Nonparametric indices of dependence between components for inhomogeneous multivariate random measures and marked sets

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Abstract

We propose new summary statistics to quantify the association between the components in coverage-reweighted moment stationary multivariate random sets and measures. They are defined in terms of the coverage-reweighted cumulant densities and extend classic functional statistics for stationary random closed sets. We study the relations between these statistics and evaluate them explicitly for a range of models. Unbiased estimators are given for all statistics and applied to simulated examples and to tropical rain forest data.

KEYWORDS

bivariate random measure, coverage-reweighted moment stationarity, cross hitting functional, empty space function, J-function, K-function, moment measure, reduced cross correlation measure, spherical contact distribution

1 | INTRODUCTION

Popular statistics for investigating the dependencies between different types of points in a multivariate point process include cross versions of the *K*-function (Ripley, 1988), the nearest-neighbour distance distribution (Diggle, 2014), or the *J*-function (van Lieshout & Baddeley, 1996). Although originally proposed under the assumption that the underlying point process distribution is invariant under translations, in the recent years, all

[†]In memory of J. Oosterhoff.

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statistics mentioned have been adapted to an inhomogeneous context. More specifically, for univariate point processes, Baddeley, Møller, and Waagepetersen (2000) proposed an inhomogeneous extension of the *K*-function, whereas van Lieshout (2011) did so for the nearest-neighbour distance distribution and the *J*-function. An inhomogeneous cross *K*-function was proposed by Møller and Waagepetersen (2004); cross nearest-neighbour distance distributions and *J*-functions were introduced by van Lieshout (2011) and studied further by Cronie and van Lieshout (2016). The *K*- and *J*-functions were extended to space–time point processes by, respectively, Gabriel and Diggle (2009) and van Lieshout (2011), Cronie and van Lieshout (2015).

Although point processes can be seen as the special class of random measures that take integer values, functional summary statistics for random measures in general do not seem to be well studied. An exception is the pioneering paper by Stoyan and Ohser (1982) in which, under the assumption of stationarity, two types of characteristics were proposed for describing the correlations between the components of bivariate random closed sets in terms of their coverage measures. The first one is based on the second-order moment measure (Daley & Vere-Jones, 2008) of the coverage measure (Molchanov, 2017), and the second one on the capacity functional (Matheron, 1975). The authors did not pursue any relations between their statistics. Our goal in this paper is, in the context of bivariate random measures, to define generalisations of the statistics of and Stoyan and Ohser (1982) that allow for inhomogeneity and to investigate the relations between them.

The paper is organised as follows. In Section 2, we review the theory of multivariate random measures. We recall the definition of the Laplace functional and Palm distribution and discuss the moment problem. We then present the notion of coverage-reweighted moment stationarity. In Section 3, we introduce new inhomogeneous counterparts to Stoyan and Ohser's reduced cross correlation measure. In the univariate case, the latter coincides with that proposed by Gallego, Ibáñez, and Simó (2016) for germ–grain models. We go on to propose a cross *J*-function and relate it to the cross hitting intensity (Stoyan & Ohser, 1982) and empty space function (Matheron, 1975) defined for stationary random closed sets and to the classic cross *J*-function for point processes defined in terms of their product densities. Next, we give explicit expressions for our functional statistics for a range of bivariate models: compound random measures including linked and balanced models, the coverage measure associated to random closed sets such as germ–grain models, and random field models with particular attention to log-Gaussian and thinning random fields. Then, in Section 5, we turn to estimators for the new statistics and apply them to simulations of the models discussed in Section 4. Finally, we use our statistics to provide empirical evidence for the hypothesis of independence between components for species abundance data in a tropical rain forest (McGill, 2010; Wiegand et al., 2012).

2 | RANDOM MEASURES AND THEIR MOMENTS

In this section, we recall the definition of a multivariate random measure (Chiu, Stoyan, Kendall, & Mecke, 2013; Daley & Vere-Jones, 2008).

Definition 1. Let $\mathcal{X} = \mathbb{R}^d \times \{1, \ldots, n\}$, for $d, n \in \mathbb{N}$, be equipped with the metric $d(\cdot, \cdot)$ defined by $d((x, i), (y, j)) = \|x - y\| + |i - j|$ for $x, y \in \mathbb{R}^d$ and $i, j \in \{1, \ldots, n\}$. Then a multivariate random measure Ψ on $\mathcal{X}$ is a measurable mapping from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the space of all locally finite Borel measures on $\mathcal{X}$ equipped with the smallest σ-algebra that makes all

$$\Psi_i(B) = \Psi(B \times \{i\}),$$

with $B \subset \mathbb{R}^d$ ranging through the bounded Borel sets and i through $\{1, \dots, n\}$ a random variable.

Below, being interested in cross statistics, we shall restrict ourselves to the bivariate case that $n = 2$. An important functional associated with a bivariate random measure is its *Laplace functional*.

Definition 2. Let $\Psi = (\Psi_1, \Psi_2)$ be a bivariate random measure. Let $u : \mathbb{R}^d \times \{1, 2\} \rightarrow \mathbb{R}^+$ be a bounded nonnegative measurable function such that the projections $u(\cdot, i) : \mathbb{R}^d \rightarrow \mathbb{R}^+$, $i = 1, 2$, have bounded support. Then,

$$L(u) = \mathbb{E} \exp \left[-\sum_{i=1}^{2} \int_{\mathbb{R}^d} u(x, i) \, \mathrm{d}\Psi_i(x) \right]$$

is the Laplace functional of Ψ evaluated at u.

The Laplace functional completely determines the distribution of the random measure Ψ (Daley & Vere-Jones, 2008, section 9.4) and is closely related to the *moment measures*. For Borel sets $B \subset \mathbb{R}^d$ and $i \in \{1, 2\}$, set

$$\mu^{(1)}(B \times \{i\}) = \mathbb{E}\Psi_i(B).$$

Provided the set function $\mu^{(1)}$ is finite for bounded Borel sets, it yields a locally finite Borel measure that is also denoted by $\mu^{(1)}$ and referred to as the *first-order moment measure* of Ψ. More generally, for $k \geq 2$, the kth order moment measure is defined by the set function

$$\mu^{(k)}((B_1 \times \{i_1\}) \times \cdots \times (B_k \times \{i_k\})) = \mathbb{E} \left(\Psi_{i_1}(B_1) \times \cdots \times \Psi_{i_k}(B_k) \right),$$

where $B_1, \dots, B_k \subset \mathbb{R}^d$ are Borel sets and $i_1, \dots, i_k \in \{1, 2\}$. If $\mu^{(k)}$ is finite for bounded B_i, it can be extended uniquely to a locally finite Borel measure on $\mathcal{X}^k$ (cf. section 9.5 in Daley and Vere-Jones , 2008).

In the sequel, we shall need the following relation between the Laplace functional and the moment measures. Let u be a bounded nonnegative measurable function $u : \mathbb{R}^d \times \{1, 2\} \rightarrow \mathbb{R}^+$ such that its projections have bounded support. Then,

$$L(u) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sum_{i_1=1}^{2} \int_{\mathbb{R}^d} \cdots \sum_{i_k=1}^{2} \int_{\mathbb{R}^d} u(x_1, i_1) \cdots u(x_k, i_k) \, \mathrm{d}\mu^{(k)} \left((x_1, i_1), \dots, (x_k, i_k) \right), \quad (1)$$

provided that the moment measures of all orders exist and that the series on the right-hand side of (1) is absolutely convergent (Daley & Vere-Jones, 2003, formula 6.1.9).

The above discussion might lead us to expect that the moment measures determine the distribution of a random measure. Such a claim cannot be made in complete generality, but Zessin (1983) derived a sufficient condition.

Theorem 1. *Let $\Psi = (\Psi_1, \Psi_2)$ be a bivariate random measure, and assume that the series*

$$\sum_{k=1}^{\infty} \mu^{(k)}((B \times C)^k)^{-1/(2k)} = \infty$$

diverges for all bounded Borel sets $B \subset \mathbb{R}^d$ and all $C \subset \{1, 2\}$. Then, the distribution of Ψ is uniquely determined by its moment measures.

The existence of the first-order moment measure implies that of a Palm distribution (Daley & Vere-Jones, 2008, proposition 13.1.IV).

Definition 3. Let $\Psi = (\Psi_1, \Psi_2)$ be a bivariate random measure for which $\mu^{(1)}$ exists as a locally finite measure. Then, Ψ admits Palm distributions $P^{(x,i)}$ that are defined uniquely up to a $\mu^{(1)}$-null set and satisfy

$$\mathbb{E}\left[\sum_{i=1}^{2}\int_{\mathbb{R}^d} g((x,i),\Psi)\,\mathrm{d}\Psi_i(x)\right] = \sum_{i=1}^{2}\int_{\mathbb{R}^d}\mathbb{E}^{(x,i)}\left[g((x,i),\Psi)\right]\mathrm{d}\mu^{(1)}(x,i) \qquad (2)$$

for any nonnegative measurable function g. Here, $\mathbb{E}^{(x,i)}$ denotes expectation with respect to $P^{(x,i)}$.

Equation (2) is sometimes referred to as the Campbell–Mecke formula.

Next, we will focus on random measures whose moment measures are absolutely continuous. Thus, suppose that

$$\mu^{(k)}((B_1\times\{i_1\})\times\cdots\times(B_k\times\{i_k\})) = \int_{B_1}\cdots\int_{B_k} p_k((x_1,i_1),\ldots,(x_k,i_k))\,\mathrm{d}x_1\cdots\mathrm{d}x_k,$$

or, in other words, that for fixed $i_1,\ldots,i_k$, $\mu^{(k)}$ is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^{kd}$ with Radon–Nikodym derivative p_k, the k-point *coverage function*. The family of p_k defines cumulant densities as follows (Daley & Vere-Jones, 2008).

Definition 4. Let $\Psi = (\Psi_1, \Psi_2)$ be a bivariate random measure, and assume that its moment measures exist and are absolutely continuous. Assume that the coverage function p_1 is strictly positive. Then, the coverage-reweighted cumulant densities ξ_k are defined recursively by $\xi_1 \equiv 1$ and, for $k \geq 2$,

$$\frac{p_k((x_1,i_1),\ldots,(x_k,i_k))}{p_1(x_1,i_1)\cdots p_1(x_k,i_k)} = \sum_{m=1}^{k}\sum_{D_1,\ldots,D_m}\prod_{j=1}^{m}\xi_{|D_j|}\left(\{(x_l,i_l): l \in D_j\}\right),$$

where the sum is over all possible partitions $\{D_1,\ldots,D_m\}$, $D_j \neq \emptyset$, of $\{1,\ldots,k\}$. Here, we use the labels $i_1,\ldots,i_k$ to define which of the components is considered and denote the cardinality of D_j by $|D_j|$.

For the special case $k = 2$,

$$\xi_2((x_1,i_1),(x_2,i_2)) = \frac{p_2((x_1,i_1),(x_2,i_2)) - p_1(x_1,i_1)\,p_1(x_2,i_2)}{p_1(x_1,i_1)\,p_1(x_2,i_2)}.$$

Consequently, ξ_2 can be interpreted as a coverage-reweighted covariance function.

An application of lemma 5.2.VI in Daley and Vere-Jones (2003) to (1) implies that

$$\log L(u) = \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{i_1=1}^{2}\int_{\mathbb{R}^d}\cdots\sum_{i_k=1}^{2}\int_{\mathbb{R}^d}\xi_k((x_1,i_1),\ldots,(x_k,i_k))\prod_{j=1}^{k}u(x_j,i_j)\,p_1(x_j,i_j)\,\mathrm{d}x_j, \qquad (3)$$

provided that the series on the right-hand side of (3) is absolutely convergent.

Indeed, up to a factor $\prod_{j=1}^{k} p_1(x_j,i_j)$, the ξ_k are the Radon–Nikodym derivatives of the cumulant measures of Ψ. If Ψ is stationary, the cumulant measures are invariant under translations. We shall need a weaker form of stationarity that can be interpreted as moment stationarity after accounting for fluctuations in the coverage function.

Definition 5. Let $\Psi = (\Psi_1, \Psi_2)$ be a bivariate random measure. Then, Ψ is called coverage-reweighted moment stationary if its coverage function exists and is bounded away from zero (i.e., $\inf p_1(x,i) > 0$) and if its coverage-reweighted cumulant densities ξ_k, $k \geq 2$,

exist and are translation invariant in the sense that for all $a \in \mathbb{R}^d$, the equation

$$\xi_k\left((x_1 + a, i_1), \ldots, (x_k + a, i_k)\right) = \xi_k\left((x_1, i_1), \ldots, (x_k, i_k)\right)$$

holds for all $i_j \in \{1, 2\}$ and almost all $x_j \in \mathbb{R}^d$.

The next result states that under the condition of Definition 5, the Palm moment measures of the coverage-reweighted random measure $\Psi(\cdot)/p_1(\cdot)$ can be expressed in terms of the k-point coverage functions of Ψ.

Theorem 2. *Let Ψ be a coverage-reweighted moment stationary bivariate random measure and $k \in \mathbb{N}$. Then, for all bounded Borel sets $B_1, \ldots, B_k$ and all $i_1, \ldots, i_k \in \{1, 2\}$, the Palm expectation is given by*

$$\mathbb{E}^{(a,i)}\left[\int_{a+B_1} \cdots \int_{a+B_k} \frac{\mathrm{d}\Psi_{i_1}(x_1)\cdots\mathrm{d}\Psi_{i_k}(x_k)}{p_1(x_1,i_1)\cdots p_1(x_k,i_k)}\right] = \int_{B_1} \cdots \int_{B_k} \frac{p_{k+1}\left((0,i),(x_1,i_1),\ldots,(x_k,i_k)\right)}{p_1(0,i)\,p_1(x_1,i_1)\cdots p_1(x_k,i_k)}\,\mathrm{d}x_1\cdots\mathrm{d}x_k$$

for $i \in \{1, 2\}$ and almost all $a \in \mathbb{R}^d$.

