VRIJE UNIVERSITEIT

Pseudo-parabolic equations with driving convection term

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor aan
de Vrije Universiteit Amsterdam,
op gezag van de rector magnificus
prof.dr. T. Sminia,
in het openbaar te verdedigen
ten overstaan van de promotiecommissie
van de faculteit der Exacte Wetenschappen
op donderdag 6 maart 2003 om 13.45 uur
in de aula van de universiteit,
De Boelelaan 1105

door

Carlota María Cuesta Romero

geboren te Madrid, Spanje

promotor: prof.dr. J. Hulshof
copromotor: prof.dr.ir. C.J. van Duijn

Pseudo-parabolic equations with driving convection term

Carlota Cuesta

The research reported in this thesis has been partially carried out at CWI, the Dutch national research laboratory for mathematics and computer science, within the theme MAS1, a subdivision of the research cluster Modelling, Analysis and Simulation.

The work was also carried out at the Vrije Universiteit Amsterdam, within the Analysis group, and at The University of Nottingham, within the division of Theoretical Mechanics.

The research reported in this thesis was financed by NWO, and also supported by the RTN network Fronts-Singularities.

ISBN 90 6196 519 5

A mi abuela C. R.

Contents

List of Figures

Introduction

This manuscript is mainly dedicated to the study of the initial value problem

$$\begin{aligned} u_t = u_{xx} + (u^2)_x + \varepsilon^2 u_{xxt} \quad &\text{on } \mathbb{R} \times \mathbb{R}^+ \\ u(x,0) = u_0(x) \quad &\text{in } \mathbb{R}. \end{aligned} \tag{1}$$

Equation (1) is a special case of

$$u_t = \{u^\alpha + u^\beta u_x + \varepsilon^2 u^\alpha (u^\gamma u_t)_x\}_x, \tag{2}$$

in which α, β and γ are non-negative constants. Equation (2) appears as a model equation for Darcy flow in porous media with a dynamic capillary pressure relation. We note that equation (2) is a nonlinear diffusion-convection equation with an additional nonlinear third order term involving two space derivatives and one time derivative. Equation (1) is simpler in the sense that only the convection term is nonlinear, but a third order term with mixed derivatives in space and time is also involved. Parabolic equations extended with such a third order term are called pseudo-parabolic equations in the literature. They often appear as regularisation of ill-posed parabolic problems or as a result of a relaxation term introduced in the model.

Every chapter in this thesis is provided with an introduction, here we give a brief summary, the derivation of equations (2) and (1), and a brief overview on the literature on pseudo-parabolic equations.

0.1 Dynamic capillary pressure and derivation of the equations

Flow in porous materials is encountered in a large number of scientific and engineering scenarios, such as flow of oil, water and gas in a petroleum reservoir, or transport of water and contaminants in aquifers. Two-phase flow is the particular case in which simultaneous flow in porous media of two immiscible fluids takes place. These fluids are also referred to as phases. Equation (2) arises in the case of unsaturated groundwater flow, where the phases involved are water and air.

To describe phenomena in a porous medium regarded as a continuum, the phenomena at the microscopic or pore level are averaged over a representative elementary volume (REV), which is also referred to as the macroscopic level. See the schematic drawing in Figure 1.

A principal characteristic of two-phase flow is saturation of each phase, this is the portion of pore volume filled with each phase. Equations for mass balance and

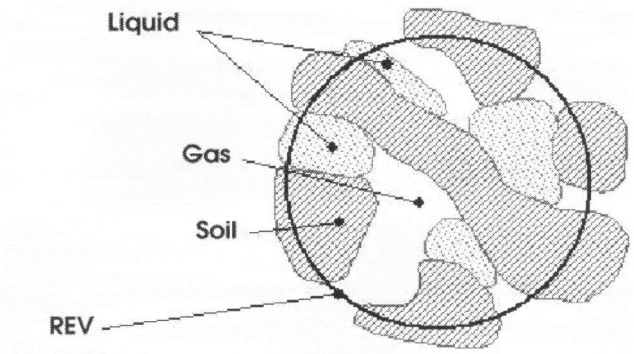


Figure 1: Schematic REV

momentum balance are formulated for each of the phases, the unknown is one of the saturations of the phases. To solve the system one considers a capillary pressure relation, that, roughly speaking, accounts for the interaction between the phases in the REV. We do not go into details on how the mass balance and momentum balance equations are postulated, this lies beyond the purpose of this introduction. We rather concentrate on the capillary pressure relation that leads to equation (2).

0.1.1 Capillary pressure and dynamic capillary pressure

When two immiscible fluids are in contact inside a narrow tube (the pore), a curved interface between them is formed, see Figure 2. The fluid with concave interface is called the wetting phase, the second fluid the non-wetting one. The force acting on the interface, that prevents the fluids from mixing, is called interfacial tension. At equilibrium, this force is related to the jump in pressure of the two fluids and to the curvature of the interface by the Young-Laplace law, see e.g. [8]. The difference of the fluid pressures is called capillary pressure, here denoted as p_c. For example, if the radius of curvature, R, of the interface is constant, then the capillary pressure is given by

$$p_c = 2\frac{\sigma}{R}, \tag{3}$$

where σ is the interfacial tension between the fluid phases. This is a microscopic description of capillary pressure, and results from considerations at the molecular level.

An REV in a porous medium is somehow a network of narrow tubes, the pore geometry is complicated, and an averaged (macroscopic) law similar to (3) is difficult to establish. Usually, in two-phase flow modelling the capillary pressure is also defined as the difference of the pressures of the two phases, so that

$$p_c = p_n - p_w, \tag{4}$$

where p_n is the pressure of the non-wetting phase, and p_w the pressure of the wetting phase. In the standard approach, capillary pressure, p_c above, is expressed

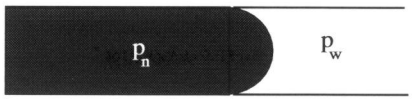


Figure 2: Capillary tube

as a function of the wetting phase saturation, S, and it has to be determined experimentally. This typically results in a function p_c which is decreasing in terms of the wetting phase saturation, see Figure 3. Of course, there are a number of model functions to replace the empirical p_c, typical examples are Brooks and Corey (1964) and van Genuchten (1980), see Bear [9, 10] or Bedrikovetsky [11]. Such expressions are based on measurements of p_c under static conditions within an REV.

It is well-known however, that the difference $p_w - p_n$ depends on other quantities, such as interfacial areas, and on dynamic effects. In order to capture all these dependences, a new definition of macroscopic capillary pressure was introduced by Gray and Hassanizadeh in [32]. This definition is based on thermodynamical considerations at the macroscale. Following [30], the new p_c is a quantity that provides an indication of the system to admit a change in the saturation of the wetting phase, and is not simply the difference of the fluid pressures. Although the new expression for capillary pressure involves many thermodynamic quantities, rather than saturations, they derived a residual thermodynamic relation (that indicates the entropy production of the system), that relates the difference of pressures and the capillary pressure measured under static conditions. The simplest approximation (linear in $\partial_t S$) for the later relation reads

$$p_n - p_w = p_c(S) - L\partial_t S. \tag{5}$$

Here p_c denotes the capillary pressure measured under static conditions, or static capillary pressure. The difference $p_n - p_w$ is defined to be the dynamic capillary pressure. L is regarded as a capillary damping coefficient, which in principle depends on saturation. This approximation tells that only under static conditions ($\partial_t S = 0$) the equality (4) holds.

Based on experiments, Stauffer in [66] postulated a similar relation. Equation (5) is compared with Stauffer's experiments and other experimental work in [30] and [31]. In the latter (5) is also compared with numerical experiments for various values of L.

Replacing (4) by (5) in the model equations leads to equations of the type (2), see the next section.

The capillary pressure relation can be further extended to capture hysteresis. From the sketch in Figure 3 we see that capillary pressure (at equilibrium) is not unique. At a particular saturation, the static capillary pressure measured following drainage of the wetting phase (the non-wetting phase displaces the wetting phase) is higher than that measured following imbibition of the wetting phase (the wetting phase displaces the wetting phase). This phenomenon is called hysteresis. The intermediate curves, or scanning curves, represent capillary pressure changes in terms of saturation of the wetting phase from imbibition to drainage, and from drainage to imbibition.

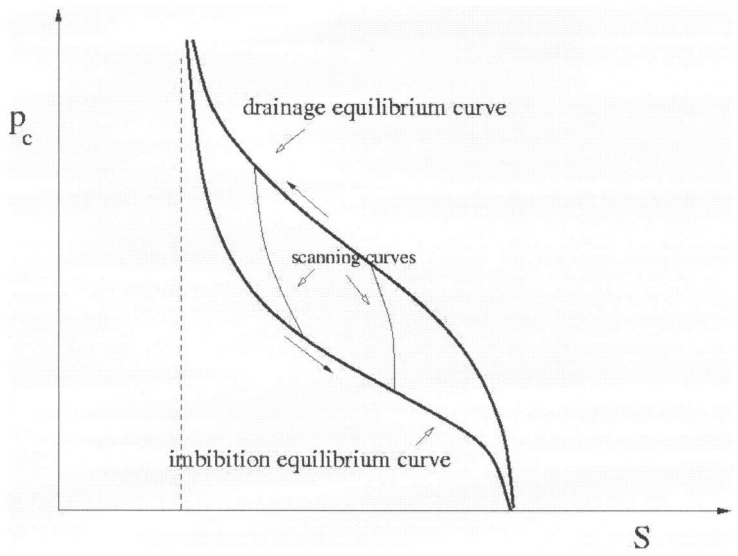


Figure 3: Typical static capillary pressure curves

Hassanizadeh and Beliaev extended equation (5) by a relation that also accounts for hysteresis, in which the scanning curves are approximated by vertical lines. We have not considered this extension. However, the mathematical difficulties are of the same order as for the equation without this effect. Numerical experiments for this extension of (5) are presented by Hassanizadeh et al. in [33], and by Schotting and Beliaev in [12].

Finally, we mention that, using homogenization techniques, Bourgeat and Panfilov in [14] derive an equation similar to (5) for the capillary pressure in a water-oil system containing small scale heterogeneities.

0.1.2 Derivation of equations (1) and (2)

In the derivation of equation (2) we confine ourselves to the particular case of unsaturated groundwater flow, where imbibition takes place under influence of gravity. The two phases are water (wetting phase) and air (non-wetting phase). Throughout we assume relative small values of the water saturation, so that regions where the porous medium is fully saturated, as described by Hulshof [37] and van Duijn and Peletier [73] for example, do not occur. According to Bear [9, 10] we have for water in a homogeneous and isotropic porous medium the momentum balance equation

$$q = -K(S)(\nabla p_w + \rho g e_z) \quad \text{(Darcy's law)}, \tag{6}$$

and the mass balance

$$\phi \partial_t(\rho S) + \operatorname{div}(\rho q) = 0. \tag{7}$$

Here q denotes volumetric water flux, S water saturation, $K(S)$ hydraulic conductivity, p_w water pressure, ρ water density, ϕ porosity and g gravity constant. Further, e_z is the unit vector in positive z-direction, i.e. against the direction of gravity. All these relations are based on the assumption that the air pressure, p_a, is constant in space and time. The equations for mass balance and momentum of the air phase are then negligible. This assumption is common practice in unsaturated groundwater flow and valid for small air viscosity. To solve equations (6) and (7) an additional relation between $p_a - p_w$ and S is needed. We then consider the capillary pressure relation proposed by Gray and Hassanizadeh

$$p_a - p_w = p_c(S) - \phi L(S)\partial_t S. \tag{8}$$

Here $L(S)$ is a nonlinear damping term.

Equations (6), (7) and (8) can be combined into a single equation for S, which reads

$$\phi\partial_t(\rho S) = \operatorname{div}\{\rho K(S)\rho g e_z + \rho K(S)\nabla(-p_c(S) + \phi L(S)\partial_t S)\}. \tag{9}$$

Assuming now a one-dimensional flow in vertical z-direction, with ρ and ϕ constant, and applying a straightforward scaling, equation (9) reduces to

$$\partial_t S = \partial_z\{K(S) + K(S)\partial_z(-p_c(S) + L(S)\partial_t S)\}. \tag{10}$$

To investigate the role of the nonlinear terms (i.e. $K(S)$, $p_c(S)$ and $L(S)$), we replace them by power-law relations. Note that this is consistent with the assumption of small water saturation. So equation (10) reduces to

$$\partial_t S = \partial_z\{S^\alpha + \beta S^{\alpha-\beta-1}\partial_z S + \varepsilon S^\alpha \partial_z(S^\gamma \partial_t S)\},$$

which after a change of notation corresponds to equation (2). Observe that the diffusion term degenerates at $S = 0$.

In a previous work, see [4] and [7], G.I. Barenblatt proposed a different model to describe the non-static situation. He modified (6) and (4) by replacing S by $S + \tau\partial_t S$ ($\tau > 0$), in $p_c(S)$ and $K(S)$. The resulting equation then is of the form

$$\partial_t S = \partial_{zz}\{\Phi_1(S + \tau\partial_t S)\} + \partial_z\{\Phi_2(S + \tau\partial_t S)\}, \tag{11}$$

where Φ_1 and Φ_2 are nonlinear positive functions and degenerate at $S + \tau\partial_t S = 0$.

We note here that if the Barenblatt Ansatz ($S \leftrightarrow S + \tau\partial_t S$) is applied only in (4), then (10) would result with

$$L(S) = -\frac{\tau}{\phi}\frac{dp_c}{dS},$$

implying that $L(S)$ becomes unbounded as $S \downarrow 0$. However, experiments carried out by Smiles et al [64], see also Hassanizadeh [30] for an overview, show that $L(S)$ vanishes as $S \downarrow 0$.

Finally, we note that equation (1) can be seen as a formal limit of equation (2) if one considers a constant initial water saturation S_0. Assuming that $S(z,t) =$

$S_0 + \varepsilon u(z,t)$ and using Taylor expansions of the nonlinear functions $K(S)$, $L(S)$ and $p_c(S)$, the dimensionless equation (10) reads

$$\begin{gathered}\varepsilon\, u_t = \{(K(S_0) + \varepsilon\, K'(S_0)\, u + \ldots) + \\ (K(S_0) + \varepsilon\, K'(S_0)\, u + \ldots)(-p_c(S_0) - \varepsilon\, p_c'(S_0)\, u - \ldots)_z + \\ (K(S_0) + \varepsilon\, K'(S_0)\, u + \ldots)(\varepsilon\, u_t(L(S_0) + \varepsilon\, L'(S_0)\, u + \ldots))_z\}_z\end{gathered}$$

The formal limit as $\varepsilon \to 0$ of the above equation is a linear equation with solution $S \equiv S_0$. In order to get other possible limit equations we scale the independent variables as $z = \frac{X}{\varepsilon^b}$ and $t = \frac{T}{\varepsilon^a}$, in particular the choice $a = 2$, $b = 1$, allows to get in the limit a convection term of the form $(u^2)_x$ as in (1). There are three cases:

If $K'(S_0) \neq 0$ the so

$$\varepsilon^3\, u_T \sim \varepsilon(\varepsilon K'(S_0)u + \frac{1}{2}\varepsilon^2 K''(S_0)\, u^2 + \ldots + \varepsilon K(S_0)(-\varepsilon p_c'(S_0)\, u + \varepsilon^3 L(S_0)\, u_T)_X + \ldots)_X.$$

This calls for the scalings $X = -\frac{1}{\varepsilon}K'(S_0)T + Z$ and $L(S_0) = \frac{1}{\varepsilon}\tilde{L}(S_0)$, wich neglecting the terms of $o(\varepsilon^3)$ gives

$$u_T = \left(\frac{1}{2}K''(S_0)\, u^2 - K(S_0)p_c'(S_0)\, u_Z + K(S_0)\tilde{L}(S_0)K'(S_0)\, u_{ZZ}\right)_Z. \tag{12}$$

If $K'(S_0) = 0$ the distinguished limit has $L(S_0) = \frac{1}{\varepsilon^2}\tilde{L}(S_0)$. In this case the limit equation is

$$u_T = \left(\frac{1}{2}K''(S_0)\, u^2 - K(S_0)p_c'(S_0)\, u_X + K(S_0)\tilde{L}(S_0)\, u_{XT}\right)_X. \tag{13}$$

If $K'(S_0) = \varepsilon\mu$ the distinguished limit has $L(S_0) = \frac{1}{\varepsilon^2}\tilde{L}(S_0)$ and setting $X = -\mu T + Z$, the limit equation is

$$u_T = \left(\frac{1}{2}K''(S_0)\, u^2 - K(S_0)p_c'(S_0)\, u_Z + K(S_0)\tilde{L}(S_0)(u_{ZT} + \mu\, u_{ZZ})\right)_Z. \tag{14}$$

Equation (13) is similar to equation (1). Equation (12) is a case of the so-called KdV-Burgers' equation. Equation (14) combines the two, and is often called BBM-Burgers' equation in the literature. These last two equations are regularisations or the dissipative versions of the KdV equation, see [43], and the BBM equation, see [13], that appear as models of long wave propagation. In this thesis we shall be concerned with (1). We shall discuss the literature for this equation on Section 0.3.

0.2 Pseudo-parabolic equations

One of the earliest pseudo-parabolic equation in the mathematical literature is

$$u_t = u_{xx} + \varepsilon u_{xxt} \quad \text{on } [0,1] \times [0,T], \quad \text{with } \varepsilon > 0, \tag{15}$$

which arises in two-temperature theory of heat conduction, see [70] by Ting. This equation admits a unique solution for given initial data. It exhibits maximum

principles and comparison principles. This can be seen by computing the fundamental solution of (15), which is positive. Unlike for the heat equation ($\varepsilon = 0$), the backward problem (or final condition problem) is also well-posed. This is because equation (15) can be written as an ODE in a Banach space. This is in fact a common feature of pseudo-parabolic equations.

Methods to investigate ill-posed parabolic problems were introduced by Lions and Lattes in [44]. This consists of perturbing the ill-posed parabolic equation so that the perturbed problem is well-posed. Then, the solution of the perturbed problem is shown to approximate a solution of the ill-posed problem. Standard perturbation terms are fourth order terms, or mixed terms: second order in space and first order in time. The latter is also called Sobolev regularisation, and gives rise to pseudo-parabolic equations. For example, Ewing in [24] introduced the Sobolev regularisation for the more general problem

$$u_t = Au, \quad 0 \leq t \leq T,$$

where A is a non-negative self-adjoint operator with dense domain $D(A)$ in a Hilbert space. It reads

$$u_t + \varepsilon A u_t = Au \quad \text{for} \quad 0 \leq t \leq T, \quad \text{with } \varepsilon > 0. \tag{16}$$

Since A is non-negative and $\varepsilon > 0$, the operator $(I + \varepsilon A)$ is invertible so that one can write the equation in evolution form as

$$u_t = (I + \varepsilon A)^{-1} Au \quad \text{for} \quad 0 \leq t \leq T, \quad \text{with } \varepsilon > 0.$$

In this case, the operator $(I + \varepsilon A)^{-1} A$ is bounded, hence the initial value problem is in fact an ODE in $D(A)$ and is well-posed. Moreover the solution approximates a solution of the ill-posed problem.

Observe that $(I+\varepsilon A)^{-1} A$ is the Hille-Yosida approximation of the operator A. In general, for accretive operators this result can be generalised. In [20], DiBenedetto and Pierre considered the class of equations

$$\frac{d}{dt}(I - \varepsilon \Delta \varphi(u)) = \Delta \psi(u), \tag{17}$$

for monotone φ and ψ, for which they prove maximum principles. The approach uses nonlinear semigroup theory, and is a generalisation of the linear case (16). Similar equations are studied in [48], [50] and [5], with a non-monotone φ, so when $\varepsilon = 0$ the problem is ill-posed. These equations have positively invariant regions: if the initial condition lies in an interval of monotonicity of ψ, then the solution will be confined to that interval for all $t > 0$. The existence of these regions is the equivalent of maximum principles.

We have brought attention to maximum principles, because the diffusion coefficient in equation (2) degenerates at $u = 0$. In fact the static capillary pressure relation, (4), would have resulted in the convection-diffusion equation

$$\partial_t S = \partial_z \{K(S) + K(S) \partial_z (-p_c(S))\}. \tag{18}$$

This is a degenerate diffusion equation if $K(S) p_c'(S) = 0$ at $S = 0$. In general, for degenerate parabolic equations, if the initial condition S_0 is non-negative and

not strictly positive, the corresponding solution will have interfaces, i.e curves on the (x,t)-plane that separate regions in which the solution is positive and regions where the solution is zero. This lack of smoothness of the solution is circumvented by defining solutions in a weak sense. Maximum and comparison principles are an essential tool to prove well-posedness. Existence and uniqueness for the initial data S_0 is achieved by first considering the family of initial conditions $S_{0,\varepsilon} = S_0 + \varepsilon > 0$. By comparison principle arguments, there is a unique solution S_ε which is positive. Also the sequence S_ε is ordered and it is then possible to pass to the limit $\varepsilon \to 0$, the unique limit of the sequence S_ε being the corresponding solution for the initial condition S_0.

Equation (2) cannot be written in the form (17), unless $K(S)$ and $L(S)$ are constant, and we cannot apply maximum principles as in [20]. It admits, however the following splitting into two equations which decouple the space and time derivatives

$$w = \partial_t S - p_c(S) \tag{19}$$

$$-\partial_z(K(S)\partial_z w) + w = \partial_z K(S) - p_c(S). \tag{20}$$

Thus equation (2) can be studied as an ODE problem in Banach spaces. The analysis is easy if $K(\cdot) > 0$ and p_c is locally Lipschitz continuous: equation (20) is an elliptic equation that supplemented with boundary conditions for w and for each piecewise continuous function S, gives a unique continuous solution w, which depends smoothly on S, and equation (19) can be written as an ODE in a Banach space. For instance, well-posedness for piecewise continuous initial data holds.

Notice that typically $K(0) = 0$ and since we cannot apply maximum principles as in [20], it is not clear how to adapt the arguments used for (18) to equation (2).

The model introduced by Barenblatt that results in equation (11), degenerates at $S + \tau S_t = 0$. This equation can be written as (17) for the unknown $w = S + \tau S_t$. Maximum principles can be applied, and the approach above described for degenerate parabolic equations can be adapted. See for instance the related model presented in [6] by Barenblatt et al., see also [7].

Maximum principles as in [20] do not apply to equation (1) either. In fact Stecher and Rundell in [67] considered the equation

$$u_t = u_{xx} + u_x + \varepsilon u_{xxt} \tag{21}$$

as an example of pseudo-parabolic equation that does not exhibit maximum principles: it is possible to choose an initial condition, and values of the parameter ε, for which the supremum norm of the corresponding solution increases in t in a small interval $[0, t_0]$. With respect to (1), we shall see numerical examples that show an increase on the supremum norm for solutions, see Chapter 3 Figure 3.5.

Although equation (1) is not degenerate and well-posedness is not difficult to get, the lack of maximum principles provides an obstacle to answer other questions like stability of travelling waves. We shall come back to this point in the next section.

Finally, we mention that pseudo-parabolic equations exhibit persistence of discontinuities in space. If the initial data has a discontinuity at a point $x = x_0$ the corresponding solution will also have a discontinuity at x_0 as long as it exist. This property will bring some advantages in the investigation of equation (1).

Other classical references on pseudo-parabolic equations can by found in the work by Showalter, for example: [62], [58], [63], [60], [59], [61].

0.3 Overview of the thesis

An interesting issue of degenerate parabolic equations is the appearance of moving fronts. For degenerate parabolic equations such as (18) moving fronts propagate with finite speed if the power α of the diffusion coefficient is positive, see e.g. Gilding [29]. By front is meant the interface separating the region where the solution is positive from the region where the solution is zero. If the front propagates with finite speed its position can be written as $\tilde{x}(t) = x_0 + ct$. Such phenomena have been widely studied for degenerate parabolic equations. Using formal asymptotic analysis J. R. King and J. Hulshof in [38] studied the behaviour of solutions near fronts for the pseudo-parabolic equation (2), without convection term and for $\gamma = 0$ (this results from taken the capillary damping coefficient constant in (5)).

Motivated by their results, in Chapter 1 we study existence of global travelling solutions of equation (2), that is to say solutions of the form $u(x,t) = f(x + ct)$, where $c > 0$ is constant, and f satisfies $f(+\infty) = A > f(-\infty) = 0$. It turns out that the profiles of these solutions, when they exists, degenerate, and at the front they behave as profiles of solutions of equation (18), i.e. the front shape does not depend on the mixed third order term. We obtain the following condition for existence of fronts

$$L(S) = o(K(S)p_c'(S)^2) \quad \text{as } S \to 0,$$

so if $K(S)$ is small near $S = 0$, then $L(S)$ must decay sufficiently fast near $S = 0$ to have fronts. This condition is agreement with the results in [38].

Apart from describing the behaviour of moving fronts, travelling wave solutions are conjectured to describe the long time behaviour of solutions of (2). In the global picture of these solutions, the third order term plays a role: travelling wave profiles are non-monotone if ε is sufficiently large. The analysis of stability of these solutions is complicated. We analyse this question for the non-degenerate equation (1). This is the aim of Chapter 2 and Chapter 4. We namely consider travelling wave solutions that connect 0 to $A = 1$. Monotonicity of the profiles depends on the third order term, in this case travelling wave solutions are monotone if $\varepsilon^2 \leq \frac{1}{4}$.

In Chapter 2 we get stability for monotone travelling waves for initial data sufficiently close to a travelling wave solution. This result also holds if the initial condition has jump discontinuities in x. We also prove well-posedness results and conservation of mass. The latter allows to integrate the equation, we then get integral estimates on the integrated equation that imply convergence to travelling wave solutions as $t \to \infty$.

In Chapter 4 we study linear stability of travelling wave solutions. We linearise the equation around a travelling wave and study the spectrum of the linearised operator by means of the Evans function.

The Evans function, denoted by $D_\varepsilon(\lambda)$, is an analytic function for λ in a domain $\Omega \subset \mathbb{C}$ that contains the right half plane. The zeros of $D_\varepsilon(\lambda)$ in Ω correspond to eigenvalues of the linearised operator. We investigate the existence of zeros with

positive real part, i.e. unstable eigenvalues. It turns out that if $\varepsilon^2 < \frac{1}{4}$ there are no eigenvalues with positive real part, this being consistent with the results of Chapter 2. The case $\varepsilon^2 \geq \frac{1}{4}$ is investigated numerically; the numerics suggest that there are no eigenvalues with positive real part. The numerical computation uses a winding number argument, and is based on the previous analysis of $D_\varepsilon(\lambda)$.

