beta(S-s)\}, \qquad s \leqq S.$$

Analogous to the proof for the discrete demand case it can be shown that the theorems 5.3 and 5.4 also hold for the continuous demand case provided that we replace $G_\beta(s-1) \geqq a_\beta^* \geqq G_\beta(s)$ by $G_\beta(s) = a_\beta^*$ and replace $p_1 > 0$ by $f(\xi) > 0$, $\xi > 0$. Further the conditions imposed on $G_\beta(k)$ can be weakened as in remark 5.2.

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