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A Note on Equivalent Systems of Linear Diophantine Equations[1])

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Summary: Another constructive proof is presented for the fact that a system of linear equations with integer coefficients in bounded integer variables is equivalent to a single equation, which is a linear combination of the original ones. The equation is obtained in a number of steps; in each step two equations are replaced by a single one. This replacement is performed subject to the condition that the remaining equations hold and a final equation with relatively small coefficients is obtained. It may be inefficient however to calculate small coefficients, as the original coefficients can be used to represent the final ones in a suitably chosen number system.

Zusammenfassung: Ein System linearer Gleichungen mit ganzzahligen Koeffizienten in beschränkten ganzzahligen Variablen ist äquivalent zu einer einzigen Gleichung, die sich als Linearkombination der ursprünglichen Gleichungen schreiben läßt. In einem neuen konstruktiven Beweis von diesem Satz wird gezeigt, wie die endgültige Gleichung in einigen Schritten gefunden werden kann. In jedem Schritt werden zwei Gleichungen von einer einzigen ersetzt unter der Voraussetzung, daß die übrigen Gleichungen gültig bleiben.

Obwohl die Koeffizienten der Endgleichung verhältnismäßig klein sind, ist es nicht immer zweckmäßig, sie in der angegebenen Weise zu berechnen, da man ein Zahlensystem anwenden kann, in dem die Koeffizienten der ursprünglichen Gleichungen zur Darstellung der neuen Koeffizienten gebraucht werden.

1. Introduction

In their paper *Elmaghraby* and *Wig* [1970] used two theorems, due to *Mathews* [1897], to aggregate a system of two linear equations with integer coefficients in bounded, integer variables, into a single, equivalent linear equation. By repeated application a system of m such equations can be reduced to a single equation which is a linear combination of the original ones.

The coefficients in the final equation however, tend to be rather large.

In the preliminary paper [*Anthonisse*, 1970] it was shown that smaller coefficients can be obtained and that the original coefficients can be used to represent the final coefficients in a suitably choosen number system.

Padberg [1970] derived some sharper results for the case of two equations.

In the present paper the results of *Padberg* [1970] are improved and combined with those of *Anthonisse* [1970].

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[1]) This note is a slightly revised version of report BW 12/71 (july 1971).

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2. The Case $m = 2$

The aggregation of a system into a single equation is based upon the following, and well-known, result.

Theorem 1:

If q_1 and q_2 are two relative prime integers, $q_i \neq 0$, then all integer solutions of the equation

$$q_1 y_1 + q_2 y_2 = 0 \tag{1}$$

are of the form

$$\left.\begin{aligned} y_1 &= t q_2 \\ y_2 &= -t q_1 , \end{aligned}\right\} \tag{2}$$

where t is any integer.

Proof:

The Eq. (1) yields $y_1 = -\frac{q_2}{q_1} y_2$. As y_1 is required to be an integer, and the greatest common divisor of q_1 and q_2 is 1, y_2 must be a multiple of q_1. This completes the proof.

Now consider the equations

$$y_i = \sum_{j=1}^{n} a_{ij} x_j - a_{i0} \qquad (i = 1,2), \tag{3a}$$

$$\left.\begin{aligned} 0 \le x_j \le b_j \\ x_j = \text{integer} \end{aligned}\right\} \qquad (j = 1, \dots, n), \tag{4}$$

where a_{ij} and b_j are assumed to be integers. Consequently, y_i is integer valued.

It is easily seen that

$$L_i \le y_i \le U_i \qquad (i = 1,2) \tag{5}$$

where

$$L_i = \sum_{\substack{j=1 \\ a_{ij} < 0}}^{n} a_{ij} b_j - a_{i0} \qquad (i = 1,2) \tag{6}$$

and

$$U_i = \sum_{\substack{j=1 \\ a_{ij} > 0}}^{n} a_{ij} b_j - a_{i0} \qquad (i = 1,2). \tag{7}$$

If $L_i = 0$ the equation $y_i = 0$ implies $x_j = b_j$ if $a_{ij} < 0$ and $x_j = 0$ if $a_{ij} > 0$. In this case these substitutions can be performed and only a single equation, in fewer variables, remains.

A similar result holds if $U_i = 0$.