Proof. By (2) with $g((a, j), \Psi) = 0$ if $j \neq i$, and

$$g((a,i),\Psi) = \frac{1_A(a)}{p_1(a,i)} \int_{a+B_1} \cdots \int_{a+B_k} \frac{1}{p_1(x_1,i_1)\cdots p_1(x_k,i_k)}\,\mathrm{d}\Psi_{i_1}(x_1)\cdots\mathrm{d}\Psi_{i_k}(x_k),$$

for some bounded Borel sets $A, B_1, \ldots, B_k \subset \mathbb{R}^d$ and any $i, i_1, \ldots, i_k \in \{1, 2\}$, one sees that

$$\mathbb{E}\left[\int_A \frac{1}{p_1(a,i)} \int_{a+B_1} \cdots \int_{a+B_k} \frac{1}{p_1(x_1,i_1)\cdots p_1(x_k,i_k)}\,\mathrm{d}\Psi_{i_1}(x_1)\cdots\mathrm{d}\Psi_{i_k}(x_k)\,\mathrm{d}\Psi_i(a)\right]$$

$$= \int_A \mathbb{E}^{(a,i)}\left[\int_{a+B_1} \cdots \int_{a+B_k} \frac{1}{p_1(a,i)\,p_1(x_1,i_1)\cdots p_1(x_k,i_k)}\,\mathrm{d}\Psi_{i_1}(x_1)\cdots\mathrm{d}\Psi_{i_k}(x_k)\right] p_1(a,i)\,\mathrm{d}a.$$

The left-hand side is equal to

$$\int_A\left[\int_{B_1} \cdots \int_{B_k} \frac{p_{k+1}\left((a,i),(a+x_1,i_1),\ldots,(a+x_k,i_k)\right)}{p_1(a,i)\,p_1(a+x_1,i_1)\cdots p_1(a+x_k,i_k)}\,\mathrm{d}x_1\cdots\mathrm{d}x_k\right]\mathrm{d}a,$$

and the inner integrand does not depend on the choice of $a \in A$ by the assumptions on Ψ. Hence, for all bounded Borel sets $A \subset \mathbb{R}^d$,

$$\int_A \mathbb{E}^{(a,i)}\left[\int_{a+B_1} \cdots \int_{a+B_k} \frac{\mathrm{d}\Psi_{i_1}(x_1)\cdots\mathrm{d}\Psi_{i_k}(x_k)}{p_1(x_1,i_1)\cdots p_1(x_k,i_k)}\right]\mathrm{d}a$$

$$= \int_A\left[\int_{B_1} \cdots \int_{B_k} \frac{p_{k+1}\left((0,i),(x_1,i_1),\ldots,(x_k,i_k)\right)}{p_1(0,i)\,p_1(x_1,i_1)\cdots p_1(x_k,i_k)}\,\mathrm{d}x_1\cdots\mathrm{d}x_k\right]\mathrm{d}a.$$

Therefore, the Palm expectation takes the same value for almost all $a \in \mathbb{R}^d$. □

3 | SUMMARY STATISTICS FOR BIVARIATE RANDOM MEASURES

3.1 | The inhomogeneous cross K-function

For the coverage measures associated to a stationary bivariate random closed set, Stoyan and Ohser (1982) defined the *reduced cross correlation measure* as follows. Let $B(x, t)$ be

the closed ball of radius $t \geq 0$ centred at $x \in \mathbb{R}^d$ and set, for any bounded Borel set B of positive volume $\ell(B)$,

$$R_{12}(t) = \frac{1}{p_1(0,1) \, p_1(0,2)} \mathbb{E}\left[\frac{1}{\ell(B)} \int_B \Psi_2(B(x,t)) \, \mathrm{d}\Psi_1(x)\right]. \tag{4}$$

Due to the assumed stationarity, the right-hand side of (4) does not depend on the choice of B. In the univariate case, Ayala and Simó (1998) called a function of this type the *K*-function in analogy to a similar statistic for point processes (Diggle, 2014; Ripley, 1977).

In order to modify (4) so that it applies to more general, and not necessarily stationary, random measures, we focus on the second-order coverage-reweighted cumulant density ξ_2 and assume it is invariant under translations. If additionally p_1 is bounded away from zero, Ψ is said to be *second-order coverage-reweighted stationary*.

Definition 6. Let $\Psi = (\Psi_1, \Psi_2)$ be a bivariate random measure that admits a second-order coverage-reweighted cumulant density ξ_2 that is invariant under translations and a coverage function p_1 that is bounded away from zero. Then, for $t \geq 0$, the cross *K*-function is defined by

$$K_{12}(t) = \int_{B(0,t)} (1 + \xi_2((0,1),(x,2))) \, \mathrm{d}x.$$

Note that the cross *K*-function is symmetric in the components of Ψ, that is, $K_{12} = K_{21}$. The next result gives an alternative expression in terms of the expected content of a ball under the Palm distribution of the coverage-reweighted random measure.

Lemma 1. *Let $\Psi = (\Psi_1, \Psi_2)$ be a second-order coverage-reweighted stationary bivariate random measure. Then,*

$$K_{12}(t) = \mathbb{E}^{(a,1)}\left[\int_{B(a,t)} \frac{1}{p_1(x,2)} \, \mathrm{d}\Psi_2(x)\right],$$

and the right-hand side does not depend on the choice of $a \in \mathbb{R}^d$.

Proof. Apply Theorem 2 for $k = 1$, $i = 1$, $B_1 = B(0,t)$, and $i_1 = 2$ to obtain

$$\mathbb{E}^{(a,1)}\left[\int_{B(a,t)} \frac{1}{p_1(x,2)} \mathrm{d}\Psi_2(x)\right] = \int_{B(0,t)} \frac{p_2((0,1),(x,2))}{p_1(0,1) \, p_1(x,2)} \, \mathrm{d}x$$

$$= \int_{B(0,t)} (1 + \xi_2((0,1),(x,2))) \, \mathrm{d}x.$$

In particular, the right-hand side does not depend on a. □

To interpret the statistic, recall that ξ_2 is equal to the coverage-reweighted covariance. Thus, if Ψ_1 and Ψ_2 are independent, then

$$K_{12}(t) = \ell(B(0,t)),$$

the Lebesgue measure of $B(0,t)$. Larger values are due to positive correlation, and smaller ones to negative correlation between Ψ_1 and Ψ_2. Furthermore, if $\Psi = (\Psi_1, \Psi_2)$ is stationary, Lemma 1 implies that

$$K_{12}(t) = \frac{1}{p_1(0,2)} \mathbb{E}^{(0,1)}\left[\Psi_2(B(0,t))\right],$$

which, by the Campbell–Mecke equation (2), is equal to

$$\frac{1}{p_1(0,1)p_1(0,2)} \mathbb{E}\left[\frac{1}{\ell(B)} \int_B \Psi_2(B(x,t)) \, d\Psi_1(x)\right]$$

for any bounded Borel set B for which $\ell(B) > 0$. Consequently, $K_{12}(t) = R_{12}(t)$, the reduced cross correlation measure of Stoyan and Ohser (1982).

3.2 | Inhomogeneous cross *J*-function

The cross *K*-function is based on the second-order coverage-reweighted cumulant density. In this section, we propose a new statistic that incorporates the coverage-reweighted cumulant densities of all orders.

Definition 7. Let $\Psi = (\Psi_1, \Psi_2)$ be a coverage-reweighted moment stationary bivariate random measure. For $t \geq 0$ and $k \geq 1$, set

$$J_{12}^{(k)}(t) = \int_{B(0,t)} \cdots \int_{B(0,t)} \xi_{k+1}\left((0,1),(x_1,2),\ldots,(x_k,2)\right) dx_1 \cdots dx_k,$$

and define the cross *J*-function by

$$J_{12}(t) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} J_{12}^{(k)}(t)$$

for all $t \geq 0$ for which the series is absolutely convergent.

Note that

$$J_{12}^{(1)}(t) = K_{12}(t) - \ell(B(0,t)).$$

The appeal of Definition 7 lies in the fact that its dependence on the cumulant densities and, furthermore, its relation to K_{12} are immediately apparent. However, being an alternating series, $J_{12}(t)$ is not convenient to handle in practise. The next theorem gives a simpler characterisation in terms of the Laplace transform.

Theorem 3. *Let $\Psi = (\Psi_1, \Psi_2)$ be a coverage-reweighted moment stationary bivariate random measure. Write $L^{(a,1)}$ for the Laplace transform under the Palm distribution $P^{(a,1)}$. Then, for $t \geq 0$ and $a \in \mathbb{R}^d$,*

$$J_{12}(t) = \frac{L^{(a,1)}\left(u_t^a\right)}{L\left(u_t^a\right)} \tag{5}$$

for $u_t^a(x,i) = 1\{(x,i) \in B(a,t) \times \{2\}\}/p_1(x,i)$, provided that the series expansions of $L(u_t^a)$ and $J_{12}(t)$ are absolutely convergent. In particular, $J_{12}(t)$ does not depend on the choice of origin $a \in \mathbb{R}^d$.

Proof. First, note that, by (3), $L(u_t^a)$ does not depend on the choice of a. Also, by Theorem 2 and the series expansion (1) of the Laplace transform for $u_t^a(x,i)$, provided that the series is

absolutely convergent,

$$L^{(a,1)}\left(u_t^a\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \mathbb{E}^{(a,1)} \left[\int_{B(a,t)} \cdots \int_{B(a,t)} \frac{\mathrm{d}\Psi_2(x_1)\cdots\mathrm{d}\Psi_2(x_k)}{p_1(x_1,2)\cdots p_1(x_k,2)} \right]$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{B(0,t)} \cdots \int_{B(0,t)} \frac{p_{k+1}\left((0,1),(x_2,2),\ldots,(x_{k+1},2)\right)}{p_1(0,1)p_1(x_2,2)\cdots p_1(x_{k+1},2)} \, \mathrm{d}x_2 \cdots \mathrm{d}x_{k+1}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{B(0,t)} \cdots \int_{B(0,t)} \sum_{m=1}^{k+1} \sum_{D_1,\ldots,D_m} \prod_{j=1}^{m} \xi_{|D_j|}\left(\{(x_l,i_l) : l \in D_j\}\right) \prod_{i=2}^{k+1} \mathrm{d}x_i,$$

where $(x_1,i_1) \equiv (0,1)$ and $i_l = 2$ for $l > 1$. By splitting the last expression into terms based on whether the sets D_j contain the index 1 (i.e., on whether $\xi_{|D_j|}$ includes $(x_1,i_1) \equiv (0,1)$), under the convention that $\sum_{k=1}^{0} = 1$, we obtain

$$L^{(a,1)}\left(u_t^a\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sum_{\Pi \in \mathcal{P}_k} J_{12}^{(|\Pi|)}(t) \sum_{m=1}^{k-|\Pi|} \sum_{\substack{D_1,\ldots,D_m \neq \emptyset \text{ disjoint} \\ \cup_{j=1}^{m} D_j = \{1,\ldots,k\}\backslash\Pi}} \prod_{j=1}^{m} I_{|D_j|},$$

where

$$I_k = \int_{B(0,t)} \cdots \int_{B(0,t)} \xi_k((x_1,2),\ldots,(x_k,2)) \, \mathrm{d}x_1 \cdots \mathrm{d}x_k,$$

$J_{12}^{(0)}(t) \equiv 1$, and $\mathcal{P}_k$ is the power set of $\{1,\ldots,k\}$. Finally, by noting that the expansion contains terms of the form $J_{12}^{(k)}(t) I_{k_1}^{m_1} \cdots I_{k_n}^{m_n}$ multiplied by a scalar and basic combinatorial arguments, we conclude that

$$L^{(a,1)}\left(u_t^a\right) = \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} J_{12}^{(k)}(t)\right) \times \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sum_{m=1}^{k} \sum_{\substack{D_1,\ldots,D_m \neq \emptyset \text{ disjoint} \\ \cup_{j=1}^{m} D_j = \{1,\ldots,k\}}} \prod_{j=1}^{m} I_{|D_j|}\right)$$

$$= J_{12}(t) L\left(u_t^a\right).$$

The right-hand side does not depend on a and is absolutely convergent as a product of absolutely convergent terms. Therefore, so is the series expansion for $L^{(a,1)}$. □

Heuristically, the cross J-function compares expectations under the Palm distribution $P^{(0,1)}$ with those under the distribution P of Ψ. If the components of Ψ are independent, conditioning on the first component placing mass at the origin, does not affect the second component, so $J_{12}(t) = 1$. A value larger than 1 means that such conditioning tends to lead to a smaller $\Psi_2(B(0,t))$ content (typical for negative association); analogously, $J_{12}(t) < 1$ suggests positive association between the components of Ψ.

4 | EXAMPLES

In this section, we calculate the cross K- and J-function for a range of well-known models that can be shown to be coverage-reweighted moment stationary and we point out some relations to familiar statistics including the empty space and spherical contact distribution functions. The explicit expressions thus obtained may be used in minimum contrast methods for parameter estimation purposes (Møller & Waagepetersen, 2004).

