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Robustness of power-law behavior in cascading line failure models

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ABSTRACT
Inspired by reliability issues in electric transmission networks, we use a probabilistic approach to study the occurrence of large failures in a stylized cascading line failure model. Such models capture the phenomenon where an initial line failure potentially triggers massive knock-on effects. Under certain critical conditions, the probability that the number of line failures exceeds a large threshold obeys a power-law distribution, a distinctive property observed in empiric blackout data. In this paper, we examine the robustness of the power-law behavior by exploring under which conditions this behavior prevails.

1. Introduction

Cascading failure models are used to describe systems of interconnected components where initial failures possibly trigger subsequent failures. Despite the deceptively simple nature, these models capture essential features of failure processes in many settings. The abstract nature allows for a wide range of applications, such as material science, traffic networks and earthquake dynamics[1]. Our main inspiration comes from major power outages in energy networks.

Due to various advances, such as the rise of renewable sources, the complexity and volatility in power transmission systems has increased tremendously in the last 15 years[2]. Large blackouts of electric power transmission systems have catastrophic consequences in modern-day society. Examples include the Northeast Blackout of 2003, the India Blackout of 2012 and the Turkey Blackout of 2015. The analysis of severe blackouts has therefore become a crucial part of transmission grid planning and operations[3,4]. Cascading failure is a key mechanism in the occurrence of severe blackouts[5]. Typically the cascading phenomenon involves long and quite complex sequences of line failures, making the evaluation of the failure propagation extremely difficult.

A possible method for analyzing involved and complex cascading failure models is a rare-event simulation methodology, such as importance sampling and splitting[4,6,7]. Importance sampling is an approach that changes the sampling distribution to make the rare event more probable, and thus easier to estimate. Re-scaling then recovers the correct probability. However, choosing an appropriate sampling distribution requires specific knowledge about the model. On the other hand, the splitting method is based on exploring system states that make the rare-event more likely. The idea is to create multiple copies for these settings in order to generate more occurrences of the event. For practical purposes, this technique is particularly advantageous as it allows for an analysis of fairly complex systems without the need of a deeper understanding of the model dynamics. We refer the reader to Ref.[8] for an excellent literature overview of rare-event simulation techniques applied to power systems.

Although the splitting method has proven to be indispensable in analyzing cascading power failures, they provide little structural insights in the cascading failure mechanism in severe blackouts. There is a strong need for a deeper, more fundamental understanding of the current power grid[9]. Therefore, we take a macroscopic view by capturing some distinctive features of severe blackouts in electric transmission systems. By taking a high-level approach, we capture the major features leading to severe blackouts while preserving the tractability of the model. This allow for a mathematical approach that justifies the results in a rigorous way. The central insights may then become a useful complement to more detailed models and simulations.

In this paper, we consider a stochastic load-dependent cascading line failure model. Specifically, we consider a star-topology where load demands, imposed on the lines, are initially exceeded by the line capacities. A possible cascading failure effect is initiated by a disturbance that additionally loads all lines. When the load demand outstrips the capacity on a particular line, it fails. Each line failure induces changes in the load distribution in the surviving network, possibly causing further lines to trip in succession and triggering knock-on effects. The cascading failure propagation continues until each line in the surviving network has enough capacity to meet its load demand again. Since the evolution of power system operation, upgrade, maintenance and design is much slower than a blackout cascade, it is reasonable to assume a fixed system during the progression of any particular cascade[10]. That is, the capacities remain fixed throughout the cascading failure process. A detailed description of the model is given in Section 2.

Historically, empirical data analyses of large blackouts in North America show that the blackout size is heavy-tailed and has a power-law dependence[11]. Work in Refs.[12–14] suggests that there is a critical loading regime where the blackout size follows a power-law distribution, which indicates that the tail decay rate is much slower than exponential. This heavy-tailed property reflects a significant risk of large blackouts occurring.

Power-law behavior appears in a wide range of contexts and applications. On one hand, motivated by empirical data, there are many models where the basic
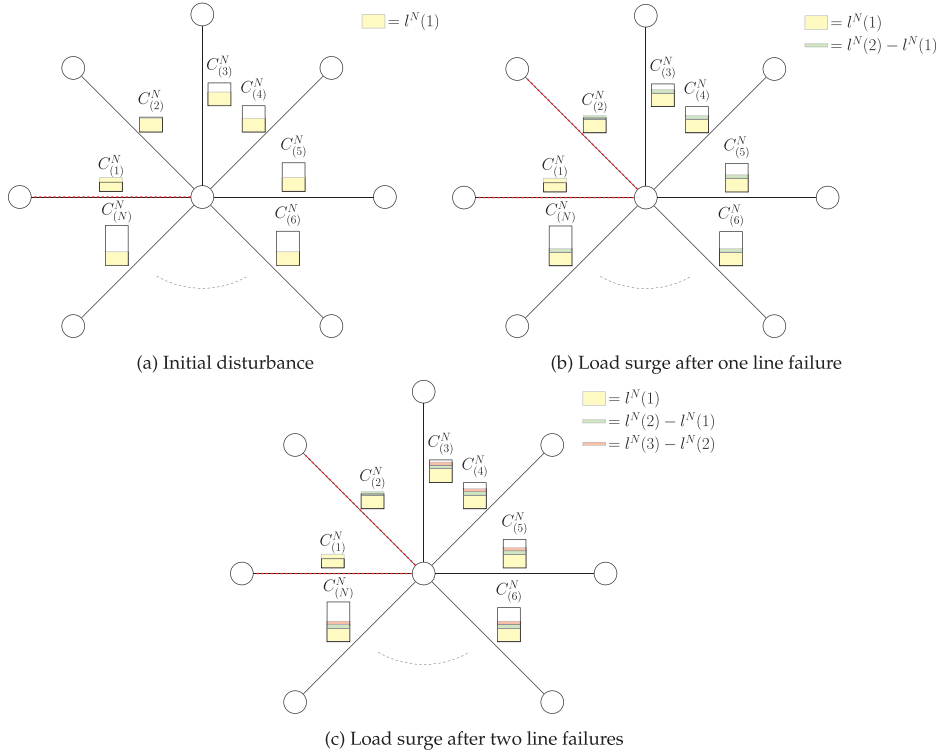
model primitives are assumed to follow a power-law distribution. Well-known examples include insurance risk models with heavy-tailed claim sizes[15], queues with heavy-tailed traffic characteristics[16] and random graphs with power-law degree distributions[17]. On the other hand, power-law dependence potentially arises from intrinsic model dynamics, even when the underlying random variables do not necessarily have heavy-tailed distributions. For instance, power-law behavior appears as the solution of random equations of multiplicative nature[18,19].

In the present paper, we are specifically interested in understanding the fundamental mechanism that leads to such power-law behavior in cascading failure models. We quantify the risk of cascading failures by the probability that the number of failed lines exceeds some threshold. The main starting point is the analytically tractable failure model introduced by Dobson et al.[5,11], where power-law behavior is observed for the blackout size under critical conditions. It turns out that this model corresponds to a particular setting in our framework. We study the robustness of the power-law behavior for this setting, and extend the results in two directions. First, we investigate whether the power-law behavior prevails when the threshold depends on the network size under similar assumptions on the surplus capacities and the load surge function as in Ref.[5] This extension provides a rigorous justification for approximations such as the probability that the network partially breaks down, e.g. the probability that at least 20% of all lines fail. Second, we allow for a wider range of settings where the power-law distribution for the number of line failure is preserved. This is outlined in the main result of our paper stated in Theorem 3.2. Whether a setting falls within our framework depends on three factors, namely the distribution of the difference between the capacity and the initial load demand, the load surge after every failure and the threshold itself. We conclude by considering particular examples and identifying the possible thresholds that yield power-law behavior.

The remainder of this paper is organized as follows. In Section 2, we describe the cascading failure model. We explain our main results in detail in Section 3, and defer the proofs to the Appendices. In Section 4, we consider illustrative examples that identify thresholds k where the power-law behavior prevails. We present a few concluding remarks and discuss possible directions for further research in Section 5.

2. Model description

We consider a star-topology consisting of N lines, see Figure 1. Each line has a limited capacity for the amount of load it can carry before it trips. To simplify the model, we consider a setting where all lines are statistically indistinguishable. We assume that the network is initially stable, i.e. all lines have capacities that exceed their initial load. We observe that what matters is the difference between load demands and line capacities, which we refer to as the *surplus capacity*. We assume that the differences between the initial load demands and capacities are random variables that are independent and identically distributed, and denote these by C_i^N for line i. Moreover, we assume that its distribution function $F(\cdot)$ is continuous with

(a) Initial disturbance

(b) Load surge after one line failure

(c) Load surge after two line failures

Figure 1. Illustration of cascading line failure process with $A_N = 2$.

a strictly positive density in zero. We will see that the continuity assumption ensures we can use the theory of order statistics in our analysis. The reason for assuming a strictly positive density will become apparent in Section 4.

In order to trigger a possible cascading failure effect, we include an initial disturbance that causes all lines to be additionally loaded. When the capacity on a line is exceeded by its load demands, that line fails. Every line failure causes an additional loading of the surviving lines in the network, which we refer to as the *load surge*. We assume that the load surge can be described through a deterministic non-decreasing function. We write $l^N(1)$ for the initial load surge on all lines, and $l^N(i)$ for the total load surge on the surviving lines when $i - 1$ lines have failed. That is, all randomness is captured by the differences between the initial load demands and the line capacities, while the load surges are deterministic. This is a modeling choice that serves to create a setting with a single source of randomness.

Our model does not explicitly account for many complexities that exist in real electric power transmission systems, such as the length of time between failure occurrences or the network topology that can lead to non-identical line capacity distributions or non-equal load distribution. Yet, this model does capture two important features of large blackouts: the initial disturbance loading the system and the cascading line failure mechanism.

The main objective in this paper involves the probability that A^N, the number of failed lines in the network, exceeds a certain threshold k as N grows large. To

express this in mathematical terms, we take a closer look at the cascading failure process. After the dummy line has tripped, a next line will fail when the smallest surplus capacity is exceeded by the load surge $l^N(1)$. If so, another line will fail if and only if $l^N(2)$ exceeds the second smallest surplus capacity and so forth. Denote by $C^N_{(1)} \leq C^N_{(2)} \leq \cdots \leq C^N_{(N)}$ the ordered surplus capacities. The above observation yields that the blackout size is given by

$$A^N = \max\left\{k \leq N : C^N_{(i)} \leq l^N(i), \quad i = 1, \ldots, k\right\} \tag{1}$$

and $A^N = 0$ if $C^N_{(1)} > l^N(1)$. The probability that the blackout size exceeds an integer k is thus given by

$$\mathbb{P}(A^N \geq k) = \mathbb{P}\left(C^N_{(i)} \leq l^N(i), \quad i = 1, \ldots, k\right). \tag{2}$$

Equation (2) can be rewritten into an expression that is easier to analyze. Let $U^N_{(i)}$ denote the standard uniformly distributed ordered statistics for $i = 1, \ldots, N$. Since $F(\cdot)$ is continuous, it follows from (2.4.1) in Ref.[20] that $(F(C^N_{(i)}))_{i=1,\ldots,N}$ and $(U^N_{(i)})_{i=1,\ldots,N}$ are equal in distribution. Therefore, (2) is equivalent to

$$\mathbb{P}(A^N \geq k) = \mathbb{P}\left(U^N_{(i)} \leq F(l^N(i)), \quad i = 1, \ldots, k\right), \tag{3}$$

where $F \circ l^N$ is a non-decreasing function in the number of failed lines with values in $[0, 1]$. Much is known on uniformly distributed order statistics (see e.g. Section 4.7 in Ref.[20]), such as the density of every order statistic. In turn, this property can be exploited to derive the asymptotic behavior of (3) in our framework.

