SUMS OF EQUAL POWERS OF POSITIVE INTEGERS

VRIJE UNIVERSITEIT TE AMSTERDAM

SUMS OF EQUAL POWERS OF POSITIVE INTEGERS

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Ik bin ek in stjerlik minske, net oars as oaren,
út it laech fan 'e earste út ierde foarme minske;
en ik bin ek flesk, foarme yn 't liif fan in mem,
tsien moanne lang, yn har bloed is it wat wirden,
troch it sied fan in man en de súver gleone
nocht en wille dy't der mei mank giet.
By myn berte haw ik sykhelle as elkenien,
út deselde lucht, en bin ik ek tolânne
kommen op 'e wrâld, dy't ús allegearre draecht;
lykas by allegearre wie skriemen myn earste lûd;
yn 'e ruften bin ik mei soargen greatbrocht.
Gjin kening dy't oars bigoun is.

(nei DE WYSHEID fan SALOMO, VII)

HET KIND EN IK

Ik zou een dag uit vissen,
ik voelde mij moedeloos.
Ik maakte tussen de lissen
met de hand een wak in het kroos.

Er steeg licht op van beneden
uit de zwarte spiegelgrond.
Ik zag een tuin onbetreden
en een kind dat daar stond.

Het stond aan zijn schrijftafel
te schrijven op een lei.
Het woord onder de griffel,
herkende ik, was van mij.

Maar toen heeft het geschreven,
zonder haast en zonder schroom,
al wat ik van mijn leven
nog ooit te schrijven droom -

En telkens als ik even
knikte dat ik het wist,
liet hij het water beven
en het werd uitgewist.

(MARTINUS NIJHOFF)

Cast thy bread upon the waters:
for thou shalt find it after many days.
Give a portion to seven, and also to eight;
for thou knowest not what evil shall be
upon the earth.
If the clouds be full of rain, they
empty themselves upon the earth:
and if the tree fall toward the south,
or toward the north, in the place where the
tree falleth, there it shall be.
He that observeth the wind shall not sow;
and he that regardeth the clouds shall not reap.
As thou knowest not what is the way of the spirit,
nor how the bones do grow in the womb of her
that is with child: even so thou knowest not
the works of God who maketh all.
In the morning sow thy seed, and in the
evening withhold not thine hand: for thou
knowest not whether shall prosper, either this
or that, or whether they both shall be alike good.
Truly the light is sweet, and a pleasant thing
it is for the eyes to behold the sun:
But if a man live many years, and rejoice in them
all; yet let him remember the days of darkness;
for they shall be many. All that cometh is vanity.
Rejoice, O young man, in thy youth; and let thy
heart cheer thee in the days of thy youth, and
walk in the ways of thine heart, and in the sight
of thine eyes: but know thou, that for all these
things God will bring thee into judgment.
Therefore remove sorrow from thy heart, and
put away evil from thy flesh:
for childhood and youth are vanity.

(ECCLESIASTES, XI)

CHAPTER 1

INEQUALITIES RELATED TO MONOTONIC APPROXIMATION OF SOME INTEGRALS

0. INTRODUCTION

We begin by giving an illustrative example (just one out of many) which led us to the subject dealt with in this chapter.
For any positive integer n consider the regular n-gon $P_1, \ldots, P_n$, where

$$(0.1) \qquad P_k := \exp(\tfrac{k}{n}\, 2\pi i), \qquad (k = 1, \ldots, n).$$

Let d_k be the distance from P_k to P_n, i.e.

$$(0.2) \qquad d_k := |P_k - 1|$$

and let the average of these distances be denoted by D_n, i.e.

$$(0.3) \qquad D_n := \frac{1}{n} \sum_{k=1}^{n} d_k .$$

Then the sequence $\{D_n\}_{n=1}^{\infty}$ tends *increasingly* to its limit $(= \frac{4}{\pi})$. Indeed, since

$$(0.4) \qquad D_n = \frac{1}{n} \sum_{k=1}^{n} |\exp(\tfrac{k}{n}\, 2\pi i) - 1| =$$

$$= \frac{1}{n} \sum_{k=1}^{n} |(\cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}) - 1| =$$

$$= \frac{1}{n} \sum_{k=1}^{n} (2 - 2 \cos \frac{2\pi k}{n})^{\frac{1}{2}} = \frac{2}{n} \sum_{k=1}^{n} \sin \frac{\pi k}{n}$$

it is clear that

$$(0.5) \qquad \lim_{n \to \infty} D_n = 2 \int_0^1 \sin \pi x \, dx = \frac{4}{\pi}.$$

In order to show that $\{D_n\}_{n=1}^{\infty}$ is *increasing* we recall that

$$(0.6) \qquad \sum_{k=1}^{m} \sin kz = \frac{\sin \frac{mz}{2} \sin \frac{(m+1)z}{2}}{\sin \frac{z}{2}}$$

by means of which it is readily seen that

$$(0.7) \qquad D_n = \frac{2}{n} \frac{\cos \frac{\pi}{2n}}{\sin \frac{\pi}{2n}} = \frac{4}{\pi} \frac{\frac{\pi}{2n}}{\tan \frac{\pi}{2n}} .$$

It is easily verified that the function $\phi : (0, \frac{\pi}{2}] \to \mathbb{R}$, defined by

$$(0.8) \qquad \phi(t) := \frac{t}{\tan t}, \qquad (0 < t < \tfrac{\pi}{2}); \qquad \phi(\tfrac{\pi}{2}) := 0$$

is *decreasing* and since

$$(0.9) \qquad D_n = \frac{4}{\pi} \phi(\tfrac{\pi}{2n})$$

the assertion follows.

The subject of this chapter was inspired by the observation that D_n may also be written as

$$(0.10) \qquad D_n = \frac{2}{n} \sum_{k=1}^{n} \frac{1}{2}\left(\sin \frac{k-1}{n}\pi + \sin \frac{k}{n}\pi\right)$$

which is equivalent to saying that $\frac{1}{2}D_n$ is the n-th *canonical trapezoidal approximation* of the integral $\int_0^1 \sin \pi x \, dx$. Thus we have proved that the sequence of canonical trapezoidal approximations of $\int_0^1 \sin \pi x \, dx$ is *increasing*. One might feel that this fact is not too much of a surprise since $\sin \pi x$ is *concave* for $0 \leq x \leq 1$.
However, from an example such as

$$(0.11) \qquad f(x) := 1 - |x| \;, \qquad (-1 \leq x \leq 1)$$

we infer that concavity of f is by no means a sufficient condition for the occurrence of this monotonicity-phenomenon. In this example f is concave but the corresponding sequence of canonical trapezoidal approximations of $\int_{-1}^{1} f(x)\, dx$ is *oscillating*.
These observations led us to the following

QUESTION: *For which (continuous) functions* $f : [0,1] \to \mathbb{R}$, *say, is the sequence* $\{T_n(f)\}_{n=1}^{\infty}$, *defined by*

$$(0.12) \qquad T_n(f) := \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2}\left(f\left(\frac{k-1}{n}\right) + f\left(\frac{k}{n}\right)\right) \;, \qquad (n \in \mathbb{N})$$

monotonic ?

It seems that there is no simple general answer to this question. Therefore we first restrict ourselves to some important special functions, viz. f_s and f_s^*, defined by

$$(0.13) \qquad f_s(x) := x^s \;, \qquad (0 \leq x \leq 1;\; s \in \mathbb{R}^+ \;,\; s \text{ fixed})$$

and

$$(0.14) \qquad f_s^*(x) := |x|^s \;, \qquad (-1 \leq x \leq 1;\; s \in \mathbb{N},\; s \geq 2,\; s \text{ fixed}).$$

In Section 1 it will be shown that $T_n := T_n(s) := T_n(f_s)$ is *increasing* for any fixed $s \in (0,1)$ and that $T_n(s)$ is *decreasing* if $s > 1$.

In Section 2 we will show that $T_n^* := T_n^*(s) := T_n(f_s^*)$ is *decreasing* for any fixed *integer* $s \geq 2$.

We think that there is little or no doubt that these statements could very well have been conjectured by any reasonably sophisticated highschool student. However, we have not been able to furnish any "really simple" proofs (compare Chapter 2). All our proofs are based upon some new *inequalities* for the sums

$$(0.15) \qquad \sigma_n(s) := \sum_{k=1}^{n} k^s \;, \qquad (n \in \mathbb{N})$$

and (also for $n \in \mathbb{N}$)

$$(0.16) \qquad \phi_n(s) := \sum_{k=0}^{n-1} (-1)^k (n-k)^s =$$

$$= n^s - (n-1)^s + \ldots + (-1)^{n-2}\, 2^s + (-1)^{n-1}$$

some of the nicest (though not necessarily the best) ones being

$$(0.17) \qquad \sigma_n(s) > \tfrac{1}{2}\frac{n^s(n+1)^{s+1} + n^{s+1}(n+1)^s}{(n+1)^{s+1} - n^{s+1}}, \qquad (s > 1)$$

and

$$(0.18) \qquad \phi_n(s) > \tfrac{1}{2}\frac{n^s(n+1)^{s+1} + n^{s+1}(n+1)^s}{(n+1)^{s+1} + n^{s+1}}, \qquad (s \in \mathbb{N},\ s \geq 2).$$

Various applications of the results obtained in this chapter will be given in Chapters 3 and 4. In Chapter 2 we will discuss some more difficult rather academic aspects of the topics dealt with in the present chapter.

1. MONOTONIC APPROXIMATIONS OF $\int_0^1 x^s\,dx$

For $n \in \mathbb{N} := \{1,2,3, \ldots\}$ and $s \in \mathbb{C}$ we define

$$(1.1) \qquad \sigma_n(s) := \sum_{k=1}^{n} k^s$$

although in this chapter we will primarily be interested in the case $s > 0$.

Comparing $\frac{\sigma_n(s)}{n^{s+1}} = \frac{1}{n}\sum_{k=1}^{n} (\frac{k}{n})^s$ with $\int_0^1 x^s\,dx$, one readily finds the inequalities

$$(1.2) \qquad \frac{n^{s+1}}{s+1} < \sigma_n(s) < \frac{n^{s+1}}{s+1} + n^s, \qquad (s > 0).$$

Similarly, considering the *canonical trapezoidal approximations* of $\int_0^1 x^s\,dx$ we easily obtain

$$(1.3) \qquad \sigma_n(s) < \frac{n^{s+1}}{s+1} + \frac{n^s}{2}, \qquad (0 < s < 1)$$

and

$$(1.4) \qquad \sigma_n(s) > \frac{n^{s+1}}{s+1} + \frac{n^s}{2}, \qquad (s > 1)$$

the case $s = 1$ being trivial: $\sigma_n(1) = n(n+1)/2$.

Next we have the less trivial

PROPOSITION 1.1.

$$(1.5) \qquad \frac{n^{s+1}\,(n+1)^s}{(n+1)^{s+1} - n^{s+1}} < \sigma_n(s) < \frac{n^s\,(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \qquad (s > 0).$$

PROOF. We first show (by induction with respect to n) that

$$(1.6) \qquad \sigma_n(s) > \frac{n^{s+1}\ (n+1)^s}{(n+1)^{s+1} - n^{s+1}}, \qquad (s > 0).$$

This inequality is easily seen to be true for $n = 1$ and all $s > 0$. Assume that (1.6) is still true for $n = 1, \ldots, N$ and all $s > 0$. Then we have

$$(1.7) \qquad \sigma_{N+1}(s) = (N+1)^s + \sigma_N(s) > (N+1)^s + \frac{N^{s+1}\ (N+1)^s}{(N+1)^{s+1} - N^{s+1}}$$

so that it suffices to show that

$$(1.8) \qquad (N+1)^s + \frac{N^{s+1}\ (N+1)^s}{(N+1)^{s+1} - N^{s+1}} \geqq \frac{(N+1)^{s+1}\ (N+2)^s}{(N+2)^{s+1} - (N+1)^{s+1}}$$

or

$$(1.9) \qquad (N+1)^s (N+2)^{s+1} - (N+1)^{2s+1} \geqq (N+1)^{s+1}\ (N+2)^s - N^{s+1}\ (N+2)^s$$

for all $N \in \mathbb{N}$ and all $s > 0$. Putting $x := 1/(N+1)$ we thus want to prove that

$$(1.10) \qquad (1+x)^{s+1} - 1 \geqq (1+x)^s - (1-x)^{s+1}\ (1+x)^s$$

i.e.

$$(1.11) \qquad \frac{(1+x)^{s+1} - 1}{x} \geqq \frac{1 - (1-x^2)^{s+1}}{x^2}.$$

Since for any (fixed) $s > 0$ the function x^{s+1} is *convex* on $\mathbb{R}^+$, (1.11) is true for *all* $x \in (0,1)$, completing the proof of (1.6).

Next we show that

$$(1.12) \qquad \sigma_n(s) < \frac{n^s\ (n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \qquad (s > 0).$$

One may verify directly that (1.12) is true for $n = 1$ and all $s > 0$. Assume that (1.12) is still true for $n = 1, \ldots, N$ and all $s > 0$. Then

$$(1.13) \qquad \sigma_{N+1}(s) = (N+1)^s + \sigma_N(s) < (N+1)^s + \frac{N^s\ (N+1)^{s+1}}{(N+1)^{s+1} - N^{s+1}}$$

so that it suffices to show that

$$(1.14) \qquad (N+1)^s + \frac{N^s\ (N+1)^{s+1}}{(N+1)^{s+1} - N^{s+1}} \leqq \frac{(N+1)^s\ (N+2)^{s+1}}{(N+2)^{s+1} - (N+1)^{s+1}}$$

for all $N \in \mathbb{N}$ and all $s > 0$. Again putting $x := 1/(N+1)$ we easily find that (1.14) is equivalent to

$$(1.15) \qquad \frac{(1-x^2)^{s+1} - (1-x)^{s+1}}{x(1-x)} \leqq \frac{1 - (1-x^2)^{s+1}}{x^2}.$$

Now observe that $(1-x^2) - (1-x) = x(1-x)$, that $0 < 1-x < 1-x^2 < 1$ and that for any (fixed) $s > 0$ the function x^{s+1} is *convex* on $\mathbb{R}^+$. It follows that (1.15) is true indeed, completing the proof. □

Defining

$$(1.16) \qquad U_n(s) := \frac{\sigma_n(s)}{n^{s+1}}, \qquad (n \in \mathbb{N};\ s \in \mathbb{C})$$

and

$$(1.17) \qquad L_n(s) := \frac{\sigma_n(s) - n^s}{n^{s+1}}, \qquad (n \in \mathbb{N};\ s \in \mathbb{C})$$

we have

PROPOSITION 1.2.1. *If* $s > 0$ *then the upper Riemann sum* $U_n(s)$ *is decreasing as a function of* $n \in \mathbb{N}$.

PROOF. In order to show that

$$(1.18) \qquad U_n(s) > U_{n+1}(s), \qquad (s > 0)$$

we may just as well show that

$$(1.19) \qquad \frac{\sigma_n(s)}{n^{s+1}} > \frac{\sigma_{n+1}(s)}{(n+1)^{s+1}}, \qquad (s > 0)$$

or

$$(1.20) \qquad (n+1)^{s+1}\,\sigma_n(s) > n^{s+1}\{(n+1)^s + \sigma_n(s)\}, \qquad (s > 0)$$

or

$$(1.20a) \qquad \sigma_n(s) > \frac{n^{s+1}\,(n+1)^s}{(n+1)^{s+1} - n^{s+1}}, \qquad (s > 0).$$

Invoking Proposition 1.1 we are done. □

Similarly, the inequality

$$(1.21) \qquad L_n(s) < L_{n+1}(s), \qquad (s > 0)$$

is equivalent to

$$(1.21a) \qquad \sigma_n(s) < \frac{n^s\,(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \qquad (s > 0)$$

so that we also have

PROPOSITION 1.2.2. *If* $s > 0$ *then the lower Riemann sum* $L_n(s)$ *is increasing (in* n*).* □

Hence, Propositions 1.2.1 and 1.2.2 together form a *restatement* of Proposition 1.1.

REMARK. For other proofs see the Problem Section of the Nieuw Archief voor Wiskunde, Vol. 23, Nr. 3 (1975) pp. 254-256. Also see [38]. In the first reference van LINT proves the following generalization: *If* $f : [a,b] \to \mathbb{R}$ *is monotonic and either convex or concave on* $[a,b]$ *then the corresponding sequence of canonical upper (lower) Riemann sums is decreasing (increasing).*

In view of this generalization it should be noted that $\{T_n(f)\}_{n=1}^{\infty}$ is not necessarily monotonic for a convex monotonic function $f:[a,b] \to \mathbb{R}$.

As an example one may take an increasing convex function whose graph consists of two (straight) line segments joined at the point $x = \frac{a+b}{2}$. Compare example (0.11).

Defining

(1.22) $\quad T_n(s) := \frac{1}{2}\{U_n(s) + L_n(s)\}$, $\quad (n \in \mathbb{N};\ s \in \mathbb{C})$

we see that if $s > 0$ then $T_n(s)$ is the n-th *canonical trapezoidal approximation* of $\int_0^1 x^s\,dx$. Concerning $T_n(s)$ we have

THEOREM 1.1. *If* $0 < s < 1$ $(s > 1)$ *then* $T_n(s)$ *is increasing (decreasing) as a function of* $n \in \mathbb{N}$.

PROOF.

Case 1. $\quad s > 1$

We have to prove that

(1.23) $\quad T_n(s) > T_{n+1}(s)$, $\quad (s > 1)$.

Since

(1.24) $$2T_n(s) = \frac{2\sigma_n(s) - n^s}{n^{s+1}}$$

we may just as well show that

(1.25) $$\frac{2\sigma_n(s) - n^s}{n^{s+1}} > \frac{2\sigma_{n+1}(s) - (n+1)^s}{(n+1)^{s+1}}, \quad (s > 1)$$

or

(1.26) $$2\sigma_n(s) > \frac{n^{s+1}(n+1)^s + n^s(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \quad (s > 1).$$

As before, we proceed by induction. If $n = 1$ we check whether

(1.27) $$2 > \frac{2^s + 2^{s+1}}{2^{s+1} - 1}, \quad (s > 1)$$

or

(1.28) $$2^s > 2, \quad (s > 1).$$

It follows that (1.26) is true for $n = 1$ and all $s > 1$. Assume that (1.26) is still true for $n = 1, \ldots, N$ and all $s > 1$. Then

(1.29) $$2\sigma_{N+1}(s) = 2(N+1)^s + 2\sigma_N(s) > 2(N+1)^s + \frac{N^{s+1}(N+1)^s + N^s(N+1)^{s+1}}{(N+1)^{s+1} - N^{s+1}}$$

so that it suffices to show that

(1.30) $$2(N+1)^s + \frac{N^{s+1}(N+1)^s + N^s(N+1)^{s+1}}{(N+1)^{s+1} - N^{s+1}} \geqq \frac{(N+1)^{s+1}(N+2)^s + (N+1)^s(N+2)^{s+1}}{(N+2)^{s+1} - (N+1)^{s+1}}$$

for all $n \in \mathbb{N}$ and all $s > 1$.

After some simplifications we see that we may just as well prove that

(1.31) $$\frac{N^s + 2(N+1)^{s+1}}{(N+1)^{s+1} - N^{s+1}} \geqq \frac{(N+1)(N+2)^s + (N+2)^{s+1}}{(N+2)^{s+1} - (N+1)^{s+1}}$$

or (as before $x := 1/(N+1)$)

$$(1.32)\qquad \frac{x(1-x)^s+2}{1-(1-x)^{s+1}} \geqq \frac{(1+x)^s+(1+x)^{s+1}}{(1+x)^{s+1}-1}$$

which may be written as

$$(1.33)\qquad 2(1-x^2)^s + x((1+x)^s - (1-x)^s) - 2 \geqq 0 .$$

In case s is an *integer* greater than 1 we may prove (1.33) as follows. Using the binomial theorem, we write the left-hand side of (1.33) as

$$(1.34)\qquad 2\sum_{r=0}^{s} \binom{s}{r}(-1)^r x^{2r} + x\sum_{r=0}^{s} \binom{s}{r} x^r (1-(-1)^r) - 2 =$$

$$= 2\{\sum_{r=1}^{\infty} \binom{s}{r}(-1)^r x^{2r} + \sum_{r=1}^{\infty} \binom{s}{2r-1} x^{2r}\}.$$

Now replace x^2 by z, $0 < z < 1$, so that it is enough to show that

$$(1.35)\qquad \sum_{r=1}^{\infty} \binom{s}{r}(-1)^r z^r + \sum_{r=1}^{\infty} \binom{s}{2r-1} z^r =$$

$$= \sum_{r=1}^{\infty} \binom{s}{2r} z^{2r} + \sum_{r=1}^{\infty} \binom{s}{2r-1}(z^r - z^{2r-1}) \geqq 0$$

for all $z \in (0,1)$. Since $0 < z < 1$ we have

$$(1.36)\qquad z^r - z^{2r-1} \geqq 0$$

for all $r \in \mathbb{N}$ and since all binomial coefficients in (1.35) are non-negative, the proof of (1.23) is complete in case $s \in \mathbb{N}$, $s \geqq 2$.

In order to prove Theorem 1.1 for a general $s > 1$ we consider two cases.

Case 1.a. $1 < s < 2$

Observe that (1.33) is equivalent to

$$(1.37)\qquad x\,\frac{(1+x)^s - (1-x)^s}{2} \geqq 1 - (1-x^2)^s, \qquad (0 < x < 1)$$

which may also be written as

$$(1.38)\qquad \frac{x}{2}\sum_{r=0}^{\infty} \binom{s}{r} x^r(1-(-1)^r) \geqq 1 - \sum_{r=0}^{\infty} \binom{s}{r}(-1)^r x^{2r} .$$

This last inequality is equivalent to

$$(1.39)\qquad \sum_{r=1}^{\infty} \binom{s}{2r-1} x^{2r} + \sum_{r=1}^{\infty} \binom{s}{r}(-1)^r x^{2r} \geqq 0$$

or, putting $x^2 = z$, $0 < z < 1$,

$$(1.40)\qquad \sum_{r=2}^{\infty} \{\binom{s}{2r-1} + (-1)^r \binom{s}{r}\} z^r \geqq 0 , \qquad (0 < z < 1).$$

Clearly (1.40) may be written as

$$(1.41)\qquad \sum_{r=1}^{\infty} \{\binom{s}{4r-1} + \binom{s}{2r}\} z^{2r} + \sum_{r=1}^{\infty} \{\binom{s}{4r+1} - \binom{s}{2r+1}\} z^{2r+1} \geqq 0.$$

Now observe that if $1 < s < 2$ then

(1.42) $\quad \binom{s}{2r} > 0 \quad$ for all $r \in \mathbb{N}$

and

(1.43) $\quad \binom{s}{2r+1} < 0 \quad$ for all $r \in \mathbb{N}$.

Moreover, it is easily seen that $|\binom{s}{r}|$ is decreasing as a function of r. Hence, for all $r \in \mathbb{N}$,

(1.44) $\quad \binom{s}{4r-1} + \binom{s}{2r} = \binom{s}{2r} - |\binom{s}{4r-1}| > 0$

and

(1.45) $\quad \binom{s}{4r+1} - \binom{s}{2r+1} = |\binom{s}{2r+1}| - |\binom{s}{4r+1}| > 0$

and the proof of Case 1.a is complete.

Case 1.b. $\quad s > 2$

Observe that

(1.46) $\quad (1-x^2)^s > 1 - sx^2 , \qquad (0 < x < 1;\ s > 1).$

Hence, in order to prove (1.33) it is sufficient to show that

(1.47) $\quad 2(1-sx^2) + x((1+x)^s - (1-x)^s) - 2 \geqq 0$

or

(1.48) $\quad (1+x)^s - (1-x)^s - 2sx \geqq 0.$

The left-hand side of (1.48) takes the value 0 at $x = 0$. Hence, it suffices to show that its derivative is positive for $0 < x < 1$, which amounts to proving that

(1.49) $\quad (1+x)^{s-1} + (1-x)^{s-1} - 2 > 0 , \qquad (0 < x < 1).$

Since (1.49) may be rewritten as

(1.50) $\quad \dfrac{(1+x)^{s-1} - 1}{x} > \dfrac{1 - (1-x)^{s-1}}{x} , \qquad (0 < x < 1)$

and, since x^{s-1} is convex for $x > 0$ if $s > 2$, we see that (1.50) is true, completing the proof of Case 1.b.

Case 2. $\quad 0 < s < 1$

In order to prove that $T_n(s) < T_{n+1}(s)$ for $0 < s < 1$ it suffices to show that

(1.51) $\quad 2(1-x^2)^s + x\{(1+x)^s - (1-x)^s\} - 2 \leqq 0 , \qquad (0 < x < 1).$

This inequality may be established similarly as (1.47) in Case 1. Just observe that if $0 < s < 1$ then for all $r \in \mathbb{N}$

(1.52) $\quad \binom{s}{2r} < 0 \quad$ and $\quad \binom{s}{2r-1} > 0$

whereas $|\binom{s}{r}|$ is again decreasing in r. This completes the proof of Theorem 1.1. □

REMARK. For another proof see the Problem Section of the Nieuw Archief voor Wiskunde, Vol. 23, Nr. 3 (1975) pp. 256-257.

From the above proof we immediately obtain

THEOREM 1.2.

$$(1.53) \qquad 2\sigma_n(s) > \frac{n^{s+1}(n+1)^s + n^s(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \qquad (s > 1)$$

and

$$(1.54) \qquad 2\sigma_n(s) < \frac{n^{s+1}(n+1)^s + n^s(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \qquad (0 < s < 1).$$

REMARK. It has been suggested to study the behaviour of $T_n(s)$ by means of the Euler-Maclaurin summation formula. In our attempt to follow up this suggestion it turned out that the use of this summation formula is *not* suitable for our purpose. For details we refer to [28; pp.13-17].

For later use we now derive some estimates for

$$(1.55) \qquad w_n(s) := \sum_{k=1}^{n} (2k-1)^s, \qquad (n \in \mathbb{N};\ s > 0).$$

Defining

$$(1.56) \qquad t_n(s) := \frac{2w_n(s)}{(2n)^{s+1}}, \qquad (n \in \mathbb{N};\ s > 0)$$

and observing that $t_n(s)$ may be interpreted as the n-th *canonical tangential approximation* of $\int_0^1 x^s\,dx$, we immediately obtain the inequalities

$$(1.57) \qquad 2\,w_n(s) > \frac{(2n)^{s+1}}{s+1}, \qquad (0 < s < 1)$$

and

$$(1.58) \qquad 2\,w_n(s) < \frac{(2n)^{s+1}}{s+1}, \qquad (s > 1).$$

We will now prove some better estimates for $w_n(s)$ in order to obtain

PROPOSITION 1.3. *If* $0 < s < 1$ $(s > 1)$ *then* $t_n(s)$ *is decreasing (increasing) as a function of* $n \in \mathbb{N}$.

PROOF.

Case 1. $0 < s < 1$

We have to prove that

$$(1.59) \qquad t_n(s) > t_{n+1}(s), \qquad (0 < s < 1).$$

One may verify that (1.59) is equivalent to

$$(1.60) \qquad w_n(s) > \frac{n^{s+1}(2n+1)^s}{(n+1)^{s+1} - n^{s+1}}, \qquad (0 < s < 1).$$

Again we proceed by induction. For n = 1 inequality (1.60) reads

$$(1.61)\qquad 1 > \frac{3^s}{2^{s+1}-1}, \qquad (0 < s < 1)$$

which may also be written as

$$(1.62)\qquad \frac{1^s + 3^s}{2} < 2^s, \qquad (0 < s < 1).$$

Since x^s is *concave* on the interval $1 \leq x \leq 3$ it follows that (1.62) is true, so that (1.60) is true for n = 1 and all $s \in (0,1)$. Assume that (1.60) is still true for $n = 1,\dots,N$ and all $s \in (0,1)$. Then

$$(1.63)\qquad w_{N+1}(s) = (2N+1)^s + w_N(s) >$$

$$> (2N+1)^s + \frac{N^{s+1}(2N+1)^s}{(N+1)^{s+1} - N^{s+1}} = \frac{(N+1)^{s+1}(2N+1)^s}{(N+1)^{s+1} - N^{s+1}}$$

so that it suffices to prove that

$$(1.64)\qquad \frac{(N+1)^{s+1}(2N+1)^s}{(N+1)^{s+1} - N^{s+1}} \geqq \frac{(N+1)^{s+1}(2N+3)^s}{(N+2)^{s+1} - (N+1)^{s+1}}$$

or

$$(1.65)\qquad \frac{(2N+1)^s}{(N+1)^{s+1} - N^{s+1}} \geqq \frac{(2N+3)^s}{(N+2)^{s+1} - (N+1)^{s+1}}$$

for all $N \in \mathbb{N}$ and all $s \in (0,1)$.
Putting $x := 1/(2N+3)$ we find that (1.65) is equivalent to

$$(1.66)\qquad \frac{(1-2x)^s}{(1-x)^{s+1} - (1-3x)^{s+1}} \geqq \frac{1}{(1+x)^{s+1} - (1-x)^{s+1}}$$

which may be written as

$$(1.67)\qquad \frac{(1+x)^{s+1} - (1-x)^{s+1}}{2x} \geqq \frac{\left(1+\frac{x}{1-2x}\right)^{s+1} - \left(1-\frac{x}{1-2x}\right)^{s+1}}{\frac{2x}{1-2x}}.$$

Now observe that

$$(1.68)\qquad \frac{(1+x)^{s+1} - (1-x)^{s+1}}{2x} = \sum_{r=0}^{\infty} \binom{s+1}{2r+1} x^{2r} = s + 1 + \sum_{r=1}^{\infty} \binom{s+1}{2r+1} x^{2r}$$

and

$$(1.69)\qquad \binom{s+1}{2r+1} < 0 \quad \text{for all } r \in \mathbb{N} \text{ and all } s \in (0,1)$$

so that (1.68) is concave for $0 \leq x < 1$.
Since $\frac{x}{1-2x} > x$ for $0 < x < \frac{1}{2}$ it follows that (1.67) is true.

Case 2. $s > 1$

Now we have to show that

$$(1.70)\qquad t_n(s) < t_{n+1}(s), \qquad (s > 1)$$

or

$$(1.71)\qquad w_n(s) < \frac{n^{s+1}(2n+1)^s}{(n+1)^{s+1} - n^{s+1}}, \qquad (s > 1).$$

For $n = 1$ this reads

$$(1.72) \qquad 1 < \frac{3^s}{2^{s+1} - 1} , \qquad (s > 1)$$

or

$$(1.73) \qquad 2^s < \frac{1^s + 3^s}{2} , \qquad (s > 1)$$

which is true because of the *convexity* of x^s on the interval $1 \leq x \leq 3$. Hence, (1.71) is true for $n = 1$ and all $s > 1$. Assume that (1.71) is still true for $n = 1, \ldots, N$ and all $s > 1$. Then

$$(1.74) \qquad w_{N+1}(s) = (2N + 1)^s + w_N(s) <$$
$$< (2N+1)^s + \frac{N^{s+1}(2N+1)^s}{(N+1)^{s+1} - N^{s+1}} = \frac{(N+1)^{s+1}(2N+1)^s}{(N+1)^{s+1} - N^{s+1}}$$

and it clearly suffices to show that

$$(1.75) \qquad \frac{(N+1)^{s+1}(2N+1)^s}{(N+1)^{s+1} - N^{s+1}} \leq \frac{(N+1)^{s+1}(2N+3)^s}{(N+2)^{s+1} - (N+1)^{s+1}}$$

for all $N \in \mathbb{N}$ and all $s > 1$.

It follows that we will be through as soon as we have shown that the function

$$(1.76) \qquad \frac{(2x+1)^s}{(x+1)^{s+1} - x^{s+1}} , \qquad (x > 0)$$

is *increasing* in x for any (fixed) $s > 1$.

Considering the derivative of (1.76) we see that it suffices to prove that

$$(1.77) \qquad (s+1)(1+2x)\{(1+x)^s - x^s\} < 2s\{(1+x)^{s+1} - x^{s+1}\}, \quad (x > 0;\ s > 1).$$

In order to prove (1.77) replace x by $\frac{t}{1-t}$, $0 < t < 1$, yielding

$$(1.78) \qquad (s+1)(1+t)(1-t^s) < 2s(1-t^{s+1}) , \qquad (0 < t < 1;\ s > 1)$$

or, equivalently,

$$(1.79) \qquad (s-1) - (s+1)t + (s+1)t^s - (s-1)t^{s+1} > 0 , \quad (0 < t < 1;\ s > 1).$$

The left-hand side of (1.79) takes the value 0 for $t = 1$ so that it suffices to show that its derivative is negative for $0 < t < 1$. Hence, we want to show that

$$(1.80) \qquad -(s+1) + s(s+1)t^{s-1} - (s-1)(s+1)t^s < 0 , \qquad (0 < t < 1)$$

or

$$(1.81) \qquad -1 + st^{s-1} - (s-1)t^s < 0 , \qquad (0 < t < 1;\ s > 1).$$

The left-hand side of (1.81) takes the value 0 at $t = 1$ so that it is enough to show that its derivative is positive for $0 < t < 1$. Hence, we want to show that

$$(1.82) \qquad s(s-1)t^{s-2} - s(s-1)t^{s-1} > 0 , \qquad (0 < t < 1;\ s > 1)$$

or

(1.83) $t^{-1} > 1$, $(0 < t < 1;\ s > 1)$.

Since this is trivially true, the proof of Proposition 1.3 is complete. □

From the above proof we immediately obtain

PROPOSITION 1.4.

(1.84) $$w_n(s) > \frac{n^{s+1}\ (2n+1)^s}{(n+1)^{s+1} - n^{s+1}}, \quad (0 < s < 1)$$

and

(1.85) $$w_n(s) < \frac{n^{s+1}\ (2n+1)^s}{(n+1)^{s+1} - n^{s+1}}, \quad (s > 1).$$

2. MONOTONIC APPROXIMATIONS OF $\int_{-1}^{1} |x|^s\, dx$

(!) Throughout this section s will denote an arbitrary though *fixed integer greater than* 1 unless stated otherwise.

DEFINITION. Let $T_n^*(s)$ $(t_n^*(s))$ be the n-th *canonical trapezoidal (tangential) approximation* of $\int_{-1}^{1} |x|^s\, dx$, i.e., more explicitly,

(2.1) $$T_n^*(s) := \frac{2}{n} \sum_{k=1}^{n} \tfrac{1}{2}\{|-1+(k-1)\tfrac{2}{n}|^s + |-1+k\tfrac{2}{n}|^s\} = \frac{2}{n} \sum_{k=1}^{n} |-1+\tfrac{2k}{n}|^s$$

and

(2.2) $$t_n^*(s) := \frac{2}{n} \sum_{k=1}^{n} |-1+\tfrac{2k-1}{n}|^s .$$

PROPOSITION 2.1.

(2.3) $$T_n^*(s) + t_n^*(s) = 4\ T_n(s)$$

where $T_n(s)$ has the same meaning as in (1.22).

PROOF. (compare (1.24))

(2.4) $$T_n^*(s) + t_n^*(s) = \frac{2}{n^{s+1}} \sum_{k=1}^{n} \{|n-2k|^s + |n+1-2k|^s\} = \frac{2}{n^{s+1}}\{2\sigma_n(s) - n^s\} = 4T_n(s).$$

Combining this result with Theorem 1.1 we obtain

PROPOSITION 2.2. $T_n^*(s) + t_n^*(s)$ *is decreasing (in* n*).*

DEFINITION. For $n \in \mathbb{N}$ let

(2.5) $$\phi_n(s) := \sum_{k=0}^{n-1} (-1)^k (n-k)^s = n^s - (n-1)^s + \ldots + (-1)^{n-2}\, 2^s + (-1)^{n-1}$$

and

(2.6) $$\delta_n^*(s) := T_n^*(s) - t_n^*(s) .$$

From (2.5) it is clear that

(2.7) $$\phi_n(s) + \phi_{n+1}(s) = (n+1)^s , \quad (n \in \mathbb{N}) .$$

Moreover, we have

PROPOSITION 2.3.

$$(2.8)\qquad \delta_n^*(s) = \frac{2}{n^{s+1}} \{2\phi_n(s) - n^s\} .$$

PROOF.

$$(2.9)\qquad \delta_n^*(s) = \frac{2}{n^{s+1}} \{ \sum_{k=1}^{n} |n-2k|^s - \sum_{k=1}^{n} |n+1-2k|^s\} = \frac{2}{n^{s+1}} \{2\phi_n(s) - n^s\} . \quad \square$$

As a counterpart of Proposition 2.2 we have

PROPOSITION 2.4. $\delta_n^*(s)$ *is decreasing (in* n*).*

PROOF. In order to prove that

$$(2.10)\qquad \delta_n^*(s) > \delta_{n+1}^*(s)$$

we may just as well prove that

$$(2.11)\qquad \frac{1}{n^{s+1}}\{ 2\phi_n(s) - n^s\} > \frac{1}{(n+1)^{s+1}}\{ 2\phi_{n+1}(s) - (n+1)^s\}$$

or

$$(2.12)\qquad (n+1)^{s+1} \{2\phi_n(s) - n^s\} > n^{s+1} \{(n+1)^s - 2\phi_n(s)\}$$

which may be written as

$$(2.13)\qquad 2\,\phi_n(s) > \frac{n^{s+1}\ (n+1)^s + n^s\ (n+1)^{s+1}}{(n+1)^{s+1} + n^{s+1}} .$$

A "direct" proof of (2.13) by induction seems to be practically unfeasible. Therefore we make the following "detour". Observe that

$$(2.14)\qquad \frac{2n^s\ (n+1)^s}{(n+1)^s + n^s} > \frac{n^{s+1}\ (n+1)^s + n^s\ (n+1)^{s+1}}{(n+1)^{s+1} + n^{s+1}} .$$

Indeed, putting $x := 1/n$, we see that (2.14) is equivalent to

$$(2.15)\qquad 2\{(1+x)^{s+1} + 1\} > (2+x)\{(1+x)^s + 1\}$$

which may be simplified to

$$(2.16)\qquad (1+x)^s > 1.$$

It follows that Proposition 2.4 is a consequence of

PROPOSITION 2.5.

$$(2.17)\qquad \frac{n^s(n+1)^s}{n^s + (n+1)^s} < \phi_n(s) \leqq \frac{n^{2s}}{(n-1)^s + n^s} , \qquad (s \in \mathbb{N},\ s \geqq 2).$$

REMARK. Note that (for any small enough positive ε) the first inequality in (2.17) is not true for $n = 2$, $s = 1+\varepsilon$ and that the second one is false for $n = 3$, $s = 1+\varepsilon$, so that (2.17) is *not* true for all real $s > 1$.