4.1 | Point processes

A point process is a random measure that takes integer values. For this special case, both cross K- and J-functions have been proposed to quantify the dependence between components (Cronie & van Lieshout, 2016; Møller & Waagepetersen, 2004; van Lieshout, 2011). However, it would be a mistake to think that the family of coverage-reweighted cumulant densities ξ_k and the notion of weak stationarity based upon it coincide with the family of n-point correlation functions and the associated notion of weak stationarity that form the theoretical foundations for the cross statistic in the context of a point process (van Lieshout, 2011).

To see why, let $N = (N_1, N_2)$ be a simple bivariate point process, that is, N almost surely does not place two points at the same location. The first-order moment measure of N seen as a random measure is given by

$$\mu^{(1)}(B \times \{i\}) = \mathbb{E}N_i(B)$$

for Borel sets $B \subset \mathbb{R}^d$. The right-hand side in the formula above is the first-order moment measure of the point process N_i. Hence, assuming absolute continuity with respect to Lebesgue measure,

$$p_1(x, i) = \lambda_i(x),$$

so the 1-point coverage function $p_1(x, i)$ coincides with the intensity function $\lambda_i(x)$ of N_i.

The second-order moment measure of the random measure N is equal to

$$\mu^{(2)}((B_1 \times \{i_1\}) \times (B_2 \times \{i_2\})) = \mathbb{E}\left[N_{i_1}(B_1)N_{i_2}(B_2)\right],$$

for Borel sets $B_1, B_2 \subset \mathbb{R}^d$ and $i_1, i_2 \in \{1, 2\}$. It can be broken up in two terms, as follows:

$$\mathbb{E}\left[\sum_{x \in N_1 \cup N_2} \sum_{x \neq y \in N_1 \cup N_2} 1\left\{x \in B_1 \cap N_{i_1}; y \in B_2 \cap N_{i_2}\right\}\right]$$

and

$$\mathbb{E}\left[\sum_{x \in N_1 \cup N_2} 1\left\{x \in (B_1 \cap N_{i_1}) \cap (B_2 \cap N_{i_2})\right\}\right]. \tag{6}$$

The first term may be absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^{2d}$, so that it can be expressed as an integral

$$\int_{B_1} \int_{B_2} \rho_2\left((x, i_1), (y, i_2)\right) \, \mathrm{d}x \mathrm{d}y$$

of *product densities* ρ_2 (Daley & Vere-Jones, 2008), but the second term is concentrated on a lower dimensional subspace and cannot be absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^{2d}$. Similar considerations apply for higher orders, and we conclude that a point process seen as a random measure is not in general coverage-reweighted moment stationary.

The cross statistic K_{12} of Definition 6, however, relies solely on the two-point coverage function evaluated at pairs of different types. Therefore, (6) may be ignored, and

$$\frac{p_2((x, 1), (y, 2))}{p_1(x, 1)p_1(y, 2)} = \frac{\rho_2((x, 1), (y, 2))}{\lambda_1(x)\lambda_2(y)}$$

is invariant under translations when the point process is second-order intensity-reweighted stationary in the sense of Møller and Waagepetersen (2004). Therefore, the cross K-function for point

processes defined by

$$\frac{1}{\ell(B)}\mathbb{E}\left[\sum_{x\in N_1}\sum_{y\in N_2}\frac{1\{x\in B;\, y\in B(x,t)\}}{\lambda_1(x)\lambda_1(y)}\right] = \frac{1}{\ell(B)}\int_B\int_{B(x,t)}\frac{\rho_2\left((x,1),(y,2)\right)}{\lambda_1(x)\lambda_1(y)}\,\mathrm{d}x\mathrm{d}y$$

reduces to

$$\int_{B(0,t)}\frac{\rho_2\left((0,1),(z,2)\right)}{\lambda_1(0)\lambda_1(z)}\,\mathrm{d}z = \int_{B(0,t)}\frac{p_2\left((0,1),(z,2)\right)}{p_1(0,1)p_1(z,2)}\,\mathrm{d}z = K_{12}(t),$$

the cross K-function of Definition 6.

A similar remark does not hold for the cross J-function, as it fundamentally relies on k-point coverage functions of all orders. Therefore, Definition 7 does not apply. Regarding the characterisation in Theorem 3, the Laplace transform of the random measure N can be expressed as

$$L\left(u_t^a\right) = G\!\left(e^{-u_t^a}\right)$$

in terms of the *generating functional G* (Daley & Vere-Jones, 2008, section 9.4) of the point process N. If we assume that the point process is intensity-reweighted moment stationary (van Lieshout, 2011; Cronie & van Lieshout, 2016) in the sense that the intensity function is bounded away from zero, the product densities ρ_k of all orders exist, and the k-point correlation functions η_k (defined in complete analogy to Definition 4 with ρ_k replacing p_k) are translation invariant, then

$$\log G\!\left(e^{-u_t^a}\right) = \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{i_1=1}^{2}\int_{\mathbb{R}^d}\cdots\sum_{i_k=1}^{2}\int_{\mathbb{R}^d}\eta_k\left((x_1,i_1),\,\ldots,(x_k,i_k)\right)\times$$

$$\times\prod_{j=1}^{k}\lambda_{i_j}(x_j)\left(1-e^{-u_t^a}(x_j,i_j)\right)\mathrm{d}x_j =$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\int_{B(a,t)}\cdots\int_{B(a,t)}\eta_k\left((x_1,2),\,\ldots,(x_k,2)\right)\prod_{j=1}^{k}\lambda_2(x_j)\left(1-e^{-1/\lambda_2(x_j)}\right)\mathrm{d}x_j,$$

provided that the series converges. Note that $G(e^{-u_t^a})$ may depend on the choice of origin a, even when all η_k are translation invariant. However, the Taylor approximation

$$1-e^{-1/\lambda_2(x_j)}\approx 1/\lambda_2(x_j)$$

ensures that the multiplier $\lambda_2(x_j)$ cancels out and the resulting approximation of $G(e^{-u_t^a})$ no longer depends on the choice of origin. For this reason, van Lieshout (2011) based inhomogeneous J-functions on the generating functional of the function

$$v_t^a(x_j,i_j) = 1 - \inf\left\{\lambda_2(x) : x\in\mathbb{R}^d\right\}\times u_t^a(x_j,i_j)$$

instead of $e^{-u_t^a}$. The scaling is needed to ensure function values in $[0,1]$. Further details may be found in Cronie and van Lieshout (2016).

The idea to take the opposite route and define a random measure version of the cross J-function by means of $L(-\log v_t^a) = G(v_t^a)$ will not hold water either because the Laplace transform is ill defined due to the unboundedness of the function $-\log v_t^a$.

4.2 | Compound random measures

Let $\Lambda = (\Lambda_1, \Lambda_2)$ be a random vector such that its components take values in $\mathbb{R}^+$ and have finite, strictly positive expectation. Set

$$\Psi = (\Lambda_1 \nu, \Lambda_2 \nu) \tag{7}$$

for some locally finite Borel measure ν on $\mathbb{R}^d$ that is absolutely continuous with density function $f_\nu \geq \epsilon > 0$. In other words, $\Psi_i(B) = \Lambda_i \int_B f_\nu(x)\,dx = \Lambda_i \nu(B)$.

Theorem 4. *The bivariate random measure* $\Psi = (\Lambda_1 \nu, \Lambda_2 \nu)$ *defined by (7) is coverage-reweighted moment stationary and*

$$K_{12}(t) = \kappa_d t^d \left(1 + \frac{\mathrm{Cov}(\Lambda_1, \Lambda_2)}{\mathbb{E}(\Lambda_1)\mathbb{E}(\Lambda_2)} \right)$$

$$J_{12}(t) = \frac{\mathbb{E}\left(\Lambda_1 \exp\left[-\Lambda_2 \kappa_d t^d / \mathbb{E}\Lambda_2 \right] \right)}{\mathbb{E}(\Lambda_1)\,\mathbb{E}\left(\exp\left[-\Lambda_2 \kappa_d t^d / \mathbb{E}\Lambda_2 \right] \right)}.$$

Here, $\kappa_d = \ell(B(0,1))$ *is the volume of the unit ball in* $\mathbb{R}^d$.

Proof. Because

$$\mathbb{E}\left[\Psi_1(B_1) \cdots \Psi_1(B_k) \Psi_2(B_{k+1}) \cdots \Psi_2(B_{k+l}) \right] = \mathbb{E}\left(\Lambda_1^k \Lambda_2^l \right) \int_{B_1} \cdots \int_{B_{k+l}} \prod_{i=1}^{k+l} f_\nu(x_i)\,dx_i,$$

for Borel sets $B_1, \ldots, B_{k+l} \subset \mathbb{R}^d$, the coverage function of Ψ is given by

$$p_{k+l}((x_1, 1), \ldots, (x_k, 1), (x_{k+1}, 2), \ldots, (x_{k+l}, 2)) = \mathbb{E}\left(\Lambda_1^k \Lambda_2^l \right) \prod_{i=1}^{k+l} f_\nu(x_i),$$

so that the coverage-reweighted cumulant densities of Ψ are translation invariant. The assumptions imply that $p_1(x, i) = \mathbb{E}(\Lambda_i) f_\nu(x)$ is bounded away from zero. Hence, Ψ is coverage-reweighted moment stationary.

Specialising to second order, one finds that

$$\xi_2((0, 1), (x, 2)) = \frac{\mathbb{E}(\Lambda_1 \Lambda_2) - \mathbb{E}(\Lambda_1)\mathbb{E}(\Lambda_2)}{\mathbb{E}(\Lambda_1)\mathbb{E}(\Lambda_2)} = \frac{\mathrm{Cov}(\Lambda_1, \Lambda_2)}{\mathbb{E}(\Lambda_1)\mathbb{E}(\Lambda_2)}$$

from which the expression for $K_{12}(t)$ follows upon integration.

As for the cross J-function, the denominator in Theorem 3 can be written as

$$L(u_t^0) = \mathbb{E} \exp\left[-\int_{B(0,t)} \frac{1}{\mathbb{E}(\Lambda_2) f_\nu(x)}\,d\Psi_2(x) \right]$$

$$= \mathbb{E} \exp\left[-\frac{1}{\mathbb{E}(\Lambda_2)} \int_{B(0,t)} \frac{1}{f_\nu(x)} \Lambda_2\,d\nu(x) \right]$$

$$= \mathbb{E} \exp\left[-\Lambda_2 \kappa_d t^d / \mathbb{E}\Lambda_2 \right],$$

using $d\nu(x) = f_\nu(x)dx$. Using a result of Daley and Vere-Jones (2008, p. 274),

$$L^{(0,1)}\left(u_t^0 \right) = \mathbb{E}\left(\Lambda_1 \exp\left[-\Lambda_2 \kappa_d t^d / \mathbb{E}\Lambda_2 \right] \right) / \mathbb{E}(\Lambda_1),$$

and the proof is complete. □

Both statistics do not depend on f_ν. To see that they capture a form of "dependence" between the components of Ψ, note that the cross K-function exceeds $\kappa_d t^d$ if and only if Λ_1 and Λ_2 are

positively correlated. For the cross J-function, recall that two random variables X and Y are *negatively quadrant dependent* if $\text{Cov}(f(X)), g(Y) \leq 0$ whenever f,g are nondecreasing functions, and *positively quadrant dependent* if $\text{Cov}(f(X)), g(Y) \geq 0$ (provided the moments exist; cf. Esary, Proschan, & Walkup, 1967; Kumar & Proschan, 1983; Lehmann, 1966). Applied to our context, it follows that if Λ_1 and Λ_2 are positively quadrant dependent, $J_{12}(t) \leq 1$ whilst $J_{12}(t) \geq 1$ if Λ_1 and Λ_2 are negatively quadrant dependent.

Let us consider two specific examples discussed by Diggle (2014).

Linked model Let $\Lambda_2 = A\Lambda_1$ and hence $\Psi_2 = A\Psi_1$ for some $A > 0$. Because, for $l_1, l_2 \in \mathbb{R}^+$,

$$\mathbb{P}(\Lambda_1 \leq l_1; \Lambda_2 \leq l_2) = \mathbb{P}(\Lambda_1 \leq \min(l_1, l_2/A)) \geq \mathbb{P}(\Lambda_1 \leq l_1)\mathbb{P}(A\Lambda_1 \leq l_2),$$

Λ_1 and Λ_2 are positively quadrant dependent (Theorem 4.4 in Esary et al., 1967) and, a fortiori, positively correlated. Therefore, $K_{12}(t) \geq \kappa_d t^d$ and $J_{12}(t) \leq 1$.

Balanced model Let Λ_1 be supported on the interval $(0, A)$ for some $A > 0$, and set $\Lambda_2 = A - \Lambda_1$. Because, for $l_1, l_2 \in (0, A)$ such that $A - l_2 \leq l_1$,

$$\mathbb{P}(\Lambda_1 \leq l_1; \Lambda_2 \leq l_2) = \mathbb{P}(\Lambda_1 \leq l_1) - \mathbb{P}(\Lambda_1 < A - l_2)$$
$$\leq \mathbb{P}(\Lambda_1 \leq l_1) - \mathbb{P}(\Lambda_1 \leq l_1)\mathbb{P}(\Lambda_1 < A - l_2) = \mathbb{P}(\Lambda_1 \leq l_1)\mathbb{P}(\Lambda_2 \leq l_2),$$

Λ_1 and Λ_2 are negatively quadrant dependent (Kumar & Proschan, 1983) and, a fortiori, negatively correlated. Therefore, $K_{12}(t) \leq \kappa_d t^d$ and $J_{12}(t) \geq 1$.