Part of this analysis shows that the limit of $D_\varepsilon(\lambda)$ as $\varepsilon \to 0$ is the Evans function corresponding to Burgers' equation, i.e. to equation (1) for $\varepsilon = 0$, see e.g. [35]. For $\varepsilon = 0$, $D_0(\lambda)$ can be explicitly computed, $\lambda = 0$ being the only zero. This is done in Chapter 5, where we consider the more general Burgers' type equation

$$u_t = u_{xx} + (u^p)_x \quad \text{with } p > 1.$$

What is important in this explicit computation, is that it shows what is the domain of analyticity of the Evans function. The domain is contained in a 4-sheeted (2-sheeted if $p = 2$) Riemann surface, each sheet of which is the complex plane $\mathbb{C}$ except for a countable number of poles. This domain is certainly larger than the domain expected from the existing theory, which says that the domain of the Evans function can be analytically extended through the cuts of a Riemann surface to a small neighbourhood of the branch points.

Coming back to equation (1), in Chapter 3 we study numerically the long time behaviour of solutions of (1). Burgers' equation, depending on the initial data, has solutions exhibiting several possible large time behaviours: convergence to a self-similar source type solution, a rarefaction wave or a travelling wave, see [39]. A scaling argument suggests that the limiting behaviour of (1) (for any $\varepsilon > 0$) is as for Burgers' equation ($\varepsilon = 0$). Numerical examples confirm the expectation, in particular we have numerical evidence for stability of travelling wave solutions for large values of ε.

As an observation, we deal numerically with jump discontinuities of solutions (that persists for all time) by imposing continuity of flux through the jumps, which we get from the conservation of mass property.

Comparison principles would have been very useful to analyse the long time behaviour of solutions. For example, in the case of travelling waves it is common to choose an initial condition that lies very close to and between two travelling wave solutions. By comparison principles the corresponding solution will travel between the travelling wave solutions and eventually converge to a travelling wave. However we cannot apply maximum principles techniques, which leads us to use integral estimates, linear stability analysis and numerical analysis to explain the long time behaviour.

Finally we mention that a combination of integral estimates, as in Chapter 2, and Fourier transform techniques are used in a number of papers dedicated to long time behaviour of solutions of BBM-Burgers' equation when a source type initial data is considered, see for instance [2] and references therein. In particular, in [47] convergence of solutions to solutions of the Burgers' equation is proved (for source type initial data). It remains to apply similar ideas to prove convergence to rarefaction waves, and improve the results of Chapter 2 for large values of ε.

Chapter 1

Equation (2): travelling waves

Preamble:
In this chapter we consider the model for non-static groundwater flow, where the saturation-pressure relation is extended by a dynamic term. This approach together with a convective term due to gravity, results in a pseudo-parabolic Burgers' type equation. We give a rigorous study of global travelling wave solutions, with emphasis on the role played by the dynamic term and the appearance of fronts [1].

1.1 Introduction

In this chapter we consider the equation

$$\partial_t S = \partial_z\{K(S) + K(S)\partial_z(-p_c(S) + L(S)\partial_t S)\}, \tag{1.1}$$

that results as the non-static model of groundwater flow described in the Introduction, see Section 0.1. To investigate the role of the nonlinear terms (i.e. $K(S)$ -hydraulic conductivity -, $p_c(S)$ -static capillary pressure - and $L(S)$ -dynamic damping coefficient -), we replace them by power-law relations. Namely we take

$$K(S) = S^\alpha \quad (\alpha > 1), \tag{1.2}$$
$$p_c(S) = -1 + S^{-\beta} \quad (\beta > 0), \tag{1.3}$$
$$L(S) = \varepsilon S^\gamma \quad (\gamma > 0), \tag{1.4}$$

where $\varepsilon > 0$ is introduced as a parameter to investigate the consequence of the third order mixed term. The parameter ranges are chosen to capture the relevant physical properties of unsaturated flow. In particular, we want K and L to be non-negative, with $L(0) = K(0) = K'(0) = 0$, and $p_c(0^+) = \infty$. Using these power law relations, equation (10) reduces to

$$\partial_t S = \partial_z\{S^\alpha + \beta S^{\alpha-\beta-1}\partial_z S + \varepsilon S^\alpha \partial_z(S^\gamma \partial_t S)\}. \tag{1.5}$$

The static capillary pressure relation, would have resulted in the convection-diffusion equation

$$\partial_t S = \partial_z\{S^\alpha + \beta S^{\alpha-\beta-1}\partial_z S\}. \tag{1.6}$$

[1] This chapter has appeared as an article in the European Journal of applied mathematics, see [19].

It is well-known, see e.g. Gilding [29], that this equation for $\varepsilon = 0$ has finite speed of propagation if and only if

$$\int_0^\delta \frac{D(S)}{S} dS < \infty \text{ for some } \delta > 0, \text{ with } D(S) = \beta S^{\alpha-\beta-1}. \tag{1.7}$$

This requires $\alpha - \beta > 1$. Because occurrence of fronts has our special interest we analyse equation (1.5) in the parameter range

$$\beta,\ \gamma > 0 \quad \text{and} \quad \beta < \alpha - 1. \tag{1.8}$$

In this chapter we analyse travelling waves solutions of (1.5). They are conjectured to describe the large time behaviour of solutions resulting from a certain class of initial conditions.

Thus we consider

$$S(z,t) = f(\eta) \quad \text{with} \quad \eta = z + ct, \tag{1.9}$$

subject to the boundary conditions

$$f(\infty) = A > 0, \quad f(-\infty) = \delta \geq 0 \quad (\delta < A). \tag{1.10}$$

Hence the fluid moves downwards whenever the wave speed c is positive. For f we obtain the equation

$$cf' = \{f^\alpha + \beta f^{\alpha-\beta-1} f' + c\varepsilon f^\alpha (f^\gamma f')'\}' \quad \text{on} \quad \mathbb{R}, \tag{1.11}$$

Integration and application of boundary conditions give the equation

$$c(f - \delta) = f^\alpha - \delta^\alpha + \beta f^{\alpha-\beta-1} f' + c\varepsilon f^\alpha (f^\gamma f')', \tag{1.12}$$

and the second order boundary value problem

$$(TW) \begin{cases} c(f-\delta) = f^\alpha - \delta^\alpha + \beta f^{\alpha-\beta-1} f' + c\varepsilon f^\alpha (f^\gamma f')' \quad \text{on} \quad \mathbb{R} \\ \\ f(-\infty) = \delta, \quad f(+\infty) = A, \end{cases}$$

where c is given by

$$c = \frac{A^\alpha - \delta^\alpha}{A - \delta}, \tag{1.13}$$

which is the Rankine-Hugoniot wave speed if we interpret (1.5) as a regularization of the hyperbolic equation $\partial_t S = \partial_x S^\alpha$. Note that $c \downarrow A^{\alpha-1}$ as $\delta \downarrow 0$, $c \uparrow \alpha A^{\alpha-1}$ as $\delta \uparrow A$, with $\frac{dc}{d\delta} > 0$ for $0 < \delta < A$.

In Section 1.2 we show existence of travelling waves for fixed positive values of ε and δ. They are unique up to translations in η. This analysis also shows an oscillatory, but non-periodic, behaviour of the profiles. Here the value of ε is crucial; for ε sufficiently small (depending on $\alpha, \beta, \gamma, \delta$ and A) we obtain monotone profiles.

In Section 1.3 we study the limit case $\varepsilon \to 0$, while $\delta > 0$ is kept fixed. Using essentially monotonicity for small ε, we obtain convergence to travelling waves of equation (1.6)

In Section 1.4 we analyse existence of front solutions to Problem TW with $\delta = 0$. It turns out that there are two relevant ranges of powers α, β, γ for which fronts appear. In the range $2\beta > \alpha - \gamma - 2$ there exists a family of solutions which degenerate at a finite value of η. When $2\beta = \alpha - \gamma - 2$ existence of fronts is shown provided $\varepsilon \leq \frac{\beta}{4A^{2(\alpha-1)}}$. Uniqueness does not hold. Nevertheless we have discerned in each of the previous cases a unique (up to translations in η) exceptional profile, which is the limit profile to (TW) when letting $\delta \to 0$. This is shown in Section 1.5. In the other cases, $2\beta = \alpha - \gamma - 2$, $\varepsilon \geq \frac{\beta}{4A^{2(\alpha-1)}}$, and $2\beta < \alpha - \gamma - 2$, bounded travelling wave solutions do not exist. These results correspond to the formal asymptotic analysis made in [38].

It is worthy to observe that the limits $\varepsilon \to 0$ and $\delta \to 0$ do not commute. We can always take the limit $\varepsilon \to 0$ followed by $\delta \to 0$. However, the converse order is only possible if $2\beta \geq \alpha - \gamma - 2$.

Remark 1.1. *Fixing $\delta \in (0, A)$, the existence is demonstrated for $\alpha > 1$ and $\beta, \gamma > 0$. All other results require in addition $\beta < \alpha - 1$.*

1.2 Existence, uniqueness and monotonicity

The main result of this section is:

Theorem 1.2. *Let $\alpha > 1$, $\beta, \gamma, \varepsilon > 0$ and $0 < \delta < A$. Further, let c be given by (1.13). Then there exists a C^∞ solution of Problem TW, unique up to translations in η.*

Proof. We transform equation (1.12) into a planar system and apply a phase plane analysis. First we set $u = f^{1+\gamma}$, which gives

$$cu^{\frac{1-\alpha}{1+\gamma}} - 1 = (c\delta - \delta^\alpha)u^{-\frac{\alpha}{1+\gamma}} + \frac{\beta}{1+\gamma}u^{(-\frac{\beta}{1+\gamma}-1)}u' + \frac{\varepsilon c}{1+\gamma}u'' \quad \text{on} \quad \mathbb{R} \tag{1.14}$$

with boundary conditions

$$u(-\infty) = \delta^{1+\gamma}, \quad u(\infty) = A^{1+\gamma}.$$

Next we put this equation in the Liénard phase-plane, by considering u and $v := \frac{\varepsilon c}{1+\gamma}u' - u^{-\frac{\beta}{1+\gamma}}$, as independent variables. This results in the system

$$(P)\begin{cases} \varepsilon u' = F(u,v) = \frac{1+\gamma}{c}\left(v + u^{-\frac{\beta}{1+\gamma}}\right) \\ \\ v' = G(u) = -1 + cu^{\frac{1-\alpha}{1+\gamma}} - (c\delta - \delta^\alpha)u^{-\frac{\alpha}{1+\gamma}}. \end{cases}$$

A solution f of (TW) is an integral curve of (P) connecting the equilibria $p_- = (\delta^{1+\gamma}, -\delta^{-\beta})$ and $p_+ = (A^{1+\gamma}, -A^{-\beta})$. The phase plane, with the isoclines $\Gamma_u = \{(u,v) : F(u,v) = 0\}$ and $\Gamma_v = \{(u,v) : G(u) = 0\}$, is drawn in Figure 1.1.

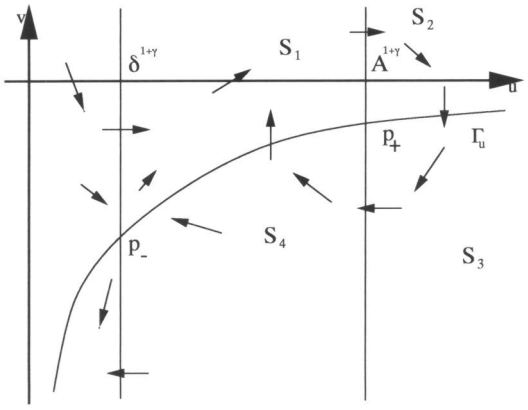


Figure 1.1: Phase plane for (P)

The matrix of the linearised system is

$$\begin{pmatrix} -\frac{\beta}{\varepsilon c}X^{-\beta-1-\gamma} & \frac{1+\gamma}{\varepsilon c} \\ \frac{X^{-\alpha-\gamma-1}}{1+\gamma}\left(c(1-\alpha)X+(cX-X^{\alpha})\alpha\right) & 0 \end{pmatrix},$$

where $X=\delta$ corresponds to the equilibrium p_-, and $X=A$ to p_+. The eigenvalues in p_- and p_+ are, expressed in X,

$$\lambda = -a \pm \sqrt{a^2+b} \text{ with } a = \frac{\beta}{2\varepsilon c}\frac{1}{X^{\beta+\gamma+1}} \text{ and } b = \frac{c-\alpha X^{\alpha-1}}{X^{\alpha+\gamma}\varepsilon c}. \tag{1.15}$$

It is clear that $a>0$ for positive X, and b has the sign of $c-X^{\alpha-1}\alpha$. Introducing its primitive

$$i(X) = cX - X^{\alpha}, \tag{1.16}$$

we note that A and δ are related by $i(A)=i(\delta)>0$, so that $b>0$ in $X=\delta$ and $b<0$ in $X=A$. Therefore at p_- the two eigenvalues are real and of opposite sign, whence p_- is a saddle point. The point p_+ is a sink and

$$\begin{aligned} &0 \le a^2+b < a^2 \text{ implies two real eigenvalues, } \lambda_1 \le \lambda_2 < 0, \text{ and} \\ &a^2+b<0 \text{ implies two complex eigenvalues, with negative real part.} \end{aligned} \tag{1.17}$$

To prove Theorem 1.2 we have to show that an orbit leaving p_- connects to p_+. Because p_- is a saddle point there exist locally two orbits containing solutions $(u(\eta),v(\eta))$ of (P), satisfying

$$\lim_{\eta\to-\infty}(u(\eta),v(\eta)) = p_-.$$

Let C be the orbit for which $u',v'>0$. Inspection of Figure 1.1 shows that the other orbit will never reach p_+. The only possibility giving existence of a travelling

wave is for C to end up in p_+. The corresponding solutions will satisfy (TW), and uniqueness up to translations in η will hold.

To show that C connects to p_+ we consider the sets, see also Figure 1.1,

$$\begin{aligned}
S_1 &= \{(u,v) \in \mathbb{R}^2 : \delta^{\gamma+1} < u < A^{\gamma+1}, v > -u^{-\frac{\beta}{1+\gamma}}\},\\
S_2 &= \{(u,v) \in \mathbb{R}^2 : u > A^{\gamma+1}, v > -u^{-\frac{\beta}{1+\gamma}}\},\\
S_3 &= \{(u,v) \in \mathbb{R}^2 : u > A^{\gamma+1}, v < u^{-\frac{\beta}{1+\gamma}}\},\\
S_4 &= \{(u,v) \in \mathbb{R}^2 : \delta^{\gamma+1} < u < A^{\gamma+1}, v < -u^{-\frac{\beta}{1+\gamma}}\}.
\end{aligned}$$

Note that the boundaries of S_1, S_2, S_3, S_4 are the isoclines of system (P). Furthermore C enters S_1 from p_-.

We have the following two possibilities.

Lemma 1.3. *The orbit C rotates around p_+, going from S_i to $S_{i+1 \bmod 4}$, or it enters p_+ from S_1 or S_3.*

Proof. Let $(u(\eta), v(\eta))$ be a solution of (P) parametrising C. Further, let $\eta_{max} \leq \infty$ be the maximum η-value for which the solution can be extended to $(-\infty, \eta_{max})$. Near points where $u'(\eta) \neq 0$, we can express any solution of (P) locally as $v = v(u)$, satisfying

$$\frac{dv}{du} = \frac{v'}{u'} = \varepsilon \frac{G(u)}{F(u,v)}. \tag{1.18}$$

Below we exhaust all possibilities.

1. Suppose $(u(\eta), v(\eta)) \in S_1$ for all $0 < \eta < \eta_{max}$. Then two cases need to be considered:

> **(i)**: $u \uparrow \bar{u}$ and $v \uparrow \bar{v}$, with $\bar{u}, \bar{v} < \infty$. This implies $\eta_{max} = +\infty$ and $(\bar{u}, \bar{v}) = p_+$.
>
> **(ii)**: $u \uparrow \bar{u}$ and $v \uparrow \infty$. Since $F(u,v)$ becomes unbounded as $v \to \infty$, while $G(u)$ remains bounded, (1.18) directly gives a contradiction.

Thus either $(u(\eta), v(\eta)) \to p_+$ as $\eta \to \infty$ or C crosses $u = A^{1+\gamma}$ at $\eta = \eta_1 \in (0, \eta_{max})$ and enters S_2.

2. Suppose $(u(\eta), v(\eta))$ remains in S_2 for all $\eta \in (\eta_1, \eta_{max})$. Again two cases are possible.

> **(i)**: $u \uparrow \bar{u}$ and $v \downarrow \bar{v}$ with $\bar{u} < \infty$ and, of course, $\bar{v} > -\infty$. Then $\eta_{max} = +\infty$ and $(\bar{u}, \bar{v})$ is an equilibrium point, which is impossible because there are no equilibria to the right of p_+.
>
> **(ii)**: $u \uparrow +\infty$ and $v \downarrow \bar{v}$, with $\bar{v} \geq 0$. Then the v-equation implies $v' \to -1$ contradicting $\bar{v} \geq 0$.

Thus, there exists $\eta_2 \in (\eta_1, \eta_{max})$ such that $(u(\eta), v(\eta))$ crosses Γ_u and enters S_3.

3. Suppose $(u(\eta), v(\eta)) \in S_3$ for all $\eta \in (\eta_2, \eta_{max})$. Completely analogous to **1.**, we find that either $(u(\eta), v(\eta)) \to p_+$ or C crosses $A^{1+\gamma}$ at some $\eta_3 \in (\eta_0, \eta_{max})$ and enters S_4.

4. Suppose $(u(\eta), v(\eta)) \in S_4$ for all $\eta \in (\eta_3, \eta_{max})$. Again we distinguish

(i): $u \downarrow \bar{u}$ and $v \uparrow \bar{v}$. As before $\eta_{max} = +\infty$ and consequently $(\bar{u}, \bar{v}) = p_-$. Thus C is a homoclinic orbit. Next we consider the domain D enclosed by C. Since its boundary is smooth, except at p_- where it is Lipschitz, we may apply Gauss' theorem. Using the fact that C is an orbit of (P) we get

$$0 = \oint_C \left(\frac{1}{\varepsilon} F(u,v)dv - G(u)du \right) = \iint_D \operatorname{div} \left(\frac{F(u,v)}{\varepsilon}, G(u) \right),$$

contradicting

$$\operatorname{div} \left(\frac{1}{\varepsilon} F(u,v), G(u) \right) = -\frac{\beta}{c\varepsilon} u^{-\frac{\beta+\gamma+1}{\gamma+1}} < 0 \quad \text{for all } u > \delta^{\gamma+1}.$$

(ii): C crosses $u = \delta^{\gamma+1}$ below p_- at $(\delta^{1+\gamma}, \tilde{v})$. Consider the closed curve $C \cup C_1$, where C_1 is the straight line segment parametrised by

$$\begin{cases} u(s) = \delta^{\gamma+1}, \\ v(s) = s \quad s \in [\tilde{v}, -\delta^{-\beta}]. \end{cases}$$

As before we call D the interior of $C \cup C_1$, and apply Gauss' theorem. This gives

$$0 > \iint_D \operatorname{div} \left(\frac{F(u,v)}{\varepsilon}, G(u) \right) = \oint_{C \cup C_1} \left(\frac{1}{\varepsilon} F(u,v)dv - G(u)du \right) =$$
$$\oint_{C_1} \left(\frac{1}{\varepsilon} F(u,v)dv - G(u)du \right) = \int_{\tilde{v}}^{-\delta^{-\beta}} \frac{1}{\varepsilon} F(\delta^{\gamma+1}, s)ds > 0,$$

a contradiction.

Hence there exists $\eta_4 \in (\eta_3, \eta_{max})$ at which $(u(\eta), v(\eta))$ crosses Γ_u and enters S_1. □

To complete the proof of the theorem we use Lemma 1.3 and the Poincaré-Bendixon theorem. This leave us with two possibilities. Either $(u(\eta), v(\eta)) \to p_+$ as $\eta \to \infty$ or C approaches a periodic orbit. The latter is impossible by the same argument as in **4(i)**, now applied to the periodic orbit instead of C. The C^∞ regularity for f follows from a bootstrap argument. □

Next we derive a sufficient condition for the travelling wave solution to have a monotone profile. This condition is related to the value of ε. Therefore we write $C = C_\varepsilon$ and $P = P_\varepsilon$ whenever appropriate.

From (1.15) we see that for all ε such that

$$\varepsilon < \varepsilon^* := \frac{\beta^2}{4c(A^{\alpha-1}\alpha - c)} A^{\alpha-2\beta-\gamma-2} \quad (\varepsilon^* > 0), \tag{1.19}$$

the eigenvalues λ_1, λ_2 are real and strictly negative, which is a necessary condition for the travelling wave profile to be monotone. Henceforth we suppose we are in

this situation. With $\lambda_2 < \lambda_1 < 0$ we call (u_1, v_1) the slow eigenvector and (u_2, v_2) the fast eigenvector at p_+, where

$$(u_{1,2}, v_{1,2}) = \left(1, \frac{1}{2(1+\gamma)} \frac{\beta \pm \sqrt{\beta^2 + 4(cA - A^\alpha\alpha)\varepsilon c A^{2\beta-\alpha+\gamma+1}}}{A^{1+\gamma+\beta}}\right).$$

By standard local analysis, see e.g. [16], there exist exactly two orbits entering p_+ tangent to the (u_2, v_2)-direction: one along (u_2, v_2), the other along $(-u_2, -v_2)$. The connecting orbit goes around p_+ at most a finite number of times.

Proposition 1.4. *Let (1.8) be satisfied. For $\delta > 0$ fixed, there exists $0 < \bar{\varepsilon} < \varepsilon^*$, such that for every $0 < \varepsilon < \bar{\varepsilon}$ the travelling wave obtained in Theorem 1.2 is strictly increasing on $\mathbb{R}$.*

Proof. Proposition 1.4 is a direct consequence of Lemma 1.5 below, in which we construct an invariant region which contains C and which itself is contained in S_1.

More specifically, for fixed $\delta \in (0, A)$ and $\mu \in (0, 1)$, let S_μ^δ denote the set enclosed by the curves

$$u = \delta^{1+\gamma}, v = -u^{-\frac{\beta}{1+\gamma}} \quad \text{and} \quad v = g_\mu(u) := (-\mu u^{-\frac{\beta}{1+\gamma}} - (1-\mu)A^{-\beta}).$$

We will show in Proposition 1.5 that for ε sufficiently small (i.e. $\varepsilon < \varepsilon_\mu^\delta$), S_μ^δ is invariant for Problem P_ε or, equivalently, $C_\varepsilon \in S_\mu^\delta$. Proposition 1.5 will also be helpful in the study of the limits $\varepsilon \to 0$ and $\delta \to 0$. In particular we want to bound ε_μ^δ away from zero as $\delta \to 0$. It will appear that this is only possible if $2\beta + \gamma + 2 - \alpha \geq 0$.

Lemma 1.5. *For any fixed δ in $(0, A)$ and $\mu \in (0, 1)$ there exists $\varepsilon_\mu^\delta \in (0, \varepsilon^*)$, such that for every $\varepsilon \in (0, \varepsilon_\mu^\delta)$, $C_\varepsilon \in S_\mu^\delta$. Further, if $2\beta + \gamma + 2 - \alpha \geq 0$,*

$$\lim_{\delta \to 0} \varepsilon_\mu^\delta = 4\mu(1-\mu) \lim_{\delta \to 0} \varepsilon^* = \mu(1-\mu) \frac{\beta^2}{4(\alpha - 1)} A^{-2\beta-\gamma-1},$$

and

$$\lim_{\delta \to 0} S_\mu^\delta \quad \textit{is invariant for Problem TW with} \quad \delta = 0.$$

Proof. Observe that the eigenvectors at p_+ satisfy

$$(u_1, v_1) \to (1, \frac{\beta}{(1+\gamma)A^{1+\gamma+\beta}}), \text{ and } (u_2, v_2) \to (1, 0) \text{ as } \varepsilon \to 0,$$

where $(1, \frac{\beta}{(1+\gamma)A^{1+\gamma+\beta}})$ is the tangent vector at p_+ to Γ_u. The invariant region S_μ^δ is below the horizontal line $v = -A^{-\beta}$, and only contains orbits entering p_+ along the slow eigenvector (u_1, v_1). Observe that $g_\mu(u) > -u^{-\frac{\beta}{1+\gamma}}$ for all $0 < u < A^{1+\gamma}$, and $g_\mu(A^{1+\gamma}) = -A^{-\beta}$. Obviously the vector field is pointing inwards at boundary points of S_μ^δ on $u = \delta^{1+\gamma}$ and Γ_u. It remains to examine the vector field on $v = g_\mu(u)$. We clearly must have

$$\frac{dv}{du} = \frac{\varepsilon c}{1+\gamma} \frac{G(u)}{g_\mu(u) + u^{-\frac{\beta}{1+\gamma}}} \leq g'_\mu(u) \quad \text{for all } u \in (\delta^{1+\gamma}, A^{1+\gamma}), \tag{1.20}$$

for S_μ^δ to be invariant. This is equivalent to

$$\varepsilon H_\delta(u) \le \mu(1-\mu) \tag{1.21}$$

where

$$H_\delta(u) = \frac{c}{\beta}\,\frac{-1 + cu^{\frac{1-\alpha}{1+\gamma}} - (c\delta - \delta^\alpha)u^{-\frac{\alpha}{1+\gamma}}}{(u^{-\frac{\beta}{1+\gamma}} - A^{-\beta})}u^{1+\frac{\beta}{1+\gamma}}. \tag{1.22}$$

Note that $H_\delta(\delta^{1+\gamma}) = 0$ and, by L'Hôspital's rule,

$$\begin{aligned} H_\delta(A^{1+\gamma}) = \lim_{u\to A^{1+\gamma}} H_\delta(u) &= c\,\frac{\alpha A^{\alpha-1} - c}{\beta^2} A^{2\beta+\gamma+2-\alpha} \\ &\to \frac{(\alpha-1)}{\beta^2} A^{2\beta+\gamma+\alpha} \text{ as } \delta \to 0. \end{aligned}$$

For $u > 0$ fixed, $\lim\limits_{\delta\to 0} H_\delta(u)$ behaves as $u^{\frac{2\beta+\gamma+2-\alpha}{1+\gamma}}$ near 0, which suggests to write $H_\delta(u)$ as

$$H_\delta(u) = \frac{c}{\beta} u^{\frac{2\beta+\gamma+2-\alpha}{1+\gamma}} h_\delta(u) \tag{1.23}$$

$$\text{where} \quad h_\delta(u) = \frac{-u^{\frac{\alpha}{1+\gamma}} + cu^{\frac{1}{1+\gamma}} - (c\delta - \delta^\alpha)}{u^{\frac{1}{1+\gamma}}(1 - A^{-\beta}u^{\frac{\beta}{1+\gamma}})}.$$

Observe that for every $u \in (0, A^{1+\gamma})$ and every $\delta > 0$, recalling (1.13),

$$h_0(u) - h_\delta(u) = \frac{(c\delta - \delta^\alpha) - u^{\frac{1}{1+\gamma}}(c - A^{\alpha-1})}{u^{\frac{1}{1+\gamma}}(1 - A^{-\beta}u^{\frac{\beta}{1+\gamma}})} > 0. \tag{1.24}$$

So that

$$h_\delta(u) < h_0(u) = \frac{-u^{\frac{\alpha}{1+\gamma}} + A^{\alpha-1}u^{\frac{1}{1+\gamma}}}{u^{\frac{1}{1+\gamma}}(1 - A^{-\beta}u^{\frac{\beta}{1+\gamma}})} = A^{\alpha-1}\frac{1 - A^{1-\alpha}u^{\frac{\alpha-1}{1+\gamma}}}{1 - A^{-\beta}u^{\frac{\beta}{1+\gamma}}}, \tag{1.25}$$

which is increasing for $u \in (0, A^{1+\gamma})$. Here we used $\beta < \alpha - 1$ from (1.8).