PROOF. One may verify that

$$(2.18)\qquad \phi_n(2) = \frac{n(n+1)}{2}$$

and

$$(2.19)\qquad \frac{n^2\,(n+1)^2}{n^2+(n+1)^2} < \frac{n(n+1)}{2} \leqq \frac{n^4}{(n-1)^2+n^2}$$

so that (2.17) is true for $s = 2$ and all $n \in \mathbb{N}$. Hence, it suffices to prove (2.17) for $s \geqq 3$ and all $n \in \mathbb{N}$. It is easily verified that (2.17) is true for $n = 1$ and all $s \geqq 3$. Assume that (2.17) is still true for $n = 1, \ldots, N$ and all $s \geqq 3$. Then

$$(2.20)\qquad \phi_{N+1}(s) = (N+1)^s - \phi_N(s) < (N+1)^s - \frac{N^s\,(N+1)^s}{N^s+(N+1)^s} = \frac{(N+1)^{2s}}{N^s+(N+1)^s}$$

so that the right-hand inequality in (2.17) is also true for $n = N+1$. We also have

$$(2.21)\qquad \phi_{N+1}(s) = (N+1)^s - \phi_N(s) \geqq (N+1)^s - \frac{N^{2s}}{(N-1)^s+N^s} =$$

$$= \frac{(N-1)^s(N+1)^s + N^s(N+1)^s - N^{2s}}{(N-1)^s+N^s}$$

so that, in order to complete the proof of (2.17), it is sufficient to prove that

$$(2.22)\qquad \frac{(N-1)^s(N+1)^s + N^s(N+1)^s - N^{2s}}{(N-1)^s+N^s} > \frac{(N+1)^s(N+2)^s}{(N+1)^s+(N+2)^s}$$

for all $N \in \mathbb{N}$. Put $x := 1/N$. Then it is easy to see that (2.22) is equivalent to

$$(2.23)\qquad \frac{(1-x)^s(1+x)^s + (1+x)^s - 1}{(1-x)^s+1} > \frac{(1+x)^s(1+2x)^s}{(1+x)^s+(1+2x)^s}\,.$$

Crossmultiplication in (2.23) and some routine simplification leads to

$$(2.24)\qquad (1-x)^s(1+x)^{2s} + (1+x)^{2s} - (1+x)^s - (1+2x)^s > 0$$

which in its turn is equivalent to

$$(2.25)\qquad (1-x^2)^s + (1+x)^s - 1 - \left(1+\frac{x}{1+x}\right)^s > 0.$$

Using the binomial theorem in (2.25), we see that we still have to show that

$$(2.26)\qquad \sum_{r=1}^{s} \binom{s}{r}\left\{(-1)^r x^{2r} + x^r - \frac{x^r}{(1+x)^r}\right\} > 0.$$

First consider the *first two terms* of this sum. One may verify that of the following inequalities each (except the last one) is a consequence of the next one

$$(2.27)\qquad s\left\{-x^2 + x - \frac{x}{1+x}\right\} + \frac{s(s-1)}{2}\left\{x^4 + x^2 - \frac{x^2}{(1+x)^2}\right\} > 0$$

$$(2.28)\qquad \left\{-x + 1 - \frac{1}{1+x}\right\} + \frac{s-1}{2}\left\{x^3 + x - \frac{x}{(1+x)^2}\right\} > 0$$

$$(2.29)\qquad \frac{-2x}{1+x} + (s-1)\left\{x^2 + 1 - \frac{1}{(1+x)^2}\right\} > 0$$

$$(2.30) \qquad \frac{-2x}{1+x} + (s-1)\{1 - \frac{1}{(1+x)^2}\} \geqq 0$$

$$(2.31) \qquad -2x(1+x) + (s-1)\{(1+x)^2 - 1\} \geqq 0$$

$$(2.32) \qquad -2 - 2x + (s-1)(2+x) \geqq 0$$

$$(2.33) \qquad (s-3)x \geqq 4-2s.$$

Since $s \geqq 3$ we have $4 - 2s < 0$ and $s - 3 \geqq 0$ so that (2.33) is true and therefore also (2.27).
For the remaining terms in (2.26), corresponding to an even r ($r = 2a$, say), we have

$$(2.34) \qquad x^{4a} + x^{2a} - \frac{x^{2a}}{(1+x)^{2a}} > 0$$

so that it suffices to show that every term in (2.26) corresponding to an odd $r \geqq 3$ ($r = 2a + 1$, say) is non-negative. Hence, we want to prove that

$$(2.35) \qquad -x^{2(2a+1)} + x^{2a+1} - \frac{x^{2a+1}}{(1+x)^{2a+1}} \geqq 0.$$

It is clear that (2.35) does *not* hold for $x = 1$ but one may verify directly that (2.25) is true in this case. Indeed, for $x = 1$, (2.25) reads

$$(2.36) \qquad 2^s - 1 - (1+\tfrac{1}{2})^s > 0$$

which is equivalent to

$$(2.37) \qquad (\tfrac{4}{3})^s > 1 + (\tfrac{2}{3})^s.$$

Since (2.37) is true for $s = 2$ it is certainly true for $s \geqq 3$. Hence, it suffices to show (2.35) for $0 < x \leqq \frac{1}{2}$. Again, one may verify that in the following list of inequalities each of them (except the last one) is a consequence of the next one (recall that $r \in \mathbb{N}$, $r \geqq 3$)

$$(2.38) \qquad -x^{2r} + x^r - \frac{x^r}{(1+x)^r} \geqq 0$$

$$(2.39) \qquad -x^r + 1 - \frac{1}{(1+x)^r} \geqq 0$$

$$(2.40) \qquad (1-x^r)(1+x)^r \geqq 1$$

$$(2.41) \qquad (1-x^r)(1+rx) \geqq 1$$

$$(2.42) \qquad r \geqq x^{r-1} + rx^r$$

$$(2.43) \qquad r \geqq x^2 + rx^3$$

$$(2.44) \qquad r \geqq (r+1)x^2$$

$$(2.45) \qquad x^2 \leqq \frac{r}{r+1} \quad (\geqq 3/4)$$

$$(2.46) \qquad 0 < x \leq \tfrac{1}{2}\sqrt{3}$$

$$(2.47) \qquad 0 < x \leq \tfrac{1}{2}.$$

Since (2.47) is true by assumption, the proof of Proposition 2.5 and hence that of Proposition 2.4 is complete. □

COROLLARY 1. $\dfrac{\phi_n(s)}{n^s}$ *is decreasing (in* n*).*

PROOF. We have to show that

$$(2.48) \qquad \frac{\phi_n(s)}{n^s} > \frac{\phi_{n+1}(s)}{(n+1)^s}$$

or

$$(2.49) \qquad (n+1)^s \phi_n(s) > n^s\{(n+1)^s - \phi_n(s)\}.$$

Since (2.49) is equivalent to

$$(2.50) \qquad \phi_n(s) > \frac{n^s(n+1)^s}{n^s + (n+1)^s}$$

the corollary follows from (2.17). □

REMARK. It is easily verified that

$$(2.51) \qquad \frac{\phi_n(1)}{n} = \frac{[\frac{n}{2}]+\frac{1}{2}(1+(-1)^{n+1})}{n}$$

so that Corollary 1 does *not* hold for s = 1. This led us to the following question: *For which* real s *is Corollary 1 true ?*

COROLLARY 2.

$$(2.52) \qquad \lim_{n\to\infty} \frac{\phi_n(s)}{n^s} = \tfrac{1}{2}$$

PROOF. This is an immediate consequence of (2.17). □

REMARK. In the formulation of Corollary 2, s is an *integer* greater than 1. However, it may be shown that (2.52) is true for any *real* s > 0, so that we also have

PROPOSITION 2.6.

$$(2.53) \qquad \lim_{n\to\infty} \frac{\phi_n(s)}{n^s} = \tfrac{1}{2}, \qquad (s > 0).$$

PROOF. See [28]. Compare PÓLYA and SZEGÖ [49; Vol. I, p. 40] . □

Returning to our study of $T_n^*(s)$ we have

THEOREM 2.1. $T_n^*(s)$ *is decreasing (in* n*).*

PROOF. Combine Propositions 2.2 and 2.4. □

COROLLARY. *For* $m,n \in \mathbb{N}$, $m \geq 2$ *we have*

$$(2.54)\qquad (2n)^{m+1}\{2\sum_{k=1}^{n-1}(2k-1)^m + (2n-1)^m\} > (2n-1)^{m+1}\{2\sum_{k=1}^{n-1}(2k)^m + (2n)^m\}$$

and

$$(2.55)\qquad (2n+1)^{m+1}\{2\sum_{k=1}^{n-1}(2k)^m + (2n)^m\} > (2n)^{m+1}\{2\sum_{k=1}^{n}(2k-1)^m + (2n+1)^m\}.$$

PROOF. (2.54) is just another way of writing $T^*_{2n-1}(m) > T^*_{2n}(m)$. Similarly, (2.55) is equivalent to $T^*_{2n}(m) > T^*_{2n+1}(m)$.

REMARK. "Direct" proofs of (2.54) and (2.55) seem to be very difficult.

We will now investigate the behaviour of the canonical upper Riemann sums $U^*_n(s)$ corresponding to f^*_s (defined on page 4). It is easily verified that

$$(2.56)\qquad U^*_{2n-1}(s) = \frac{2}{(2n-1)^{s+1}}\{-1 + 2w_n(s)\}$$

and

$$(2.57)\qquad U^*_{2n}(s) = \frac{2\sigma_n(s)}{n^{s+1}}\ .$$

From the last two formulas it is easily deduced that

$$(2.58)\qquad U^*_{2n}(s) = T^*_{2n}(s) + \frac{1}{n}$$

and

$$(2.59)\qquad U^*_{2n+1}(s) = T^*_{2n+1}(s) + \frac{2}{2n+1} - \frac{1}{(2n+1)^{s+1}}\ .$$

We are now in a suitable position to prove

PROPOSITION 2.7. *The sequence* $\{U^*_n(s)\}_{n=2}^{\infty}$ *is decreasing.*

PROOF. First consider the inequality

$$(2.60)\qquad U^*_{2n}(s) > U^*_{2n+1}(s)$$

or

$$(2.61)\qquad T^*_{2n}(s) + \frac{1}{n} > T^*_{2n+1}(s) + \frac{2}{2n+1} - \frac{1}{(2n+1)^{s+1}}\ .$$

Since $T^*_{2n}(s) > T^*_{2n+1}(s)$, it suffices to show that

$$(2.62)\qquad \frac{1}{n} \geq \frac{2}{2n+1} - \frac{1}{(2n+1)^{s+1}}\ .$$

Since this is trivially true we are done with (2.60). Next consider

$$(2.63)\qquad U^*_{2n-1}(s) > U^*_{2n}(s)\ ,\qquad (n \geq 2)$$

or

$$(2.64)\qquad T^*_{2n-1}(s) + \frac{2}{2n-1} - \frac{1}{(2n-1)^{s+1}} > T^*_{2n}(s) + \frac{1}{n}\,, \qquad (n \geqq 2).$$

In view of Theorem 2.1 it clearly suffices to show that

$$(2.65)\qquad \frac{2}{2n-1} - \frac{1}{(2n-1)^{s+1}} \geqq \frac{1}{n}\,, \qquad (n \geqq 2)$$

or

$$(2.66)\qquad (2n-1)^s \geqq n\,, \qquad (n \geqq 2).$$

Since this is obviously true the proof is complete. □

In order to be able to deal with the canonical *lower* Riemann sums corresponding to f^*_s we first prove

PROPOSITION 2.8. $t^*_{2n+1}(s)$ *is increasing (in* n*).*

PROOF. First observe that

$$(2.67)\qquad t^*_{2n+1}(s) = \frac{2^{s+2}}{(2n+1)^{s+1}}\,.\,\sigma_n(s)\,.$$

Hence, we may just as well prove that

$$(2.68)\qquad \frac{\sigma_n(s)}{(2n+1)^{s+1}} < \frac{\sigma_{n+1}(s)}{(2n+3)^{s+1}}$$

which can be shown to be equivalent to

$$(2.69)\qquad \sigma_n(s) < \frac{(n+1)^s(2n+1)^{s+1}}{(2n+3)^{s+1} - (2n+1)^{s+1}}\,.$$

We prove (2.69) by induction. For n = 1, (2.69) is equivalent to

$$(2.70)\qquad 5^{s+1} - 3^{s+1} < 3.6^s\,.$$

It may be verified directly that (2.70) is true for s = 2.
Now observe that (2.70) is equivalent to

$$(2.71)\qquad 1 - \left(\frac{3}{5}\right)^{s+1} < \frac{3}{5}\left(1+\frac{1}{5}\right)^s$$

so that it suffices to show that

$$(2.72)\qquad \frac{3}{5}\left(1+\frac{1}{5}\right)^s \geqq 1\,, \qquad (s \geqq 3).$$

Since

$$(2.73)\qquad \left(1+\frac{1}{5}\right)^s > 1 + \frac{s}{5} + \frac{s(s-1)}{2.5^2} \geqq 1 + \frac{3}{5} + \frac{3}{25} = \frac{43}{25} \geqq \frac{5}{3}$$

it follows that (2.69) is true for n = 1 and all s ≧ 2.
In (2.69) we replace n by n−1, so that we still have to show that

$$(2.74)\qquad (\sigma_{n-1}(s) =)\ \sigma_n(s) - n^s < \frac{n^s(2n-1)^{s+1}}{(2n+1)^{s+1} - (2n-1)^{s+1}}$$

or

$$(2.75) \qquad \sigma_n(s) < \frac{n^s(2n+1)^{s+1}}{(2n+1)^{s+1} - (2n-1)^{s+1}} \quad \text{for all } n \in N.$$

We continue the proof by showing that (2.75) holds for all *real* $s \geqq 1$. Observe that (2.75) is obviously true for $n = 1$. Assume that (2.75) is still true for $n = 1, \ldots, N$ and all (real) $s \geqq 1$. Then

$$(2.76) \qquad \sigma_{N+1}(s) = (N+1)^s + \sigma_N(s) < (N+1)^s + \frac{N^s(2N+1)^{s+1}}{(2N+1)^{s+1} - (2N-1)^{s+1}}$$

so that it suffices to show that

$$(2.77) \qquad (N+1)^s + \frac{N^s(2N+1)^{s+1}}{(2N+1)^{s+1} - (2N-1)^{s+1}} \leqq \frac{(N+1)^s(2N+3)^{s+1}}{(2N+3)^{s+1} - (2N+1)^{s+1}}$$

for all $N \in \mathbb{N}$. Putting $x := 1/N$ we arrive at the equivalent inequality

$$(2.78) \qquad (1+x)^s + \frac{(2+x)^{s+1}}{(2+x)^{s+1} - (2-x)^{s+1}} \leqq \frac{(1+x)^s(2+3x)^{s+1}}{(2+3x)^{s+1} - (2+x)^{s+1}} .$$

In (2.78) replace x by $2x$ (so that from now on $0 < x \leqq \frac{1}{2}$) in order to arrive at

$$(2.79) \qquad (1+2x)^s + \frac{(1+x)^{s+1}}{(1+x)^{s+1} - (1-x)^{s+1}} \leqq \frac{(1+2x)^s(1+3x)^{s+1}}{(1+3x)^{s+1} - (1+x)^{s+1}} .$$

After crossmultiplication and some simplifications it turns out that we may just as well prove that

$$(2.80) \qquad (1+2x)^s \{(1+x)^{s+1} - (1-x)^{s+1}\} \geqq (1+3x)^{s+1} - (1+x)^{s+1}$$

which in its turn is equivalent to

$$(2.81) \qquad \frac{(1+x)^{s+1} - (1-x)^{s+1}}{x} \geqq \frac{\left(1+\frac{x}{1+2x}\right)^{s+1} - \left(1-\frac{x}{1+2x}\right)^{s+1}}{\frac{x}{1+2x}} .$$

Since $x > \frac{x}{1+2x}$ for $x > 0$ it follows that the proof is complete as soon as we can show that the function

$$(2.82) \qquad \psi(x) := \frac{(1+x)^{s+1} - (1-x)^{s+1}}{x}$$

is increasing on the interval $0 < x \leqq \frac{1}{2}$. Observe that for $x > 0$

$$(2.83) \qquad \psi'(x) = \frac{x\{(s+1)(1+x)^s + (s+1)(1-x)^s\} - (1+x)^{s+1} + (1-x)^{s+1}}{x^2}$$

so that it suffices to show that

$$(2.84) \qquad (sx+x-1-x)(1+x)^s + (sx+x+1-x)(1-x)^s \geqq 0$$

or

$$(2.85) \qquad h(x) := (sx-1)(1+x)^s + (sx+1)(1-x)^s \geqq 0 .$$

Since $h(0) = 0$, it suffices to show that for $x > 0$

(2.86) $$h'(x) = s(1+x)^s + (sx-1)s(1+x)^{s-1} + s(1-x)^s-(sx+1)s(1-x)^{s-1} \geqq 0$$

or

(2.87) $$(s+sx+s^2x-s)(1+x)^{s-1} + (s-sx-s^2x-s)(1-x)^{s-1} =$$
$$= (s+s^2)x\{(1+x)^{s-1} - (1-x)^{s-1}\} \geqq 0 \ .$$

Since (2.87) is clearly true for $s \geqq 1$ and $0 < x < 1$ the proof of Proposition 2.8 is complete. □

From the above proof we immediately obtain

PROPOSITION 2.9.

(2.88) $$\sigma_n(s) < \frac{n^s(2n+1)^{s+1}}{(2n+1)^{s+1} - (2n-1)^{s+1}} \ , \ ((n \in \mathbb{N};\ s \textit{ real} \text{ and } \geqq 1).$$

In the remaining part of this section we will investigate the behaviour of the *canonical lower* Riemann sums $L_n^*(s)$ corresponding to f_s^*. One may verify that

(2.89) $$L_{2n-1}^*(s) = \frac{4}{2n-1} \sum_{k=1}^{n-1} (\frac{2k-1}{2n-1})^s = \frac{4\{w_n(s) - (2n-1)^s\}}{(2n-1)^{s+1}}$$

and

(2.90) $$L_{2n}^*(s) = 2L_n(s) \ .$$

PROPOSITION 2.10. *The sequence* $\{L_n^*(s)\}_{n=1}^{\infty}$ *is increasing.*

PROOF. We first consider

(2.91) $$L_{2n+1}^*(s) < L_{2n+2}^*(s)$$

which may be shown to be equivalent to

(2.92) $$2(n+1)^{s+1} w_n(s) < (2n+1)^{s+1} \sigma_n(s) \ .$$

In view of (1.53) and (1.85) it suffices to show that

(2.93) $$4(n+1)^{s+1} \frac{n^{s+1}(2n+1)^s}{(n+1)^{s+1} - n^{s+1}} < (2n+1)^{s+1} \frac{n^{s+1}(n+1)^s + n^s(n+1)^{s+1}}{(n+1)^{s+1} - n^s}$$

or, equivalently, that $4n(n+1) < (2n+1)^2$. It follows that (2.91) is true.

Next we consider

(2.94) $$L_{2n}^*(s) < L_{2n+1}^*(s)$$

or

$$(2.95) \qquad (2n+1)^{s+1}\,\sigma_{n-1}(s) < 2n^{s+1}\,w_n(s)$$

which may also be written as

$$(2.96) \qquad \frac{\sigma_n(s)}{n^{s+1}} - \frac{\sigma_{2n}(s)}{(2n)^{s+1}} < \frac{\sigma_n(s)}{n^{s+1}} \cdot \frac{1-(1+\frac{1}{2n})^{s+1}}{2} + \frac{1}{2n}(1+\frac{1}{2n})^{s+1} .$$

A "direct" proof of either of the last three inequalities seems to be quite cumbersome. Therefore we proceed as follows. We already know that $U^*_{2n-1}(s) > U^*_{2n}(s)$, $(n \geqq 2)$. Inequality (2.96) is equivalent to

$$(2.97) \qquad \frac{1}{(2n-1)^{s+1}}\{-1 + 2w_n(s)\} > \frac{\sigma_n(s)}{n^{s+1}}$$

which may be written as

$$(2.98) \qquad \frac{\sigma_n(s)}{n^{s+1}} - \frac{\sigma_{2n}(s)}{(2n)^{s+1}} < \frac{\sigma_n(s)}{n^{s+1}} \cdot \frac{1-(1-\frac{1}{2n})^{s+1}}{2} - \frac{1}{2(2n)^{s+1}} .$$

Hence, it suffices to show that

$$(2.99) \qquad \frac{\sigma_n(s)}{n^{s+1}} \cdot \frac{1-(1-\frac{1}{2n})^{s+1}}{2} - \frac{1}{2(2n)^{s+1}} < \frac{\sigma_n(s)}{n^{s+1}} \cdot \frac{1-(1+\frac{1}{2n})^{s+1}}{2} + \frac{1}{2n}(1+\frac{1}{2n})^{s+1}$$

or

$$(2.100) \qquad \sigma_n(s) < \frac{n^s + n^s(2n+1)^{s+1}}{(2n+1)^{s+1} - (2n-1)^{s+1}} .$$

Since (2.100) is true by Proposition 2.9, the proof of Proposition 2.10 is complete. □

REMARK. A number of direct applications of the main results of this chapter are given in [28]. Some related topics are dealt with in [37].

CHAPTER 2

CONVEX AND LOG-CONVEX TRAPEZOIDAL APPROXIMATION OF SOME ELEMENTARY INTEGRALS

0. INTRODUCTION

A good deal of the results presented in this chapter were obtained in cooperation with N.M. TEMME, R. TIJDEMAN and M. VOORHOEVE.

The topics treated here are closely related to those discussed in Chapter 1. As in Chapter 1 we consider the n-th canonical trapezoidal approximations $T_n := T_n(s) := T_n(f_s)$ of $\int_0^1 x^s\,dx$, where s is some (fixed) positive real number. In Chapter 1 we showed that the sequence $\{T_n\}_{n=1}^{\infty}$ is *decreasing* for $s > 1$.

In Section 1 we prove that for every (fixed) $m \in \mathbb{N}$ the sequence $\{T_n(m)\}_{n=1}^{\infty}$ is *convex*, i.e.

$$(0.1) \qquad 2T_n(m) \leqq T_{n-1}(m) + T_{n+1}(m), \qquad (n \geqq 2).$$

The convexity of $\{T_n(m)\}_{n=1}^{\infty}$ is established by defining a suitable function $\phi(y)$ such that

$$(0.2) \qquad \phi(n) = T_n(m), \qquad (n \in \mathbb{N})$$

and proving that $\phi''(y) > 0$ for $y > 0$, so that ϕ is convex.

In Section 1 we state the conjecture that for any *real* $s > 1$ the sequence $\{T_n(s)\}_{n=1}^{\infty}$ is *logarithmically convex*, i.e. $\log T_n(s)$ is convex.

In Section 2 we prove (by complex analytical means) the correctness of this conjecture for the intervals $1 < s < 3$ and $5 < s < 7$.

In Section 3 we show the *convexity* of $\{T_n(s)\}_{n=1}^{\infty}$ for $3 < s < 4$.

In Section 4 we establish the *log-convexity* of the sequence of canonical trapezoidal approximations of $\int_\alpha^\beta x^{-s}\,dx$, where α,β and s are fixed and positive ($\alpha < \beta$).

1. CONVEX TRAPEZOIDAL APPROXIMATION OF $\int_0^1 x^m\,dx$

1.1. Preliminaries; statement of the Theorem

Let $f : [0,1] \to \mathbb{R}$ be twice differentiable with continuous second derivative. Then we have by the Euler-Maclaurin summation formula

$$(1.1.1) \qquad T_n(f) := \frac{1}{n}\{-\tfrac{1}{2}f(0) + \sum_{k=0}^{n} f(\tfrac{k}{n}) - \tfrac{1}{2}f(1)\} =$$

$$= \int_0^1 f(x)\,dx + \frac{1}{n}\int_0^n (x - [x] - \tfrac{1}{2})\,df(\tfrac{x}{n}).$$

Let the function $\theta(t)$ be defined by

$$(1.1.2) \qquad \theta(t) = -\int_0^t (x - [x] - \tfrac{1}{2})\,dx, \qquad (t \in \mathbb{R}).$$

Since $\theta(t) = 0$ for $t \in \mathbb{Z}$ we can write

$$(1.1.3) \qquad T_n(f) = \int_0^1 f(x)\,dx - \frac{1}{n^2}\int_0^n f'(\tfrac{x}{n})\,d\theta(x) =$$

$$= \int_0^1 f(x)\,dx + \frac{1}{n^3} \int_0^n f''(\tfrac{x}{n})\theta(x)\,dx.$$

Now define

(1.1.4) $$\phi_f(t) = \frac{1}{t^3} \int_0^t f''(\tfrac{x}{t})\theta(x)\,dx, \qquad (t > 0).$$

If f is four times continuously differentiable and if $f''(1) = f^{(3)}(1) = 0$, then $\phi_f(t)$ has a continuous second derivative for $t > 0$

(1.1.5) $$\phi_f''(t) = \frac{1}{t^4} \int_0^1 (12f''(u) + 8uf^{(3)}(u) + u^2 f^{(4)}(u))\theta(tu)\,du.$$

Let $m \in \mathbb{N}$, $m \geq 5$ and put

(1.1.6) $$g_m(x) := (1-x)^{m-1}.$$

Note that, by symmetry, $\overset{*}{T}_n(m) := T_n(x^{m-1}) = T_n(g_m(x))$, so that

(1.1.7a) $$\overset{*}{T}_n(m) = \frac{1}{m} + \frac{1}{n^3} \int_0^n g_m''(\tfrac{x}{n})\theta(x)\,dx.$$

Since $g_m''(1) = g_m^{(3)}(1) = 0$ the corresponding function $\phi_m(t) := \phi_{g_m}(t)$ satisfies

(1.1.7b) $$\frac{t^4 \phi_m''(t)}{(m-1)(m-2)} = \int_0^1 \{(m^2+m)u^2 - 8mu + 12\}(1-u)^{m-5}\theta(tu)\,du.$$

We intend to prove

THEOREM 1. *For every* $m \in \mathbb{N}$, *the sequence* $\{T_n(x^m)\}_{n=1}^{\infty}$ *is convex.*

We shall prove this theorem by showing that the right-hand side of (1.1.7b), and thus $\phi_m''(t)$, is positive for $m \geq 9$ and $t > 0$. Since by Taylor's theorem

(1.1.8) $$\phi_m(n+1) + \phi_m(n-1) = 2\phi_m(n) + \tfrac{1}{2}(\phi_m''(t_1) + \phi_m''(t_2))$$

where $t_1 \in (n-1,n)$ and $t_2 \in (n,n+1)$, this implies Theorem 1 for $m \geq 8$. For $s = 2, \ldots, 8$ we express $\overset{*}{T}_n(s)$ by means of the Bernoulli polynomials

(1.1.9) $$\overset{*}{T}_n(s) = \frac{1}{s} \sum_{0 \leq k \leq \frac{1}{2}n} \binom{s}{2k} B_{2k}\, n^{-2k} \quad (= T_n(x^{s-1})).$$

For $m = 1, \ldots, 7$ the theorem can be verified directly by this formula. Hence, it is sufficient to show that for $t > 0$ and $m \geq 9$

(1.1.10) $$\int_0^1 \{(m^2+m)u^2 - 8mu + 12\}(1-u)^{m-5}\theta(tu)\,du > 0.$$

1.2. Some lemmas

LEMMA 1.1. *Let* $\theta(t)$ *be defined by* (1.1.2). *Then*

a) θ *is periodic with period* 1

b) $\theta(t) = \frac{1}{2}t(1-t)$ *for* $0 \leq t < 1$

c) $\theta(t) \leq \frac{1}{8}$ *for all* $t \in \mathbb{R}$; $\theta(t) \leq \frac{1}{16}t$ *for* $t \geq 2$

d) $\int_0^n (\theta(t) - \frac{1}{12})\, dt = 0$ *for* $n \in \mathbb{Z}$

e) $\int_0^x (\frac{1}{12} - \theta(t))\, dt \leqq \frac{\sqrt{3}}{216} < \frac{1}{120}$.

PROOF. By direct verification from (1.1.2). □

LEMMA 1.2. *If* $0 \leqq a \leqq \frac{1}{2}$ *and* $0 \leqq t \leqq 2$, *then* $\theta(at) \geqq \frac{1}{2}t\theta(2a)$.

PROOF. Since $0 \leqq at \leqq 2a \leqq 1$, we have $\theta(at) = \frac{1}{2}at(1-at)$ and $\theta(2a) = a(1-2a)$. Since $0 \leqq t \leqq 2$, we thus have

(1.1.11) $$\theta(at) = \frac{1}{2}at(1-at) \geqq \frac{1}{2}at(1-2a) = \frac{1}{2}t\theta(2a). \; \square$$

LEMMA 1.3. *If* $0 \leqq a \leqq \frac{1}{3}$ *and* $2 \leqq t \leqq 6$, *then* $\theta(at) \leqq \frac{1}{2}t\theta(2a)$.

PROOF. If $at < 1$ we have by $t \geqq 2$

(1.1.12) $$\theta(at) = \frac{1}{2}at(1-at) \leqq \frac{1}{2}at(1-2a) = \frac{1}{2}t\theta(2a).$$

If $at \geqq 1$, then, since $1 \leqq at \leqq 2$ and $0 \leqq 2a < 1$

(1.1.13) $$t\theta(2a) - 2\theta(at) = at(1-2a) - (at-1)(2-at) =$$

$$= (at)^2 - 2(1+a)at + 2 \geqq (at)^2 - 2(1+a)at + (1+a)^2 \geqq 0. \; \square$$

LEMMA 1.4. *If* $a \geqq \frac{1}{4}$ *and* $0 \leqq x \leqq 2$, *then*

(1.1.14) $$\chi_a(x) := \int_0^x \theta(at)\, dt \geqq x^2/32.$$

PROOF. Suppose $0 \leqq ax \leqq 1$. Then we have

(1.1.15) $$\chi_a(x) = \int_0^x \theta(at)\, dt = \frac{1}{a}\int_0^{ax} \theta(u)\, du = \frac{1}{a}\int_0^{ax} \tfrac{1}{2}u(1-u)\, du =$$

$$= \frac{1}{12}x(3ax - 2(ax)^2) = \frac{1}{12}x^2a(3-2ax) \geqq \frac{1}{12}ax^2$$

so that the lemma is true if $a \geqq 3/8$. But if $a < 3/8$ we have $ax \leqq 6/8$, so that $(3-2ax) \geqq 3/2$. Hence

(1.1.16) $$\int_0^x \theta(at)\, dt = \frac{1}{12}ax^2(3-2ax) \geqq \frac{1}{48}x^2 . \frac{3}{2} = \frac{1}{32}x^2.$$

Suppose that $ax > 1$. Then by Lemma 1.1 ((d) and (e))

(1.1.17) $$\int_0^x \theta(at)\, dt = \frac{x}{12} + \frac{1}{a}\int_0^{ax} (\theta(u) - \frac{1}{12})\, du =$$

$$= \frac{x}{12} - \frac{1}{a}\int_{[ax]}^{ax} (\frac{1}{12} - \theta(u))\, du \geqq \frac{x}{12} - \frac{1}{120a} \geqq x(\frac{1}{12} - \frac{1}{120}) \geqq x^2/32$$

since $x \leqq 2$. □

LEMMA 1.5. *For* $m \geq 9$ *we have*

$$I(m) := \int_0^6 t(t-2)(t-6)(1-t/m)^{m-5}\,dt > 0. \tag{1.1.18}$$

PROOF. Integration by parts reveals that

$$I(m) = \frac{m^2}{(m-3)(m-4)}\{12 - 24(1-\tfrac{6}{m})^{m-3} - \frac{16m}{m-2} - \frac{20m}{m-2}(1-\tfrac{6}{m})^{m-2} + \tag{1.1.19}$$
$$+ \frac{6m^2}{(m-1)(m-2)} - \frac{6m^2}{(m-1)(m-2)}(1-\tfrac{6}{m})^{m-1}\}.$$

By direct calculation one may verify that $I(m) > 0$ for $m = 9, \ldots, 20$. Since $(1-6/m)^m$ *increases* to its limit e^{-6} we have

$$I(m) > \frac{m^2}{(m-3)(m-4)}\{2 - \frac{14}{m-1} - \frac{8}{(m-1)(m-2)} + \tag{1.1.20}$$
$$- (\frac{24m^3}{(m-6)^3} + \frac{20m^3}{(m-2)(m-6)^2} + \frac{6m^3}{(m-1)(m-2)(m-6)})e^{-6}\}.$$

Since the form in curly brackets { } is *increasing* in m and positive for m = 21, the proof is complete. □

1.3. PROOF of Theorem 1

Put $h_m(t) := (t-2)(t-6)(1-t/m)^{m-5}$. We shall prove that for $a > 0$ and $m \geq 9$

$$\int_0^6 h_m(t)\theta(at)\,dt > 0. \tag{1.1.21}$$

Since $h_m(t) \geq 0$ for $t \geq 6$ and since $\theta(t) \geq 0$ for all t, (1.1.21) implies

$$\int_0^m h_m(t)\theta(at)\,dt > 0 \tag{1.1.22}$$

so that, putting $u = t/m$ and $y = am$, we obtain

$$\int_0^1 (m^2y^2 - 8mu + 12)(1-u)^{m-5}\theta(uy)\,du > 0 \tag{1.1.23}$$

which implies (1.1.10) and hence Theorem 1. Hence, it is sufficient to show (1.1.21). Now suppose that $0 < a < \frac{1}{4}$. By Lemmas 1.2 and 1.3 we have

$$\int_0^6 h_m(t)\theta(at)\,dt \geq \{\int_0^2 + \int_2^6\}\, h_m(t).\tfrac{1}{2}t\theta(2a)\,dt = \tfrac{1}{2}\theta(2a)\int_0^6 th_m(t)\,dt \tag{1.1.24}$$

which is *positive* by Lemma 1.5.
So, let $a \geq \frac{1}{4}$ and, as before, put $\chi_a(x) = \int_0^x \theta(at)\,dt$. Since $\chi_a(0) = 0 = h_m(2)$, we have

$$I_1(m) := \int_0^2 h_m(t)\theta(at)\,dt = \int_0^2 h_m(t)\,d\chi_a(t) = -\int_0^2 \chi_a(t)\,dh_m(t). \tag{1.1.25}$$

Observe that $h_m(t)$ is *decreasing* for $0 \leq t \leq 2$. We thus have by Lemma 1.4

$$(1.1.26) \qquad I_1(m) \geq -\int_0^2 \frac{t^2}{32}\, dh_m(t) = \int_0^2 h_m(t)\frac{t}{16}\, dt.$$

Since $h_m(t) \leq 0$ for $2 \leq t \leq 6$ we have by Lemma 1.1

$$(1.1.27) \qquad I_2(m) := \int_2^6 h_m(t)\theta(at)\, dt \geq \int_2^6 h_m(t)\frac{t}{16}\, dt.$$

Hence, by Lemma 1.5,

$$(1.1.28) \qquad \int_0^6 h_m(t)\theta(at)\, dt = I_1(m) + I_2(m) \geq \frac{1}{16}\int_0^6 t h_m(t)\, dt > 0$$

for $m \geq 9$, completing the proof of Theorem 1. □

COROLLARY.

$$(1.1.29) \qquad T_n(m) \geq \tfrac{1}{2}\frac{n^{m-1}(n+1)^m + 2(n+1)^{m-1}(n-1)^{m-1} + n^{m-1}(n-1)^m}{(n+1)^m n^m - 2(n+1)^m(n-1)^m + n^m(n-1)^m}.$$

REMARK. The method in this section can be applied to all functions x^s with (real) $s \geq 9$. For $s < 9$ and $s \notin \mathbb{N}$ we cannot prove anything (yet). However, numerical evidence supports the following stronger

CONJECTURE. For any (fixed) real $s > 1$ the sequence $\{T_n(s)\}_{n=1}^{\infty}$ is *logarithmically convex*. For a still *deeper conjecture* we refer to [38]. (Also see [34].)

Although we are still unable to prove this conjecture (for *all* $s > 1$), we have established the following

THEOREM 1.2. *For any constant* $a \in (0,1)$ *the sequence*

$$(1.1.30) \qquad \{\sum_{m=0}^{\infty} (T_n(m) - \frac{1}{m+1})a^m\}_{n=1}^{\infty}$$

is log-convex. $(T_n(0) := 1)$

PROOF. See [38] in combination with Theorem 2.3.1. □

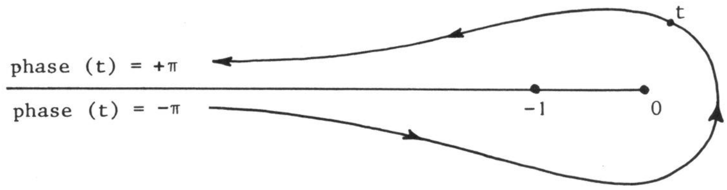
2. LOG-CONVEX TRAPEZOIDAL APPROXIMATION OF $\int_0^1 x^s\, dx$

2.1. Preliminaries

Our starting point is Hankel's integral representation of the reciprocal of Euler's gamma function (cf. WHITTAKER & WATSON [60; pp.244-245] or SANSONE & GERRETSEN [52; pp. 201-204])

$$(2.1.1) \qquad \frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\oint e^t t^{-s}\, dt, \qquad (s \in \mathbb{C})$$

where $\oint$ denotes integration along a contour as depicted below



For any $p > 0$ we substitute $t = pw$ in (2.1.1), replace s by $s+1$ and obtain

$$p^s = \frac{\Gamma(s+1)}{2\pi i} \oint e^{pw} w^{-s-1}\, dw, \qquad (s \in \mathbb{C}). \tag{2.1.2}$$

Setting $p = \frac{k}{n}$, $k = 1, \ldots, n$, we obtain by summation over k

$$T_n = \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w - 1}{w} \frac{w}{2n} \frac{e^{\frac{w}{n}} + 1}{e^{\frac{w}{n}} - 1} w^{-s-1}\, dw, \qquad (s > 0). \tag{2.1.3}$$

Making $n \to \infty$ we get

$$\frac{1}{s+1} = \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w - 1}{w} w^{-s-1}\, dw, \qquad (s > 0)$$

(a result obtainable in various other ways; compare Section 2.4) so that (2.1.3) may be rewritten as

$$T_n = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w - 1}{w} H(\tfrac{w}{n}) w^{-s-1}\, dw, \qquad (s > 0) \tag{2.1.4}$$

where

$$H(z) := \frac{z}{2} \frac{e^z + 1}{e^z - 1} - 1 = z\left(\frac{1}{e^z - 1} - \frac{1}{z} + \frac{1}{2}\right).$$

It is well known that (cf. SANSONE & GERRETSEN [52; p. 88])

$$\frac{1}{e^z - 1} - \frac{1}{z} + \frac{1}{2} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{|B_{2k}|}{(2k)!} z^{2k-1}, \qquad (|z| < 2\pi) \tag{2.1.5}$$

from which it is clear that the *(even)* function $H(z)$ has a zero of order 2 at $z = 0$. With this in mind we rewrite (2.1.4) as follows

$$T_n = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w - 1}{w} \left(\frac{1}{w^2} H(\tfrac{w}{n})\right) w^{1-s}\, dw, \qquad (s > 0). \tag{2.1.6}$$

2.2. The case $1 < s < 2$

For $1 < s < 2$ (so that $-1 < 1-s < 0$) we may, by the regularity of $w^{-2} H(\frac{w}{n})$ at $w = 0$, contract the contour of integration in (2.1.6) to the negative real axis so that by a standard argument, using the fact that $H(z)$ is an *even* function,

$$(2.2.1) \qquad T_n = \frac{1}{s+1} + \frac{\Gamma(s+1)\sin(s-1)\pi}{\pi} \int_0^\infty \frac{1-e^{-x}}{x} H(\tfrac{x}{n}) x^{-s-1}\, dx, \qquad (1 < s < 2).$$

Substituting $x = nu$ and writing $\frac{1-e^{-nu}}{nu} = \int_0^1 e^{-nuv}\, dv$ we may write (2.2.1) as

$$(2.2.2) \qquad n^s(T_n - \frac{1}{s+1}) = \frac{\Gamma(s+1)\sin(s-1)\pi}{\pi} \int_0^\infty \left(\int_0^1 e^{-nuv}\, dv \right) H(u) u^{-s-1}\, du.$$

Since $\sin(s-1)\pi > 0$ for $1 < s < 2$ and $H(u) > 0$ for $u > 0$, we find, by the general theory of log-convex functions (cf. ARTIN [2]) that the sequence $\{n^s(T_n - \frac{1}{s+1})\}_{n=1}^\infty$ is log-convex, a result which is even *stronger* than the previously announced assertion that $\{T_n\}_{n=1}^\infty$ is log-convex for all (fixed) $s \in (1,2)$. □

REMARK. Similarly one may show that $\{\frac{1}{s+1} - T_n\}_{n=1}^\infty$ is log-convex for all (fixed) $s \in (0,1)$.