4.3 | Coverage measure of random closed sets

Let $X = (X_1, X_2)$ be a bivariate random closed set. Then, by Robbins′ theorem (Molchanov, 2017, p. 97), the Lebesgue content

$$\ell(X_i \cap B) = \int_B 1\{x \in X_i\}\, dx$$

of $X_i \cap B$ is a random variable for every Borel set $B \subset \mathbb{R}^d$ and every component X_i, $i = 1, 2$. Letting B and i vary, one obtains a bivariate random measure denoted by Ψ. Clearly, Ψ is locally finite.

Reversely, a bivariate random measure $\Psi = (\Psi_1, \Psi_2)$ defines a bivariate random closed set by the supports

$$\text{supp}(\Psi_i) = \bigcap_{n=1}^{\infty} \text{cl}\left(\left\{x_j \in \mathbb{Q}^d : \Psi_i\left(B\left(x_j, 1/n\right)\right) > 0\right\}\right),$$

where $B(x_j, 1/n)$ is the closed ball around x_j with radius $1/n$, and $\text{cl}(B)$ is the topological closure of the Borel set B. In other words, if $x \in \text{supp}(\Psi_i)$, then every ball that contains x has strictly positive Ψ_i mass. By proposition 1.9.22 in Molchanov (2017), the supports are well-defined random closed sets whose joint distribution is uniquely determined by that of the random measures.

Indeed, Ayala, Ferrandiz, and Montes (1991) proved the following result.

Theorem 5. *Let $X = (X_1, X_2)$ be a bivariate random closed set. Then, the distribution of X is completely determined by that of $\Psi = (\ell(X_1 \cap \cdot), \ell(X_2 \cap \cdot))$ if and only if X is distributed as the (random) support of Ψ.*

From now on, assume that X is stationary. Then, Stoyan and Ohser (1982) showed that

$$T_{12}(t) = \mathbb{E}\left[\frac{1}{\ell(B)}\int_B 1\{X_2 \cap B(x, t) \neq \emptyset\}\, d\Psi_1(x)\right]$$

does not depend on the choice of B from the family of bounded Borel sets with positive volume $\ell(B)$ and called $T_{12}(t)$ the *hitting intensity* at range t. The hitting intensity is similar in spirit to another classic statistic, the *empty space function* (Matheron, 1975) defined by

$$F_2(t) = \mathbb{P}(X_2 \cap B(a, t) \neq \emptyset),$$

which can also be shown to not depend on the choice of origin a. The related *cross spherical contact distribution* can be defined as

$$H_{12}(t) = \mathbb{P}(X_2 \cap B(a, t) \neq \emptyset \mid a \in X_1)$$

in analogy to the classical univariate definition (Chiu et al., 2013). Again, the expression on the right-hand side does not depend on the choice of $a \in \mathbb{R}^d$ due to the assumed stationarity. In order to relate T_{12} and F_2 to our J_{12}-function, we need the concept of "scaling". Let $s > 0$. Then, the scaling of X by s results in $sX = (sX_1, sX_2)$, where $sX_i = \{sx : x \in X_i\}$.

Theorem 6. *Let $X = (X_1, X_2)$ be a stationary bivariate random closed set with strictly positive volume fractions $p_1(0, i) = \mathbb{P}(0 \in X_i)$, $i = 1, 2$. Then, the associated random coverage measure Ψ is coverage-reweighted moment stationary and the following hold.*

1. *The cross statistics are*

$$K_{12}(t) = \frac{\mathbb{E}\left(\ell\left(X_2 \cap B(0, t)\right) \mid 0 \in X_1\right)}{p_1(0, 2)};$$

$$J_{12}(t) = \frac{\mathbb{E}\left(1\{0 \in X_1\} \exp\left[-\ell(X_2 \cap B(0, t))/p_1(0, 2)\right]\right)}{p_1(0, 1)\,\mathbb{E}\left(\exp\left[-\ell(X_2 \cap B(0, t))/p_1(0, 2)\right]\right)}.$$

2. *Use a subscript sX to denote that the statistic is evaluated for the scaled random closed set sX, and let u_t^0 be as in Theorem 3. Then,*

$$\lim_{s \to \infty} L^{(0,1)}\left(s^d u_t^0\right) = 1 - \frac{T_{12}(t)}{p_1(0, 1)}$$

and, for $t > 0$,

$$\lim_{s \to \infty} J_{12;sX}(st) = \frac{\mathbb{P}(X_2 \cap B(0, t) = \emptyset \mid 0 \in X_1)}{\mathbb{P}(X_2 \cap B(0, t) = \emptyset)} = \mathbb{E}\left[\frac{1\{0 \in X_1\}}{p_1(0, 1)} \middle| X_2 \cap B(0, t) = \emptyset\right]$$

whenever $\mathbb{P}(X_2 \cap B(0, t) = \emptyset) \neq 0$.

In words, the scaling limit of the cross *J*-function compares the empty space function with the cross spherical contact distribution.

Proof. First, note that

$$\mu^{(k)}((B_1 \times \{i_1\}) \times \cdots \times (B_k \times \{i_k\})) = \mathbb{E}\left(\ell\left(X_{i_1} \cap B_1\right) \times \cdots \times \ell\left(X_{i_k} \cap B_k\right)\right),$$

which, by (1.5.11) in Molchanov (2017, p. 98) is equal to

$$\int_{B_1} \cdots \int_{B_k} \mathbb{P}\left(x_1 \in X_{i_1}; \ldots; x_k \in X_{i_k}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

Here, $k \in \mathbb{N}$ and $B_1, \ldots, B_k$ are Borel subsets of $\mathbb{R}^d$. Hence, Ψ admits moment measures of all orders, and the probabilities $\mathbb{P}(x_1 \in X_{i_1}; \ldots; x_k \in X_{i_k}) = p_k((x_1, i_1), \ldots, (x_k, i_k))$ define the coverage functions. By assumption, p_1 is bounded away from zero, so the stationarity of X implies that Ψ is coverage-reweighted moment stationary.

By Chiu et al. (2013, p. 288), the Palm distribution amounts to conditioning on having a point of the required component at the origin, and the expression for the cross K-function follows from Lemma 1, whereas that for the cross J-function follows from Theorem 3.

Use a subscript sX to denote that a statistic is evaluated for the scaled set. To see the effect of scaling on J_{12} to obtain $J_{12;sX}$, observe that, because

$$\mathbb{P}\left(x_1 \in sX_{i_1}; \dots ; x_k \in sX_{i_k}\right) = \mathbb{P}\left(x_1/s \in X_{i_1}; \dots ; x_k/s \in X_{i_k}\right),$$

the k-point coverage probabilities of sX are related to those of X by $p_{k;sX}((x_1, i_1), \dots, (x_k, i_k)) = p_X((x_1/s, i_1), \dots, (x_k/s, i_k))$. Similarly, $\xi_{k;sX}((x_1, i_1), \dots, (x_k, i_k)) = \xi_{k;X}((x_1/s, i_1), \dots, (x_k/s, i_k))$, and consequently, $J_{12;sX}^{(k)}(t) = s^{dk} J_{12;X}^{(k)}(t/s)$. Also, scaling the balls $B(0, t)$ by s to fix the coverage fraction, one obtains $J_{12;sX}^{(k)}(st) = s^{dk} J_{12;X}^{(k)}(t)$ and $K_{12;sX}(st) = J_{12;X}^{(1)}(st) + \kappa^d(st)^d = s^d K_{12;X}(t)$. The numerator in the expression of J_{12} in terms of Laplace functionals (cf. Theorem 3) after such scaling reads as follows. Define, for $x \in \mathbb{R}^d$ and $i \in \{1, 2\}$,

$$u_{st;sX}(x, i) = \frac{1\{(x, i) \in B(0, st) \times \{2\}\}}{p_{1;sX}(x, i)} = \frac{1\{(x/s, i) \in B(0, t) \times \{2\}\}}{p_{1;X}(x/s, i)}.$$

Then,

$$L_{sX}^{(0,1)}(u_{st;sX}) = \mathbb{E}\left[\exp\left(-\int_{B(0,st)} \frac{1\{x \in sX_2\}}{p_{1;sX}(x, 2)}\, dx\right)\middle| 0 \in sX_1\right] = L_X^{(0,1)}(s^d u_{t;X}).$$

For $t > 0$, as $s \to \infty$,

$$L_X^{(0,1)}(s^d u_{t;X}) \to \mathbb{P}(X_2 \cap B(0, t) = \emptyset \mid 0 \in X_1)$$

by the monotone convergence theorem.

Turning to $T_{12}(t)$, note that

$$\mathbb{E}\left[\frac{1}{\ell(B)}\int_B 1\{X_2 \cap B(x, t) \neq \emptyset; x \in X_1\}\, dx\right] = \frac{1}{\ell(B)}\int_B \mathbb{P}(X_2 \cap B(x, t) \neq \emptyset; x \in X_1)\, dx$$

by Robbins' theorem. Because the volume fractions are strictly positive, we may condition on having a point at any $x \in \mathbb{R}^d$, so that

$$\mathbb{P}(X_2 \cap B(x, t) \neq \emptyset; x \in X_1) = \mathbb{P}(X_2 \cap B(0, t) \neq \emptyset \mid 0 \in X_1)\mathbb{P}(0 \in X_1)$$

upon using the stationarity of X. We conclude that $L_X^{(0,1)}(s^d u_{t;X}) \to 1 - T_{12}(t)/p_1(0, 1)$ as claimed.

Finally, consider the effect of scaling on the denominator in (5). Now,

$$L_{sX}(u_{st;sX}) = \mathbb{E}\left[\exp\left(-\ell\left(sX_2 \cap B(0, st)\right)/p_1(0, 2)\right)\right] = L_X(s^d u_{t;X}).$$

For $t > 0$,

$$\lim_{s \to \infty} L_X(s^d u_{t;X}) = \mathbb{P}\left(X_2 \cap B(0, t) = \emptyset\right)$$

by the monotone convergence theorem. Combining numerator and denominator, the theorem is proved. □

The case $t = 0$ is special. Indeed, both the spherical contact distribution and empty space function may have a "nugget" at the origin. In contrast, $J_{12}(0) \equiv 1$.

Before specialising to germ–grain models, let us make a few remarks. First, note that the moment measures of Ψ have a nice interpretation. Indeed, by Fubini's theorem, the k-point coverage function coincides with the k-point coverage probabilities of the underlying random closed set. Moreover, because $\mu^{(k)}((B \times \{1, 2\})^k) \leq (2\ell(B))^k$, the Zessin condition holds (cf. Theorem 1).

Secondly, if X_1 and X_2 are independent, $J_{12}(t) \equiv 1$. More generally, if $\ell(X_2 \cap B(0, t))$ and $1\{0 \in X_1\}$ are negatively quadrant dependent, $J_{12}(t) \geq 1$. If the two random variables are positively quadrant dependent, then $J_{12}(t) \leq 1$. A similar interpretation holds for the cross K-function: If $\ell(X_2 \cap B(0, t))$ and $1\{0 \in X_1\}$ are negatively correlated, $K_{12}(t) \leq \kappa_d t^d$; if the two random variables are positively correlated, then $K_{12}(t) \geq \kappa_d t^d$.

Germ–grain models Let $N = (N_1, N_2)$ be a stationary bivariate point process. Placing closed balls of radius $r > 0$ around each of the points defines a bivariate random closed set

$$(X_1, X_2) = (U_r(N_1), U_r(N_2)),$$

where, for every locally finite configuration $\phi \subset \mathbb{R}^d$,

$$U_r(\phi) = \bigcup_{x \in \phi} B(x, r).$$

Theorem 7. *Let $N = (N_1, N_2)$ be a stationary bivariate point process and X the associated germ–grain model for balls of radius $r > 0$. Write, for $x \in \mathbb{R}^d$, $t_1, t_2 \in \mathbb{R}^+$,*

$$F_N(t_1, t_2; x) = \mathbb{P}(d(0, N_1) \leq t_1; d(x, N_2) \leq t_2)$$

for the joint empty space function of N at lag x, and let F_{N_i} be the marginal empty space function of N_i, $i = 1, 2$. Here, $d(x, N_i)$ denotes the distance from x to N_i. If $F_{N_i}(r) > 0$ for $i = 1, 2$, the random coverage measure Ψ of X is coverage-reweighted moment stationary with

$$K_{12}(t) = \frac{1}{F_{N_1}(r) F_{N_2}(r)} \int_{B(0,t)} F_N(r, r; x) \, \mathrm{d}x$$

and, for $t > 0$,

$$\lim_{s \to \infty} J_{12;sX}(st) = \frac{F_{N_1}(r) - F_N(r, r + t; 0)}{F_{N_1}(r)(1 - F_{N_2}(r + t))}$$

whenever $F_{N_1}(r) > 0$ and $F_{N_2}(r + t) < 1$.

Hence, the cross statistic of the germ–grain model can be expressed entirely in terms of the joint empty space function of the germ processes; the radius of the grains translates itself in a shift.