Setting

$$M := \sup_{u\in(0,A^{1+\gamma})} h_0(u) = \frac{(\alpha-1)A^{\alpha-1}}{\beta},$$

an upper bound for $H_\delta(u)$ is given by

$$H_\delta(u) < M\frac{c}{\beta}A^{2\beta+\gamma+2-\alpha} \quad \text{if} \quad 2\beta+\gamma+2-\alpha \ge 0,$$

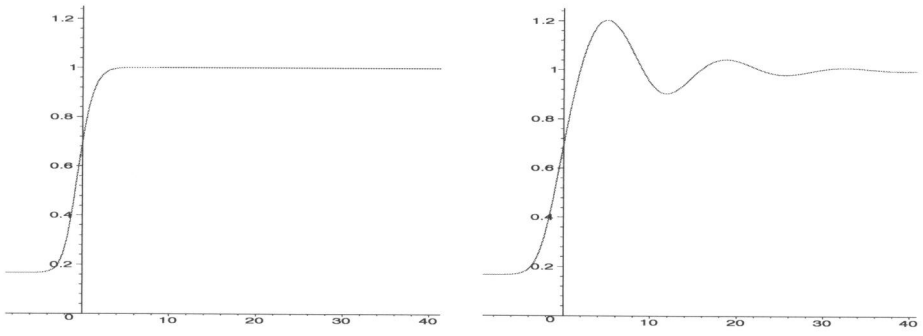
and

$$H_\delta(u) < M\frac{c}{\beta}\delta^{2\beta+\gamma+2-\alpha} \quad \text{if} \quad 2\beta+\gamma+2-\alpha \le 0.$$

Thus a sufficient condition for (1.21) to hold is

$$\varepsilon < \varepsilon_\mu^\delta := \begin{cases} \mu(1-\mu)\frac{\beta^2}{c(\alpha-1)}A^{-2\beta-\gamma-1} & \text{if} \quad 2\beta+\gamma+2-\alpha \ge 0 \\ \mu(1-\mu)\frac{\beta^2}{c(\alpha-1)A^{\alpha-1}}\delta^{\alpha-2\beta-\gamma-2} & \text{if} \quad 2\beta+\gamma+2-\alpha \le 0. \end{cases} \tag{1.26}$$

This completes the proof of the first statement. The statements about the $\delta \to 0$ limit follow immediately from (1.26) with $2\beta \geq \alpha - \gamma - 2$, (1.19) and (1.13). □



(a) monotone profile: $\varepsilon = \varepsilon^* - 0.01 \approx 0.048$

(b) oscillatory profile: $\varepsilon = 1$

Figure 1.2: Travelling wave solutions for different values of ε, where $f(0) = \frac{A+\delta}{2}$, $\alpha = \frac{4}{3}$, $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$, $A = 1$, $\delta = \frac{1}{6}$

1.3 The $\varepsilon \to 0$ limit case

Let $\delta \in (0, A)$ be fixed and (1.8) be satisfied. In this section we examine the behaviour of the connecting orbit C_ε and that of the corresponding travelling wave $f = f_\varepsilon$ as $\varepsilon \to 0$. For $\varepsilon < \bar{\varepsilon}$ we denote C_ε by

$$v = \varphi_\varepsilon(u), \quad \delta^{\gamma+1} \leq u \leq A^{\gamma+1}. \tag{1.27}$$

As a first convergence result we have

Proposition 1.6. *$\varphi_\varepsilon(u) \to -u^{-\frac{\beta}{1+\gamma}}$ uniformly on $[\delta^{1+\gamma}, A^{1+\gamma}]$ as $\varepsilon \to 0$.*

Proof. Lemma 1.5 implies

$$-u^{\frac{-\beta}{1+\gamma}} < \varphi_\varepsilon(u) < g_\mu(u) \tag{1.28}$$

for all $u \in (\delta^{1+\gamma}, A^{1+\gamma})$ and for all $\varepsilon \in (0, \varepsilon_\mu^\delta)$. Since $g_\mu(u) \to -u^{-\frac{\beta}{1+\gamma}}$ as $\mu \uparrow 1$, the result is immediate. □

For the travelling waves f_ε we have

Theorem 1.7. *Translate f_ε so that $f_\varepsilon(0) = \frac{\delta+A}{2}$ for all $\varepsilon > 0$. Then $f_\varepsilon \to f \in C^\infty(\mathbb{R})$ as $\varepsilon \to 0$, uniformly on $\mathbb{R}$, where f satisfies Problem TW with $\varepsilon = 0$.*

Proof. First we employ the scaling $\eta = \varepsilon\tau$, so that in the τ-variable (P_ε) reads

$$(\tilde{P}_\varepsilon)\begin{cases} \dot{u} = F(u,v), \\ \dot{v} = \varepsilon G(u). \end{cases} \tag{1.29}$$

Unlike (P_0) the limit system $(\tilde{P}_0)$, is well defined. The one-dimensional manifold of critical points

$$M_0 = \{F(v,u) = 0\} = \{v = -u^{\frac{-\beta}{1+\gamma}}\}$$

is invariant and normally hyperbolic in the sense of geometric singular perturbation theory, see [65], because for $(\tilde{P}_0)$ the only pure imaginary eigenvalue is zero, and has a one-dimensional eigenspace tangential to M_0. Let K be a neighbourhood of $\{(u,v) \in M_0 : \delta^{1+\gamma} \le u \le A^{1+\gamma}\}$, and choose $0 < \delta_1 < \delta^{1+\gamma} < A^{1+\gamma} < A_1$ such that $\{(u,v) \in M_0 : \delta_1 \le u \le A_1\} \subset\subset K$. By Fenichel's invariant manifold theorem [65], there exists, for given $k \in \mathbb{N}$, a number $\varepsilon_0 > 0$ and a function $h \in C^k([\delta_1, A_1]\times[0,\varepsilon_0])$ with $h(u,0) = -u^{-\frac{\beta}{1+\gamma}}$, such that for every $0 < \varepsilon < \varepsilon_0$

$$M_\varepsilon = \{(u,v) \in K : \ v = h(u,\varepsilon),\ \delta_1 \le u \le A_1\}$$

is locally invariant. The manifold M_ε is not uniquely determined. However between $u = \delta^{1+\gamma}$ and $u = A^{1+\gamma}$ it must coincide with the connecting orbit $v = \varphi_\varepsilon(u)$, because this is the only orbit which remains close to $\{v = -u^{-\frac{\beta}{1+\gamma}} : \ \delta_1 < u < A_1\}$.

Using (1.27) and the v-equation in (P_ε), we note that $u_\varepsilon = f_\varepsilon^{1+\gamma}$ satisfies,

$$u' = \frac{G(u)}{\varphi_\varepsilon'(u)}, \tag{1.30}$$

and connects the two zeros of G.

Since $h \in C^k$ and $h(u,\varepsilon) = \varphi_\varepsilon(u)$ for $\delta^{1+\gamma} \le u \le A^{1+\gamma}$, we have as a result of Proposition 1.6, that $\varphi_\varepsilon'(u) \to \frac{\beta}{1+\gamma} u^{-\frac{\beta}{1+\gamma}-1}$ uniformly on $[\delta^{1+\gamma}, A^{1+\gamma}]$ and thus

$$\frac{G(u)}{\varphi_\varepsilon'(u)} \to \frac{G(u)}{\frac{\beta}{1+\gamma} u^{-\frac{\beta}{1+\gamma}-1}} \quad \text{uniformly}$$

on $[\delta^{1+\gamma}, A^{1+\gamma}]$ as $\varepsilon \to 0$. In this limit the differential equation (1.30) is identical to equation (1.14) with $\varepsilon = 0$. Using the fact that $u_\varepsilon(0)$ is fixed for all $\varepsilon > 0$, standard arguments imply that u_ε converges uniformly on $\mathbb{R}$ to the corresponding solution of the limit equation. □

1.4 $\delta = 0$ system

In this section we consider the limit case $\delta = 0$ directly. Thus we study the system

$$(P_\varepsilon^0)\begin{cases} \varepsilon u' = F_0(u,v) = \frac{1+\gamma}{c}\left(v + u^{-\frac{\beta}{1+\gamma}}\right), \\ v' = G_0(u) = -1 + c u^{\frac{1-\alpha}{1+\gamma}}, \end{cases}$$

where $c = A^{\alpha-1}$, and we look for orbits connecting $u = 0$ to $u = A^{1+\gamma}$. The critical point corresponding to the latter now has real eigenvalues, see also (1.19), for

$$0 < \varepsilon \leq \varepsilon^* = \frac{\beta^2}{4(\alpha-1)} A^{-\alpha-2\beta-\gamma}.$$

The phase plane, see Figure 1.1, clearly implies, that the desired orbit has to be originated from the segment $\{(u,v) : u = 0, v \leq 0\}$ where the equations are singular. Since we are interested in (P_ε^0) as limit of (P_ε^δ), and in particular of a possible limit orbit of the connecting orbit C, we expect such a limit orbit, if it exists, to behave as $v \sim -d\,u^{-\frac{\beta}{1+\gamma}}$, $0 < d \leq 1$, as $u \to 0$. Thus a convenient new dependent variable is $Z = u^q v$, where q, for later purposes, is not fixed yet. Whenever $u' \neq 0$, Z satisfies the equation

$$u\frac{dZ}{du} = qZ + \frac{\varepsilon c}{1+\gamma} u^{1+2q+\frac{1-\alpha}{1+\gamma}} \frac{c - u^{\frac{\alpha-1}{1+\gamma}}}{Z + u^{q-\frac{\beta}{1+\gamma}}}. \tag{1.31}$$

Below we investigate the solvability of (1.31) for $0 < u < A^{1+\gamma}$. The analysis and results critically depend on the value of the parameters α, β and γ. In particular the value of $2\beta - \alpha + \gamma + 2$ plays a crucial role, which is to be expected considering the results of the formal analysis by Hulshof and King [38]. With q appropriately chosen, we consider the cases:

1.4.1 $2\beta > \alpha - \gamma - 2$.

Here we take $q = \frac{\beta}{1+\gamma}$, and set $W = u^{\frac{2\beta+\gamma+2-\alpha}{1+\gamma}}$. Then (1.31) becomes

$$(2\beta + \gamma + 2 - \alpha)\frac{dZ}{dW} = \frac{\beta Z}{W} + \frac{\varepsilon c(c - W^{\frac{\alpha-1}{2\beta+\gamma+2-\alpha}})}{Z+1}. \tag{1.32}$$

We look for solutions of (1.32) with $Z > -1$ as $W \to 0$ (i.e. $u' > 0$ as $u \to 0$). In Figure 1.3 we sketch the (W,Z)-phase plane. Equation (1.32) and the phase plane imply that $Z \to Z_0 \in \{0, -1\}$ as $W \to 0$, where orbits with $Z_0 = 0$ have $v = o(-u^{-\frac{\beta}{1+\gamma}})$ while orbits with $Z_0 = -1$ have $v \sim -u^{-\frac{\beta}{1+\gamma}}$.

Proposition 1.8. *For $2\beta > \alpha - \gamma - 2$ there is a unique orbit C^0 with $u \to 0$ and $v \sim -u^{-\frac{\beta}{1+\gamma}}$ as η decreases. This orbit reaches $(u,v) = (0, -\infty)$ at some finite η-value, implying the existence of a travelling wave with a front. The local behavior of the front is determined by the relation*

$$f' \sim \frac{c}{\beta} f^{\beta+2-\alpha} \quad \text{as} \quad f \to 0.$$

Proof. We first prove existence. Choose W_0 small and denote the solution of (1.32) with $Z = \xi$ in $W = W_0$ by $Z = Z(W, \xi)$. Let

$$\begin{aligned} S_+ &= \{\xi \in (-1,0) : \ Z(W,\xi) \to 0 \text{ as } W \downarrow 0\} \\ S_- &= \{\xi \in (-1,0) : \exists W_* \in (0, W_0)\ Z(W,\xi) \to -1 \text{ as } W \downarrow W_* > 0\} \\ S_0 &= \{\xi \in (-1,0) : \ Z(W,\xi) \to -1 \text{ as } W \downarrow 0\}. \end{aligned}$$

By standard arguments we have for W_0 sufficiently small that

$$(0,-1) = S_- \cup S_0 \cup S_+$$

and S_- and S_+ are nonempty and open. Hence S_0 is nonempty, which gives existence. We observe that for such an orbit, see (1.32),

$$Z + 1 \sim aW \text{ as } W \downarrow 0 \text{ where } a = \frac{\varepsilon c^2}{\beta}. \tag{1.33}$$

Next we prove uniqueness. Suppose there are two solutions $Z = Z_1(W)$ and $Z = Z_2(W)$ with $Z \to -1$ as $W \downarrow 0$. Since

$$\frac{1}{Z_1+1} - \frac{1}{Z_2+1} = -\frac{1}{(\tilde{Z}+1)^2}(Z_1 - Z_2),$$

where $\tilde{Z}$ lies between Z_1 and Z_2, we have for $Y = Z_1 - Z_2 > 0$, say,

$$\frac{dY}{dW} - b\frac{Y}{W} \sim -b'\frac{Y}{W^2} \quad \text{as } W \to 0,$$

where $b = \frac{\beta}{2\beta+\gamma+2-\alpha}$, and $b' = \frac{\beta}{\varepsilon c^2(2\beta+\gamma+2-\alpha)}$. Here we used (1.33). Hence $Y \to \infty$ as $W \downarrow 0$, contradicting (1.33).

Expressing W and Z in terms of f we observe that (1.33) implies the behaviour

$$f' \sim \frac{c}{\beta} f^{\beta+2-\alpha} \quad \text{as } f \to 0.$$

Since $\beta + 2 - \alpha < 1$ we find that f reaches zero at some finite η-value; i.e. the travelling wave has a finite front. □

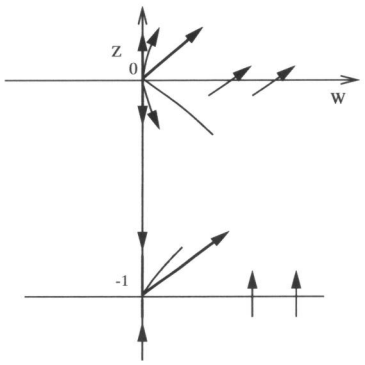


Figure 1.3: Phase plane related to (1.32), with $q = \frac{\beta}{1+\gamma}$.

Note that we did not put any restriction on ε. Thus the conclusion about the behaviour as $f \to 0$, provided $2\beta > \alpha - \gamma - 2$, is valid for any value of $\varepsilon \geq 0$. This ε-uniformity is lost in the next case.

Indeed, the behaviour of the travelling wave near the front when it vanishes corresponds to criterion (1.7) in the case where the capillary damping is absent.

1.4.2 $2\beta = \alpha - \gamma - 2$.

Now we take $q = \frac{\beta}{1+\gamma}$, and set $W = u^{\frac{\alpha-1}{1+\gamma}}$. Then (1.31) becomes

$$(\alpha - 1)\frac{dZ}{dW} = \left(\beta Z + \frac{\varepsilon c(c - W)}{(Z + 1)}\right)\frac{1}{W}. \tag{1.34}$$

Again we look for solutions satisfying $Z > -1$ and $Z \to Z_0$ as $W \to 0$. It follows that

$$Z_0^{\pm} = -\frac{1}{2} \pm \frac{1}{2}\left(1 - \frac{4\varepsilon c^2}{\beta}\right)^{\frac{1}{2}},$$

implying $\varepsilon \leq \frac{\beta}{4c^2}$. With reference to Figure 1.4 we have

Proposition 1.9. *(i) If $\varepsilon < \frac{\beta}{4c^2}$, there exists a family of orbits satisfying $v \sim Z_0^+ u^{-\frac{\beta}{1+\gamma}}$ as $u \to 0$, and a unique orbit, denoted by C^0, which satisfies $v \sim Z_0^- u^{-\frac{\beta}{1+\gamma}}$ as $u \to 0$. All cases give travelling waves with finite fronts. In particular the orbit C^0 implies*

$$f' \sim \frac{1}{\varepsilon c}\left(1 - \left(1 - \frac{4\varepsilon c^2}{\beta}\right)^{\frac{1}{2}}\right) f^{\beta+2-\alpha}, \quad \text{as } f \to 0.$$

(ii) If $\varepsilon = \frac{\beta}{4c^2}$ there exists a family of orbits having $v \sim -\frac{1}{2}u^{-\frac{\beta}{1+\gamma}}$ as $u \to 0$. In particular there is a unique orbit, again denoted by C^0, satisfying $u^{\frac{\beta}{1+\gamma}}v \uparrow -\frac{1}{2}$ as $u \to 0$. The orbit C^0 implies again a traveling wave with a finite front, such that

$$f' \sim \frac{2c}{\beta} f^{\beta+2-\alpha}, \quad \text{as } f \to 0.$$

(iii) If $\varepsilon > \frac{\beta}{4c^2}$, there is no orbit with $v > -u^{\frac{-\beta}{1+\gamma}}$ and $u \to 0$.

Remark 1.10. *(i) Comparing (1.19) and (1.26) we observe that*

$$\varepsilon_{\frac{1}{2}}^0 = \varepsilon^* = \frac{\beta^2}{4(\alpha - 1)c^2} < \frac{\beta}{4c^2},$$

implying that, depending on the ε-value, monotone and oscillatory waves with finite fronts occur.

(ii) Since now $\beta + 2 - \alpha = -\beta - \gamma < 0$, finite front waves have $f' \to \infty$ as $f \to 0$.

Proof. *(i)* It is immediate from the phase-plane in Figure 1.4 that $(0, Z_0^+)$ is a source, and $(0, Z_0^-)$ is a saddle, with one unique orbit $Z = Z(W)$ leaving in the direction $W > 0$. It behaves as $Z - Z_0^- \sim \left(\frac{1}{2} - \frac{1}{2}(1 - \frac{4\varepsilon c^2}{\beta})^{\frac{1}{2}}\right)$ as $W \to 0$.

(ii) Now $Z_0 = -\frac{1}{2}$ at local analysis shows that $(0, Z_0)$ is a saddle-node, with a unique orbit in the direction of $W > 0$. This orbit does not cross the isocline, and behaves as $Z - Z_0 \sim \frac{1}{2}W$.

(iii) Now the segment $\{(W, Z) : W = 0, Z \in (-1, 0)\}$ is disconnected from the isocline and hence no connecting orbit exists. □

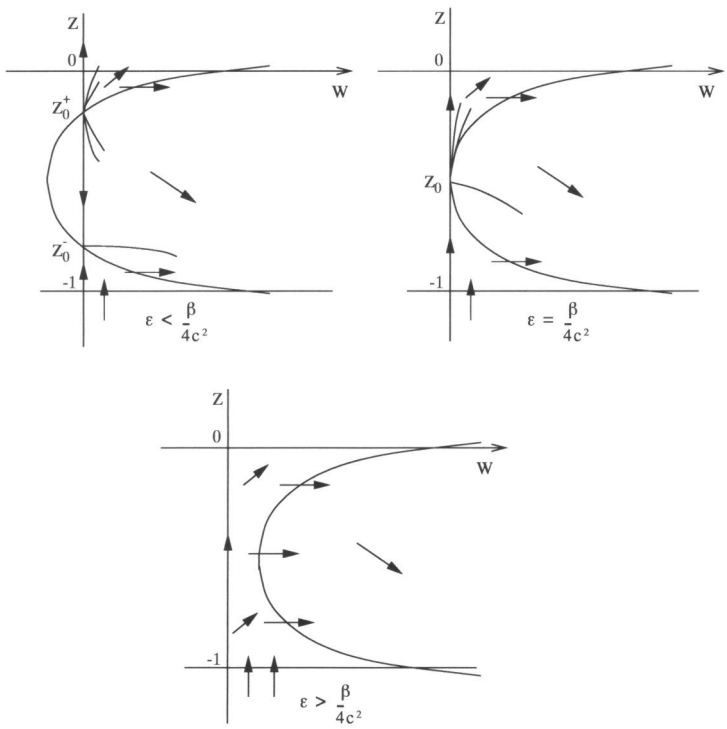


Figure 1.4: $q = \frac{\beta}{1+\gamma}$, $2\beta = \alpha - \gamma - 2$ phase-plane for different ε-values.

1.4.3 $2\beta < \alpha - \gamma - 2$.

Here we choose $q = \frac{\alpha-2-\gamma}{2(1+\gamma)} > \frac{\beta}{1+\gamma}$ and $W = u^{q-\frac{\beta}{1+\gamma}}$, yielding the equation

$$(\alpha - \gamma - 2 - 2\beta)\frac{dZ}{dW} = (\alpha - \gamma - 2)\frac{Z}{W} + \frac{2\varepsilon c(c - W^{\frac{2(\alpha-1)}{\alpha-\gamma-2-2\beta}})}{(W+Z)W} \tag{1.35}$$

Proposition 1.11. *There exists no orbits with $v > -u^{-\frac{\beta}{1+\gamma}}$ and $u \to 0$.*

Proof. Suppose such an orbit exists. Then we would have

$$(\alpha - \gamma - 2 - 2\beta)\frac{dZ}{dW} \sim \frac{1}{W}\left((\alpha - \gamma - 2)Z + \frac{2\varepsilon c^2}{Z}\right),\ Z > 0,$$

as $W \downarrow 0$. Since $(\alpha - \gamma - 2)Z + \frac{2\varepsilon c^2}{Z}$ is negative and bounded away from zero, this gives a contradiction. □

1.5 $\delta \to 0$ limit

For the purpose of this section we denote the connecting orbit of (P_ε^δ) by $v = \varphi_\delta(u)$ and, in the cases for which C^0 from Proposition 1.8 and 1.9 exists, we call its graph $v = \varphi_0(u)$.

Lemma 1.12. *There exists $\delta^* > 0$ such that, with $c = c(\delta) = \frac{A^\alpha - \delta^\alpha}{A - \delta}$,*

$$\mathcal{F}_u(\delta) = c(\delta)G_\delta(u) = c(\delta)\left(-1 + c(\delta)u^{\frac{1-\alpha}{1+\gamma}} - c(\delta)A - A^\alpha u^{-\frac{\alpha}{1+\gamma}}\right)$$

is decreasing in $0 \le \delta < \delta^*$ for any fixed $0 < u \le \left(\frac{A}{2}\right)^{1+\gamma}$.

Proof. Since

$$\frac{d\mathcal{F}_u}{d\delta} = \frac{dc}{d\delta}\left(-1 + 2cu^{\frac{1-\alpha}{1+\gamma}} + (A^\alpha - 2cA)u^{-\frac{\alpha}{1+\gamma}}\right),$$

and $\frac{dc}{d\delta} > 0$, we need to show that the term between brackets is negative for small δ. At $\delta = 0$ it becomes

$$-1 + A^{\alpha-1}u^{-\frac{\alpha}{1+\gamma}}(2u^{\frac{1}{1+\gamma}} - A) < 0,$$

for all $0 < u < \left(\frac{A}{2}\right)^{1+\gamma}$. □

Proposition 1.13. *For α, β, γ such that $2\beta > \alpha - \gamma - 2$, or $2\beta = \alpha - \gamma - 2$ and $\varepsilon \in (0, \frac{\beta}{4c^2})$ fixed, translate f_δ such that $f_\delta(0) = \frac{A}{2}$ for all $\delta \in (0, \delta^*)$. Then $f_\delta \to f \in C^\infty(\mathbb{R})$ uniformly on $\mathbb{R}$. Hence f satisfies Problem TW with $\delta = 0$.*

Proof. It will be sufficient to show that $\varphi_\delta(u) \to \varphi_0(u)$ locally uniformly. By Lemma 1.12 we have for any $0 < \bar{\delta} \le \delta^*$

$$-u^{-\frac{\beta}{1+\gamma}} < \varphi_{\delta_1}(u) < \varphi_{\delta_2}(u) < \varphi_0(u)$$

for $0 < \delta_1 < \delta_2 < \delta^*$ and $u \in (\bar{\delta}, \left(\frac{A}{2}\right)^{1+\gamma})$. Also

$$\frac{\varepsilon c(\delta)}{1+\gamma}\frac{G_\delta(u)}{F_\delta(u,v)} \to \frac{\varepsilon A^{\alpha-1}}{1+\gamma}\frac{G_0(u)}{F_0(u,v)} \text{ as } \delta \to 0$$

uniformly on $[\bar{\delta}^{1+\gamma}, \left(\frac{A}{2}\right)^{1+\gamma}]$. Therefore

$$\varphi_\delta(u) \uparrow \bar{\varphi}(u) \le \varphi_0(u)$$

where $v = \bar{\varphi}_0(u)$ is a solution of (P_ε^0). The reasoning above holds for every $0 < \bar{\delta} \le \delta^*$, which implies that $\bar{\varphi}_0(u)$ exists for all $u \in (0, \left(\frac{A}{2}\right)^{1+\gamma})$. In view of Section 1.4 and $\bar{\varphi} \le \varphi_0$ this implies that $\bar{\varphi}_0(u) = \varphi_0(u)$.

Using that $f_\delta(0) = \frac{A}{2}$ is fixed for all $0 < \delta < \delta^*$, standard arguments imply that f_δ converges uniformly on $(-\infty, \frac{A}{2})$ to the corresponding solution of the limit equation. Existence of global travelling waves and uniqueness of the initial value problem for all $0 \le \delta < \delta^*$, implies the uniform convergence on $\mathbb{R}$. □

1.6 Concluding remarks

In this chapter we present a study of a model for unsaturated groundwater flow which includes expression (8) for the non-static phase pressure difference. Replacing the nonlinearities in the transport equation by power-law expressions we arrive at (1.5). We study travelling wave solutions representing moisture profiles moving downwards due to gravity.

For positive initial saturation ($\delta > 0$) we demonstrate existence and uniqueness (up to translations). Small values of the damping coefficient ε result in monotone saturation profiles. Large values of ε result in profiles which exhibit oscillatory behaviour near the injection saturation A.

When initially no moisture is present ($\delta = 0$) existence of bounded travelling waves depends critically on the exponents of the power-law expressions. This is related to the occurrence of finite fronts in the moisture profiles: i.e. descending planes (in the direction of gravity) below which the water saturation remains zero. Related to equation (1.5) we have shown the following.