Our framework can be seen as an extension of the model presented by Dobson et al.[5]. Their model comprises loaded components that are independent and identically uniformly distributed, where components fail when a fixed load limit (larger than all possible initial loads) is exceeded. Due to the properties of the uniform distribution, it is shown in Ref.[5] how one can normalize this case to one with standard uniformly distributed initial loads and a fixed load limit of one. In this normalized setting, there is an initial disturbance additionally loading all components by θ/N and every line failure causes a load surge of λ/N on the remaining components. The probability that the number of failed lines A^N_d exceeds a threshold k is thus given by

$$\begin{aligned}
\mathbb{P}(A^N_d \geq k) &= \mathbb{P}\left(U^N_{(N-i+1)} + \frac{\theta + (i-1)\lambda}{N} \geq 1, \quad i = 1, \ldots, k\right) \\
&= \mathbb{P}\left(1 - U^N_{(N-i+1)} \leq \frac{\theta + (i-1)\lambda}{N}, \quad i = 1, \ldots, k\right) \\
&= \mathbb{P}\left(U^N_{(i)} \leq \frac{\theta + (i-1)\lambda}{N}, \quad i = 1, \ldots, k\right).
\end{aligned}$$

Comparing this to (3), we observe this is equivalent to the case in our framework with

$$F(l^N(i)) = \frac{\theta + (i-1)\lambda}{N}, \quad i \geq 1 \tag{4}$$

with constants $\theta > 0$ and $\lambda > 0$. The main result in Ref.[5] is that the number of failures follows a quasi-binomial distribution. Moreover, it is indicated that as $N \to \infty$, the quasi-binomial distribution converges to a generalized Poisson distribution. This latter distribution also appears in the setting of branching processes, where it corresponds to the number of offspring. In Ref.[11], Dobson et al. use the branching process relation as an approximation for the blackout size, and note that $\lambda = 1$ corresponds to a critical window where a power-law dependence manifests itself. In fact, it yields the approximation

$$\mathbb{P}(A^N = k) \approx \frac{\theta}{\sqrt{2\pi}} k^{-3/2} \tag{5}$$

for all large k independent of N.

In view of (4), we refer to the particular setting in Ref.[5] with $\lambda = 1$ as the affine case. In this paper, we aim to find a broader set of scenarios for which this power-law behavior prevails. First, we will extend to thresholds that allow for a dependency on the network size. Second, we will explore how the assumptions on the load surge function and the surplus capacity distribution can be relaxed while preserving the power-law behavior for the line failure distribution.

3. Tail of the number of line failures

The main object of interest in this paper is the probability that the number of failed lines A^N exceeds network-dependent thresholds $k := k(N)$ that satisfy both $k \to \infty$ and $N - k \to \infty$ as $N \to \infty$. For compactness, we suppress the dependence of the threshold k on N in the remainder of the paper.

3.1. Affine case

We first examine the robustness of the power-law behavior of the affine model. As indicated in Section 2, this covers all cases with

$$F(l^N(i)) = \frac{\theta + i - 1}{N} \tag{6}$$

for some constant $\theta > 0$. That is, the composition $F \circ l^N$ needs to be linearly increasing with step increments $1/N$. For example, Equation (6) holds for standard exponentially distributed surplus capacities, i.e. $F(x) = e^{-x}$ for all $x \geq 0$, and load surge function $l^N(i) = -\log((\theta + i - 1)/N)$ for all $1 \leq i \leq k$.

In Ref.[11], it is shown that a branching process approach yields the approximation (5). This method essentially uses a double limit approach: first the asymptotic behavior is derived as $N \to \infty$ for fixed k, followed by considering the behavior as $k \to \infty$. Specifically, (5) originates from the result

$$\lim_{k \to \infty} k^{3/2} \lim_{N \to \infty} \mathbb{P}(A^N = k) = \frac{\theta}{\sqrt{2\pi}}. \tag{7}$$

Table 1. Asymptotic behavior affine case.

	k fixed	k growing	$k = N - l, l$ fixed
$\mathbb{P}(A^N = k)$	$\frac{\theta(\theta+k)^{k-1}}{k!} e^{-(\theta+k)}$	$\frac{\theta}{\sqrt{2\pi}} \frac{k^{-3/2}}{\sqrt{1-k/N}}$	$\frac{\theta(l-\theta)^l}{l!} e^{-(l-\theta)} N^{-1}$

We provide a mathematically rigorous derivation of (7) in Appendix B. However, this approach does not allow thresholds that grow large with the network size, such as $k = \alpha N$ with $\alpha \in (0, 1)$. Our result accounts for the dependency between k and N, and justifies an approximation for such thresholds as well.

Proposition 3.1. *Let $F \circ l^N$ be as in (6) with constant $\theta > 0$ for each $N \in \mathbb{N}$. Let $k_\star := k_\star(N)$ and $k^\star := k^\star(N)$ be functions of N with $k^\star \geq k_\star$, $k_\star \to \infty$ and $N - k^\star \to \infty$ as $N \to \infty$. Then,*

$$\lim_{N \to \infty} \sup_{k \in [k_\star, k^\star]} \left| k^{3/2} \sqrt{1 - k/N} \mathbb{P}(A^N = k) - \frac{\theta}{\sqrt{2\pi}} \right| = 0. \tag{8}$$

For network-size dependent thresholds k, we thus obtain the approximation

$$\mathbb{P}(A^N = k) \approx \frac{\theta}{\sqrt{2\pi}} \frac{1}{\sqrt{1 - k/N}} k^{-3/2}. \tag{9}$$

Particularly, if $k = \alpha N$ for some fixed coefficient $\alpha \in (0, 1)$, our approximation leads to a different prefactor than the branching process approximation (5). The proof of Proposition 3.1 is given in Appendix B.

Dobson et al. show in Ref.[5] that the blackout size for the affine case follows a quasi-binomial distribution, and the proof of Proposition 3.1 relies heavily on the explicit form of that distribution function. We note that the same technique can be used for fixed k, which yields the generalized Poisson distribution, or for $k = N - l$ with $l > \theta$ a fixed integer. This gives rise to the results summarized in Table 1.

Proposition 3.1 can be used to derive the asymptotic behavior of the tail of the blackout size distribution.

Theorem 3.1. *Let $F \circ l^N$ be of the form (6) with constant $\theta > 0$ for each $N \in \mathbb{N}$, and $k := k(N) < N$ a positive function of N such that $k \to \infty$ and $N - k \to \infty$ as $N \to \infty$. Then,*

$$\lim_{N \to \infty} \sqrt{\frac{kN}{N-k}} \mathbb{P}(A^N \geq k) = \frac{2\theta}{\sqrt{2\pi}}.$$

Theorem 3.1 yields the approximation

$$\mathbb{P}(A^N \geq k) \approx \frac{2\theta}{\sqrt{2\pi}} \sqrt{1 - k/N} k^{-1/2}. \tag{10}$$

We conclude from Theorem 3.1 that the power-law behavior for the affine model extends to thresholds k that are appropriately growing functions of the network size. We refer the reader to Appendix B for the proof.

3.2. Main result

Theorem 3.1 shows that the probability to exceed a network-dependent threshold has a power-law distribution when the composition $F \circ l^N$ is of the specific form (6). To understand why this leads to power-law behavior, recall (3). In expectation, the difference between two consecutive uniformly distributed order statistics is $1/(N + 1) \approx 1/N$. When $F \circ l^N$ is of the form as in (6), the step increments are $1/N$ and thus (nearly) equal to the expected difference of two consecutive uniform order statistics. It is this critical correspondence that leads to heavy-tailed behavior for the exceedance probability.

The goal of this paper is to determine what other forms of the composition $F \circ l^N$ lead to power-law behavior for the number of failed lines. To do so, we include additive perturbations with respect to (6). Specifically, we consider compositions $F \circ l^N$ of the form

$$F(l^N(i)) = \frac{\theta + i - 1 + \Delta(i, N)}{N}, \tag{11}$$

where $\Delta(\cdot, \cdot)$ represents the perturbation with respect to the corresponding affine case. We impose suitable conditions on the magnitude of the perturbations such that the power-law behavior prevails as $N \to \infty$.

Intuitively, these conditions can be understood as follows. The limiting behavior of the perturbations $\Delta(i, N)$ can be of any (finite) size as long as i is fixed. Yet, for larger values of $i \leq k$ that grow to infinity with N, the perturbations need to become much smaller. In fact, they need to be zero in this domain. These conditions ensure that for most values up to k, the step increment of $F \circ l^N$ still equals the expected difference of two consecutive uniform order statistics, and consequently remains in the framework where power-law behavior appears.

Theorem 3.2. *Let $k := k(N) < N$ be a positive function of N such that $k \to \infty$ and $N - k \to \infty$ as $N \to \infty$. Let $F \circ l^N$ be as in (11) with $\Delta(\cdot, \cdot)$ satisfying the following properties:*

(A) $\Delta(i, N) \to \Delta(i)$ pointwise as $N \to \infty$ for some well-defined function $\Delta(\cdot) :$ $\mathbb{N} \to \mathbb{R}$ with $\lim_{i \to \infty} \Delta(i) = 0$;

(B) For all $i(N) \leq k$ satisfying $\lim_{N \to \infty} i(N) = \infty$, it must hold that $\lim_{N \to \infty} \Delta(i(N), N) = 0$.

Then, there exists a constant $V(\theta, \Delta) \in (0, \infty)$ such that

$$\lim_{N \to \infty} \sqrt{\frac{kN}{N - k}} \mathbb{P}(A^N \geq k) = V(\theta, \Delta).$$

That is, if $\Delta(\cdot, \cdot)$ satisfies (A) and (B) specified in Theorem 3.2, then there exists a finite, strictly positive constant $V(\theta, \Delta)$ (not depending on k) such that

$$\mathbb{P}(A^N \geq k) \approx V(\theta, \Delta) \sqrt{\frac{N - k}{kN}}$$

for large N. The proof of Theorem 3.2 is given in Appendix A.

The main point to take from this result is that the perturbations only change the prefactor, while the more relevant decay rate remains the same. For instance, for all $k = \alpha N$ with $\alpha \in (0, 1)$, the power of the exponent of the power-law remains $1/2$.

The constant $V(\theta, \Delta)$ is generally difficult to compute explicitly, but we can approximate its value with arbitrary precision. The idea is to account for only the first $M < \infty$ perturbations, where M is chosen large enough, and take the corresponding prefactor $V_M(\theta, \Delta)$ as an approximation of the true value. The definition of the approximation $V_M(\theta, \Delta)$ requires some notation. Write $c_{i,N} = NF(l^N(i)) = \theta + i - 1 + \Delta(i, N)$ and for every fixed $i \in \mathbb{N}$,

$$c_i = \lim_{N \to \infty} NF(l^N(i)) = \theta + i - 1 + \Delta(i). \tag{12}$$

Let β_i, $i \in \mathbb{N}$, be defined as

$$\beta_i = \sum_{j=1}^{i} \frac{(-1)^{j+1}}{j!} \beta_{i-j}(c_{i-j+1})^j, \quad \beta_0 = 1, \tag{13}$$

where

$$\sigma_M(y) = \max\{i \in \mathbb{N} : i \leq M, c_i < y\}.$$

Finally, let $\gamma(\cdot, \cdot)$ denote the lower incomplete gamma distribution:

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} \, dt.$$

Then, the value of $V_M(\theta, \Delta)$ can be expressed as

$$V_M(\theta, \Delta) = \frac{2}{\sqrt{2\pi}} \left(\frac{\theta}{(M-1)!} \gamma(M, c_M) + \frac{(c_M)^M}{(M-1)!} e^{-c_M} \right.$$
$$+ \sum_{j=1}^{M-1} \beta_{j-1} e^{-c_j} \frac{\Delta(j)}{(M-j)!} \gamma(M-j+1, c_M - c_j)$$
$$\left. - \sum_{j=1}^{M-1} \beta_{j-1} \frac{(c_M - c_j)^{M-j+1}}{(M-j)!} e^{-c_M} \right). \tag{14}$$

The constant $V(\theta, \Delta)$ is defined as

$$V(\theta, \Delta) = \lim_{M \to \infty} V_M(\theta, \Delta). \tag{15}$$

The idea to approximate $V(\theta, \Delta)$ by $V_M(\theta, \Delta)$ for a sufficiently large M gives rise to Algorithm 1.

