AARHUS UNIVERSITET

PRINCIPALLY POLARIZED ABELIAN VARIETIES
OF DIMENSION TWO OR THREE ARE JACOBIAN VARIETIES

Frans Oort and Kenji Ueno

February 1973

Preprint Series
1972/73 No. 38

MATEMATISK INSTITUT

6

PRINCIPALLY POLARIZED ABELIAN VARIETIES OF DIMENSION TWO OR THREE ARE JACOBIAN VARIETIES

Frans Oort and Kenji Ueno

(Amsterdam/Aarhus and Tokyo/Bonn)

In this note we prove that a principally polarized abelian variety of dimension $g \leq 3$ is the canonically polarized Jacobian variety of a (possibly reducible) algebraic curve; for $g = 2$ this result was proved by A. Weil. Using results of P. Deligne and D. Mumford on the irreducibility of moduli spaces of stable curves, we thus derive the irreducibility of certain moduli spaces of abelian varieties.

For $g \geq 4$ the number $\frac{1}{2}g(g+1)$ of moduli for abelian varieties of dimension g is bigger than the number $3g-3$ of moduli of algebraic curves of genus g; this explains the restriction on g we are making.

1. Definition. An algebraic curve C over an algebraically closed field K is called a *good curve*, if it is an irreducible non-singular curve, or a stable curve (in the sense of [1], Definition 1.1) such that its (generalized) Jacobian variety $\mathrm{Jac}(C) = \underline{\mathrm{Pic}}^0(C)$ is an abelian variety.

Equivalently: C is irreducible and non-singular, or, if C is not irreducible, C is connected, all its irreducible components have multiplicity one and are non-singular curves of genus at least one, and the components are connected like a *bamboo* (i.e. for any component we will associate the symbol o, and if two components intersect we write o—o; then the curve C has a configuration o—o ... —o—o).

A stable curve $\pi: C \rightarrow S$ in the sense of [1], Definition 1.1 is called a good curve if all geometric fibres are good curves in the sense explained above (an explanation why we choose the word "good" is given in Lemma 6 below, in connection with the reduction theory for abelian varieties).

2. Definition. Let X be an abelian variety over a field K, let X^t be its dual (i.e. X^t is the Picard variety of X, in some literature denoted by $\hat{X}$); a polarization

$$\lambda: X \longrightarrow X^t$$

(cf. [4], Definition 6.2) is called principal if λ is an isomorphism.

3. Notation. Let C be a good curve, C_i its components; we denote by $J = \mathrm{Jac}(C)$ the product $J = \prod \mathrm{Jac}(C_i)$, and by Θ we denote the related divisor

$$\Theta = \sum_i (J_1 \times \ldots \times J_{i-1} \times \Theta_i \times J_{i+1} \times \ldots)$$

(cf. [3], p.416); as usual Θ gives rise to a principal polarization $\lambda_\Theta: J \longrightarrow J^t$; a morphism of polarized abelian varieties is a homomorphism which commutes with the polarizations.

4. Theorem. Let $0 \leqq g \leqq 3$, and suppose K is an algebraically closed field. An abelian variety X of dimension g over K which carries a principal polarization $\lambda: X \longrightarrow X^t$ is isomorphic with the canonically polarized Jacobian variety of a good curve: $(X,\lambda) \cong (\mathrm{Jac}(C),\lambda_\Theta)$.

5. Remark. The theorem is trivial for $g = 0$ and $g = 1$, well known for $g = 2$ (cf. [8], Satz 2, p.37), but we could not find a reference for the case $g = 3$ (cf. [8], p.38; the case $\mathrm{char}(K) = 0$ immediately follows from [3]).

Proof of the theorem. Let $\underline{M}_g$ be the coarse moduli scheme of good curves denoted by $\tilde{\underline{M}}$ in [5] and let $\underline{A}_g^{(1)}$ be the coarse moduli scheme of g-dimensional principally polarized abelian varieties (moduli schemes over the ring of integers in the sense of [4], p.99, p.139, Theorem 7.10, and p.143, Corollary 7.14). As was proved by analytic methods, $\underline{A}_g^{(1)} \otimes \mathbb{C}$ is irreducible (e.g. cf. [5], p.459); moreover, the image of the Jacobian mapping

$$j \otimes \mathbb{C}: \underline{M}_g \otimes \mathbb{C} \longrightarrow \underline{A}_g^{(1)} \otimes \mathbb{C}$$

is closed as was proved by Hoyt (cf. [3], Theorem 2); if K is a field and $0 \leq g \leq 3$,

$$\dim \underline{M}_g \otimes K = \dim \underline{A}_g^{(1)} \otimes K = \tfrac{1}{2}g(g+1)$$

(the left hand side equals 0, resp. 1 in case $g = 0$, resp. $g = 1$, and equals $3g-3$ if $g \geq 2$; the last equality follows from the Riemann symmetry conditions: $g^2 - \frac{1}{2}g(g-1) = \frac{1}{2}g(g+1)$; in case $\mathrm{char}(K)$ is arbitrary, use the fact that the coarse and fine moduli schemes $\underline{A}_{g,1,1}$ and $A_{g,1,n}$ have the same dimension (cf. [4], 7.3), and use the fact that separable polarizations yield smooth formal deformation spaces as was proved by Grothendieck (cf. [6], Theorem 2.4.1, and Theorem 2.3.3; note the words "and only if" have to be deleted in the cited Theorem 2.4.1); thus the theorem follows for fields of characteristic zero.