If $L_i > 0$ or $U_i < 0$ the equation $y_i = 0$ has no solution and the system is infeasible.

Define

$$S_0 = \{(u_1, u_2) \,|\, L_i \le u_i \le U_i, u_i = \text{integer}\,(i = 1,2)\}. \tag{8}$$

Now consider the system

$$y_i = \sum_{j=1}^{n} a_{ij} x_j - a_{i0} = 0 \qquad (i = 1,2), \tag{3}$$

$$0 \leq x_j \leq b_j, \quad x_j = \text{integer} \qquad (j = 1, \ldots, n). \tag{4}$$

Theorem 2:

For any two relative prime integers q_2 and q_1 such that

and

$$\left.\begin{array}{l}(q_2, -q_1) \notin S_0 \\ (-q_2, q_1) \notin S_0,\end{array}\right\} \tag{9}$$

the unique solution of

$$q_1 y_1 + q_2 y_2 = 0 \tag{1}$$

$$\left.\begin{array}{l}L_i \leq y_i \leq U_i \\ y_i = \text{integer}\end{array}\right\} \quad (i = 1,2) \tag{10}$$

is $y_1 = y_2 = 0$.

Proof:

Each solution of (1) is of the form $y_1 = tq_2$, $y_2 = -tq_1$.

If $t \neq 0$ then (9) leads to the conclusion that $(y_1, y_2) \notin S_0$, contradicting (10). This completes the proof.

As a consequence of the above theorem system (3), (4) is equivalent to:

$$\sum_{j=1}^{n} (q_1 a_{1j} + q_2 a_{2j}) x_j - (q_1 a_{10} + q_2 a_{20}) = 0 \tag{11}$$

$$\left.\begin{array}{l}0 \leq x_j \leq b_j \\ x_j = \text{integer}\end{array}\right\} \quad (j = 1, \ldots, n), \tag{4}$$

for any two relative prime integers satisfying (9). Valid choices are:

	$q_1 \geq U_2 + 1$	and	$q_2 \geq U_1 + 1$,
or	$q_1 \geq -L_2 + 1$	and	$q_2 \geq -L_1 + 1$,
or	$q_1 \geq 1$	and	$q_2 \geq 1 + \max(U_1, -L_1)$,
or	$q_1 \geq 1 + \max(U_2, -L_2)$	and	$q_2 \geq 1$.

The above results were given by *Padberg* [1970].

It should be noted that theorem 2 is based upon the integrality of y_i. The fact that y_i represents a linear function will be exploited to obtain smaller $|q_i|$.

Define $v(p)$ = the minimum and $w(p)$ = the maximum of

$$\sum_{j=1}^{n} a_{2j} x_j - a_{20} \tag{12}$$

subject to

$$\left.\begin{array}{l}\sum_{j=1}^{n} a_{1j} x_j - a_{10} = p \\ 0 \leq x_j \leq b_j \quad (j = 1, \ldots, n),\end{array}\right\} \tag{13}$$

for all p such that

$$L_1 \leq p \leq U_1 .$$

Both $v(p)$ and $w(p)$ are piece-wise linear functions, $v(p)$ is convex, $w(p)$ is concave.

Define furthermore,

$$S_1 = \{(u_1,u_2) | L_1 \leq u_1 \leq U_1, v(u_1) \leq u_2 \leq w(u_1), u_i = \text{integer}\} . \tag{14}$$

Theorem 3:

For any two relative prime integers q_2 and q_1 such that

$$(q_2, -q_1) \notin S_1 \quad \text{and} \quad (-q_2, q_1) \notin S_1 , \tag{15}$$

system (3), (4) is equivalent to system (11), (4).

Proof:

As (11) is a linear combination of (3) each solution of (3), (4) obviously solves (11), (4). Assume $x_j = \underline{x}_j$ $(j = 1, \ldots, n)$ satisfies (11), (4) but not (3).

Define

$$\underline{y}_i = \sum_{j=1}^{n} a_{ij} \underline{x}_j - a_{i0} \quad (i = 1,2) , \tag{16}$$

then (11) yields

$$q_1 \underline{y}_1 + q_2 \underline{y}_2 = 0 ,$$

so $\underline{y}_1 = t q_2$ and $\underline{y}_2 = -t q_1$, with $t \neq 0$.