If $2\beta > \alpha - \gamma - 2$, then travelling wave solutions with fronts exist for all $\varepsilon > 0$. In other words, for $S(z,t) = f(\eta)$, with $\eta = z + ct$, there exists $\eta_0 \in \mathbb{R}$ such that $f(\eta) = 0$ for all $\eta \leq \eta_0$. Moreover near $\eta = \eta_0$ the profile satisfies

$$f' \sim \frac{A^{\alpha-1}}{\beta} f^{\beta+2-\alpha}.$$

This corresponds to the front behaviour of solutions of the convection diffusion equation under static conditions (equation (1.6)); i.e. ε and γ are absent in this asymptotic expression.

If $2\beta = \alpha - \gamma - 2$ we obtain a similar result provided the damping coefficient ε is sufficiently small: i.e. $\varepsilon \leq \frac{\beta}{4A^{2(\alpha-1)}}$. For larger values of ε, no waves exist satisfying $f(-\infty) = 0$. Finally, if $2\beta < \alpha - \gamma - 2$, again no such waves exist, regardless the value of $\varepsilon > 0$.

Let us interpret this in terms of the nonlinear functions, $K(S)$, $p_c(S)$ and $L(S)$, as they were approximated in (1.2), (1.3) and (1.4). If we write the condition for existence of travelling wave solution with fronts as $\gamma > \alpha - 2(\beta + 1)$, this implies that if $\alpha < 2(\beta + 1)$, then for any positive value of γ fronts exist. But instead, if $\alpha > 2(\beta + 1)$, then γ needs to be sufficiently large in order to have fronts solutions. In other words, if $K(S)$ is small near $S = 0$ in the sense that $\alpha > 2(\beta + 1)$, then $L(S)$ must decay sufficiently fast near $S = 0$, i.e. $\gamma > \alpha - 2(\beta + 1)$, to have fronts. We can express the condition for existence of travelling wave solutions with fronts as

$$L(S) = o(K(S)\, p_c'(S)^2) \text{ as } S \to 0.$$

We also investigate the limit $\varepsilon \to 0$ (for $\delta > 0$, fixed) and $\delta \to 0$ (for $\varepsilon > 0$ fixed). In particular the latter provides a uniqueness criterion for the degenerate case when $\delta = 0$. We also note that the limits $\varepsilon \to 0$ and $\delta \to 0$ do not commute: $\varepsilon \to 0$ followed by $\delta \to 0$ is always possible, while $\delta \to 0$ followed by $\varepsilon \to 0$ is only possible when $2\beta \geq \alpha - \gamma - 2$.

Chapter 2

Equation (1): well-posedness and stability of monotone travelling waves

Preamble:
We investigate stability of travelling wave solutions of the pseudo-parabolic Burgers' equation (1), for which we first obtain well-posedness results [1].

2.1 Introduction

In this chaper we consider the pseudo-parabolic Burgers' equation

$$u_t = u_{xx} + 2uu_x + \varepsilon^2 u_{xxt} \quad \text{on} \quad \mathbb{R} \times [0, T] \tag{2.1}$$

with initial data

$$u(x, 0) = u_0(x) \quad \text{in} \quad \mathbb{R}. \tag{2.2}$$

In this chapter we aim for a better understanding of the effect of the third order term on the dynamics of diffusion and convection. Intuitively one expects that this effect is more notable if the large time behaviour of solutions of the diffusion-convection equation is characterised by profiles which do not become flat (in terms of their dependence on x). This is why we restrict our study of the large time behaviour to the travelling wave case, meaning that $S(+\infty) > S(-\infty) \geq 0$. In fact we shall only consider the large time behaviour of solutions of (2.1), the simplest pseudo-parabolic equation allowing convection-driven travelling waves, with initial data satisfying

$$u_0(-\infty) = 0, \quad u_0(+\infty) = 1. \tag{2.3}$$

We will show that such solutions converge to a travelling wave solution

$$u(x, t) = \phi(x + t), \tag{2.4}$$

provided the travelling wave profile ϕ is monotone. This depends on $\varepsilon > 0$: travelling wave solutions connecting zero to one and travelling with speed one exist for all $\varepsilon > 0$, but only when $0 < \varepsilon < \frac{1}{2}$ the profiles are monotone, see Section 2.3.

[1] This chapter is to appear as an article in Nonlinear analysis TMR, see [18].

Our stability result is of a global character and therefore we first require well-posedness results for the initial value problem. To this end we reformulate equation (2.1) in Section 2.2 as

$$u_t = F_\varepsilon(u) = A_\varepsilon u + B_\varepsilon u^2, \tag{2.5}$$

where A_ε, B_ε are linear operators defined by

$$A_\varepsilon u = (I - \varepsilon^2 \frac{d^2}{dx^2})^{-1} u_{xx}\,; \quad B_\varepsilon u = (I - \varepsilon^2 \frac{d^2}{dx^2})^{-1} u_x, \tag{2.6}$$

and study local well-posedness of the ODE (2.5) in several Banach spaces, namely in $L^1 \cap L^2$, $L^1 \cap H^1$, L^∞, L^2 and H^1. Here $L^p(\mathbb{R}) = L^p$ with norm $u \to |u|_p$ and $H^1 = H^1(\mathbb{R})$ is the Sobolev space with norm $u \to ||u|| = \sqrt{|u|_2^2 + |u'|_2^2}$. We note that although formally equation (2.1) preserves the integral (conservation of mass), the map $u \to B_\varepsilon u^2$ is not well defined on L^1, hence the choice of $L^1 \cap L^2$ with norm $u \to |u|_{1,2} = |u|_1 + |u|_2$, and $L^1 \cap H^1$ with norm $u \to ||u||_{1,2} = |u|_1 + ||u||$.

Since travelling wave solutions do not belong to L^p if $1 \leq p < \infty$, we also consider (2.1) in affine spaces of the form $\mathcal{H} + X$, where $\mathcal{H}$ is any smooth function such that $\mathcal{H}(-\infty) = 0$, $\mathcal{H}(+\infty) = 1$. It is no restriction to assume that $\mathcal{H}'$ is nonnegative and compactly supported. In Section 2.4 we obtain local well-posedness in $\mathcal{H} + X$ for $X = L^2, L^1 \cap L^2, H^1, L^1 \cap H^1$.

In Section 2.5 we establish mass conservation: if u_1 and u_2 are solutions of (2.1) with $u_1 - u_2$ in $L^1 \cap L^2$ then

$$\frac{d}{dt} \int_{\mathbb{R}} (u_1(x,t) - u_2(x,t))dx = 0 \text{ for all } t.$$

This allows us to follow [39] by introducing

$$v(x,t) = \int_{-\infty}^{x} (u(s,t) - \phi(s+t))ds. \tag{2.7}$$

The function v is well defined if u is in $\mathcal{H} + L^1$. For solutions u with values in $\mathcal{H} + X$, $X = L^1 \cap L^2$, shifting either ϕ or u_0 we may restrict attention to solutions v of

$$v_t = v_{xx} + v_x^2 + 2v_x\phi + \varepsilon^2 v_{xxt}, \tag{2.8}$$

with $v(+\infty) = 0$.

Equation (2.8) is derived in Section 2.6 and analysed in Section 2.7. Again we establish local well-posedness in several natural spaces, in particular in H^1 and $H^2 = \{v \in L^2;\, v', v'' \in L^2\}$. In Section 2.8 we prove the identities

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} (v^2 + \varepsilon^2 v_x^2)dx = -\int_{\mathbb{R}} (1 - v)v_x^2 dx - \int_{\mathbb{R}} \phi' v^2 dx, \tag{2.9}$$

and

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}} (v_x^2 + \varepsilon^2 v_{xx}^2)dx = -\int_R v_{xx}^2 dx + \int_{\mathbb{R}} \phi' v_x^2 dx, \tag{2.10}$$

and obtain a global well-posedness result for solutions of (2.8) in H^2. To formulate and prove stability results we need $\phi' \geq 0$ and $v(x,0)$ sufficiently small in H^1 guaranteeing $v < 1$. Using (2.9) this gives convergence of the integral $\int_0^\infty ||v(\cdot,t)||^2 dt$ and

thereby of $\int_0^\infty \int_{-\infty}^\infty |u(x,t) - \phi(x+t)|^2 dxdt$. If in addition $v(x,0)$ is in H^2 we adapt methods from [52] and show that $v(\cdot,t) \to 0$ in H^2 whence $||u(\cdot,t) - \phi(\cdot+t)|| \to 0$ as $t \to \infty$.

A natural question is of course whether the monotonicity of ϕ is essential. In the context of Korteweg-de Vries type equations there are examples where a switch from monotone to oscillatory behaviour of the travelling wave leads to instability, but this depends on the exponent in the nonlinearity, see [53]. Numerical experiments so far have been inconclusive.

So far we have formulated asymptotic stability in $\mathcal{H}+H^1$, in particular we assume solutions to be continuous in space. This can be relaxed a little bit. In Section 2.9 we examine solutions with jump discontinuities and show that jumps are fixed in space and decay in time. Moreover we extend our stability results to solutions having one single jump in $x = 0$.

We conclude this introduction with the observation that we have avoided a transformation of the problem to travelling wave variables. Such a change is common in the study of stability properties of travelling wave solutions of Burgers' and other "normal" equations. Here it would lead to an equation with yet another third order term involving three space derivatives, which cannot be seen as an ODE in a function space.

2.2 Local well-posedness in Banach spaces

In this section we show that the initial value problem for the ODE (2.5), which as we recall reads

$$u_t = F_\varepsilon(u) = A_\varepsilon u + B_\varepsilon u^2,$$

is locally well-posed in the Banach spaces $L^1 \cap L^2$, $L^1 \cap H^1$, L^∞, L^2 and H^1. The operators A_ε and B_ε may be rewritten as

$$A_\varepsilon u = (I - \varepsilon^2 \frac{d^2}{dx^2})^{-1} u_{xx} = \frac{1}{\varepsilon^2} \left((I - \varepsilon^2 \frac{d^2}{dx^2})^{-1} - I \right) u = \frac{1}{\varepsilon^2} (g_\varepsilon * u - u), \quad (2.11)$$

and

$$B_\varepsilon u = (I - \varepsilon^2 \frac{d^2}{dx^2})^{-1} u_x = g_\varepsilon * u_x, \quad (2.12)$$

where

$$g_\varepsilon(x) = \frac{1}{2\varepsilon} e^{-\frac{|x|}{\varepsilon}} \quad (2.13)$$

is the Green's function for the problem

$$w - \varepsilon^2 w'' = f. \quad (2.14)$$

That is to say,

$$w(x) = (G_\varepsilon f)(x) = (g_\varepsilon * f)(x) = \int_{\mathbb{R}} g_\varepsilon(x-y) f(y) dy \quad (2.15)$$

is the solution of (2.14). Since for any $u \in L^p$ with $1 \leq p \leq \infty$, the derivative $(g_\varepsilon * u)_x$ is well defined and $(g_\varepsilon * u)_x = g'_\varepsilon * u$ we have

$$B_\varepsilon u(x) = (g'_\varepsilon * u)(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}} g_\varepsilon(x-y) \int_x^y u(s) ds dy \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.16}$$

Theorem 2.1. *Let X denote any of the Banach spaces $L^1 \cap L^2$, $L^1 \cap H^1$, L^∞, L^2 and H^1. Then for all $u_0 \in X$ there exists $T > 0$ such that there exists a unique solution $u \in C^1([0,T]; X)$ of (2.5).*

Theorem 2.1 will follow from Picard's Theorem: if X is a Banach space and $F : X \to X$ is Lipschitz continuous in a neighbourhood of $u_0 \in X$, then there exists $T > 0$ and a unique solution $u \in C^1([0,T];\ X)$ of $u_t = F(u)$ with $u(0) = u_0$. We shall show that F_ε is locally Lipschitz continuous on each of the Banach spaces listed above. This is done within the next five lemmas and is based on the following properties of g_ε:

$$|g_\varepsilon|_1 = \int_{\mathbb{R}} g_\varepsilon(y) dy = 1, \quad |g_\varepsilon|_2^2 = \int_{\mathbb{R}} (g_\varepsilon(y))^2 dy = \frac{1}{4\varepsilon}, \tag{2.17}$$

$$|g'_\varepsilon|_1 = \int_{\mathbb{R}} |g'_\varepsilon(y)| dy = \frac{1}{\varepsilon}, \quad |g'_\varepsilon|_2^2 = \int_{\mathbb{R}} (g'_\varepsilon(y))^2 dy = \frac{1}{4\varepsilon^3}, \tag{2.18}$$

$$\int_{\mathbb{R}} g_\varepsilon(y)|y| dy = \varepsilon, \quad \int_{\mathbb{R}} g'_\varepsilon(y) dy = 0. \tag{2.19}$$

Throughout $B(X,Y)$ denotes the Banach space of bounded linear operators from a Banach space X to a Banach space Y, with the usual convention that $B(X) = B(X,X)$.

Lemma 2.2. *The map F_ε is locally Lipschitz continuous in L^∞.*

Proof. Since $A_\varepsilon = \frac{1}{\varepsilon^2}(G_\varepsilon - I)$ it follows from (2.17) that A_ε is a bounded linear operator in L^∞ with $\|A_\varepsilon\|_{B(L^\infty)} \leq \frac{2}{\varepsilon^2}$. Thus it is uniformly Lipschitz continuous on L^∞. By (2.16) and (2.19) we have

$$|(B_\varepsilon u)(x)| \leq \frac{1}{\varepsilon^2} \int_{-\infty}^{\infty} g_\varepsilon(x-y)\, |x-y|\, |u|_\infty\, dy = \frac{1}{\varepsilon} |u|_\infty,$$

which implies that the operator B_ε is a bounded linear operator in L^∞. The map $u \to u^2$ clearly maps L^∞ into L^∞, and is locally Lipschitz continuous: if $u_1, u_2 \in L^\infty$ are such that $\|u_i\|_\infty \leq R$, for some $R > 0$, then

$$|u_1^2 - u_2^2|_\infty = |(u_1 + u_2)(u_1 - u_2)|_\infty \leq 2R|u_1 - u_2|_\infty.$$

□

Lemma 2.3. *The map F_ε is locally Lipschitz continuous in L^2.*

Proof. Since A_ε is the Hille-Yosida approximation of the operator $A = -\frac{d^2}{dx^2}$, which is maximal monotone in the Hilbert space L^2, A_ε is bounded linear operator in L^2 with $\|A_\varepsilon\|_{B(L^2)} \leq \frac{1}{\varepsilon^2}$, see [15]. In view of $B_\varepsilon u = g'_\varepsilon * u$, the inequality $|f*g|_p \leq |f|_p\,|g|_1$ and (2.18), the linear operator B_ε is bounded as an operator from L^1 to L^2, with

$$\|B_\varepsilon\|_{B(L^1,L^2)} \leq \left(\int_{\mathbb{R}} |g'_\varepsilon(y)|^2 dy\right)^{\frac{1}{2}} = \frac{1}{2\varepsilon^{\frac{3}{2}}}. \tag{2.20}$$

For $u_1,\ u_2 \in L^2$ with $|u_i|_2 \leq R$ we have

$$|B_\varepsilon(u_1^2 - u_2^2)|_2 \leq \|B_\varepsilon\|_{B(L^1,L^2)}|(u_1+u_2)(u_1-u_2)|_1 \leq \frac{R}{\varepsilon^{\frac{3}{2}}}|u_1-u_2|_2.$$

□

Lemma 2.4. *The map F_ε is locally Lipschitz continuous in H^1.*

Proof. Since $A = -\frac{d^2}{dx^2}$ is maximal monotone on the Hilbert space H^1, its Hille-Yosida approximation A_ε is a bounded linear operator in H^1 with $\|A_\varepsilon\|_{B(H^1)} \leq \frac{1}{\varepsilon^2}$. For u_1, u_2 in H^1 with $||u_i|| \leq R$ we now have

$$|B_\varepsilon(u_1^2 - u_2^2)|_2 \leq \frac{R}{\varepsilon^{\frac{3}{2}}}||u_1 - u_2||,$$

and

$$\begin{aligned} |\left(B_\varepsilon(u_1^2-u_2^2)\right)_x|_2 \leq \|B_\varepsilon\|_{B(L^1,L^2)}\,|\,\left((u_1+u_2)(u_1-u_2)\right)_x|_1 \leq \\ \|B_\varepsilon\|_{B(L^1,L^2)}(\,|(u_1+u_2)_x|_2|u_1-u_2|_2 + |u_1+u_2|_2|(u_1-u_2)_x|_2\,) \leq \\ \frac{R}{\varepsilon^{\frac{3}{2}}}||u_1-u_2||. \end{aligned}$$

Thus $u \to B_\varepsilon u^2$ is locally Lipschitz continuous. □

Lemma 2.5. *The map F_ε is locally Lipschitz continuous in $L^1 \cap L^2$.*

Proof. The inequality $|g_\varepsilon * u|_p \leq |g_\varepsilon|_1\,|u|_p$ for all $1 \leq p \leq \infty$ and (2.17) imply that A_ε is a bounded linear operator in L^1 with $\|A_\varepsilon\|_{B(L^1)} \leq \frac{2}{\varepsilon^2}$. Consequently A_ε is also bounded in $L^1 \cap L^2$ with $\|A_\varepsilon\|_{B(L^1\cap L^2)} \leq \frac{3}{\varepsilon^2}$. Now B_ε is a bounded linear operator in $L^1(\mathbb{R})$ with

$$\|B_\varepsilon\|_{B(L^1)} \leq \int_{\mathbb{R}} |g'_\varepsilon(y)|dy = \frac{1}{\varepsilon}, \tag{2.21}$$

so by (2.21) and (2.20) $u \to B_\varepsilon u^2$ maps $L^1 \cap L^2$ to itself. Let $u_1,\ u_2 \in L^1 \cap L^2$ with $|u_i|_{1,2} \leq R$. Then

$$\begin{aligned} |B_\varepsilon(u_1^2-u_2^2)|_{1,2} \leq \|B_\varepsilon\|_{B(L^1,L^2)}|(u_1+u_2)(u_1-u_2)|_1 + \\ \|B_\varepsilon\|_{B(L^1)}|(u_1+u_2)(u_1-u_2)|_1 \leq \\ 2R\left(\frac{1}{\varepsilon}+\frac{1}{2\varepsilon^{\frac{3}{2}}}\right)|u_1-u_2|_2 \leq 2R\left(\frac{1}{\varepsilon}+\frac{1}{2\varepsilon^{\frac{3}{2}}}\right)|u_1-u_2|_{1,2}. \end{aligned}$$

□

Lemma 2.6. *The map F_ε is locally Lipschitz continuous in $L^1 \cap H^1$.*

Proof. Again A_ε is a bounded linear operator on $L^1 \cap H^1$ with $\|A_\varepsilon\|_{B(L^1\cap H^1)} \leq \frac{3}{\varepsilon^2}$. As in Lemma 2.5 one has that $u \to B_\varepsilon u^2$ is locally Lipschitz continuous in $L^1 \cap H^1$ with Lipschitz constant $2R\left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon^{\frac{3}{2}}}\right)$ on the ball with radius R in $L^1 \cap H^1$. □

2.3 Travelling waves

The analysis of travelling wave solutions of (2.1) is similar to the analysis of travelling wave solutions of (1.5) in Chapter 1. Substituting

$$u(x,t) = \phi(x+ct) \tag{2.22}$$

in (2.1), we have for $\phi(x)$, after an integration in x, that

$$c(\phi(x) - \phi(-\infty)) = \phi'(x) + (\phi(x)^2 - \phi(-\infty)^2) + \varepsilon^2 c\phi''(x), \tag{2.23}$$

so that

$$c = \frac{\phi(\infty)^2 - \phi(-\infty)^2}{\phi(\infty) - \phi(-\infty)}. \tag{2.24}$$

Restricting attention to $\phi(-\infty) = 0$ and $\phi(\infty) = 1$ we have $c = 1$ and (2.23) can be written as a Lienard type system of two equations:

$$\begin{cases} \varepsilon^2\phi' = \psi - \phi \\ \psi' = \phi(1-\phi). \end{cases}$$

The travelling wave solutions connecting $\phi(-\infty) = 0$ to $\phi(\infty) = 1$ are unique up to translation and correspond to a unique orbit connecting the saddle $(0,0)$ to the sink $(1,1)$. Note that $(0,0)$ has eigenvalues

$$\lambda_{1,2} = -\frac{1}{2\varepsilon^2}\left(1 \pm \sqrt{1+4\varepsilon^2}\right), \quad \lambda_1 < 0 < \lambda_2, \tag{2.25}$$

and $(1,1)$ has eigenvalues

$$\mu_{1,2} = -\frac{1}{2\varepsilon^2}(1 \pm \sqrt{1-4\varepsilon^2}), \quad \mathrm{Re}\mu_{1,2} < 0. \tag{2.26}$$

The unique orbit coming out of $(0,0)$ into the first quadrant connects to $(1,1)$. This follows from arguments very similar to the arguments in [19] and relies in particular on the negativity of the divergence of the Lienard vector field.

If $\varepsilon^2 < \frac{1}{4}$ the eigenvalues at $(1,1)$ are negative: $(1, -\varepsilon^2\mu_2)$ is an eigenvector of the slow eigenvalue μ_1 and $(1, -\varepsilon^2\mu_1)$ is an eigenvector of the fast eigenvalue μ_2. The set $\{\phi > 0,\ 0 < \psi < -\varepsilon^2\mu_2\phi + (1+\varepsilon^2\mu_2)\}$, contained in the region where $\phi' > 0$, is then invariant and contains the connecting orbit. Therefore ϕ is monotone if $\varepsilon^2 \leq \frac{1}{4}$. In this case the invariant region gives an explicit upper bound for ϕ', namely

$$\phi' = \frac{\psi - \phi}{\varepsilon^2} \leq \frac{1+\varepsilon^2\mu_2}{\varepsilon^2} = \frac{1-\sqrt{1-4\varepsilon^2}}{2\varepsilon^2}. \tag{2.27}$$

Theorem 2.7. *Equation (2.1) has a travelling wave solution connecting $u = 0$ in $x = -\infty$ to $u = 1$ in $x = \infty$. This solution is unique up to translation and of the form $u(x,t) = \phi(x+t)$. If $\varepsilon^2 < \frac{1}{4}$ the profile ϕ is monotone increasing and its derivative is bounded by (2.27).*

2.4 Local well-posedness in affine Banach spaces

In this section we show that the initial value problem for the ODE (2.5) is locally well-posed in the affine Banach spaces $Y = \mathcal{H}+X$, $X = L^1 \cap L^2, L^1 \cap H^1, L^2, H^1$. We recall that $\mathcal{H}$ is a smooth function with $\mathcal{H}(-\infty) = 0$, $\mathcal{H}(\infty) = 1$ and $\mathcal{H}'$ nonnegative and compactly supported. We say that u is a solution of (2.5) in $C^1([0,T];\ Y)$ if $\bar{u} = u - \mathcal{H}$ is a solution in $C^1([0,T];\ X)$ of the equation

$$\bar{u}_t = F_\varepsilon(\bar{u}) + 2B_\varepsilon(\mathcal{H}\bar{u}) + F_\varepsilon(\mathcal{H}). \tag{2.28}$$

Theorem 2.8. *Let $Y = \mathcal{H}+X$, where X is any of the spaces $L^1 \cap L^2, L^1 \cap H^1, L^2, H^1$. Then for all $u_0 \in Y$ there exists $T > 0$ and a unique solution of problem (2.1) $u \in C^1([0,T];\ Y)$.*

Proof. If we show that the operator

$$\bar{u} \to F_\varepsilon(\bar{u}) + 2B_\varepsilon(\mathcal{H}\bar{u}) + F_\varepsilon(\mathcal{H})$$

is locally Lipschitz from X to X, the theorem follows again from Picard's theorem. From Section 2.2 we know that $\bar{u} \to F_\varepsilon(\bar{u})$ is locally Lipschitz in X for each of the choices of X. We only need to prove that $F_\varepsilon(\mathcal{H}) \in X$ and that the linear map $\bar{u} \to 2B_\varepsilon(\mathcal{H}\bar{u})$ is a bounded operator in X.

Clearly $F_\varepsilon(\mathcal{H}) = A_\varepsilon\mathcal{H} + B_\varepsilon\mathcal{H}^2 = G_\varepsilon\mathcal{H}'' + G_\varepsilon(\mathcal{H}^2)'$. $F_\varepsilon(\mathcal{H})$ is in X for any of the choices of X, because $\mathcal{H}''$ and $(\mathcal{H}^2)'$ are compactly supported smooth functions. As for $\bar{u} \to 2B_\varepsilon(\mathcal{H}\bar{u})$, we saw in (2.21) that B_ε is a bounded linear operator in L^1. It is also bounded in L^2 and H^1 with

$$\|B_\varepsilon\|_{B(L^2)} \leq \int_{\mathbb{R}} |g_\varepsilon'(y)|dy = \frac{1}{\varepsilon}, \tag{2.29}$$

$$\|B_\varepsilon\|_{B(H^1)} \leq \int_{\mathbb{R}} |g_\varepsilon'(y)|dy = \frac{1}{\varepsilon}, \tag{2.30}$$

respectively. Thus B_ε is bounded in $L^1 \cap L^2$ and in $L^1 \cap H^1$. Finally if we set $|\mathcal{H}|_\infty = 1$, $|\mathcal{H}'|_\infty < \infty$,

$$|\mathcal{H}\bar{u}|_2 \leq |\bar{u}|_2,$$

$$\begin{aligned} \|\mathcal{H}\bar{u}\| \ &\leq\ |\mathcal{H}\bar{u}|_2 + |\mathcal{H}\bar{u}_x|_2 + |\mathcal{H}'\bar{u}|_2 \leq \\ |\bar{u}|_2 + |\bar{u}_x|_2 + |\mathcal{H}'|_\infty\, |\bar{u}|_2 \ &\leq\ \|\bar{u}\|_{H^1}\,(1 + |\mathcal{H}'|_\infty). \end{aligned}$$

Combining (2.29) and (2.30) with the above estimates we get that $\bar{u} \to 2B_\varepsilon(\mathcal{H}\bar{u})$ is a bounded linear operator on each X. This completes the proof. □

2.5 Conservation of mass

In this section we prove that equation (2.1) preserves the integral if we consider solutions in $\mathcal{H} + L^1 \cap L^2$. Note that unlike in the case of the Burgers' equation, this is not the same as being contracting in L^1. Again this is due to the absence of a comparison principle.

Proposition 2.9. *Let u_1, u_2 be two solutions of equation (2.5) in $C^1([0,T];\ \mathcal{H}+L^1\cap L^2)$. Then*

$$\int_{\mathbb{R}}(u_1(x,t)-u_2(x,t))dx=\int_{\mathbb{R}}(u_1(x,0)-u_2(x,0))dx \quad \textit{for all} \quad t\in[0,T]. \tag{2.31}$$

Proof. Consider the composite map

$$\mathcal{F}: t\in[0,T]\to u(\cdot,t)\in L^1\cap L^2\to \int_{\mathbb{R}}u(x,t)dx\in\mathbb{R}$$

where $u=u_1-u_2$. Since by definition $u\in C^1([0,T];L^1\cap L^2)$, we have $\mathcal{F}\in C^1([0,T],\mathbb{R})$. The chain rule implies that

$$\mathcal{F}'(t)=\int_{\mathbb{R}}A_\varepsilon(u_1(x,t)-u_2(x,t))dx+\int_{\mathbb{R}}B_\varepsilon(u_1(x,t)^2-u_2(x,t)^2)dx. \tag{2.32}$$

We claim that both right hand side terms in (2.32) are zero for all $t\in[0,T]$.