Proof. Because the coverage probabilities

$$p_1(0, i) = \mathbb{P}(0 \in X_i) = \mathbb{P}(d(0, N_i) \leq r) = F_{N_i}(r)$$

are strictly positive by assumption, Theorem 6 implies that Ψ is coverage-reweighted moment stationary. By stationarity,

$$K_{12}(t) = \frac{1}{F_{N_1}(r) F_{N_2}(r)} \int_{B(0,t)} \mathbb{P}(0 \in X_1; x \in X_2) \, \mathrm{d}x.$$

The observation that

$$\mathbb{P}(0 \in X_1; x \in X_2) = \mathbb{P}(d(0, N_1) \leq r; d(x, N_2) \leq r) = F_N(r, r; x)$$

implies the claimed expression for the cross K-function. Furthermore,

$$\mathbb{P}(X_2 \cap B(0, t) \neq \emptyset) = \mathbb{P}(d(0, N_2) \leq r + t) = F_{N_2}(r + t)$$

and

$$\mathbb{P}(X_2 \cap B(0, t) = \emptyset \mid 0 \in X_1) = \frac{\mathbb{P}(N_1 \cap B(0, r) \neq \emptyset; N_2 \cap B(0, r + t) = \emptyset)}{\mathbb{P}(N_1 \cap B(0, r) \neq \emptyset)}$$

$$= \frac{F_{N_1}(r) - F_N(r, r + t; 0)}{F_{N_1}(r)}$$

can be expressed in terms of the joint empty space function of (N_1, N_2). The claim for the scaling limit of J_{12} follows from Theorem 6. □

For the special case $t = 0$, note that although $J_{12}(0) = 1$, in the limit, $F_{N_1}(r) - F_N(r, r; 0)$ is not necessarily equal to $F_{N_1}(r) - F_{N_1}(r)F_{N_2}(r)$ unless N_1 and N_2 are independent.

The reader may wonder why we made the strong assumption of stationarity for the "germ" point processes. The answer is that a germ–grain model built on an inhomogeneous germ process in general is not coverage-reweighted moment stationary. Consider, for example, a Boolean model (Molchanov, 1997) obtained as the union set X of closed balls of radius $r > 0$ centred at the points of a Poisson process with intensity function $\lambda(\cdot)$. For this model, the first- and second-order k-point coverage functions can be calculated explicitly and are given by

$$p_1(x) = 1 - \exp\left[-\int \lambda(z) \mathbf{1}\{z \in B(x, r)\} \, dz\right];$$

$$p_2(x, y) = p_1(x) + p_1(y) - 1 + \exp\left[-\int \lambda(z) \mathbf{1}\{z \in B(x, r) \cup B(y, r)\} \, dz\right].$$

Hence, the second-order coverage-reweighted cumulant density $\xi_2(x, y)$ is not necessarily invariant under translations, contrary to the claim by Gallego et al. (2016).

Even in the stationary case, that is, for constant $\lambda(\cdot)$, the Laplace transform $L(u_t^0) = \mathbb{E}\exp[-\ell(X \cap B(0, t))/p_1(0)]$ is intractable, being the partition function of an area-interaction process with interaction parameter $\log \gamma = 1/p_1(0)$ and range r observed in the ball $B(0, t)$ (Baddeley & van Lieshout, 1995).

4.4 | Random field models

Inhomogeneity may be introduced into the coverage measure associated to a random closed set by means of a random weight function. Let $X = (X_1, X_2)$ be a bivariate random closed set and $\Gamma = (\Gamma_1, \Gamma_2)$ a bivariate random field taking almost surely nonnegative values. Suppose that X and Γ are independent, and set $\Psi = (\Psi_1, \Psi_2)$, where

$$\Psi_i(B) = \int_B \Gamma_i(x) \mathbf{1}\{x \in X_i\} \, dx. \tag{8}$$

The univariate case was dubbed a random field model by Ballani, Kabluchko, and Schlather (2012) for which, under the assumption that both X and Γ are stationary, Koubek, Pawlas, Brereton, Kriesche, and Schmidt (2016) employed the R_{12}-function for testing purposes.

Theorem 8. *Let $\Psi = (\Psi_1, \Psi_2)$ with*

$$\Psi_i(B) = \int_B \Gamma_i(x) \mathbf{1}\{x \in X_i\} \, dx$$

as in (8) be a bivariate random field model, and suppose that Γ admits a continuous version and that its associated random measure is coverage-reweighted moment stationary. Furthermore, assume that X is stationary and has strictly positive volume fractions. Then, the

random field model is coverage-reweighted moment stationary, and writing c_{12}^X and c_{12}^Γ for the coverage-reweighted cross covariance functions of X and Γ, respectively, the following holds:

$$K_{12}(t) = \int_{B(0,t)} \left(c_{12}^X(0,x) + 1 \right) \left(c_{12}^\Gamma(0,x) + 1 \right) \, \mathrm{d}x;$$

$$J_{12}(t) = \frac{\mathbb{E}\left[\Gamma_1(0) \exp\left(-\frac{1}{\mathbb{P}(0 \in X_2)} \int_{B(0,t) \cap X_2} \frac{\Gamma_2(x)}{\mathbb{E}\Gamma_2(x)} \, \mathrm{d}x \right) \middle| \, 0 \in X_1 \right]}{\mathbb{E}\Gamma_1(0)\, \mathbb{E} \exp\left(-\frac{1}{\mathbb{P}(0 \in X_2)} \int_{B(0,t) \cap X_2} \frac{\Gamma_2(x)}{\mathbb{E}\Gamma_2(x)} \, \mathrm{d}x \right)}.$$

Proof. First, with p_k^X for the k-point coverage probabilities of X and $B_1, \ldots, B_{k+l}$ Borel subsets of $\mathbb{R}^d$,

$$\mathbb{E}\left[\Psi_1(B_1) \cdots \Psi_1(B_k) \Psi_2(B_{k+1}) \cdots \Psi_2(B_{k+l}) \right]$$

$$= \mathbb{E}\left[\int_{B_1} \cdots \int_{B_k} \int_{B_{k+1}} \cdots \int_{B_{k+l}} \left(\prod_{i=1}^k 1\{x_i \in X_1\} \Gamma_1(x_i) \, \mathrm{d}x_i \right) \left(\prod_{i=1}^l 1\{y_i \in X_2\} \Gamma_2(y_i) \, \mathrm{d}y_i \right) \right]$$

$$= \int_{B_1} \cdots \int_{B_k} \int_{B_{k+1}} \cdots \int_{B_{k+l}} p_{k+l}^X((x_1, 1), \ldots, (x_k, 1), (y_1, 2), \ldots, (y_l, 2)) \times$$

$$\times \mathbb{E}\left[\prod_{i=1}^k \Gamma_1(x_i) \prod_{i=1}^l \Gamma_2(y_i) \right] \mathrm{d}x_1 \cdots \mathrm{d}x_k \, \mathrm{d}y_1 \cdots \mathrm{d}y_l$$

by Fubini's theorem and the independence of X and Γ (recalling that the moment measures are locally finite). Hence, $\mu^{(k+l)}$ is absolutely continuous, and its Radon–Nikodym derivative p_{k+l} satisfies

$$\frac{p_{k+l}((x_1, 1), \ldots, (x_k, 1), (y_1, 2), \ldots, (y_l, 2))}{p_1(x_1, 1) \cdots p_1(x_k, 1) \, p_1(y_1, 2) \cdots p_1(y_l, 2)}$$

$$= \frac{p_{k+l}^X((x_1, 1), \ldots, (x_k, 1), (y_1, 2), \ldots, (y_l, 2))}{p_1^X(x_1, 1) \cdots p_1^X(x_k, 1) \, p_1^X(y_1, 2) \cdots p_1^X(y_l, 2)} \frac{\mathbb{E}\left[\prod_{i=1}^k \Gamma_1(x_i) \prod_{i=1}^l \Gamma_2(y_i) \right]}{\prod_{i=1}^k \mathbb{E}\Gamma_1(x_i) \prod_{i=1}^l \mathbb{E}\Gamma_2(y_i)}.$$

Here, p_{k+l}^X denotes the $k+l$-point coverage probability of X. Because X is stationary and Γ coverage-reweighted moment stationary, translation invariance follows. Moreover, the function

$$p_1(x, i) = p_1^X(x, i)\, \mathbb{E}\Gamma_i(x) = p_1^X(0, i)\, \mathbb{E}\Gamma_i(x)$$

is bounded away from zero because X has strictly positive volume fractions, and Γ is coverage-reweighted moment stationary by assumption. For $k = 2$, we have

$$\xi_2((x, 1), (y, 2)) = \frac{p_2^X((x, 1), (y, 2))}{p_1^X(x, 1)\, p_1^X(y, 2)} \frac{\mathbb{E}[\Gamma_1(x)\Gamma_2(y)]}{\mathbb{E}\Gamma_1(x)\, \mathbb{E}\Gamma_2(y)} - 1$$

from which the claimed form of the cross K-function follows. For the cross J-function, one needs the Palm distribution. By the Campbell–Mecke formula, for any Borel set $A \subset \mathbb{R}^d$, $i = 1, 2$, and for any measurable F,

$$\int_A \mathbb{P}^{(x,i)}(F)\, p_1(x, i)\, \mathrm{d}x = \mathbb{E}\left[\int_{A \cap X_i} 1_F(\Psi) \Gamma_i(x)\, \mathrm{d}x \right] = \int_A \frac{\mathbb{E}\left[1_F(\Psi) \Gamma_i(x) \mid x \in X_i \right]}{\mathbb{E}\Gamma_i(x)}\, p_1(x, i)\, \mathrm{d}x$$

by Fubini's theorem. Therefore, for p_1-almost all x and $i = 1, 2$,

$$\mathbb{P}^{(x,i)}(F) = \frac{\mathbb{E}\left[1_F(\Psi)\Gamma_i(x) \mid x \in X_i\right]}{\mathbb{E}\Gamma_i(x)},$$

and the proof is complete. □

Note that if the covariance functions of both the random closed set X and the random field Γ are nonnegative, $K_{12}(t) \geq \kappa_d t^d$; if there is nonpositive correlation, $K_{12}(t) \leq \kappa_d t^d$. Similarly, if the random variables $\Gamma_1(0)1\{0 \in X_1\}$ and

$$\int_{B(0,t) \cap X_2} \frac{\Gamma_2(x)}{\mathbb{E}\Gamma_2(x)} \, dx$$

are positively quadrant dependent, $J_{12}(t) \leq 1$ and, reversely, $J_{12}(t) \geq 1$ when they are negatively quadrant dependent.

Log-Gaussian random field model A flexible choice is to take $\Gamma_i = e^{Z_i}$ for some bivariate Gaussian random field $Z = (Z_1, Z_2)$ with mean functions m_i, $i = 1, 2$, and (valid) covariance function matrix $(c_{ij})_{i,j \in \{1,2\}}$. Because Ψ involves integrals over Γ, conditions on m_i and c_{ij} are needed. Therefore, we shall assume that m_1 and m_2 are continuous, bounded functions, for example, taking into account covariates. For the covariance function, sufficient conditions are given in theorem 3.4.1 of Adler (1981). Further details and examples can be found in Møller, Syversveen, and Waagepetersen (1998) or in section 5.8 of Møller and Waagepetersen (2004).

Theorem 9. *Consider a bivariate random field model for which Γ is log-Gaussian with bounded continuous mean functions and translation invariant covariance functions $\sigma_{ij}^2 r_{ij}(\cdot)$ such that Γ admits a continuous version. Furthermore, assume that X is stationary and has strictly positive volume fractions. Then, the random field model is coverage-reweighted moment stationary and the following holds. The cross K-function is equal to*

$$K_{12}(t) = \int_{B(0,t)} \left(1 + c_{12}^X(0, x)\right) \exp\left[\sigma_{12}^2 r_{12}(x)\right] dx,$$

where c_{12}^X is the coverage-reweighted cross covariance function of X; the cross J-function reads

$$J_{12}(t) = \frac{\mathbb{E}\left[\exp\left(Y_1(0) - \frac{1}{\mathbb{P}(0 \in X_2)}\int_{B(0,t) \cap X_2} e^{Y_2(x)} dx\right) \middle| 0 \in X_1\right]}{\mathbb{E}\exp\left[-\frac{1}{\mathbb{P}(0 \in X_2)}\int_{B(0,t) \cap X_2} e^{Y_2(x)} dx\right]}$$

$$= \frac{\mathbb{E}\left[\exp\left(-\frac{1}{\mathbb{P}(0 \in X_2)}\int_{B(0,t) \cap X_2} e^{Y_2(x) + \sigma_{12}^2 r_{12}(x)} dx\right) \middle| 0 \in X_1\right]}{\mathbb{E}\exp\left[-\frac{1}{\mathbb{P}(0 \in X_2)}\int_{B(0,t) \cap X_2} e^{Y_2(x)} dx\right]},$$

where $Y_i(x) = Z_i(x) - m_i(x) - \sigma_{ii}^2/2$.

Proof. For a log-Gaussian random field model,

$$\mathbb{E}\exp\left[\sum_{i=1}^{k} Z_1(x_i) + \sum_{i=1}^{l} Z_2(y_i)\right] = \exp\left[\sum_{i=1}^{k} m_1(x_i) + \sum_{i=1}^{l} m_2(y_i) + \frac{k}{2}\sigma_{11}^2 + \frac{l}{2}\sigma_{22}^2\right]$$

$$\times \exp\left[\sigma_{11}^2 \sum_{1 \leq i < j \leq k} r_{11}(x_j - x_i) + \sigma_{22}^2 \sum_{1 \leq i < j \leq l} r_{22}(y_j - y_i) + \sigma_{12}^2 \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq l} r_{12}(y_j - x_i)\right],$$

so that, with notation as in the proof of Theorem 8, $\mu^{(k+l)}$ is absolutely continuous, and its Radon–Nikodym derivative p_{k+l} satisfies

$$\frac{p_{k+l}((x_1,1),\ldots,(x_k,1),(y_1,2),\ldots,(y_l,2))}{p_1(x_1,1)\cdots p_1(x_k,1)p_1(y_1,2)\cdots p_1(y_l,2)}$$
$$=\frac{p_{k+l}^X((x_1,1),\ldots,(x_k,1),(y_1,2),\ldots,(y_l,2))}{p_1^X(x_1,1)\cdots p_1^X(x_k,1)p_1^X(y_1,2)\cdots p_1^X(y_l,2)}\times$$
$$\times\exp\left[\sigma_{11}^2\sum_{1\le i<j\le k}r_{11}(x_j-x_i)+\sigma_{22}^2\sum_{1\le i<j\le l}r_{22}(y_j-y_i)+\sigma_{12}^2\sum_{1\le i\le k}\sum_{1\le j\le l}r_{12}(y_j-x_i)\right].$$

Because X is stationary, translation invariance follows.