This implies that $(\underline{y}_1, \underline{y}_2) \notin S_1$.

Both relations $\underline{y}_1 < L_1$ and $\underline{y}_1 > U_1$ contradict (4).

If $L_1 \leq \underline{y}_1 \leq U_1$ then either $\underline{y}_2 < v(\underline{y}_1)$ or $\underline{y}_2 > w(\underline{y}_1)$, contradicting the definition of $v(p)$ or $w(p)$ respectively.

This completes the proof.

As $S_1 \subset S_0$ theorem 3 may lead to smaller $|q_i|$ than can be obtained by theorem 2.

It is not difficult to determine the functions $v(p)$ and $w(p)$. Consider the example from [*Mathews*, 1897].

$$y_2 = -x_1 + 3x_2 - 5x_3 - x_4 + 4x_5 + x_6 + 2 = 0$$
$$y_1 = 2x_1 - 6x_2 + 3x_3 + 2x_4 - 2x_5 + x_7 = 0$$
$$x_i \in \{0,1\} \quad (i = 1, \ldots, 5)$$
$$0 \leq x_6 \leq 5, \quad 0 \leq x_7 \leq 8, \quad \text{integer}.$$

First substitute $x_2 = 1 - x_2'$, $x_5 = 1 - x_5'$, then rearrange the variables in such a way that the ratio a_{2j}/a_{1j} is non-increasing $(j = 1, \ldots, n)$.

$$y_2 = x_6 - x_1 - 3x_2' - x_4 - 5x_3 - 4x_5' + 9 = 0$$
$$y_1 = x_7 + 2x_1 + 6x_2' + 2x_4 + 3x_3 + 2x_5' - 8 = 0$$

$v(p)$ is found by working from the right to the left:

p		$v(p)$	
-8	$= -8$	$+9$	$= +9$
$-8+2$	$= -6$	$+9-4$	$= +5$
$-8+2+3$	$= -3$	$+9-4-5$	$= 0$
$-8+2+3+2$	$= -1$	$+9-4-5-1$	$= -1$
$-8+2+3+2+6$	$= +5$	$+9-4-5-1-3$	$= -4$
$-8+2+3+2+6+2$	$= +7$	$+9-4-5-1-3-1$	$= -5$
$-8+2+3+2+6+2+8$	$= +15$	$+9-4-5-1-3-1$	$= -5$

Similarly, $w(p)$ is found by working in the opposite direction:

p		$w(p)$	
$-8 =$	-8	$+9+5 =$	$+14$
$-8+8 =$	0	$+14+0 =$	$+14$
$0+2 =$	$+2$	$+14-1 =$	$+13$
$+2+6 =$	$+8$	$+13-3 =$	$+10$
$+8+2 =$	$+10$	$+10-1 =$	$+9$
$+10+3 =$	$+13$	$+9-5 =$	$+4$
$+13+2 =$	$+15$	$+4-4 =$	0

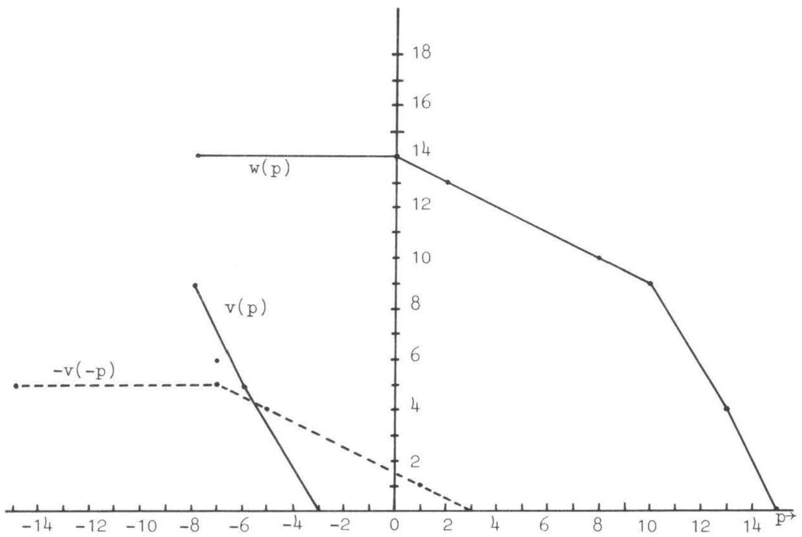

Fig. 1

The situation is depicted in Fig. 1, where the functions $v(p)$ and $w(p)$ are also drawn for non-integer p, and $-v(-p)$ is given if $v(p) < 0$. Evidently, $(-7,6) \notin S_1$, and $(7,-6) \notin S_1$, the choice $q_2 = 7$, $q_1 = 6$ leads to:

$$7x_6 + 6x_7 + 5x_1 + 15x_2' + 5x_4 - 17x_3 - 16x_5' + 15 = 0,$$

or, with $x_3 = 1 - x_3'$, $x_5' = 1 - x_5$,

$$7x_6 + 6x_7 + 5x_1 + 15x_2' + 5x_4 + 17x_3' + 16x_5 = 18.$$

$q_2 = 11$, $q_1 = 6$ yields:

$$11x_6 + 6x_7 + x_1 + 3x_2' + x_4 + 37x_3' + 32x_5 = 18,$$

so $x_3 = 1$ and $x_5 = 0$ in any solution, what was not evident at the first choice.

This leads to the conclusion that it may be worthwhile to select other than the 'minimal' q_2 and q_1.

3. The General Linear Case

A system of m equations

$$\sum_{j=1}^{n} a_{ij}x_j - a_{i0} = 0 \qquad (i = 1,\dots,m), \tag{17}$$

$$\left.\begin{array}{l} 0 \le x_j \le b_j \\ x_j = \text{integer} \end{array}\right\} \qquad (j = 1,\dots,n), \tag{4}$$

where b_j and a_{ij} denote integers, can be aggregated into a single, equivalent equation

$$\sum_{j=1}^{n} a_j x_j - a_0 = 0, \tag{18}$$

$$\left.\begin{array}{l} 0 \le x_j \le b_j \\ x_j = \text{integer} \end{array}\right\} \quad (j = 1,\dots,n). \tag{4}$$

by $m - 1$ applications of theorem 3. Each coefficient a_j $(j = 0,1,\dots,n)$ in (18) is a linear combination of the coefficients $a_{1j},\dots,a_{mj}$. The $|a_j|$ may be very large. The next theorem leads to smaller coefficients.

Define L_{m-1} = the minimum, and U_{m-1} = the maximum of

$$y_{m-1} \tag{19}$$

subject to

$$\left.\begin{array}{ll} y_i = 0 & (i = 1,\dots,m-2), \\ 0 \le x_j \le b_j & (j = 1,\dots,n), \end{array}\right\} \tag{20}$$

where y_i denotes the function $\sum_{j=1}^{n} a_{ij}x_j - a_{i0}$.

It should be noted that (20) does not restrict the x_j to integer values.

Define $v(p)$ = the minimum, and $w(p)$ = the maximum of

$$y_m \tag{21}$$

subject to

$$\left.\begin{array}{ll} y_i = 0 & (i = 1,\dots,m-2) \\ y_{m-1} = p & \end{array}\right\} \tag{22}$$

for all p such that

$$L_{m-1} \leq p \leq U_{m-1} .$$

Again, both $v(p)$ and $w(p)$ are piece-wise linear, $v(p)$ is convex, $w(p)$ is concave.
Finally, define

$$S_2 = \{(u_1, u_2) | L_{m-1} \leq u_1 \leq U_{m-1}, v(u_1) \leq u_2 \leq w(u_1), u_i = \text{integer}\} . \quad (23)$$

Both $v(p)$ and $w(p)$ are required for integer values of p only, thus S_2 can be determined by solving a sequence of ordinary linear programming problems.