To see that the first term is zero, we recall that $A_\varepsilon=\frac{1}{\varepsilon^2}(G_\varepsilon-I)$ and note that $\int_{\mathbb{R}}G_\varepsilon u=\int_{\mathbb{R}}u$ for all $u\in L^1$. This is immediate from the definition of G_ε as convolution with the Green's function for (2.14): if $f\in L^1$ then both w and w' are in L^1 and $\int_{\mathbb{R}}w=\int_{\mathbb{R}}f$.

Before proving

$$\int_{\mathbb{R}}B_\varepsilon(u_1(x,t)^2-u_2(x,t)^2)dx=0,$$

we observe that it is well-defined. Indeed, if $u_i=\bar{u}_i+\mathcal{H}$, $i=1,2$ then $u_1^2-u_2^2=\bar{u}_1^2-\bar{u}_2^2+2\mathcal{H}(\bar{u}_1-\bar{u}_2)$. Since $\bar{u}_1^2-\bar{u}_2^2\in L^1$ and $2\mathcal{H}(\bar{u}_1-\bar{u}_2)\in L^1\cap L^2$ we have $u_1^2-u_2^2\in L^1$ and from Lemma 2.5 (2.21) we have $B_\varepsilon\in B(L^1)$. Thus the second right hand side term in (2.32) is well defined.

Now let $w\in L^1$ and consider the integral

$$\int_{\mathbb{R}}B_\varepsilon w(x)dx=\int_{\mathbb{R}}\left(\int_{\mathbb{R}}g_\varepsilon'(x-y)w(y)dy\right)dx.$$

Applying Fubini's theorem to $g_\varepsilon'(x-y)w(y)\in L^1(\mathbb{R}\times\mathbb{R})$ we obtain, in view of (2.19),

$$\int_{\mathbb{R}}B_\varepsilon w(x)dx=\int_{\mathbb{R}}w(y)\left(\int_{\mathbb{R}}g_\varepsilon'(x-y)dx\right)dy=0,$$

which in particular holds for $w=u_1(\cdot,t)^2-u_2(\cdot,t)^2\in L^1$. Thus also the second term in (2.32) is zero. □

2.6 The integrated equation

Now that we have conservation of the integral we may adapt ideas from [52]. Rather than solving (2.1) for the unknown $u(x,t)$ we consider an equation for the unknown $v(x,t)$, which as we recall is formally defined as

$$v(x,t)=\int_{-\infty}^{x}(u(s,t)-\phi(s+t))ds. \tag{2.33}$$

Thus if u is a solution of (2.1) in $C^1([0,T];\ \mathcal{H}+L^1\cap L^2)$, then by Proposition 2.9

$$\int_{\mathbb{R}}(u(s,t)-\phi(s+t))ds=\int_{\mathbb{R}}(u(s,0)-\phi(s))ds \quad \text{for all} \quad t\in[0,T], \tag{2.34}$$

which, without loss of generality, we take equal to zero, just by shifting ϕ. This allows us to work with function spaces for v having $v(\pm\infty,t)=0$ in some weak or strong sense. The equation for $v(x,t)$ is obtained by formally integrating (2.1). This yields (2.8), which, rewritten as an ODE in similar fashion as (2.5), reads

$$v_t+\frac{1}{\varepsilon^2}v=G_\varepsilon\left(\frac{1}{\varepsilon^2}v+v_x^2+2v_x\phi\right). \tag{2.35}$$

Proposition 2.10. *Let u be a solution in $C^1([0,T];\ \mathcal{H}+L^1\cap L^2)$. Then v defined by (2.33) is a solution of (2.35) defined on $[0,T]$.*

Proof. We rewrite equation (2.5) as

$$u_t+\frac{1}{\varepsilon^2}u=G_\varepsilon\left(\frac{1}{\varepsilon^2}u+(u^2)_x\right). \tag{2.36}$$

Subtracting from (2.36) the same equation for $\phi(x+t)$ we arrive at

$$z_t+\frac{1}{\varepsilon^2}z=\frac{1}{\varepsilon^2}G_\varepsilon z+G_\varepsilon(z^2+2z\phi)_x. \tag{2.37}$$

for $z(x,t)=u(x,t)-\phi(x+t)$. We define the operator $J:L^1\to L^\infty$ by

$$(Jf)(x)=\int_{-\infty}^{x}f, \tag{2.38}$$

and apply J to (2.37). Then

$$(Jz)_t+\frac{1}{\varepsilon^2}Jz=\frac{1}{\varepsilon^2}JG_\varepsilon z+JG_\varepsilon(z^2+2z\phi)_x,$$

and $v=Jz$ satisfies (2.35), provided $JG_\varepsilon f=G_\varepsilon Jf$ and $JG_\varepsilon f_x=G_\varepsilon f$ for all $f\in L^1$.

We note that J commutes with G_ε. Indeed, if $f\in L^1$, then $w=G_\varepsilon f$ has w, w' and w'' in L^1 and satisfies the equation $-\varepsilon^2w''+w=f$. Thus $Jw=JG_\varepsilon f$ satisfies

$$-\varepsilon^2(Jw)''+Jw=Jf,$$

whence $G_\varepsilon Jf=JG_\varepsilon f$. Finally $JG_\varepsilon f_x=G_\varepsilon f$ for all $f\in L^1(\mathbb{R})$ because

$$JG_\varepsilon f_x=J(g_\varepsilon'*f)=(Jg_\varepsilon')*f=G_\varepsilon f,$$

where, if we write the integrals explicitly, we have used Fubini's theorem applied to $(s,y)\to g_\varepsilon'(s-y)f(y)$ on $(-\infty,x)\times\mathbb{R}$. □

Remark 2.11. *Note that if $u \in L^1 \cap H^1$ we only need G_ε commuting with J in the argument above. Applying J to (2.37) gives*

$$(J(u-\phi))_t + \frac{1}{\varepsilon^2} J(u-\phi) = JG_\varepsilon(u-\phi) + JG_\varepsilon(u^2-\phi^2)_x.$$

Since $(u^2-\phi^2)_x \in L^1$ this implies

$$v_t + \frac{1}{\varepsilon^2} v = G_\varepsilon v + G_\varepsilon(u^2-\phi^2),$$

and the result follows because $u^2-\phi^2 = (u-\phi)(u+\phi) = v_x^2 + 2\phi v_x$.

2.7 Local well-posedness of the integrated equation

By Proposition 2.10 a solution u of (2.5) in $C^1([0,T];\ \mathcal{H} + L^1 \cap L^2)$ defines a solution v of (2.35) in the Banach space $X = \{v \in L^\infty :\ v_x \in L^1 \cap L^2\}$ with norm $||v||_X = |v|_\infty + |v_x|_{1,2}$. In this section we give a direct proof of local well-posedness of (2.35) in a number of Banach spaces.

Proposition 2.12. *The initial value problem for equation (2.35) is well-posed in each of the following Banach spaces.*

(i) $X = \{v \in L^\infty, v_x \in L^1 \cap L^2\}$ *with norm* $|v|_\infty + |v_x|_{1,2}$.

(ii) $X = \{v \in L^\infty, v_x \in L^2\}$ *with norm* $|v|_\infty + |v_x|_2$.

(iii) $X = H^1$ *with norm* $|v|_2 + |v_x|_2$.

(iv) $X = \{v \in L^2,\ v_x \in L^1 \cap L^2\}$ *with norm* $|v|_2 + |v_x|_{1,2}$.

For each of these spaces it is also well-posed in $X_1 = \{v \in X,\ v_{xx} \in L^2\}$ *with norm* $\|v\|_{X_1} = \|v\|_X + |v_{xx}|_2$.

Proof. We rewrite the equation (2.35) as

$$v_t - A_\varepsilon v = G_\varepsilon(v_x^2 + 2v_x\phi). \tag{2.39}$$

We first observe that the linear operator A_ε is bounded in X and in X_1. This follows from $(A_\varepsilon v)_x = A_\varepsilon v_x$ and $(A_\varepsilon v)_{xx} = A_\varepsilon v_{xx}$ and the boundedness of A_ε on L^∞, L^1 and L^2, see Section 2.2.

Next we prove that the operator $v \to G_\varepsilon(v_x\phi)$ is bounded in X and in X_1. It is bounded on L^2 and L^∞ because for all $v \in L^p$ $(1 < p \le \infty)$ we have, writing $\phi v_x = (\phi v)_x - \phi_x v$,

$$|G_\varepsilon(\phi v_x)|_p \le |g_\varepsilon' * (\phi v)|_p + |g_\varepsilon * (\phi_x v)|_p \le \frac{1}{\varepsilon}|\phi v|_p + |\phi_x v|_p \le (\frac{1}{\varepsilon}|\phi|_\infty + |\phi'|_\infty)|v|_p.$$

Moreover,

$$|(G_\varepsilon(\phi v_x))_x|_p = |g_\varepsilon' * (\phi v_x)|_p \le |g_\varepsilon'|_1 |\phi v_x|_p \le \frac{1}{\varepsilon}|\phi|_\infty |v_x|_p.$$

Thus the operator $v \to G_\varepsilon(v_x\phi)$ is bounded in X for each of the choices of X. Finally if $v_x \in L^2$ and $v_{xx} \in L^2$, then

$$|(G_\varepsilon(\phi v_x))_{xx}|_2 \le |g'_\varepsilon * (\phi v_{xx})|_2 + |g'_\varepsilon * (\phi_x v_x)|_2 \le \frac{1}{\varepsilon}(|\phi|_\infty |v_{xx}|_2 + |\phi'|_\infty |v_x|_2),$$

so $v \to G_\varepsilon(v_x\phi)$ is also bounded in each X_1.

It remains to show that the map $v \to G_\varepsilon v_x^2$ is locally Lipschitz continuous in X and in X_1. It is well defined on X because with $f = v_x^2 \in L^1$ the solution $w = G_\varepsilon v_x^2$ of (2.14) has $w, w_x \in L^1$ and $w, w_x \in L^2$. If in addition $v_{xx} \in L^2$, then $f_x = (v_x^2)_x = 2v_x v_{xx} \in L^1$, so that w_x has the same properties as just formulated for w and in particular $w_{xx} \in L^2$. The local Lipschitz continuity in each X follows from the estimates

$$|G_\varepsilon((v_1)_x^2 - (v_2)_x^2)|_p \le |g_\varepsilon|_p |(v_1 + v_2)_x|_2 \, |(v_1 - v_2)_x|_2,$$

which we use for $p = 2$ and $p = \infty$,

$$|(G_\varepsilon((v_1)_x^2 - (v_2)_x^2))_x|_1 = |g'_\varepsilon * ((v_1)_x^2 - (v_2)_x^2)|_1 = |B_\varepsilon((v_1)_x^2 - (v_2)_x^2)|_1 \le \\ ||B_\varepsilon||_{B(L^1)} |(v_1 + v_2)_x|_2 |(v_1 - v_2)_x|_2,$$

and

$$|(G_\varepsilon((v_1)_x^2 - (v_2)_x^2))_x|_2 = |g'_\varepsilon * ((v_1)_x^2 - (v_2)_x^2)|_2 = |B_\varepsilon((v_1)_x^2 - (v_2)_x^2)|_2 \le \\ ||B_\varepsilon||_{B(L^1,L^2)} |(v_1 + v_2)_x|_2 |(v_1 - v_2)_x|_2.$$

The local Lipschitz continuity in each X_1 , i.e. the estimate for the L^2-norm of the difference of the second order derivatives, is left to the reader. □

Remark 2.13. *As long as $v_x \in L^2$ the operator $v \to G_\varepsilon v_x^2$ is Lipschitz continuous in X. Thus if $v_x \in L^2$ for all $t > 0$ then the solution of (2.35) for $v_0 \in X$ exists globally. As for $v_0 \in X_1$, v exists globally if $v_x \in L^2$ and $v_{xx} \in L^2$ for all $t > 0$.*

2.8 Global existence and stability

In this section we establish two equalities for solutions of the integrated equation (2.35) and deduce from them global existence for small initial data and stability properties of the zero solution of equation (2.35). The first comes from testing the equation with v.

Lemma 2.14. *Any solution v of equation (2.35) in $C^1([0,T]; H^1)$ satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (v^2 + \varepsilon^2 v_x^2) = -\int_{\mathbb{R}} \left((1 - v)v_x^2 + \phi_x v^2\right). \tag{2.40}$$

Proof. We use again (2.39). Let $w = v_t - A_\varepsilon v$ and $f = v_x^2 + 2v_x\phi$. Then by assumption f is in $L^1 + L^2$ and thus $w \in H^1$ is the (weak) solution of $-\varepsilon^2 w'' + w = f$, i.e.

$$\varepsilon^2 \int_{\mathbb{R}} w_x \varphi_x + \int_{\mathbb{R}} w\varphi = \int_{\mathbb{R}} f\varphi \quad \text{for all} \quad \varphi \in H^1.$$

Taking $\varphi = v$ and observing that

$$\int_{\mathbb{R}} fv = \int_{\mathbb{R}} vv_x^2 - \int_{\mathbb{R}} \phi_x v^2, \quad \varepsilon^2 \int_{\mathbb{R}} (A_\varepsilon v)_x v_x + \int_{\mathbb{R}} (A_\varepsilon v)v = -\int_{\mathbb{R}} v_x^2,$$

we arrive at

$$\int_{\mathbb{R}} (vv_t + \varepsilon^2 v_x v_{xt}) = -\int_{\mathbb{R}} \left((1-v)v_x^2 + \phi_x v^2\right).$$

This equality is valid for each $t \in [0,T]$. □

The second equality is derived testing with v_x.

Lemma 2.15. *For any choice of X_1 in Proposition 2.12 any solution v of equation (2.35) in $C^1([0,T];X_1)$ satisfies*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (v_x^2 + \varepsilon^2 v_{xx}^2) = -\int_{\mathbb{R}} v_{xx}^2 + \int_{\mathbb{R}} \phi_x v_x^2 \tag{2.41}$$

Proof. We follow the proof of Lemma 2.14 above. Differentiating with respect to x we have with the same notation that $f_x = v_{xx}v_x + 2\phi v_{xx} + 2\phi_x v$ is in $L^1 + L^2$, $w_x = G_\varepsilon f_x$ is in H^1, so that

$$\varepsilon^2 \int_{\mathbb{R}} w_{xx}\varphi_x + \int_{\mathbb{R}} w_x \varphi = \int_{\mathbb{R}} f_x \varphi \quad \text{for all} \quad \varphi \in H^1.$$

Taking $\varphi = v_x$ and observing that

$$\int_{\mathbb{R}} f_x v_x = \int_{\mathbb{R}} \phi_x v_x^2, \quad \varepsilon^2 \int_{\mathbb{R}} (A_\varepsilon v)_{xx} v_{xx} + \int_{\mathbb{R}} (A_\varepsilon v)_x v_x = -\int_{\mathbb{R}} v_{xx}^2,$$

we arrive at

$$\int_{\mathbb{R}} (v_x v_{xt} + \varepsilon^2 v_{xx} v_{xxt}) = -\int_{\mathbb{R}} v_{xx}^2 + \int_{\mathbb{R}} \phi_x v_x^2.$$

This equality is again valid for each $t \in [0,T]$. □

Since the L^∞-norm is controlled by the H^1-norm, see (2.43) below, the first equality tells us that a solution in H^1 can be continued as long as $|v_x(\cdot)|_2$ remains bounded. The second equality shows that this is also the criterion for solutions in X_1 to be continued. In terms of u the condition for global existence is therefore that $|u(\cdot,t) - \phi(\cdot + t)|_2$ does not blow up in finite time. Next we show that this can be assured by a smallness condition on the initial data. It will be convenient to use the norm

$$||v||_\varepsilon = \left(\int_{\mathbb{R}} (v^2 + \varepsilon^2 v_x^2)\right)^{\frac{1}{2}},$$

which is equivalent to the standard H^1-norm:

$$||v||_\varepsilon \leq ||v|| \leq \frac{1}{\varepsilon}||v||_\varepsilon. \tag{2.42}$$

Estimate (2.40) implies stability of the null solution in H^1, provided the travelling wave profile ϕ is monotone.

Proposition 2.16. *Let $\varepsilon^2 \leq \frac{1}{4}$. For every initial value $v_0 \in H^1$ with $||v_0||_\varepsilon < \sqrt{\varepsilon}$ there exists a unique solution $v : [0,\infty) \to H^1$ with $||v(\cdot,t)||_\varepsilon$ decreasing for all $t \geq 0$. Moreover*

$$\int_0^\infty |v_x(\cdot,t)|_2^2 dt < \infty,$$

whence also

$$\int_0^\infty |v(\cdot,t)|_\infty^4 dt < \infty.$$

This result is better than stability but slightly weaker than asymptotic stability: we do not get that $v(\cdot,t) \to 0$ as $t \to \infty$ in H^1.

Proof. In view of $\phi_x \geq 0$ and the estimate

$$|v^2|_\infty \leq \int_{\mathbb{R}} |2vv_x| \leq 2|v|_2|v_x|_2 \leq \frac{1}{\varepsilon}|v|_2^2 + \varepsilon|v_x|_2^2 = \frac{1}{\varepsilon}||v||_\varepsilon^2, \tag{2.43}$$

the assertion follows immediately from (2.40). In particular the solution has $1 - v$ bounded away from zero by a positive constant, C, so that upon integrating (2.40) we find the first estimate:

$$C\int_0^\infty \int_{\mathbb{R}} v_x^2 < \frac{1}{2}||v_0||_\varepsilon^2.$$

Combining $|v^2|_\infty \leq 2|v|_2\,|v_x|_2$ and the estimate above with the boundedness of $|v(\cdot,t)|_2$ gives the second estimate in the theorem. □

Next we obtain an asymptotic stability result using the stronger norms with $v_{xx} \in L^2$ and a combination of (2.40) and (2.41).

Proposition 2.17. *Let $\varepsilon^2 \leq \frac{1}{4}$ and $0 < \alpha < \frac{1}{\max \phi'}$. For every initial value $v_0 \in H^2$ with $||v_0||_\varepsilon < \sqrt{(1 - \alpha \max \phi')\varepsilon}$ (no assumption on the size of $|v_0''|_2$) there exists a unique solution $v : [0,\infty) \to H^2$ with not only*

$$\int_{\mathbb{R}} (v^2 + \varepsilon^2 v_x^2),$$

but also

$$\int_{\mathbb{R}} (v^2 + (\varepsilon^2 + \alpha)v_x^2 + \varepsilon^2 \alpha v_{xx}^2)$$

decreasing for all $t \geq 0$. Moreover, $t \to ||v_x(\cdot,t)||_\varepsilon$ is square integrable and converges to zero as $t \to \infty$. Finally, $|v(\cdot,t)|_\infty \to 0$ and $|v_x(\cdot,t)|_\infty \to 0$.

Proof. Combining (2.40) and (2.41) we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} (v^2 + (\varepsilon^2 + \alpha)v_x^2 + \varepsilon^2 \alpha v_{xx}^2) = -\int_{\mathbb{R}} \phi_x v^2 - \int_{\mathbb{R}} (1 - v - \alpha\phi_x)v_x^2 - \alpha \int_{\mathbb{R}} v_{xx}^2. \tag{2.44}$$

As in the proof of Proposition 2.16, the first assertion follows immediately from (2.44). In particular the solution has $1 - v - \alpha\phi_x$ bounded away from zero by a positive constant. Note that Proposition 2.16 applies here as well.

To establish the asymptotic behaviour we note that we now have two decreasing functions. Taking the difference it follows that the function

$$t \to \int_{\mathbb{R}} (v_x(x,t)^2 + \varepsilon^2 v_{xx}(x,t)^2)dx$$

has a finite limit as $t \to \infty$. Integrating (2.44) over $(0,\infty)$ on t, it is also integrable over $(0,\infty)$ and thus the limit is zero. This proves the statement about $||v_x(\cdot,t)||_\varepsilon$. The remaining assertion follows again using (2.43). □

We list the consequences that Proposition 2.16 and Proposition 2.17 have for solutions of (2.1).

Theorem 2.18. *Let $\varepsilon^2 \leq \frac{1}{4}$ and let $u_0 \in \mathcal{H} + L^1 \cap L^2$ be such that $v_0 \in L^2$, where*

$$v_0(x) = \int_{-\infty}^{x} (u(s,t) - \phi(s+t))ds. \tag{2.45}$$

If v_0 is sufficiently small the solution exists globally and

$$t \to \int_{-\infty}^{\infty} |u(x,t) - \phi(x+t)|^2 dx$$

is both integrable and bounded on $[0,\infty)$. If in addition $v_0 \in H^2$ is sufficiently small, the solution has $(u(\cdot,t) - \phi(\cdot+t)) \to 0$ as $t \to \infty$ in H^1 and therefore also in L^∞ as $t \to \infty$. Without any restriction on ε and the norm of v_0 the solution is global if $v_0 \in H^2$.

2.9 Discontinuities

In this section we show that jump discontinuities persist as t evolves. The jumps do not move and decrease exponentially as t increases. We then focus on the case of initial data with a single jump at $x = 0$ and establish convergence of the solution to a travelling wave.

Theorem 2.19. *If $u_0 \in L^1 \cap L^\infty$ or $u_0 \in \mathcal{H} + L^1 \cap L^\infty$ has a jump discontinuity in $x = 0$, then so does the solution $u(\cdot,t)$ as long as it exists. Moreover*

$$u(0^+,t) - u(0^-,t) = \exp\left(\frac{-t}{\varepsilon^2}\right)\left(u_0(0^+) - u_0(0^-)\right),$$

in which the superscripts $-$ and $+$ indicate left and right limits.

Proof. We only discuss the case $u_0 \in \mathcal{H} + L^1 \cap L^\infty$ and write $u = \mathcal{H} + \bar{u}$, thus $\bar{u}$ satisfies the equation

$$\bar{u}_t + \frac{1}{\varepsilon^2}\bar{u} = \Phi_\varepsilon(\bar{u})$$

where $\Phi_\varepsilon(\bar{u}) = \frac{1}{\varepsilon^2}G_\varepsilon\bar{u} + B_\varepsilon\bar{u}^2 + 2B_\varepsilon(\mathcal{H}\bar{u}) + F_\varepsilon(\mathcal{H})$. Clearly

$$\Phi_\varepsilon : L^1 \cap L^\infty \to H^1 \text{ is continuous.} \tag{2.46}$$

By using the variation of constants formula we get

$$\bar{u}(\cdot,t) = \bar{u}(\cdot,0)\exp\left(\frac{-t}{\varepsilon^2}\right) + \int_0^t \Phi_\varepsilon(\bar{u}(\cdot,t))\,\exp\left(\frac{-(t-s)}{\varepsilon^2}\right)ds,$$

so $t \to \left(\bar{u}(\cdot,t) - \bar{u}_0\exp\left(\frac{-t}{\varepsilon^2}\right)\right)$ is a continuously differentiable H^1-valued function. Since $u \in H^1 \mapsto u(0^+)$ and $u \in H^1 \mapsto u(0^-)$ are bounded linear functionals, the theorem follows. □

In what follows we assume that $u_0 \in \mathcal{H} + L^1 \cap L^\infty$ and that u_0' is the sum of an L^2-function and a multiple of the δ-distribution with a point mass at the origin. Following the steps of the proof of Proposition 2.17 for the integrated equation (2.35) for (2.33) we shall prove that

$$|u - \phi|_\infty \to 0 \text{ as } t \to \infty.$$

Reasoning as in the proof of Theorem 2.19 we have $v_{xx} \in L^2(-\infty,0)$ and $v_{xx} \in L^2(0,\infty)$, as long as the solution exists, and that $v_t + \frac{1}{\varepsilon^2}v$ is continuous. However Lemma 2.15 does not apply because $v \notin H^2$. Instead we have the next lemma.

Lemma 2.20. *As long as the solution exists it satisfies*

$$\frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^0 (v_x^2 + \varepsilon^2 v_{xx}^2) + \int_0^\infty (v_x^2 + \varepsilon^2 v_{xx}^2)\right) = \\ -\int_{-\infty}^0 v_{xx}^2 - \int_0^\infty v_{xx}^2 + \int_{\mathbb{R}} \phi_x v_x^2 + R(t), \tag{2.47}$$

where

$$R(t) = \exp\left(\frac{-t}{\varepsilon^2}\right)(u_0^+ - u_0^-)\left(\frac{1}{3}((v_x^+)^2 + v_x^+ v_x^- + (v_x^-)^2) + \phi(t)(v_x^+ + v_x^-) - v_t(0,t)\right),$$

the superscripts $-$ and $+$ denoting the left and right limits at $x = 0$ for fixed t.

Proof. The function $z = u - \phi$ satisfies equation (2.37) and thus $w = z_t + \frac{1}{\varepsilon^2}z$ is a solution of (2.14) with $f = \frac{1}{\varepsilon^2}z + (z^2)_x + 2(\phi z)_x$. Using the formula

$$\varepsilon^2\int_0^\infty w'\varphi' + \varepsilon^2 w'(0^+)\varphi(0) + \int_0^\infty w\varphi = \int_0^\infty f\varphi$$

with $\varphi = z$ leads, after integrating by parts, to

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty (z^2 + \varepsilon^2 z_x^2) = -\int_0^\infty z_x^2 + \int_0^\infty \phi_x z^2 + R^+(t), \tag{2.48}$$

where

$$R^+(t) = -\frac{2}{3}(z^+)^3 - \phi(t)(z^+)^2 - \varepsilon^2 z^+(z_{xt}^+ + \frac{1}{\varepsilon^2}z_x^+).$$

Along the same lines we have

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^0 (z^2 + \varepsilon^2 z_x^2) = -\int_{-\infty}^0 z_x^2 + \int_{-\infty}^0 \phi_x z^2 + R^-(t), \tag{2.49}$$

where

$$R^-(t) = \frac{2}{3}(z^-)^3 + \phi(t)(z^-)^2 + \varepsilon^2 z^-(z^-_{xt} + \frac{1}{\varepsilon^2} z^-_x).$$

Using $v_x = z$ we express $R^+(t)$ and $R^-(t)$ in terms of v. Since

$$\varepsilon^2 v^\pm_{xxt} + v^\pm_{xx} + \frac{2}{3}(v^\pm)^2_x + \phi(t)v^\pm_x = v^\pm_t - \frac{1}{3}(v^\pm)^2_x - \phi(t)v^\pm_x$$

it follows

$$R^+(t) = -(v^+_t v^+_x - \frac{1}{3}(v^+_x)^3 - \phi(t)(v^+_x)^2),$$
$$R^-(t) = (v^-_t v^-_x - \frac{1}{3}(v^-_x)^3 - \phi(t)(v^-_x)^2).$$

Setting $R(t) = R^+(t) + R^-(t)$ completes the proof. □

With Lemma 2.14 and Lemma 2.20 we can now adapt Proposition 2.17.