For $k=1$ and $k=2$, we have

$$p_1(x,i)=p_1^X(0,i)\exp\left[m_i(x)+\sigma_{ii}^2/2\right]$$

and

$$\xi_2((x,1),(y,2))=\frac{p_2^X((x,1),(y,2))}{p_1^X(x,1)p_1^X(y,2)}\exp\left[\sigma_{12}^2 r_{12}(y-x)\right]-1.$$

The function $p_1(x,i)$ is bounded away from zero because X has strictly positive volume fractions and the m_i are bounded. The form of the cross K-function follows from that of ξ_2, and the first expression for $J_{12}(t)$ is an immediate consequence of Theorem 8.

Finally, consider the ratio of $p_{1+k+l}((a,1),(x_1,1),\ldots,(x_k,1),(y_1,2),\ldots,(y_l,2))$ and $p_1(a,1)\prod_{i=1}^k p_1(x_i,1)\prod_{i=1}^l p_1(y_i,2)$, which can be written as

$$\frac{\mathbb{P}(x_i\in X_1,i=1,\ldots,k;y_i\in X_2,i=1,\ldots,l\mid a\in X_1)}{\prod_{i=1}^k\mathbb{P}(x_i\in X_1)\prod_{i=1}^l\mathbb{P}(y_i\in X_2)}$$
$$\times\frac{p_{k+l}^\Gamma((x_1,1),\ldots,(x_k,1),(y_1,2),\ldots,(y_l,2))}{\prod_{i=1}^k p_1^\Gamma(x_i,1)\prod_{i=1}^l p_1^\Gamma(y_i,2)}\times\prod_{i=1}^k e^{\sigma_{11}^2 r_{11}(x_i-a)}\prod_{i=1}^l e^{\sigma_{12}^2 r_{12}(y_i-a)}.$$

Hence, $L^{(a,1)}(u_t^a)$ (cf. Theorem 3) becomes the Laplace functional L evaluated for the function

$$\tilde{u}_t^a(x,i)=1\{(x,i)\in B(a,t)\times\{2\}\}\exp\left[\sigma_{12}^2 r_{12}(x-a)\right]/p_1(x,2)$$

after conditioning on $a\in X_1$, an observation that completes the proof. □

In the context of a point process, Coeurjolly, Møller, and Waagepetersen (2017) proved the stronger result that the Palm distribution of a log-Gaussian Cox process is another log-Gaussian Cox process.

Random thinning field model Consider the following random field model (Diggle, 2014) with intercomponent dependence modelled by means of a (deterministic) nonnegative function $r_i(x)$, $i=1,2$, on $\mathbb{R}^d$ such that $r_1+r_2\equiv 1$. Let Γ_0 be a nonnegative random field, and assume that the components $\Gamma_i(x)=r_i(x)\Gamma_0(x)$ are integrable on bounded Borel sets. As before, X is a stationary bivariate random closed set, and a random measure is defined through (8). Heuristically speaking, the $r_i(x)$ can be thought of as location-dependent retention probabilities for X_i.

For the model just described,

$$1+c_{12}^\Gamma(0,x)=\frac{\mathbb{E}\left[\Gamma_0(0)\Gamma_0(x)\right]}{\mathbb{E}\Gamma_0(0)\,\mathbb{E}\Gamma_0(x)}=1+c^{\Gamma_0}(0,x)$$

and similarly for higher orders so that Γ is coverage-reweighted moment stationary precisely when Γ_0 is. Hence, Theorem 8 holds with the Γ_i replaced by Γ_0.

5 | ESTIMATION

For notational convenience, introduce the random measure $\Phi = (\Phi_1, \Phi_2)$ defined by

$$\Phi_i(A) = \int_A \frac{1}{p(x, i)} \, \mathrm{d}\Psi_i(x)$$

for Borel sets $A \subset \mathbb{R}^d$.

Theorem 10. *Let $\Psi = (\Psi_1, \Psi_2)$ be a coverage-reweighted moment stationary bivariate random measure that is observed in a compact set $W \subset \mathbb{R}^d$ whose erosion $W_{\ominus t} = \{w \in W : B(w, t) \subset W\}$ has positive volume $\ell(W_{\ominus t}) > 0$. Then, under the assumptions of Theorem 3,*

$$\widehat{L_2(t)} = \frac{1}{\ell(W_{\ominus t})} \int_{W_{\ominus t}} e^{-\Phi_2(B(x,t))} \, \mathrm{d}x \tag{9}$$

is an unbiased estimator for $L(u_t^0)$,

$$\widehat{K_{12}(t)} = \frac{1}{\ell(W_{\ominus t})} \int_{W_{\ominus t}} \Phi_2(B(x, t)) \, \mathrm{d}\Phi_1(x) \tag{10}$$

is an unbiased estimator for $K_{12}(t)$, and

$$\widehat{L_{12}(t)} = \frac{1}{\ell(W_{\ominus t})} \int_{W_{\ominus t}} e^{-\Phi_2(B(x,t))} \, \mathrm{d}\Phi_1(x) \tag{11}$$

is unbiased for $L^{(0,1)}(u_t^0)$.

Proof. First, note that for all $x \in W_{\ominus t}$, the mass $\Phi_2(B(x, t))$ can be computed from the observation because $B(x, t) \subset W$. Moreover,

$$\mathbb{E}\left[e^{-\Phi_2(B(x,t))}\right] = L(1_{B(x,t) \times \{2\}}(\cdot)/p_1(\cdot))$$

regardless of x by an appeal to Theorem 3. Consequently, (9) is unbiased.

Turning to (11), by (2) with

$$g((x, i), \Psi) = \frac{1_{W_{\ominus t} \times \{1\}}(x, i)}{p_1(x, i)} \exp[-\Phi_2(B(x, t))],$$

we have

$$\ell(W_{\ominus t})\mathbb{E}\widehat{L_{12}(t)} = \int_{W_{\ominus t}} \frac{L^{(x,1)}(1_{B(x,t) \times \{2\}}(\cdot)/p_1(\cdot))}{p_1(x, 1)} p_1(x, 1) \, \mathrm{d}x.$$

Because $L^{(x,1)}(1_{B(x,t) \times \{2\}}(\cdot)/p_1(\cdot))$ does not depend on x by Theorem 3, the estimator is unbiased. The same argument for

$$\tilde{g}((x, i), \Psi) = \frac{1_{W_{\ominus t} \times \{1\}}(x, i)}{p_1(x, i)} \Phi_2(B(x, t))$$

proves the unbiasedness of $\widehat{K_{12}(t)}$. $\qquad \square$

A few remarks are in order. In practise, the integrals will be approximated by Riemann sums. Moreover, in accordance with the Hamilton principle (Stoyan & Stoyan, 2000), the denominator

$\ell(W_{\ominus t})$ in $\widehat{K_{12}(t)}$ and $\widehat{L_{12}(t)}$ can be replaced by $\Phi_1(W_{\ominus t})$. Finally, we assumed that the coverage function is known. If this is not the case, a plug-in estimator may be used.

6 | THEORETICAL EXAMPLES

In this section, we illustrate the use of our statistics on simulated realisations of some of the models discussed in Section 4.

Widom–Rowlinson germ–grain model First, consider the Widom–Rowlinson (1970) germ–grain model defined as follows. Let (N_1, N_2) be a bivariate point process whose joint density with respect to the product measure of two independent unit rate Poisson processes is

$$f_{\mathrm{mix}}(\phi_1, \phi_2) \propto \beta_1^{|\phi_1|} \beta_2^{|\phi_2|} 1\{d(\phi_1, \phi_2) > r\},$$

writing $|\cdot|$ for the cardinality and $d(\phi_1, \phi_2)$ for the smallest distance between a point of ϕ_1 and one of ϕ_2. In other words, points of different components are not allowed to be within distance r of one another. Placing closed balls of radius $r/2$ around each of the points yields a bivariate random closed set, the Widom–Rowlinson germ–grain model. It is important to note that N_i, $i = 1, 2$, cannot, in general, be reconstructed from $U_r(N_i)$, because some discs could be hidden behind others (van Lieshout, 1997).

A sample from the germ process can be obtained by coupling from the past (Häggström, van Lieshout, & Møller, 1999; Kendall & Møller, 2000; van Lieshout & Stoica, 2006). We used the `mpplib` library (Steenbeek, van Lieshout, & Stoica, 2002) to generate a realisation with $\beta_1 = \beta_2 = 1$ and $r = 1$ on $W = [0, 10] \times [0, 20]$. To avoid edge effects, we sampled on $[-1, 11] \times [-1, 21]$ and clipped the result to W. Grains in the form of a ball of radius $r/2$ were then placed around the germs to obtain a realisation from the germ–grain model, an example of which is shown in Figure 1. Note that

$$U_{r/2}(\phi_1) \cap U_{r/2}(\phi_2) = \emptyset,$$

so that there is a negative association between the two components as illustrated in Figure 2.

The estimated cross statistic for 20 samples are shown in Figure 3. The graphs of $\widehat{J_{ij}(t)}$ lie above one, reflecting the inhibition between the components. The graphs of $\widehat{K_{ij}(t)}$ lie below that of the function $t \to \pi t^2$, which confirms the negative correlation between the components.

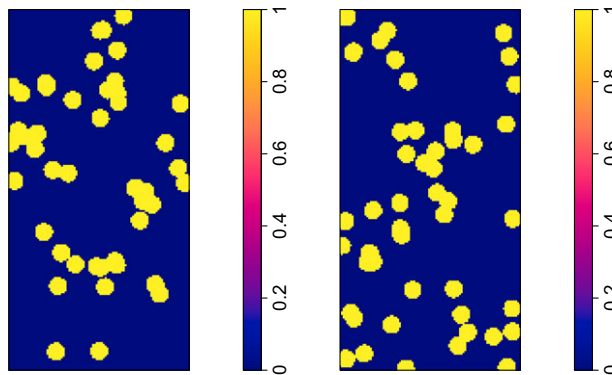**FIGURE 1** Images of $U_{r/2}(\phi_1)$ (left) and $U_{r/2}(\phi_2)$ (right) for a realisation (ϕ_1, ϕ_2) of the Widom–Rowlinson germ process with $\beta_1 = \beta_2 = 1$ on $W = [0, 10] \times [0, 20]$ for $r = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]
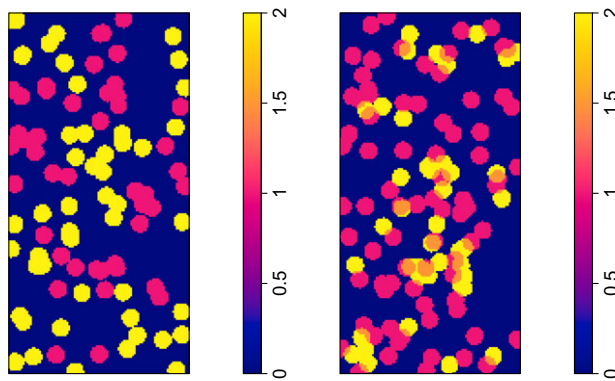
FIGURE 2 Superposition image of the components in Figure 1 (left) and Figure 4 (right). In red: $U_{r/2}(\phi_1)\backslash U_{r/2}(\phi_2)$; in yellow: $U_{r/2}(\phi_2)\backslash U_{r/2}(\phi_1)$; in orange: $U_{r/2}(\phi_2) \cap U_{r/2}(\phi_1)$ [Colour figure can be viewed at wileyonlinelibrary.com]
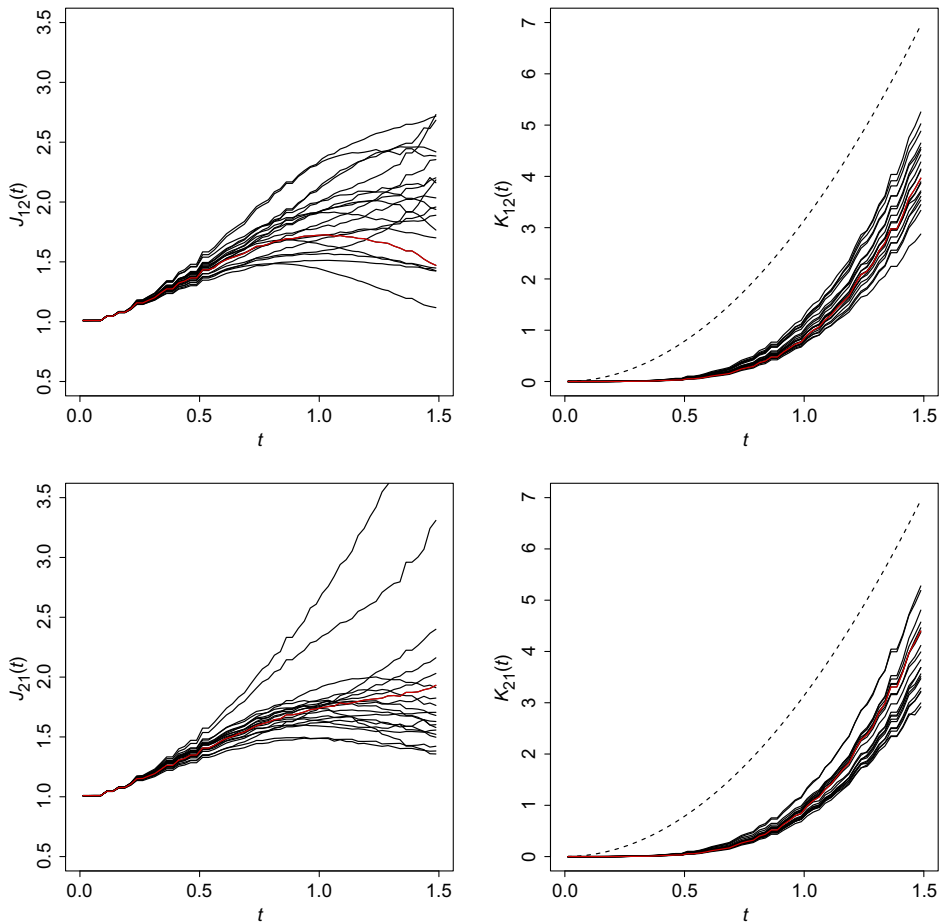