2.3. Intermezzo: A special property of $H(u) := u(\frac{1}{e^u - 1} - \frac{1}{u} + \frac{1}{2})$

In the previous section we transformed (2.2.1) into (2.2.2) and then concluded that $\{T_n - \frac{1}{s+1}\}_{n=1}^\infty$ is log-convex for all $s \in (1,2)$. In this section we show that this result may also be obtained directly from (2.2.1) by observing that the function $H(\frac{1}{x})$, $x > 0$, has the remarkable property of being log-convex on $\mathbb{R}^+$. As a matter of fact we will prove the following more general

THEOREM 2.3.1. *There exists a constant* $\alpha_0 > 2.863$ *such that for every (fixed)* $\alpha \in (0,\alpha_0]$ *the function* $\phi_\alpha : \mathbb{R}^+ \to \mathbb{R}^+$, *defined by* $\phi_\alpha(x) := H(x^{-\alpha})$, $x > 0$, *is log-convex on* $\mathbb{R}^+$.

PROOF. Writing

$$(2.3.1) \qquad \psi(x) := \log \phi_\alpha(x) = \log H(x^{-\alpha})$$

we have

$$(2.3.2) \qquad \psi''(x) = \alpha u^2 + \frac{AB-C^2}{A^2}$$

where

$$(2.3.3) \qquad u := \frac{1}{x},$$

$$v := u^\alpha,$$

$$A := (e^v-1)^{-1} - \frac{1}{v} + \frac{1}{2},$$

$$B := 2\alpha^2u^2v^2e^{2v}(e^v-1)^{-3}\, 2 - \alpha^2u^2v^2e^v(e^v-1)^{-2} + $$

$$- \alpha(\alpha+1)u^2ve^v(e^v-1)^{-2} - \alpha(\alpha-1)\frac{u^2}{v},$$

$$C := \alpha uve^v(e^v-1)^{-2} - \alpha\frac{u}{v}.$$

It clearly suffices to show that $\phi''(x) > 0$ for all $x \in \mathbb{R}^+$ so that (since $\alpha > 0$) we may just as well prove that

$$(2.3.4) \qquad \frac{\psi''(x)}{\alpha u^2} = 1 + \frac{A_1 B_1 - \alpha C_1^2}{A_1^2} > 0$$

where (u and v being defined as above)

$$(2.3.5) \qquad A_1 := A \text{ (as defined above)},$$

$$B_1 := 2\alpha v^2 e^{2v}(e^v-1)^{-3} - \alpha v^2 e^v (e^v-1)^{-2} + - (\alpha+1) v e^v (e^v-1)^{-2} - \frac{\alpha-1}{v},$$

$$C_1 := v e^v (e^v-1)^{-2} - \frac{1}{v}.$$

Hence, it suffices to show that for all $x \in \mathbb{R}^+$

$$(2.3.6) \qquad A_1^2 + A_1 B_1 > \alpha C_1^2 .$$

Multiplying both sides of this inequality by $v^2(e^v-1)^4$ we arrive at the equivalent inequality

$$(2.3.7) \qquad v^2(e^v-1)^2 + (v^2-2v)(e^v-1)^3 + (1-\tfrac{v}{2})^2(e^v-1)^4 + + \{v+(\tfrac{v}{2}-1)(e^v-1)\}\{2\alpha v^3 e^{2v} - \alpha v^3 e^v (e^v-1) - (\alpha+1) v^2 e^v (e^v-1) + - (\alpha-1)(e^v-1)^3\} > \alpha\{v^4 e^{2v} - 2v^2 e^v (e^v-1)^2 + (e^v-1)^4\} .$$

This inequality may be written in the equivalent form

$$(2.3.8) \qquad \sum_{k=0}^{4} P_k(v) e^{kv} > 0$$

where

$$(2.3.9) \qquad P_0(v) = \frac{\alpha+1}{2} + \frac{v}{4},$$

$$P_1(v) = -(\alpha+1) + (3\alpha+1)v + \frac{3\alpha+1}{2}v^2 + \frac{\alpha}{2}v^3,$$

$$P_2(v) = -\frac{5+12\alpha}{2}v,$$

$$P_3(v) = \alpha + 1 + (3\alpha+1)v - \frac{3\alpha+1}{2}v^2 + \frac{\alpha}{2}v^3,$$

$$P_4(v) = -\frac{\alpha+1}{2} + \frac{v}{4}.$$

Now we write the left-hand side of (2.3.8) in the form $\Sigma_{n=0}^{\infty} c_n v^n$ and observe that $c_0 = c_1 = 0$ for all α. For $n \geqq 2$ one may verify that

$$(2.3.10) \qquad n!c_n = (\alpha+1)(-1+3^n-2^{2n-1}) + n\{(3\alpha+1)-(12\alpha+5)2^{n-2}+(3\alpha+1)3^{n-1}+4^{n-2}\} + + (3\alpha+1)\frac{n(n-1)}{2}(1-3^{n-2}) + \frac{\alpha}{2}n(n-1)(n-2)(1+3^{n-3}) = = \alpha . a(n) + b(n)$$

where

$$(2.3.11) \qquad a(n) := -1 + 3^n - 2^{2n-1} + 3n - 3n2^n + n3^n + \frac{n(n-1)}{2}(3-3^{n-1}) + \frac{n(n-1)(n-2)}{2}(1+3^{n-3})$$

and

$$(2.3.12) \qquad b(n) := -1 + 3^n - 2^{2n-1} + n - 5n2^{n-2} + n3^{n-1} + n4^{n-2} + \frac{n(n-1)}{2}(1-3^{n-2}) .$$

It is a matter of routine to show that

$$((2.3.13) \qquad \begin{aligned} a(n) &= 0 \text{ for } n \leq 8 \\ a(n) &< 0 \text{ for } n \geq 9 \\ b(n) &= 0 \text{ for } n \leq 6 \\ b(n) &> 0 \text{ for } n \geq 7 \end{aligned}$$

and

$$(2.3.14) \qquad \min_{n\geq 9} -\frac{b(n)}{a(n)} = -\frac{b(24)}{a(24)} = 2.863\ 921 \ldots$$

from which it follows that for $0 < \alpha < 2.8639$ we have $c_n = 0$ for $n \leq 6$ and $c_n > 0$ for $n \geq 7$. This proves the theorem. □

REMARK. It is not known to us which α_0^* is the largest number such that $H(x^{-\alpha})$ is log-convex on $\mathbb{R}^+$ for all $\alpha \in (0,\alpha_0^*]$. Numerical computations show that $H(x^{-3})$ is *not* log-convex on *all* of $\mathbb{R}^+$ so that $(2.863<)\ \alpha_0^* < 3$.

2.4. Further preparations

In order to carry our analysis somewhat further we need some auxiliary formulas. In (2.1.2) let $p \downarrow 0$ (keeping s fixed and > 0) and it follows that

$$(2.4.1) \qquad \oint w^{-s-1}\, dw = 0, \qquad (s > 0).$$

Besides directly considering a primitive of w^{-s-1} we may prove (2.4.1) also as follows. In (2.1.3) put $n = 1$ so that (for $s > 0$)

$$(2.4.2) \qquad T_1(s) = \frac{1}{2} = \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w+1}{2} w^{-s-1}\, dw =$$

$$= \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2} \oint e^w w^{-s-1}\, dw + \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2} \oint w^{-s-1}\, dw =$$

$$= \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2}\,\frac{2\pi i}{\Gamma(s+1)} + \frac{\Gamma(s+1)}{2\pi i}\,\frac{1}{2} \oint w^{-s-1}\, dw$$

and it follows again that $\oint w^{-s-1}\, dw = 0$ for $s > 0$.

Our next important auxiliary result is

LEMMA 2.4.1. *For any positive integer* N *we have*

$$(2.4.3) \qquad H(z) = z\left(\frac{1}{e^z-1} - \frac{1}{z} + \frac{1}{2}\right) = P_N(z) + (-1)^N z^{2N+2} \mu_N(z)$$

where

$$(2.4.4) \qquad P_N(z) := \sum_{k=1}^{N} (-1)^{k-1} \frac{|B_{2k}|}{(2k)!} z^{2k}$$

and

$$(2.4.5) \qquad \mu_N(z) := \sum_{m=1}^{\infty} \frac{2}{(z^2+4\pi^2m^2)(2\pi m)^{2N}} .$$

PROOF. In order to prove this lemma we apply Taylor's formula as described in WHITTAKER & WATSON [60; p. 93]. We observe that (compare (2.1.5))

$$(2.4.6) \qquad P_N(z) = \sum_{k=1}^{N} \frac{H^{(2k)}(0)}{(2k)!} z^{2k} \quad \text{and} \quad H^{(2k+1)}(0) = 0$$

so that

$$(2.4.7) \qquad (-1)^N \mu_N(z) = \frac{1}{2\pi i} \oint \frac{H(w)}{(w-z)w^{2N+2}} dw$$

where $\oint$ denotes counter clockwise integration along a closed contour containing the points $w = 0$ and $w = z$ in its interior and such that it does not encircle any of the points $w = k2\pi i$, $k \in \mathbb{Z} \setminus \{0\}$. A standard application of the calculus of residues then yields

$$(2.4.8) \qquad (-1)^N \mu_N(z) = \frac{1}{2\pi i} \oint \frac{\frac{1}{e^w-1} - \frac{1}{w}+\frac{1}{2}}{(w-z)w^{2N+1}} dw =$$

$$= - \sum_{m=1}^{\infty} \left\{ \frac{1}{(2\pi im-z)(2\pi im)^{2N+1}} + \frac{1}{(-2\pi im-z)(-2\pi im)^{2N+1}} \right\} =$$

$$= (-1)^N \sum_{m=1}^{\infty} \frac{2}{(z^2+4\pi^2m^2)(2\pi m)^{2N}}$$

and the lemma follows. □

REMARKS.

1) We note that Lemma 2.4.1 is also true for $N = 0$. In this case we have the well known formula

$$(2.4.9) \qquad H(z) = \sum_{m=1}^{\infty} \frac{2z^2}{z^2+4\pi^2m^2} .$$

2) As an immediate consequence of Lemma 2.4.1 we have for any fixed $N > 0$

$$(2.4.10) \qquad \mu_N(x) = O(x^{-2}), \qquad (x \to \infty).$$

3) $\mu_N(z)$ is regular at $z = 0$.

LEMMA 2.4.2. *For any fixed* $N > 0$ *the function* $\mu_N(\frac{1}{x})x^{-2N-2}$ *is log-convex on* $\mathbb{R}^+$.

PROOF. In order to see this we write

$$(2.4.11) \qquad \mu_N(\tfrac{1}{x})x^{-2N-2} = 2\sum_{m=1}^{\infty} \frac{x^{-2N-2}}{(x^{-2}+4\pi^2m^2)(2\pi m)^{2N}} = 2\sum_{m=1}^{\infty} \frac{1}{(1+4\pi^2m^2x^2)(2\pi mx)^{2N}}$$

and observe that every term of this series is log-convex on $\mathbb{R}^+$. Indeed, for any (fixed) $a > \frac{1}{8}$, the function

$$(2.4.12) \qquad \phi_a(x) := -\log(1+x^2) - 2a\log x$$

is convex on $\mathbb{R}^+$. □

2.5. The case $2 < s < 3$

From

$$(2.5.1) \qquad T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} H(\tfrac{w}{n})w^{-s-1}\,dw$$

we obtain by means of the results of the previous section (for $2<s<3$)

$$(2.5.2) \qquad T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} (H(\tfrac{w}{n})-P_1(\tfrac{w}{n}))w^{-s-1}\,dw + $$
$$+ \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} P_1(\tfrac{w}{n})w^{-s-1}\,dw .$$

Since $P_1(z) = z^2/12$ we thus find that

$$(2.5.3) \qquad T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} (\tfrac{w}{n})^4(-1)^1\mu_1(\tfrac{w}{n})w^{-s-1}\,dw + $$
$$+ \frac{\Gamma(s+1)}{2\pi i}\frac{1}{12n^2} \oint (e^w-1)w^{-s}\,dw =$$
$$= \frac{1}{s+1} + \frac{s}{12n^2} - \frac{\Gamma(s+1)}{\pi}\sin(s-3)\pi \int_0^\infty \frac{1-e^{-t}}{t}(\tfrac{t}{n})^4\mu_1(\tfrac{t}{n})t^{-s-1}\,dt .$$

In Section 2.4 it was shown that $x^{-4}\mu_1(x^{-1})$ is log-convex on $\mathbb{R}^+$ so that for any $t > 0$, $(t/n)^4\mu_1(t/n)$ is log-convex as a function of $n \in \mathbb{N}$. Since $\sin(s-3)\pi < 0$ for $2 < s < 3$ it follows that $\{T_n - 1/(s+1) - s/(12n^2)\}_{n=1}^{\infty}$ is log-convex (in n) for any fixed $s \in (2,3)$, a result which is *even stronger* than the previously announced log-convexity of $\{T_n\}_{n=1}^{\infty}$.

2.6. Some remarks on the general case: $2N < s < 2(N+1)$

Similarly as before we have

$$(2.6.1) \qquad T_n(s) = \frac{1}{s+1} + \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} P_N(\tfrac{w}{n})w^{-s-1}\,dw + $$
$$+ (-1)^N\frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w} (\tfrac{w}{n})^{2N+2}\mu_N(\tfrac{w}{n})w^{-s-1}\,dw =$$
$$= \frac{1}{s+1} + I_1(n) + I_2(n), \text{ say.}$$

According to the preliminaries in Section 4 we have

$$(2.6.2)\qquad I_1(n) = \frac{\Gamma(s+1)}{2\pi i} \oint \frac{e^w-1}{w}\left\{\sum_{k=1}^{N}(-1)^{k-1}\frac{|B_{2k}|}{(2k)!}\left(\frac{w}{n}\right)^{2k}\right\}w^{-s-1}\,dw =$$

$$= \sum_{k=1}^{N}(-1)^{k-1}\frac{|B_{2k}|}{(2k)!}\,\frac{\Gamma(s+1)}{\Gamma(s-2k+2)}\,\frac{1}{n^{2k}}$$

and, similarly as before,

$$(2.6.3)\qquad I_2(n) = (-1)^N\frac{\Gamma(s+1)}{2\pi i}\oint \frac{e^w-1}{w}\left(\frac{w}{n}\right)^{2N+2}\mu_N\left(\frac{w}{n}\right)w^{-s-1}\,dw =$$

$$= (-1)^N\frac{\Gamma(s+1)\sin(s-2N-1)\pi}{\pi}\int_0^\infty \frac{1-e^{-t}}{t}\left(\frac{t}{n}\right)^{2N+2}\mu_N\left(\frac{t}{n}\right)t^{-s-1}\,dt$$

the last integral being convergent at $t = 0$ since $(2N+2) - s - 1 > -1$ and at $t = \infty$ since $-1 + (2N+2) - 2 - s - 1 < -1$. We now observe that

$$(2.6.4)\qquad \begin{aligned} &N \text{ even and } 2N+1 < s < 2N+2 \Rightarrow (-1)^N\sin(s-2N-1)\pi > 0\,,\\ &N \text{ even and } 2N < s < 2N+1 \Rightarrow \quad '' \quad < 0\,,\\ &N \text{ odd and } 2N < s < 2N+1 \Rightarrow \quad '' \quad > 0\,,\\ &N \text{ odd and } 2N+1 < s < 2N+2 \Rightarrow \quad '' \quad < 0\,.\end{aligned}$$

Hence, whenever we can show that $\{I_2(n)\}_{n=1}^\infty$ is log-convex then $\{T_n\}_{n=1}^\infty$ is log-convex if $(-1)^N\sin(s-2N-1)\pi > 0$. *It follows that our approach can only be successful if* $2N+1 < s < 2N+3$, *where* N *is even.*

2.7. The case $5 < s < 7$

We first assume $5 < s < 6$ so that

$$(2.7.1)\qquad T_n(s) - \frac{1}{s+1} = \frac{s}{12n^2} - \frac{s(s-1)(s-2)}{720n^4} + \text{log-convex (in } n).$$

Hence, in order to show the log-convexity of $\{T_n - 1/(s+1)\}_{n=1}^\infty$ it suffices to show the log-convexity of $\{s/(12n^2)-s(s-1)(s-2)/(720n^4)\}_{n=1}^\infty$. Since $5 < s < 6$ it is easily seen that this in its turn is a consequence of the log-convexity of $\{1/(n^2) - 1/(3n^4)\}_{n=1}^\infty$, the verification of which is a matter of routine.

Now let $6 < s < 7$, so that by the results of Section 2.6 it suffices to show the log-convexity of

$$(2.7.2)\qquad \left\{\frac{s}{12n^2} - \frac{s(s-1)(s-2)}{720n^4} + \frac{s(s-1)(s-2)(s-3)(s-4)}{42720n^6}\right\}_{n=1}^\infty$$

which, using the assumption $6 < s < 7$, is an easy consequence of the log-convexity of $\left\{\frac{1}{n^2} - \frac{7}{12n^4} + \frac{3}{89n^6}\right\}_{n=1}^\infty$, a (though tedious) matter of routine.

REMARK. For $9 < s < 10$ we would have to verify the log-convexity of

$$(2.7.3)\qquad \left\{\frac{s}{12n^2} - \frac{s(s-1)(s-2)}{720n^4} + \frac{s(s-1)\dots(s-4)}{42720n^6} - \frac{s(s-1)\dots(s-6)}{1209600n^8}\right\}_{n=1}^{\infty}$$

whereas for still larger values of s it seems practically unfeasible (*if true*) to prove the log-convexity (in n) of forms of such a complexity. However, splitting up the s-interval in appropriate subintervals and proceeding as above we may obtain further results.

3. CONVEXITY OF $T_n(s)$, $3 < s < 4$

THEOREM 3.1. *If* $3 < s < 4$ *then the sequence* $\{T_n(s)\}_{n=1}^{\infty}$ *is convex.*

Our proof is based upon the following

LEMMA 3.1. *For* $s > 0$ *we have*

$$(3.1)\qquad g_n(s) = \sum_{k=2}^{\infty} (-1)^k \frac{g_n(k)}{\Gamma(1+s-k)}$$

where

$$(3.2)\qquad g_n(s) := \frac{T_n(s) - \frac{1}{s+1}}{\Gamma(s+1)}, \qquad s > 0.$$

For the sake of clarity we postpone the proof of this lemma and first give the

PROOF of Theorem 3.1.
Recall that in Section 1 we proved the convexity of $\{T_n(k)\}_{n=1}^{\infty}$ for every $k \in \mathbb{N}$, from which it is clear that $\{g_n(k)\}_{n=2}^{\infty}$ is *convex* for every $k \in \mathbb{N}$. Now observe that for $3 < s < 4$

$$(3.3)\qquad \frac{(-1)^k}{\Gamma(1+s-k)} > 0 \qquad \text{for all } k \geqq 4,\ k \in \mathbb{N}$$

so that

$$(3.4)\qquad \sum_{k=4}^{\infty} (-1)^k \frac{g_n(k)}{\Gamma(1+s-k)}$$

is convex (in n) for any fixed $s \in (3,4)$. Hence, we are through if we can prove the convexity of

$$(3.5)\qquad \frac{g_n(2)}{\Gamma(s-1)} - \frac{g_n(3)}{\Gamma(s-2)} = \frac{g_n(2)}{(s-2)\Gamma(s-2)} - \frac{\frac{1}{2}g_n(2)}{\Gamma(s-2)} =$$

$$= \frac{1}{\Gamma(s-2)}\left(\frac{1}{s-2} - \frac{1}{2}\right)g_n(2)\ .$$

Since $3 < s < 4$ we have $1/(s-2) - 1/2 > 0$ and $\Gamma(s-2) > 0$; since $g_n(2)$ is convex our proof is complete. Note that we even showed the convexity of $\{T_n(s) - 1/(s+1)\}_{n=1}^{\infty}$. □

PROOF of Lemma 3.1.

For $s > 0$ we have (cf. KOWALEWSKI [21; p. 115]) for $0 \leq t \leq 1$

$$(3.6)\qquad (1-t)^s = \sum_{k=0}^{\infty} (-1)^k \binom{s}{k} t^k = 1+\Gamma(s+1) \sum_{k=1}^{\infty} (-1)^k \frac{t^k}{k!\Gamma(s-k+1)} .$$

Since T_n acts as a linear operator we easily find that

$$(3.7)\qquad \frac{T_n(s)}{\Gamma(s+1)} = \sum_{k=2}^{\infty} (-1)^k \frac{T_n(k)}{k!\Gamma(s-k+1)} .$$

Substituting $t = 1$ in (3.6) and replacing s by $s+1$ we obtain

$$(3.8)\qquad \frac{1}{(s+1)\Gamma(s+1)} = \sum_{k=2}^{\infty} (-1)^k \frac{1/(k+1)}{k!\Gamma(s-k+1)}$$

so that, in combination with (3.7), we get (3.1). □

REMARKS.

1) We may derive (3.1) directly from (3.6) by observing that $\Omega_n := T_n - \int_0^1$ acts as a linear operator.

2) For $s = 3$ we obtain from (3.1) that

$$(3.9)\qquad g_n(3) = \frac{g_n(2)}{\Gamma(2)} - \frac{g_n(3)}{\Gamma(1)}$$

so that

$$(3.10)\qquad g_n(3) = \tfrac{1}{2} g_n(2)$$

a result tacitly used in the proof of Theorem 3.1.

3) If $g_n(k)$ were log-convex (in n) for every $k \in \mathbb{N}$, then it would follow that $g_n(s)$ is log-convex for every $s \in (3,4)$.

4) During the preparation of this chapter we succeeded (in cooperation with N.M. TEMME) in showing that $\{T_n(s) - 1/(s+1)\}_{n=1}^{\infty}$ is *log-convex* for $3 < s < 4$.

4. LOG-CONVEX APPROXIMATION OF $\int_\alpha^\beta x^{-s}\, dx$

4.1. Preliminaries

LEMMA 4.1. *Let* $\{A_n(t)\}_{n=1}^{\infty}$ *be a log-convex sequence for each (fixed)* $t \in [\alpha,\beta]$. *If* $p(t) \geq 0$, *then the sequence* $\{b_n\}_{n=1}^{\infty}$, *defined by*

$$(4.1)\qquad b_n := \int_\alpha^\beta p(t) A_n(t)\, dt$$

is log-convex.

PROOF. See ARTIN [2]. □

REMARK. This is the "general theory" of log-convex functions alluded to in Section 2.

Let $f : [\alpha,\beta] \to \mathbb{R}$ be continuous. Similarly as in Chapter 1 we consider the n-th canonical trapezoidal approximations $T_n(f;\alpha,\beta)$ of $\int_\alpha^\beta f(x)\,dx$ defined by

$$(4.2)\qquad T_n(f;\alpha,\beta) := \frac{1}{n}\{-\frac{1}{2}f(\alpha) + \sum_{k=0}^{n} f(\alpha+k\frac{\beta-\alpha}{n}) - \frac{1}{2}f(\beta)\}.$$

4.2. Log-convexity of $\{T_n(e^{-\lambda x};\alpha,\beta)\}_{n=1}^{\infty}$

The following lemma is essential in our discussion.

LEMMA 4.2. *The function*

$$(4.3)\qquad K(x) := \frac{1}{x}\,\frac{e^{\frac{1}{x}}+1}{e^{\frac{1}{x}}-1}, \qquad (x \in \mathbb{R}^+)$$

satisfies

$$(4.4)\qquad (\log K(x))'' > 0 \qquad \text{(so that } K(x) \text{ is log-convex).}$$

PROOF. Define $\phi(x) := \log K(x)$ and observe that

$$(4.5)\qquad \phi''(x) = \frac{1}{x^2} - \frac{4}{x^3(e^{\frac{1}{x}} - e^{-\frac{1}{x}})} + \frac{2(e^{\frac{1}{x}} + e^{-\frac{1}{x}})}{x^4(e^{\frac{1}{x}} - e^{-\frac{1}{x}})^2}\ .$$

Setting $u = \frac{1}{x}$ we want to show that for $u > 0$

$$(4.6)\qquad 1 - \frac{4u}{e^u - e^{-u}} + \frac{2u^2(e^u + e^{-u})}{(e^u - e^{-u})^2} > 0$$

or

$$(4.7)\qquad e^{4u} - 2e^{2u} + 1 - 4u(e^{3u} - e^u) + 2u^2(e^{3u} + e^u) > 0.$$

The left-hand side is an entire function of u with power series expansion

$$(4.8)\qquad \sum_{n=0}^{\infty} c_n u^n\ , \text{ say.}$$

Now observe that $c_0 = c_1 = 0$ and that for $n \geqq 2$

$$(4.9)\qquad c_n = \frac{1}{n!}(4^n - 2^{n+1} - 4n3^{n-1} + 4n + 2n(n-1)3^{n-2} + 2n(n-1)).$$

Hence $c_2 = 0$, $c_3 = 0$, $c_4 = 2$, $c_5 = 4$, $c_6 = 77/18$. For $n \geqq 7$ we have

$$(4.10)\qquad n!c_n > -4n3^{n-1} + 2n(n-1)3^{n-2} = 2n(n-7)3^{n-2} \geqq 0$$

so that $c_n > 0$ for $n \geqq 7$.

This proves (4.7) and hence Lemma 4.2. □

We now prove

THEOREM 4.1. *Let* $\lambda \in \mathbb{R}$ *be fixed and let* $[\alpha,\beta] \subset \mathbb{R}$. *Then the sequence*

$$\{T_n(e^{\lambda x};\alpha,\beta)\}_{n=1}^{\infty} \tag{4.11}$$

is logarithmically convex (in n*).*

PROOF. Put $\Delta = \beta-\alpha$. Then we have

$$T_n(e^{\lambda x};\alpha,\beta) = \frac{1}{2n}\sum_{k=0}^{n-1}\{e^{\lambda(\alpha+k\Delta/n)} + e^{\lambda(\alpha+(k+1)\Delta/n)}\} = \tag{4.12}$$

$$= \frac{1}{2}e^{\lambda\alpha}\,\frac{e^{\lambda\Delta}-1}{\lambda\Delta}\,\frac{\lambda\Delta}{n}\,\frac{e^{\lambda\Delta/n}+1}{e^{\lambda\Delta/n}-1}\ .$$

Since $\frac{1}{2}e^{\lambda\alpha}(e^{\lambda\Delta}-1)/\lambda\Delta$ is positive, we must show that the sequence

$$\{K(\tfrac{n}{\lambda\Delta})\}_{n=1}^{\infty} \tag{4.13}$$

is log-convex. For $\lambda > 0$ this follows from Lemma 4.2. For $\lambda < 0$ observe that $K(x) = K(-x)$. For $\lambda = 0$ the theorem is trivial. □

4.3. The main Theorem (of Section 4)

THEOREM 4.2. *Let* $s > 0$ *be fixed and let* $\beta > \alpha > 0$. *Then the sequence*

$$\{T_n(x^{-s};\alpha,\beta)\}_{n=1}^{\infty} \tag{4.14}$$

is log-convex.

PROOF. For $s > 0$ and $x > 0$ we have

$$\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}\,du = x^s\int_0^\infty e^{-xt}t^{s-1}\,dt \tag{4.15}$$

so that

$$x^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-xt}t^{s-1}\,dt\ . \tag{4.16}$$

Since T_n acts as a linear operator, we have for $0 < \alpha < \beta$

$$T_n(x^{-s};\alpha,\beta) = \frac{1}{\Gamma(s)}\int_0^\infty T_n(e^{-xt};\alpha,\beta)t^{s-1}\,dt\ . \tag{4.17}$$

Since each sequence $T_n(e^{-xt};\alpha,\beta)$ is log-convex, the theorem follows directly from Lemma 4.1. □

REMARK. Theorem 4.2 can be generalized as follows. Let $\{c_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that

$$f(x) := \sum_{k=1}^{\infty} c_k x^{-k} \tag{4.18}$$

is convergent for $x \in [\alpha,\beta]$.

Then

$$(4.19)\qquad T_n(f;\alpha,\beta) = \sum_{k=1}^{\infty} c_k T_n(x^{-k};\alpha,\beta) = \sum_{k=1}^{\infty} \frac{c_k}{\Gamma(k)} \cdot \int_0^{\infty} T_n(e^{-xt};\alpha,\beta) t^{k-1}\, dt = \int_0^{\infty} g(t)\, T_n(e^{-xt};\alpha,\beta)\, dt$$

where $g(t) := \sum_{k=1}^{\infty} \frac{c_k}{\Gamma(k)} t^{k-1}$.

If $g(t)$ converges for $t \in \mathbb{R}^+$ and is non-negative on $\mathbb{R}^+$, then it follows that $T_n(f)$ is *log-convex*.

EXAMPLE. Let

$$(4.20)\qquad f(x) := \sum_{k=1}^{2m+1} (-1)^{k+1}\, x^{-k}\ .$$

Then

$$(4.21)\qquad g(t) = 1 - \frac{t}{1!} + \frac{t^2}{2!} - \dots + \frac{t^{2m}}{(2m)!}\ .$$

Since $e^{-t} = g(t) - \frac{t^{2m+1}}{(2m+1)!} e^{-\eta t}$ for some $\eta \in (0,1)$ by Taylor's theorem, it is clear that $g(t) > 0$ for all $t \in \mathbb{R}^+$. Hence $\{T_n(f;\alpha,\beta)\}_{n=1}^{\infty}$ is log-convex. The above argument can be directly extended to functions of the form

$$(4.22)\qquad f(x) = \sum_{k=1}^{\infty} c_k x^{-s_k}$$

where $0 < s_1 < s_2 < \dots$ are real numbers, the c_k's satisfying similar conditions as above.

The reader will have no difficulties in constructing an integral analogue of the above generalization of Theorem 4.2.

CHAPTER 3

ON A CONJECTURE OF ERDÖS

0. INTRODUCTION

In [12] ERDÖS proposed the following problem (compare OGILVY [47]). Prove that if m and n are positive integers satisfying

$$(0.0)\qquad (1-\frac{1}{m})^n > \frac{1}{2} > (1-\frac{1}{m-1})^n$$

then

$$(0.1)\qquad 1^n + 2^n + \ldots + (m-2)^n < (m-1)^n$$

and

$$(0.2)\qquad 1^n + 2^n + \ldots + m^n > (m+1)^n.$$

Show also that

$$(0.3)\qquad 1^n + 2^n + \ldots + (m-1)^n < m^n$$

in infinitely many instances and that

$$(0.4)\qquad 1^n + 2^n + \ldots + (m-1)^n > m^n$$

in infinitely many instances.

There is little or no doubt that the origin of this problem must be sought in the closely related (still open) question whether the *diophantine* equation

$$(*)\qquad 1^n + 2^n + \ldots + m^n = (m+1)^n$$

is solvable for $n > 1$. (See SIERPIŃSKI [53].)
In MOSER [46] it is shown that if (*) has a solution ($m \in \mathbb{N}$) with $n > 1$ then $m \geq 10^{1,000,000}$. In this paper Moser refers to a letter in which Erdös conjectured that (*) is satisfied only by n=1, m=2.

In 1975, using some of the results described in Chapter 1, we [29] were able to give a partial solution of Erdös' problem by showing that (0.1) is true indeed. Shortly afterwards Tijdeman (cf. [29]) gave a much simpler proof of this result and it is his proof which we will present in Section 1. ......
In Section 3 we prove the validity (under condition (0.0)) of the related inequality

$$(0.5)\qquad 1^n + 2^n + \ldots + (m+1)^n > (m+2)^n .$$

Our method of proof (also using some of the inequalities given in Chapter 1) is very similar to *our first proof* of (0.1).
In Section 4 we give the rather technical preparations for our further results: (Note that by Lemma 2.1 there exists for every $n \in \mathbb{N}$, $n \geq 2$, precisely one m satisfying (0.0))

1) The set of all $n \in \mathbb{N}$ for which inequality (0.2) is false has natural density zero.
2) The set of all $n \in \mathbb{N}$ for which inequality (0.3) is false has natural density zero.

In Section 6 we devote a few words to the solvability of the diophantine equation

$$(0.6) \qquad 1^n + 2^n + \ldots + m^n = G.(m+1)^n$$

where G is some given positive rational number.
Finally, in Section 7 we give a brief summary of the results of BEST & te RIELE [4] concerning the inequalities (0.3) and (0.4).

1. PROOF OF THE FIRST INEQUALITY

We succeeded in proving the first part of Erdös' conjecture, namely the inequality (0.1), by the use of one of the inequalities we proved in Chapter 1. But this proof, although elementary, was rather intricate and a very simple proof was shown to us by R. Tijdeman. As regards the inequality (0.1) we, therefore, restrict ourselves to Tijdeman's proof. However, the methods we developed in our proof were useful in our dealing with some related inequalities.

If

$$(1.1) \qquad \left(1 - \frac{1}{m-1}\right)^n < \frac{1}{2}$$

then also

$$(1.2) \qquad \left(1 - \frac{1}{k}\right)^n < \frac{1}{2}, \qquad (1 \leqq k < m-1)$$

so that

$$(1.3) \qquad (k-1)^n < \frac{1}{2} k^n, \qquad (1 \leqq k < m-1).$$

Hence

$$(1.4) \qquad \begin{array}{l} (m-2)^n < \frac{1}{2}(m-1)^n, \\ (m-3)^n < \frac{1}{2}(m-2)^n < (\frac{1}{2})^2(m-1)^n, \\ (m-4)^n < \frac{1}{2}(m-3)^n < (\frac{1}{2})^2(m-2)^n < (\frac{1}{2})^3(m-1)^n \\ \ldots\ldots\ldots\ldots\ldots \text{ etc. } \ldots\ldots\ldots\ldots\ldots \end{array}$$

so that certainly

$$(1.5) \qquad 1^n + 2^n + \ldots + (m-2)^n < (m-1)^n . \sum_{r=1}^{\infty} (\tfrac{1}{2})^r = (m-1)^n$$

proving (0.1). □

2. PREPARATIONS

If $n = 1$ then (0.0) reads

$$(2.1) \qquad 1 - \frac{1}{m} > \frac{1}{2} > 1 - \frac{1}{m-1}$$

which is equivalent to

$$(2.2) \qquad 2 < m < 3.$$

Since m is a positive integer (2.2) is impossible so that we cannot have $n = 1$. Therefore, from now on we assume $n \geqq 2$.

LEMMA 2.1. *If* $n \geq 2$ *then there is precisely one* $m(= m(n))$ *satisfying* (0.0), *to wit*

$$m(n) = [\lambda(n)] + 1 \tag{2.3}$$

where $[x]$ *denotes the greatest integer not exceeding* x *and*

$$\lambda(n) = \frac{1}{1-2^{-1/n}} . \tag{2.4}$$

PROOF. It is easy to see that (0.0) is equivalent to

$$\lambda(n) < m < \lambda(n) + 1. \tag{2.5}$$

Since $2^{1/n}$ is irrational for $n \geq 2$ it follows from (2.4) that the positive number $\lambda(n)$ is *irrational* so that there is *precisely one* positive integer $m(n)$ $(= [\lambda(n)])$ between $\lambda(n)$ and $\lambda(n) + 1$. Since $\lambda(n)$ is increasing it follows from (2.5) that

$$m(n) > \lambda(n) \geq \lambda(2) = 3.4\ldots \tag{2.6}$$

so that

$$m(n) \geq 4 \quad \text{for all } n \geq 2. \tag{2.7}$$

LEMMA 2.2. *For any pair* (m,n) *of positive integers we have*

$$\sigma_m(n) > \frac{m^{n+1}(m+1)^n}{(m+1)^{n+1} - m^{n+1}} . \tag{2.8}$$

PROOF. This follows from (the left-hand side of) inequality (1.5) in Section 1 of Chapter 1. □

LEMMA 2.3. *If* $x > 0$ *and* $n \in \mathbb{N}$ *then*

$$1 + \frac{x}{n+1} > (1+x)^{\frac{1}{n+1}} . \tag{2.9}$$

PROOF. This is an immediate consequence of Bernoulli's inequality

$$(1+a)^{n+1} > 1 + (n+1)a, \quad (n \in \mathbb{N};\ a > 0). \ \square \tag{2.10}$$

LEMMA 2.4. *For any* $x > 0$ *we have*

$$\frac{1}{x} - \frac{1}{2} < \frac{1}{e^x-1} < \frac{1}{x} - \frac{1}{2} + \frac{x}{12} . \tag{2.11}$$

PROOF. It is well known that (cf. KNOPP [19; p. 378])

$$\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2} = \sum_{r=1}^{\infty} \frac{2x}{x^2 + 4r^2\pi^2} . \tag{2.12}$$

Hence, if $x > 0$, it follows that

$$\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2} > 0 \tag{2.13}$$

and

$$\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2} < \sum_{r=1}^{\infty} \frac{2x}{4r^2\pi^2} = \frac{x}{2.\pi^2} \sum_{r=1}^{\infty} r^{-2} = \frac{x}{2.\pi^2}\frac{\pi^2}{6} = \frac{x}{12} . \ \square \tag{2.14}$$

3. A RELATED INEQUALITY

In this section we prove

THEOREM 3.1. *If* $n \geqq 2$ *and* m *is determined by* (0.0) *then*

$$1^n + 2^n + \ldots + (m+1)^n > (m+2)^n \, . \tag{3.1}$$

PROOF. In order to prove this we may just as well show that

$$1^n + 2^n + \ldots + (m+1)^n + (m+2)^n > 2.(m+2)^n \tag{3.2}$$

or

$$\sigma_{m+2}(n) > 2.(m+2)^n \, . \tag{3.3}$$

In view of Lemma 2.2 it suffices to show that

$$\frac{(m+2)^{n+1}(m+3)^n}{(m+3)^{n+1} - (m+2)^{n+1}} \geqq 2.(m+2)^n \tag{3.4}$$

which is easily seen to be equivalent to

$$\frac{m+2}{m+3} \geqq 2 - 2\left(\frac{m+2}{m+3}\right)^{n+1} \tag{3.5}$$

which in its turn may be rewritten as

$$2\left(1 - \frac{1}{m+3}\right)^{n+1} \geqq 1 + \frac{1}{m+3} \, . \tag{3.6}$$

Hence, since $m(n) > \lambda(n)$ by (2.6), it is enough to show that

$$2\left(1 - \frac{1}{\lambda(n)+3}\right)^{n+1} \geqq 1 + \frac{1}{\lambda(n)+3} \tag{3.7}$$

or

$$2^{\frac{1}{n+1}}\left(1 - \frac{1}{\lambda(n)+3}\right) \geqq \left(1 + \frac{1}{\lambda(n)+3}\right)^{\frac{1}{n+1}} \, . \tag{3.8}$$

Hence, in view of Lemma 2.3, it suffices to show that

$$2^{\frac{1}{n+1}}\left(1 - \frac{1}{\lambda(n)+3}\right) \geqq 1 + \frac{1}{(n+1)(\lambda(n)+3)} \tag{3.9}$$

which may be written as

$$2^{\frac{1}{n+1}} - 1 \geqq \frac{1}{\lambda(n)+3}\left(2^{\frac{1}{n+1}} + \frac{1}{n+1}\right) \tag{3.10}$$

or

$$3 + \lambda(n) \geqq \frac{2^{\frac{1}{n+1}} + \frac{1}{n+1}}{2^{\frac{1}{n+1}} - 1} \, . \tag{3.11}$$

Using the definition of $\lambda(n)$ we easily see that (3.11) is equivalent to

$$3 + \frac{1}{2^{\frac{1}{n}} - 1} \geqq \frac{1 + \frac{1}{n+1}}{2^{\frac{1}{n+1}} - 1} \tag{3.12}$$

so that, in view of Lemma 2.4, it suffices to show that

$$(3.13) \qquad 3 + \left(\frac{n}{\log 2} - \frac{1}{2}\right) \geqq \left(1 + \frac{1}{n+1}\right)\left(\frac{n+1}{\log 2} - \frac{1}{2} + \frac{\log 2}{12(n+1)}\right)$$

which may be reduced to

$$(3.14) \qquad 3 - \frac{2}{\log 2} \geqq \frac{1}{n+1}\left(\frac{\log 2}{12} - \frac{1}{2} + \frac{\log 2}{12(n+1)}\right) .$$

Since

$$(3.15) \qquad 3 - \frac{2}{\log 2} > 0$$

and

$$(3.16) \qquad \frac{\log 2}{12} - \frac{1}{2} + \frac{\log 2}{12(n+1)} \leqq \frac{\log 2}{12} - \frac{1}{2} + \frac{\log 2}{36} = \frac{\log 2}{9} - \frac{1}{2} < 0$$

it follows that (3.14) is true, completing the proof of Theorem 3.1. □

4. FURTHER PREPARATIONS AND THE MAIN AUXILIARY THEOREM

The results presented in Sections 4 and 5 were obtained in cooperation with H.J.J. te RIELE (cf. [29] and [30]). In these sections we discuss inequalities (0.2) and (0.3) in Erdös' conjecture. In Section 5 we show that the natural density of the set of all $n \in \mathbb{N}$ for which (0.2) is true equals 1 and that the natural density of the set of all $n \in \mathbb{N}$ for which (0.3) is true also equals 1, so that (0.3) certainly holds in infinitely many instances.
We begin our considerations with some technicalities.