Proposition 2.21. *Let $\varepsilon^2 < \frac{1}{4}$, $0 < \alpha < \frac{1}{\max \phi'}$. Let $u_0 \in \mathcal{H} + L^1 \cap L^\infty$ be such that u_0' is the sum of an L^2 function and a multiple of the δ-distribution with a point mass at $x = 0$. If v_0 is given by (2.45) and $\|v_0\|_\varepsilon < \sqrt{\varepsilon}$, there exists a unique global solution v of (2.35) with*

$$\|v\|^2_\varepsilon = \int_{\mathbb{R}} (v^2 + \varepsilon^2 v_x^2)$$

decreasing for all $t \geq 0$. Moreover

$$\int_{\mathbb{R}} \left(v^2 + (\varepsilon^2 + \alpha)v_x^2\right) + \varepsilon^2 \alpha \left(\int_0^\infty v_{xx}^2 + \int_{-\infty}^0 v_{xx}^2\right)$$

is eventually decreasing and

$$t \to \int_{\mathbb{R}} v_x^2 + \varepsilon^2 \int_0^\infty v_{xx}^2 + \varepsilon^2 \int_{-\infty}^0 v_{xx}^2$$

is integrable on $[0, \infty)$ and converges to zero as $t \to \infty$. Finally $|v(\cdot, t)|_\infty \to 0$ and $|v_x(\cdot, t)|_\infty = |u(\cdot, t) - \phi(\cdot + t)|_\infty \to 0$ as $t \to \infty$.

Proof. Proposition 2.16 implies global existence and the monotonicity of $\|v\|_\varepsilon$. Combining (2.40) and (2.47) we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}} \left(v^2 + (\alpha + \varepsilon^2)v_x^2\right) + \varepsilon^2 \alpha \int_{-\infty}^0 v_{xx}^2 + \varepsilon^2 \alpha \int_0^\infty v_{xx}^2\right) \leq$$
$$-\int_{\mathbb{R}} (1 - v - \alpha \max \phi')v_x^2 - \int_{\mathbb{R}} \phi_x v^2 - \alpha \int_{-\infty}^0 v_{xx}^2 - \alpha \int_0^\infty v_{xx}^2 + \alpha\, |R(t)|. \quad (2.50)$$

If $|R(t)| \to 0$ as $t \to \infty$ we have

$$\lim_{t \to \infty} \left(\int_{\mathbb{R}} \left(v^2 + (\alpha + \varepsilon^2)v_x^2\right) + \varepsilon^2 \alpha \int_{-\infty}^0 v_{xx}^2 + \varepsilon^2 \alpha \int_0^\infty v_{xx}^2\right) < \infty,$$

which implies the rest of the statements by the same arguments as in Proposition 2.17.

Thus it remains to show that $|R(t)| \to 0$ as $t \to \infty$. We observe that $|R(t)|$ is bounded for all $t \geq 0$,

$$|R(t)| \leq \exp\left(\frac{-t}{\varepsilon^2}\right)|u_0^+ - u_0^-|\left(|v_x^2|_\infty + 2|\phi|_\infty|v_x|_\infty + |v_t(0,t)|\right).$$

We shall prove that the last factor cannot increase as fast as $\exp\left(\frac{t}{\varepsilon^2}\right)$ as $t \to \infty$.

The last term inside brackets, $|v_t(0,t)|$, is bounded. Indeed, v satisfies equation (2.35), so, by the maximum principle we get

$$v_t + \frac{1}{\varepsilon^2}v \leq |v_x^2|_\infty + 2|\phi|_\infty|v_x|_\infty + \frac{1}{\varepsilon^2}|v|_\infty$$

for all $(x,t) \in \mathbb{R} \times [0,T]$. In particular

$$|v_t(0,t)| \leq |v_x^2|_\infty + 2|\phi|_\infty|v_x|_\infty + \frac{2}{\varepsilon^2},$$

where we have used that $\|v\|_\varepsilon$ decreases with t and $\|v_0\|_\varepsilon < \sqrt{\varepsilon}$. Thus

$$|R(t)| \leq \exp\left(\frac{-t}{\varepsilon^2}\right)|u_0^+ - u_0^-|\left(2|v_x|_\infty^2 + 4|\phi|_\infty|v_x|_\infty + \frac{2}{\varepsilon^2}\right). \tag{2.51}$$

Note that

$$\varepsilon|v_x|_\infty^2 \leq y(t) \stackrel{\text{def}}{=} \left(\int_{-\infty}^0 (v_x^2 + \varepsilon^2 v_{xx}^2) + \int_0^\infty (v_x^2 + \varepsilon^2 v_{xx}^2)\right),$$

so, in view of (2.51)

$$|R(t)| \leq \exp\left(\frac{-t}{\varepsilon^2}\right)|u_0^+ - u_0^-|\left(\frac{2}{\varepsilon}y(t) + \frac{4}{\sqrt{\varepsilon}}|\phi|_\infty\sqrt{y(t)} + \frac{2}{\varepsilon^2}\right). \tag{2.52}$$

From (2.47) we also get

$$\frac{1}{2}y'(t) \leq |\phi'|_\infty y(t) + |R(t)|. \tag{2.53}$$

Next we obtain an estimate for $y(t)$ as $t \to \infty$ which, combined with (2.52) gives the behaviour of $|R(t)|$ as $t \to \infty$. By (2.53) and (2.52),

$$\frac{1}{2}y'(t) \leq |\phi'|_\infty y(t) + \exp\left(\frac{-t}{\varepsilon^2}\right)|u_0^+ - u_0^-|\left(\frac{2}{\varepsilon}y(t) + \frac{4}{\sqrt{\varepsilon}}|\phi'|_\infty\sqrt{y(t)} + \frac{2}{\varepsilon^2}\right). \tag{2.54}$$

With $\sigma(t) = 4\varepsilon\exp\left(\frac{-t}{\varepsilon^2}\right)|u_0^+ - u_0^-|$ as integrating factor,. $w(t) = e^{\sigma(t)}y(t)$ satisfies

$$w'(t) \leq 2|\phi'|_\infty w(t) + 2\exp\left(\frac{-t}{\varepsilon^2} + \frac{\sigma(t)}{2}\right)|u_0^+ - u_0^-|\left(\sqrt{w(t)} + \frac{2}{\varepsilon^2}\right).$$

Since $\exp\left(\frac{-t}{\varepsilon^2}+\frac{\sigma(t)}{2}\right)\to 0$ as $t\to\infty$, there exist $\tilde{t}>0$ and a positive constant C such that

$$w(t)\le \exp\left(2t|\phi'|_\infty\right)C \quad \text{for all } t\ge\tilde{t},$$

and, possibly with a larger constant,

$$y(t)\le \exp\left(2t|\phi'|_\infty\right)C \quad \text{for all } t\ge\tilde{t}. \tag{2.55}$$

By substituting (2.55) into (2.52), we get

$$|R(t)|\le \exp\left(\frac{-t}{\varepsilon^2}+2t|\phi'|_\infty\right)|u_0^+-u_0^-|\,r(t) \text{ for all } t\ge\tilde{t},$$

where

$$r(t)=\left(\frac{2}{\varepsilon}C+|\phi|_\infty\exp\left(-t|\phi'|_\infty\right)\sqrt{C}+\frac{2}{\varepsilon^2}\exp\left(-2t|\phi'|_\infty\right)\right)$$

is uniformly bounded in t. The upper bound (2.27) for ϕ' implies $2|\phi'|_\infty<\frac{1}{\varepsilon^2}$, thus $|R(t)|\to 0$ as $t\to\infty$. □

Chapter 3

Long time behaviour of (1): numerical analysis

Preamble: In this chapter we approximate equation (1) numerically. Numerical examples that reflect the long time behaviour of solutions are presented. The chapter is complemented with analysis of the equation posed on bounded interval and a discussion of the numerical methods and convergence results.

3.1 Introduction

In this chapter we investigate numerically the long time behaviour of solutions of the Cauchy problem

$$\begin{aligned} u_t = u_{xx} + (u^2)_x + \varepsilon^2 u_{xxt} \quad &\text{on } \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad &\text{in } \mathbb{R}. \end{aligned} \tag{3.1}$$

To this end we consider numerical solutions of the problem

$$u_t = u_{xx} + (u^2)_x + \varepsilon^2 u_{xxt} \quad \text{on } (-l,l) \times [0,T], \tag{3.2}$$

with initial condition satisfying

$$u_0(-l) = u^-, \; u_0(l) = u^+ \tag{3.3}$$

and boundary conditions imposed on the pressure $w := u + \varepsilon^2 u_t$, namely we take

$$w(-l,t) = u^- \quad w(l,t) = u^+ \quad \text{for } t \in [0,T]. \tag{3.4}$$

Here u^- and u^+ are non-negative numbers, and $l > 0$ is sufficiently large.

Equation (3.2) subject to initial data u_0 satisfying $u_0(-\infty) = 0 < u_0(+\infty) = 1$ is investigated in Chapter 2, where stability of travelling waves is proved in different spaces. Nevertheless, these results rely on monotonicity of travelling waves, i.e. ε^2 must be sufficiently small. Two natural questions appear from this former analysis, namely whether the non-monotone travelling wave solutions are stable, and what is the long time behaviour of solutions obtained for initial data satisfying $-\infty <$

$u_0(+\infty) < u_0(-\infty) < \infty$, and also $u_0(+\infty) = u_0(-\infty) = 0$, for both small and large values of ε^2.

Comparison and maximum principles are generally used to answer these questions. Maximum principles for a class of pseudo-parabolic equations were studied in [20]. However these results are not applicable here. For instance, the linear version of (3.1) is known not to exhibit a maximum principle, see [67]. Therefore we concentrate on numerical experiments.

Two numerical schemes are used, one first order explicit, and the second first order implicit in time. Spatial discretisation is achieved by first order up-wind schemes, see [45]. In conservative form, equation (3.2) reads

$$u_t = F_x, \tag{3.5}$$

with the flux $F = u^2 + w_x$, and the pressure w satisfies the elliptic equation

$$-\varepsilon^2 w_{xx} + w = u + \varepsilon^2 (u^2)_x.$$

The conservation form is convenient for discontinuous initial data. As shown in Chapter 2, if u_0 has a jump discontinuity at some $x_0 \in \mathbb{R}$, then so does the solution for every $t \geq 0$. Conservation of mass allows one to impose flux continuity at the location of the jump of the solution, and deal with it numerically. Similar techniques can be found in [71] or [72], where interface conditions between different homogeneous porous layers are considered.

Numerical methods for equations of this type are considered, for example, in [13] and [25], where the finite element method (FEM) is used. These papers are motivated by the so-called Benjamin-Bona-Mahony (BBM) equation that arises in the context of long wave motion, see [13] for a derivation of the model. The analysis carried out in these works results in error estimates, but no jump discontinuities are considered. We finally mention the work by Schotting, Beliaev and Hassanisadeh [33], where a numerical treatment of the original problem for a capillary pressure relation accounting for hysteresis is presented.

This chapter is organised as follows. In Section 3.2 we indicate qualitative properties of solutions to the problem (3.2)-(3.4). We prove well-posedness, derive conservation of mass and state persistence in time of jump discontinuities of solutions. Our numerical schemes take into account these properties. A global existence result is also proved.

Section 3.3 describes the numerical schemes, including the treatment of jump discontinuities. For the implicit scheme we apply an iterative scheme, which is shown to converge in Section 3.5.

Before presenting the examples, in Section 3.4 we give an heuristic argument for the expected long time behaviour. To this end we refer to Burgers' equation with and without diffusion and apply a scaling argument. Extending this argument to equation (3.2), we immediately conclude that the long time behaviour is similar to the one for Burgers' equation. Hence if the initial data satisfies $u^- < u^+$, we expect asymptotically stable travelling wave solutions. If $u^- > u^+$, solutions should rather exhibit a rarefaction wave profile. Finally, if $u^+ = u^-$ the expected limiting profiles are approximations of N-waves. The numerical examples given at the end of Section 3.4 sustain the predictions above.

It is worth mentioning here that numerical examples also confirm the absence of maximum principles. When $u^- \leq u_0 \leq u^-$ and ε^2 is sufficiently large the solutions evolve to profiles that oscillate around the constant state u^+. Also when $u^- \geq u_0 \geq u^+ = 0$ with ε^2 large enough, the numerical solutions become non-positive at early time steps.

Throughout this chapter $||\cdot||$ denotes the L^2-norm, $||\cdot||_1$ the H^1-norm and $||\cdot||_\infty$ the L^∞-norm. $(\cdot,\cdot)$ stands for the usual inner product of L^2. We introduce the following coercive ($\varepsilon^2 > 0$) bilinear form in H^1,

$$a_\varepsilon(u,v) := (u,v) + \varepsilon^2(u_x, v_x),$$

and denote the associated norm by $||\cdot||_\varepsilon$. The norms $||\cdot||_1$ and $||\cdot||_\varepsilon$ are equivalent, with

$$||u||_\varepsilon \leq C_\varepsilon ||u||_1, \quad ||u||_1 \leq c_\varepsilon ||u||_\varepsilon, \tag{3.6}$$

where $C_\varepsilon = 1$ and $c_\varepsilon = \frac{1}{\varepsilon^2}$ if $\varepsilon \leq 1$, respectively $C_\varepsilon = \varepsilon^2$ and $c_\varepsilon = 1$ if $\varepsilon > 1$.

3.2 Analytical results

In this section we give some analytical results that are analogous to those proved in Chapter 2: well-posedness, persistence in time of jump discontinuities, conservation of mass, and global existence of solutions in $H^1(-l,l)$.

To prove well-posedness problem (3.2)-(3.4) is formulated by introducing the unknown $w = u + \varepsilon^2 u_t$. Formally w satisfies the elliptic equation

$$-\varepsilon^2 w_{xx} + w = u + \varepsilon^2 (u^2)_x \quad \text{on } (-l,l). \tag{3.7}$$

with boundary conditions that we take to be $w(-l) = u^-$ and $w(l) = u^+$. Note that since u itself is time dependent, t appears only as a parameter in equation (3.7).

In a rigorous manner, w is defined by the non-linear operator

$$w := W(u) + \bar{w} = G_\varepsilon \left(u + \varepsilon^2 (u^2)_x\right) + \bar{w}, \tag{3.8}$$

where G_ε is the Green function associated with the operator $(I - \varepsilon^2 \frac{d^2}{dx^2})$, on the domain $(-l,l)$ (with boundary conditions $w(\pm l) = 0$), and $\bar{w}$ is a solution of $-\varepsilon^2 w_{xx} + w = 0$ in $[-l,l]$ with boundary conditions $\bar{w}(\pm l) = u^\pm$. In this way we end up with the initial value problem

$$u_t = \frac{1}{\varepsilon^2}(W(u) - u) + \frac{1}{\varepsilon^2}\bar{w}, \quad \text{on } (-l,l) \times [0,T] \tag{3.9}$$

$$u(\cdot,0) = u_0(\cdot) \quad \text{in } (-l,l), \quad \text{and } u_0(\pm l) = u^\pm. \tag{3.10}$$

For the problem above we have

Theorem 3.1. *Let $X = L^2(-l,l), H^1(-l,l)$. If $u_0 \in X$, a $T > 0$ exists so that problem (3.9)-(3.10) has a unique solution $u \in C^1(0,T;X)$.*

Proof. The proof follows the ideas in Chapter 2 for the Cauchy problem on $\mathbb{R}$. The operator $\mathcal{L}(u) = \frac{1}{\varepsilon^2}(W(u) - u)$ maps X to X, and is locally Lipschitz continuous. Then by Picard's theorem for ordinary differential equations in Banach spaces, equation (3.9) has a unique solution in $C^1(0,T;X)$. □

Notice that (3.9)-(3.10) is an initial value problem rather than a boundary value one. As for the diffusive Burgers' equation the long time behaviour is expected to depend on the boundary values of the solution. The next lemma shows that the solution has the same boundary values as the initial condition. The lemma also accounts for evolution of discontinuities on the initial condition.

By the variation of constants formula, (3.9) is solved by

$$u(\cdot,t) = u_0(\cdot)\exp\left(\frac{-t}{\varepsilon^2}\right) + \frac{1}{\varepsilon^2}\int_0^t w(\cdot,t)\exp\left(\frac{-(t-s)}{\varepsilon^2}\right) \quad \text{for } t \in [0,T], \tag{3.11}$$

w given by (3.8). From this we obtain the following

Lemma 3.2. *Let u_0 in $L^2(-l,l)$, then*

(i) If u_0 has a jump discontinuity at x_0, then so does the corresponding unique solution u. Moreover the jump decreases according to

$$u(x_0^-,t) - u(x_0^+,t) = \exp\left(\frac{-t}{\varepsilon^2}\right)(u_0(x_0^-) - u_0(x_0^+)) \quad \textit{for all } t \in [0,T], \tag{3.12}$$

where by $u(x_0^\pm)$ we mean the left and right limits of u at x_0.

(ii) If $\exists \lim_{x\to -l} u_0(x) = u^-$ and $\lim_{x\to l} u_0(x) = u^+$, then u solving (3.9) also satisfies

$$u(-l,t) = u^- \quad \textit{and} \quad u(l,t) = u^+ \quad \textit{for all } t \in [0,T]$$

Proof. *(i)* The proof is analogous to the proof of Theorem 2.19 in Chapter 2. We use continuity of $w \in H^1(-l,l)$, and (3.11).

(ii) Let $\mu > 0$ be small enough such that u_0 is continuous on $(-l, -l+\mu)$ and $(l-\mu, l)$. By *(i)*, u is continuous on these intervals as well, so passing to the limit in (3.11) as $x \searrow -l$ and $x \nearrow l$ makes sense. To conclude the proof we use the boundary conditions on w and on u_0 in (3.4). □

Lemma 3.2-*(i)* covers the case of initial conditions with jump discontinuities, in the sense that $\frac{d}{dx}u_0 = v_0 + \sum_{i=0}^N C_i\delta_{x_i}$, with $v_0 \in L^2$, $C \in \mathbb{R}$, δ_{x_i} denoting the Dirac distribution in x_i. In Section 3.4 we consider initial conditions of this form with a single jump. As for the diffusive Burgers' equation, mass is conserved if equation (3.1) is taken on $\mathbb{R}$ (see Chapter 2). A similar property holds for problem (3.9)-(3.10). The numerical schemes for discontinuous initial conditions relies on this property. We namely have the following

Proposition 3.3. *If $u_0 \in L^2(-l,l)$, then the flux $F = w_x + u^2$ is continuous. Moreover, the solution u of (3.9)-(3.10) satisfies*

$$\int_{-l}^{l} u(x,t)dx = \int_{-l}^{l} u_0(x)dx - \int_0^t (F(-l,s) - F(l,s))ds \quad \textit{for } t \in [0,T]. \tag{3.13}$$

Proof. We first notice by (3.8) (or (3.7)) and (3.9) that the flux F defined in (3.5) satisfies

$$F_x = w_{xx} + (u^2)_x = (w - u)/\varepsilon^2 \tag{3.14}$$

in distribution sense. By Theorem 3.1, if $u_0 \in L^2(-l, l)$, then $u \in C^1(0, T; L^2(-l, l))$ and thus $w \in C(0, T; L^2(-l, l))$. It follows from (3.14) that $F_x \in C(0, T; L^2(-l, l))$. Also by the definition of F, $F \in C(0, T; L^2(-l, l))$ thus $F \in C(0, T; H^1(-l, l))$, and hence the flux is continuous everywhere. This also holds at the position of a jump discontinuity.

For the second part we test equation (3.9) with a family of functions $\varphi_\delta \in H_0^1(-l, l)$ and obtain

$$\int_0^t \int_{-l}^{l} u_t(y, s)\varphi_\delta(y)\, ds dy = \frac{1}{\varepsilon^2} \int_0^t \int_{-l}^{l} (w(y, s) - u(y, s))\varphi_\delta(y)\, dy\, ds.$$

By (3.7) we obtain

$$\begin{aligned} \int_0^t \int_{-l}^{l} u_t(y, s)\varphi_\delta(y)\, ds\, dy &= \int_0^t \int_{-l}^{l} (w_y(y, s) + u^2(y, s))_y \varphi_\delta(y)\, dy\, ds \\ &= -\int_0^t \int_{-l}^{l} (w_y(y, s) + u^2(y, s))\partial_y \varphi_\delta(y)\, dy\, ds. \end{aligned} \tag{3.15}$$

Since F is H^1 and $u_t \in L^2$ we can take $\{\varphi_\delta\}_\delta$ converging to 1 (the constant function) as $\delta \searrow 0$. Convergence is understood strongly in $L^2(-l, l)$ and therefore holds almost everywhere. This also means that $\partial_y \varphi_\delta \to \delta(1) - \delta(-1)$ in H^{-1} sense. Now passing to the limit in (3.15) gives (3.13). □

The next proposition gives stability estimates for equation (3.2), which ensure global existence of solutions in $H^1(-l, l)$. We first need the following lemma

Lemma 3.4. *Assume that the boundary values $u^+, u^- \geq 0$, then there exists a unique stationary solution h of equation (3.2) such that $h(-l) = u^-$ and $h(l) = u^+$, and h is monotone in $[-l, l]$.*

Proof. A stationary solution of (3.2) must satisfy

$$h' + h^2 = C, \quad \text{in } [-l, l], \quad \text{with } C \in \mathbb{R}. \tag{3.16}$$

The initial value problem of (3.16) with initial condition u^- at $x = -l$ is well-posed for any $C \in \mathbb{R}$. Let us see that C can be chosen uniquely so that $h(l) = u^+$. Let h_C denote the family of solutions of (3.16) such that $h_C(-l) = u^-$. Note that h_C depends continuously on C. Clearly, if $u^+ = u^-$, $h_C \equiv u^+$ with $C > 0$ such that $u^+ = \sqrt{C}$ is the only possibility. Let us consider next the other cases.

If $u^- < u^+$ we assume that $C > 0$ so that $h' > 0$, and $h \geq 0$. In this case the equation implies that $h''(x) = -2h(x)h'(x) < 0$ for all $x > -l$, and $h''(-l) \leq 0$, i.e. every h_C with $C > 0$ is concave in $(-l, l]$. Now if $C_1 < C_2$, then by (3.16) $h'_{C_1}(-l) < h'_{C_2}(-l)$, but h_{C_1} and h_{C_2} are concave, hence $h_{C_1}(x) < h_{C_2}(x)$ for all $x > -l$. Therefore h_C is monotone increasing in C. This and the continuity in C implies that there exists a unique C such that $h_C(l) = u^+$.

Similarly, if $u^+ < u^-$, then from (3.16), if $C > 0$ such that $u^+ > \sqrt{C}$ then $h' < 0$. By similar arguments we then have that $h''(x) > 0$ for all $x > -l$, and $h''(-l) \geq 0$. Again this implies that $h_C(-l)$ is monotone increasing in C, so that there exist a unique C such that $h(l) = u^+$. If $u^+ = 0$ then C is negative, and similar arguments apply. □

Proposition 3.5. *Assume, $u^+, u^- \geq 0$ and that $u_0 \in \{u \in H^1(-l,l) : u(-l) = u^-, u(l) = u^+\}$. Let h be the stationary solution of (3.2) such that $h(-l) = u^-$ and $h(l) = u^+$, then the solution u of (3.2) satisfies the estimates.*

(i) If $u^- \geq u^+$, then $t \to ||u(t) - h||_\varepsilon^2$ is decreasing, hence

$$||u(t) - h||_\varepsilon^2 \leq ||u_0 - h||_\varepsilon^2 \quad \text{for all } t \geq 0.$$

(ii) If $u^- < u^+$, then

$$||u(t) - h||_\varepsilon^2 \leq \exp(Kt)||u_0 - h||_\varepsilon^2 \quad \text{for all } t \geq 0.$$

where $K > 0$ is a constant given below.

Proof. Using the weak formulation of (3.7), and equation (3.9), we get equation (3.2) in weak form as

$$\varepsilon^2 \int_{-l}^{l} u_{tx}\varphi_x + \int_{-l}^{l} u_t\varphi = -\int_{-l}^{l} u_x\varphi_x - \int_{-l}^{l} u^2\varphi_x \quad \text{for all } \varphi \in H_0^1(-l,l). \tag{3.17}$$

Set $\tilde{u} := u - h$, then $\tilde{u} \in H_0^1(-l,l)$ and satisfies the equation

$$\int_{-l}^{l} \tilde{u}_t\varphi + \varepsilon^2 \int_{-l}^{l} \tilde{u}_{xt}\varphi_x = -\int_{-l}^{l} \tilde{u}_x\varphi_x + \int_{-l}^{l} (\tilde{u})_x^2\varphi, \quad \text{for } \varphi \in H_0^1(-l,l). \tag{3.18}$$

Testing (3.18) with $\varphi = \tilde{u}$ we get

$$\frac{1}{2}\frac{d}{dt}\int_{-l}^{l} (\tilde{u}^2 + \varepsilon^2\tilde{u}_x^2) = -\int_{-l}^{l} \tilde{u}_x^2 + \int_{-l}^{l} \tilde{u}^2 h_x. \tag{3.19}$$

Then if $u^+ \geq u^-$, Lemma 3.4 and equation (3.19) immediately imply the statement.

If $u^- < u^+$, by Lemma 3.4, $h_x(x) \geq 0$ for $x \in [-l,l]$, and it is bounded by $C - (u^-)^2$, where $C > 0$ as in the proof of the lemma, then application of Gronwall's lemma implies *(ii)* with $K = C - (u^-)^2$. □

Remark 3.6. *As we have seen in Proposition 3.5, (3.2) admits stationary solutions, and those are stable in H^1 if $u^- \geq u^+$. We then expect numerical solutions to approach the stationary solutions. However if the interval $[-l,l]$ is long enough the solution will exhibit the long time behaviour of the problem posed on $\mathbb{R}$, until the solution is affected by the boundary values.*

3.3 The numerical schemes

In this section we describe the numerical schemes we use to approximate equation (3.2).We use the following notation. Let $-l = x_0 < \ldots < x_{n+1} = l$ be a uniform partition of the spatial interval $I = [-l, l]$, with $h = x_{i+1} - x_i$. Also let $0 = t_0 < t_1 < \ldots < t_{m+1} = T$ be a uniform partition of the time interval $[0, T]$, and $\tau = t_{k+1} - t_k$ respectively. u^k and w^k are the numerical approximations of u and w solving (3.2) and (3.7)respectively, at $t = k\tau$. Their values at a grid point x_i are denoted by u_i^k and w_i^k.