FIGURE 3 Estimated cross statistics for 20 samples from the Widom–Rowlinson germ–grain model with parameters as in Figure 1. Top row: $\widehat{J_{12}(t)}$ plotted against t (left); $\widehat{K_{12}(t)}$ (solid) and πt^2 (dotted) plotted against t (right). Bottom row: $\widehat{J_{21}(t)}$ plotted against t (left); $\widehat{K_{21}(t)}$ (solid) and πt^2 (dotted) plotted against t (right). The graphs for the data shown in Figure 1 are displayed in red [Colour figure can be viewed at wileyonlinelibrary.com]
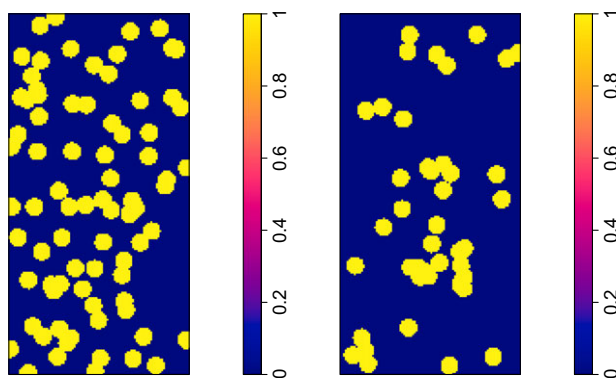
FIGURE 4　Images of $U_{r/2}(\phi_1)$ (left) and $U_{r/2}(\phi_2)$ (right) for a realisation (ϕ_1, ϕ_2) of the dual Widom–Rowlinson germ process with $\beta_1 = \beta_2 = 1/4$ on $W = [0, 10] \times [0, 20]$ for $r = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

Dual Widom–Rowlinson germ–grain model The dual Widom–Rowlinson germ–grain model is based on a bivariate point process with joint density

$$f_{\text{mix}}(\phi_1, \phi_2) \propto \beta_1^{n(\phi_1)} \beta_2^{n(\phi_2)} 1\{\phi_2 \subset U_r(\phi_1)\}$$

with respect to the product measure of two independent unit rate Poisson processes. Because the germs of the second component lie in $U_r(\phi_1)$, that is, within distance r of a germ from the first component, the model exhibits positive association. Placing balls of radius $r/2$ around the components yields a germ–grain model. Again, in general, the N_i, $i = 1, 2$, cannot be reconstructed from $U_r(N_i)$ due to occlusion of a grain by other grains.

Exact samples from this model can be obtained in three steps. First, generate an area-interaction point process (Baddeley & van Lieshout, 1995) with parameter β_1 and $\gamma = e^{-\beta_2}$ using coupling from the past (Kendall & Møller, 2000) by means of the `mpplib` (2002–2003) library. Then, conditionally on the first component being ϕ_1, generate a Poisson process of intensity β_2 and accept only those points that fall in $U_r(\phi_1)$ to obtain ϕ_2. Finally, place balls of radius $r/2$ around the points of ϕ_1 and ϕ_2.

Figure 4 shows a realisation of the components $U_{r/2}(\phi_1)$ and $U_{r/2}(\phi_2)$ for $\beta_1 = \beta_2 = 1/4$ and $r = 1$ on $W = [0, 10] \times [0, 20]$. To avoid edge effects, we sampled the germs on $[-1, 11] \times [-1, 21]$ and clipped the result to W. The positive association due to the tendency of germs in ϕ_2 to fall near germs in ϕ_1 is inherited by the germ–grain model. Indeed, in Figure 2, the clumps of yellow balls tend to overlap a red clump. The interaction is not symmetric though, and one may find red clumps without an overlapping yellow cluster of balls.

The estimated cross statistics for 20 samples are shown in Figure 5. The graphs of $\widehat{J_{ij}(t)}$ lie below one, reflecting the attraction between the components. It is interesting to note that the lack of symmetry in the model is reflected in a stronger attraction from the perspective of the second component than from that of the first. The graphs of $\widehat{K_{ij}(t)}$ lie above that of the function $t \to \pi t^2$, which confirms the positive correlation between the components.

Boolean model marked by linked log-Gaussian field Our last illustrations concern random field models based on Gaussian random fields. Thus, let Γ_0 be a Gaussian random field with mean function $m(\cdot)$ and exponential covariance function

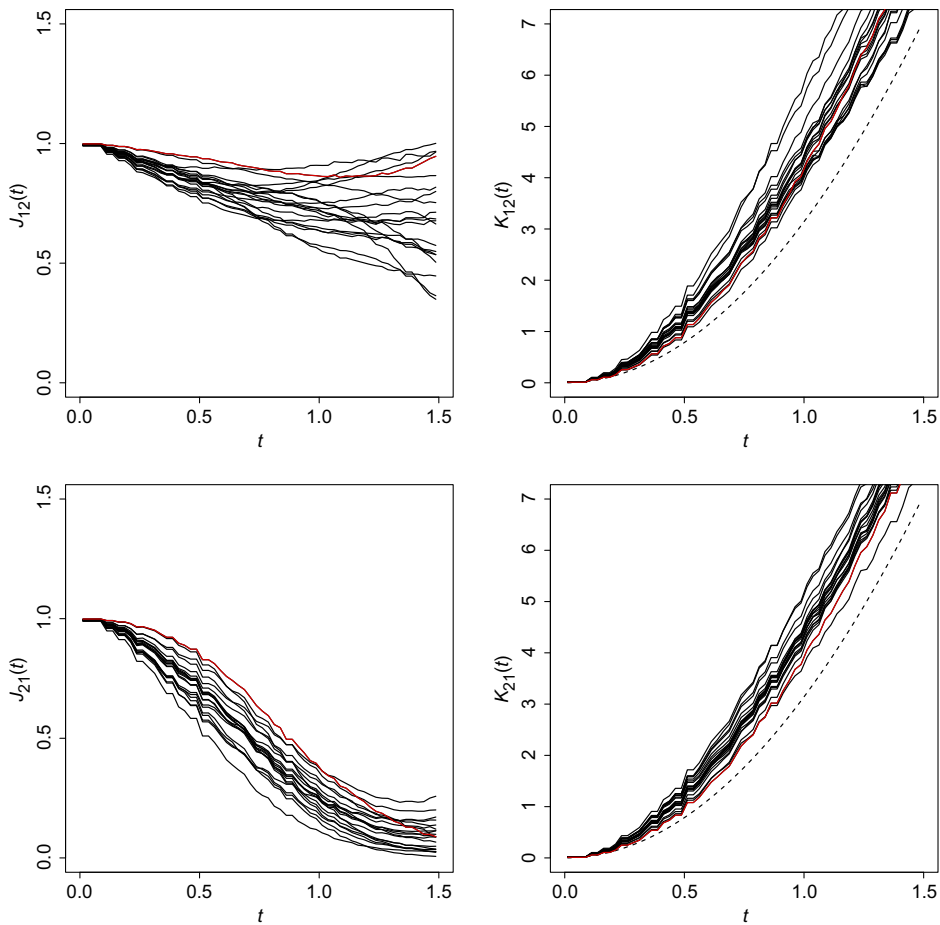$$\sigma^2 \exp[-\beta \|x - y\|]. \tag{12}$$

FIGURE 5 Estimated cross statistics for 20 samples from the dual Widom–Rowlinson germ–grain model with parameters as in Figure 4. Top row: $\widehat{J_{12}(t)}$ plotted against t (left); $\widehat{K_{12}(t)}$ (solid) and πt^2 (dotted) plotted against t (right). Bottom row: $\widehat{J_{21}(t)}$ plotted against t (left); $\widehat{K_{21}(t)}$ (solid) and πt^2 (dotted) plotted against t (right). The graphs for the data shown in Figure 4 are displayed in red [Colour figure can be viewed at wileyonlinelibrary.com]

The package `fields` (Nychka, Furrer, Paige, & Sain, 2015) can be used to obtain approximate realisations. An example on $W = [0, 10] \times [0, 20]$ with

$$m(x, y) = \frac{x + y}{10}$$

and parameters $\sigma^2 = 1$, $\beta = 0.8$ viewed through independent Boolean models is depicted in Figure 6. More precisely, for a linked random field model, let (X_1, X_2) consist of two independent stationary Boolean models with balls as primary grains, and set

$$(\Psi_1, \Psi_2) = \left(\int_{X_1} e^{\Gamma_0(x)} dx, \int_{X_2} e^{\Gamma_0(x)} dx \right).$$

Here, the common random field, although viewed through independent prisms, causes positive association between the components of Ψ.

The estimated cross statistics for 20 samples are shown in Figure 8 for Γ_0 as in Figure 6 and Boolean models having germ intensity $1/2$ and grain radius $r = 1/2$. The graphs of $\widehat{J_{ij}(t)}$ lie
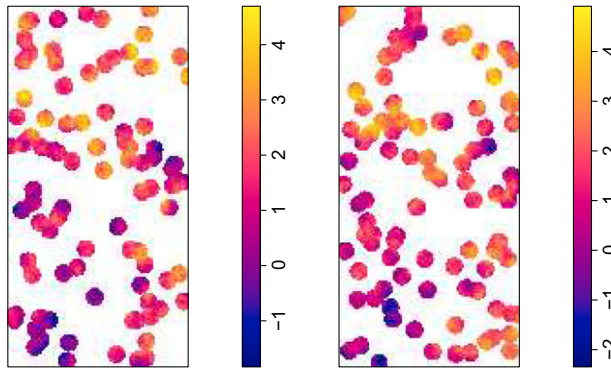
FIGURE 6 Images of a Gaussian random field on $W = [0, 10] \times [0, 20]$ with mean function $m(x, y) = (x + y)/10$ and covariance function $\sigma^2 \exp[-\beta\| \cdot \|]$ for $\beta = 0.8$ and $\sigma^2 = 1$ viewed through independent Boolean models X_1 (left) and X_2 (right) with germ intensity $1/2$ and primary grain radius $1/2$ [Colour figure can be viewed at wileyonlinelibrary.com]

below one for small t, reflecting the attraction between the components. The graphs of $\widehat{K_{ij}(t)}$ mostly lie above that of the function $t \rightarrow \pi t^2$, which confirms the positive correlation between the components.

An example of a random thinning field on $W = [0, 10] \times [0, 20]$ with

$$1 - r_2(x, y) = r_1(x, y) = \frac{y}{20}$$

applied to $\exp[\Gamma_0(\cdot)]$, with Γ_0 having mean zero and covariance function (12) for $\sigma^2 = 1$ and $\beta = 0.8$, and X consisting of independent Boolean models as described above is shown in Figure 7. Note that the first component of the corresponding random measure Ψ tends to place larger mass towards the top of W (left panel), whereas the second component tends to place its mass near the bottom (right panel of Figure 7).