Whether conditions (A) and (B) on the perturbations $\Delta(\cdot, \cdot)$ are satisfied, depends on the surplus capacity distribution $F(\cdot)$, the load surge function $l^N(\cdot)$ and the threshold k. The proof of Theorem 3.2 uses an equivalent, but more technical, condition to (A) and (B). However, conditions (A) and (B) are more tractable for examples and therefore stated in the theorem. In the next section, we consider some

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Algorithm 1: Approximation scheme for $V(\theta, \Delta)$.

Input: Target error $\delta > 0$, constant $\theta > 0$ and perturbations $\Delta(\cdot, \cdot)$ satisfying
      (A) and (B) in Theorem 3.2.

Output: Approximation $V_{M_\epsilon}(\theta, \Delta)$ such that $\left|V(\theta, \Delta) - V_{M_\epsilon}(\theta, \Delta)\right| < \delta$.

   1. Determine $\epsilon > 0$ such that $\frac{8\epsilon(1+\epsilon)}{\sqrt{2\pi}} \leq \delta$.
   2. Determine pair (M_ϵ, N_ϵ) such that $|\Delta(i, N)| < \epsilon$ for all $N \geq N_\epsilon$ and all
      $M_\epsilon \leq i \leq k(N)$.
   3. Return $V_{M_\epsilon}(\theta, \Delta)$ defined in Proposition (14).

---

examples where we identify thresholds k such that the power-law behavior prevails. A particularly compelling example involves the case where the loads of the failed lines are equally redistributed over the remaining lines, see e.g., Ref.[6]. Every time a line fails, the total load is redistributed over all surviving lines. Theorem 3.2 implies that in this case for power-law behavior to prevail, the step increments of the composition $F \circ l^N$ should become approximately $1/N$ up to the k'th failure from a certain point on. In Example 4.2 in the next section, we explain that a Taylor expansion of the composition $F \circ l^N$ suggests for which settings this holds, indicating that power-law behavior only prevails in a remarkably narrow window.

4. Identifying thresholds where power-law behavior prevails

The purpose of this section is to illustrate the use of Theorem 3.2. When the surplus capacity distribution and/or load surge function are given, we would like to know what (growing) thresholds $k := k(N)$, if any, yield power-law behavior for the exceedance probability. A sufficient condition is provided by Theorem 3.2 and accordingly, we need to determine the thresholds k such that (A) and (B) are satisfied. As we will illustrate in the two examples, the key approach involves a Taylor expansion. To conclude this section, we consider an approach that can be used to numerically explore the asymptotic behavior for settings that do not fall in the framework of Theorem 3.2.

Example 4.1. First, we illustrate the impact of a different surplus capacity distribution. That is, suppose

$$l^N(i) = \frac{i}{N}, \quad i \geq 1,$$

and let the surplus capacities be exponentially distributed with mean one. Using a Taylor expansion we obtain

$$F(l^N(i)) = 1 - e^{-\frac{i}{N}} = \frac{i}{N} + O\left(\left(\frac{i}{N}\right)^2\right),$$

where $O(\cdot)$ denotes the big-O notation in relation with $N \to \infty$. Hence, $\Delta(i, N) = i^2/N$ for all $(i, N) \in \mathbb{N} \times \mathbb{N}$ and condition (A) is satisfied. We note that for (B) to hold, we need $k = o(\sqrt{N})$. All thresholds that satisfy $k = o(\sqrt{N})$ thus result in power-law behavior for the exceedance probability.

The final claim holds generally for any surplus capacity distribution with a positive density in zero. That is, in general, Taylor expansion yields

$$F(l^N(i)) = F'(0)l^N(i) + O((l^N(i))^2). \qquad (16)$$

We observe that (16) leads to an approximation of the composition that only requires information on the value $F'(0)$ and the load surge function. That is, the only property of the surplus capacity distribution we need for checking whether power-law behavior prevails, is its behavior near its minimum. In particular, the mean of the surplus capacity does not play any role.

If the load surge function is given by

$$l^N(i) = \frac{i}{F'(0)N}, \quad i \geq 1,$$

and $k = o(\sqrt{N})$, we thus remain in the setting described in Theorem 3.2.

Example 4.2. Next, we verify and formalize the claims for the model in Ref.[6] that we discussed in Section 3. Specifically, the load surge function is given by

$$l^N(i) = \frac{aN}{N-i} - a = \frac{ai}{N-i},$$

and suppose that $a = 1/F'(0)$. Then, applying the Taylor expansion (16), we obtain

$$\Delta(i, N) = O\left(\left(\frac{i}{N}\right)^2\right) + O\left(N\left(\frac{i}{N-i}\right)^2\right)$$

for all $(i, N) \in \mathbb{N} \times \mathbb{N}$. Again, we have pointwise convergence $\Delta(i) = 0$ for all $i \in \mathbb{N}$. In addition, we require that $k = o(\sqrt{N})$ for condition (B) to hold for all $i \leq k$.

We close this section by setting the threshold k to a certain fixed integer, which allows us to analyze cases where the perturbations do not satisfy conditions (A) and (B). We suggest a method to explore the asymptotic behavior numerically for these cases.

Example 4.3. Observe that if the value $\Delta(1, N)$ tends too close to its lower bound as $N \to \infty$, the system does not perceive an initial disturbance and no line will fail. On the other hand, if $\Delta(k, N)$ becomes too large as $N \to \infty$, the system cannot deal with such a strong increase of load and the threshold k will certainly be exceeded. If $\Delta(1, N)$ is not too small and $\Delta(k, N)$ is not too large as $N \to \infty$, we obtain a non-degenerate limit for the exceedance probability.

Proposition 4.1. Let $c_{i,N} := N \cdot F(l^N(i))$ for $(i, N) \in \mathbb{N} \times \mathbb{N}$ and $c_i = \lim_{N \to \infty} c_{i,N}$ for $i \in \mathbb{N}$, which is a non-decreasing sequence. If $c_1 > 0$ and $c_k = O(1)$, then

$$\lim_{N \to \infty} \mathbb{P}(A^N \geq k) = 1 - \sum_{j=1}^{k} \beta_{j-1} e^{-j}$$

where β_i, $i \in \mathbb{N}$ are defined as in (13).

This result can be proven by applying results from extreme value theory, see Appendix B. Proposition 4.1 thus provides a method to determine the asymptotic exceedance probability for every fixed k. This can be used to numerically explore how the asymptotic tail of the number of failed lines decays as k grows large.

5. Summary and outlook

The model of Dobson et al.[5] shows power-law dependence for the exceedance probability when the system is critically loaded. In this paper, we identify settings where the power law prevails by extending the setting of Ref.[5] in two directions. First, we show that the threshold can grow with the network size. Second, we consider broader load surge functions and surplus capacity distributions. We show that the power-law distribution prevails when the composition of the surplus capacity distribution function and the load surge function ultimately tends to a linearly increasing function with critical slope.

We emphasize that there are many aspects that exist in reality and are not accounted for by this model. For example, the propagation of the cascading failure process often depends heavily on the network topology. In particular in an energy transmission system, when a part of the network becomes disconnected to the power generators, all the lines in that part fail immediately. Given a particular network structure, we are highly interested in finding all combinations of surplus capacity distributions, load surge functions, network-dependent thresholds that yield power-law behavior for the blackout size. We intend to pursue this complex problem in future research.

Appendices

A. Proofs for the affine case

When relation (6) holds, the blackout size follows a quasi-binomial distribution[5], ensuring an analytic expression for the probability distribution of the blackout size. A double limit method to derive the asymptotic behavior used in Ref.[11] yields an incorrect prefactor.

Proof of (7). The probability distribution of the number of failed lines is given by[5]

$$
\mathbb{P}(A^N = k) = \begin{cases} \binom{N}{k}\frac{\theta}{N}\left(\frac{\theta+k}{N}\right)^{k-1}\left(1-\frac{\theta+k}{N}\right)^{N-k}, & \text{if } k \leq N-\theta, \\ 0, & \text{if } N-\theta < k < N, \\ \sum_{i=\lfloor N-\theta \rfloor +1}^{N}\binom{N}{i}\frac{\theta}{N}\left(\frac{\theta+i}{N}\right)^{i-1}\left(1-\frac{i+\theta}{N}\right)^{N-i}, & \text{if } k = N. \end{cases} \quad \text{(A.1)}
$$

This distribution converges to a generalized Poisson distribution, i.e.[5],

$$
\lim_{N\to\infty}\mathbb{P}(A^N = k) = \theta\,\frac{(\theta+k)^{k-1}}{k!}\mathrm{e}^{-(\theta+k)}.
$$

Applying Stirling's approximation, we obtain

$$\lim_{k\to\infty} k^{3/2} \lim_{N\to\infty} \mathbb{P}(A^N = k) = \lim_{k\to\infty} k^{3/2}\theta \frac{(\theta + k)^{k-1}}{k!} e^{-(\theta+k)}$$
$$= \lim_{k\to\infty} \frac{\theta}{\sqrt{2\pi}} \left(1 + \frac{\theta}{k}\right)^{k-1} e^{-\theta} = \frac{\theta}{\sqrt{2\pi}}.$$

$\square$

Yet, when accounting for the dependency of k on N, the analytic expression for the probability distribution function of the number of failed lines can be exploited to derive the correct prefactor.

Proof of Proposition 3.1. Recall that the probability distribution of the number of failed lines is given by (A.1). The idea of the proof is to use Stirling's approximation to derive an upper and lower bound for (A.1) and show they asymptotically coincide.

Since we consider $k \in [k_\star, k^\star]$, we are only concerned with the probability for $k \leq N - \theta$. With Stirling's approximation (formula (6.1.38) of Ref.[21]) we have that for every integer $m > 0$

$$m! = \sqrt{2\pi} m^{m+1/2} e^{-m + \frac{y(m)}{12m}},$$

for some $0 < y(m) < 1$. So the binomial term is bounded by

$$\binom{N}{k} \geq \frac{1}{\sqrt{2\pi}} \frac{N^N}{k^k (N-k)^{N-k}} \sqrt{\frac{N}{k(N-k)}} e^{-\frac{1}{12k}} e^{-\frac{1}{12(N-k)}},$$

$$\binom{N}{k} \leq \frac{1}{\sqrt{2\pi}} \frac{N^N}{k^k (N-k)^{N-k}} \sqrt{\frac{N}{k(N-k)}} e^{\frac{1}{12N}}.$$

Using these bounds for the binomial term in (A.1) yields

$$k^{3/2}\sqrt{1 - k/N}\mathbb{P}(A^N = k) \geq \frac{\theta}{\sqrt{2\pi}} \left(1 + \frac{\theta}{k}\right)^{k-1} \left(1 - \frac{\theta}{N-k}\right)^{N-k} e^{-\frac{1}{12k}} e^{-\frac{1}{12(N-k)}}$$

and

$$k^{3/2}\sqrt{1 - k/N}\mathbb{P}(A^N = k) \leq \frac{\theta}{\sqrt{2\pi}} \left(1 + \frac{\theta}{k}\right)^{k-1} \left(1 - \frac{\theta}{N-k}\right)^{N-k} e^{\frac{1}{12N}}$$

for any $k_\star \leq k \leq k^\star$. Note that for every constant $\theta > 0$ the functions $(1 + \theta/x)^{x-1}$ and $(1 - \theta/x)^x$ are both monotone increasing in $x > 0$. Moreover, the function $e^{-1/(12x)}$ is monotone increasing in $x > 0$. Therefore, we obtain the lower bound

$$\sup_{k\in[k_\star,k^\star]} k^{3/2}\sqrt{1 - k/N}\mathbb{P}(A^N = k) \geq \frac{\theta}{\sqrt{2\pi}} \sup_{k\in[k_\star,k^\star]}\left(1 + \frac{\theta}{k}\right)^{k-1} e^{-\frac{1}{12k}} \sup_{k\in[k_\star,k^\star]}\left(1 - \frac{\theta}{N-k}\right)^{N-k} e^{-\frac{1}{12(N-k)}}$$

$$= \frac{\theta}{\sqrt{2\pi}} \left(1 + \frac{\theta}{k^\star}\right)^{k^\star-1} \left(1 - \frac{\theta}{N-k_\star}\right)^{N-k_\star} e^{-\frac{1}{12k^\star}} e^{-\frac{1}{12(N-k_\star)}}.$$