The discretisation in space will differ for continuous initial data and initial data with a jump discontinuity. We shall take discrete initial data for which a single jump discontinuity occurs at the mid-point of the interval $[x_j, x_{j+1}]$ for some $j \in \{1, \ldots, n\}$. This point is denoted by $x_{j+\frac{1}{2}}$. By Lemma 3.2, the jump persists at $x_{j+\frac{1}{2}}$ as k increases. Therefore we assume that at $x_{j+\frac{1}{2}}$ the numerical solution u^k takes two different values $u^{k,-}$ and $u^{k,+}$, which denote the left and right value of u^k at $x_{j+\frac{1}{2}}$ respectively. The same might be assumed for the numerical solution w^k at time step k, but w is continuous in $(-l, l)$, since it solves the linear elliptic equation (3.7). Hence we simply let W^k denote the value of w^k at $x_{j+\frac{1}{2}}$.

We have implemented the following schemes.

3.3.1 Explicit time discretisation

We write equation (3.2) in conservation form

$$u_t = F_x, \quad \text{with } F = w_x + u^2 \tag{3.20}$$

with w solving the elliptic equation

$$-\varepsilon^2 w_{xx} + w = u + \varepsilon^2 (u^2)_x \quad \text{on } (-l, l). \tag{3.21}$$

Knowing u^k at a given time step t_k, we first solve (3.21) numerically, and obtain the corresponding pressure w^k. Next we use u^k and w^k to obtain u^{k+1} explicitly.

A first order upwind discretisation - that we adopt for gaining in stability - of (3.21) at $t = t_k$ reads

$$w_i^k - \frac{\varepsilon^2}{h^2}(w_{i-1}^k - 2w_i^k + w_{i+1}^k) = u_i^k + \frac{\varepsilon^2}{h}((u_{i+1}^k)^2 - (u_i^k)^2), \tag{3.22}$$

with boundary conditions $w_0^k = u^-$ and $w_{n+1}^k = u^+$. Next u^{k+1} is computed by

$$u_i^{k+1} - u_i^k = \frac{\tau}{h}(F_{i+\frac{1}{2}}^k - F_{i-\frac{1}{2}}^k), \quad \text{for } i = 0 \ldots n, \tag{3.23}$$

with the discrete upwind flux given by

$$F_{i+\frac{1}{2}}^k := (u_{i+1}^k)^2 + \frac{1}{h}(w_{i+1}^k - w_i^k), \quad \text{for } i = 0 \ldots n.$$

It is easy to check that this one-side discretisation is conservative, see [45], i.e. it satisfies the discrete version of (3.13).

When the initial data has a jump discontinuity at $x_{j+\frac{1}{2}}$, we modify the scheme as follows. We define

$$F^{k,-}_{j+\frac{1}{2}} = (u^{k,-})^2 + \frac{2}{h}(W^k - w^k_j),$$
$$F^{k,+}_{j+\frac{1}{2}} = (u^k_{j+1})^2 + \frac{2}{h}(w^k_{j+1} - W^k),$$

to be the fluxes at the left and right sides of the jump respectively.

We modify the discretisation of (3.22) at the right and left adjacent grid points of $x_{j+\frac{1}{2}}$:

$$w^k_j - \frac{\varepsilon^2}{h^2}(w^k_{j-1} - 2w^k_j + W^k) = u^k_j + \frac{\varepsilon^2}{2h}((u^k_{j+1})^2 + (u^{k,-})^2 - 2(u^k_j)^2) \tag{3.24}$$

and

$$w^k_{j+1} - \frac{\varepsilon^2}{h^2}(W^k - 2w^k_{j+1} + w^k_{j+2}) = u^k_{j+1} + \frac{\varepsilon^2}{2h}(2(u^k_{j+2})^2 - (u^k_{j+1})^2 - (u^{k,-})^2). \tag{3.25}$$

The right hand sides in the above equations are due to the upwind strategy. For example, $(u^2)_x$ is approximated in (3.24) as $(((u^k_{j+1})^2 + (u^{k,-})^2)/2 - (u^k_j)^2)/h$. In this way at the jump discontinuity we take into account the contribution of u from both sides. This approach is also consistent with an up-wind finite element formulation.

To determine W^k (the pressure at the jump) we impose flux continuity at $x_{j+\frac{1}{2}}$: $F^{k,-}_{j+\frac{1}{2}} = F^{k,+}_{j+\frac{1}{2}}$ for all k, which gives W^k as

$$W^k = \frac{h}{4}\left((u^k_{j+1})^2 + \frac{2}{h}(w^k_{j+1} + w^k_j) - (u^{k,-})^2\right) \quad \text{for each } k. \tag{3.26}$$

Thus the system of equations (3.22) for $i \neq j, j+1$, (3.24), (3.25) and (3.26), gives w^k at every x_i and at $x_{j+\frac{1}{2}}$.

To get u^k away from the discontinuity we use (3.23), while at x_j and x_{j+1} we take at the left side of the jump

$$u^{k+1}_j - u^k_j = \frac{\tau}{h}(F^{k,-}_{j+\frac{1}{2}} - F^k_{j-\frac{1}{2}}),$$

and at the right side of the jump (3.23) we take

$$u^{k+1}_{j+1} - u^k_{j+1} = \frac{\tau}{h}(F^k_{j+\frac{3}{2}} - F^{k,+}_{j+\frac{1}{2}}).$$

Finally we have to determine the values of u at the discontinuity, i.e. $u^{k+1,\pm}$. To do so we use the definition of w, $w = u + \varepsilon u_t$. Explicit discretisation of u_t gives

$$u^{k+1,\pm} = u^{k,\pm} + \frac{\tau}{\varepsilon}(W^k - u^{k,\pm}).$$

Observe that at each time step u^+ is not used to solve the equations at the each time step. This is due to the right up-wind discretisation. However, we compute these values and use them in the examples.

3.3.2 Implicit time discretisation

For the implicit scheme we consider equations (3.7) and (3.9). Assuming that u^{k-1} and w^{k-1} are given, at grid points not adjacent to the jump location the fully discrete equations read

$$w_i^k - \frac{\varepsilon^2}{h^2}(w_{i-1}^k - 2w_i^k + w_{i+1}^k) = u_i^k + \frac{\varepsilon^2}{h}((u_{i+1}^k)^2 - (u_i^k)^2), \tag{3.27}$$

and

$$u_i^k - u_i^{k-1} = \frac{\tau}{\varepsilon^2}(w_i^k - u_i^k). \tag{3.28}$$

The above scheme is nonlinear, due to the convection term. A straightforward semi-implicit linearization of (3.27) reads

$$w_i^k - \frac{\varepsilon^2}{h^2}(w_{i-1}^k - 2w_i^k + w_{i+1}^k) = u_i^k + \frac{\varepsilon^2}{h}(u_{i+1}^{k-1}u_{i+1}^k - u_i^{k-1}u_i^k). \tag{3.29}$$

In order to get a better approximation to the fully implicit scheme (3.27)-(3.28) we use the following iteration procedure

$$\begin{aligned} w_i^{k,s} - \tfrac{\varepsilon^2}{h^2}(w_{i-1}^{k,s} - 2w_i^{k,s} + w_{i+1}^{k,s}) &= u_i^{k,s} + \tfrac{\varepsilon^2}{h}(u_{i+1}^{k,s-1}u_{i+1}^{k,s} - u_i^{k,s-1}u_i^{k,s}), \\ u_i^{k,s} - u_i^{k-1} &= \tfrac{\tau}{\varepsilon^2}(w_i^{k,s} - u_i^{k,s}), \end{aligned} \tag{3.30}$$

where $s > 0$ is the iteration step. Initially we take $u^{k,0} = u^{k-1}$ and $w^{k,0} = w^{k-1}$, i.e. the first iteration solves the semi-implicit scheme (3.28)-(3.29). Under some restrictions on the discretisation parameters, convergence of $u^{k,s}$ to the solution of the fully implicit scheme (3.27)-(3.28) u^k (as $s \nearrow \infty$) is shown in Section 3.5.4.

For grid points in the neighborhood of the jump location we mention only the modifications that are specific to the semi-implicit discretisation (3.28)-(3.29), iteration procedure and fully implicit scheme are treated similarly. At x_j and x_{j+1}, (3.29) becomes

$$w_j^k - \frac{\varepsilon^2}{h^2}(w_{j-1}^k - 3w_j^k + 2W^k) - u_j^k - \frac{2\varepsilon^2}{h}(u^{k-1,-}u^{k,-} - u_{j+1}^{k-1}u_{j+1}^k) = 0, \tag{3.31}$$

and

$$w_{j+1}^k - \frac{\varepsilon^2}{h^2}(2W^k - 3w_{j+1}^k + w_{j+2}^k) - u_{j+1}^k - \frac{2\varepsilon^2}{h}(u_{j+2}^{k-1}u_{j+2}^k - u_{j+1}^{k-1}u_{j+1}^k) = 0. \tag{3.32}$$

Observe that now, the flux function reads:

$$F_{i+\frac{1}{2}}^k := u_{i+1}^{k-1}u_{i+1}^k + \frac{1}{h}(w_{i+1}^k - w_i^k), \quad \text{for } i \neq j.$$

At the jump we consider the left and right flux functions

$$\begin{aligned} F_{j+\frac{1}{2}}^{k,-} &= u^{k-1,-}u^{k,-} + \frac{2}{h}(W^k - w_j^k), \\ F_{j+\frac{1}{2}}^{k,+} &= u_{j+1}^{k-1}u_{j+1}^k + \frac{2}{h}(w_{j+1}^k - W^k). \end{aligned}$$

Then continuity of flux gives the following expression for W^k

$$W^k = \frac{h}{4}\left(u_{j+1}^{k-1}\, u_{j+1}^{k} + \frac{2}{h}(w_{j+1}^{k} + w_j^{k}) - u^{k-1,-}\, u^{k,-}\right) \quad \text{for all } k. \tag{3.33}$$

Again the definition of w (3.9) gives equations for u^+ and u^-, in this case

$$\left(\frac{\tau+\varepsilon^2}{\varepsilon^2}\right) u^{k,\pm} - \frac{\tau}{\varepsilon^2} W^k = -u^{k-1,\pm}. \tag{3.34}$$

In this way we end up with an algebraic system. This includes equations (3.29) for $i = 1 \ldots j-1$ and $i = j+2, \ldots, n$, to which we add equations (3.31) and (3.32), finally, equation (3.33), the equations (3.28) for $i = 1, \ldots, n$, and (3.34) close the system.

Solving this system we obtain a solution for the semi-implicit scheme. Finally, the iteration procedure is performed for discontinuous initial data with the obvious changes.

3.4 Asymptotic behaviour

This section is divided in two parts. The first part gives the preliminaries for understanding the asymptotic behaviour that is observed numerically. The second part comprises the numerical examples.

3.4.1 Preliminaries

In this section we briefly discuss the long time behaviour that should be expected for the Cauchy problem of equation (3.1). We first give a review on large time behaviour for the inviscid Burgers' equation and the diffusive (viscous) Burgers' equation, from where we conclude formally the asymptotic behaviour for equation (3.35).

First we consider the scalar conservation law

$$u_t = (u^2)_x \quad \text{on } \mathbb{R}. \tag{3.35}$$

Observe that this equation is invariant under the group of scaling transformations $x \to \lambda x$ and $t \to \lambda t$, so that if $u(x,t)$ is a solution of (3.35), the family of functions

$$u_\lambda(x,t) = u(x\lambda, t\lambda), \quad \text{for } \lambda \in \mathbb{R} \tag{3.36}$$

satisfies (3.35) as well. In fact there exist solutions of the form $u(x,t) = f(\frac{x}{t})$.

For this equation subject to the Riemann condition

$$\begin{cases} u^+ & \text{if } x > 0 \\ u^- & \text{if } x \le 0, \end{cases} \tag{3.37}$$

it is well-known that if $1 = u^- > u^+ = 0$ the weak entropy solution is a rarefaction

wave, a solution of the form $u(x,t) = f(\frac{x}{t})$, which is given by

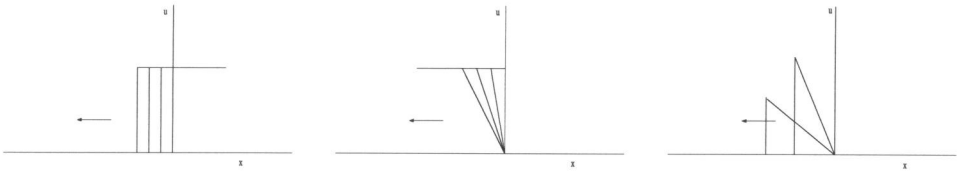
$$r\left(\frac{x}{t}\right) = \begin{cases} 1 & \text{if } \frac{x}{t} \leq -2 \\ -\frac{1}{2}\frac{x}{t} & \text{if } -2 \leq \frac{x}{t} \leq 0 \\ 0 & \text{if } \frac{x}{t} \geq 0. \end{cases} \tag{3.38}$$

Equation (3.35) is also invariant under translations in space and time. In fact if $0 = u^- < u^+ = 1$ in (3.37) then the weak entropy solution is a travelling shock wave, namely

$$g(x+t) = \begin{cases} 0 & \text{if } x+t < 0 \\ 1 & \text{if } x+t \geq 0. \end{cases} \tag{3.39}$$

which is also of the form $u(x,t) = f(\frac{x}{t})$.

Solutions of the Cauchy problem of (3.35) with bounded compactly supported initial data, tend to a so-called N-wave, see [46], a solution of (3.35), also of the form $u(x,t) = f(\frac{x}{t})$. In fact, N-waves combine both travelling shock and rarefaction wave behaviour, in a way that mass is conserved. The graphs of this solutions are drawn in Figure 3.1 for completeness.



(a) Travelling shock. (b) Rarefaction. (c) N-wave.

Figure 3.1: Entropy solutions of (3.35)

These three types of solutions describe the large time behaviour of more general solutions. One way to prove that is by observing that the scaling (3.36) transforms the limit problem $t \to \infty$ into the limit problem $\lambda \to \infty$. Indeed, it can be proved that if $u_0 \in L^\infty(\mathbb{R})$, and the family $u_{0,\lambda} \to v_0$ as $\lambda \to \infty$, where v_0 is the initial data that gives one of the above solutions of the form $v(x,t) = f(\frac{x}{t})$, then the family of solutions u_λ converges to that solution v as $\lambda \to \infty$. In particular

$$u(\lambda x, \lambda \tau) \to v(x,\tau) \quad \text{as } \lambda \to \infty$$

for each fixed $\tau \in (0,T]$, and by setting $\tau = 1$, $\lambda = t$ and $\tilde{x} = \frac{x}{t}$

$$u(y,t) \to v(\frac{y}{t}, 1) \quad \text{as } t \to \infty.$$

The diffusive (viscous) Burgers equation

$$u_t = u_{xx} + (u^2)_x \tag{3.40}$$

is invariant under the groups of transformations $x \to \mu x$, $t \to \mu^2 t$ and $u \to u/\mu$, and under translation in x and t. It is not invariant under the scaling (3.36). In fact the family u_λ satisfies the equation

$$u_{\lambda,t} = \frac{1}{\lambda} u_{\lambda,xx} + (u_\lambda^2)_x,$$

which limit equation for $\lambda \to \infty$ is (3.35). Similarly the limit $\lambda \to \infty$ transforms to the limit $t \to \infty$, i.e for initial data such that $u^+ > u^-$ solutions tend to an approximation of a travelling shock, which is a travelling wave solution of (3.40), this being consistent with translation invariance. For initial data with $u^+ < u^-$ solutions tend to an approximation of a rarefaction wave. Finally for initial data with $u^+ = u^- = 0$ solutions tend to an approximation of an N-wave, in this case a self-similar solution of equation (3.40), which is consistent with the invariance under the transformations $x \to \mu x$, $t \to \mu^2 t$ and $u \to u/\mu$.

These results can be found in [39] and [35]. See also [74] for a more general theory on asymptotic behaviour of parabolic equations and conservation laws.

Assuming that the same argument can be applied to equation (3.1) we scale equation (3.1) according to (3.36), then the family u_λ satisfies the equation

$$u_{\lambda,t} = \frac{1}{\lambda} u_{\lambda,xx} + (u_\lambda^2)_x + \frac{\varepsilon^2}{\lambda^2} u_{\lambda,xxt}.$$

Thus taking the limit $\lambda \to \infty$ we expect the limiting behaviour as $t \to \infty$ to be described by the formal *limit* equation (3.35). Then in the case $u^- < u^+$ travelling wave solutions are expected to describe the long time behaviour for any value of ε^2. In the other two cases we expect solutions to approximate rarefaction waves and N-waves respectively, as $t \to \infty$.

In view of the third order term in the rescaled equation, we expect this convergence to take longer as ε^2 gets larger, since as long as $\lambda < \varepsilon^2$ the third order term dominates.

3.4.2 Numerical examples

In this section we present numerical experiments that illustrate the long time behaviour exhibited by solutions of (3.1).

There is no visual difference in the numerical solutions obtained by the explicit discretisation (3.22)-(3.23), and solutions obtained by the iterative procedure (3.30)-(3.28); note that the rate of convergence for both method is of the same order, see Section 3.5. Since the explicit method is less time consuming, we chose it to generate the examples.

We consider three cases, depending on whether u^+ is larger, smaller of equal than u^-. The asymptotic behaviour is of the following type:

Travelling waves

We take $u^+ = 1$ and $u^- = 0$, and the following step function as initial condition,

$$u_0(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{3.41}$$

Solutions are represented in the travelling wave coordinate $\eta = x + t$, each graph corresponds to the profile at a time step. We have taken the half length of the interval to be $l = 100$. The spatial step size is $h = 0.5$ and the temporal step size is $\tau = 0.01$.

In Figure 3.2-(a) we plot results for $\varepsilon^2 = 0.2$ at time steps $t = 5, 10, 15$ and 20. In Figure 3.2-(b) solutions are plot for $\varepsilon^2 = 5$, at time steps $t = 10, 15, 20, 25$ and 30. It is easily observed that the profiles tend to overlap (in the travelling coordinate) as t increases. This convergence takes longer for $\varepsilon^2 = 5$ than for $\varepsilon^2 = 0.2$. Finally we observe that the profile of the solution oscillate around $u^+ = 1$ for $\varepsilon^2 = 5$, this being consistent with oscillatory travelling waves solutions found for $\varepsilon^2 > \frac{1}{4}$, see Section 2.3.

We finally point out that the example with $\varepsilon^2 = 5 > \frac{1}{4}$ confirms the absence of a maximum principle, since the oscillations make the solution exceed the maximum of the initial condition.

Rarefaction waves

We take $u^- = 1$ and $u^+ = 0$ for simplicity, and as initial condition the step function

$$u_0(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0. \end{cases} \tag{3.42}$$

We have taken the half length of the interval to be $l = 200$. The spatial step size is $h = 0.5$ and the temporal step size is $\tau = 0.01$.

The solutions are plot in the rarefaction coordinate $\frac{x}{2t}$ at each time step. Figure 3.3 shows results for $\varepsilon^2 = 0.2$ and $\varepsilon^2 = 5$ at time steps $t = 10, 20, 30$ and 40. For both values of ε^2 the profiles of the solution tend to overlap as t increases.

N-waves

In this section we consider examples for continuous compactly supported initial data. We namely take the following initial condition

$$u_0(x) = \begin{cases} \frac{-x}{25} & \text{if } -25 < x < 0 \\ \frac{x}{25} + 2 & \text{if } -50 \leq x < -25 \\ 0. & \text{otherwise} \end{cases} \tag{3.43}$$

We have taken the half length of the interval to be $l = 200$. The spatial step size is $h = 0.5$ and the temporal step size is $\tau = 0.1$. The solution is plot in self-similar variables, i.e. we plot $\frac{x}{\sqrt{t}}$ against $u(x,t)\sqrt{t}$ for each time step. In Figure 3.4 the corresponding results for $\varepsilon^2 = 0.2$ and $\varepsilon^2 = 5$ are drawn at time steps $t = 50, 100, 150, 200$ and 250.

Observe that in this example, with continuous initial condition, we have taken much larger time step. This is because if we were to compute solutions for step function intial data as for the previous cases, with steps sizes as in this case, we would have seen the effect of the boundary values at $t = 200$. Also, in the travelling wave case, the profiles of the solution (in the travelling coordinate) overlap already at approximately $t = 20$.

Remark 3.7. *Figure 3.5 shows initial jump discontinuities decreasing with time. The initial data are the heaviaside function, H, and reverse heavisisde function, $1-H$, as in (3.41) and (3.42) respectively. In both cases $\varepsilon^2 = 5$. We have taken $\tau = 0.01$, $h = 0.5$, here $l = 200$. At each time step, $t = 4, 12$ and 20, the profile of the solution is plot against the spatial coordinate x.*

In Figure 3.5-(a), for small values of t the discontinuity and the oscillations of the solution generate a peak in the profile. This, however, disappears as t increases due to the decrease of the jump discontinuity.

In Figure 3.5-(b), the solution becomes non-positive at early time steps. But the decrease on the jump, pushes up the solution as t increases. In particular the gives an example of non-positivity.

3.5 Convergence results

In this section we give error estimates for both explicit and implicit discretisation schemes, assuming the initial data is in H^1, and the boundary values are homogeneous. Convergence results for less regular initial data can be obtained in a similar fashion, leading eventually to lower convergence orders, but this lies beyond the purpose of this chapter. In what follows, we let $V_h \subset H_0^1(-l, l)$ denote the space of piecewise linear finite elements. These are considered on a uniform spatial grid of size h and vanish at the boundaries.

The following inequalities will be used in the sequel, their proof being elementary.

$$||u||_\infty^2 \leq 2||u||\,||\partial_x u|| \leq ||u||^2 + ||\partial_x u||^2 \leq \varepsilon^{-1}(||u||^2 + \varepsilon^2||\partial_x u||^2) \tag{3.44}$$

$$||u||^2 \leq 4l||u||\,||\partial_x u|| \Rightarrow ||u|| \leq 4l||\partial_x u|| \tag{3.45}$$

3.5.1 Error estimates

We first consider only the spatial discretisation of equation (3.2). This consists in seeking $U \in C^1([0,T]; V_h)$ such that, for all $\chi \in V_h$ and all $t > 0$

$$(U_t, \chi) + \varepsilon^2(U_{xt}, \chi_x) = -(U_x, \chi_x) + ((U^2)_x, \chi), \tag{3.46}$$

with initial data given by $(U(0), \chi) = (u_0, \chi)$ for all $\chi \in V_h$.

Observe that the weak formulation of equation (3.7) allows to write problem (3.9)-(3.10) in a weak form similar to the above, the solution of which is compared to the semidiscrete solution. Then we obtain the following estimates.

Theorem 3.8. *The semi-discrete solution U of problem (3.46) is defined for all $t \geq 0$, and satisfies the stability estimate*

$$||U(t)||_1 \leq C||u_0||_1 \tag{3.47}$$

for all $t > 0$, where $C = C_\varepsilon c_\varepsilon$ (the constants defined in (3.6)). The approximation error is bounded by

$$||U(t) - u(t)||_s \leq C(||u_0||_1)h^{1-s}\left(1 + \int_0^t (||u_t(\nu)||_1 + ||u(\nu)||_1)d\nu\right) \tag{3.48}$$

for all $t > 0$ and $s = 0, 1$ respectively.

For a proof we refer to [3], where this result is shown in a more general framework, or to Appendix A.1. The estimates are obtained for a standard finite element formulation. For stability reasons we have considered an up-wind discretisation of the convection term. In this case, in which U and $\partial_t U$ are H^1, this approach does not affect the above results, see [69] and [56], and Appendix A.2.

3.5.2 Time explicit discretisation

Applying a forward Euler discretisation to equation (3.46) we look for $\{U^k\}_{k=\overline{1,m}} \subset V_h$ such that

$$\left(U^k - U^{k-1}, \chi\right) + \varepsilon^2 \left(U_x^k - U_x^{k-1}, \chi_x\right) = -\tau(U_x^{k-1}, \chi) - \tau((U^{k-1})^2, \chi_x) \tag{3.49}$$

for all $\chi \in V_h$, with $U^0 = U(0) \in V_h$. In this case we obtain the following

Theorem 3.9. *Let U^k and u solving (3.49), respectively (3.2), then the approximation error is bounded by*

$$||u(t_k) - U^k||_s \leq h^{1-s} C_1 + \tau C_2 \int_0^{t_k} ||U_{tt}(\nu)||_1 \, d\nu \tag{3.50}$$

where

$$C_1 = C(||u_0||_1) \left(1 + \int_0^1 (||u_t(\nu)||_1 + ||u(\nu)||_1) d\nu\right),$$

and C_2 is a positive constant that depends on the uniform bound of the solutions on $[0, T]$ and on T.

3.5.3 Time implicit discretisation

The implicit scheme can be defined in a similar manner. We seek for $\{U^k\}_{k=\overline{1,m}} \subset V_h$ such that, for all $\chi \in V_h$,

$$(U^k - U^{k-1}, \chi) + \varepsilon^2 (U_x^k - U_x^{k-1}, \chi_x) = -\tau(U_x^k, \chi_x) - \tau((U^k)^2, \chi_x). \tag{3.51}$$

Testing into this with $\chi = U^k$ and using (3.52) and (3.44) we can immediately prove the following a-priori estimates.

Lemma 3.10. *Let U^k be a solution of (3.51), then*

$$||U^k||_\varepsilon \leq ||U^{k-1}||_\varepsilon \tag{3.52}$$

for $k = 1, \ldots, m$. In particular

$$||U^k||_\infty \leq \frac{1}{\varepsilon^{1/2}} ||u_0||_\varepsilon \tag{3.53}$$

for all m.

The error estimates for (3.51) are obtained in the same fashion as for the explicit scheme. Details are omitted here.

Theorem 3.11. *Let U^{k+1} and u solving (3.51) and (3.2) respectively, then*

$$||u(t_k) - U^k||_s \leq h^{1-s}C_1 + \tau C_2 \int_0^{t_k} ||U_{tt}||_1 \, d\nu \tag{3.54}$$

where

$$C_1 = C(||u_0||_1)\left(1 + \int_0^1 (||u_t(\nu)||_1 + ||u(\nu)||_1)d\nu\right),$$

and C_2 is a positive constant that depends on T and $||u_0||_1$.

3.5.4 Iterative process

Here we prove that the iteration procedure described by (3.30) converges to the solution of the implicit scheme, assuming the boundary conditions are homogeneous and $u^0 \in H_0^1(-l,l)$. To this end we denote $M_U := ||U^0||_\varepsilon^2$, then Lemma 3.10 and (3.44) give $||U^k||_\infty^2 \leq \varepsilon^{-1}M_U$ for any $k \geq 0$. Now the iterative procedure can be written, as follows. Fix $k > 0$ and let U^{k-1} solve (3.51). For any $i > 0$ find $U^{k,i} \in H_0^1(-l,l)$ such that for all $\chi \in H_0^1(-l,l)$ we have

$$(U^{k,s},\chi) + (\varepsilon^2+\tau)(\partial_x U^{k,s}, \partial_x\chi) + \tau(U^{k,s-1}U^{k,s}, \partial_x\chi) = (U^{k-1},\chi) + \varepsilon(\partial_x U^{k-1}, \partial_x\chi), \tag{3.55}$$

with $U^{k,0} = U^{k-1}$.