In Section 2 it was already shown that we may assume $n \geqq 2$ and that (0.0) entails that for any given $n \in \mathbb{N}$ the number $m(= m(n))$ is uniquely determined by

$$(4.1) \qquad \lambda(n) < m(n) < \lambda(n) + 1$$

or

$$(4.2) \qquad m(n) = [\lambda(n)] + 1$$

where

$$(4.3) \qquad \lambda(n) = \frac{1}{1 - 2^{-1/n}} = 1 + \frac{1}{2^{1/n} - 1} .$$

From (4.1), (4.3) and Lemma 2.4 it follows that

$$(4.4) \qquad m(n) > \lambda(n) = 1 + \frac{1}{\exp(\frac{\log 2}{n}) - 1} > 1 + \left\{\frac{n}{\log 2} - \frac{1}{2}\right\} > \frac{n}{\log 2}$$

so that

$$(4.5) \qquad \frac{n}{m(n)} < \log 2 .$$

Similarly we obtain

$$(4.6) \qquad m(n) < 1 + \lambda(n) = 2 + \frac{1}{\exp(\frac{\log 2}{n}) - 1} <$$

$$< 2 + \left\{\frac{n}{\log 2} - \frac{1}{2} + \frac{\log 2}{12n}\right\} \leqq \frac{n}{\log 2} + \frac{3}{2} + \frac{\log 2}{24} .$$

Since $m(2) = 4$ and

$$(4.7) \qquad \frac{n}{\log 2} + \frac{3}{2} + \frac{\log 2}{24} < 2n, \qquad (n \geq 3)$$

it follows that

$$(4.8) \qquad \frac{n}{m(n)} \geq \frac{1}{2}, \qquad (n \geq 2).$$

Moreover, from (4.4) and (4.6) it is clear that

$$(4.9) \qquad \lim_{n\to\infty} \frac{n}{m(n)} = \log 2 .$$

Similarly as in Chapter 1 we define

$$(4.10) \qquad \sigma_m(n) := \sum_{k=1}^{m} k^n, \qquad (m,n \in \mathbb{N}).$$

In Chapter 1 (Proposition 1.1) it was shown that for all $m,n \in \mathbb{N}$

$$(4.11) \qquad \frac{m^{n+1}(m+1)^n}{(m+1)^{n+1} - m^{n+1}} < \sigma_m(n) < \frac{m^n(m+1)^{n+1}}{(m+1)^{n+1} - m^{n+1}} .$$

We now define $\theta := \theta(m,n)$ implicitly by

$$(4.12) \qquad \sigma_m(n) = \frac{m^n(m+1)^n(m+\theta)}{(m+1)^{n+1} - m^{n+1}}$$

or, more explicitly, by

$$(4.13) \qquad \theta(m,n) = - m + (m+1)\frac{\sigma_m(n)}{m^n}\{1 - (\frac{m}{m+1})^{n+1}\}$$

so that by (4.11) we have

$$(4.14) \qquad 0 < \theta(m,n) < 1.$$

Since, by Theorem 1.2 in Chapter 1

$$(4.15) \qquad \sigma_m(n) > \frac{1}{2}\, \frac{m^{n+1}(m+1)^n+m^n(m+1)^{n+1}}{(m+1)^{n+1} - m^{n+1}} = \frac{m^n(m+1)^n(m+\frac{1}{2})}{(m+1)^{n+1} - m^{n+1}}, \qquad (m,n \in \mathbb{N})$$

we even have

$$(4.16) \qquad \frac{1}{2} \leq \theta(m,n) < 1.$$

Concerning the function $\theta(m,n)$ we have the

(MAIN AUXILIARY) THEOREM 4. *If for* $n \geq 2$ *the number* $m = m(n)$ *is determined by* (0.0) *then*

$$(4.17) \qquad \lim_{n\to\infty} \theta(m,n) = 2(1 - \log 2) = 0.613\,705 \ldots .$$

Before proving this we first examine the sums $\sigma_m(n)$ somewhat closer. By means of the Euler-Maclaurin summation formula we readily obtain (cf. KNOPP [19; p. 527])

$$(4.18) \qquad \sigma_m(n) = \frac{m^{n+1}}{n+1} + \frac{1}{2}\, m^n + \sum_{r=1}^{[\frac{n}{2}]} \frac{B_{2r}}{2r} \binom{n}{2r-1} m^{n-2r+1}$$

or

$$(4.19)\qquad \frac{\sigma_m(n)}{m^n} = \frac{m}{n+1} + \frac{1}{2} + \sum_{r=1}^{[\frac{n}{2}]} \frac{B_{2r}}{2r} \binom{n}{2r-1} m^{-2r+1}$$

where the Bernoulli numbers B_k are defined by

$$(4.20)\qquad \frac{z}{e^z-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k, \qquad (|z| < 2\pi).$$

It is well known that for any real $\alpha \neq 0$ (cf. KNOPP [19; p. 528])

$$(4.21)\qquad \frac{1}{e^{\alpha}-1} = \frac{1}{\alpha} - \frac{1}{2} + \sum_{r=1}^{k} \frac{B_{2r}}{(2r)!} \alpha^{2r-1} + R_k(\alpha)$$

with

$$(4.22)\qquad R_k(\alpha) = \frac{\alpha^{2k+1}}{e^{\alpha}-1} \int_0^1 P_{2k+1}(x)\, e^{\alpha x}\, dx$$

so that

$$(4.23)\qquad \frac{1}{e^{n/m}-1} - \frac{m}{n} + \frac{1}{2} = \sum_{r=1}^{k} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} + R_k\left(\frac{n}{m}\right).$$

Taking $k = [\frac{n}{2}]$ in (4.23) we obtain from (4.19) and (4.23) that

$$(4.24)\qquad \left(\frac{1}{e^{n/m}-1} - \frac{m}{n} + \frac{1}{2}\right) - \left(\frac{\sigma_m(n)}{m^n} - \frac{m}{n+1} - \frac{1}{2}\right) =$$

$$= \frac{1}{e^{n/m}-1} + 1 - \frac{m}{n(n+1)} - \frac{\sigma_m(n)}{m^n} =$$

$$= \sum_{r=2}^{k} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \left(1 - \frac{n(n-1)\ldots(n-2r+2)}{n^{2r-1}}\right) + R_k\left(\frac{n}{m}\right) =$$

$$= \sum_{r=2}^{k} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \delta_n(2r-2) + R_k\left(\frac{n}{m}\right)$$

where δ_n is defined by

$$(4.25)\qquad \delta_n(a) := 1 - \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{a}{n}\right), \qquad (a \in \mathbb{N}).$$

From (4.25) it is easily seen that for any fixed $a \in \mathbb{N}$

$$(4.26)\qquad \lim_{n\to\infty} n.\delta_n(a) = 1 + 2 + \ldots + a = \frac{1}{2}\, a(a+1).$$

By mathematical induction it is easily shown that

$$(4.27)\qquad (0 <)\ \delta_n(a) < \frac{2^a}{n}, \qquad (1 \leq a < n;\ n \geq 2)$$

and as a consequence we have

$$(4.28)\qquad \left|\frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \delta_n(2r-2)\right| \leq \frac{|B_{2r}|}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} \frac{2^{2r-2}}{n} = \frac{1}{2n} \frac{|B_{2r}|}{(2r)!} \left(\frac{2n}{m}\right)^{2r-1}$$

so that, in view of (4.5),

$$(4.29)\qquad \left|\frac{B_{2r}}{(2r)!} \left(\frac{n}{m(n)}\right)^{2r-1} n\, \delta_n(2r-2)\right| < \frac{1}{2} \frac{|B_{2r}|}{(2r)!} (2 \log 2)^{2r-1}$$

the right-hand side of (4.29) being the general term of a convergent series with positive terms (see (4.20) and note that $\log 2 < \pi$). Hence, by a uniform convergence argument (or by Lebesgue's dominated convergence theorem) we obtain

$$(4.30) \qquad \lim_{n\to\infty} \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m}\right)^{2r-1} n\,\delta_n(2r-2) =$$

$$= \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r)!} (\log 2)^{2r-1} \tfrac{1}{2}(2r-2)(2r-1) =$$

$$= \tfrac{1}{2}(\log 2)^2 \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r)!} (2r-1)(2r-2)(\log 2)^{2r-3} =$$

$$= \tfrac{1}{2}(\log 2)^2 \frac{d^2}{dx^2}\Big\{ \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r)!} x^{2r-1}\Big\}_{x=\log 2} .$$

Now observe that (cf. KNOPP [19; p. 204])

$$(4.31) \qquad x \cot x = 1 - \frac{B_2}{2!}(2x)^2 + \frac{B_4}{4!}(2x)^4 - \ldots + \ldots$$

from which it is easily seen that

$$(4.32) \qquad \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r)!} x^{2r-1} = \frac{i}{2} \cot \frac{ix}{2} - \frac{1}{x} - \frac{x}{12}$$

so that

$$(4.33) \qquad \frac{d^2}{dx^2}\Big\{ \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r)!} x^{2r-1}\Big\} = \frac{d^2}{dx^2}\Big\{\frac{i}{2} \cot \frac{ix}{2} - \frac{1}{x} - \frac{x}{12}\Big\} =$$

$$= -\frac{2}{x^3} + \frac{d}{dx}\Big\{\frac{1}{4(\sin \frac{ix}{2})^2}\Big\} = -\frac{2}{x^3} - \frac{e^{-x} - e^{x}}{(e^{-x/2} - e^{x/2})^4}$$

which, for $x = \log 2$, takes the value

$$(4.34) \qquad \frac{d^2}{dx^2}\Big\{\frac{i}{2} \cot \frac{ix}{2} - \frac{1}{x} - \frac{x}{12}\Big\}_{x=\log 2} = \frac{-2}{(\log 2)^3} + 6 .$$

Hence, defining

$$(4.35) \qquad \rho(n) := n \sum_{r=2}^{k} \frac{B_{2r}}{(2r)!} \left(\frac{n}{m(n)}\right)^{2r-1} \delta_n(2r-2)$$

we deduce from (4.24) that for $m = m(n)$

$$(4.36) \qquad \frac{\sigma_m(n)}{m^n} = \frac{1}{e^{n/m}-1} + 1 - \frac{m}{n(n+1)} - \frac{\rho(n)}{n} - R_k\left(\frac{n}{m}\right)$$

where, in view of (4.30), (4.33) and (4.34),

$$(4.37) \qquad \lim_{n\to\infty} \rho(n) = \tfrac{1}{2}(\log 2)^2\left(\frac{-2}{(\log 2)^3} + 6\right) = -\frac{1}{\log 2} + 3(\log 2)^2 .$$

As to $R_k(\frac{n}{m})$ we have the following estimate

$$(4.38) \qquad \left|R_k\left(\frac{n}{m}\right)\right| \leq \frac{\left(\frac{n}{m}\right)^{2k+1}}{e^{n/m} - 1} \int_0^1 |P_{2k+1}(x)|\, e^{\frac{nx}{m}}\, dx .$$

Since

$$(4.39)\qquad \max_{0 \leq x \leq 1} |P_{2k+1}(x)| \leq \frac{4}{(2\pi)^{2k+1}}, \qquad \text{(cf. KNOPP [19; p. 527])}$$

and

$$(4.40)\qquad 2k+1 = 2[\tfrac{n}{2}] + 1 \geq n$$

it follows from (4.5), (4.8) and (4.38) that

$$(4.41)\qquad |R_k(\tfrac{n}{m})| \leq (\frac{\log 2}{2\pi})^n \frac{8}{\exp(\frac{1}{2}) - 1}$$

so that $R_k(\frac{n}{m})$ tends exponentially fast to zero as $n \to \infty$.
As a simple consequence of (4.36), (4.37) and (4.41) we have

$$(4.42)\qquad \lim_{n\to\infty} \frac{\sigma_m(n)}{m^n} = \frac{1}{e^{\log 2} - 1} + 1 = 2$$

(a relation which may also be proved by much simpler means).

PROOF of Theorem 4. From (4.13) it follows that

$$(4.43)\qquad \theta(m,n) = m\{\frac{\sigma_m(n)}{m^n}(1 - (1 - \frac{1}{m+1})^{n+1}) - 1\} + \frac{\sigma_m(n)}{m^n}(1 - (1 - \frac{1}{m+1})^{n+1}).$$

Since

$$(4.44)\qquad \lim_{n\to\infty}(1 + \frac{\alpha(n)}{n})^n = e^{\alpha} \quad \text{if} \quad \lim_{n\to\infty}\alpha(n) = \alpha$$

it follows from (4.9) and (4.42) that

$$(4.45)\qquad \lim_{n\to\infty} \frac{\sigma_m(n)}{m^n}(1 - (1 - \frac{1}{m+1})^{n+1}) = 2.(1 - \frac{1}{2}) = 1$$

so that, in order to determine $\lim_{n\to\infty}\theta(m,n)$, we only need to study the asymptotic behaviour of

$$(4.46)\qquad m\{\frac{\sigma_m(n)}{m^n}(1 - (1 - \frac{1}{m+1})^{n+1}) - 1\} =$$

$$= m\{(\frac{1}{e^{n/m} - 1} + 1 - \frac{m}{n(n+1)} - \frac{\rho(n)}{n} - R_k(\tfrac{n}{m}))(1 - (1 - \frac{1}{m+1})^{n+1}) - 1\} =$$

$$= - m\{\frac{m}{n(n+1)} + \frac{\rho(n)}{n}\}\{1 - (1 - \frac{1}{m+1})^{n+1}\} +$$

$$+ m\{(\frac{1}{e^{n/m} - 1} + 1)(1 - (1 - \frac{1}{m+1})^{n+1}) - 1\} +$$

$$- m\, R_k(\tfrac{n}{m})\{1 - (1 - \frac{1}{m+1})^{n+1}\}.$$

Since $R_k(\frac{n}{m})$ tends exponentially fast to zero as $n \to \infty$ and $m(n) = \mathcal{O}(n)$ we obtain

$$(4.47)\qquad \lim_{n\to\infty} m\, R_k(\tfrac{n}{m})\{1 - (1 - \frac{1}{m+1})^{n+1}\} = 0 .$$

Next observe that

$$(4.48)\qquad \lim_{n\to\infty} m\{\frac{m}{n(n+1)} + \frac{\rho(n)}{n}\} = \lim_{n\to\infty}\{\frac{m^2}{n(n+1)} + \frac{m}{n}\rho(n)\} =$$

$$= \frac{1}{(\log 2)^2} + \frac{1}{\log 2}\{- \frac{1}{\log 2} + 3(\log 2)^2\} = 3 \log 2$$

so that

$$(4.49) \quad \lim_{n\to\infty} - m\{\frac{m}{n(n+1)} + \frac{\rho(n)}{n}\}\{1 - (1 - \frac{1}{m+1})^{n+1}\} = - \frac{3}{2} \log 2 \ .$$

Finally we have

$$(4.50) \quad m\{(\frac{1}{e^{n/m} - 1} + 1)(1 - (1 - \frac{1}{m+1})^{n+1}) - 1\} = m \frac{1 - e^{n/m}(1 - \frac{1}{m+1})^{n+1}}{e^{n/m} - 1} =$$

$$= \frac{m}{e^{n/m} - 1}\{1 - \exp(\frac{n}{m} + (n+1)\log(1 - \frac{1}{m+1}))\} =$$

$$= - \frac{m}{e^{n/m}-1} \cdot \frac{\exp(\frac{n}{m} + (n+1)\log(1 - \frac{1}{m+1})) - 1}{\frac{n}{m} + (n+1)\log(1 - \frac{1}{m+1})}\{\frac{n}{m} + (n+1)\log(1 - \frac{1}{m+1})\}$$

so that, in view of

$$(4.51) \quad \lim_{n\to\infty} \{\frac{n}{m} + (n+1)\log(1 - \frac{1}{m+1})\} = \log 2 + \log \frac{1}{2} = 0$$

we obtain

$$(4.52) \quad \lim_{n\to\infty} (4.50) = - \lim_{n\to\infty} m\{\frac{n}{m} + (n+1)\log(1 - \frac{1}{m+1})\} =$$

$$= - \lim_{n\to\infty} m\{\frac{n}{m} - (n+1)(\frac{1}{m+1} + \frac{1}{2(m+1)^2} + O(m^{-3}))\} =$$

$$= - \lim_{n\to\infty} m\{\frac{n}{m} - \frac{n+1}{m+1} - \frac{n+1}{2(m+1)^2}\} = - \lim_{n\to\infty}\{\frac{n-m}{m+1} - \frac{m(n+1)}{2(m+1)^2}\} =$$

$$= - (\log 2 - 1 - \frac{1}{2} \log 2) = 1 - \frac{1}{2} \log 2 \ .$$

Combining (4.45) through (4.52) with (4.43) we find that

$$(4.53) \quad \lim_{n\to\infty} \theta(m,n) = 1 + 0 - \frac{3}{2} \log 2 + (1 - \frac{1}{2} \log 2) = 2(1 - \log 2)$$

completing the proof of Theorem 4. □

5. SOME FURTHER RESULTS

THEOREM 5.1. *The set of all* $n \in \mathbb{N}$ *for which inequality* (0.2) *is false has natural density equal to zero.*

Before proving this theorem we study the numbers $m(n)-\lambda(n)$ somewhat closer.

LEMMA 5.1. *If the (real) sequence* $\{\alpha(n)\}_{n=1}^{\infty}$ *is uniformly distributed modulo* 1 *(u.d. mod* 1*, for short) and if* $\{\beta(n)\}_{n=1}^{\infty}$ *is any (real) convergent sequence then also* $\{\alpha(n) + \beta(n)\}_{n=1}^{\infty}$ *is u.d. mod* 1.

PROOF. See KUIPERS & NIEDERREITER [22; p. 3]. □

LEMMA 5.2. *The (real) sequence* $\{\alpha(n)\}_{n=1}^{\infty}$ *is u.d. mod* 1 *if and only if the sequence* $\{-\alpha(n)\}_{n=1}^{\infty}$ *is u.d. mod* 1.

PROOF. Easy. □

LEMMA 5.3. *The sequence* $\{m(n) - \lambda(n)\}_{n=2}^{\infty}$ *is uniformly distributed on the interval* $(0,1)$.

PROOF. Since $m(n) \in \mathbb{N}$ and $\lambda(n) < m(n) < \lambda(n) + 1$ it suffices to show that $\{-\lambda(n)\}_{n=2}^{\infty}$ is u.d. mod 1. In view of Lemma 5.2 it therefore suffices to show that $\{\lambda(n)\}_{n=2}^{\infty}$ is u.d. mod 1. Observing that

$$(5.1) \qquad \lambda(n) = 1 + \frac{1}{2^{1/n} - 1} = 1 + \frac{1}{\exp(\frac{\log 2}{n}) - 1} =$$

$$= 1 + (\frac{n}{\log 2} - \frac{1}{2} + O(\frac{1}{n})) = \frac{n}{\log 2} + \frac{1}{2} + O(\frac{1}{n}), \qquad (n \to \infty)$$

we obtain from Lemma 5.1 and the irrationality of log 2 that $\{\lambda(n)\}_{n=2}^{\infty}$ is u.d. mod 1 (compare KOKSMA [20; p. 92, Satz 9]), proving the lemma. □

Next we have the simple (though crucial)

LEMMA 5.4. *If* $\{\alpha(n)\}_{n=1}^{\infty}$ *is uniformly distributed on the interval* $(0,1)$ *and* $\{\alpha(n_k)\}_{k=1}^{\infty}$ *is any convergent subsequence then the natural density of* $\{n_k\}_{k=1}^{\infty}$ *is equal to zero.*

PROOF. Easy. □

PROOF of Theorem 5.1. If (0.2) is false for only finitely many $n \in \mathbb{N}$ then we are done. Therefore, we assume (0.2) to be false for *infinitely* many n. For *these* n we have

$$(5.2) \qquad 1^n + 2^n + \ldots + m^n \leqq (m+1)^n$$

or

$$(5.3) \qquad \sigma_m(n) \leqq (m+1)^n .$$

Hence, writing θ instead of $\theta(m,n)$,

$$(5.4) \qquad \frac{m^n(m+1)^n(m+\theta)}{(m+1)^{n+1} - m^{n+1}} \leqq (m+1)^n$$

so that

$$(5.5) \qquad m^n(m+\theta) \leqq (m+1)^{n+1} - m^{n+1}$$

or

$$(5.6) \qquad 2 + \frac{\theta}{m} \leqq (1 + \frac{1}{m})^{n+1}$$

which may be rewritten as

$$(5.7) \qquad m \leqq \frac{1}{(2 + \frac{\theta}{m})^{1/(n+1)} - 1} .$$

From this it follows that

$$(5.8) \qquad 0 < m(n) - \lambda(n) = -1 + m(n) - \frac{1}{2^{1/n} - 1} \leqq$$

$$\leqq -1 + \frac{1}{(2+\frac{\theta}{m})^{1/(n+1)} - 1} - \frac{1}{2^{1/n} - 1} =$$

$$= -1 + \frac{1}{\exp(\frac{1}{n+1}\log(2+\frac{\theta}{m})) - 1} - \frac{1}{\exp(\frac{1}{n}\log 2) - 1} =$$

$$= -1 + \{\frac{n+1}{\log(2+\frac{\theta}{m})} - \frac{1}{2} + O(\frac{1}{n})\} - \{\frac{n}{\log 2} - \frac{1}{2} + O(\frac{1}{n})\} =$$

$$= -1 + \{\frac{1}{\log(2+\frac{\theta}{m})} + n\{\frac{1}{\log(2+\frac{\theta}{m})} - \frac{1}{\log 2}\} + O(\frac{1}{n}) =$$

$$= -1 + \frac{1}{\log(2+\frac{\theta}{m})} - \frac{n\log(1+\frac{\theta}{2m})}{\log 2\,\log(2+\frac{\theta}{m})} + O(\frac{1}{n}) .$$

In view of Theorem 4 we have

$$(5.9)\qquad \lim_{n\to\infty} n\log(1+\frac{\theta}{2m}) = \lim_{n\to\infty}\log(1+\frac{\frac{n\theta}{2m}}{n})^n = \log(\exp(\lim_{n\to\infty}\frac{n\theta}{2m})) =$$

$$= \lim_{n\to\infty}\frac{n\theta}{2m} = (1-\log 2).\log 2$$

so that, if n runs through those positive integers for which (0.2) is false

$$(5.10)\qquad 0 \leqq \limsup_{n\to\infty}\{m(n)-\lambda(n)\} \leqq -1 + \frac{1}{\log 2} - \frac{(1-\log 2)\log 2}{(\log 2)^2} = 0$$

from which it is clear that

$$(5.11)\qquad \lim_{n\to\infty}\{m(n)-\lambda(n)\} = 0, \quad \text{(where n is such that (0.2) is } \textit{false}).$$

From this and Lemmas 5.3 and 5.4 it follows that the set of all n for which (0.2) is false, has natural density equal to zero, completing the proof of Theorem 5.1. □

THEOREM 5.2. *The set of all* $n \in \mathbb{N}$ *for which inequality* (0.3) *is false has natural density equal to zero.*

PROOF. Suppose that the inequality in question is false for infinitely many $n \in \mathbb{N}$. For *these* n we have

$$(5.12)\qquad 1^n + 2^n + \ldots + (m-1)^n \geqq m^n$$

or

$$(5.13)\qquad \sigma_{m-1}(n) \geqq m^n .$$

Writing θ instead of $\theta(m-1,n)$ we have, in view of (4.12), that

$$(5.14)\qquad (m-1)^n(m-1+\theta) \geqq m^{n+1} - (m-1)^{n+1}$$

which may be written as

$$(5.15)\qquad m \geqq 1 + \frac{1}{(2+\frac{\theta}{m-1})^{1/(n+1)} - 1} .$$

It follows that

$$(5.16)\qquad 1 > m(n) - \lambda(n) \geq 1 + \frac{1}{(2+\frac{\theta}{m-1})^{1/(n+1)} - 1} - (1 + \frac{1}{2^{1/n}-1}) =$$

$$= \frac{1}{(2+\frac{\theta}{m-1})^{1/(n+1)} - 1} - \frac{1}{2^{1/n} - 1}$$

and, similarly as in the proof of Theorem 5.1, we get

$$(5.17)\qquad \lim_{n\to\infty} \{m(n) - \lambda(n)\} = 1$$

where n is such that (0.3) is *false*. Again, utilizing Lemmas 5.3 and 5.4, this completes the proof of Theorem 5.2. □

Consequently, Erdös' conjecture about inequality (0.3) is true.

6. A RELATED PROBLEM

Let G be any rational constant greater than $(e^{2\pi} - 1)^{-1}$ and consider the diophantine equation

$$(6.1)\qquad 1^n + 2^n + \ldots + m^n = G.(m+1)^n, \qquad (m,n \in \mathbb{N}).$$

By applying techniques resembling those used in the previous two sections it can be shown that the set of all $n \in \mathbb{N}$ for which (6.1) has a solution $m \in \mathbb{N}$, has natural density zero. A (quite technical) proof may be found in [32].

7. THE FINISHING TOUCH ...

For the sake of completeness we present in this section a summary of the main results (with slight modifications) concerning Erdös' conjecture, obtained by BEST & te RIELE (B & R, for short).
For the proofs we refer to their report [4].

THEOREM 7.1. If m and n are integers such that $m \geq 2$ and

$$(7.1)\qquad n > (m - \frac{3}{2} - \frac{1}{12m})\log 2$$

then

$$(7.2)\qquad (1 - \frac{1}{m-1})^n < \frac{1}{2}\ .$$

THEOREM 7.2. If m and n are positive integers such that

$$(7.3)\qquad n < (m - \frac{3}{2} - \frac{1}{256m})\log 2$$

then

$$(7.4)\qquad (1 - \frac{1}{m})^n > \frac{1}{2}$$

and

$$(7.5)\qquad 1^n + 2^n + \ldots + (m-1)^n > m^n\ .$$

From these two theorems B & R derive the following

THEOREM 7.3. If m and n are positive integers such that

$$(7.6) \qquad (1 - \frac{1}{m-1})^n \geqq \frac{1}{2}$$

then

$$(7.7) \qquad 1^n + 2^n + \ldots + (m-1)^n > m^n .$$

CONSEQUENCE: In Theorem 7.3 replace m by m+1 and it is easily seen that (0.2) is true under condition (0.0). Hence, the only still unsettled part of Erdös' conjecture consists of (0.4).
The following analysis is mainly devoted to this part of the conjecture.

As a further consequence of the previous theorems B & R obtain

THEOREM 7.4. If m and n are positive integers such that

$$(7.8) \qquad \frac{1}{128mn} < \frac{2m-3}{n} - \frac{2}{\log 2} < \frac{1}{6mn}$$

then the conditions of Theorems 7.1 and 7.2 are satisfied so that (0.0) is true, i.e.

$$(7.9) \qquad (1 - \frac{1}{m})^n > \frac{1}{2} > (1 - \frac{1}{m-1})^n$$

and (0.4) is true, i.e.

$$(7.10) \qquad 1^n + 2^n + \ldots + (m-1)^n > m^n.$$

In order to put this theorem effectively to work B & R prove

THEOREM 7.5. Let $(b_0; b_1, b_2, b_3, \ldots)$ be the regular continued fraction expansion of 2/log 2. If p_k/q_k (in its lowest terms) denotes the corresponding k-th approximation of 2/log 2 and if, for some $k \geqq 5$, p_k is odd and b_{k+1} satisfies

$$(7.11) \qquad 9 \leqq b_{k+1} \leqq 182$$

then the positive integers

$$(7.12) \qquad m := (p_k+3)/2 \quad \text{and} \quad n := q_k$$

satisfy (7.8).

By means of this theorem B & R found 33 pairs (m,n) of positive integers satisfying (0.0) and (0.4), the smallest being

$$(7.13) \qquad m = 1\ 121\ 626\ 023\ 352\ 385 \quad \text{and} \quad n = 777\ 451\ 915\ 729\ 368.$$

Next we mention

THEOREM 7.6. If m is an integer $\geqq 2$ and the *real* number s is defined by

$$(7.14)\qquad (1 - \frac{1}{m-1})^s = \frac{1}{2}$$

then

$$(7.15)\qquad s = (m - \frac{3}{2} - (\frac{1}{12} + o(1))m^{-1})\log 2, \qquad (m \to \infty).$$

THEOREM 7.7. If m is an integer $\geqq 2$ and the *real* number s is defined by

$$(7.16)\qquad 1^s + 2^s + \ldots + (m-1)^s = m^s$$

then

$$(7.17)\qquad s = (m - \frac{3}{2} - (\frac{25}{12} - 3\log 2 + o(1))m^{-1})\log 2, \qquad (m \to \infty).$$

By means of the last two theorems B & R derive the following *improvement* of Theorem 5.2.

THEOREM 7.8. If M is the set of all positive integers $\geqq 2$ for which there exists an $n \in \mathbb{N}$ such that $(1 - \frac{1}{m-1})^n < \frac{1}{2}$ and $1^n+2^n+\ldots+(m-1)^n \geqq m^n$, then the number of $m \in M$ with the property $m \leqq x$ is $O(\log x)$ as $x \to \infty$.

From Theorems 7.6 and 7.7 B & R also derive

THEOREM 7.9. If there exist infinitely many pairs (m,n) of positive integers such that $(1 - \frac{1}{m-1})^n < \frac{1}{2}$ and $1^n+2^n+\ldots+(m-1)^n > m^n$ then there exist infinitely many partial denominators b_k in the regular continued fraction expansion $(b_0; b_1, b_2, b_3, \ldots)$ of $2/\log 2$ such that

$$(7.18)\qquad 7 \leqq b_k \leqq 185.$$

REMARK. In view of this last theorem it occurs to us that a *complete* proof of Erdös' conjecture is beyond the scope of present day mathematics. Compare R.K. GUY, *Unsolved problems in number theory*, Springer (1981) pp. 85-86.

CHAPTER 4

ON THE ZEROS OF "SECTIONS OF RIEMANN's ZETA FUNCTION"

0. INTRODUCTION

We begin this chapter by recalling Wiener's general tauberian theorem for the real line. For any $f \in L^1(\mathbb{R})$ and any $h \in \mathbb{R}$ let $f_h \in L^1(\mathbb{R})$ be the h-translate of f, i.e.

$$(0.1) \qquad f_h(x) := f(x+h) , \qquad (x \in \mathbb{R})$$

and let S_f denote the span of the set of all translates of f, i.e.

$$(0.2) \qquad S_f := \{ \sum_{n=1}^{N} c_n f_{h_n} \mid N \in \mathbb{N};\ c_n \in \mathbb{C};\ h_n \in \mathbb{R}\}.$$

Then Wiener's general tauberian theorem states that S_f is dense in $L^1(\mathbb{R})$ if and only if the Fourier transform $\hat{f}$ of f does not vanish on $\mathbb{R}$, i.e.

$$(0.3) \qquad \hat{f}(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \neq 0 \quad \text{for all } t \in \mathbb{R}.$$

In WISKUNDIGE OPGAVEN [62] KOREVAAR showed that Wiener's theorem may be translated into the following theorem for $L^1(0,1)$. If $f \in L^1(0,1)$ then the span of the set of functions

$$(0.4) \qquad \{x^{\lambda-1} f(x^{\lambda})\}_{\lambda \in \mathbb{R}^+} , \qquad (0 < x < 1)$$

is dense in $L^1(0,1)$ if and only if

$$(0.5) \qquad \int_0^1 f(x)\, (\log \tfrac{1}{x})^{it}\, dx \neq 0 \quad \text{for all } t \in \mathbb{R}.$$

In Sections 1,2 and 3 we examine whether condition (0.5) is satisfied by the function

$$(0.6) \qquad f_N(x) := \frac{1 - x^N}{1 - x} = 1 + x + \ldots + x^{N-1}, \qquad (0 < x < 1)$$

where N is some (fixed) positive integer.
Therefore we first compute the integral

$$(0.7) \qquad \int_0^1 \frac{1 - x^N}{1 - x} (\log \tfrac{1}{x})^{it}\, dx = \int_0^1 (1+x+\ldots+x^{N-1})(\log \tfrac{1}{x})^{it}\, dx =$$

$$= \int_0^{\infty} (1+e^{-u}+\ldots+e^{-(N-1)u})\, u^{it} e^{-u}\, du = \sum_{n=1}^{N} \int_0^{\infty} e^{-nu} u^{it}\, du =$$

$$= \sum_{n=1}^{N} \int_0^{\infty} e^{-v} (\tfrac{v}{n})^{it} \frac{dv}{n} = \Gamma(1+it) \cdot \sum_{n=1}^{N} \frac{1}{n^{1+it}}$$

from which it is clear that f_N satisfies condition (0.5) if and only if

$$(0.8) \qquad \zeta_N(1+it) := \sum_{n=1}^{N} \frac{1}{n^{1+it}} \neq 0 \quad \text{for all } t \in \mathbb{R}.$$

Since for all $t \in \mathbb{R}$

$$(0.9) \qquad \zeta_1(1+it) = 1$$

$$(0.10) \qquad |\zeta_2(1+it)| \geqq 1 - \frac{1}{2} = \frac{1}{2}$$

$$(0.11) \qquad |\zeta_3(1+it)| \geqq 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

it is clear that (0.8) is true for N = 1,2 and 3.

Following JESSEN, we have for N = 4 (cf. TURÁN [59])

$$(0.12) \qquad \mathrm{Re}\ \zeta_4(1+it) = \sum_{n=1}^{4} \frac{1}{n} \cos(t \log n) \geqq \qquad \ldots (x := t \log 2) \ldots$$

$$\geqq 1 + \frac{1}{2}\cos x - \frac{1}{3} + \frac{1}{4}\cos 2x = 1 - \frac{1}{3} + \frac{1}{2}\cos x + \frac{1}{4}\{2\cos^2 x - 1\} =$$

$$= \frac{5}{12} + \frac{1}{2}(u+u^2) \geqq \frac{5}{12} + \frac{1}{2} \min_{|u|\leq 1} (u+u^2) \qquad \ldots (u := \cos x) \ldots$$

so that

$$(0.13) \qquad \mathrm{Re}\ \zeta_4(1+it) \geqq \frac{5}{12} + \frac{1}{2}\left(-\frac{1}{4}\right) = \frac{7}{24}.$$

Consequently

$$(0.14) \qquad \zeta_4(1+it) \neq 0 \quad \text{for all } t \in \mathbb{R}.$$

From the above calculations it is also clear that

$$(0.15) \qquad \mathrm{Re}\ \zeta_5(1+it) \geqq \mathrm{Re}\ \zeta_4(1+it) - \frac{1}{5} \geqq \frac{7}{24} - \frac{1}{5} = \frac{11}{120}$$

so that also

$$(0.16) \qquad \zeta_5(1+it) \neq 0 \quad \text{for all } t \in \mathbb{R}$$

a result also due to JESSEN (cf. TURÁN [59]).

In Sections 1,2 and 3 we prove similar results for N = 6, N = 8, N = 9 and N = 10.

Using the results of Sections 1,2 and 3 we present in Section 4 a new device for showing that for N = 2, ..., 6 and N = 8, 9, 10 the entire functions $\zeta_N(s)$ have no zeros in the halfplane $\sigma > 1$.

By SPIRA [54] it was shown that for $N \geqq 2$ all zeros of $\zeta_N(s)$ satisfy

$$(0.17) \qquad \sigma > 1 - N.$$

By means of one of our inequalities derived in Chapter 1 we will show in Section 5 that (0.17) may be improved to

$$(0.18)\qquad \sigma > 1 - \frac{\log 2}{\log(1+\frac{1}{N-\frac{1}{2}})} .$$

Moreover, in Section 5 it will be shown that (roughly speaking) the south-west corner of the halfstrip

$$(0.19)\qquad \sigma > 1 - \frac{\log 2}{\log(1+\frac{1}{N-\frac{1}{2}})} , \qquad (\mathrm{Im}(s) =: t \geqq 0)$$

is a zero-free region for $\zeta_N(s)$. For a more precise formulation see Theorem 5.2 and the Figure on page 82.

In Section 6 we present a systematic procedure for finding *special zeros* (zeros in $\sigma > 1$) of $\zeta_N(s)$. (Compare [35] and/or [36].)

We give a brief account of the history of this kind of zeros. In 1949 TURÁN [59] showed that the Riemann hypothesis for $\zeta(s)$ is true if there exist constants $N_0 \in \mathbb{N}$ and $c \in \mathbb{R}^+$ such that for every $N > N_0$ the entire function

$$(0.20)\qquad \zeta_N(s) := \sum_{n=1}^{N} n^{-s} , \qquad (s \in \mathbb{C},\ s = \sigma+it)$$

has no zeros in the halfplane $\sigma > 1+c.N^{-\frac{1}{2}}$. Compare TITCHMARSH [58; p. 328]. In 1958 HASELGROVE [15] implicitly showed (compare SPIRA [55; Section 3]) that there *exist* $N \in \mathbb{N}$ such that $\zeta_N(s) = 0$ for some s with $\sigma > 1$.

In 1968 SPIRA [55] proved, computationally, that, for N = 19, N = 22, ... 27 and N = 29, ..., 50, $\zeta_N(s)$ has zeros with $\sigma > 1$, although he was unable to locate any of them numerically. Also see SPIRA [56].

Until the appearance of [35] (by the present author & te RIELE in 1977) no special zero of any $\zeta_N(s)$ had ever been located numerically. We shall present a *simple* method for the explicit numerical computation of special zeros of $\zeta_N(s)$. We even believe that this method produces all special zeros with imaginary part in a given interval.

Finally, in Section 7 we present a *selection* of special zeros of $\zeta_N(s)$ which were actually computed by our method and/or *another one* described by the present author & te RIELE in [35].