Moreover, since $(1 + \theta/x)^{x-1} \leq e^\theta$ and $(1 - \theta/x)^x \leq e^{-\theta}$ for all $x > 0$, we have the upper bound

$$\sup_{k\in[k_\star,k^\star]} k^{3/2}\sqrt{1 - k/N}\mathbb{P}(A^N = k) \leq \frac{\theta}{\sqrt{2\pi}} e^{\frac{1}{12N}}.$$

We observe that both the upper bound and the lower bound converge to $\theta/\sqrt{2\pi}$ as $N \to \infty$ under the given assumptions on $k_\star$ and $k^\star$, implying that (8) holds. $\square$

Next, we turn to the asymptotic behavior of the probability that the blackout size exceeds the threshold k. For this, we bound the discrete density function of the blackout size by two continuous functions that grow arbitrarily close to one another for all $i \in [k, N - \log(N - k)]$, see

$$i = 0.75N \qquad\qquad i$$
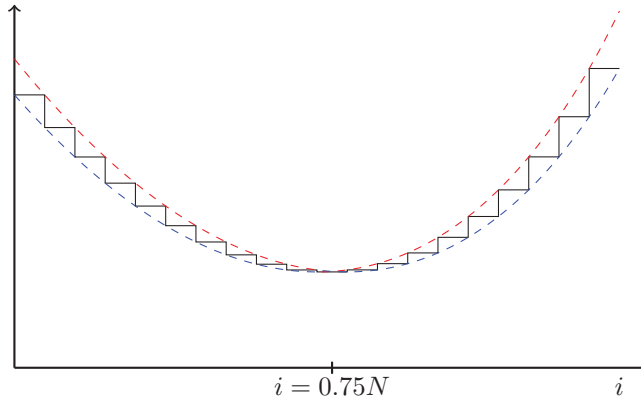
Figure 2. Continuous bounds for $\mathbb{P}(A^N = i)$.

Figure 2. We conclude the proof by deriving the integral counterparts of the continuous functions and showing that $\mathbb{P}(A^N \geq N - \log(N - k))$ is asymptotically negligible.

Proof of Theorem 3.1. Set $k^\star = N - \log(N - k)$. Observe that $k^\star \geq k$ and note that this choice ensures $k^\star = N - o(N)$ and $N - k^\star \to \infty$ as $N \to \infty$. Due to Proposition 3.1, it follows that for every $\epsilon > 0$ there is a $N_\epsilon > 0$ such that for all $N \geq N_\epsilon$ and $i \in [k, k^\star]$

$$\frac{\theta}{\sqrt{2\pi}}\, i^{-3/2}(1 - i/N)^{-1/2}(1 - \epsilon) \leq \mathbb{P}(A^N = i) \leq \frac{\theta}{\sqrt{2\pi}} i^{-3/2}(1 - i/N)^{-1/2}(1 + \epsilon).$$

Next, we use this observation to bound the exceedance probability from above and below and show that these bounds coincide as $\epsilon \downarrow 0$.

An upper bound for the exceedance probability is given by

$$\mathbb{P}(A^N \geq k) \leq \mathbb{P}(A^N = k) + (1 + \epsilon) \sum_{i=k+1}^{k^\star} \frac{\theta}{\sqrt{2\pi}} i^{-3/2}(1 - i/N)^{-1/2} + \sum_{i=k^\star+1}^{N} \mathbb{P}(A^N = i)$$

We consider the first term separately from the second term, because this results in a nicer expression for the second term and the contribution of $\mathbb{P}(A^N = k)$ is asymptotically negligible. That is, for every integer m, Stirling's bound[21] yields $\sqrt{2\pi}\, m^{m+1/2}e^{-m} \leq m! \leq e\sqrt{2\pi}\, m^{m+1/2}e^{-m}$. Therefore,

$$\sqrt{\frac{kN}{N-k}}\mathbb{P}(A^N = k) \leq e\sqrt{\frac{N}{N-k}}\sqrt{\frac{N}{N-k}}\frac{\theta}{\theta + k}\left(1 + \frac{\theta}{k}\right)^k\left(1 - \frac{\theta}{N-k}\right)^{N-k}$$

$$\leq e\frac{N}{k(N-k)}\frac{\theta}{\theta/k + 1} \longrightarrow 0$$

as $N \to \infty$.

For the second term, we consider the integral

$$\int_k^N x^{-3/2}\left(1 - \frac{x}{N}\right)^{-1/2} \mathrm{d}x = \int_{\arcsin(\sqrt{k/N})}^{\pi/2} N^{-3/2}\sin(u)^{-3}(1 - \sin(u)^2)^{-1/2}2N\sin(u)\cos(u)\,\mathrm{d}u$$

$$= 2N^{-1/2}\int_{\arcsin(\sqrt{k/N})}^{\pi/2} \sin(u)^{-2}\,\mathrm{d}u$$

$$= 2N^{-1/2}\frac{\sqrt{1 - k/N}}{\sqrt{k/N}} = 2\sqrt{\frac{N-k}{kN}},$$

where we applied the variable substitution $x = N \sin(u)^2$. Then, the second term is bounded by

$$\sum_{i=k+1}^{k^\star} \frac{\theta}{\sqrt{2\pi}} i^{-3/2} (1 - i/N)^{-1/2} \leq \int_k^N \frac{\theta}{\sqrt{2\pi}} i^{-3/2} (1 - i/N)^{-1/2} \, \mathrm{d}i = \frac{2\theta}{\sqrt{2\pi}} \sqrt{\frac{N-k}{kN}}.$$

For the third term,

$$\sum_{i=k^\star+1}^{N} \mathbb{P}(A^N = i) \leq (N - k^\star) \sup_{i \in (k^\star, N]} \mathbb{P}(A^N = i).$$

To determine the supremum, we take a closer look at (A.1). For all $i \in (N - \theta, N)$, if any, $\mathbb{P}(A^N = i) = 0$. Moreover, for all integers $i \in (k^\star, N - \theta]$, Stirling's bound yields

$$\binom{N}{i} \frac{\theta}{N} \left(\frac{\theta + i}{N}\right)^{i-1} \left(1 - \frac{\theta + i}{N}\right)^{N-i} \leq e\sqrt{\frac{N}{i(N-i)}} \frac{\theta}{\theta + i} \left(1 + \frac{\theta}{i}\right)^i \left(1 - \frac{\theta}{N-i}\right)^{N-i}$$

$$\leq e\sqrt{\frac{N}{N - \theta}} \frac{\theta}{\theta + k^\star}.$$

Therefore, $\sup_{i \in (k^\star, N-\theta]} \mathbb{P}(A^N = i) \leq c_1/k^\star$ for some constant $c_1 > 0$, and

$$\mathbb{P}(A^N = N) = \frac{\theta}{N}\left(1 + \frac{\theta}{N}\right)^{N-1} + \sum_{i=\lfloor N-\theta \rfloor + 1}^{N-1} \binom{N}{i}\frac{\theta}{N}\left(\frac{\theta+i}{N}\right)^{i-1}\left(1 - \frac{\theta+i}{N}\right)^{N-i}$$

$$\leq \frac{\theta e^\theta}{N} + \theta e \sqrt{\frac{N}{N-1}} \frac{\theta}{\theta + k^\star} \leq \frac{c_2}{k^\star}$$

for some constant $c_2 > 0$. Recall $k \leq k^\star = N - \log(N - k)$, and set $c = \max\{c_1, c_2\}$. This yields

$$\sqrt{\frac{kN}{N-k}}(N - k^\star) \sup_{i \in (k^\star, N]} \mathbb{P}(A^N = i) \leq c\frac{\sqrt{kN}}{k^\star}\frac{N - k^\star}{\sqrt{N-k}} = c\underbrace{\frac{\sqrt{k/N}}{1 - \log(N-k)/N}}_{=O(1)}\underbrace{\frac{\log(N-k)}{\sqrt{N-k}}}_{=o(1)}$$

as $N \to \infty$, since $N - k \to \infty$ as $N \to \infty$. We conclude that

$$\limsup_{N \to \infty} \sqrt{\frac{kN}{N-k}} \mathbb{P}(A^N \geq k) \leq (1 + \epsilon)\frac{2\theta}{\sqrt{2\pi}}.$$

A lower bound is given by

$$\mathbb{P}(A^N \geq k) \geq (1 - \epsilon) \sum_{i=k}^{k^\star} \frac{\theta}{\sqrt{2\pi}} i^{-3/2}(1 - i/N)^{-1/2} \geq (1 - \epsilon) \int_k^{k^\star} \frac{\theta}{\sqrt{2\pi}} i^{-3/2}(1 - i/N)^{-1/2} \, \mathrm{d}i$$

$$= (1 - \epsilon)\frac{2\theta}{\sqrt{2\pi}}\left(\sqrt{\frac{N-k}{kN}} - \sqrt{\frac{N-k^\star}{k^\star N}}\right).$$

It follows that

$$\liminf_{N \to \infty} \sqrt{\frac{kN}{N-k}} \mathbb{P}(A^N \geq k) \geq \liminf_{N \to \infty}(1 - \epsilon)\frac{2\theta}{\sqrt{2\pi}}\left(1 - \underbrace{\sqrt{\frac{k}{k^\star}}}_{=O(1)}\underbrace{\sqrt{\frac{N-k^\star}{N-k}}}_{=o(1)}\right) = (1 - \epsilon)\frac{2\theta}{\sqrt{2\pi}}.$$

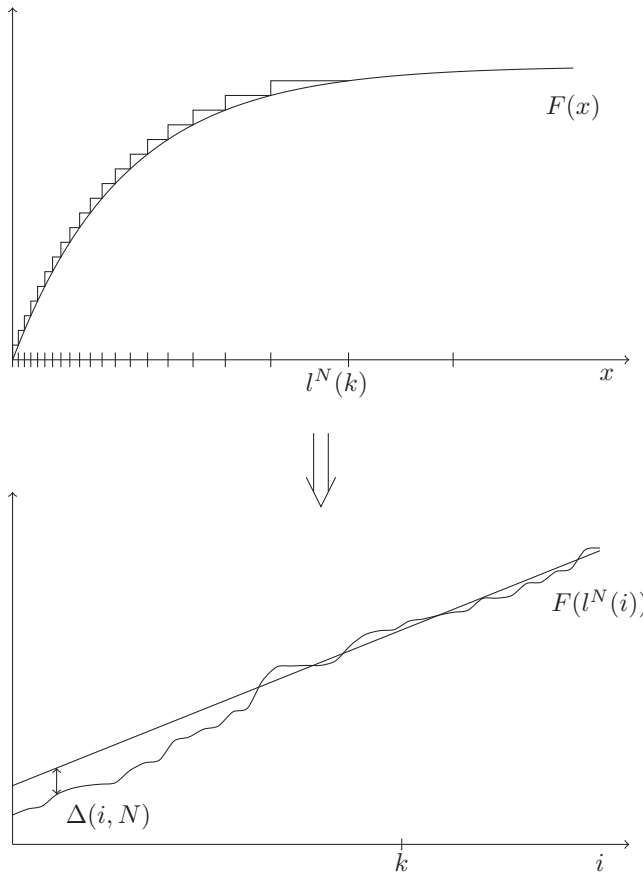As $\epsilon \downarrow 0$, the limsup and liminf coincide. $\qquad\square$

B. Proofs for perturbations of the composition

Whether we obtain power-law behavior for the black-out size distribution depends on the surplus capacity distribution, the load surge function and the threshold k. Due to relation (3), we observe that the relation between the surplus capacity distribution and the load surge function is captured by the composition $F \circ l^N$, see Figure 3. In this section, we prove that if $F \circ l^N$ has a form as in (11) with perturbations $\Delta(\cdot, \cdot)$ satisfying (A) and (B) as in Theorem 3.2, the power-law behavior for the exceedance probability prevails.

Recall (12) and note that by conditions (A) and (B), c_i, $i \in \mathbb{N}$, is a well-defined non-decreasing sequence that tends to the function $\theta + i - 1$ as i grows large. We will show that these conditions result in power-law behavior for the exceedance probability. To do so, we leverage two basic asymptotic properties formulated in the following two lemmas.