Theorem 4:
For any two relative prime integers q_m and q_{m-1} such that

$$(q_m, -q_{m-1}) \notin S_2 \quad \text{and} \quad (-q_m, q_{m-1}) \notin S_2 , \quad (24)$$

system (17), (4) is equivalent to

$$\left.\begin{aligned} y_i &= 0 \quad (i = 1, \ldots, m-2) \\ q_{m-1} y_{m-1} + q_m y_m &= 0 \end{aligned}\right\} \quad (25)$$

$$\left.\begin{aligned} 0 \leq x_j \leq b_j \\ x_j = \text{integer} \end{aligned}\right\} \quad (j = 1, \ldots, n) . \quad (4)$$

Proof:
Assume $x_j = \underline{x}_j$ solves (25), (4), but not (17). Then

$$\underline{y}_i = \sum_{j=1}^{n} a_{ij} \underline{x}_j - a_{i0} \quad (i = m-1, m)$$

satisfy $\underline{y}_{m-1} = t q_m$ and $\underline{y}_m = -t q_{m-1}$, with $t \neq 0$.
This implies $(\underline{y}_{m-1}, \underline{y}_m) \notin S_2$.
But $\underline{x}_j$ satisfies (25), so $L_{m-1} \leq \underline{y}_{m-1} \leq U_{m-1}$ and $v(\underline{y}_{m-1}) \leq \underline{y}_m \leq w(\underline{y}_{m-1})$ hold by the definitions of L_{m-1}, U_{m-1} and $v(p)$, $w(p)$ respectively.
This contradiction completes the proof.
As $S_2 \subset S_1$ the resulting q_{m-1} and q_m may be, in absolute value, much smaller than those obtained by the application of theorem 3 to the equations $y_{m-1} = 0$, $y_m = 0$.
Theorem 3 can be used with S_2 replaced by

$$S_3 = \{(u_1, u_2) | L_{m-1} \leq u_1 \leq U_{m-1}, L_m \leq u_2 \leq U_m, u_i = \text{integer}\} ,$$

where L_m and U_m denote the minimum and maximum of y_m subject to (20), respectively.
Consider the problem

$$\begin{aligned} y_1 &= x_2 - 2x_3 + x_4 + x_5 + x_8 + 1 \\ y_2 &= 2x_1 - 6x_2 + 3x_3 + 2x_4 - 2x_5 + x_7 \\ y_3 &= -x_1 + 3x_2 - 5x_3 - x_4 + 4x_5 + x_6 + 2 \\ &x_i \in \{0, 1\} \quad (i = 1, \ldots, 5) \\ &0 \leq x_6 \leq 5 , \quad 0 \leq x_7 \leq 8 , \quad 0 \leq x_8 \leq 1 , \quad \text{integer} \end{aligned}$$

which is the previous example with an additional constraint. This example was used in [*Anthonisse*, 1970]. $y_1 = 0$ implies $x_3 = 1$, after this substitution it is seen that $x_2 = 1$ in any solution, this substitution yields $x_4 = x_5 = x_8 = 0$. The only solutions are:

$$x_1 = x_6 = 0\,, \quad x_7 = 3 \quad \text{and} \quad x_1 = x_6 = x_7 = 1\,.$$

Now the system will be aggregated into a single equation.

$$y_1 = 0 \quad \text{implies} \quad -3 \leq y_2 \leq 15\,.$$

Fig. 2 contains the $-v(-p)$ and $w(p)$ of y_3 subject to $y_1 = 0, y_2 = p\,(p = -3, \ldots, 15)$. As $(-4, 5) \notin S_2$ and $(4, -5) \notin S_2$ the system is equivalent to

$$y_1 = 0$$

$$5y_2 + 4y_3 = 0\,,$$

or

$$y_1 = x_2 + 2x_3' + x_4 + x_5 + x_8 - 1 = 0$$

$$y_4 = 6x_1 - 18x_2 + 5x_3' + 6x_4 + 6x_5 + 4x_6 + 5x_7 + 3 = 0\,.$$

Minimizing and maximizing y_4 subject to $y_1 = p$ yields

p	$v(p)$	$w(p)$
-1	$+3$	$+69$
0	-15	$+75$
1	-15	$+81$
2	-12.5	$+83.5$
3	-10	$+86$
4	-4	$+86$
5	$+2$	$+68$,

see Fig. 3.