For the present state of affairs concerning the zeros of $\zeta_N(s)$ we refer to APOSTOL [1], LEVINSON [26] and MONTGOMERY [44] and [45].

1. THE CASE N = 6

The case N = 6 is somewhat less transparant than the cases N < 6. In [54] SPIRA *claims* to have shown that

$$(1.1) \qquad \operatorname{Re} \zeta_N(s) > 0, \qquad (\operatorname{Re}(s) \geqq 1)$$

for N = 6 and N = 8 and promises to return to these matters in [55]. However, in [55] SPIRA presents *machine proofs*.
In this section we present a *theoretical* proof of

PROPOSITION 1. $\operatorname{Re} \zeta_6(1+it) > 33/1000$ for all $t \in \mathbb{R}$.

PROOF. In this proof we write

$$(1.2) \qquad x := t \log 2 \ ; \qquad y := t \log 3$$

$$(1.3) \qquad u := \cos x \ ; \qquad v := \cos y \ .$$

Observe that

$$(1.4) \qquad \operatorname{Re} \zeta_6(1+it) = \sum_{n=1}^{6} \frac{1}{n} \cos(t \log n) =$$

$$= 1 + \frac{1}{2}\cos x + \frac{1}{3}\cos y + \frac{1}{4}\cos 2x + \frac{1}{5}\cos(t \log 5) + \frac{1}{6}\cos(x+y) \geqq$$

$$\geqq 1 + \frac{1}{2}\cos x + \frac{1}{3}\cos y + \frac{1}{4}(2\cos^2 x - 1) - \frac{1}{5} + \frac{1}{6}(\cos x \cos y - \sin x \sin y) \geqq$$

$$\geqq \frac{11}{20} + \frac{1}{2}(u+u^2) + \frac{v}{3} + \frac{uv}{6} - \frac{1}{6}(1-u^2)^{\frac{1}{2}}.(1-v^2)^{\frac{1}{2}} =: \phi(u,v), \qquad (-1 \leqq u,v \leqq 1).$$

It is clear that ϕ is continuous on the compact square $-1 \leqq u,v \leqq 1$ so that ϕ assumes an absolute minimum. The partial derivatives of ϕ in the open square $-1 < u,v < 1$ may be written as

$$(1.5) \qquad \frac{\partial \phi}{\partial u} = \frac{1}{2} + u + \frac{v}{6} + \frac{u}{6}\left(\frac{1-v^2}{1-u^2}\right)^{\frac{1}{2}}$$

and

$$(1.6) \qquad \frac{\partial \phi}{\partial v} = \frac{1}{3} + \frac{u}{6} + \frac{v}{6}\left(\frac{1-u^2}{1-v^2}\right)^{\frac{1}{2}}.$$

We first prove that ϕ assumes its minimal value in the *interior* of the square $-1 \leqq u,v \leqq 1$.

a. On the segment $v = 1$, $-1 \leq u \leq 1$ we have

$$(1.7) \qquad \phi(u,v) = \phi(u,1) = \frac{53}{60} + \frac{2}{3}u + \frac{1}{2}u^2$$

which is minimal for $u = -\frac{2}{3}$ with minimal value $\frac{119}{180}$. At $(u,v) = (-\frac{2}{3},1)$ we have $\frac{\partial\phi}{\partial v} = +\infty$ so that ϕ does *not* assume its minimal value on the segment $v = 1$, $-1 \leq u \leq 1$.

b. On the segment $u = 1$, $-1 \leq v \leq 1$ we have

$$(1.8) \qquad \phi(u,v) = \phi(1,v) = \frac{31}{20} + \frac{v}{2}$$

which is minimal for $v = -1$ with minimal value $\frac{21}{20}$ $(> \frac{119}{180})$ so that ϕ does *not* assume its minimal value on the segment $u = 1$, $-1 \leq v \leq 1$.

c. For $v = -1$, $-1 \leq u \leq 1$ we have

$$(1.9) \qquad \phi(u,v) = \phi(u,-1) = \frac{13}{60} + \frac{u}{3} + \frac{u^2}{2}$$

which is minimal for $u = -\frac{1}{3}$ with minimal value $\frac{29}{180}$.
Since $\frac{\partial\phi}{\partial v}(-\frac{1}{3},-1) = -\infty$ it follows that ϕ does *not* assume its minimal value on the segment $v = -1$, $-1 \leq u \leq 1$.

d. Finally, for $u = -1$, $-1 \leq v \leq 1$ we have

$$(1.10) \qquad \phi(u,v) = \phi(-1,v) = \frac{11}{20} + \frac{v}{6}$$

which is minimal for $v = -1$ with minimal value $\frac{23}{60}$ $(> \frac{29}{180})$ so that ϕ does *not* assume its minimal value on the segment $u = -1$, $-1 \leq v \leq 1$.

Consequently we may restrict ourselves to the *open* square $-1 < u,v < 1$. In order to show that the minimal value of ϕ is positive we consider the equations

$$(1.11) \qquad \frac{\partial\phi}{\partial u} = 0 \quad \text{and} \quad \frac{\partial\phi}{\partial v} = 0, \qquad (-1 < u,v < 1).$$

Observe that if $u \geq 0$ then

$$(1.12) \qquad \frac{\partial\phi}{\partial u} \geq \frac{1}{2} + \frac{v}{6} > \frac{1}{2} - \frac{1}{6} > 0$$

and that if $v \geq 0$ then

$$(1.13) \qquad \frac{\partial\phi}{\partial v} \geq \frac{1}{3} + \frac{u}{6} > \frac{1}{3} - \frac{1}{6} > 0$$

so that we only need to consider $\phi(u,v)$ in the open square $-1 < u,v < 0$. The equations $\frac{\partial\phi}{\partial u} = 0$ and $\frac{\partial\phi}{\partial v} = 0$ are equivalent to, respectively,

$$(1.14) \qquad 3 + 6u + v + u\left(\frac{1 - v^2}{1 - u^2}\right)^{\frac{1}{2}} = 0$$

and

$$(1.15) \qquad 2 + u + v\left(\frac{1 - u^2}{1 - v^2}\right)^{\frac{1}{2}} = 0$$

from which we obtain

$$(1.16) \qquad \frac{3 + 6u + v}{u} = \frac{v}{2 + u}$$

or

$$(1.17) \qquad v = -\frac{1}{2}(6 + 15u + 6u^2).$$

Since $6+15u+6u^2 = 0$ for $u = -\frac{1}{2}$ and $u = -2$, and $-\frac{1}{2}(6+15u+6u^2) = -1$ for

$$(1.18) \qquad u = \frac{-15 + \sqrt{129}}{12} \quad \text{and} \quad u = \frac{-15 - \sqrt{129}}{12} \; (< -1)$$

we find that ϕ is minimal on the curve

$$(1.19) \qquad v = -\frac{1}{2}(6 + 15u + 6u^2) \quad \text{where} \quad -\frac{1}{2} < u < \frac{-15 + \sqrt{129}}{12} \; (< -\frac{1}{4}).$$

From $\frac{\partial\phi}{\partial v} = 0$ it follows that

$$(1.20) \qquad \frac{1}{3} + \frac{u}{6} = -\frac{v}{6}\left(\frac{1 - u^2}{1 - v^2}\right)^{\frac{1}{2}}$$

from which we obtain subsequently

$$(1.21) \qquad \frac{1}{3} + \frac{1}{6}(-\frac{1}{2}) < \frac{-v}{6\sqrt{(1-v^2)}}$$

$$(1.22) \qquad \frac{3}{2} < \frac{-v}{\sqrt{(1-v^2)}}$$

$$(1.23) \qquad \frac{9}{4} < \frac{v^2}{1-v^2} = -1 + \frac{1}{1-v^2}$$

$$(1.24) \qquad v^2 > \frac{9}{13}$$

$$(1.25) \qquad -v = |v| > \frac{3}{\sqrt{13}} \; (> \frac{3}{4}).$$

Hence, if u and v satisfy the restrictions described above we have

(1.26) $\phi(u,v) > \frac{11}{20} + \frac{1}{2}\inf(u+u^2) - \frac{1}{3} + \frac{1}{6}(\inf|u|)(\inf|v|) - \frac{1}{6}\sqrt{(1-\frac{1}{16})}\sqrt{(1-\frac{9}{13})} >$

$> \frac{11}{20} - \frac{1}{8} - \frac{1}{3} + \frac{1}{6}.\frac{1}{4}.\frac{3}{4} - \frac{1}{12}\frac{\sqrt{15}}{\sqrt{13}} > 33/1000$

so that Re $\zeta_6(1+it) > 33/1000$ for all $t \in \mathbb{R}$, proving the Proposition.□

REMARK. Numerical computations indicate that ϕ assumes its minimal value 0.1197... at the point $u = -0.3266...$, $v = -0.8705...$. Computing Re $\zeta_7(1+it)$ for $t = n.10^{-1}$, $(n = 1,2,3,...)$, we found (using a programmable pocket calculator)

(1.27) Re $\zeta_7(1+it) = -0.0136...$ for $t = 1009$

so that Proposition 1 is *not* true when ζ_6 is replaced by ζ_7.

CONJECTURE. For every positive integer N

(1.28) $\zeta_N(1+it) \neq 0$ for all $t \in \mathbb{R}$.

SOME MOTIVATION. Pick *any* $g \in L^1(\mathbb{R}^+)$. Then, for any $N \in \mathbb{N}$, the function $G_N : \mathbb{R} \to \mathbb{C}$, defined by

(1.29) $G_N(x) := e^{-x} \sum_{n=1}^{N} g(n.e^{-x}), \quad (x \in \mathbb{R})$

belongs to $L^1(\mathbb{R})$ and hence admits of a Fourier transform

(1.30) $\hat{G}_N(t) := \int_{-\infty}^{\infty} e^{txi} G_N(x)\, dx$.

A simple calculation shows that

(1.31) $\hat{G}_N(-t) = \int_{-\infty}^{\infty} e^{-txi} e^{-x} \sum_{n=1}^{N} g(n.e^{-x})\, dx = \sum_{n=1}^{N} \int_{-\infty}^{\infty} e^{-x(1+it)} g(n.e^{-x})\, dx =$

$= \sum_{n=1}^{N} \int_{0}^{\infty} (\frac{u}{n})^{1+it} g(u) \frac{-du}{u} = -\zeta_N(1+it) . \int_{0}^{\infty} u^{it} g(u)\, du.$

However, it seems hard to believe that there exists an $N \in \mathbb{N}$ such that $\zeta_N(1+it_0) = 0$ for some $t_0 \in \mathbb{R}$, since, as a consequence, for *every* $g \in L^1(\mathbb{R})$, we would have

(1.32) $\hat{G}_N(t_0) = 0$.

2. THE CASE N = 8

The results of Sections 2 and 3 were obtained in cooperation with H.J.J. te RIELE. See [31] and [33].

Defining

(2.1) $R_N(t) := \mathrm{Re}\ \zeta_N(1+it), \qquad (t \in \mathbb{R})$

we have

THEOREM 2. $R_8(t) > 12/1000$.

PROOF. We will use the following notation

(2.2) $u := t \log 2;\ v := t \log 3;\ x := \cos u;\ y := \cos v$.

It is clear that

$$(2.3) \qquad R_8(t) = \sum_{n=1}^{8} \frac{1}{n}\cos(t \log n) \geqq 1 + \frac{1}{2}\cos u + \frac{1}{3}\cos v + \frac{1}{4}\cos 2u +$$

$$- \frac{1}{5} + \frac{1}{6}\cos(u+v) - \frac{1}{7} + \frac{1}{8}\cos 3u \geqq 1 + \frac{x}{2} + \frac{y}{3} + \frac{1}{4}(2x^2-1) - \frac{1}{5} +$$

$$+ \frac{1}{6}(xy - \sqrt{(1-x^2)}\sqrt{(1-y^2)}) - \frac{1}{7} + \frac{1}{8}(4x^3-3x) = 1 - \frac{1}{4} - \frac{1}{5} - \frac{1}{7} +$$

$$+ \frac{x}{8} + \frac{x^2}{2} + \frac{x^3}{2} + \frac{y}{3} + \frac{xy}{6} - \frac{1}{6}\sqrt{(1-x^2)}\sqrt{(1-y^2)} =: \phi(x,y), \qquad (-1 \leqq x,y \leqq 1).$$

Note that

$$(2.4) \qquad 1 - \frac{1}{4} - \frac{1}{5} - \frac{1}{7} > 0.407$$

and that for $-1 < x,y < 1$ we have

$$(2.5) \qquad \frac{\partial\phi}{\partial x} = \frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6} + \frac{x}{6}\left(\frac{1-y^2}{1-x^2}\right)^{\frac{1}{2}}$$

and

$$(2.6) \qquad \frac{\partial\phi}{\partial y} = \frac{1}{3} + \frac{x}{6} + \frac{y}{6}\left(\frac{1-x^2}{1-y^2}\right)^{\frac{1}{2}} .$$

We will first show that $\phi(x,y)$ assumes its minimal value in the *interior* of the square $[-1,1] \times [-1,1]$. First, observe that the minimum of $\phi(x,y)$ over the vertices of the square $[-1,1] \times [-1,1]$ lies in the vertex $[-1,-1]$. Furthermore, it is easily verified that the minimum of $\phi(x,y)$ on the *edge* $y = -1$ is *not* assumed at the point $x = -1$, so that $\phi(x,y)$ does not assume its minimum in one of the vertices of the square $[-1,1] \times [-1,1]$.

Next, observe that for $-1 < x < 1$ we have

$$(2.7) \qquad \lim_{y \uparrow 1} \frac{\partial \phi}{\partial y} = +\infty \quad \text{and} \quad \lim_{y \downarrow -1} \frac{\partial \phi}{\partial y} = -\infty$$

so that $\phi(x,y)$ does *not* assume its minimum on the edges $y = \pm 1$. Similarly, it follows that $\phi(x,y)$ does *not* assume its minimum on the edges $x = \pm 1$, so that indeed its minimal value is assumed in the *interior* of the square $[-1,1] \times [-1,1]$. It follows that $\phi(x,y)$ is minimal at a point (x,y) satisfying $-1 < x,y < 1$ and

$$(2.8) \qquad \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0$$

or, more explicitly,

$$(2.9) \qquad \frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6} + \frac{x}{6}\left(\frac{1 - y^2}{1 - x^2}\right)^{\frac{1}{2}} = 0$$

and

$$(2.10) \qquad \frac{1}{3} + \frac{x}{6} + \frac{y}{6}\left(\frac{1 - x^2}{1 - y^2}\right)^{\frac{1}{2}} = 0 .$$

Since $x > -1$ we have

$$(2.11) \qquad \frac{1}{3} + \frac{x}{6} > \frac{1}{3} - \frac{1}{6} > 0$$

so that in view of (2.10) we must have

$$(2.12) \qquad y < 0.$$

It is easily verified that (2.9) and (2.10) do not admit of $x = 0$. From (2.9), (2.10) and the fact that $x \neq 0$ it follows that

$$(2.13) \qquad \frac{6}{x}\left(\frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6}\right) = -\left(\frac{1 - y^2}{1 - x^2}\right)^{\frac{1}{2}}$$

and (recall that $y \neq 0$ by (2.12))

$$(2.14) \qquad \frac{6}{y}\left(\frac{1}{3} + \frac{x}{6}\right) = -\left(\frac{1 - x^2}{1 - y^2}\right)^{\frac{1}{2}}$$

so that

$$(2.15) \qquad \frac{6}{x}\left(\frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6}\right) = \frac{y}{2 + x}$$

or

$$(2.16) \qquad -y = \frac{3}{2}(2+x)(3x^2 + 2x + \frac{1}{4}).$$

Now observe that

$$(2.17) \qquad -y < 1$$

so that we must have

$$(2.18)\qquad \frac{3}{2}(2+x)(3x^2 + 2x + \frac{1}{4}) < 1.$$

Since $2+x > 0$ this may also be written as

$$(2.19)\qquad \frac{3}{2}(3x^2 + 2x + \frac{1}{4}) - \frac{1}{2 + x} < 0$$

Since the left hand side of (2.19) is increasing for $x > 0$ and takes a positive value at $x = 0.04$, we may conclude that

$$(2.20)\qquad x < 0.04 .$$

From (2.10) it follows that

$$(2.21)\qquad 1 < 2 + x = -y(\frac{1 - x^2}{1 - y^2})^{\frac{1}{2}} \leq \frac{-y}{\sqrt{(1-y^2)}}$$

so that

$$(2.22)\qquad 1 < \frac{y^2}{1-y^2}$$

from which we obtain (recall that $y < 0$)

$$(2.23)\qquad y < -\frac{1}{2}\sqrt{2}.$$

From (2.9) it is clear that

$$(2.24)\qquad \frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6} = \frac{-x}{6}(\frac{1 - y^2}{1 - x^2})^{\frac{1}{2}}$$

so that, in case $x < 0$, we must have

$$(2.25)\qquad \frac{1}{8} + x + \frac{3}{2}x^2 + \frac{y}{6} > 0$$

so that in view of (2.23)

$$(2.26)\qquad \frac{3}{2}x^2 + x + (\frac{1}{8} - \frac{1}{12}\sqrt{2}) > 0.$$

From this it is easily seen that

$$(2.27)\qquad x < -0.659 \quad \text{or} \quad x > -0.008 .$$

Case a. $\quad -0.008 < x < 0.04.$

In this case we have

$$(2.28)\qquad \phi(x,y) > 0.407 + \frac{x}{8} - \frac{1}{6}|x||y| + \frac{y}{3} - \frac{1}{6}\sqrt{(1-y^2)} >$$

$$> 0.407 - \frac{1}{8}*0.008 - \frac{1}{6}*0.04 + \frac{1}{6}\{2y - \sqrt{(1-y^2)}\}.$$

Defining

$$(2.29)\qquad f(y) := 2y - \sqrt{(1-y^2)}\ , \qquad (-1 < y < -\tfrac{1}{2}\sqrt{2})$$

we find that

$$(2.30)\qquad \min_{-1<y<-\frac{1}{2}\sqrt{2}} f(y) = f(-\tfrac{2}{\sqrt{5}}) = -\sqrt{5}$$

so that

$$(2.31)\qquad \phi(x,y) > 0.407 - 0.001 - 0.007 - \tfrac{1}{6}\sqrt{5} > 0.026\ .$$

Case b. $-1 < x < -0.659$

In this case we have

$$(2.32)\qquad \phi(x,y) > 0.407 + \frac{x}{8} + \frac{xy}{6} + \frac{y}{3} - \frac{1}{6}\sqrt{(1-x^2)}\sqrt{(1-y^2)} >$$

$$> 0.407 + (\frac{x}{8} + \frac{xy}{6} + \frac{y}{3}) - \frac{1}{6}\{(1-(0.659)^2)(1-(\tfrac{1}{2}\sqrt{2})^2)\}^{\frac{1}{2}}.$$

Defining

$$(2.33)\qquad \psi(x,y) := \frac{x}{8} + \frac{xy}{6} + \frac{y}{3}$$

for $-1 \leqq x \leqq -0.659$ and $-1 \leqq y \leqq -\frac{1}{2}\sqrt{2}$, we have

$$(2.34)\qquad \frac{\partial\psi}{\partial y} = \frac{x}{6} + \frac{1}{3} > 0$$

so that $\psi(x,y)$ is minimal on the edge $y = -1$.

Since

$$(2.35)\qquad \psi(x,-1) = -\frac{1}{3} - \frac{x}{24}$$

we obtain

$$(2.36)\qquad \psi(x,y) \geqq -\frac{1}{3} + \frac{1}{24}*0.659 > -0.306$$

so that, in view of (2.32), it follows that

$$(2.37)\qquad \phi(x,y) > 0.407 - 0.306 - 0.089 = 0.012\ .$$

This completes the proof of Theorem 2. □

REMARK. Substituting (2.16) in $\phi(x,y)$ we found *numerically* that $\phi(x,y)$ assumes its minimal value 0.03419... at the point (x_0,y_0) where $x_0 = 0.02204...$ and $y_0 = -0.89641...$.

3. THE CASES N = 9 AND N = 10

Using techniques very similar to those in the previous sections we obtained

THEOREM 3. *There exist positive constants* c_9 *and* c_{10} *such that*

(3.1) $R_9(t) > c_9$ *and* $R_{10}(t) > c_{10}$ *for all* $t \in \mathbb{R}$.

PROOF. See [33].

REMARKS. In Section 1 we already mentioned that Re $\zeta_7(1+it)$ has a real zero. Furthermore, numerical computations indicate that Re $\zeta_N(1+it)$ has real zeros for every $N \geq 11$. Hence, in order to prove the *conjecture* that for *all* $N \in \mathbb{N}$

(3.2) $\zeta_N(1+it) \neq 0$ for all $t \in \mathbb{R}$

one will have to search for a method of proof also involving the imaginary part of $\zeta_N(1+it)$. To the best of our knowledge this is still an *unsolved problem*.

In the table below we list the *smallest positive* zero $t_1(N)$ of (the even function) $R_N(t) := \text{Re}\ \zeta_N(1+it)$ for N = 7 and N = 11,..., 100, the *machine proof* of the table being based on the following (almost trivial) "*maximum slope principle*": If the differentiable function $f : \mathbb{R} \to \mathbb{R}$ is such that

(3.3) $f(t_0) > 0$ for some $t_0 \in \mathbb{R}$

and

(3.4) $|f'(t)| < h$ for all $t \in \mathbb{R}$

where h is a constant, then

(3.5) $f(t) > 0$ for $t_0 \leq t \leq t_0 + \dfrac{f(t_0)}{h}$.

TABLE

N=	$t_1(N)\cong$	N=	$t_1(N)\cong$	N=	$t_1(N)\cong$
7	1008.9095				
11	1180.3887	41	1.0124	71	0.8580
12	3098.0590	42	1.0044	72	0.8547
13	1919.3622	43	0.9968	73	0.8514
14	1379.8280	44	0.9894	74	0.8483
15	1.5897	45	0.9823	75	0.8452
16	1.5120	46	0.9754	76	0.8421
17	1.4566	47	0.9688	77	0.8392
18	1.4114	48	0.9625	78	0.8362
19	1.3727	49	0.9563	79	0.8334
20	1.3388	50	0.9504	80	0.8306
21	1.3086	51	0.9446	81	0.8278
22	1.2814	52	0.9390	82	0.8251
23	1.2567	53	0.9336	83	0.8225
24	1.2342	54	0.9284	84	0.8199
25	1.2135	55	0.9233	85	0.8173
26	1.1943	56	0.9183	86	0.8148
27	1.1765	57	0.9135	87	0.8124
28	1.1599	58	0.9089	88	0.8100
29	1.1444	59	0.9043	89	0.8076
30	1.1298	60	0.8999	90	0.8052
31	1.1161	61	0.8956	91	0.8029
32	1.1032	62	0.8914	92	0.8007
33	1.0910	63	0.8873	93	0.7985
34	1.0794	64	0.8833	94	0.7963
35	1.0684	65	0.8794	95	0.7941
36	1.0580	66	0.8757	96	0.7900
37	1.0480	67	0.8720	97	0.7899
38	1.0385	68	0.8683	98	0.7879
39	1.0294	69	0.8648	99	0.7859
40	1.0208	70	0.8613	100	0.7839

4. EXTENSION OF THE PROBLEM; SOME NEW PROOFS

In Sections 1,2 and 3 we have considered the functions ζ_N on the line $s = 1+it$, $t \in \mathbb{R}^+$. From now on we will remove this restriction and consider the entire functions

$$(4.1) \qquad \zeta_N(s) := \sum_{n=1}^{N} n^{-s}, \qquad (s = \sigma+it;\ N = 2,3,4,\ \ldots).$$

We will prove

THEOREM 4.1. *For* $N = 2, \ldots, 6$ *and* $N = 8, 9, 10$

$$(4.2) \qquad \zeta_N(s) \neq 0 \qquad \textit{for} \quad \sigma \geqq 1.$$

It is clear that this theorem is an immediate consequence of

THEOREM 4.2. *For* $N = 2, \ldots, 6$ *and* $N = 8, 9, 10$

$$R_N(\sigma,t) > 0, \quad (\sigma \geqq 1) \tag{4.3}$$

where

$$R_N(\sigma,t) := \operatorname{Re} \zeta_N(\sigma+it) = \sum_{n=1}^{N} \frac{\cos(t \log n)}{n^\sigma}. \tag{4.4}$$

PROOF. We recall that for the N mentioned in the last theorem there exist positive constants c_N such that for all $t \in \mathbb{R}$

$$R_N(t) := \operatorname{Re} \zeta_N(1+it) \geqq c_N. \tag{4.5}$$

As to the function $R_N(\sigma,t)$ we clearly have

$$R_N(\sigma,0) = \sum_{n=1}^{N} n^{-\sigma} \geqq 1, \quad (\sigma \in \mathbb{R}) \tag{4.6}$$

and

$$R_N(\sigma,t) \geqq 1 - \sum_{n=2}^{N} n^{-\sigma} > 2 - \zeta(\sigma) \geqq 2 - \zeta(2) = 2 - \frac{\pi^2}{6} > 0 \tag{4.7}$$

for all $\sigma \geqq 2$ and all $t \in \mathbb{R}$.

Now observe that $R_N(t)$ is an *almost-periodic function* so that for every $\varepsilon > 0$ there exists an increasing positive sequence $\{T_k\}_{k=1}^{\infty}$ such that $T_k \to \infty$ as $k \to \infty$ and

$$R_N(T_k) > R_N(0) - \varepsilon. \tag{4.8}$$

If we choose ε small enough it follows that in the sum

$$R_N(T_k) = \sum_{n=1}^{N} \frac{1}{n} \cos(T_k \log n) \tag{4.9}$$

all cosines must be close to 1 (and hence positive) so that also

$$R_N(\sigma,T_k) = \sum_{n=1}^{N} \frac{1}{n^\sigma} \cos(T_k \log n) > 0, \quad (\sigma \in \mathbb{R}). \tag{4.10}$$

Finally observe that $R_N(\sigma,t)$ is a *harmonic* function so that by the *minimum principle* for harmonic functions on compact domains we obtain the theorem. □

Similarly one may prove

THEOREM 4.3. *If* $R_N(\sigma_0,t) > 0$, $(\forall t \in \mathbb{R})$, *then* $\zeta_N(s) \neq 0$ *for* $\sigma \geqq \sigma_0$.

REMARK. In order to prove Theorem 4.2 it suffices to show that $R_N(\sigma,t) > 0$, $(1 \leqq \sigma \leqq 2,\ t \geqq 0)$ for all N stated in the theorem. Indeed, by (4.7) we already know that $R_N(\sigma,t) > 0$, $(\sigma \geqq 2;\ t \in \mathbb{R})$. Observing that

$$\frac{\partial R_N(\sigma,t)}{\partial\sigma} = -\sum_{n=2}^{N} \frac{\log n}{n^\sigma}\cos(t\log n) \tag{4.11}$$

we have

$$\left|\frac{\partial R_N(\sigma,t)}{\partial\sigma}\right| \leqq \sum_{n=2}^{N} \frac{\log n}{n^\sigma} \leqq \sum_{n=2}^{N}\frac{\log n}{n} \tag{4.12}$$

for $\sigma \geqq 1$ and all $t \in \mathbb{R}$.

Hence, by the *maximum slope principle* (see page 76), if

$$\sum_{n=2}^{N}\frac{\log n}{n} < \sum_{n=1}^{N}\frac{1}{n} - \varepsilon = R_N(0) - \varepsilon \tag{4.13}$$

then

$$R_N(\sigma,T_k) \geqq R_N(T_k) - (\sigma-1)\sum_{n=2}^{N}\frac{\log n}{n} > \tag{4.14}$$

$$> R_N(0) - \varepsilon - \sum_{n=2}^{N}\frac{\log n}{n}, \qquad (1 \leqq \sigma \leqq 2).$$

Since $\varepsilon > 0$ may be chosen as small as we please it is enough to show that

$$L_N := \sum_{n=2}^{N}\frac{\log n}{n} < \sum_{n=1}^{N}\frac{1}{n} = R_N(0), \qquad (2 \leqq N \leqq 10). \tag{4.15}$$

It is easily verified (see the table below) that (4.15) is true indeed for N = 2, ..., 10 so that we have obtained (again using the minimum principle for harmonic functions) an alternative proof of Theorem 4.1.

TABLE

N	L_N	$R_N(0)$
2	.3466	1.5000
3	.7128	1.8333
4	1.0594	2.0833
5	1.3812	2.2833
6	1.6799	2.4500
7	1.9579	2.5929
8	2.2178	2.7179
9	2.4619	2.8290
10	2.6922	2.9290
11	2.9102	3.0199
12	3.1172	3.1032

5. NARROWING THE ZERO-FREE STRIP OF $\zeta_N(s)$

THEOREM 5.1. *If* $\zeta_N(s) = 0$ *then*

$$(5.1)\qquad \sigma := \mathrm{Re}(s) > 1 - \frac{\log 2}{\log(1+\frac{2}{2N-1})} .$$

In order to prove this we recall from Chapter 1 (formula (2.75)) that

$$(5.2)\qquad \sigma_n(p) := \sum_{k=1}^{n} k^p < \frac{n^p(2n+1)^{p+1}}{(2n+1)^{p+1} - (2n-1)^{p+1}} , \qquad (p \geqq 1)$$

PROOF of Theorem 5.1. The case N = 2 being trivial we assume $N \geqq 3$. Let

$$(5.3)\qquad p := \frac{\log 2}{\log(1+\frac{2}{2N-1})} - 1$$

and assume the theorem to be false. Then there exists an $N \geqq 3$ and an $s = \sigma+it \in \mathbb{C}$ such that

$$(5.4)\qquad \sigma \leqq -p$$

and

$$(5.5)\qquad \zeta_N(s) = \sum_{n=1}^{N} n^{-s} = 0$$

so that

$$(5.6)\qquad \sum_{n=1}^{N-1} n^{-s} = -N^{-s} .$$

It follows that

$$(5.7)\qquad -1 = \sum_{n=1}^{N-1} (\tfrac{N}{n})^s$$

so that

$$(5.8)\qquad 1 \leqq \sum_{n=1}^{N-1} (\tfrac{N}{n})^{\sigma} .$$

In (5.8) we have $1 \leqq n \leqq N-1$ so that $\frac{N}{n} > 1$. Since $\sigma \leqq -p$ we have

$$(5.9)\qquad 1 \leqq \sum_{n=1}^{N-1} (\tfrac{N}{n})^{-p} = \sum_{n=1}^{N-1} (\tfrac{n}{N})^p$$

or

$$(5.10)\qquad 2.N^p \leqq \sigma_N(p).$$

It is easily verified that $p > 1$ for $N \geqq 3$. Hence, we may apply (5.2) to (5.10) in order to obtain

$$(5.11)\qquad 2.N^p < \frac{N^p(2N+1)^{p+1}}{(2N+1)^{p+1} - (2N-1)^{p+1}}$$

from which it is easily seen that

$$(5.12)\qquad p + 1 < \frac{\log 2}{\log(1+\frac{2}{2N-1})}$$

contradicting the definition of p and hence proving the theorem. □

REMARK. In Chapter 1 it was shown (formula (1.5)) that for all real $p > 0$ we have

$$\sigma_N(p) < \frac{N^p(N+1)^{p+1}}{(N+1)^{p+1} - N^{p+1}} . \tag{5.13}$$

Similarly as above we derive from (5.13) that if $\zeta_N(s) = 0$ then

$$\sigma > 1 - \frac{\log 2}{\log(1+\frac{1}{N})} \tag{5.14}$$

a result just a little less sharp than (5.1) .

We conclude this section by proving

THEOREM 5.2. *$\zeta_N(s)$ has no zeros in the domain G_N described by*

$$\begin{cases} s = \sigma + a\sigma i; \quad a,\sigma \in \mathbb{R} \\ \sigma \leqq -1 \\ |s-1| \leqq \dfrac{2(N-1)}{1+\sqrt{(1+a^2)}} . \end{cases} \tag{5.15}$$

PROOF. If $\zeta_N(s) = 0$ then we have

$$0 = \zeta_N(s) = \sum_{n=1}^{N} n^{-s} = \int_{1-}^{N+} x^{-s}\, d[x] = \int_1^N x^{-s}\, dx - \int_{1-}^{N+} x^{-s}\, d\phi_1(x) = \tag{5.16}$$

(where $\phi_1(x) := x - [x] - \frac{1}{2}$)

$$= \frac{N^{-s+1}-1}{-s+1} + \frac{1}{2}(N^{-s}+1) - s\int_1^N \frac{\phi_1(x)}{x^{s+1}}\, dx$$

so that

$$\frac{1-N^{s-1}}{s-1} = \frac{1}{2N} + \frac{1}{2}N^{s-1} - sN^{s-1}\int_1^N \frac{\phi_1(x)}{x^{s+1}}\, dx . \tag{5.17}$$

It follows that

$$\frac{1-N^{\sigma-1}}{|s-1|} \leqq \frac{1}{2N} + \frac{1}{2}N^{\sigma-1} + |s|\, N^{\sigma-1}\int_1^N \frac{|\phi_1(x)|}{x^{\sigma+1}}\, dx \leqq \tag{5.18}$$

$$\leqq \frac{1}{2N} + \frac{1}{2}N^{\sigma-1} + |s|\, N^{\sigma-1}\, \frac{N^{-\sigma}-1}{-\sigma} = \frac{1}{2N} + \frac{1}{2}\, N^{\sigma-1} + \frac{|s|}{-2\sigma}(\frac{1}{N} - N^{\sigma-1}).$$

Now let $s = \sigma+a\sigma i$ with $\sigma \leqq -1$ and $a \in \mathbb{R}$. Then (5.18) yields

$$\frac{1-N^{-2}}{|s-1|} < \frac{1}{2N} + \frac{1}{2N^2} + \frac{|s|}{2N|\sigma|} \tag{5.19}$$

so that

$$\frac{N^2-1}{|s-1|} < \frac{N+1}{2} + \frac{N}{2}\sqrt{(1+a^2)} < \frac{N+1}{2}(1+\sqrt{(1+a^2)}). \tag{5.20}$$

Hence

$$|s-1| > \frac{2(N-1)}{1+\sqrt{(1+a^2)}} \tag{5.21}$$

proving the theorem. □

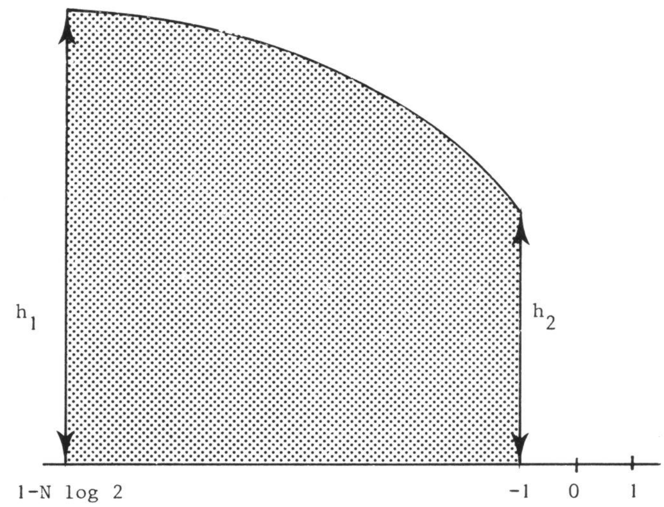
REMARK. Since

(5.22) $$1 - N \log 2 < 1 - \frac{\log 2}{\log(1+\frac{1}{N-\frac{1}{2}})} \quad \text{for all } N \in \mathbb{N}$$

we derive from Theorems 5.1 and 5.2 that the intersection of the domain G_N in Theorem 5.2 and the domain

(5.23) $$\sigma > 1 - N \log 2 \ ; \quad t \geq 0$$

is a *zero-free region* for $\zeta_N(s)$. This region may be depicted as the shaded area in the figure below. One should compare these results with the empirical observations made by SPIRA [54; Section 4].



From (5.2) it is easily seen that if N is large enough then (in the picture above)

(5.24) $$h_1 > \frac{N}{2}$$

and

(5.25) $$h_2 \cong \sqrt{(2N)} \ .$$

6. EXPLICIT NUMERICAL COMPUTATION OF SPECIAL ZEROS OF $\zeta_N(s)$

6.1. Some generalities on the zero-curves of the real and imaginary parts of $\zeta_N(s)$

Before explaining the heuristic principle for finding special zeros of $\zeta_N(s)$ we give a global description of the zero-curves of the real and imaginary parts of $\zeta_N(s)$ in the complex plane.

Defining

$$(6.1.1) \qquad R_N(\sigma,t) := \text{Re}\ \zeta_N(s) = \sum_{n=1}^{N} \frac{\cos(t \log n)}{n}$$

and

$$(6.1.2) \qquad I_N(\sigma,t) := \text{Im}\ \zeta_N(s) = -\sum_{n=1}^{N} \frac{\sin(t \log n)}{n}$$

we obviously have $\zeta_N(s) = 0$ if and only if both $R_N(\sigma,t) = 0$ and $I_N(\sigma,t) = 0$. It is easy to see that

$$(6.1.3) \qquad R_N(\sigma,t) > 0 \quad \text{for} \quad \sigma \geqq 2$$

so that the entire zero-set of $\zeta_N(s)$ is contained in the halfplane $\sigma < 2$. Now let N ($\geqq 3$) be fixed and consider the zero-set of $R_N(\sigma,t)$ in the halfplane $\sigma < 0$. If $R_N(\sigma_0,t_0) = 0$ then

$$(6.1.4) \qquad -N^{-\sigma_0} \cos(t_0 \log N) = \sum_{n=1}^{N-1} n^{-\sigma_0} \cos(t_0 \log n)$$

so that

$$(6.1.5) \qquad |\cos(t_0 \log N)| \leqq \sum_{n=1}^{N-1} (\tfrac{n}{N})^{-\sigma_0} < N \int_0^1 x^{-\sigma_0}\, dx = \frac{N}{1-\sigma_0} .$$

Choose a small $\varepsilon > 0$ ($\varepsilon = 1/N$ is sufficient) and take $\sigma_0 < 1 - N/\varepsilon$. Then we have $|\cos(t_0 \log N)| < \varepsilon$ so that $t_0 \log N = \frac{\pi}{2} + k\pi$, for some $k \in \mathbb{Z}$ or, equivalently, $t_0 = (2k+1)\pi/(2 \log N)$, for some $k \in \mathbb{Z}$. From this it follows that the zero-set of $R_N(\sigma,t)$ in the halfplane $\sigma < 1 - N/\varepsilon$ consists of simple zero-curves having $-\infty + (2k+1)\pi i/(2 \log N)$, ($k \in \mathbb{Z}$), as asymptotical points.
For $\sigma = 1$ (as well as for any other fixed $\sigma \in \mathbb{R}$) $R_N(\sigma,t)$ is an almost periodic function of t and since

$$(6.1.6) \qquad \max_{t \in \mathbb{R}} R_N(1,t) = R_N(1,0) = \sum_{n=1}^{N} \frac{1}{n}$$

there exist arbitrarily large values t^* of t such that for any $\varepsilon > 0$

$$(6.1.7) \qquad R_N(1,t^*) > -\varepsilon + \sum_{n=1}^{N} \frac{1}{n}$$

or

$$(6.1.8) \qquad \sum_{n=1}^{N} \frac{1}{n} \cos(t^* \log n) > -\varepsilon + \sum_{n=1}^{N} \frac{1}{n} .$$

Choosing $\varepsilon > 0$ small enough we find that all cosines in (6.1.8) are close to 1 and hence positive so that for these particular values t^* we have

$$(6.1.9) \qquad R_N(\sigma,t^*) = \sum_{n=1}^{N} n^{-\sigma} \cos(t^* \log n) > 0 \text{ for all } \sigma \in \mathbb{R} .$$

Hence, the horizontal lines $t = t^*$ act as barriers for the zero-curves of $R_N(\sigma,t)$. Since the zero-curves of any harmonic function on the entire plane cannot have endpoints, it follows that a zero-curve of $R_N(\sigma,t)$ starting at a point

$$(6.1.10) \qquad -\infty + \frac{(2k+1)\pi i}{2 \log N}$$

must return to some other asymptotical point of the same form (possibly not a neighbouring one (!?)).