Lemma B.1. Let k be a function of N such that both $k \to \infty$ and $N - k \to \infty$ as $N \to \infty$. Then for every fixed $M_1, M_2 \in \mathbb{N}$,

$$\lim_{N \to \infty} \sqrt{\frac{kN}{N-k}} \mathbb{P}\left(U_{(i)}^{N-M_1} \leq \frac{\theta + i - 1}{N - M_1}, \quad \forall i \leq k - M_2\right) = \frac{2\theta}{\sqrt{2\pi}}.$$

Figure 3. Relation surplus capacity distribution function and load surge function.

Proof. Recall (3) and (6), and observe that the case with $M_1 = M_2 = 0$ is implied by Theorem 3.1. The general case follows from a straightforward calculation:

$$\sqrt{\frac{kN}{N-k}}\mathbb{P}\left(U_{(i)}^{N-M_1} \le \frac{\theta + i - 1}{N - M_1}, \quad i = 1, \dots, k - M_2\right)$$

$$= \underbrace{\sqrt{\frac{k}{k - M_2}}}_{\to 1} \underbrace{\sqrt{\frac{N}{N - M_1}}}_{\to 1} \underbrace{\sqrt{\frac{N - k + M_2 - M_1}{N - k}}}_{\to 1}$$

$$\cdot \sqrt{\frac{(k - M_2)(N - M_1)}{N - k + M_2 - M_1}}\mathbb{P}\left(U_{(i)}^{N-M_1} \le \frac{\theta + i - 1}{N - M_1}, \quad i = 1, \dots, k - M_2\right) \overset{N \to \infty}{\longrightarrow} \frac{2\theta}{\sqrt{2\pi}}.$$

The last convergence follows from noting (3) and applying Theorem 3.1 to a network with $N - M_1$ lines and threshold $k - M_2$. $\qquad\square$

Lemma B.2. Let k be a function of N such that both $k \to \infty$ and $N - k \to \infty$ as $N \to \infty$. Then for every fixed $M \in \mathbb{N}$

$$\lim_{N \to \infty} \sqrt{\frac{kN}{N-k}}\mathbb{P}\left(U_{(i)}^{N-M} \le \frac{\theta + i - 1}{N}, \quad i = 1, \dots, k\right) = \frac{2\theta}{\sqrt{2\pi}}. \qquad (\text{B.1})$$

Proof. Note that

$$\limsup_{N \to \infty} \sqrt{\frac{kN}{N-k}}\mathbb{P}\left(U_{(i)}^{N-M} \le \frac{\theta + i - 1}{N}, \quad i = 1, \dots, k\right)$$

$$\le \limsup_{N \to \infty} \sqrt{\frac{kN}{N-k}}\mathbb{P}\left(U_{(i)}^{N} \le \frac{\theta + i - 1}{N}, \quad i = 1, \dots, k\right) = \frac{2\theta}{\sqrt{2\pi}}.$$

To obtain a lower bound, we first consider the case $M = 1$. Consider a Poisson process with unit rate where the epoch of the i'th event is denoted by $S_i = \sum_{j=1}^{i} E_j$ with E_j standard independent exponential random variables for all $j \ge 1$. Note that, given $S_i = t$, the joint distribution of $(S_1, \dots, S_{i-1})$ is the same as the joint distribution of $i - 1$ ordered independent uniform random variables on $(0, t)$. Therefore, Equation (B.1) with $M = 1$ is equivalent to

$$\lim_{N \to \infty} \sqrt{\frac{kN}{N-k}}\mathbb{P}\left(\frac{S_i}{S_N} \le \frac{\theta + i - 1}{N}, \quad i = 1, \dots, k\right) = \frac{2\theta}{\sqrt{2\pi}}.$$

We observe that for every $\epsilon > 0$,

$$\mathbb{P}\left(\frac{S_i}{S_{N+1}} \le \frac{\theta + i - 1}{N}, \quad \forall i \le k\right) \le \mathbb{P}\left(S_i \le \frac{(\theta + i - 1)S_{N+1}}{N}, \quad \forall i \le k; E_{N+1} \le \epsilon S_N\right)$$

$$+ \mathbb{P}\left(E_{N+1} > \epsilon S_N\right)$$

$$\le \mathbb{P}\left(S_i \le (\theta + \epsilon + i - 1)\frac{S_N}{N}, \quad \forall i \le k\right) + \mathbb{P}\left(E_{N+1} > \epsilon S_N\right).$$

Since

$$\mathbb{P}\left(E_{N+1} > \epsilon S_N\right) = \mathbb{E}\left(e^{-\epsilon S_N}\right) = \mathbb{E}\left(e^{-\epsilon S_1}\right)^N = \left(\frac{1}{1+\epsilon}\right)^N,$$

it follows that

$$\liminf_{N\to\infty} \sqrt{\frac{kN}{N-k}}\mathbb{P}\left(\frac{S_i}{S_N} \leq \frac{\theta+i-1}{N}, \quad \forall i \leq k\right)$$

$$\geq \liminf_{N\to\infty} \sqrt{\frac{kN}{N-k}}\left(\mathbb{P}\left(\frac{S_i}{S_{N+1}} \leq \frac{\theta-\epsilon+i-1}{N}, \quad \forall i \leq k\right) - \left(\frac{1}{1+\epsilon}\right)^N\right) = \frac{2(\theta-\epsilon)}{\sqrt{2\pi}}$$

for every $\epsilon > 0$. The result for $M = 1$ follows by letting $\epsilon \downarrow 0$. Using induction yields the result for any fixed $M > 0$. □

In view of (1), (3) and (11), it is convenient to introduce the stopping times

$$\tau_{\theta,\Delta}^N = \min\left\{i \in \mathbb{N} : U_{(i)}^N > \frac{\theta+i-1+\Delta(i,N)}{N}\right\} - 1 \tag{B.2}$$

for all constants $\theta \in \mathbb{R}$ and functions $\Delta : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$. In particular, if the constant θ and function $\Delta(\cdot,\cdot)$ are chosen as in (11), then $A^N = \tau_{\theta,\Delta}^N$. Yet, the advantage of the notation as in (B.2) appears when we compare the asymptotic exceedance probability for different constants θ and functions $\Delta(\cdot,\cdot)$.

Our derivation of the asymptotic behavior of the exceedance probability makes use of similar arguments multiple times in the proof. We present these arguments separately by means of the next two lemmas.

Lemma B.3. Let $k := k(N) \leq N$ be a positive function of N such that $k \to \infty$ and $N - k \to \infty$ as $N \to \infty$. Let c_i, $i \in \mathbb{N}$ be as in (12) and for some fixed $M \in \mathbb{N}$, suppose $\Delta(i,N) = 0$ for all $i \geq M$ and $N \geq N_0$ for some $N_0 \in \mathbb{N}$. For all constants $a, b \in \mathbb{R}_{\geq 0}$ with $a \leq b$,

$$\lim_{N\to\infty} \sqrt{\frac{kN}{N-k}}\mathbb{P}\left(\tau_{\theta,\Delta}^N \geq k; U_{(M)}^N \in \left[\frac{a}{N}, \frac{b}{N}\right]\right)$$

$$= \frac{2}{\sqrt{2\pi}}\int_a^b \mathbb{P}\left(U_{(i)}^{M-1} \leq \frac{c_i}{y}, \quad \forall i \leq M-1\right)(\theta+M-y)\frac{y^{M-1}}{(M-1)!}e^{-y}\,\mathrm{d}y. \tag{B.3}$$

Proof. The density of the M'th order statistic of a sample of N standard uniformly distributed random variables is given by a beta distribution[20, 78,79]

$$f_{U_{(M)}^N}(x) = \frac{N!}{(M-1)!(N-M)!}x^{M-1}(1-x)^{N-M}.$$

Conditioning on the M'th order statistic yields

$$\mathbb{P}(\tau_{\theta,\Delta}^N \geq k) = \int_{\frac{a}{N}}^{\frac{b}{N}} \mathbb{P}\left(U_{(i)}^N \leq \frac{c_{i,N}}{N}, \quad \forall i \leq k \,\middle|\, U_{(M)}^N = x\right)f_{U_{(M)}^N}(x)\,\mathrm{d}x$$

$$= \int_a^b \mathbb{P}\left(U_{(i)}^N \leq \frac{c_{i,N}}{N}, \quad \forall i \leq k \,\middle|\, U_{(M)}^N = \frac{y}{N}\right)\frac{f_{U_{(M)}^N}\left(\frac{y}{N}\right)}{N}\,\mathrm{d}y$$

$$= \int_a^b \mathbb{P}\left(U_{(i)}^{M-1} \leq \frac{c_{i,N}}{y}, \quad \forall i \leq M-1\right)$$

$$\cdot \mathbb{P}\left(U_{(i)}^{N-M} \leq \frac{\theta+M-y+i-1}{N(1-\frac{y}{N})}, \quad \forall i \leq k-M\right)\frac{f_{U_{(M)}^N}\left(\frac{y}{N}\right)}{N}\,\mathrm{d}y.$$

The latter equality follows from the Markov property: Given that $U_{(M)}^N = y/N$, the first $M-1$ order statistics are independent of the other order statistics and distributed as $M-1$ uniformly distributed random variables on the interval $[0, y/N]$. Similarly, the other order statistics are independent of the first M order statistics, and have the same law as $N-M$ uniformly distributed random variables on the interval $[y/N, 1]$. Rescaling the intervals results in the above expression.

Next, we show that an interchange of limit and integration is justified by bounding all three terms within the integral form above. First, we observe that for all $y \in [a, b]$,

$$\frac{f_{U^N_{(M)}}\left(\frac{y}{N}\right)}{N} = \frac{(N-1)!}{(M-1)!(N-M)!}\left(\frac{y}{N}\right)^{M-1}\left(1-\frac{y}{N}\right)^{N-M} \leq \frac{N^{M-1}}{(M-1)!}\left(\frac{y}{N}\right)^{M-1}$$

$$\leq \frac{b^{M-1}}{(M-1)!} < \infty.$$

Second, we show that the second term multiplied with $\sqrt{kN/(N-k)}$ is also bounded. Let $M^\star = \lceil b \rceil$, and hence for all $y \in [a, b]$, $N - M^\star \leq N - y \leq N$,

$$\mathbb{P}\left(U^{N-M}_{(i)} \leq \frac{\theta + M - y + i - 1}{N(1 - \frac{y}{N})}, \quad \forall i \leq k - M\right)$$

$$\geq \mathbb{P}\left(U^{N-M}_{(i)} \leq \frac{\theta + M - y + i - 1}{N}, \quad \forall i \leq k - M\right)$$

and

$$\mathbb{P}\left(U^{N-M}_{(i)} \leq \frac{\theta + M - y + i - 1}{N(1 - \frac{y}{N})}, \quad \forall i \leq k - M\right)$$

$$\leq \mathbb{P}\left(U^{N-M}_{(i)} \leq \frac{\theta + M - y + i - 1}{N - M^\star}, \quad \forall i \leq k - M\right).$$

Applying Lemmas B.1 and B.2 and subsequently the squeeze theorem yields

$$\lim_{N\to\infty} \sqrt{\frac{kN}{N-k}}\mathbb{P}\left(U^{N-M}_{(i)} \leq \frac{\theta + M - y + i - 1}{N(1 - \frac{y}{N})}, \quad \forall i \leq k - M\right)$$

$$= \liminf_{N\to\infty} \sqrt{\frac{kN}{N-k}}\mathbb{P}\left(U^{N-M}_{(i)} \leq \frac{\theta + M - y + i - 1}{N}, \quad \forall i \leq k - M\right)$$

$$= \limsup_{N\to\infty} \sqrt{\frac{kN}{N-k}}\mathbb{P}\left(U^{N-M}_{(i)} \leq \frac{\theta + M - y + i - 1}{N - M^\star}, \quad \forall i \leq k - M\right)$$

$$= \frac{2(\theta + M - y)}{\sqrt{2\pi}}.$$

We find that the second term multiplied with $\sqrt{kN/(N-k)}$ is indeed bounded, since every converging sequence is bounded. Finally, the first term is trivially bounded by one, and therefore the dominated convergence theorem justifies an interchange of limit and integration. Since $U^{M-1}_{(i)}$, $i = 1, \ldots, M - 1$, have a density not depending on N, it holds that

$$\lim_{N\to\infty} \mathbb{P}\left(U^{M-1}_{(i)} \leq \frac{c_{i,N}}{y}, \quad \forall i \leq M - 1\right) = \mathbb{P}\left(U^{M-1}_{(i)} \leq \frac{c_i}{y}, \quad \forall i \leq M - 1\right),$$

and moreover,

$$\lim_{N\to\infty} \frac{1}{N}f_{U^N_{(M)}}\left(\frac{y}{N}\right) = \frac{y^{M-1}}{(M-1)!}e^{-y}.$$

We conclude that (B.3) holds. □

To obtain a more quantitative handle on the integral expression in (B.3), we need to have a deeper understanding of the probability term within the integral. The second lemma expresses this probability by means of a recursive formula.