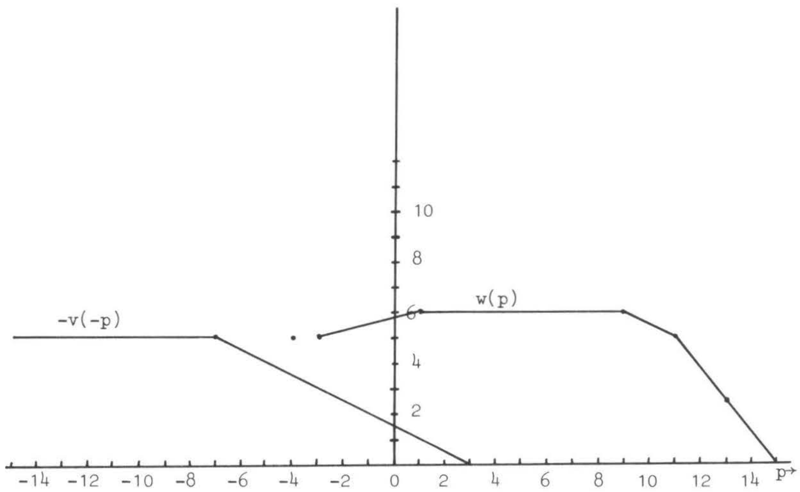

Fig. 2

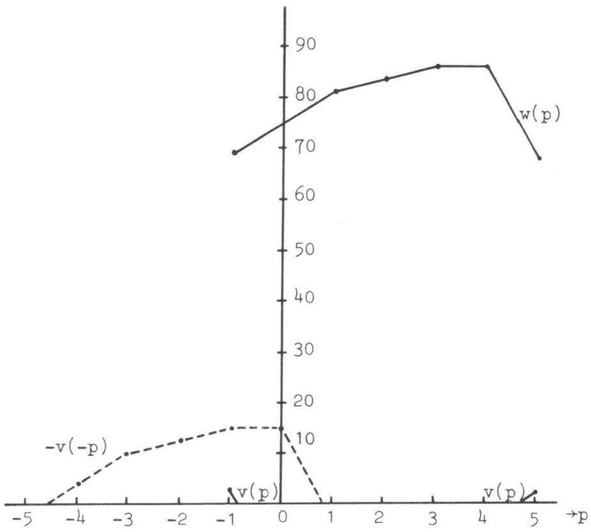

Fig. 3

Now $(-5,1) \notin S_1$ and $(5,-1) \notin S_1$, leading to $y_1 + 5y_4 = 0$ or:

$$30x_1 + 89x_2' + 27x_3' + 31x_4 + 20x_6 + 25x_7 + x_8 = 75.$$

4. A Generalisation

Part of the previous results can be obtained without using the integrality of the functions y_i. Let y_1 and y_2 denote real valued functions, which are defined over an arbitrary domain.

Theorem 5:

If the function y_1 is bounded, i.e. $|y_i| \leq B$, and $y_2 \neq 0$ implies $|y_2| \geq \varepsilon > 0$ then, for any q_1 satisfying $|q_1| > B/\varepsilon$, the system of equations

$$\left.\begin{aligned} y_1 &= 0 \\ y_2 &= 0 \end{aligned}\right\} \tag{26}$$

is equivalent to

$$y_1 + q_1 y_2 = 0. \tag{27}$$

Proof:

Obviously, any solution of (26) satisfies (27). If (27) holds and $y_2 = 0$ then $y_1 = 0$, if $y_2 \neq 0$ then

$$|y_1| = |q_1 y_2| \geq |q_1| \varepsilon > B.$$

This completes the proof.

Consider the system of equations

$$y_i = 0 \quad (i = 1, \ldots, m), \tag{28}$$

where $y_i \neq 0$ implies $|y_i| \geq 1$ $(i = 2, \ldots, m)$.

Define
$$C_i = \sup(|y_i| \,|\, y_k = 0\,(k = 1,\dots,i-1))\,, \qquad (i = 1,\dots,m-1)\,. \tag{29}$$

Theorem 6:

For any q_i satisfying $|q_i| \geq C_i + 1$ $(i = 1,\dots,m-1)$ and $q_0 = 1$, the system (28) is equivalent to
$$\sum_{i=1}^{m} q_1 q_2 \cdots q_{i-1} y_i = 0\,. \tag{30}$$

Proof:

It is easily seen that system (28) is equivalent to
$$\left.\begin{aligned} y_i &= 0 \qquad (i = 1,\dots,m-2) \\ y_{m-1} + q_{m-1} y_m &= 0\,. \end{aligned}\right\} \tag{31}$$
The first $m-2$ equations imply $|y_{m-1}| < |q_{m-1}|$. If $y_m \neq 0$ the last one implies
$$|y_{m-1}| = |q_{m-1}| \cdot |y_m| \geq |q_{m-1}|\,.$$
Now assume $m > 2$.