Next we consider the zero-curves of $I_N(\sigma,t)$. If $I_N(\sigma_0,t_0) = 0$ then

$$(6.1.11) \qquad N^{-\sigma_0} \sin(t_0 \log N) = - \sum_{n=2}^{N-1} n^{-\sigma_0} \sin(t_0 \log n)$$

so that for $\sigma_0 < 0$

$$(6.1.12) \qquad |\sin(t_0 \log N)| \leqq \sum_{n=2}^{N-1} (\tfrac{n}{N})^{-\sigma_0} < \frac{N}{1-\sigma_0} .$$

Similarly as before, we choose a small $\varepsilon > 0$ and take $\sigma_0 < 1 - \frac{N}{\varepsilon}$ so that $|\sin(t_0 \log N)| < \varepsilon$. Consequently, $t_0 \log N \cong k\pi$, or equivalently, $t_0 \cong k\pi/\log N$, for some $k \in \mathbb{Z}$. Hence, the zero-set of $I_N(\sigma,t)$ in the halfplane $\sigma < 1 - \frac{N}{\varepsilon}$ consists of a system of simple zero-curves having the points $-\infty+k\pi i/\log N, (k \in \mathbb{Z})$, as asymptotical points.
For large positive σ we have in case of a zero of $I_N(\sigma,t)$

$$(6.1.13) \qquad 2^{-\sigma_0} \sin(t_0 \log 2) = - \sum_{n=3}^{N} n^{-\sigma_0} \sin(t_0 \log n)$$

and hence $|\sin(t_0 \log 2)| \leqq \sum_{n=3}^{N} (\frac{2}{n})^{\sigma_0}$. Taking a small $\varepsilon > 0$ and taking $\sigma_0 > \log(N/\varepsilon)/\log(3/2)$ we thus have $|\sin(t_0 \log 2)| < \varepsilon$ so that $t_0 \log 2 \cong k\pi$, for some $k \in \mathbb{Z}$, or, equivalently, $t_0 \cong k\pi/\log 2$, for some $k \in \mathbb{Z}$. It follows that the zero-set of $I_N(\sigma,t)$ in the halfplane $\sigma > \log(N/\varepsilon)/\log(3/2)$ consists of simple zero-curves having the points $+\infty + k\pi i/\log 2$, $(k \in \mathbb{Z})$, as asymptotical points.
It can be shown that every zero-curve of $I_N(\sigma,t)$ starting at some asymptotical point $+\infty + k\pi i/\log 2$ is somehow connected with some asymptotical point $-\infty + l\pi i/\log N$. In other words: such a zero-curve traverses the s-plane more or less horizontally. Moreover, every zero-curve of $I_N(\sigma,t)$ starting at $-\infty + k\pi i/\log N$ is connected either with an asymptotical point $+\infty + l\pi i/\log 2$ or with an asymptotical point of the form $-\infty + m\pi i/\log N$.
Drawing the zero-curves of $I_N(\sigma,t)$ as dotted lines, we see that the zero-curves of $I_N(\sigma,t)$ and $R_N(\sigma,t)$ have a typical pattern as sketched in Figure 1. This *sketch* is based on actual computations of the signs of R_N and I_N for various values of N.
We recall that earlier in this chapter it was shown that for $2 \leqq N \leqq 10$, $N \neq 7$, the zero-curves of $R_N(\sigma,t)$ do not intersect the

vertical $\sigma = 1$. Hence, in order to find a special zero of $\zeta_N(s)$, it stands to reason that N should be taken fairly large (compare Section 4).

REMARK. The combinatorial interplay of the zero-curves of $R_N(\sigma,t)$ and $I_N(\sigma,t)$ remains a mystery.

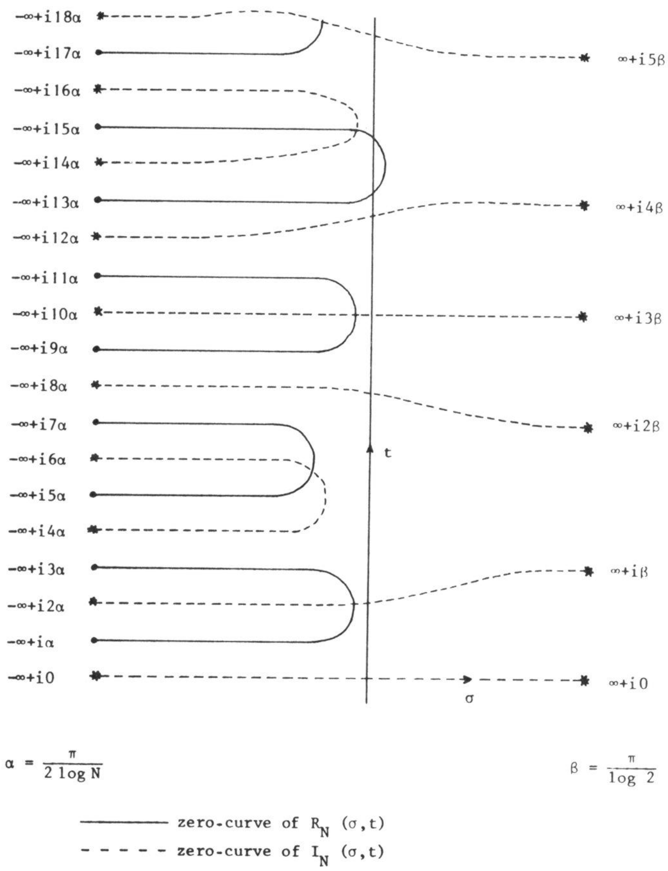


Figure 1

6.2. The heuristic principle

In case of a special zero s_0 of $\zeta_N(s)$ we expect to have a zero-curve pattern as sketched in Figure 2. Here (in accordance with numerical observations) we have tacitly assumed that all zeros of $\zeta_N(s)$ are *simple*, so that at s_0 the zero-curves of R_N and I_N are perpendicular. In order to detect such a pattern "all" zeros of $R_N(1,t)$ were computed for $t > 0$, yielding the increasing sequence $t_1 < t_2 < \ldots$ of zeros of $R_N(1,t)$. Once the zeros t_{2l-1} and t_{2l} were located we checked whether $I_N(1,t)$ had a zero between t_{2l-1} and t_{2l}. If so, it was a simple matter to locate the corresponding special zero of $\zeta_N(s)$ with great accuracy (by Newton's method, for example).

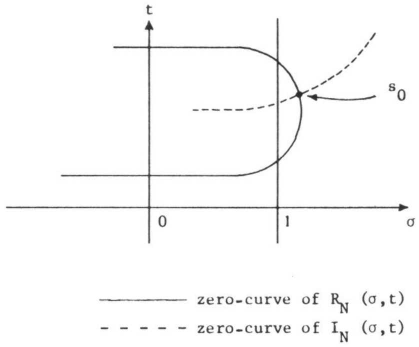


Figure 2.

6.3. The generalized maximum slope principle

The systematic search for all zeros of $R_N(1,t)$ was carried out by means of the *generalized maximum slope principle*.
Let $f(t)$ be a differentiable function on $t \geqq t_0$ and suppose that

(6.3.1) $\quad |f'(t)| \leqq M_0$ for all $t \geqq t_0$.

The (direct) maximum slope principle is the simple observation that if $f(t_0) > 0$ and if f is not linear then $f(t) > 0$ for $t \in [t_0, t_1]$, where

(6.3.2) $$t_1 := t_0 + \frac{f(t_0)}{M_0}.$$ Compare Section 3, page 76.

If f has a zero, let t^* be the smallest one. Starting at t_0 the principle provides us with a new lower bound (t_1) for t^* and repeating the principle, t^* may be approximated as accurately as desired.
Note that every application of this principle requires an evaluation of f. If the definition of f contains functions like sin, cos or log (see the definitions of R_N and I_N) it may be considerably more efficient to apply the following *modification*.
Suppose that for some positive integer k

(6.3.3) $\quad |f^{(k+1)}(t)| \leqq M_k \quad$ for all $\quad t \in \mathbb{R}$.

Then (see Figure 3), from the Taylor expansion of $f(t)$ around t_0, we have for all $t \geqq t_0$

(6.3.4) $$f(t) \geqq P_k(t_0,t) := \sum_{r=0}^{k} \frac{f^{(r)}(t_0)}{r!}(t-t_0)^r - \frac{M_k}{(k+1)!}(t-t_0)^{k+1}.$$

An evaluation of $P_k(t_0,t)$ is quite often *considerably cheaper* than an evaluation of $f(t)$ and then it is preferable to apply the maximum slope principle to P_k rather than to f. This we call the *generalized maximum slope principle*. It stands to reason that $f^{(r)}$ should be easy to compute for all r occurring in (6.3.4) whereas k usually has to be determined experimentally. When applying this principle one obtains an increasing sequence of points $t_{0,j}$ defined by

$$(6.3.5) \qquad t_{0,0} := t_0 \text{ and } t_{0,j+1} := t_{0,j} + \frac{P_k(t_0, t_{0,j})}{M_0}, \quad (j=0,1,2,\ldots).$$

The procedure is interrupted at $t = t_{0,n}$ if $P_k(t_0, t_{0,n}) \leqq \varepsilon (= 10^{-6}, \text{say})$. Note that $t_{0,j} < t^*$ for $j = 0, 1, \ldots, n$. Now we compute $f(t_{0,n})$. If $f(t_{0,n}) > \varepsilon$ then put $t_1 := t_{0,n}$ and set up a *new polynomial* $P_k(t_1, t)$ and continue as above. This yields a *finite* sequence $t_{1,0} := t_1, t_{1,1}, t_{1,2}, \ldots$, and at the next repetition we get $t_{2,0} := t_2, t_{2,1}, t_{2,2}, \ldots$. We continue until we find an m and a corresponding $n = n(m)$ such that $P_k(t_m, t_{m,n}) < \varepsilon$ *and* $f(t_{m,n}) < \varepsilon$. At such an instance we compute $f(t_{m,n}+\delta)$ (with $\delta = 10^{-2}$, say). The values of ε and δ given above were determined experimentally such that always $f(t_{m,n}) \cdot f(t_{m,n}+\delta) < 0$. The next sign change of f(t) is determined similarly, starting at $t_0 := t_{m,n}+\delta$ (see Figure 3). (The above principle was also applied in [39].)

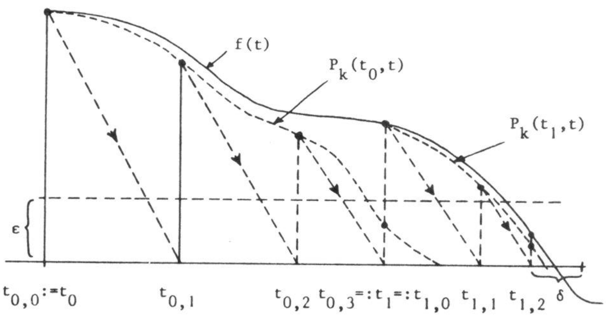


Figure 3.

6.4. The systematic search for special zeros

In order to apply the generalized maximum slope principle to $R_N(1,t)$ we need suitable estimates for $\sup_{t\in\mathbb{R}} |R_N^{(k+1)}(1,t)| =: M_{k,N}$, $(k=0,1,2,\ldots)$. Since

$$(6.4.1) \qquad R_N'(1,t) = - \sum_{n=2}^{N} \frac{\log n}{n} \sin(t \log n)$$

we have

$$(6.4.2) \qquad M_{0,N} = \sup_{t\in\mathbb{R}} \left| \sum_{n=2}^{N} \frac{\log n}{n} \sin(t \log n) \right| \leqq \sum_{n=2}^{N} \frac{\log n}{n}$$

which yields, e.g., $M_{0,22} < 4.78$. By using the prime decomposition of all $n \in [2,N]$ and the linear independence of the logarithms of the primes over the rationals (compare Sections 1,2,3 of Chapter 3), we

derived the sharper bound $M_{0,22} < 4.275$. However, this improvement did not speed up the systematic search significantly.
For the higher derivatives we used the estimates

$$(6.4.3) \qquad M_{k,N} := \sup_{t\in\mathbb{R}} |R_N^{(k+1)}(1,t)| \leq \sum_{n=2}^{N} \frac{(\log n)^{k+1}}{n} .$$

Similar estimates were used for finding zeros of $I_N(1,t)$.
For N = 23 the procedure sketched in Section 6.2 led very quickly to the special zero

$$(6.4.4) \qquad \sigma = 1.008\,496\,93\ldots, \quad t = 8645.524\,423\,32\ldots .$$

It took considerably more time to find a special zero for N = 19 :

$$(6.4.5) \qquad \sigma = 1.001\,095\ldots, \quad t = 600\,884.203\,427\ldots .$$

SPIRA's investigations [55] show that N = 19, 22 and 23 are the smallest candidates for yielding special zeros. We did not succeed in finding a special zero of $\zeta_{22}(s)$ in the range $0 \leq t \leq 75\,000\,000$ (note that 22 is composite). Various experiments showed that $k \cong 14$ was the optimal choice in this case (N = 22).
In [35] the present author and te RIELE showed by means of a different method that $\zeta_{22}(\sigma+it) = 0$ for

$$(6.4.6) \qquad \sigma = 1.002\,890\ldots, \quad t = 558\,159\,406.148\,225\ldots .$$

6.5. A selection of special zeros

Below we present a table of a number of special zeros of $\zeta_N(s)$ (computed by the systematic and/or the almost-period method described in [35]). The inherent programming for the CDC CYBER 6600 was taken care of by H.J.J. te RIELE.

Table

N	σ	t
19	1.00109551	600884.20342778
22	1.00289095	558159406.14822557
23	1.00849693	8645.52442332
23	1.00519091	938296.18122556
24	1.00404187	32520751.78599510
25	1.00044920	32520751.80223907
26	1.00147172	3202110.43537085
27	1.00041028	61242054160408938.59968064
29	1.00370506	2589158977352418.11781520
30	1.00035753	2589158977352418.10546556
31	1.00710369	52331955.65876128

REMARK. The "smaller examples" in this table may even be (and actually were) verified by means of a programmable pocket calculator.

CHAPTER 5

RIGOROUS HIGH SPEED SEPARATION OF THE NON-TRIVIAL ZEROS OF RIEMANN's ZETA FUNCTION

0. INTRODUCTION

The work described in this chapter was done in cooperation with H.J.J. te RIELE and D.T. WINTER. See [7], [40], [41] and [42].

Riemann's zeta function is the meromorphic function $\zeta : \mathbb{C}\backslash\{1\} \to \mathbb{C}$, which, for $\mathrm{Re}(s) > 1$, may be represented explicitly by

$$(0.1) \qquad \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad (s = \sigma+it).$$

It is well known (see TITCHMARSH [58; Chapters II and X]) that

$$(0.2) \qquad \xi(s) := \tfrac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{s}{2})\zeta(s)$$

is an entire function of order 1, satisfying the functional equation

$$(0.3) \qquad \xi(s) = \xi(1-s)$$

so that

$$(0.4) \qquad \Xi(z) := \xi(\tfrac{1}{2}+iz), \qquad (z \in \mathbb{C})$$

being an *even entire* function of order 1, has an infinity of zeros. The Riemann-Hypothesis (cf. RIEMANN [50; p. 148]) is the statement that all zeros of $\Xi(z)$ are real, or, equivalently, that all non-real zeros of $\zeta(s)$ lie on the so called critical line $\sigma = \frac{1}{2}$. Since $\zeta(\bar{s}) = \overline{\zeta(s)}$ we may restrict ourselves to the halfplane $t > 0$.
To this day, Riemann's Hypothesis has neither been proved nor disproved. Numerical investigations concerning this unsolved problem were initiated by Riemann (cf. [11; pp. 155-156]) himself and later on continued more systematically by the writers listed below (including their progress).

Investigator	Year	The first N complex zeros of $\zeta(s)$ are simple and lie on $\sigma = \frac{1}{2}$
GRAM [14]	1903	N = 15
BACKLUND [3]	1914	N = 79
HUTCHINSON [17]	1925	N = 138
TITCHMARSH [57]	1935/36	N = 1,041

Those listed above used the Euler-Maclaurin summation formula and performed their computations by hand or desk calculator. (As to the work of GRAM [14] also see LINDELÖF [27] who worked with Plana's summation formula.) Those listed below applied the Riemann-Siegel formula in conjunction with electronic computing devices.

LEHMER [24, 25]	1956	N = 25,000
MELLER [43]	1958	N = 35,337
LEHMAN [23]	1966	N = 250,000
ROSSER, YOHE & SCHOENFELD [51]	1968	N = 3,500,000
BRENT [6]	1979	N = 81,000,001

An *excellent explanatory account* of most of these computations may be found in EDWARDS [11]. This chapter is a *continuation* of BRENT [6], BRENT et al. [7], and [42].

In practice, the numerical verification of the Riemann-Hypothesis (RH, for short) in a given range consists of *separating* the zeros of the well known real function Z(t), described in Section 2.0, or, equivalently, of finding sufficiently many sign changes of Z(t). Our program (aiming at a fast separation of these zeros) is based, essentially, on the modification of LEHMER's [24] method introduced by ROSSER et al. [51]. However, we have developed a more efficient strategy of searching for sign changes of Z(t) in Gram blocks of length $L \geqq 2$ (see Section 1). This enabled us to bring down the average number of Z-evaluations, needed to separate a zero from its predecessor, from about 1.41 (cf. BRENT [6]) to about 1.19. From the statistics in Section 3 it follows that this figure could *not* have been reduced below 1.136 by any program evaluating Z(t) at all Gram points. It may be noted here that about 98 percent of the computer's running time was spent on evaluating Z(t). Our (FORTRAN/COMPASS) program was executed on a CDC CYBER 170-750 and ran about ten times as fast as the UNIVAC 1100/42 program of BRENT [6]. This is roughly what could be expected in view of the relative speeds of the different machines.

1. THE STRATEGY FOR FINDING THE REQUIRED NUMBER OF SIGN CHANGES OF Z(t) IN A GRAM BLOCK OF LENGTH L $\geq$ 2

We recall some definitions. Let $\theta(t)$ be the real continuous function defined by

(1.1) $\theta(t) := \arg(\pi^{-\frac{1}{2}it}\Gamma(\frac{1}{4}+\frac{1}{2}it))$, $(t > 0)$; $\theta(0) := 0$.

The j-th *Gram point* g_j $(j = -1,0,1,2, \ldots)$ is defined as the unique number satisfying $\theta(g_j) = j\pi$, $g_j > 7$. A Gram point g_j is called *good* if $(-1)^j Z(g_j) > 0$ and *bad* otherwise. An interval $[g_j, g_{j+1})$ is called a *Gram interval*. A *Gram block* of length L ($\geq$ 1) is an interval $B_j = [g_j, g_{j+L})$ such that g_j and g_{j+L} are good Gram points and $g_{j+1}, \ldots, g_{j+L-1}$ are bad Gram points.
A Gram block B_j of length L is said to satisfy "Rosser's rule" if Z(t) has *at least* L zeros in B_j.
The strategy of Brent for finding the required number of sign changes of Z(t) is based on this rule. We refined this strategy in order to reduce the number of Z-evaluations as much as we could. This will be described here in some detail.
In order to reduce the number of Z-evaluations, we first observe that after having determined a Gram block B_j of length L $\geq$ 2, *we already have implicitly detected* L - 2 *sign changes of* Z(t). Hence, the problem reduces to finding the "missing two" sign changes. Next we observe that these missing two (if existing) must both lie in one and the same Gram interval of the block B_j. Some preliminary experiments with our program revealed that in the majority of cases the missing two are situated in one of the *outer* Gram intervals of B_j. Therefore, we first search in (g_j, g_{j+1}) or (g_{j+L-1}, g_{j+L}) according as which of $abs(Z(g_j) + Z(g_{j+1}))$ and $abs(Z(g_{j+L-1}) + Z(g_{j+L}))$ is the smallest. In the selected interval an efficient parabolic interpolation search routine (SRCH2A; see the program in [41]) is invoked.

Here is one of the main improvements upon Brent's method, which used random search rather than parabolic interpolation. If this routine terminates without having detected the missing two zeros, the other *outer* Gram interval of the block is treated in the same manner. In case the missing two are still not found, another search routine is called, depending on the length L of the block $B_j = [g_j, g_{j+L})$.
If L = 2, the interval (g_j, g_{j+2}) is scanned again (SRCH2B; see the program in [41]) and if L > 2 we continue to search in the interval (g_{j+1}, g_{j+L-1}) (SRCH3; see the program in [41]). In both cases, the search is performed by means of a refinement of a search routine described by LEHMAN [23].
SRCH2B searches for the missing two by putting a grid of increasing refinement on the Gram block $B_j = [g_j, g_{j+2})$ until one (and hence *two*) sign change is found, or the maximal search depth is reached. In the latter case the missing two were not found or, in case of non-existence, we have encountered an exception to Rosser's rule (which occurs very rarely; see Table 3.2).
SRCH3 searches for the missing two in the *inner* L - 2 Gram intervals of B_j by means of an adaption of a search routine described by LEHMAN [23]. SRCH3 is always applied to the composing Gram intervals of the block and *never* to the block as a whole. In addition, this search is performed in a zig-zag manner, moving from the periphery of the block towards its centre. Such a zig-zag cycle is repeated a number of times with a grid of increasing refinement. For more details we refer to the source text in [41]. If at some instant one of these search routines has detected the missing two, a new Gram block is set up and we continue as described above. In the opposite case (which occurs very rarely) the program prints a message and a "plot" of Z(t) corresponding to the whole Gram block under investigation and proceeds by pretending (!) that the missing two were found indeed. These plots of Z(t) are inspected afterwards (if necessary) "by hand". So far, the missing two were always easily found either in the Gram block under

consideration or in an adjacent Gram block (compare BRENT [6; Section 4]).

After having covered the range $[g_{300,000,000}, g_{400,000,000})$ we ran the computation a little further, and found 4 Gram blocks in $[g_{400,000,000}, g_{400,000,005})$, all of them satisfying Rosser's rule.

By applying Theorem 3.2 of BRENT [6] we were able to prove that the first 400,000,001 complex zeros of $\zeta(s)$ are simple and lie on $\sigma = \frac{1}{2}$.

2. COMPUTATION OF Z(t) AND ERROR ANALYSIS

2.0. Introduction

The *unambiguous determination of the sign* of $Z(t)$ requires a rigorous bound for the error, made when actually computing $Z(t)$ on a computer. In our program we used two methods (A and B) for evaluating $Z(t)$.

Method A is a very fast and efficient method which usually gives the correct sign of $Z(t)$.

Method B is a comparatively slow, though very accurate method which is invoked when $|Z(t)|$ is too small for method A. Until now not a single t was met for which method B could not determine the sign of $Z(t)$ rigorously.

We used the well known Riemann-Siegel formula (with two correction terms in either case):

$$(2.1) \qquad Z(t) = 2 \sum_{k=1}^{m} k^{-\frac{1}{2}} \cos(t.\log k - \theta(t)) + (-1)^{m-1} \tau^{-\frac{1}{4}} \sum_{j=0}^{1} \Phi_j(z)\, \tau^{-\frac{j}{2}} + R_1(t)$$

where

$$(2.1.a) \qquad m = [\tau^{\frac{1}{2}}], \quad \tau = t/(2\pi), \quad z = 1-2(\tau^{\frac{1}{2}} - m), \qquad (\text{note that } -1 < z \leq 1)$$

$$(2.2) \qquad \theta(t) = \arg(\pi^{-\frac{1}{2}it}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)), \qquad \theta(t) \text{ continuous and } \theta(0) = 0$$

$$(2.3) \qquad \Phi_0(z) = \cos(\pi(4z^2+3)/8)/\cos(\pi z) =: \sum_{k=0}^{\infty} c_{2k}^{(0)} z^{2k}$$

and

$$(2.4) \qquad \Phi_1(z) = \Phi_0^{(3)}(z)/(12\pi^2) =: \sum_{k=0}^{\infty} c_{2k+1}^{(1)} z^{2k+1} .$$

The error term $R_1(t)$ will be dropped in our actual computations (but not in the error analysis). GABCKE [13] and BRENT & SCHOENFELD [8] have given bounds on $R_n(t)$ (here, n+1 denotes the number of terms in the second sum in (2.1)). We used the bound (GABCKE [13])

(2.5) $$|R_1(t)| < 0.053\ t^{-5/4} < 0.0054\ \tau^{-5/4} \quad \text{for } t \geq 200.$$

The *floating point machine approximations* of Z by means of methods A and B will be denoted by $\tilde{Z}_A$ and $\tilde{Z}_B$, respectively. More generally, throughout this section, the result of the floating point machine approximation of some quantity q will be denoted by $\tilde{q}$.

We present an error analysis which accounts for *all* possible errors in $\tilde{Z}$, for any t (resp. τ) in the range

(2.6) $$3.5 \times 10^7 < t < 3.72 \times 10^8 \quad (\text{or } 5.5 \times 10^6 < \tau < 5.92 \times 10^7).$$

This covers the range between zero # 81,000,001 and zero # 1,000,000,000 of $\zeta(s)$ in the critical strip, which we originally planned to investigate ($\gamma_{81,000,001} \cong 35,018,261.166$, $\gamma_{1,000,000,000} \cong 371,870,203.837$). The computations were carried out on a CDC CYBER 175-750 having a 60-bit word, and single-precision (SP) and double-precision (DP) floating point arithmetic using 48- and 96- bit binary fractions, respectively. In the sequel we will frequently work with the unit roundoffs $\varepsilon_s := 2^{-47}$ and $\varepsilon_d := 2^{-95}$.

2.1. Computation of Z(t)

At the start of our program four tables are precomputed:

a. $\log k$ for $1 \leq k \leq m_0$ in DP, where m_0 is large enough to cover the range of the current job;

b. $k^{-\frac{1}{2}}$ for $1 \leq k \leq m_0$ in DP, truncated to SP;

c. $\cos(2\pi k 2^{-13})$ for $0 \leq k \leq 2^{13} + 1$ in DP, truncated to SP;

d. $\cos(2\pi(k+1)2^{-13}) - \cos(2\pi k 2^{-13})$ for $0 \leq k \leq 2^{13}$ in DP, truncated to SP.

Methods A and B run essentially as follows.

Method A. Given a τ as a DP floating point number, $t = 2\pi\tau$ and $\theta(t)$ are computed in DP; $f^{(1)} := \text{frac}(\theta(t)(2\pi)^{-1})$ is computed in DP, and truncated to SP. Next, the main loop (corresponding to the first sum in (2.1)) is executed. This loop has been written in COMPASS (machine language of the CYBER) and optimized using the specific properties of the CYBER's central processing units. One cycle of the loop executes in about 2.1 μ seconds.

$f^{(2)} := \mathrm{frac}(\tau \log k)$ (where log k is looked up in the precomputed table) is computed as follows: the DP product of τ and log k is decreased with the integer part of the SP product of τ and log k and the result is truncated to SP. This programming "trick" (see ZFUNC in the program in [41]) saves a considerable amount of time in the main loop. Further, $x = \mathrm{abs}(f^{(1)} - f^{(2)})$ is computed in SP, and $\cos(2\pi x)$ is approximated by linear interpolation from the precomputed cos-table, using the precomputed cos-difference table. The result is multiplied by the precomputed $k^{-\frac{1}{2}}$ and the product is accumulated in an SP sum. End of the main loop. Next, the two terms in the asymptotic expansion of the Riemann-Siegel formula (2.1) are approximated using the truncated Taylor series expansions (for the definition of z see (2.1.a))

$$(2.7) \qquad \Phi_0(z) \cong \sum_{k=0}^{N_0} c_{2k}^{(0)} z^{2k} \quad \text{and} \quad \Phi_1(z) \cong \sum_{k=0}^{N_1} c_{2k+1}^{(1)} z^{2k+1} .$$

The total correction is computed and added to 2 times the sum obtained in the main loop. The computations after the main loop are carried out in SP.

Method B. The same as method A, with *all* computations in DP. The value of $\cos(2\pi x)$ is computed using the available standard DP library function DCOS.

2.2. Error analysis

In our error analysis we *assume* that τ is *exactly representable* as a floating point number. The positive integer m (= $[\tau^{\frac{1}{2}}]$) is *exactly* computed from τ by testing the inequalities $m^2 \leq \tau < (m+1)^2$ and by correcting the machine-computed value, if necessary. Now let

$$(2.8) \qquad s(t) := 2 \sum_{k=1}^{m} k^{-\frac{1}{2}} \cos(t.\log k - \theta(t)) , \qquad (t = 2\pi\tau).$$

By $\tilde{s}(\tilde{t})$ we denote the computed value of $s(t)$, where errors may have been introduced in the computation of t, $\log k$, $\theta(t)$, $t.\log k - \theta(t)$, $\cos(.)$, $k^{-\frac{1}{2}}$ and the final inner product. The following lemma accounts for all these errors.

LEMMA 2.1. *Suppose that* $|t-\tilde{t}| \leq \delta_0 t$, $|\log k - L(k)| \leq \delta_1 \log k$ *for* $k = 1,2, \ldots, m$, *and* $|\theta(u)-\tilde{\theta}(u)| \leq \delta_2\theta(u)$; *let* $f_k := \mathrm{frac}\{\tau L(k)-\widetilde{\tilde{\theta}(\tilde{t})(2\pi)^{-1}}\}$ *and suppose that* $|f_k-\tilde{f}_k| \leq \delta_3$ *for* $k = 1,2, \ldots, m$. *Moreover, suppose that* $|\cos(x)-\tilde{c}(x)| \leq \delta_4$ *for* $0 \leq x \leq 2\pi + h$, *where* h *is fixed* (the reason for the occurrence of this (small) number h in this lemma will be clarified in section (2.3)), $|k^{-\frac{1}{2}} - \widetilde{k^{-\frac{1}{2}}}| \leq \delta_5 k^{-\frac{1}{2}}$ *for* $k = 1,2, \ldots, m$, *and that the inner product of the two vectors with components* $(1 \leq k \leq m)$ $\widetilde{k^{-\frac{1}{2}}}$ *and* $\tilde{c}(\widetilde{2\pi\tilde{f}_k})$, *respectively, is computed in floating point arithmetic, with a relative error in the basic arithmetic operations* (+,-, * *and* /) *bounded by* ε. *Then we have*

$$(2.9)\qquad |s(t)-\tilde{s}(\tilde{t})| \leq 4\pi\tau^{5/4}\log(\tau)\{2\delta_0+\delta_1(1+\delta_0) + \delta_2\} + $$
$$+ 4\tau^{\frac{1}{4}}\{2\pi\delta_3 + \delta_4 + (1+\delta_4)\{\delta_5+(1+\delta_5)((1+\varepsilon)^m-1)\}\}.$$

This lemma is similar to lemma 5.3 of BRENT [6], the difference being that we explicitly account for all possible errors in the computation of $s(t)$. The proof is routine and uses the technique of backward error analysis (cf. WILKINSON [61]) for the inner product computation (cf. PARLETT [48; pp. 30-32]) and for the other basic arithmetic operations. Let

$$(2.10)\qquad \chi(\tau) := (-1)^{m-1}\,\tau^{-\frac{1}{4}}\{\Phi_0(z) + \tau^{-\frac{1}{2}}\Phi_1(z)\}\ .$$

By $\tilde{\chi}(\tau)$ we denote the computed value of $\chi(\tau)$ where errors may have been introduced in the computation of $\tau^{-\frac{1}{2}}$, $\tau^{-\frac{1}{4}}$, z, $\Phi_0(z)$, $\Phi_1(z)$, and in the other arithmetic operations. The following lemma accounts for all these errors.

LEMMA 2.2. *Let ε be as in Lemma 2.1 and let the relative error in the square root computation be bounded by* $a\varepsilon$. *Moreover, suppose that* $|z-\tilde{z}| \leq \delta_6$ *and that* $\Phi_0(z)$ *and* $\Phi_1(z)$ *are approximated by* $\tilde{\Phi}_0(z) := \sum_{k=0}^{N_0} \tilde{c}_{2k}^{(0)} z^{2k}$ *and* $\tilde{\Phi}_1(z) := \sum_{k=0}^{N_1} \tilde{c}_{2k+1}^{(1)} z^{2k+1}$ *respectively, where*

$|c_{2k}^{(0)} - \tilde{c}_{2k}^{(0)}| \leq \delta_7$ *and* $|c_{2k+1}^{(1)} - \tilde{c}_{2k+1}^{(1)}| \leq \delta_7$. *Then*

$$(2.11) \qquad |\chi(\tau)-\tilde{\chi}(\tau)| \leq \tau^{-\frac{1}{4}}\{2\delta_6+2\delta_7(N_0+1)+\frac{1}{(N_0+1)!}(\tfrac{\pi}{2})^{N_0+1}+(5N_0+2a+4)\varepsilon\} + $$
$$+ \tau^{-3/4}\{\tfrac{1}{4}\delta_6+2\delta_7(N_1+1)+\frac{1}{6}\frac{N_1+\frac{5}{2}}{(N_1+1)!}(\tfrac{\pi}{2})^{N_1+1}+(5N_1+3a+7)\varepsilon\} .$$

In the proof of this lemma, which we shall omit, we have used the inequalities $|\Phi_0(z)| \leq 1$, $|\Phi_1(z)| \leq 1$, $|\Phi_0'(z)| \leq 1$ and $|\Phi_1'(z)| \leq \frac{1}{4}$ for $|z| \leq 1$ (see GABCKE [13; Theorem 1, p. 60]) and of the bounds given in GABCKE [13; Theorem 2, p. 62] on the error induced by truncating the infinite series in (2.3) and (2.4).

2.3. Derivation of $\delta_0, \ldots, \delta_7$ for methods A and B

δ_0. Since τ is (assumed to be) exact and we work with a DP value of t $(= 2\pi\tau)$, for both methods, one may easily show that $\delta_0 = 1.01 \times 10^{-28}$ is a safe choice.

δ_1. The values of log k, $1 \leq k \leq 7700$, where k covers the range corresponding to (2.6), are computed in DP and then compared with the corresponding numbers computed by means of Brent's multiple-precision package [5] with an accuracy of 40 digits. Taking into account the maximal observed relative error, we can safely take $\delta_1 = 5.1 \times 10^{-29}$ for both methods.

δ_2. The function $\theta(t)$ was computed in DP from the formula

$$(2.12) \qquad \theta(t) = \frac{t}{2}(\log(\tau)-1) - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + \frac{31}{80640t^5} + R\theta(t)$$

where $|R\theta(t)| < \frac{t^{-7}}{3322}$ (see GABCKE [13; p. 4]). According to the CDC-manual [9] the maximum relative error made by the DP standard FORTRAN

library function DLOG (for the computation of log(.)) is bounded by 1.8×10^{-27} ($< 128\varepsilon_d$). In Table 2.1 we present, in detailed steps, a backward error analysis of the algorithm for the computation of $\theta(t)$, following the FORTRAN-source text of our program in [41].
Here, we use the following notation: $x(1+\varepsilon^{(j)})$ denotes an approximation of x, with $|\varepsilon^{(j)}| \leqq j\varepsilon_d$. If $\square$ stands for any of the basic DP arithmetic operations +,−,* and /, and if we want to compute $\alpha\square\beta$, then we may assume in our analysis that the resulting machine number is $(\alpha\square\beta)(1+\varepsilon^{(1)})$. From the last line in Table 2.1 it follows that it suffices to take $\delta_2 = 3.6 \times 10^{-27}$ ($> 139\varepsilon_d$).

δ_3. *Method A.* Before the main loop is started, the value of $f^{(1)} = \text{frac}\{\tilde{\theta}(\tilde{t})(2\pi)^{-1}\}$ is precomputed in DP, and truncated to SP. The absolute error introduced should be $\leqq \varepsilon_s$; however, in our program we first computed $\tilde{\theta}(\tilde{t})$ mod 2π, and next $f^{(1)}$ by multiplying by $(2\pi)^{-1}$. Therefore, a safe bound on this error is $5\varepsilon_s$. In the main loop the value of $f^{(2)} = \text{frac}(\tau.L(k))$ is computed with an absolute error bounded by $2\varepsilon_s$. Using these two bounds it follows that we can safely choose $\delta_3 = 5 \times 10^{-14}$ ($> 7\varepsilon_s$). Due to the programming "trick" mentioned in Section 2.1 we must take into account the possibility that the *computed* value of $f^{(2)}$ may be (slightly) larger than 1 by an amount which is bounded by $2.5\varepsilon_s\tau L(k)$. In the t-range (2.6) this excess is bounded by 10^{-5}. Instead of correcting $f^{(2)}$ by subtracting 1 (which is necessary only very rarely) we use *one extra element* in the cos-interpolation table beyond $\cos(2\pi)$, viz. $\cos(2\pi+h)$, where $h = 2\pi.2^{-13} \simeq 7.7 \times 10^{-4}$ ($> 10^{-5}$).

δ_3. *Method B.* For method B the absolute error in the DP computed value of $\tau L(k) - \tilde{\theta}(\tilde{t})$ can be shown to be bounded by $139\varepsilon_d|\tilde{\theta}(\tilde{t})-\tau L(k)| \leqq$ $\leqq 139\varepsilon_d|\tilde{\theta}(\tilde{t})|$. Since $\theta(t) < \pi\tau\log\tau$ and $\tau < 5.93 \times 10^7$ it follows that we may take $\delta_3 = 1.2 \times 10^{-17}$, for method B.

δ_4. In method A the cos-values are approximated by linear interpolation in the table of $\cos(2\pi\frac{k}{8192})$, $0 \leqq k \leqq 8192$, using a second table

TABLE 2.1

Backward error analysis of the computation of $\theta(t)$ (τ exact); all computations are in DP.

FORTRAN-text	value obtained in the machine	comment
1. DTWOPIN	$(2\pi)^{-1}(1+\varepsilon^{(2)})$	precomputed in main program
2. DTAU	τ	τ is assumed to be exactly representable in the machine
3. DT=DTAU*DTWOPI	$t(1+\varepsilon^{(4)})$	2 and 3 actually do not occur in the source-text, but the analysis is expressed in this way because τ is supposed to be exact, instead of t
4. DTINV=1.DO/DT	$t^{-1}(1+\varepsilon^{(6)})$	
5. DLOG(DTAU)	$\ln(\tau)(1+\varepsilon^{(128)})$	
6. -1.DO	$(\ln(\tau)-1)(1+\varepsilon^{(130)})$	subtract 1 from $\ln(\tau)$
7. *.5DO	$\frac{1}{2}(\ln(\tau)-1)(1+\varepsilon^{(130)})$	exact multiplication by $\frac{1}{2}$
8. *DT	$\frac{1}{2}t(\ln(\tau)-1)(1+\varepsilon^{(135)})$	multiplication by t
9. -DPISL8	$(\frac{1}{2}t(\ln(\tau)-1)-\frac{\pi}{8})(1+\varepsilon^{(137)})$	subtract $\frac{\pi}{8}$ (precomputed)
10. DCNST1	$\frac{1}{48}(1+\varepsilon^{(1)})$	
11. DCNST2	$\frac{7}{5760}(1+\varepsilon^{(1)})$	precomputed (items 10–12)
12. DCNST3	$\frac{31}{80640}(1+\varepsilon^{(1)})$	
13. DCNST3*DTINV	$\frac{31}{80640t}(1+\varepsilon^{(9)})$	
14. *DTINV	$\frac{31}{80640t^2}(1+\varepsilon^{(17)})$	
15. +DCNST2	$(\frac{31}{80640t^2}+\frac{7}{5760})(1+\varepsilon^{(19)})$	
16. *DTINV	$(\frac{31}{80640t^3}+\frac{7}{5760t})(1+\varepsilon^{(27)})$	
17. *DTINV	$(\frac{31}{80640t^4}+\frac{7}{5760t^2})(1+\varepsilon^{(35)})$	
18. +DCNST1	$(\frac{31}{80640t^4}+\frac{7}{5760t^2}+\frac{1}{48})(1+\varepsilon^{(37)})$	
19. *DTINV	$(\frac{31}{80640t^5}+\frac{7}{5760t^3}+\frac{1}{48t})(1+\varepsilon^{(45)})$	
20. + "9"	$\theta(t)(1+\varepsilon^{(139)})$	

of cos-differences. The interpolation error is bounded by $\frac{1}{8}(2\pi\times 2^{-13})^2$ (with exact arithmetic). The error induced by the inexact arithmetic in the SP evaluation of the linear interpolation formula is bounded in absolute value by 8×2^{-47}. So $\delta_4 = 7.36 \times 10^{-8}$ is a safe choice for method A.