Lemma B.4. Let $M \in \mathbb{N}$ be fixed, and suppose $\Delta(i, N) = 0$ for all $i \geq M$ and $N \geq N_0$ for some $N_0 \in \mathbb{N}$. Let c_i, $i \in \mathbb{N}$ be as in (12) and for every $y \in \mathbb{R}_{\geq 0}$, define $\sigma_M(y) = 0$ if $c_1 > y$ and otherwise

$$\sigma_M(y) = \max\{i \in \mathbb{N} : i \leq M, c_i < y\}.$$

Then,

$$\mathbb{P}\left(U_{(i)}^M \leq \frac{c_i}{y}, \quad \forall i \leq M\right) = 1 - \frac{M!}{y^M} \sum_{j=1}^{\sigma_M(y)} \beta_{j-1} \frac{(y - c_j)^{M-j+1}}{(M - j + 1)!},$$

where β_i are defined as in (13).

The proof of Lemma B.4 uses the two following identities.

Lemma B.5. For β_k, $k \geq 1$ defined as in (13),

$$\beta_k = \int_{-c_1}^{0} \int_{-c_2}^{y_1} \cdots \int_{-c_k}^{y_{k-1}} dy_k \cdots dy_1.$$

Proof. The proof is by induction. For $k = 1$, we indeed have $\int_{-c_1}^{0} dy_1 = c_1 = \beta_1$. Suppose the lemma holds for all integers strictly smaller than k. Then,

$$\int_{-c_1}^{0} \int_{-c_2}^{y_1} \cdots \int_{-c_k}^{y_{k-1}} dy_k \cdots dy_1 = \int_{-c_1}^{0} \int_{-c_2}^{y_1} \cdots \int_{-c_{k-1}}^{y_{k-2}} y_{k-1} \, dy_{k-1} \cdots dy_1 + c_k \beta_{k-1}$$

$$= \int_{-c_1}^{0} \int_{-c_2}^{y_1} \cdots \int_{-c_{k-2}}^{y_{k-3}} \frac{(y_{k-2})^2}{2} \, dy_{k-2} \cdots dy_1 - \frac{(c_{k-1})^2}{2} \beta_{k-2} + c_k \beta_{k-1}$$

$$= \int_{-c_1}^{0} \frac{1}{(k-1)!} y_1^{k-1} \, dy_1 + \sum_{j=1}^{k-1} \frac{(-1)^{j+1}}{j!} \beta_{k-j} (c_{k-j+1})^j$$

$$= \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j!} \beta_{k-j} (c_{k-j+1})^j = \beta_k.$$

$\square$

Lemma B.6. Let $(c_i)_{i \in \mathbb{N}}$ be a non-negative, non-decreasing sequence and β_k, $k \geq 0$ defined as in (13). If $x \geq c_k$, then

$$\beta_k + \sum_{j=1}^{k} \beta_{j-1} \sum_{l=0}^{k+1-j} \frac{(x - c_j)^l}{l!} = \sum_{l=0}^{k} \frac{x^k}{k!}.$$

Proof. Particularly, we note that the identity is true for $k = 0$. For $k \geq 1$, we find that due to the binomial formula,

$$\sum_{j=1}^{k} \beta_{j-1} \sum_{l=0}^{k+1-j} \frac{(x - c_j)^l}{l!} = \sum_{j=1}^{k} \sum_{l=0}^{k+1-j} \sum_{m=0}^{l} \beta_{j-1} \binom{l}{m} \frac{x^m (-c_j)^{l-m}}{l!}$$

$$= \sum_{j=1}^{k} \sum_{m=0}^{k+1-j} \sum_{l=m}^{k+1-j} \beta_{j-1} \frac{x^m (-c_j)^{l-m}}{m!(l-m)!}$$

$$= \sum_{j=1}^{k} \sum_{l=0}^{k+1-j} \beta_{j-1} \frac{(-c_j)^l}{l!} + \sum_{m=1}^{k} \frac{x^m}{m!} \sum_{j=1}^{k+1-m} \sum_{l=m}^{k+1-j} \beta_{j-1} \frac{(-c_j)^{l-m}}{(l-m)!}$$

$$= \sum_{j=1}^{k} \beta_{j-1} + \sum_{j=1}^{k} \sum_{l=1}^{k+1-j} \beta_{j-1} \frac{(-c_j)^l}{l!} + \sum_{m=1}^{k} \frac{x^m}{m!} \sum_{j=1}^{k+1-m} \beta_{j-1} + \sum_{m=1}^{k} \frac{x^m}{m!} \sum_{j=1}^{k-m} \sum_{l=1}^{k+1-m-j} \beta_{j-1} \frac{(-c_j)^l}{l!}.$$

In the second term and the fourth term, we observe a double summation for all pairs of integers in a triangle. We apply the variable substitution $u = l + j - 1$ and $v = l$ to sum over all pairs in the triangle via the diagonal lines. For the second term this yields

$$\sum_{j=1}^{k} \sum_{l=1}^{k+1-j} \beta_{j-1} \frac{(-c_j)^l}{l!} = \sum_{u=1}^{k} \sum_{v=1}^{u} \beta_{u-v} \frac{(-c_{u-v+1})^v}{v!} = -\sum_{u=1}^{k} \beta_u.$$

Similarly, this argument can be applied to the fourth term. In the end, we obtain

$$\sum_{j=1}^{k} \beta_{j-1} \sum_{l=0}^{k+1-j} \frac{(x-c_j)^l}{l!} = \sum_{j=1}^{k} \beta_{j-1} - \sum_{u=1}^{k} \beta_u + \sum_{m=1}^{k} \frac{x^m}{m!} \left(\sum_{j=1}^{k+1-m} \beta_{j-1} - \sum_{u=1}^{k-m} \beta_u \right)$$

$$= \beta_0 - \beta_k + \beta_0 \sum_{m=1}^{k} \frac{x^m}{m!} = -\beta_k + \sum_{m=0}^{k} \frac{x^m}{m!}.$$

$\square$

Proof of Lemma B.4. First, note that if $\sigma_M(y) = 0$, then $y \le c_i$ for all $i = 1, \ldots, M$ and the probability equals one, and hence the identity holds in this case.

To show the result for $\sigma_M(y) \in (0, M)$, we need the joint density of M order statistics. This is given by the constant $M!$[20, p. 11], yielding

$$\mathbb{P}\left(U_{(i)}^M \le \frac{c_i}{y}, \quad \forall i \le M \right) = \int_0^{c_1/y} \int_{u_1}^{c_2/y} \cdots \int_{u_{\sigma_M(y)-1}}^{c_{\sigma_M(y)}/y} \int_{u_{\sigma_M(y)}}^1 \cdots \int_{u_{M-1}}^1 M! \, du_M \cdots du_2 \, du_1$$

$$= \frac{M!}{y^M} \int_0^{c_1} \int_{v_1}^{c_2} \cdots \int_{v_{\sigma_M(y)-1}}^{c_{\sigma_M(y)}} \int_{v_{\sigma_M(y)}}^y \cdots \int_{v_{M-1}}^y 1 \, dv_M \cdots dv_2 \, dv_1$$

$$= \frac{M!}{y^M} \int_0^{c_1} \int_{v_1}^{c_2} \cdots \int_{v_{\sigma_M(y)-1}}^{c_{\sigma_M(y)}} \frac{(y - v_{\sigma_M(y)})^{M-\sigma_M(y)}}{(M - \sigma_M(y))!} \, dv_{\sigma_M(y)} \cdots dv_2 \, dv_1$$

$$= -\frac{M!}{y^M} \beta_{\sigma_M(y)-1} \frac{(y - c_{\sigma_M(y)})^{M-\sigma_M(y)+1}}{(M - \sigma_M(y) + 1)!}$$

$$+ \frac{M!}{y^M} \int_0^{c_1} \int_{v_1}^{c_2} \cdots \int_{v_{\sigma_M(y)-2}}^{c_{\sigma_M(y)-1}} \frac{(y - v_{\sigma_M(y)})^{M-\sigma_M(y)+1}}{(M - \sigma_M(y) + 1)!} \, dv_{\sigma_M(y)-1} \cdots dv_2 \, dv_1$$

$$= -\frac{M!}{y^M} \sum_{j=2}^{\sigma_M(y)} \beta_{j-1} \frac{(y - c_j)^{M-j+1}}{(M - j + 1)!} + \frac{M!}{y^M} \int_0^{c_1} \frac{(y - v_1)^{M-1}}{(M - 1)!} \, dv_1$$

$$= 1 - \frac{M!}{y^M} \sum_{j=1}^{\sigma_M(y)} \beta_{j-1} \frac{(y - c_j)^{M-j+1}}{(M - j + 1)!},$$

where we used the change of variable $u_i = v_i/y$ for $i = 1, \ldots, M$ and then applied Lemma B.5 multiple times.

For $\sigma_M(y) = M$, we observe that $y > c_i$ for all $i \le M$, and thus requires a separate analysis. Note

$$\frac{y^M}{M!} \mathbb{P}\left(U_{(i)}^M \le \frac{c_i}{y}, \quad \forall i \le M \right) = \int_0^{c_1} \int_{v_1}^{c_2} \cdots \int_{v_{M-1}}^{c_M} dv_M \cdots dv_2 \, dv_1$$

$$= \beta_M = \sum_{j=0}^{M} \frac{y^j}{j!} - \sum_{j=1}^{M} \beta_{j-1} \sum_{l=0}^{M+1-j} \frac{(y - c_j)^l}{l!}$$
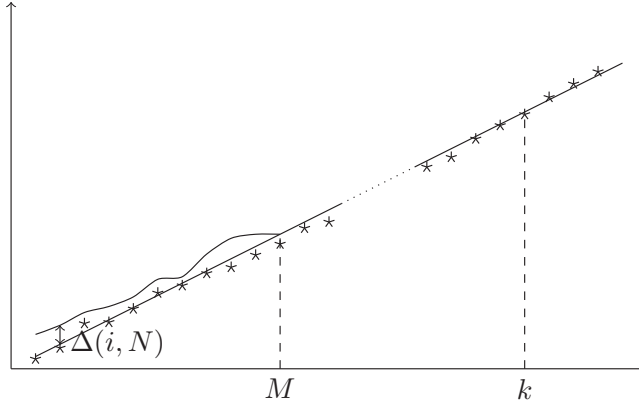
Figure 4. Effect of perturbations for the truncated case.

$$
= \frac{y^M}{M!} - \sum_{j=1}^{M} \beta_{j-1} \frac{(y - c_j)^{M+1-j}}{(M + 1 - j)!}
$$

$$
+ \sum_{j=0}^{M-1} \frac{y^j}{j!} - \sum_{j=1}^{M-1} \beta_{j-1} \sum_{l=0}^{M-j} \frac{(y - c_j)^l}{l!} - \beta_{M-1}
$$

$$
= \frac{y^M}{M!} - \sum_{j=1}^{M} \beta_{j-1} \frac{(y - c_j)^{M+1-j}}{(M + 1 - j)!},
$$

where we applied Lemma B.6 twice. □

Next, we use these results to prove Theorem 3.2. As a first step, we consider a scenario with only finitely many perturbations, see Figure 4.