Although the first-order structures—as displayed in Figures 7 and 6—of the random thinning field and the linked random field model are completely different, their interaction structures coincide and so do their cross statistics (cf. Figure 8).
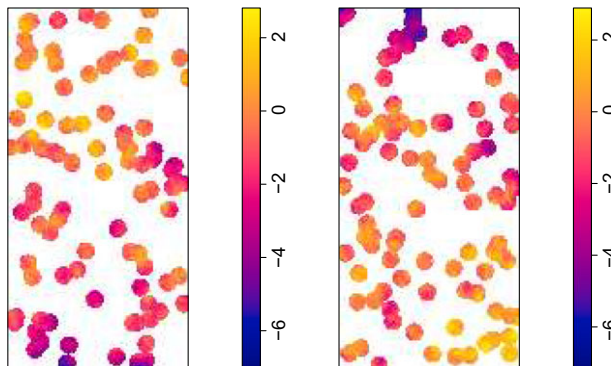
FIGURE 7 Images of realisations $(\log \psi_1, \log \psi_2)$ of a random thinning field model (Ψ_1, Ψ_2) on $W = [0, 10] \times [0, 20]$ with $r_1(x, y) = y/20$, $\log \Gamma_0$ a mean zero Gaussian random field with covariance function $\sigma^2 \exp[-\beta\| \cdot \|]$ for $\beta = 0.8$ and $\sigma^2 = 1$, and X as in Figure 6 [Colour figure can be viewed at wileyonlinelibrary.com]
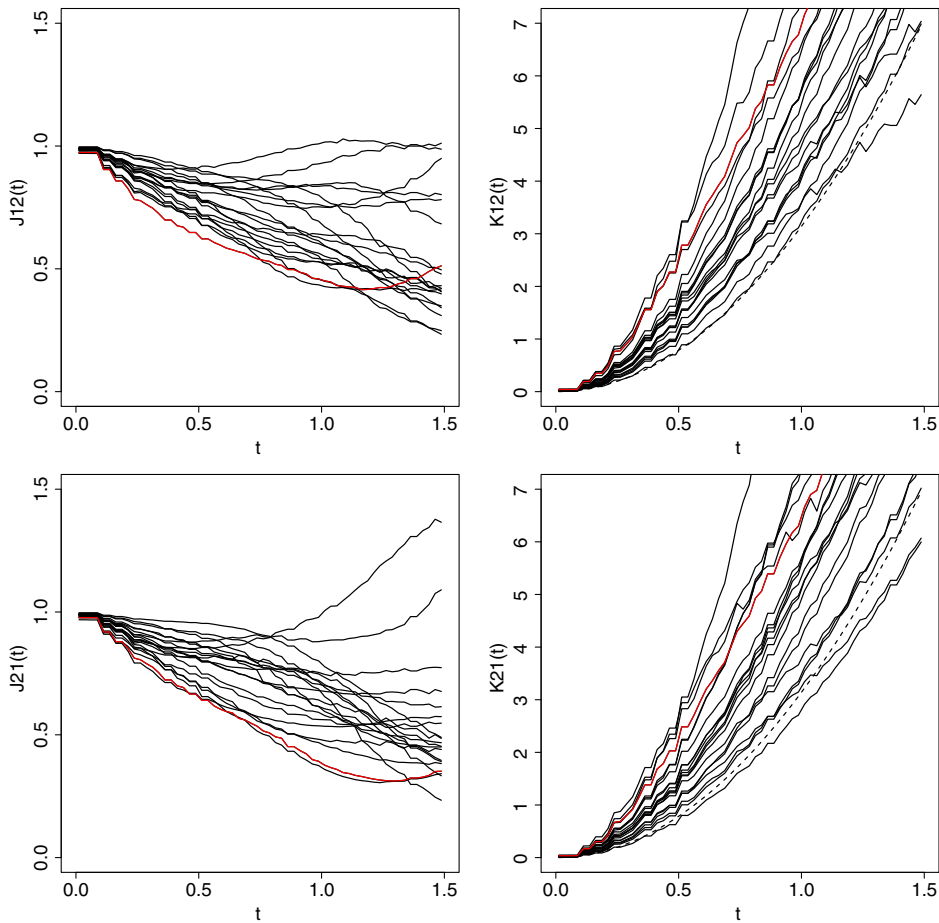
FIGURE 8 Estimated cross statistics for 20 samples from a random field model on $W = [0, 10] \times [0, 20]$ defined by X and Γ as follows: $\Gamma_1(x) = \Gamma_2(x) = \exp[Z(x)]$ where Z is a Gaussian random field with mean function $m(x, y) = (x + y)/10$ and covariance function $\sigma^2 \exp[-\beta \|\cdot\|]$ for $\beta = 0.8$ and $\sigma^2 = 1$; the components of X are independent Boolean models with germ intensity $1/2$ and primary grain radius $1/2$, cf. Figure 6. Top row: $\widehat{J_{12}(t)}$ plotted against t (left); $\widehat{K_{12}(t)}$ (solid) and πt^2 (dotted) plotted against t (right). Bottom row: $\widehat{J_{21}(t)}$ plotted against t (left); $\widehat{K_{21}(t)}$ (solid) and πt^2 (dotted) plotted against t (right). The graphs for the data shown in Figure 6 are displayed in red [Colour figure can be viewed at wileyonlinelibrary.com]

7 | APPLICATION TO ECOLOGY

The stochastic geometry of biodiversity according to McGill (2010) relies on three axioms: clustering within a species, the coexistence of many rare species with a few common ones, and independence between species. The third axiom is addressed by Wiegand et al. (2012) who noted that in this context, classic point process analyses of bivariate spatial patterns are challenging "because they require complete mapping and because of difficulties in teasing apart two major, yet contrasting factors: habitat association and direct species interactions." Moreover, plots of a bivariate point pattern with tens of thousands of trees are hard to interpret visually. The authors proceed by a Monte Carlo approach based on *homogeneous* summary statistics and random *local* translations of the points of the second component to show that species-rich forests approximate

species independence. Still, due to the erroneous stationarity assumption, this approach cannot fully separate heterogeneity and interaction; the statistics proposed in this paper can. Indeed, they do not require stationarity, and the coverage reweighting accounts explicitly for habitat association. Moreover, being based on random measures, they only require smoothed quadrat counts instead of a full map.

As an illustration, let us consider data on the spatial distribution of stems of a large number of woody trees and shrub species measuring at least 1 cm in diameter found in a 50-hectare plot on Barro Colorado Island, Panama (Hubbell & Foster, 1983; cf. https://dx.doi.org/10.5479/data. bci.20130603) and analysed in, amongst others, Volkov, Banavar, Hubbell, and Maritan (2009); Wiegand et al. (2012); Waagepetersen, Guan, Jalilian, and Mateu (2016). Wiegand et al. (2012) reported that the species *Oenocarpus mapora* and *Trichilia tuberculata* displayed the most significant interactions with other species, mostly negative association. An explicit list of large negative covariances compiled using entropy methods is given in Volkov et al. (2009) and includes the pair *Trichilia tuberculata* and *Mouriri myrtilloides*.

Based on the considerations outlined above, we restrict ourselves from now on to the stems of *Mouriri myrtilloides* and *Trichilia tuberculata* that were alive during the seventh census in 2010. After weeding out multiple stems belonging to the same tree, we are left with a multivariate point pattern of which the first component contains 7,241 trees, and the second one 11,293 points. Because in a mapped pattern of this size, no details are visible, we use the random measure framework and show the smoothed abundance plots normalised into a probability density in Figure 9. A Gaussian smoother was used with a bandwidth chosen according to the rule of thumb proposed by Cronie and van Lieshout (2018). To estimate the coverage function p_1, we use the data on trees that had died by 2010. The result is shown in Figure 10. One observes that the *Trichilia* exhibit a strong preference for the top right corner of the stand, a region little favoured by *Mouriri* shrubs who prefer the left half of the stand.
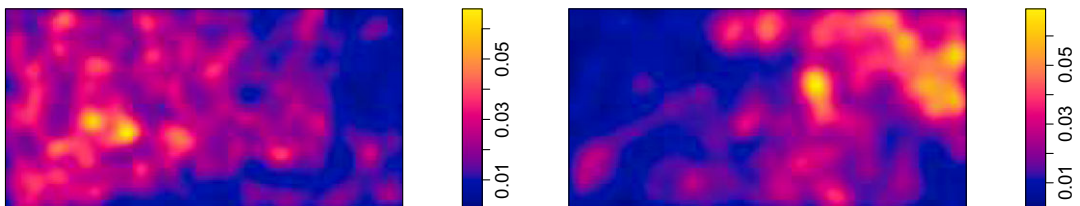

FIGURE 9 Normalised abundance plots of alive trees of *Mouriri myrtilloides* (left) and *Trichilia tuberculata* (right) in a $[0, 10] \times [0, 5]$ units of 100-m region based on the seventh census in 2010 [Colour figure can be viewed at wileyonlinelibrary.com]
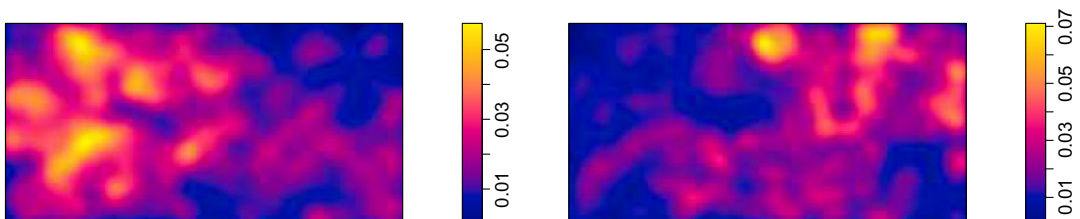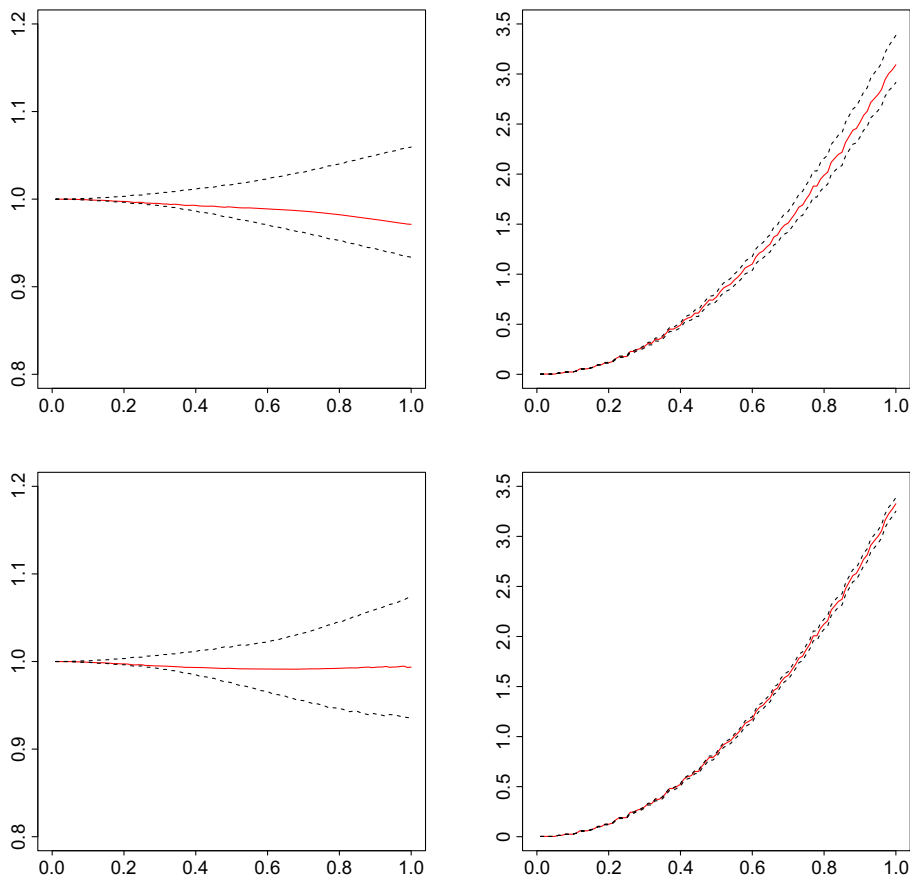

FIGURE 10 Normalised abundance plots of dead trees of *Mouriri myrtilloides* (left) and *Trichilia tuberculata* (right) in a $[0, 10] \times [0, 5]$ units of 100-m region based on the seventh census in 2010 [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 11 Estimated cross statistics for the Barro Colorado data. Top row: The solid red lines are the graphs of $\widehat{J_{12}(t)}$ (left) and $\widehat{K_{12}(t)}$ (right) plotted against t. Bottom row: The solid red lines are the graphs of $\widehat{J_{21}(t)}$ (left) and $\widehat{K_{21}(t)}$ (right) plotted against t. The dotted lines are envelopes based on 99 torus translations of the second component (*Trichilia tuberculata*) [Colour figure can be viewed at wileyonlinelibrary.com]

The estimated cross statistics are plotted in Figure 11 together with upper and lower envelopes of 99 torus translations of the second component in the spirit of Lotwick and Silverman (1982) and Cronie and van Lieshout (2016). Because the graphs of $\widehat{K_{ij}(t)}$ and $\widehat{J_{ij}(t)}$ fall within the envelopes, we find no indication of negative association between the two species. Therefore, we conclude that a failure to take into account spatial inhomogeneity leads to misleading conclusions. Moreover, our analysis provides strong empirical evidence for McGill's theory of biodiversity.

To conclude this section, note that there are many more species, and further analysis should be model based. One step in this direction is, for example, the recent paper of Waagepetersen et al. (2016), but the field of inference for many variate random measures is still in its infancy and requires further work to mature.

8 | CONCLUSION

In this paper, we introduced summary statistics to quantify the correlation between the components of coverage-reweighted moment stationary bivariate random measures inspired by the

F-, *G*-, and *J*-function for point processes (Cronie & van Lieshout, 2016; van Lieshout, 2011; van Lieshout & Baddeley, 1999). The role of the generating functional in these papers is taken over by the Laplace functional and that of the product densities by the coverage functions. Our statistics can also be seen as generalisations of the correlation measures introduced by Stoyan and Ohser (1982) for stationary random closed sets.

To the best of our knowledge, such cross statistics for inhomogeneous marked sets have not been proposed before. Under the strong assumption of stationarity, however, some statistics were suggested. Foxall and Baddeley (2002) defined a cross *J*-function for the dependence of a random closed set *X*—a line segment process in their geological application—on a point pattern *Y* by

$$J(t) = \frac{\mathbb{P}^0(d((0, X) > t)}{\mathbb{P}(d(0, X) > t)},$$

where $\mathbb{P}^0$ is the Palm distribution of *Y*, whereas Kleinschroth, van Lieshout, Mortier, and Stoica (2013) replaced the numerator by

$$\mathbb{P}^{(0,i)}(\Psi_j(B(0, t)) = 0)$$

for the random length-measures Ψ_j associated to a bivariate line segment process. It is not clear, though, how to generalise the resulting statistics to nonhomogeneous models, as the moment measure of the random length-measure may not admit a Radon–Nikodym derivative.

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