In method B the cos-values are approximated by the DP standard FORTRAN library function DCOS. According to the CDC-manual [9], a safe choice is $\delta_4 = 1.5 \times 10^{-27}$ for method B.

δ_5. In method A the values of $k^{-\frac{1}{2}}$ are computed in DP and truncated to SP, so that $\delta_5 = 7.2 \times 10^{-15}$ ($> \varepsilon_s$). In method B the DP values of $k^{-\frac{1}{2}}$ are used. A comparison with the corresponding 40D values obtained with Brent's multiple-precision package [5] shows that we may safely take $\delta_5 = 1.01 \times 10^{-28}$.

δ_6. In both methods, the number $z = 1-2(\tau^{\frac{1}{2}}-[\tau^{\frac{1}{2}}])$ is computed in DP. In method A it is truncated to SP, so that we may take $\delta_6 = 7.2 \times 10^{-15}$ ($> \varepsilon_s$). In method B the absolute error in z can be safely bounded by $10\varepsilon_d\tau^{\frac{1}{2}} < 2.0 \times 10^{-24}$ so that we may take $\delta_6 = 2.0 \times 10^{-24}$ for method B.

δ_7. The coefficients $c_{2k}^{(0)}$ and $c_{2k+1}^{(1)}$ in the Taylor series expansions of $\Phi_0(z)$ and $\Phi_1(z)$, respectively, were borrowed from CRARY & ROSSER [10] (and compared with HASELGROVE & MILLER [16] and GABCKE [13]) in such a way that for method A a safe choice is $\delta_7 = 5 \times 10^{-14}$ and for method B, $\delta_7 = 5 \times 10^{-28}$.

In Table 2.2 we display the values of $\delta_0, \ldots, \delta_7$ for both methods.

TABLE 2.2

Values of $\delta_0, \ldots, \delta_7$ for methods A and B

Method	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7
A	1.01×10^{-28}	5.1×10^{-29}	3.6×10^{-27}	5×10^{-14}	7.36×10^{-8}	7.2×10^{-15}	7.2×10^{-15}	5×10^{-14}
B	1.01×10^{-28}	5.1×10^{-29}	3.6×10^{-27}	1.2×10^{-17}	1.5×10^{-27}	1.01×10^{-28}	2×10^{-24}	5×10^{-28}

2.4. The error bounds on $\tilde{Z}$ for methods A and B

In order to complete the error analysis we apply Lemmas 2.1 and 2.2 with $\delta_0, \ldots, \delta_7$ as given in Table 2.2, $\varepsilon = \varepsilon_s = 2^{-47}$, $a = 10$, $N_0 = 16$ and $N_1 = 17$ for method A, and $\varepsilon = \varepsilon_d = 2^{-95}$, $a = 10$, $N_0 = N_1 = 29$ for method B; including the inherent error (2.5) this yields

$$(2.13)\quad |Z(t)-\tilde{Z}_A(\tilde{t})| \leq 3\times 10^{-7}\tau^{\frac{1}{4}}+8.6\times 10^{-12}\tau^{-\frac{1}{4}}+5.4\times 10^{-3}\tau^{-\frac{5}{4}}+5\times 10^{-26}\tau^{\frac{5}{4}}\log\tau$$

and

$$(2.14)\quad |Z(t)-\tilde{Z}_B(\tilde{t})| \leq 5.4\times 10^{-3}\tau^{-\frac{5}{4}}+3.1\times 10^{-16}\tau^{\frac{1}{4}}+4.1\times 10^{-24}\tau^{-\frac{1}{4}}+5\times 10^{-26}\tau^{\frac{5}{4}}\log\tau\ .$$

Instead of these variable bounds, we actually used in our program the extremely conservative *fixed* bounds $\varepsilon_1 = 10^{-4}$ or (in the experimental phase) even 2×10^{-4} in (2.13) and $\varepsilon_2 = 2.5\times 10^{-6}$ in (2.14). In the τ-range (2.6) we could safely have taken $\varepsilon_1 = 2.7\times 10^{-5}$ and $\varepsilon_2 = 2\times 10^{-11}$. In case $|\tilde{Z}_A(\tilde{t})|$ was less than ε_1 method B was called *immediately* rather than trying a few shifts with $\tilde{t}$ (see [40; p. 12]). To give an impression of the actual accuracy of methods A and B we present in Table 2.3 a difference table of $\tilde{Z}_B(76{,}969{,}020.001+k\times 0.001)$ for $k = 0,1, \ldots, 24$, with 15 differences. In addition, since $|\tilde{Z}_B(\tilde{t})-\tilde{Z}_A(\tilde{t})|$ is a very accurate approximation of the error in $\tilde{Z}_A(\tilde{t})$, we have computed this difference for 10,000 "random" values of t. The maximal difference was less than 2×10^{-6} (compare our bound $\varepsilon_1 = 10^{-4}$).

3. SOME STATISTICS

In this section we present some statistics concerning the interval $[g_{300,000,000}, g_{400,000,000})$.
The counts given here are *exact* (in contrast with our counts given in [40] and/or [7]). This was realized by immediately computing $Z_B(g_n)$ in case $|Z_A(g_n)|$ was too small (rather than computing $Z_A(g_n-\delta)$ for a few small values of δ, as we did in the references just mentioned. We never met a value of t for which our method B could not

Table 2.3

Difference table

t	$\tilde{Z}_B(t)$	DIFFERENCES														
		first	second	third	fourth											
76969020.001	-.28955711764714243821															
		-.6175E-02														
76969020.002	-.29573213549545573483		.1183E-03													
		-.6057E-02		.3185E-06												
76969020.003	-.30178890090261723635		.1186E-03		-.9351E-08											
		-.5938E-02		.3092E-06		-.1720E-10										
76969020.004	-.30772709531999851450		.1189E-03		-.9368E-08		.6569E-12									
		-.5819E-02		.2998E-06		-.1654E-10		.1031E-14								
76969020.005	-.31354640955018108511		.1192E-03		-.9385E-08		.6579E-12		-.4388E-16							
		-.5700E-02		.2904E-06		-.1588E-10		.9875E-15		.8428E-18						
76969020.006	-.31924654376415536475		.1195E-03		-.9401E-08		.6589E-12		-.4304E-16		-.5507E-17					
		-.5581E-02		.2810E-06		-.1523E-10		.9445E-15		-.4664E-17		.1666E-16				
76969020.007	-.32482720751786273581		.1198E-03		-.9416E-08		.6599E-12		-.4771E-16		.1116E-16		-.4087E-16			
		-.5461E-02		.2716E-06		-.1457E-10		.8968E-15		.6491E-17		-.2421E-16		.9040E-16		
76969020.008	-.33028811976807968868		.1200E-03		-.9431E-08		.6608E-12		-.4121E-16		-.1306E-16		.4952E-16		-.1882E-15	
		-.5341E-02		.2622E-06		-.1390E-10		.8556E-15		-.6565E-17		.2531E-16		-.9778E-16		.3767E-15
76969020.009	-.33562900888764305346		.1203E-03		-.9445E-08		.6616E-12		-.4778E-16		.1226E-16		-.4826E-16		.1885E-15	
		-.5221E-02		.2528E-06		-.1324E-10		.8078E-15		.5690E-17		-.2295E-16		.9069E-16		-.3564E-15
76969020.010	-.34084961268001537672		.1205E-03		-.9458E-08		.6624E-12		-.4209E-16		-.1069E-16		.4243E-16		-.1679E-15	
		-.5100E-02		.2433E-06		-.1258E-10		.7657E-15		-.5003E-17		.1948E-16		-.7725E-16		.4184E-15
76969020.011	-.34594967839318954661		.1208E-03		-.9470E-08		.6632E-12		-.4709E-16		.8785E-17		-.3483E-16		.2504E-15	
		-.4979E-02		.2338E-06		-.1192E-10		.7186E-15		.3782E-17		-.1535E-16		.1732E-15		-.1919E-14
76969020.012	-.35092896273293181065		.1210E-03		-.9482E-08		.6639E-12		-.4331E-16		-.6563E-17		.1383E-15		-.1668E-14	
		-.4858E-02		.2243E-06		-.1125E-10		.6753E-15		-.2781E-17		.1230E-15		-.1495E-14		.1188E-13
76969020.013	-.35578723187536237847		.1212E-03		-.9494E-08		.6646E-12		-.4609E-16		.1164E-15		-.1357E-14		.1021E-13	
		-.4737E-02		.2148E-06		-.1059E-10		.6292E-15		.1137E-15		-.1234E-14		.8716E-14		-.5077E-13
76969020.014	-.36052426147887284377		.1215E-03		-.9504E-08		.6652E-12		.6756E-16		-.1117E-14		.7360E-14		-.4056E-13	
		-.4616E-02		.2053E-06		-.9924E-11		.6968E-15		-.1004E-14		.6126E-14		-.3184E-13		.1519E-12
76969020.015	-.36513983669537970707		.1217E-03		-.9514E-08		.6659E-12		-.9360E-15		.5009E-14		-.2448E-13		.1114E-12	
		-.4494E-02		.1958E-06		-.9258E-11		-.2392E-15		.4005E-14		-.1836E-13		.7954E-13		-.3341E-12
76969020.016	-.36963375218091332361		.1219E-03		-.9523E-08		.6657E-12		.3069E-14		-.1335E-13		.5506E-13		-.2227E-12	
		-.4372E-02		.1863E-06		-.8592E-11		.2830E-14		-.9342E-14		.3670E-13		-.1431E-12		.5567E-12
76969020.017	-.37400581210554164760		.1220E-03		-.9532E-08		.6685E-12		-.6273E-14		.2335E-13		-.8808E-13		.3340E-12	
		-.4250E-02		.1768E-06		-.7924E-11		-.3442E-14		.1401E-13		-.5138E-13		.1908E-12		-.7157E-12
76969020.018	-.37825583016262807580		.1222E-03		-.9540E-08		.6650E-12		.7740E-14		-.2803E-13		.1028E-12		-.3817E-12	
		-.4128E-02		.1672E-06		-.7259E-11		.4297E-14		-.1401E-13		.5138E-13		-.1908E-12		
76969020.019	-.38238362957742362968		.1224E-03		-.9547E-08		.6693E-12		-.6274E-14		.2336E-13		-.8809E-13			
		-.4005E-02		.1577E-06		-.6589E-11		-.1977E-14		.9344E-14		-.3670E-13				
76969020.020	-.38638904311499064611		.1225E-03		-.9554E-08		.6674E-12		.3070E-14		-.1335E-13					
		-.3883E-02		.1481E-06		-.5922E-11		.1093E-14		-.4004E-14						
76969020.021	-.39027191308746141885		.1227E-03		-.9560E-08		.6685E-12		-.9341E-15							
		-.3760E-02		.1386E-06		-.5253E-11		.1589E-15								
76969020.022	-.39403209136062749381		.1228E-03		-.9565E-08		.6686E-12									
		-.3637E-02		.1290E-06		-.4585E-11										
76969020.023	-.39766943935986159461		.1230E-03		-.9569E-08											
		-.3514E-02		.1194E-06												
76969020.024	-.40118382807537108562		.1231E-03													
		-.3391E-02														
76969020.025	-.40457513806678281355															

determine the sign of Z(t) rigorously, with the bounds given in Section 2.2.
In Table 1 we present a list of the 137 exceptions to Rosser's rule of length 2 and the 6 of length 3, in the range $[g_{300,000,000}, g_{400,000,000})$, including their types. Note the occurrence of a type 4-exception of length 3 (the first one observed of this kind), viz. $G_{368,714,780}$.
Table 2 gives a survey of the various types of exceptions to Rosser's rule and their frequencies in $[g_{300,000,000}, g_{400,000,000})$.
We note that an exception to Rosser's rule of length 3 and type 3 has not been observed in the interval $0 \leq t \leq g_{400,000,000}$.
Table 3 gives the numbers of Gram blocks of length 8 in $[g_{300,000,000}, g_{400,000,000})$ for strings of 10^7 successive zeros. The last line gives the totals for the whole range. The average Gram block length for this range is 1.2062, against 1.2039, 1.2003 and 1.1900 for the ranges $[g_{200,000,000}, g_{300,000,000})$, $[g_{100,000,000}, g_{200,000,000})$ and $[g_0, g_{100,000,000})$, respectively. Note that the number of Gram blocks of length 1 is slowly decreasing in favour of the number of Gram blocks of lengths ≥ 2 and ≤ 7.
In Table 4 we present the number of Gram intervals $G_j = [g_j, g_{j+1})$, $n \leq j < n+10^7$, which contain exactly m zeros of Z(t), $0 \leq m \leq 4$. Note that the number of Gram intervals with precisely one zero is slightly decreasing in favour of the number of Gram intervals with no zeros and those with 2 zeros.
In Table 5 we list the number of Gram blocks of type (j,k), $1 \leq j \leq 8$, $1 \leq k \leq j$, in the range $[g_{300,000,000}, g_{400,000,000})$. We also give the number of Gram blocks of length 2 with zero-pattern "0 0" and "2 2", and those of length 3 with zero-pattern "0 1 0". The "0 0"-blocks correspond to the 137 length 2-exceptions to Rosser's rule of types 1, 2, 3 and 5, 6, and the "2 2"-blocks correspond to the length 2-exceptions of types 5 and 6. The "0 1 0"-blocks correspond to the 6 length 3-exceptions of types 1, 2 and 4 (compare Table 2).
The entries in parentheses denote the percentages with respect to the total number of blocks of length j, given in the final column.
These percentages are nearly the same as those given in the corresponding table for the range $g_{200,000,000}, g_{300,000,000})$ in [40] and [42], and we conclude that our strategy of dealing with Gram blocks of length $j \geq 2$ is successful. For a detailed description of our strategy we refer to the program in [41].

TABLE 1

The 143 *exceptions to Rosser's rule in* $[g_{300,000,000}, g_{400,000,000})$

<u>137 of length 2.</u> Notation: n(type),

where n is the index of the Gram block

$B_n = [g_n, g_{n+2})$ with zero-pattern "0 0"

300746959(2)	328123173(2)	351826159(2)	378091746(2)
301097423(1)	328182212(1)	352228580(1)	379139523(2)
301834209(1)	328257499(2)	352376245(2)	380279160(1)
302554791(1)	328315836(1)	352853758(1)	380619443(2)
303497446(2)	328800974(1)	355110439(1)	381244232(2)
304165345(2)	328998510(2)	355808090(3)	382357074(2)
305302352(1)	329725370(1)	355941557(2)	383545480(2)
306785997(2)	332080601(1)	356360231(1)	384363766(1)
307051444(2)	332221246(1)	356586658(2)	384401788(6)
307481540(2)	332299899(1)	356892926(1)	385198213(2)
308605570(2)	332532822(1)	356908233(2)	385824476(1)
309237611(2)	333334544(5)	357738762(1)	385908195(2)
310509287(1)	333881267(2)	357912731(2)	386946807(2)
310554058(2)	334703268(2)	358120345(2)	387592177(6)
310646346(2)	334875139(2)	359044096(1)	388329293(1)
311274896(1)	336531452(2)	360819358(2)	388679567(2)
311894273(2)	336825907(1)	362361315(1)	388832143(2)
312269470(1)	336993167(1)	363610112(1)	390087103(1)
312306601(3)	337493999(2)	363964805(2)	390190926(5)
312683194(2)	337861035(2)	364527375(1)	390331208(2)
314499805(2)	337899191(1)	365090327(1)	391674496(2)
314636802(1)	337958123(1)	365414540(2)	391937832(2)
314689898(2)	342331983(2)	366738475(2)	391951632(5)
314721320(2)	342676069(2)	368831545(1)	392963986(1)
316132891(2)	347063782(2)	368902387(1)	393007922(2)
316465706(2)	347697349(2)	370109770(2)	393373211(2)
316542790(1)	347954320(2)	370963334(2)	393759572(1)
320822348(2)	348162776(2)	372681562(1)	394036662(1)
321733243(2)	349210702(1)	373009410(1)	395813866(1)
324413970(1)	349212914(2)	373458971(2)	395956691(2)
325890638(1)	349248650(1)	375648659(2)	396031671(2)
325950140(1)	349913501(2)	376834729(2)	397076434(2)
326675884(1)	350891530(2)	377119945(1)	397470602(2)
326704209(2)	351089324(2)	377335703(1)	398289459(2)
327596248(2)			

<u>6 of length 3.</u> Notation: n(type),

where n is the index of the Gram block

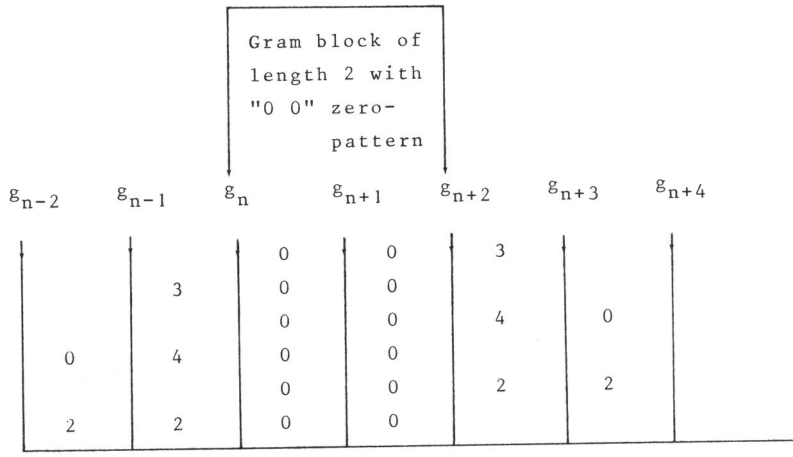
$B_n = [g_n, g_{n+3})$ with zero-pattern "0 1 0"

304790219(2)	361399662(1)	372541137(2)
316217470(1)	368714780(4)	382327446(1)

(For the definition of the types in case of length 3-exceptions, see Table 2.)

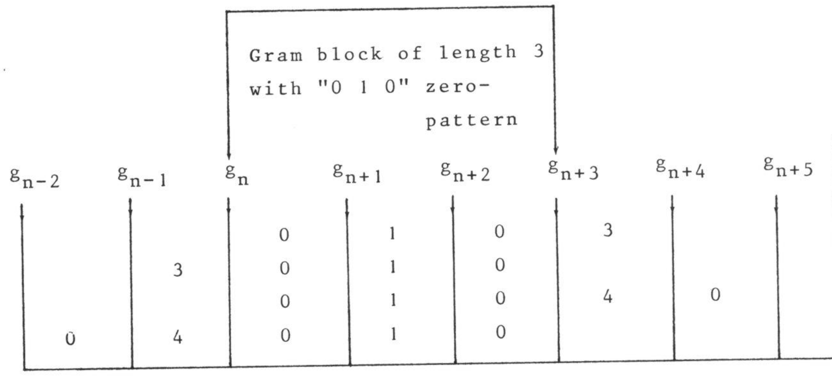
TABLE 2

Various types of exceptions to Rosser's rule and their frequencies in $[g_{300,000,000}, g_{400,000,000})$



LENGTH = 2

type	frequency
1	53
2	77
3	2
4	0
5	3
6	2



LENGTH = 3

type	frequency
1	3
2	2
3	0
4	1*)

*) B_n, for n = 368,714,780 (the first observed occurrence of an exception to Rosser's rule of length 3, type 4).

TABLE 3

Number of Gram blocks of given length (= k)

$J'(k,n) := J(k,n + 10^7) - J(k,n)$

n	J'(1,n)	J'(2,n)	J'(3,n)	J'(4,n)	J'(5,n)	J'(6,n)	J'(7,n)	J'(8,n)
300,000,000	6,945,581	1,058,070	239,335	47,699	5,451	354	13	1
310,000,000	6,943,007	1,058,373	239,824	47,727	5,511	362	20	0
320,000,000	6,940,839	1,058,446	239,719	48,273	5,520	367	30	1
330,000,000	6,940,868	1,057,898	240,014	48,174	5,673	343	25	0
340,000,000	6,936,658	1,058,965	240,413	48,317	5,674	393	23	2
350,000,000	6,937,443	1,057,946	240,752	48,345	5,718	382	21	0
360,000,000	6,935,064	1,058,830	240,475	48,582	5,800	375	39	0
370,000,000	6,933,157	1,058,460	240,990	48,851	5,825	369	30	0
380,000,000	6,931,022	1,058,000	241,422	49,267	5,832	379	30	0
390,000,000	6,930,808	1,058,343	241,524	49,143	5,771	384	29	0
Totals	69,374,447	10,583,331	2,404,468	484,378	56,775	3,708	260	4

TABLE 4

Number of Gram intervals $G_j = [g_j, g_{j+1})$ *in the range* $[g_{300,000,000}, g_{400,000,000})$ *containing exactly m zeros*

n	m = 0	m = 1	m = 2	m = 3	m = 4
300,000,000	1,368,713	7,280,364	1,333,133	17,790	0
310,000,000	1,369,780	7,278,404	1,333,853	17,962	1
320,000,000	1,370,347	7,277,297	1,334,365	17,991	0
330,000,000	1,369,919	7,277,954	1,334,335	17,792	0
340,000,000	1,371,716	7,274,497	1,335,858	17,929	0
350,000,000	1,370,897	7,275,940	1,335,430	17,732	1
360,000,000	1,372,063	7,273,837	1,336,138	17,961	1
370,000,000	1,372,392	7,273,083	1,336,658	17,867	0
380,000,000	1,372,772	7,272,298	1,337,088	17,842	0
390,000,000	1,372,979	7,271,827	1,337,409	17,785	0
Totals	13,711,578	72,755,501	13,354,267	178,651	3

TABLE 5

Number of Gram blocks of type (j,k), $1 \leq j \leq 8$, $1 \leq k \leq j$, *in the range* $[g_{300,000,000}, g_{400,000,000})$

↓j \ k→	1	2	3	4	5	6	7	8	Totals
1	69,374,447								69,601,860
2	5,292,964	5,290,225	(137 "0 0" - blocks)						10,569,849
	(50)	(50)	(5 "2 2" - blocks)						
3	1,139,688	124,597	1,140,177	(6 "0 1 0" - blocks)					2,404,468
	(47.4)	(5.2)	(47.4)						
4	222,268	20,137	20,025	221,948					484,378
	(45.9)	(4.2)	(4.1)	(45.8)					
5	22,743	4,424	2,126	4,535	22,947				56,775
	(40.1)	(7.8)	(3.7)	(8.0)	(40.4)				
6	650	916	311	298	885	648			3,708
	(17.5)	(24.7)	(8.4)	(8.0)	(23.9)	(17.5)			
7	1	79	53	15	36	75	1		260
8	0	0	2	0	0	2	0	0	4

4. SOME LOCAL GRAPHS OF Z(t)

In order to give the reader an impression of the erratic behaviour of Z(t), we give in Figures 1.1—1.8 some graphs of Z(t) in the neighbourhood of the first (observed) exceptions to Rosser's rule of lengths 2 and 3 and of various types. We have plotted $Z(g_x)$ with x as a continuous independent variable. The exceptional Gram block is marked by two arrows pointing *downwards*. The adjacent Gram block where the "missing two" zeros are situated is marked by two arrows pointing *upwards*. A magnification of the latter block is shown in an accompanying graph. Some "critical" values of Z(t) are explicitly mentioned. The most recently found length 3-exception of type 4 is given in the form of a table.