Let k be an algebraically closed field of characteristic $p \neq 0$, let (X_o, λ_o) be a principally polarized abelian variety over k. Then (cf. [6], Theorem 2.4.1) there exists an integral domain R of characteristic zero, a reduction homomorphism $R \to k$, and an abelian scheme $\underline{X}$ over $\mathrm{Spec}(R)$ with a (principal) polarization $\lambda: \underline{X} \to \underline{X}^t$, so that

$$(\underline{X}, \lambda) \otimes_R k \cong (X_o, \lambda_o);$$

we can choose R so that R is a discrete valuation ring with residue class field k and field of fractions K, field of characteristic zero, such that $(X, \lambda) = (\underline{X}, \lambda) \otimes_R K$ is isomorphic with the canonically polarized Jacobian variety of a good curve C over K if $g = \dim(X) = \dim(X_o) \leq 3$; thus the proof of the theorem is concluded using the following lemma ("a curve has good reduction if and only if its Jacobian has good reduction"):

6. Lemma. Let R be a discrete valuation ring, k its residue class field, K its field of fractions, $\underline{X}$ an abelian scheme over $\mathrm{Spec}(R)$ with a polarization λ, and C a good curve over K so that $(\underline{X}, \lambda) \otimes_R K = (\mathrm{Jac}(C), \lambda_\Theta)$. Then there exists a good curve $\underline{C}$ over $\mathrm{Spec}(R)$, and an isomorphism between $\underline{X}$ and $\underline{\mathrm{Pic}}^o(\underline{C})$ extending the given isomorphism $\underline{X} \otimes K \cong \mathrm{Jac}(C)$; moreover $(X_o, \lambda_o) := (\underline{X}, \lambda) \otimes_R k$ is isomorphic with the canonically polarized Jacobian of $\underline{C} \otimes_R k$.

Proof. Using "flat extensions" ("adherence", cf. EGA.IV[2], 2.8.5), we construct subschemes $\underline{C}$ and $\underline{\Theta}$ of $\underline{X}$ which are flat over $\mathrm{Spec}(R)$ with generic fibres $C, \Theta \subset X$; thus we can define

$$\alpha(\underline{C}, \underline{\Theta}) \in \mathrm{End}_R(\underline{X}),$$

(as in [3], p.415), which is possible because $\underline{X}$ is proper and smooth over Spec(R), and we deduce immediately that

$$\alpha(\underline{C}, \underline{\Theta}) \otimes_R K = \alpha(C, \Theta) \in \mathrm{End}_K(X);$$

as (X,λ) is the canonically polarized Jacobian variety of C, we know $\alpha(C,\Theta) = \mathrm{id}$ (cf. [3], Theorem 1.1), thus by rigidity (cf. [4], Corollary 6.2) we conclude $\alpha(\underline{C},\underline{\Theta}) = \mathrm{id}_{\underline{X}}$, thus again using the result of Hoyt, we conclude the last statement of the lemma; the first claim of the lemma now easily follows.

Q.E.D.

Alternative proof: As X has good reduction, C has stable reduction (cf. [1], Theorem 2.4), and it then readily follows C has good reduction: it is easily seen that the Jacobian variety of the normalization of C_0 maps onto X_0, moreover $\dim X_0 =$ = genus (C_0), and C_0 is a good curve if and only if its normalization has the same genus, C_0 being connected and reduced. We could also have used results of Raynaud (cf. [7]).

7. Remark. The Jacobi functor

$$j:\ \underline{M}_g \longrightarrow \underline{A}_g^{(1)}$$

is injective for geometric points in $\underline{M}_g$ corresponding to irreducible curves (Torelli theorem), it is injective for geometric points in case $g = 2$, but it is not injective if $g = 3$ (Torelli theorem does not hold for arbitrary good curves of genus 3, e.g. cf. P.A.Griffiths - Seminar on degeneration of algebraic varieties, p.15: "joining an elliptic curve and a curve of genus 2 by one varying point, the resulting curve with nodes varies, while the (already compact) Jacobian is unchanged").

8. Corollary. For $g \leq 3$, and for any field k the following moduli schemes over k are irreducible: $\underline{A}_{g,1,1}$ any $8n$, (cf. [4], pp.129 and 139), $\underline{A}_g^{(\delta)}$ and $\underline{A}_n^*$ (any $\delta = (\delta_1, \ldots, \delta_g)$, cf. [5], and $p = \mathrm{char}(k) \nmid \delta_g, n$).

Proof. The theorem specializes the main results of the paper by Deligne and Mumford (cf. [1], Section 3, and Theorem 5.15) into the statements of the corollary.

REFERENCES

1. P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus. Publ.Math. No. 36 (Volume dedicated to O. Zariski), IHES, 1969, pp.75-109.
2. A. Grothendieck and J. Dieudonné, Éléments de géometrie algébrique Chap. IV^2. Publ.Math. No. 24, IHES, 1965 (cited as EGA).
3. W.L. Hoyt, On the products and algebraic families of Jacobian varieties, Ann.Math. 77 (1963), pp.415-423.
4. D. Mumford, Geometric invariant theory. Ergebnisse der Math. Neue F. Bd.34, Springer Verlag, 1965.
5. D. Mumford, The structure of the moduli spaces of curves and abelian varieties. Actes, Congrès intern.math., 1970, Vol.I, pp.457-465.
6. F. Oort, Finite group schemes, local moduli for abelian varieties, and lifting problems. Comp.Math. 23 (1971), pp.265-296 (also: Algebraic geometry, Oslo, 1970, Wolters-Noordhoff, 1972).
7. M. Raynaud, Spécialisation du foncteur de Picard. Publ.Math. No.38, IHES, 1970, pp.27-76.
8. A. Weil, Zum Beweis des Torellischen Satzes. Nachr.Akad. Wissensch. Göttingen, Math.-Phys.Kl., 1957, pp.33-53.

/UE