Proposition B.1. *Let $k := k(N) \leq N$ be a positive function of N such that $k \to \infty$ and $N - k \to \infty$ as $N \to \infty$. Let $F \circ l^N$ be as in (11) with $\Delta(i, N) = 0$ for all $i \geq M$ and $N \geq N_0$ for some fixed $M \in \mathbb{N}$ and $N_0 \in \mathbb{N}$, and let $c_i, i \in \mathbb{N}$ be as in (12). Then, there exists a constant $V_M(\theta, \Delta) \in (0, \infty)$ such that*

$$
\lim_{N \to \infty} \sqrt{\frac{kN}{N - k}} \mathbb{P}(A^N \geq k) = V_M(\theta, \Delta).
$$

Let $\beta_i, i \in \mathbb{N}$, be as in (13) and let $\gamma(\cdot, \cdot)$ denote the lower incomplete gamma distribution. The value of $V_M(\theta, \Delta)$ can be expressed as in (14), i.e.

$$
V_M(\theta, \Delta) = \frac{2}{\sqrt{2\pi}} \left(\frac{\theta}{(M - 1)!} \gamma(M, c_M) + \frac{(c_M)^M}{(M - 1)!} e^{-c_M} \right.
$$

$$
\left. + \sum_{j=1}^{M-1} \beta_{j-1} e^{-c_j} \frac{\Delta(j)}{(M - j)!} \gamma(M - j + 1, c_M - c_j) - \sum_{j=1}^{M-1} \beta_{j-1} \frac{(c_M - c_j)^{M-j+1}}{(M - j)!} e^{-c_M} \right).
$$

Proof. Noting (3) and (B.2), applying Lemma B.3 with $a = 0$ and $b = c_M$ and subsequently invoking Lemma B.4 yields

$$\lim_{N \to \infty} \sqrt{\frac{kN}{N-k}} \mathbb{P}(A^N \geq k) = \frac{2}{\sqrt{2\pi}} \int_0^{c_M} \mathbb{P}\left(U_{(i)}^{M-1} \leq \frac{c_i}{y}, \forall i \leq M-1\right) (\theta + M - y)$$

$$\frac{y^{M-1}}{(M-1)!} e^{-y} \, dy$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{c_M} (\theta + M - y) \frac{y^{M-1}}{(M-1)!} e^{-y} \, dy$$

$$- \int_0^{c_M} \frac{2(\theta + M - y)}{\sqrt{2\pi}} \sum_{j=1}^{\sigma_{M-1}(y)} \beta_{j-1} \frac{(y - c_j)^{M-j}}{(M-j)!} e^{-y} \, dy.$$

The first term can also be expressed as

$$\int_0^{c_M} (\theta + M - y) \frac{y^{M-1}}{(M-1)!} e^{-y} \, dy = \frac{(\theta + M)\gamma(M, c_M)}{(M-1)!} - \frac{M\gamma(M, c_M)}{(M-1)!} + \frac{(c_M)^M}{(M-1)!} e^{-c_M}$$

$$= \frac{\theta}{(M-1)!} \gamma(M, c_M) + \frac{(c_M)^M}{(M-1)!} e^{-c_M}.$$

The second term yields

$$\int_0^{c_M} (\theta + M - y) \sum_{j=1}^{\sigma_{M-1}(y)} \beta_{j-1} \frac{(y - c_j)^{M-j}}{(M-j)!} e^{-y} \, dy$$

$$= \sum_{m=1}^{M-1} \sum_{j=1}^{m} \int_{c_m}^{c_{m+1}} (\theta + M - y) \beta_{j-1} \frac{(y - c_j)^{M-j}}{(M-j)!} e^{-y} \, dy$$

$$= \sum_{j=1}^{M-1} \int_{c_j}^{c_M} \beta_{j-1} (\theta + M - y) \frac{(y - c_j)^{M-j}}{(M-j)!} e^{-y} \, dy$$

$$= \sum_{j=1}^{M-1} \beta_{j-1} e^{-c_j} \int_0^{c_M - c_j} \frac{(\theta + M - c_j - u) u^{M-j}}{(M-j)!} e^{-u} \, du,$$

and similarly as for the first term, this can also be expressed as

$$\int_0^{c_M} (\theta + M - y) \sum_{j=1}^{\sigma_{M-1}(y)} \beta_{j-1} \frac{(y - c_j)^{M-j}}{(M-j)!} e^{-y} \, dy$$

$$= \sum_{j=1}^{M-1} \beta_{j-1} e^{-c_j} \left(\frac{\theta + M - c_j}{(M-j)!} \gamma(M - j + 1, c_M - c_j) \right.$$

$$\left. - \frac{M - j + 1}{(M-j)!} \gamma(M - j + 1, c_M - c_j) + \frac{(c_M - c_j)^{M+j-1}}{(M-j)!} e^{-(c_M - c_j)} \right)$$

$$= \sum_{j=1}^{M-1} \beta_{j-1} e^{-c_j} \frac{-\Delta(j)}{(M-j)!} \gamma(M - j + 1, c_M - c_j) + \sum_{j=1}^{M-1} \beta_{j-1} \frac{(c_M - c_j)^{M+j-1}}{(M-j)!} e^{-c_M}.$$

Subtracting the second term from the first concludes the proof. □

Next, we allow for all perturbations that satisfy conditions (A) and (B) as in Theorem 3.2. It turns out that for the proof it is more convenient to use the following equivalent condition: for every $\epsilon > 0$ there exists a pair $(M_\epsilon, N_\epsilon) \in \mathbb{N} \times \mathbb{N}$ such that $|\Delta(i, N)| < \epsilon$ for all $N \geq N_\epsilon$ and all $M_\epsilon \leq i \leq k(N)$. Conditions (A) and (B) are more intuitive and tractable when considering examples, such as given in Section 4.
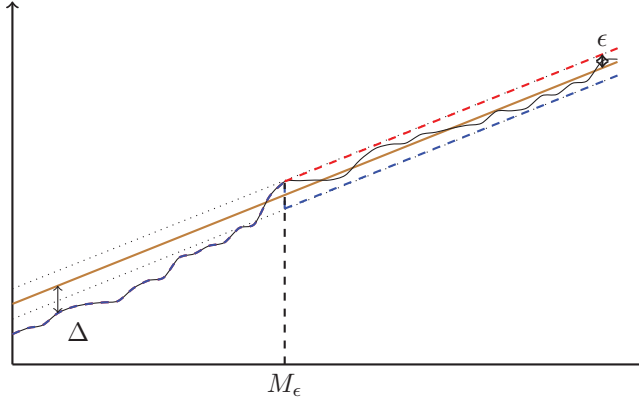
Figure 5. Illustration of infinitely many perturbations setting.

Lemma B.7. Conditions (A) and (B) for perturbations $\Delta(\cdot, \cdot)$ defined in Theorem 3.2 are equivalent to the following: For every $\epsilon > 0$ there exists a pair $(M_\epsilon, N_\epsilon) \in \mathbb{N} \times \mathbb{N}$ such that $|\Delta(i, N)| < \epsilon$ for all $N \geq N_\epsilon$ and all $M_\epsilon \leq i \leq k(N)$.

Proof. ($\Rightarrow$) The boundedness of $\Delta(\cdot)$ is an immediate consequence of the boundedness of $\Delta(\cdot, \cdot)$. By definition of $\Delta(\cdot)$, we can pick a $\hat{N}_{i,\epsilon} \geq N_{\epsilon/2}$ for every $\epsilon > 0$ and for all $i \geq M_{\epsilon/2}$ such that $|\Delta(i, N) - \Delta(i)| < \epsilon/2$ for all $N \geq \hat{N}_{i,\epsilon}$. Then,

$$|\Delta(i)| \leq |\Delta(i, \hat{N}_{i,\epsilon})| + \epsilon/2 < \epsilon$$

for all $i \geq M_{\epsilon/2}$, showing that $\lim_{i \to \infty} \Delta(i) = 0$.

For condition (B) to hold, suppose $\epsilon > 0$ and let $\tilde{N}_\epsilon \in \mathbb{N}$ be such that $i(N) \geq M_\epsilon$ for all $N \geq \tilde{N}_\epsilon$ and $\tilde{N}_\epsilon \geq N_\epsilon$. Then, by assumption we obtain $|\Delta(i(N), N)| < \epsilon$ for all $N \geq \tilde{N}_\epsilon$.

($\Leftarrow$) If not, then $\exists \epsilon > 0$ such that for every $(M_\epsilon, N_\epsilon) \in \mathbb{N} \times \mathbb{N}$ there exists an $i > M_\epsilon$ and $N \geq N_\epsilon$ such that $|\Delta(i, N)| \geq \epsilon$. In particular, if we choose $M_\epsilon = k(N_\epsilon)/2$, then there exists an $\epsilon > 0$ such that for every $N_\epsilon \in \mathbb{N}$ there are a $k(N_\epsilon)/2 \leq i \leq k(N_\epsilon)$ and $N \geq N_\epsilon$ such that $|\Delta(i, N)| \geq \epsilon$, contradicting condition (B). □

We show that the exceedance probability times $\sqrt{kN/(N-k)}$ still converges to a constant by considering the bounds illustrated by the dashed lines in Figure 5 for every fixed $\epsilon > 0$. That is, for an upper bound, we consider the exceedance probability in case of an initial disturbance $(\theta + \epsilon)/N$ and allowing for the first $M_\epsilon - 1$ perturbations. Indeed, this yields an upper bound for all $N \geq N_\epsilon$: the values are the same for all pairs (i, N) with $i \leq M_\epsilon - 1$, and for $i \geq M_\epsilon$, we have $\theta + i - 1 + \Delta(i, N) \leq \theta + \epsilon + i - 1$ for all $N \geq N_\epsilon$. Similarly, for a lower bound we consider the case with initial disturbance $(\theta - \epsilon)/N$ where we allow for the first $M_\epsilon - 1$ perturbations. By applying Proposition B.1, we can determine the asymptotic behavior of the bounds explicitly. We show that as $\epsilon \downarrow 0$, the upper and lower bound converges to the same constant $V(\theta, \Delta) \in (0, \infty)$ defined as in (15).

Proof of Theorem 3.2. By assumption and Lemma B.7, we know that $\forall \epsilon > 0$ there exists a pair $(M_\epsilon, N_\epsilon) \in \mathbb{N} \times \mathbb{N}$ such that $|\Delta(i, N)| < \epsilon$ for every $N \geq N_\epsilon$ and $M_\epsilon \leq i \leq k(N)$. Fix $\epsilon > 0$, and define for all $(i, N) \in \mathbb{N} \times \mathbb{N}$,

$$\Delta_1(i, N) = \begin{cases} \Delta(i, N) - \epsilon & \text{if } i < M_\epsilon, \\ 0 & \text{if } i \geq M_\epsilon, \end{cases}$$

and

$$\Delta_2(i, N) = \begin{cases} \Delta(i, N) + \epsilon & \text{if } i < M_\epsilon, \\ 0 & \text{if } i \geq M_\epsilon. \end{cases}$$

Recall the definition of the stopping times defined in (B.2) and particularly, $\tau_{\theta,\Delta}^N = A^N$. Observe that the case of the upper and lower bound described above thus correspond to stopping times $\tau_{\theta+\epsilon,\Delta_1}^N$ and $\tau_{\theta-\epsilon,\Delta_2}^N$ respectively. Applying Proposition B.1 to these cases with $M = M_\epsilon$ yields

$$\lim_{N\to\infty} \sqrt{\frac{kN}{N-k}} \mathbb{P}\left(\tau_{\theta+\epsilon,\Delta_1}^N \geq k\right) = V_{M_\epsilon}(\theta + \epsilon, \Delta_1),$$

$$\lim_{N\to\infty} \sqrt{\frac{kN}{N-k}} \mathbb{P}\left(\tau_{\theta-\epsilon,\Delta_2}^N \geq k\right) = V_{M_\epsilon}(\theta - \epsilon, \Delta_2).$$