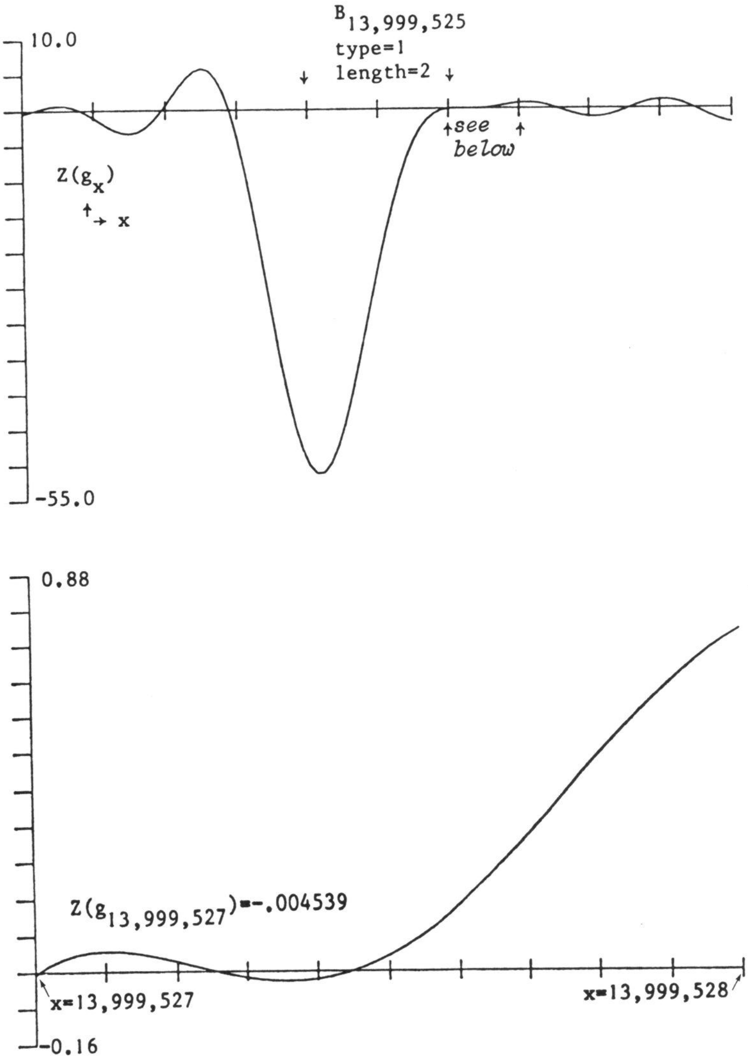


Fig. 1.1 $B_{13,999,525}$

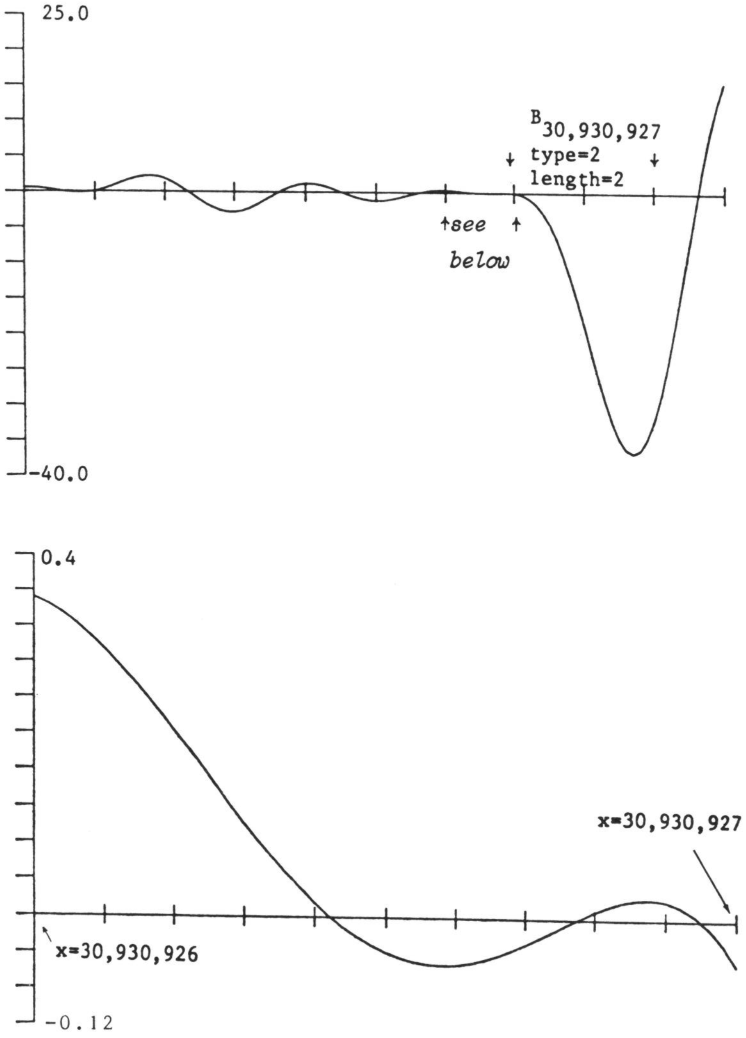


Fig. 1.2 $B_{30,930,927}$

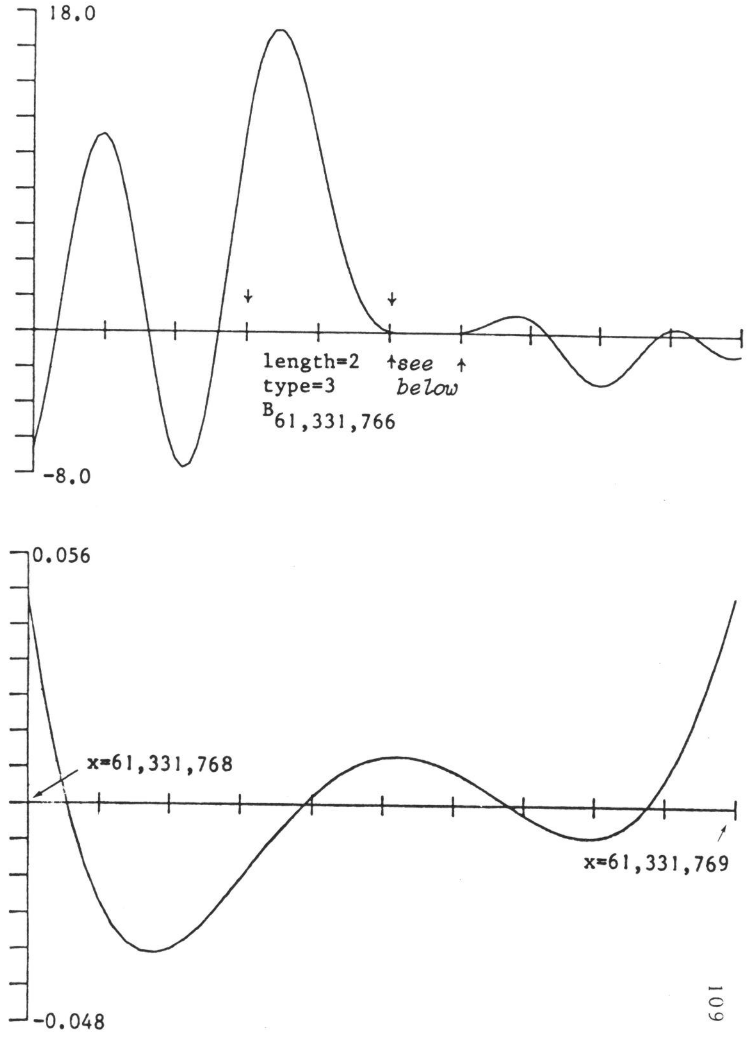


Fig. 1.3 $B_{61,331,766}$

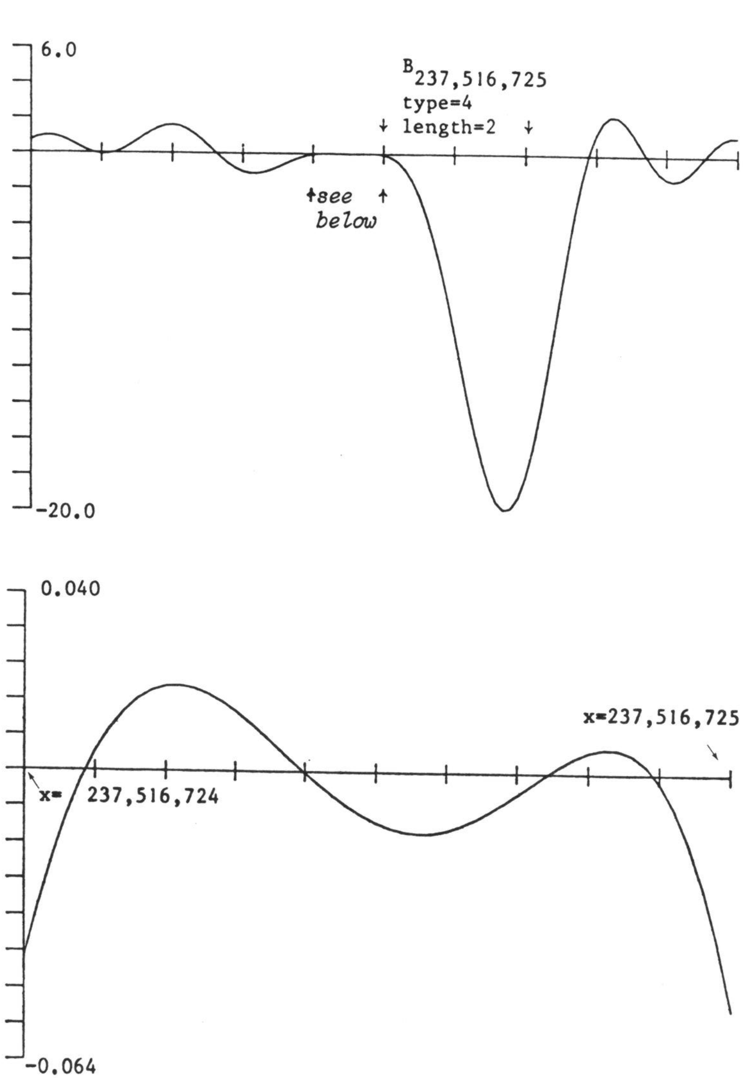


Fig. 1.4 $B_{237,516,725}$

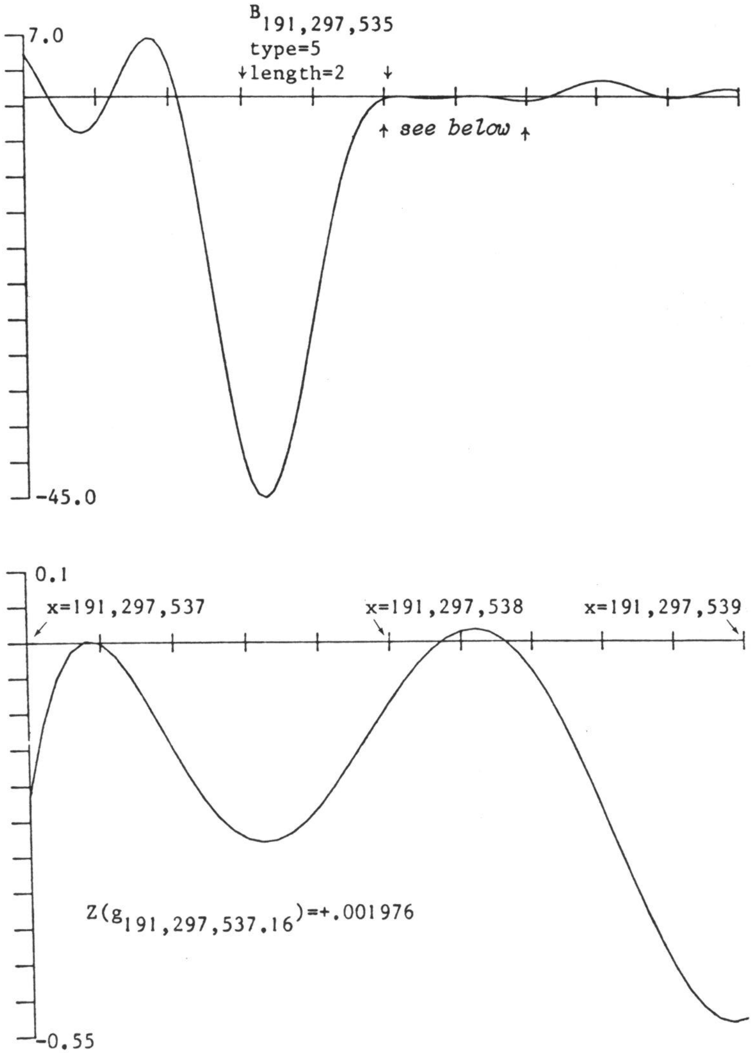


Fig. 1.5 $B_{191,297,535}$

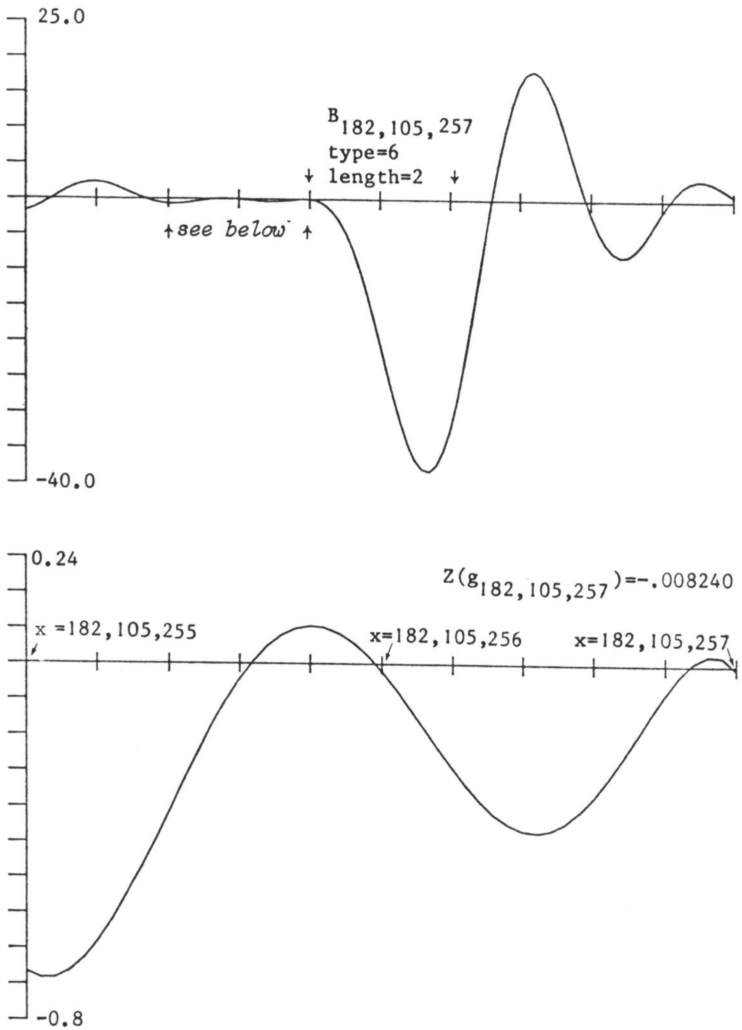


Fig. 1.6 $B_{182,105,257}$

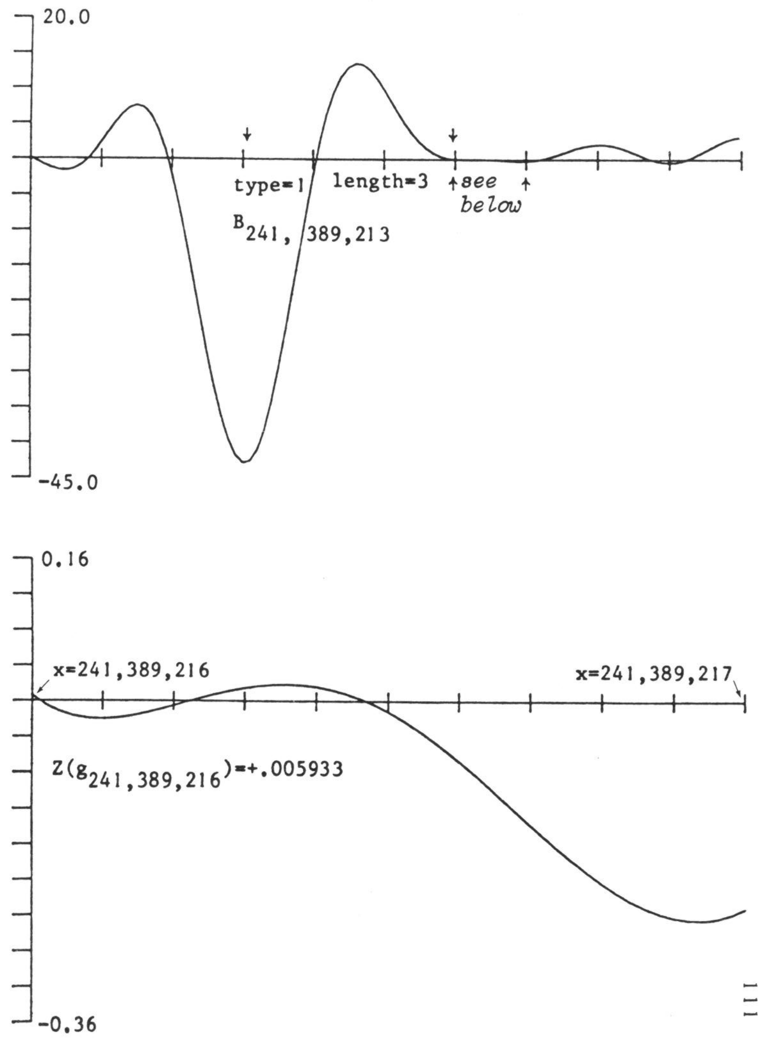


Fig. 1.7 $B_{241,389,213}$

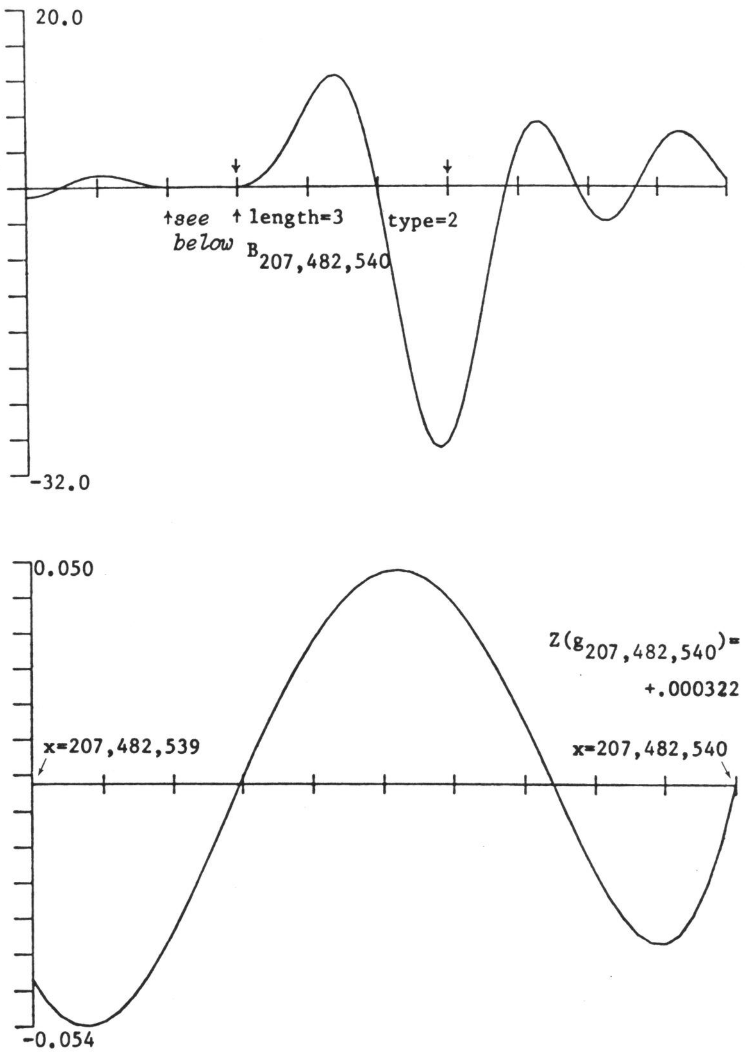


Fig. 1.8 $B_{207,482,540}$

Table of some values of Z(t) *in the neighbourhood of* $B_{368,714,780}$

x	g_x	$Z(g_x)$
368714778.000	145195813.58154	2.48878
.125	.62787	2.40938
.250	.67419	2.17013
.375	.72051	1.81343
.500	.76683	1.39438
.625	.81315	0.97104
.750	.85947	0.59466
.875	.90579	0.30160
368714779.000	.95211	0.10832
.125	.99843	0.01048
.250	145195814.04475	-0.01374
.375	.09107	0.00337
.500	.13739	0.02836
.625	.18371	0.03665
.750	.23003	0.02104
.875	.27635	-0.00297
368714780.000	.32267	0.00353
.125	.36899	0.09823
.250	.41531	0.34809
.375	.46164	0.81597
.500	.50796	1.54548
.625	.55428	2.54659
.750	.60060	3.78457
.875	.64692	5.17458
368714781.000	.69324	6.58377
.125	.73956	7.84123
.250	.78588	8.75552
.375	.83220	9.13782
.500	.87852	8.82797
.625	.92484	7.71992
.750	.97116	5.78274
.875	145195815.01748	3.07410
368714782.000	.06380	-0.25679
.125	.11012	-3.97768
.250	.15644	-7.79325
.375	.20276	-11.37474
.500	.24908	-14.39585
.625	.29541	-16.57093
.750	.34173	-17.69034
.875	.38805	-17.64856
368714783.000	.43437	-16.46104

Table of some values of Z(t) *in the Gram interval* $G_{368714779}$

x	g_x	$Z(g_x)$
368714779.000	145195813.952109	0.108319
.025	.961373	0.081597
.050	.970637	0.058621
.075	.979902	0.039234
.100	.989166	0.023254
.125	.998430	0.010479
.150	145195814.007694	0.000683
.175	.016958	-0.006375
.200	.026222	-0.010951
.225	.035486	-0.013313
.250	.044750	-0.013737
.275	.054014	-0.012504
.300	.063279	-0.009898
.325	.072543	-0.006201
.350	.081807	-0.001689
.375	.091071	0.003367
.400	.100335	0.008710
.425	.109599	0.014096
.450	.118863	0.019299
.475	.128127	0.024115
.500	.137391	0.028360
.525	.146656	0.031881
.550	.155920	0.034549
.575	.165184	0.036270
.600	.174448	0.036979
.625	.183712	0.036650
.650	.192976	0.035289
.675	.202240	0.032944
.700	.211504	0.029698
.725	.220768	0.025675
.750	.230033	0.021037
.775	.239297	0.015986
.800	.248561	0.010763
.825	.257825	0.005646
.850	.267089	0.000951
.875	.276353	-0.002970
.900	.285617	-0.005734
.925	.294881	-0.006924
.950	.304145	-0.006095
.975	.313410	-0.002777
368714780.000	145195814.322674	0.003525

5. A FEW WORDS ON THE FORTRAN/COMPASS PROGRAM

Our original program was written in FORTRAN 66 and carefully tested in all possible ways we could reasonably think of. After this rather simple minded setting the "heavy" arithmetical part of the program was converted into COMPASS (machine code for the CDC CYBER 170-750) by D.T. WINTER and it stands to reason that after intensive optimization of this COMPASS version the running time was reduced considerably. A complete listing of this program was given in: J. van de LUNE, H.J.J. te RIELE & D.T. WINTER, *Rigorous high speed separation of zeros of Riemann's zeta function*, Mathematical Centre, Amsterdam, Report NW 113 (1981). After this program had been run for quite a long time a few improvements of our strategy were built in with the effect that the average number of Z-evaluations needed to separate a zero from its predecessor was brought down from about 1.21 to about 1.19. This new version was published in: J. van de LUNE & H.J.J. te RIELE, *Rigorous high speed separation of zeros of Riemann's zeta function*, II, Mathematical Centre, Amsterdam, Report NN 26 (1982). It should be noted that there are a few harmless misprints/minor errors *in the comment lines* of REAL FUNCTION Z(T) in NW 113 and REAL FUNCTION Z(DT) in NN 26.
Finally we mention that our program has also been extensively tested on a CRAY 1 and two different versions of the CDC CYBER 205 and we intend to continue our computations in the near future.

Added in proof: By means of the CDC CYBER 205 installed at SARA (Stichting Academisch Rekencentrum Amsterdam) we have verified the Riemann hypothesis up to the 750,000,000-th non-trivial zero of $\zeta(s)$. Details will be published elsewhere.

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SUMMARY

In this dissertation we almost exclusively deal with *sums of equal powers of positive integers* of the form $1^s+2^s+3^s+\ldots+N^s+\ldots$.
In the first four chapters we truncate this sum after the term N^s whereas in the last chapter we consider the "full" sum and its analytic continuation. Further, in the first three chapters s is real whereas in the last two chapters s is complex.
In Chapter 1 we prove that sequences of canonical integral-approximations for some elementary functions are monotonic by deriving some new inequalities for the above truncated sums.
In Chapter 2 we treat some more difficult rather academic, though not less interesting, topics closely related to those dealt with in Chapter 1.
By means of some of the previously found inequalities and some additional devices we give in Chapter 3 a fairly complete solution of a hitherto unsolved problem proposed by P. Erdős in 1949.
In Chapter 4 we discuss aspects of zeros of the above sum considered as an entire function of the complex variable s. In particular we present a procedure for systematically finding special zeros (with $Re(s) < -1$), the non-existence of which would have implied the truth of the Riemann Hypothesis.
In Chapter 5 we give a description of an improved method for separating the non-trivial zeros of Riemann's zeta function (the analytic continuation of the "full" sum with s replaced by -s).
Applying this method we were able to show, by extensive numerical computations, that the first 400 000 000 non-trivial zeros of Riemann's zeta function are all simple and lie on the so-called critical line.

SAMENVATTING

In dit academisch proefschrift houden we ons vrijwel uitsluitend bezig met *sommen van gelijke machten van positieve gehele getallen* van de vorm $1^s + 2^s + 3^s + \ldots + N^s + \ldots$.
In de eerste vier hoofdstukken breken we deze som af na de term N^s terwijl we in het laatste hoofdstuk de "volledige" som en haar analytische voortzetting beschouwen. Verder is s in de eerste drie hoofdstukken reëel en complex in de laatste twee.
In Hoodstuk 1 bewijzen we dat sommige rijen van canonieke integraal-approximaties voor bepaalde elementaire funkties monotoon zijn door een aantal nieuwe ongelijkheden af te leiden voor de bovenstaande afgeknotte sommen.
In Hoofdstuk 2 behandelen we een aantal wat moeilijker tamelijk academische, maar daarom niet minder interessante, onderwerpen die nauw verwant zijn met de in Hoofdstuk 1 behandelde.
Met behulp van enkele eerder gevonden ongelijkheden en wat andere hulpmiddelen geven we in Hoofdstuk 3 een tamelijk volledige oplossing van een tot nu toe onopgelost probleem dat in 1949 door P. Erdös gesteld werd.
In Hoofdstuk 4 bespreken we een aantal verschillende aspecten van de nulpunten van bovenstaande som, maar dan beschouwd als gehele funktie van de complexe variabele s. In het bijzonder beschrijven we een procedure voor het systematisch bepalen van speciale nulpunten (met $Re(s) < -1$), het niet bestaan waarvan de juistheid van de Riemann Hypothese tot gevolg gehad zou hebben.
In Hoofdstuk 5 geven we een beschrijving van een verbeterde methode om de niet-triviale nulpunten van de zeta funktie van Riemann (de analytische voortzetting van de "volledige" som met s vervangen door -s) te scheiden. Door deze methode toe te passen konden we, na zeer uitvoerige numerieke berekeningen, aantonen dat de eerste 400 000 000 niet-triviale nulpunten van de zeta funktie van Riemann allemaal enkelvoudig zijn en op de zogenaamde critieke lijn liggen.

CURRICULUM VITAE

De schrijver van dit proefschrift werd geboren op 13 januari 1937 te Hallum (FRL) als zoon van Tjerk van de Lune en Trijntje van de Lune-Kingma.
In 1954 behaalde hij het HBS-B diploma aan de Christelijke Hogere Burgerschool te Leeuwarden, waarna hij wiskunde ging studeren aan de Vrije Universiteit te Amsterdam.
In 1960 trad hij in dienst van het Mathematisch Centrum te Amsterdam op de afdeling Mathematische Statistiek.
In 1963 behaalde hij de Akte MO-A (Wiskunde, Staatsexamen, Den Haag) en in 1968 de Akte MO-B (Wiskunde, Vrije Universiteit, Amsterdam).
Van 1971 tot 1973 studeerde hij in de Verenigde Staten van Amerika aan de Kansas State University te Manhattan, Kansas. In 1973 behaalde hij aldaar, aan de faculteit der wiskunde, onder supervisie van Prof. dr. Robert E. Dressler, de graad van Doctor of Philosophy na, onder meer, het schrijven van een dissertatie getiteld: Some observations in number theory and analysis.
Daarna keerde hij terug naar het Mathematisch Centrum (het huidige Centrum voor Wiskunde en Informatica) om daar als wetenschappelijk medewerker werkzaam te zijn op de afdeling Zuivere Wiskunde.

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ALONE

From childhood's hour I have not been
As others were - I have not seen
As others saw - I could not bring
My passions from a common spring.
From the same source I have not taken
My sorrow; I could not awaken
My heart to joy at the same tone;
And all I lov'd, *I* lov'd alone.
Then - in my childhood - in the dawn
Of a most stormy life - was drawn
From ev'ry depth of good and ill
The mystery which binds me still:
From the torrent, or the fountain,
From the red cliff of the mountain,
From the sun that round me roll'd,
In it's autumn tint of gold -
From the lightning in the sky
As it pass'd me flying by -
From the thunder and the storm,
And the cloud that took the form
(When the rest of Heaven was blue)
Of a demon in my view.

(EDGAR ALLAN POE)

... Dat was het wat mij ontbrak, de ervaring een *volkomen menselijk leven* te leiden en niet alleen maar een leven van kennis zodat ik daarmee de ontwikkeling van mijn gedachten niet zou baseren op - nu ja, op iets dat men objectief noemt, iets dat in ieder geval niet van mijzelf is, maar op iets dat samenhangt met de diepste wortels van mijn ziel waarmee ik om zo te zeggen vastgegroeid ben in de goddelijke natuur en waaraan ik vastgekleefd ben, ook al zou de hele wereld in elkaar storten. Dit is het, dat mij *ontbreekt* en *daarnaar streef ik...* Waar het op aankomt, is dit handelen van binnenuit, dit goddelijke in de mens, niet op een massa erkende feiten, want die zullen wel uit zichzelf komen en dan niet meer blijken te zijn dan een aantal dingen die toevallig bij elkaar gekomen zijn, of een rij bijzonderheden, de een naast de ander, zonder systeem of brandpunt waarin alle stralen samenkomen. Zo'n brandpunt heb ik ook inderdaad gezocht. Zowel op de bodemloze zee van het genot als in de diepten der kennis heb ik tevergeefs een ankerplaats gezocht; ik heb de bijna onweerstaanbare macht gevoeld waarmee het ene genot het andere de hand reikt; ik heb het onechte enthousiasme gevoeld, dat het te voorschijn kan roepen; ik heb ook de verveling gevoeld en de verscheurdheid welke er op volgt. Ik heb de vruchten van de boom der kennis geproefd en mij dikwijls over hun smaak verheugd. Maar deze vreugde was er slechts op het moment van de kennis en liet geen dieper spoor in mij achter. Het komt mij voor dat ik niet van de beker der wijsheid gedronken heb, maar dat ik eerder erin ben gevallen.

(SØREN AABYE KIERKEGAARD)

Het is met opzet dat ik mij herhaal;
Ik heb dit vroeger al gezegd
Maar nooit zo op den man af en zo kaal,
Zo vrij van het bedrog der mooie taal.
Wie dient, zij het de schoonheid, is een knecht.
Ik heb geen opdrachtgevers en ik dwaal
Voor eigen rekening; wie noemt dat slecht?
Men zal mij niet voor Ikaros zien spelen
Om door de schijnmanoeuvres te verhelen
Hoezeer ik ben, o Aarde, aan u gehecht.

Veel liever dan met leugens om te gaan
Zou 'k zwijgen tot mijn laatsten nood.
Ik spreek niet van het mannetje in de maan
Of van een blanken ridder met een zwaan;
Ik ken alléén mezelf, gekleed of bloot,
Verheugd of droevig, hoopvol of ontdaan,
Nu eens onmachtig, dan weer goed op stoot.
Er is geen kleine achterdeur meer open
Ik kan den heer J. .... niet ontlopen,
Ik ben aan hem gekluisterd tot mijn dood.

Maar ergens ver, in Sappemeer of Weert,
Bestaat misschien een jonge man
Die eenzaam uit mijn rijmen heeft geleerd
Hoe waardeloos de waarde is die men eert;
Een opgewonden knaap die walgend van
Het huis waar hij tot knecht werd gedresseerd
Door mij voor goed bevrijd is uit zijn clan
En die nu met zijn hart en oogen open
Door de natuur in vollen bloei durft lopen
Niet bang voor wat hem overkomen kan.

Hier sta ik mager in mijn hemd ten toon
En geef me zonder schaamte prijs,
Vrijwillig offer, aan den zuren hoon
Der zedemeesters van 't verdiende loon.
Geneer u niet, heer censor, gil en krijs,
Ik ben uw kouwe drukte allang gewoon,
Mijn hart is taai, al zijn mijn haren grijs.
Er zuchten nog te veel gedweeë stakkers
Onder den druk der schouten met hun rakkers:
Hen maakt misschien mijn schade en schande wijs!

En de olifant komt met zijn langen snuit
Hij blaast naar overouden trant
't Vertelsel en het kleine leven uit.
Dag lieve zon, ik ga in de kajuit.
Bespaar mij snotterstukjes in de krant,
Den spreker aan de groeve, hol en luid;
Stop me maar stiekum ergens onder 't zand:
Geen kransen en geen uitgehouwen zerken,
Geen commentaren, geen complete werken,
Alléén een vrouw die snikt achter haar hand.

(naar JAN GRESHOFF)

DE WUIVENDE

Mijn vrouw is de wuivende, die met haar zakdoek
in 't licht langs het korenveld gaat.
Zij zendt mij een uiterste teken van liefde
nu zij mij, gedwongen, verlaat.

Wie weet voor hoelang zij vertrekt? Ik blijf eenzaam
doch jubel slaat op in mijn bloed.
Ik voel mij niet langer gevangen; rondom mij
is alom haar wuivende groet.

Mijn God in den hemel, die 't ziet, en die weet
hoe ik nooit voor mijzelven iets vroeg
- Al wat Gij mij gaaft heb ik dankbaar aanvaard
En Gij gaaft mij geluk genoeg! -

Verhoor voor vandaag en de rest van mijn leven
één enkele bede van mij:
Dat altijd mijn vrouw als Uw teken van liefde
voor mij deze wuivende zij.

Haar simpel bewegen der hand bij haar afscheid
zond mij het geheim tegemoet,
Waarom Gij Uw engel zijn boodschap liet zeggen
beginnende met "Wees Gegroet!"

Want al wat beweegt, hier op aarde, in de zee
langs Uw heemlen vol heerlijkheid
Is niets dan een wuivende groet aan de ziel
om te zeggen, hoe goed Gij zijt.

Wie God wil begrijpen die heeft niet genoeg
aan ons vorsende mensenverstand.
Hij zie naar het dansen van sterren en golven
en 't wuiven der dierbaarste hand.

Al wat ik geloof en belijd vat ik samen
In deze, mijn opperste wet:
Mijn ziel zij een wuivende groet aan mijn God
want ik heb geen volmaakter gebed.

Mijn ziel zij een riet aan de stroom der genade
en een wuivende golfslag, die spoelt
Langs de zoelheid der kust, en een graanveld in zon,
dat den tocht van den zomerwind voelt.

Mijn ziel zij gelijk aan de ziel van de vrouw
die mij toezond Uw Godlijken groet.
Want zij is de wuivende, die Gij mij gaaft
en ik dank U, het leven is goed.

(ANTON VAN DUINKERKEN)

Al spriek ik mei de tongen fan ingels en minsken -
as ik de leafde net hie, wie ik in bounzjend
bekken of in galmjende gong.
Al wie it my jown om to profetearjen en koe ik
alle geheimen en alle wittenskip en al hie ik
alle leauwen om bergen to forsetten -
as ik de leafde net hie, wie ik neat.
Al joech ik al myn goed wei foar de earmen,
ja, al joech ik myn eigen lichem
om forbarnd to wirden; as ik de leafde net hie,
joech it my allegearre neat.
De leafde is langmoedich, freonlik is de leafde,
hja is net oergunstich.
De leafde pronket net, hja is net greatsk.
Hja is net ûnfoech, hja siket harsels net,
hja is net gau rekke en bringt it kwea net yn rekken.
Hja hat gjin nocht oan ûnrjocht,
mar hja is bliid mei de wierheit.
Alles kin hja forneare, alles leauwt hja,
alles hopet hja, alles kin hja drage.
De leafde giet nea forlern.
Profetearjen, it hâldt in kear op,
sprekken yn frjemde talen, it forstommet in kear;
kennisse, it hâldt in kear op.
Want ûnfolslein is ús kennen en ûnfolslein
is ús profetearjen.
As nou it folsleine komt, is it dien
mei it ûnfolsleine.
Doe't ik in lyts bern wie, prate ik as in bern,
tocht ik as in bern, redenearre ik as in bern.
Nou't ik in man wirden bin,
haw ik it bernlike oan 'e kant dien.
Nou sjogge wy noch ûndúdlik yn in spegel,
mar dan each yn each.
Nou ken ik ûnfolslein, mar dan sil ik folslein kenne,
lykas God my ken.
Yn ien wird, trije dingen bliuwe foar altiten:
leauwen, hope en leafde;
mar de greatste dêrfan is de leafde.

(I KORINTIËRS, XIII)

*

STELLINGEN

1.1. Als, in overeenstemming met de notatie in Paragraaf 3 van Hoofdstuk 2,

$$g_n(k) := \frac{T_n(k) - \frac{1}{k+1}}{k!}, \qquad (k,n \in \mathbb{N})$$

dan voldoet voor elk positief geheel getal n de *gehele* funktie

$$G_n(z) := \sum_{k=2}^{\infty} g_n(k)\, z^k, \qquad (z \in \mathbb{C})$$

aan de funktionaal-vergelijking

$$G_n(z) = e^z\, G_n(-z).$$

Verder heeft $G_n(z)$ oneindig veel nulpunten, *alle* gelegen op de imaginaire as.

1.2. Voor $x > 1$ geldt

$$\sum_{k=2}^{\infty} \frac{1}{k} \left(T_n(k) - \frac{1}{k+1}\right) x^{-k} = \frac{\mu(nx-n) - \mu(nx)}{n}$$

waarbij $\mu(s)$ de (uit de theorie van de Gammafunktie bekende) funktie van Binet is.

Zie: J. van de LUNE & M. VOORHOEVE, *Some problems on log-convex approximation of certain integrals*, Mathematisch Centrum, Amsterdam, Rapport ZN 85 (1978).

2. De differentie-differentiaal-vergelijking $f'(x) = f(x) - f(x-1)$ laat oplossingen toe die *niet constant* zijn, zelfs gehele analytische. Bovendien geldt voor een oplossing onder zeer algemene voorwaarden

$$\lim_{x\to\infty} f(x) = 2 \int_0^1 t.f(t-1)\, dt.$$

Zie: J. van de LUNE, *Cursus Computerwiskunde*, *Analyse*, Mathematisch Centrum, Amsterdam, Rapport ZC 82 (1971) pp. 84-93.

3. Onder het voorbehoud van de betrouwbaarheid van de computer is het mogelijk vast te stellen dat het 10^{12}-de niet-triviale nulpunt van $\zeta(s)$ enkelvoudig is en op de kritieke lijn $\sigma = \frac{1}{2}$ ligt op een hoogte van ongeveer $t = 267\ 653\ 395\ 648{,}625\ 948$.

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4.1. Voor de rij $\{H_n\}_{n=1}^{\infty}$ van door Ramanujan ingevoerde *Superior Highly Composite Numbers* is het nog geen uitgemaakte zaak dat voor elke $n \in \mathbb{N}$ het quotient H_{n+1}/H_n een priemgetal is.

4.2. Er is reden te vermoeden dat, als we in de standaard priemontbinding van elk positief geheel getal $n \geq 2$ elk priemgetal p (in de "basis") vervangen door log p, we een over $\mathbb{Q}$ lineair onafhankelijke verzameling verkrijgen. Uitgaande van de juistheid van dit vermoeden laat het zich aanzien dat voor elke $n \in \mathbb{N}$ het quotient H_{n+1}/H_n inderdaad een priemgetal is.

4.3. De begrippen "Highly Composite Number" en "Superior Highly Composite Number" hebben analogieën in de theorie van de gehele getallen in $\mathbb{Q}(i)$, de zogenaamde roosterpunten van Gauss.

Zie: S. RAMANUJAN, *Highly composite numbers*, Proc. London Math. Soc., 2, 14 (1915) pp. 347-409.

Collected papers of SRINIVASA RAMANUJAN, Cambridge University Press (1927) pp. 78-128 (in het bijzonder p. 114).

5.1. Zij voor $\alpha \in \mathbb{R}$ en $N \in \mathbb{N}$

$$S_N(\alpha) := \sum_{n=1}^{N} (-1)^{[n\alpha]}.$$

Dan is voor elke $N \in \mathbb{N}$

$$S_N(\alpha) \geq 0$$

als en slechts als de regelmatige kettingbreukontwikkeling $(a_0; a_1, a_2, \ldots)$ van α de eigenschap $a_{2n} \equiv 0 \pmod 2$ heeft voor alle $n \in \mathbb{N} \cup \{0\}$.

5.2. Computerberekeningen doen vermoeden dat voor $\alpha = \sqrt{2}$ geldt

$$\limsup_{N \to \infty} \frac{S_N(\alpha)}{\log N} = \frac{1}{2\log(1+\sqrt{2})} \cong 0{,}567\ 296$$

en het ziet er naar uit dat "soortgelijke" uitspraken waar zijn voor alle kwadratisch irrationale α's.

Zie: A.E. BROUWER & J. van de LUNE, *A note on certain oscillating sums*, Mathematisch Centrum, Amsterdam, Rapport ZW 90 (1976).

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6. Voor $n \in \mathbb{N}$, $n \geq 2$, definiëren we $g(n)$ (respectievelijk $s(n)$) als de grootste (respectievelijk kleinste) priemdeler van n en $g(1) := s(1) := 1$. Als p en q priemgetallen voorstellen dan gelden de volgende uitspraken:

a) $$\sum_{n \leq x} g(n)/s(n) \sim c_1 \cdot \frac{x^2}{\log x}, \quad (x \to \infty)$$

waarbij
$$c_1 := \frac{\pi^2}{12} \sum_p \left(p^{-3} \prod_{q<p} (1-q^{-2})\right).$$

b) $$\sum_{n \leq x} 1/s(n) \sim c_2 \cdot x, \quad (x \to \infty)$$

waarbij
$$c_2 := \sum_p \left(p^{-2} \prod_{q<p} (1-q^{-1})\right).$$

c) Voor elke $a \in \mathbb{R}$ geldt
$$\sum_{n \leq x} 1/g(n) = O(x(\log x)^{-a}), \quad (x \to \infty).$$

d) Er bestaat *geen* $a < 1$ met de eigenschap
$$\sum_{n \leq x} 1/g(n) = O(x^a), \quad (x \to \infty).$$

e) $$\sum_{n \leq x} s(n)/g(n) = o(x), \quad (x \to \infty).$$

(In 1982 werd het laatste resultaat verscherpt door ERDÖS & van LINT.)

Zie: J. van de LUNE, *Some sums involving the largest and smallest prime divisor of a natural number*, Mathematisch Centrum, Amsterdam, Rapport ZW 25 (1974).

P. ERDÖS & J.H. van LINT, *On the average ratio of the smallest and largest prime divisor of n*, Proc. KNAW, A 85 (2) (1982) pp. 127-132.

7. Als σ_0 het supremum van de verzameling
$$\{\sigma \in \mathbb{R} \mid \operatorname{Re} \zeta(\sigma+it) < 0 \text{ voor zekere } t \in \mathbb{R}\}$$
voorstelt dan is σ_0 de (enige) oplossing van de vergelijking
$$\sum_p \arcsin(p^{-x}) = \frac{\pi}{2}, \quad (x > 1)$$

****

waarbij, in de sommatie $\sum_p$, p de priemgetallen doorloopt.
Numeriek kan worden aangetoond dat $\sigma_0 > 1{,}192$.
Voor $\sigma = \sigma_0$ geldt dat $\mathrm{Re}\ \zeta(\sigma+it) > 0$ voor alle $t \in \mathbb{R}$.

Zie: J. van de LUNE, *Some observations concerning the zero-curves of the real and imaginary parts of Riemann's zeta function*, Mathematisch Centrum, Amsterdam, Rapport ZW 201 (1983).

8.1. Veronderstel omtrent de continue funktie $p : \mathbb{R}^+ \to \mathbb{R}^+$ dat $P(s) := \int_0^\infty u^s\, p(u)\, du$ eindig is voor alle $s \in \mathbb{R}^+$. Fixeer een $\alpha \in (0,1)$ en definieer de funktie $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ impliciet door

$$\int_0^{\theta(s)} u^s\, p(u)\, du = \alpha P(s), \qquad (s \in \mathbb{R}^+).$$

Dan is θ *monotoon stijgend* op $\mathbb{R}^+$.

8.2. Computerberekeningen doen vermoeden dat

$$\frac{1}{n(n+1)^4} \sum_{i1=0}^{n} \sum_{i2=0}^{n} \sum_{j1=0}^{n} \sum_{j2=0}^{n} \{(i1-j1)^2 + (i2-j2)^2\}^{\frac{1}{2}}, \qquad (n \in \mathbb{N})$$

voor toenemende n *convex daalt*.

Het ziet er naar uit dat voor elke Euclidische ruimte van willekeurig hoge eindige dimensie een analoge uitspraak gedaan kan worden.

8.3. Zij α een positieve constante. Als $s_0 := 1$ en

$$s_n := \frac{s_0}{n^\alpha} + \frac{s_1}{(n-1)^\alpha} + \ldots + \frac{s_{n-2}}{2^\alpha} + s_{n-1}, \qquad (n \in \mathbb{N})$$

dan is de rij $\{s_n\}_{n=0}^\infty$ *logarithmisch convex*.

Zie: *Problem Section*, Nieuw Archief voor Wiskunde, Vierde Serie, Deel 2, No. 1 (maart 1984).

J. van de LUNE, *Average distances in/on certain n-dimensional bodies*, Mathematisch Centrum, Amsterdam, Rapport ZN 84 (1978).

J. van de LUNE, *On some sequences defined by recurrence relations of increasing length*, Mathematisch Centrum, Amsterdam, Rapport ZW 174 (1982).

9. Kies bij elk priemgetal p een reëel getal τ_p. Laat $n = \prod_p p^{e_p(n)}$ de canonieke priemfactorisatie voorstellen van $n \in \mathbb{N}$ en definieer het "natuurlijke polynoom" $n(\tau)$ door

$$n(\tau) := \prod_p (p+\tau_p)^{e_p(n)}$$

waarbij $\tau := (\tau_2, \tau_3, \tau_5, \ldots)$.

*****

Als, bijvoorbeeld, de getallen τ_p zodanig gekozen worden dat

$$|\tau_p| \leq K.p^{\alpha} \text{ voor alle voldoend grote } p$$

en

$$p + \tau_p > 1 \text{ voor alle } p$$

waarbij α en K constanten zijn die voldoen aan $0 \leq \alpha < 1$ en $K > 0$, dan is de funktie

$$\zeta_\tau(s) := \sum_{n=1}^{\infty} (n(\tau))^{-s}, \quad (s := \sigma + it \in \mathbb{C},\ \sigma > 1)$$

te beschouwen als een generalisatie van de zeta-funktie van Riemann.
$\zeta_\tau(s)$ heeft een analytische voortzetting tot op het halfvlak $\sigma > \alpha$ met slechts een enkelvoudige pool in $s = 1$.
Verder heeft $\zeta_\tau(s)$ op $\sigma > \alpha$ precies *hetzelfde nulpunten-patroon* (qua ligging en multipliciteit) als $\zeta(s)$.
Nadere beschouwing van $\zeta_\tau(s)$ in de omgeving van $s = 1$ leidt tot vermoedens zoals (hier stelt $\mu(n)$ de funktie van Möbius voor)

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n(u)} \overset{?!}{=\!=} 0, \quad (\tau_p = u > -1 \text{ voor alle } p)$$

en

$$\sum_{n=1}^{\infty} \left(\frac{\mu(n)\log n}{n} \sum_{p/n} \frac{1}{p}\right) \overset{?!}{=\!=} \sum_{p} \frac{1}{(p-1)p}.$$

De klassieke funkties $L(s,\chi)$ van Dirichlet laten zich eveneens generaliseren zoals hierboven voor $\zeta(s)$ expliciet is aangegeven en ook hun nulpunten-patroon is invariant onder deze generalisatie.

10. Als a en b willekeurige reële constanten zijn dan hebben de vergelijkingen

$$e^z = \frac{a+z}{a-z} \quad \text{en} \quad e^z = \frac{a+z}{a-z} \cdot \frac{b+z}{b-z}, \quad (z \in \mathbb{C})$$

al hun oplossingen op het assenstelsel van het complexe vlak.
Voor $n \in \mathbb{N}$, $n \geq 3$, heeft de vergelijking

$$e^z = \left(\frac{n+z}{n-z}\right)^n, \quad (z \in \mathbb{C})$$

in het *open* eerste kwadrant van $\mathbb{C}$

$$\left[\frac{\frac{3}{2}\pi + 1}{2\pi} n\right] - \left[\frac{n}{2}\right] \quad (> 0)$$

oplossingen. Hierbij geeft $[x]$ het grootste gehele getal $\leq x$ aan.
In tegenstelling tot wat men op het eerste gezicht wellicht zou kunnen denken neemt dit aantal oplossingen *niet monotoon* toe als n stijgt.

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11. Toen P. Erdös 64 jaar oud werd moet hij naar verluidt opgemerkt hebben dat 2^6 (= 64) "net zo" begint als de exponent (6, in dit geval). Verdere voorbeelden, naast 2^{10} = 1024, waren toen niet bekend. Inmiddels hebben computerberekeningen uitgewezen dat de exponenten 1542, 77075, 113939 en 1122772 dezelfde eigenschap hebben (als 6 en 10).

    Zie: J. van de LUNE, *A note on a problem of Erdös*, Mathematisch Centrum, Amsterdam, Rapport ZN 87 (1978).

12. In de boekhandel is geen Fries etymologisch woordenboek te koop.

13.1. In computer-terminal-ruimten zouden "tekenen aan de wand" moeten worden aangebracht om bij "downs" de terminal-gebruikers mee te delen hoe lang de storing naar schatting zal duren.

13.2. Tenslotte, het is gewenst er rekening mee te houden dat de zogenaamde idle time van computers vaak nuttig kan worden gebruikt.

o-o

GEARFETTING

Yn dit akademyske proefskrift hâlde wy ûs omtrint inkeld en allinne dwaende mei *sommen fan gelyke machten fan hiele getallen* fan'e foarm $1^s + 2^s + 3^s + \dots + N^s + \dots$.

Yn'e earste fjouwer haedstikken brekke wy dizze som ôf nei de term N^s wylst wy yn it lêste haedstik de som yn syn hiele hear en fear, mei syn analytyske foartsetting biskouwe. Dêr komt noch by dat yn'e earste trije haedstikken s reëel is en yn'e lêste twa kompleks.

Yn Haedstik 1 liede wy foar dizze som in stikmennich nije ûngelykheden ôf troch to bispeuren dat guon rijen fan kanonyke integrael-approksimaesjes monotoan binne foar guon elementaire funksjes.

Haedstik 2 is in rillik dreech akademysk stik, mar dêrom net minder nijsgjirrich, dat wol hwat to meitsjen hat mei de ynhâld fan Haedstik 1.

Mei in pear earder foune ûngelykheden en noch hwat oar genifel jouwe wy yn Haedstik 3 in rillik folsleine oplossing fan in yn 1949 troch P. Erdös bitocht fraechstik hwer't ont nou ta noch gjin oplossing foar foun wie.

Yn Haedstik 4 bisprekke wy in stikmennich forskaete aspekten fan'e nulpunten fan boppesteande som, mar dan biskouwd as in hiele funksje fan'e komplekse fariabele s. Binammen biskriuwe wy in methoade foar it systematysk finen fan spesiale nulpunten (mei $\mathrm{Re}(s) < -1$). Hwannear't dizze nulpunten der net west hienen hie dat meibrocht dat de Hypothese fan Riemann mei wissichheit wier west hie.

Yn Haedstik 5 jouwe wy in biskriuwing fan in nije en bettere methoade om de net-triviale nulpunten fan'e zeta funksje fan Riemann (de analytyske foartsetting fan de som sûnder ein mei s forfongen troch $-s$) fan inoar to skieden. Troch dizze methoade ta to passen koenen wy, nei tige tiidforslinende en gâns wiidweidige birekkeningen, ûtmeitsje dat de earste 400 000 000 net-triviale nulpunten fan dizze funksje fan Riemann net allinnich inkelfâldich binne mar ek allegearre op'e saneamde krityke lijn lizze. Meikoarten sille wy witte hoe't it der foarstiet mei de earste 1 000 000 000 net-triviale nulpunten.

ERRATA in

SUMS OF EQUAL POWERS OF POSITIVE INTEGERS

by J. van de LUNE

page	line	instead of	read
33	3↑	2–	–
35	7↓	((2.	(2.
47	14↑	1.	1.
49	10↓	$[\lambda(n)]$	$[\lambda(n)]+1$
67	12↓	hypothesis	Hypothesis
96	16↓	175	170
100	12↓	$(2\pi)^{-1}$	$\widetilde{(2\pi)}^{-1}$
100	14↓	mod 2π	mod $\widetilde{2\pi}$
100	14↓	$(2\pi)^{-1}$	$\widetilde{(2\pi)}^{-1}$
104	4↑	g_{200},	$[g_{200}$,
107	13↑	69,601,860	69,374,447
107	12↑	10,569,849	10,583,331
115	2↑	hypothesis	Hypothesis