If $y_{m-1} = 0$ or $y_m = 0$ then $y' = y_{m-1} + q_{m-1}\ y_m \neq 0$ implies $|y'| \geq 1$. If $y_{m-1} \neq 0$ and $y_m \neq 0$ then $|y_{m-1}| \leq C_{m-1}$ and $|q_{m-1} y_m| \geq C_{m-1} + 1$ yield the same implication.

This completes the proof.

The above theorem leads to the conclusion that y_1 should be bounded, y_2 should be bounded on that part of the domain where $y_1 = 0$, y_3 bounded on that part where $y_1 = y_2 = 0$ and so on. y_m, however, may be unbounded.

During the computation of C_i it might be found that $y_k = 0$ $(k = 1,\dots,i-1)$ implies $y_i \neq 0$. In this case the system is infeasible.

If $C_i = 0$ the equation $y_i = 0$ is redundant.

5. Numerical Aspects

This discussion is restricted to the linear case, with integer coefficients. It is easily seen that the aggregation of a system of equations may lead to rather large coefficients in the final equation. The coefficients can be decreased by using small q_i but these can be obtained at a rather high computational price only.

If theorem 6 is used, the coefficients of the original system can be transformed into a representation of the final coefficients.

The original system is (17), (4). Let the integers q_i $(i = 1,\dots,m-1)$ satisfy
$$q_i \geq 1 + \max(|y_i| \,|\, y_k = 0\,(k = 1,\dots,i-1), 0 \leq x_j \leq b_j, (j = 1,\dots,n))\,,$$
and $q_0 = 1$.

Then the system is equivalent to
$$\sum_{j=1}^{n} a_j x_j = a_0 \tag{18}$$

$$\left.\begin{array}{l} 0 \le x_j \le b_j \\ x_j = \text{integer} \end{array}\right\} \quad (j = 1, \dots, n), \tag{4}$$

where

$$a_j = \sum_{i=1}^{m} q_1 \cdots q_{i-1} a_{ij} \qquad (j = 0, 1, \dots, n).$$

If $0 \le a_{ij} < q_i$ $(i = 1, \dots, m-1)$ and $a_{mj} \ge 0$ then the j-th column $(a_{1j}, \dots, a_{mj})$ from the matrix (a_{ij}) can be interpreted as the representation of a_j in a, possibly unfamiliar, number system determined by the q_i.

This number system has q_i 'digits' for the i-th position $(i = 1, \dots, m-1)$, the number of digits for the m-th position is unbounded.

If $-q_i < a_{ij} \le 0$ $(i = 1, \dots, m-1)$ and $a_{mj} \le 0$ then $(a_{1j}, \dots, a_{mj})$ represents a_j in the same system.

With the convention that all 'digits' are either non-negative or non-positive any integer has a unique representation in the system.

Thus a_j can be computed by transforming the column $(a_{1j}, \dots, a_{mj})$ into the representation of a_j. This is not difficult as

$$\begin{aligned} q_1 \cdots q_{i-1} a_{ij} + q_1 \cdots q_i a_{i+1,j} &= q_1 \cdots q_{i-1}(a_{ij} + q_i) + q_1 \cdots q_i(a_{i+1,j} - 1) \\ &= q_1 \cdots q_{i-1}(a_{ij} - q_i) + q_1 \cdots q_i(a_{i+1,j} + 1). \end{aligned}$$

Valid choices for q_i are

$$q_i = 1 + \max(|y_i| \,|\, 0 \le x_j \le b_j, (j = 1, \dots, n)) \qquad (i = 1, \dots, m-1)$$

and

$$q_i = q = 1 + \max_i \max(|y_i| \,|\, 0 \le x_j \le b_j, (j = 1, \dots, n)).$$

In the latter case

$$a_j = \sum_{i=1}^{m} q^{i-1} a_{ij} \qquad (j = 0, \dots, n).$$

6. References

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