Couple $\tau_{\theta+\epsilon,\Delta_1}^N$, $\tau_{\theta,\Delta}^N = A^N$ and $\tau_{\theta-\epsilon,\Delta_2}^N$. Then the inequalities $\tau_{\theta-\epsilon,\Delta_2}^N \leq A^N \leq \tau_{\theta+\epsilon,\Delta_1}^N$ hold, and hence we obtain

$$\lim_{N\to\infty} \sqrt{\frac{kN}{N-k}} \mathbb{P}\left(A^N \geq k\right) \in \left[V_{M_\epsilon}(\theta - \epsilon, \Delta_2), V_{M_\epsilon}(\theta + \epsilon, \Delta_1)\right].$$

Next, we show that the limits of the upper and lower bound coincide as $\epsilon \downarrow 0$, i.e.

$$\lim_{\epsilon\downarrow 0} \left[V_{M_\epsilon}(\theta + \epsilon, \Delta_1) - V_{M_\epsilon}(\theta - \epsilon, \Delta_2)\right] = 0.$$

For this, we condition on the value of $U_{(M_\epsilon)}^N$:

$$V_{M_\epsilon}(\theta + \epsilon, \Delta_1) - V_{M_\epsilon}(\theta - \epsilon, \Delta_2) = \lim_{N\to\infty} \sqrt{\frac{kN}{N-k}} \left(\mathbb{P}\left(\tau_{\theta+\epsilon,\Delta_1}^N \geq k\right)\right.$$
$$\left. - \mathbb{P}\left(\tau_{\theta-\epsilon,\Delta_2}^N \geq k\right)\right) \leq z_1(\epsilon) + z_2(\epsilon),$$

where

$$z_1(\epsilon) = \lim_{N\to\infty} \sqrt{\frac{kN}{N-k}} \left(\mathbb{P}\left(\tau_{\theta+\epsilon,\Delta_1}^N \geq k; U_{(M_\epsilon)}^N \in I_1\right) - \mathbb{P}\left(\tau_{\theta-\epsilon,\Delta_2}^N \geq k; U_{(M_\epsilon)}^N \in I_1\right)\right),$$

and

$$z_2(\epsilon) = \lim_{N\to\infty} \sqrt{\frac{kN}{N-k}} \mathbb{P}\left(\tau_{\theta+\epsilon,\Delta_1}^N \geq k; U_{(M_\epsilon)}^N \in I_2\right)$$

with $I_1 = [0, \frac{\theta-\epsilon+M_\epsilon-1}{N}]$ and $I_2 = [\frac{\theta-\epsilon+M_\epsilon-1}{N}, \frac{\theta+\epsilon+M_\epsilon-1}{N}]$.

Note that for all $i < M_\epsilon$ and $N \in \mathbb{N}$, $c_{i,N}$ are the same for $\tau_{\theta+\epsilon,\Delta_1}^N$ and $\tau_{\theta+\epsilon,\Delta_1}^N$ by definition of Δ_1 and Δ_2. Applying Lemma B.3 to $z_1(\epsilon)$, we obtain

$$z_1(\epsilon) = \frac{2}{\sqrt{2\pi}} \int_0^{\theta-\epsilon+M_\epsilon-1} \mathbb{P}\left(U_{(i)}^{M_\epsilon-1} \leq \frac{c_i}{y}, \quad \forall i \leq M_\epsilon - 1\right)$$
$$\cdot \left((\theta + \epsilon + M_\epsilon - y)\frac{y^{M_\epsilon-1}}{(M_\epsilon-1)!}e^{-y} - (\theta - \epsilon + M_\epsilon - y)\frac{y^{M_\epsilon-1}}{(M_\epsilon-1)!}e^{-y}\right) dy$$
$$\leq \frac{4\epsilon}{\sqrt{2\pi}} \int_0^{\theta+\epsilon+M_\epsilon-1} \frac{y^{M_\epsilon-1}}{(M_\epsilon-1)!}e^{-y} dy \leq \frac{4\epsilon}{\sqrt{2\pi}}.$$

For every fixed $M \in \mathbb{N}$, differentiating $y^M e^{-y}$ with respect to y and determining its roots shows that this function has one maximum attained at $y = M$ and hence, $y^M e^{-y} \leq M^M e^{-M}$. Using

Lemma B.3, the previous argument and Stirling's bound yields

$$z_2(\epsilon) = \frac{2}{\sqrt{2\pi}} \int_{\theta-\epsilon+M_\epsilon-1}^{\theta+\epsilon+M_\epsilon-1} \mathbb{P}\left(U_{(i)}^{M-1} \leq \frac{c_i}{y}, \quad \forall i \leq M-1\right)(\theta+\epsilon+M_\epsilon-y)\frac{y^{M-1}}{(M-1)!}e^{-y}\,dy$$

$$\leq \frac{2}{\sqrt{2\pi}} \int_{\theta-\epsilon+M_\epsilon-1}^{\theta+\epsilon+M_\epsilon-1} (1+2\epsilon)\frac{(M-1)^{M-1}}{(M-1)!}e^{-(M-1)}\,dy \leq \frac{4\epsilon(1+2\epsilon)}{\sqrt{2\pi}}.$$

Consequently,

$$V_{M_\epsilon}(\theta+\epsilon,\Delta_1) - V_{M_\epsilon}(\theta-\epsilon,\Delta_2) \leq \frac{8\epsilon(1+\epsilon)}{\sqrt{2\pi}},$$

and since the difference is non-negative, it must converge to zero as $\epsilon \downarrow 0$.

What remains to be shown is that the limit of $V_{M_\epsilon}(\theta+\epsilon,\Delta_1)$ exists, and thus also $V_{M_\epsilon}(\theta-\epsilon,\Delta_2)$, and is the same as $V(\theta,\Delta)$ defined in (15). The existence of the limit follows from monotonicity. That is, $V_{M_\epsilon}(\theta+\epsilon,\Delta_1)$ is non-decreasing and bounded from below by a strictly positive constant, for example $V_{M_\epsilon}(\theta-\epsilon,\Delta_2)$ with $\epsilon = 1$. Since every monotone bounded function in a complete metric space converges, it follows that the limit exists as $\epsilon \downarrow 0$. Moreover, since $V(\theta,\Delta) \in [V(\theta-\epsilon,\Delta_2), V(\theta+\epsilon,\Delta_1)]$ for every $\epsilon > 0$, the value of the limit must in fact be $V(\theta,\Delta)$. □

Suppose that for a fixed $\epsilon > 0$, we determined the pair (M_ϵ, N_ϵ) such that $|\Delta(i,N)| < \epsilon$ for all $N \geq N_\epsilon$ and all $M_\epsilon \leq i \leq k(N)$. Since $V(\theta,\Delta)$ lies between $V(\theta-\epsilon,\Delta_2)$ and $V(\theta+\epsilon,\Delta_1)$, it follows from the proof of Theorem 3.2 that

$$\left|V(\theta,\Delta) - V_{M_\epsilon}(\theta,\Delta)\right| \leq \frac{8\epsilon(1+\epsilon)}{\sqrt{2\pi}}.$$

This observation explains why Algorithm 1 can be used to find an approximation for $V(\theta,\Delta)$ that is within a preset distance from its true value.

C. Asymptotic tail behavior for fixed thresholds

Proof of Proposition 4.1. It is known that the distribution function of a standard uniformly distributed random variable is contained in the maximum domain of attraction of a Weibull distribution:

$$\mathbb{P}\left(N(U_{(N)}^N - 1) \leq x\right) = \mathbb{P}\left(U_{(N)}^N \leq 1 + \tfrac{x}{N}\right) \longrightarrow \begin{cases} e^x & x \leq 0, \\ 1 & x > 0, \end{cases}$$

as $N \to \infty$.

Then for every fixed $k \in \mathbb{N}$, the first k order statistics converge in distribution to[20, Chapter 8]

$$\left(N(U_{(N-i+1)}^N - 1)\right)_{i=1,\ldots,k} \xrightarrow{d} \left(Y^{(i)}\right)_{i=1,\ldots,k}$$

as $N \to \infty$, where the joint density of $(Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)})$ is given by

$$\psi_1(x_1,\ldots,x_k) = e^{x_k}, \quad x_k < \cdots < x_1 < 0.$$

This observation is essential to determine the asymptotic exceedance probability, which we derive next.

First suppose that $c_{i,N}$ does not depend on N, i.e. $c_{i,N} = c_i$ for all $N \in \mathbb{N}$. Then, the proof follows by induction. For $k = 1$, the statement holds, since

$$\lim_{N\to\infty} \mathbb{P}\left(U_{(1)}^N \leq \frac{c_1}{N}\right) = \lim_{N\to\infty} 1 - \left(1 - \frac{c_1}{N}\right)^N = 1 - e^{-c_1}.$$

Suppose the statement holds for all integers strictly smaller than k. Then,

$$\lim_{N \to \infty} \mathbb{P}\left(U_{(i)}^N \le \frac{c_i}{N}, \quad \forall i \le k\right)$$

$$= \lim_{N \to \infty} \mathbb{P}\left(U_{(N-i+1)}^N > 1 - \frac{c_i}{N}, \quad \forall i \le k\right)$$

$$= \mathbb{P}\left(Y^{(i)} > -c_i, \quad \forall i \le k\right)$$

$$= \int_{-c_1}^0 \int_{-c_2}^{y_1} \cdots \int_{-c_k}^{y_{k-1}} e^{y_k} \, dy_k \cdots dy_1$$

$$= \int_{-c_1}^0 \int_{-c_2}^{y_1} \cdots \int_{-c_{k-1}}^{y_{k-1}} e^{y_{k-1}} \, dy_{k-1} \cdots dy_1 - e^{-c_k} \int_{-c_1}^0 \int_{-c_2}^{y_1} \cdots \int_{-c_{k-1}}^{y_{k-2}} dy_{k-1} \cdots dy_1$$

$$= 1 - \sum_{j=1}^{k-1} \beta_{j-1} e^{-j} - e^{c_k} \beta_{k-1} = 1 - \sum_{j=1}^{k} \beta_{j-1} e^{-j}.$$

By induction, the statement thus holds for all $k \ge 1$.

Next, suppose $c_{i,N}$ does depend on N, i.e. there is at least one $N \in \mathbb{N}$ such that $c_{i,N} \ne c_i$. Then,

$$\mathbb{P}(A^N \ge k) = \lim_{N \to \infty} \mathbb{P}\left(U_{(i)}^N \le \frac{c_{i,N}}{N}, \quad \forall i \le k\right)$$

$$\xrightarrow{N \to \infty} \mathbb{P}\left(N(U_{(N-i+1)}^N - 1) \ge -c_i(1 + o(1)), \; \forall i \le k\right).$$

Note that for every $\epsilon > 0$ (small enough) there exists a $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$ and $1 \le i \le k$,

$$\mathbb{P}(A^N \ge k) \ge \mathbb{P}\left(N(U_{(N-i+1)}^N - 1) \ge -c_i + \epsilon, \quad \forall i \le k\right)$$

and

$$\mathbb{P}(A^N \ge k) \le \mathbb{P}\left(N(U_{(N-i+1)}^N - 1) \ge -c_i - \epsilon, \quad \forall i \le k\right).$$

Write V_1, V_2 for the integration area of the upper bound and the lower bound respectively, and V for the integration area corresponding to $c_1, \ldots, c_k$. Since $e^x < 1$ for all $x < 0$, it follows that

$$\limsup_{N \to \infty} \mathbb{P}(A^N \ge k) = \int_{V_1} e^y \, dy \le \int_V e^y \, dy + \int_{V_1 \setminus V} 1 \, dy \le 1 - \sum_{j=1}^k \beta_{j-1} e^{-j} + k(c_k + \epsilon)^{k-1} \epsilon.$$

Similarly, for the lower bound, we have

$$\liminf_{N \to \infty} \mathbb{P}(A^N \ge k) \ge \int_V e^y \, dy - \int_{V \setminus V_2} 1 \, dy \ge 1 - \sum_{j=1}^k \beta_{j-1} e^{-j} - k(c_k)^{k-1} \epsilon.$$

Letting $\epsilon \downarrow 0$ we obtain that both the upper bound and the lower bound converge to $1 - \sum_{j=1}^k \beta_{j-1} e^{-j}$. □

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