For each s, existence and uniqueness of a solution is provided by standard arguments (monotone perturbation of bounded and coercive bilinear forms). Moreover, the resulting array can be bounded a-priori.

Lemma 3.12. *Denote $\alpha = 16l^2/(16l^2+\varepsilon) \in (0,1)$ and assume τ satisfying*

$$\tau \leq \frac{\alpha\varepsilon^3}{8l^2M_U} = \frac{2\varepsilon^3}{M_U(16l^2+\varepsilon)}. \tag{3.56}$$

If U^{k-1} solves (3.51), then for each $i \geq 0$ the solution of (3.55) satisfies

$$||U^{k,s}||^2 + \varepsilon||\partial_x U^{k,s}||^2 \leq M_U\frac{(1-\alpha^{s+1})}{1-\alpha}. \tag{3.57}$$

Proof. The proof will be done by mathematical induction. For $U^{k,0} = U^{k-1}$ (3.57) obviously holds. Now fix $s > 0$ and assume (3.57) for $U^{k,s-1}$. Denoting M_s the H^1 equivalent norm of $U^{k,s}$

$$M_s := ||U^{k,s}||^2 + \varepsilon||\partial_x U^{k,s}||^2,$$

by (3.44) we have

$$||U^{k,s-1}||_\infty^2 \leq \varepsilon^{-1}M_{s-1}. \tag{3.58}$$

Taking $\chi = U^{k,s}$ into (3.55) and using Cauchy's inequality yields

$$\begin{aligned} ||U^{k,s}||^2 + (\varepsilon+\tau)||\partial_x U^{k,s}||^2 \leq\ & \tau\varepsilon^{-1/2}M_{s-1}^{1/2}||U^{k,s}||\,||\partial_x U^{k,s}|| \\ & + ||U^{k-1}||\,||U^{k,s}|| + \varepsilon||\partial_x U^{k-1}||\,||\partial_x U^{k,s}||. \end{aligned}$$

Applying the mean inequality $2|ab| \leq \nu|a|^2 + |b|^2/\nu$ for any reals a, b and $\nu > 0$ (with $\nu_1 = \varepsilon^{-1/2} M_{s-1}^{1/2}/2$, $\nu_2 = 1$ and $\nu_3 = 1$), after multiplying by two we end up with

$$2||U^{k,s}||^2 + 2(\varepsilon+\tau)||\partial_x U^{k,s}||^2 \leq \tfrac{\tau\varepsilon^{-1}M_{s-1}}{2}||U^{k,s}||^2 + 2\tau||\partial_x U^{k,s}||^2 + ||U^{k-1}||^2 + ||U^{k,s}||^2 + \varepsilon||\partial_x U^{k-1}||^2 + \varepsilon||\partial_x U^{k,s}||^2.$$

This can be rewritten as

$$\left(1 - \tfrac{\tau\varepsilon^{-1}M_{s-1}}{2}\right)||U^{k,s}||^2 + \varepsilon||\partial_x U^{k,s}||^2 \leq ||U^{k-1}||^2 + \varepsilon||\partial_x U^{k-1}||^2. \tag{3.59}$$

The choice of α and (3.56) ensures that the factor multiplying $||U^{k,s}||^2$ above is positive. Therefore Lemma 3.10 gives

$$\varepsilon||\partial_x U^{k,s}||^2 \leq M_U,$$

which, together with (3.45) leads to

$$||U^{k,s}||^2 \leq 16l^2 M_U/\varepsilon.$$

Applying the last inequality into (3.59) gives

$$M_s = ||U^{k,s}||^2 + \varepsilon||\partial_x U^{k,s}||^2 \leq M_U\left(1 + \tau 8l^2\varepsilon^{-3}M_{s-1}\right), \tag{3.60}$$

which, together with (3.56) proves the induction assumption. □

Remark 3.13. *The lemma above guarantees that, under the given restrictions on τ, the iteration array $\{U^{k,s}\}_{s\geq 0}$ is bounded in H^1. In fact, by (3.44), for all $s \geq 0$ we have*

$$||U^{k,s}||^2 + \varepsilon||\partial_x U^{k,s}||^2 \leq \frac{M_U}{1-\alpha} \quad \text{and} \quad ||U^{k,s}||_\infty \leq \frac{M_U}{\sqrt{\varepsilon}(1-\alpha)}.$$

In this way we have shown that the iteration array is uniformly bounded in both H^1 and L^∞. We will use this result for proving that the iteration (3.55) converges to U^k.

Theorem 3.14. *Let $k > 0$ be fixed and assume τ satisfies both (3.56) and*

$$\tau < \frac{\sqrt{2}\varepsilon^4}{M_U[\varepsilon^4 + (16l^2+\varepsilon)^2]^{1/2}}. \tag{3.61}$$

Then the iteration array $\{U^{k,s}\}_{s\geq 0}$ defined by (3.55) converges to the solution U^k of (3.51) strongly in H^1.

Remark 3.15. *Note that both restrictions imposed on τ do not depend on k or s.*

Proof. In what follows we denote the error at the iteration s by $e^s := U^k - U^{k,s}$. Subtracting (3.55) from (3.51) gives

$$(e^s, \chi) + (\varepsilon+\tau)(\partial_x e^s, \partial_x \chi) + \tau(U^k e^s + U^{k,s} e^{s-1}, \partial_x \chi) = 0. \tag{3.62}$$

Setting $\chi = e^s$, Cauchy's inequality gives

$$||e^s||^2 + (\varepsilon + \tau)||\partial_x e^s||^2 \leq \tau\varepsilon^{-1} M_U ||e^s||\,||\partial_x e^s|| + \tau\varepsilon^{-3} M_U (16l^2 + \varepsilon)||e^{s-1}||\,||\partial_x e^s||,$$

where we have also used Lemma 3.10 and Remark 3.13. Using now the mean inequality yields

$$\begin{aligned} ||e^s||^2 + (\tau + \varepsilon)||\partial_x e^s||^2 \leq{}& \frac{\tau M_U}{2\varepsilon^{\frac{1}{4}}} \left(\delta_1 ||e^s||^2 + \frac{1}{\delta_1} ||e_x^s||^2 \right) \\ &+ \frac{\tau M_U}{2\varepsilon(16l^2 + \varepsilon^2)} \left(\delta_2 ||e^{s-1}||^2 + \frac{1}{\delta_2} ||e_x^s||^2 \right). \end{aligned}$$

We choose $\delta_1 = \frac{\tau M_U}{\varepsilon^3}$ and $\delta_2 = \frac{M_U(16l^2+\varepsilon)\tau}{\varepsilon^5}$, then

$$\left(1 - \frac{\tau^2 M_U^2}{2\varepsilon^4}\right) ||e^s||^2 + \tau ||\partial_x e^s||^2 \leq \frac{\tau^2 M_U^2 (16l^2 + \varepsilon)^2}{2\varepsilon^8} ||e^{s-1}||^2. \tag{3.63}$$

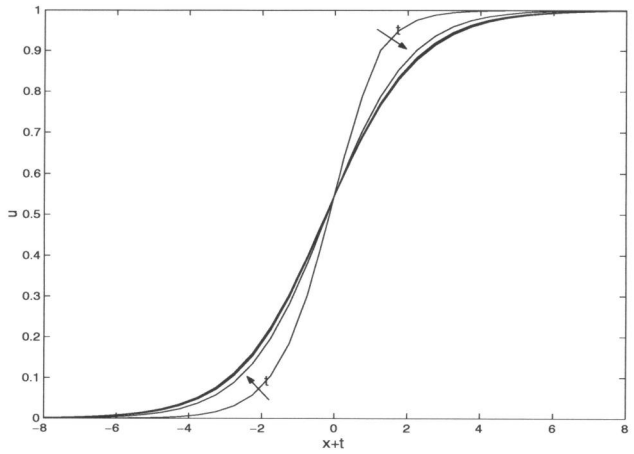
Since τ satisfies (3.61) it follows that

$$\left(1 - \frac{\tau^2 M_U^2}{2\varepsilon^4}\right) > \frac{\tau^2 M_U^2 (16l^2 + \varepsilon)^2}{2\varepsilon^8} > 0, \tag{3.64}$$

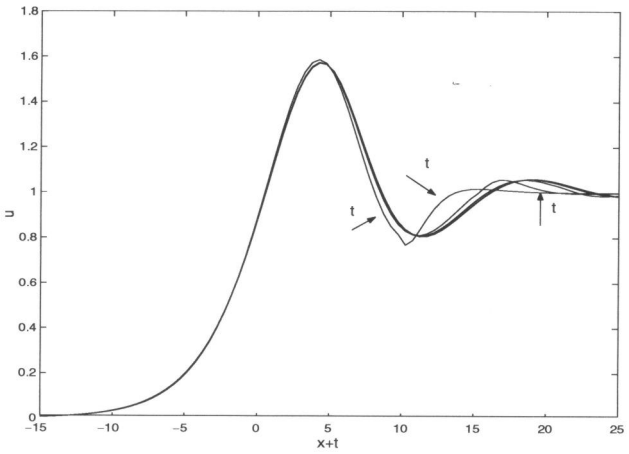
and hence

$$||e^s||^2 + \frac{2\tau\varepsilon^4}{2\varepsilon^4 - \tau^2 M_U^2} ||\partial_x e^s||^2 \leq \frac{\tau^2 M_U^2 (16l^2 + \varepsilon)^2}{\varepsilon^4 (2\varepsilon^4 - \tau^2 M_U^2)} \left(||e^{s-1}||^2 + \frac{2\tau\varepsilon^4}{2\varepsilon^4 - \tau^2 M_U^2} ||\partial_x e^{s-1}||^2 \right).$$

By (3.64), the multiplication factor on the right is less than one, which immediately implies that $e^s \to 0$ strongly in H^1. □

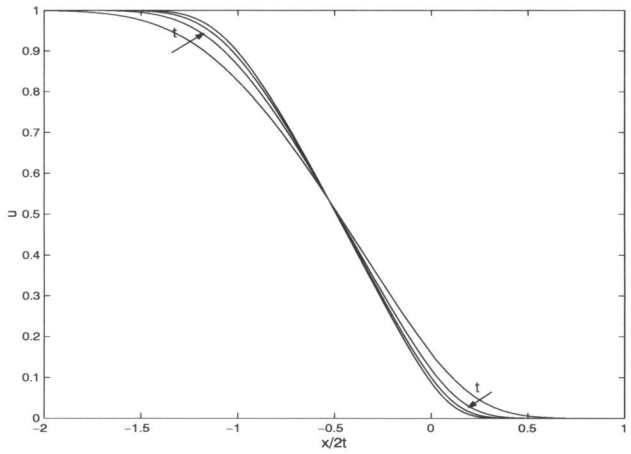


(a) $\varepsilon^2 = 0.2$, $t = 5, 10, 15$ and 20.

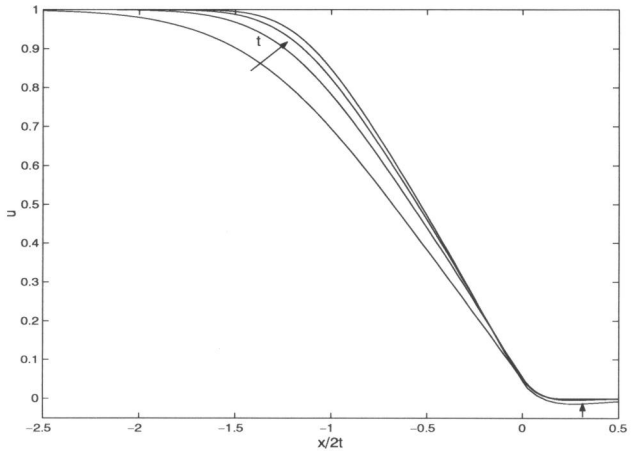


(b) $\varepsilon^2 = 5$, $t = 10, 15, 20, 25$ and 30.

Figure 3.2: Travelling wave type limiting profile.

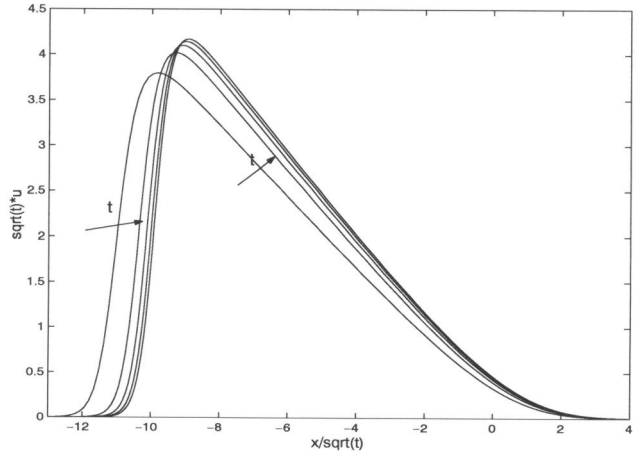


(a) $\varepsilon^2 = 0.2$, $t = 10, 20, 30$ and 40.

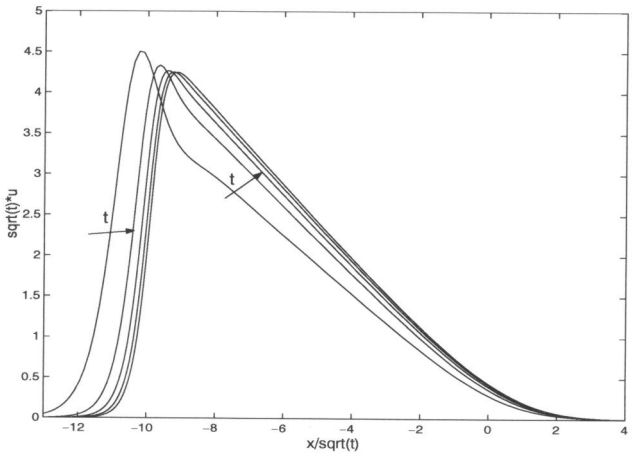


(b) $\varepsilon^2 = 5$, $t = 10, 20, 30$ and 40.

Figure 3.3: Rarefaction wave type limiting profile.

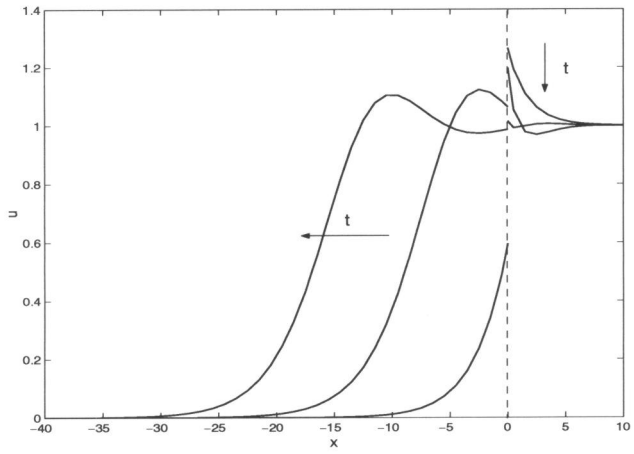


(a) $\varepsilon^2 = 0.2$, $t = 50, 100, 150, 200$ and 250.

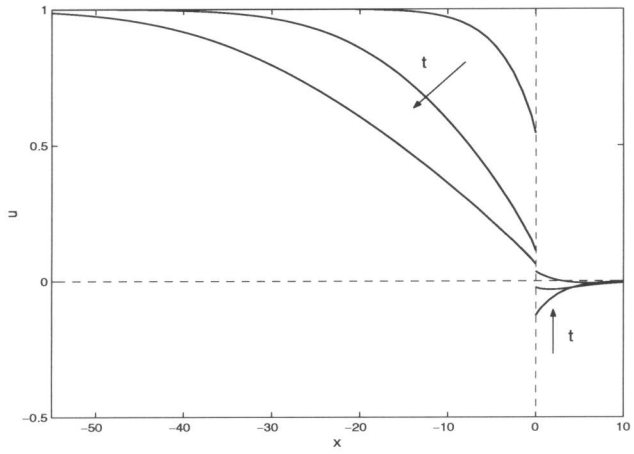


(b) $\varepsilon^2 = 5$, $t = 50, 100, 150$ and 250.

Figure 3.4: N-wave type limiting profile.



(a) $\varepsilon^2 = 5$, $t = 4, 12$ and 20.



(b) $\varepsilon^2 = 5$, $t = 4, 12$ and 20.

Figure 3.5: Persistence of discontinuities in x.

Chapter 4

Linear stability analysis of travelling waves: Evans function.

Preamble: We investigate the linear stability of travelling wave solutions of the pseudo-parabolic Burgers' equation (1), introduced in the introduction. We search for eigenvalues with positive real part of the linearised operator obtained from linearisation around travelling wave solutions. For that we use the Evans function and analyse the appearance of zeros in the right half plane. The analysis is divided in two parts. We analyse the problem for small ε, where we prove that the Evans function is a continuous perturbation in ε of the Evans function for Burgers' equation. The analysis of the Evans function is completed by a numerical search of zeros for large values of ε. The numerical examples yield the conclusion that no zeros with positive real part appear.

4.1 Introduction

In this chapter we analyse the linear stability of travelling wave solutions to the problem

$$u_t = u_{xx} + 2uu_x + \varepsilon^2 u_{xxt} \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^+, \tag{4.1}$$

with initial condition

$$u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R},$$

where u_0 is a bounded function and satisfies

$$u_0(-\infty) = 0, \quad u_0(+\infty) = 1. \tag{4.2}$$

Equation (4.1) was introduced as a pilot-problem of the model of unsaturated groundwater flow presented in Chapter 1. The question of stability of these special solutions was partially answered in Chapter 2. From the integral identities (2.40) and (2.41) we obtained stability for monotone travelling waves, i.e. for $\varepsilon^2 \leq \frac{1}{4}$. In Chapter 3 a numerical investigation shows convergence to travelling wave solutions when $\varepsilon^2 > \frac{1}{4}$ (see Fig 3.2 where $\varepsilon^2 = 5$). In order to understand this behaviour we now concentrate on the linear stability analysis.

In this chapter we define the Evans function $D(\lambda)$ for the eigenvalue problem of the linearised operator corresponding to equation (4.1). The definition follows the idea of defining the Evans function as a transmission coefficient, see [23] and [54]. In Section 4.2 we formulate the eigenvalue problem for the linearised operator. We prove for completeness that this operator generates a C_0-semigroup. Next we locate the essential spectrum and define the Evans function. In Section 4.3 we prove that the Evans function converges to the Evans function corresponding to the diffusive Burgers' equation (obtained by setting $\varepsilon = 0$ in (4.1)) as $\varepsilon \to 0$. To prove the limit we use geometric singular perturbation theory, see [27]. In Section 4.4 the limit $D(\lambda) \to 1$ as $|\lambda| \to \infty$ is proved for $\varepsilon > 0$ fixed. This result and the continuity of the Evans function in ε help us carry out a numerical computation to find eigenvalues. In Section 4.6 we described the numerical method and give examples for different values of ε. The examples lead to the conclusion that no eigenvalues with positive real part appear for large values of ε, and hence stability of travelling wave solutions is expected.

Preliminaries

First we give a brief overview of the general method. Let us consider a general initial value problem

$$\begin{aligned} u_t &= Bu + F(u, u_x) \quad \text{on } \mathbb{R} \times [0, T] \\ u(0, x) &= u_0(x) \quad \text{in } \mathbb{R} \end{aligned} \tag{4.3}$$

where B is a linear operator, and $F(U, V)$ is nonlinear. We assume that problem (4.3) is well-posed in a Banach space X, which is typically $L^\infty(\mathbb{R})$, or the space of uniformly continuous functions. We also assume that (4.3) admits travelling wave solutions, i.e solutions of the form $u(x, t) = \phi(x + ct)$, with $c > 0$, that connect two constant states: $\phi(+\infty) = \phi^+$ to $\phi(-\infty) = \phi^-$.

Before performing the linearisation it is convenient to transform the equation to the travelling wave coordinate $\eta = x + ct$, so that a travelling wave $\phi(\eta)$ is a stationary solution of the resulting equation. Let $z := u - \phi$ and (4.3) becomes

$$z_t + cz_\eta = Bz + B\phi + F(z + \phi, z_\eta + \phi') - c\phi'. \tag{4.4}$$

Since ϕ satisfies the equation

$$c\phi' = B\phi + F(\phi, \phi'),$$

the linear part of (4.4) reads

$$z_t = Bz + \frac{\partial F(\phi, \phi')}{\partial U} z + \frac{\partial F(\phi, \phi')}{\partial V} z_\eta - cz_\eta := Lz. \tag{4.5}$$

We further assume that the operator L is an infinitesimal generator of a C_0-semigroup $T(t)$. Then an estimate of the form

$$||T(t)z_0|| < Me^{\omega t} \quad \text{for } t \geq 0 \tag{4.6}$$

holds for the solution $z(t) = T(t)z_0$ of (4.5), see [51]. The infimum of all possible ω's such that (4.6) holds is called the type of the semigroup $T(t)$. Clearly if this is

negative, travelling wave solutions are asymptotically linearly stable, in the sense that $||z|| \to 0$ as $t \to \infty$, where z is a solution of the linearised equation (4.5). If further the nonlinear operator is locally Lipschitz continuous in X then linear stability implies stability.

The classical method of finding the type of the semigroup relies on the fact that for most of these problems, the linearised operator L is sectorial and hence it generates an analytic semigroup $T(t)$, see [51], or equivalently there exist $a \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$ and $M \geq 0$, such that the sector of the complex plane

$$\mathcal{S} := \{\lambda \in \mathbb{C} : \ 0 \leq |\arg(\lambda - a)| \leq \theta, \ \lambda \neq a\}$$

is contained in the resolvent set of L, and the resolvent operator satisfies the estimate

$$||(\lambda I - L)^{-1}|| \leq \frac{M}{|\lambda - a|} \quad \text{on } \mathcal{S}. \tag{4.7}$$

The last estimate allows to get the following representation of the semigroup

$$T(t) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - L)^{-1} e^{\lambda t} d\lambda, \tag{4.8}$$

where Γ is a contour of the spectrum $\sigma(L)$ of L. The contour Γ can be taken with $\arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$, which together with the estimate (4.7) implies that the integral (4.8) exists. As a consequence of (4.8) and the semigroup being analytic the following form of the spectral mapping theorem holds:

$$e^{\sigma(L)t} = \sigma\left(T(t)\right) - \{0\} \quad \text{for all } t \geq 0. \tag{4.9}$$

The proof can be found in [42].

In general if the spectral mapping theorem (4.9) holds for a semigroup and its infinitesimal generator, then the type of the semigroup coincides with the spectral bound of L, i.e. with $\sup\{\mathrm{Re}(\lambda) : \ \lambda \in \sigma(L)\}$. The analysis of the stability for the zero solution of the linearised problem (4.5) then reduces to finding the sign of the spectral bound of L.

We also observe that if L generates a C_0-semigroup, the spectral inclusion

$$e^{\sigma(L)t} \subset \sigma\left(T(t)\right), \quad \text{for } t \geq 0,$$

holds, and therefore the type of the semigroup is less than or equal to the spectral bound of L. In particular, the spectral bound of L being positive is a sufficient condition for (linear) instability.

The first difficulty in the application of these ideas to our problem, is the fact that the linearised operator for equation (4.1) does not generate an analytic semigroup. Moreover, it is not known whether the spectral mapping theorem holds in this case.

Recently, Howard and Zumbrun in [36] concluded stability results of dispersive-diffusive waves by first estimating the resolvent operator of the linearised operator. These estimates are later used to prove that an evolutionary Green's function of the form (4.8) can be constructed. Further estimates on this Green's function then give the stability and instability results. We postpone a similar analysis on the resolvent

operator in our case, and conjecture the spectral mapping theorem for the moment; in this chapter we analyse the spectrum of the linearised operator related to (4.1).

The first step of the analysis is to locate the essential spectrum, i.e. the spectrum of L aside from isolated eigenvalues with finite multiplicity. Even though in most cases the essential spectrum is contained in the left half plane, instability can still originate from the appearance of isolated eigenvalues in the right half plane; so after locating the essential spectrum it is necessary to look at the eigenvalue problem

$$(L - \lambda I)\zeta = 0. \tag{4.10}$$

In L^∞ the eigenvalues are those values of λ for which there exist non-trivial bounded solutions of (4.10), see [34]. In order to find eigenvalues equation (4.10) is written as a system of first order linear ODEs

$$Y' = A(\lambda, \eta)Y, \tag{4.11}$$

where $A(\lambda, \eta)$ is an $n \times n$ matrix, n being the order of equation (4.5), and Y is the column vector $(\zeta, \zeta' \dots \zeta^{(n)})$.

The coefficients of $A(\eta, \lambda)$ depend on ϕ and ϕ'. This implies that the matrix $A(\lambda, \eta)$ tends to constant matrices $A^{\pm}(\lambda)$ as $\eta \to \pm\infty$. By standard results on asymptotic behaviour of ordinary differential equations, see [17], solutions of (4.5) behave as solutions of the constant coefficient equations $Y' = A^{\pm}Y$ as η approaches $\pm\infty$, hence bounded solutions of (4.11) must decay exponentially to 0 at both $\eta = \pm\infty$. This can be measured in terms of a vanishing determinant of a set of solutions of (4.11). If λ is away from the essential spectrum, then a set of solutions of (4.11) can be formed by k independent solutions that decay to 0 at $\eta = -\infty$, and $n - k$ independent solutions that decay to 0 as $\eta = \infty$. When the determinant (or Wronskian) of this set is zero, a linear combination of these solutions give a bounded solution, hence λ is an eigenvalue. This determinant is the so-called Evans function, see [41], [1] and [54] for more precise definitions. Thus the Evans function has the properties of being analytic in λ aside from the essential spectrum, and its zeros on this domain are isolated eigenvalues of L.

Finally, observe that $\lambda = 0$ is always an eigenvalue since translation invariance of the equation (4.5) implies $L\phi_\eta = 0$. If $\lambda = 0$ is isolated, and the rest of the spectrum lies in the left half plane one can consider the projection on $X_1 := \ker(L) = \text{span}(\phi')$, so that $X = X_1 \oplus X_2$. This allows to pose the linear problem in X_2, where the spectral bound is strictly negative. An estimate of the form (4.6) holds on X_2 and hence stability also holds, see [22] or [34]. If zero is contained in the essential spectrum, a weighted norm might be introduced in a way that the spectrum of the operator in the weighted space is pushed off the imaginary axis, leaving the zero eigenvalue isolated. A typical example in which stability is studied in weighted spaces is the diffusive Burgers' equation, see [34] and [57].

4.2 The linearised operator and the Evans function

In this section we formulate the eigenvalue problem for the linearised operator L resulting from equation (4.1). We first prove that the linearised operator generates a

C^0-semigroup. Next we define the Evans function and locate the essential spectrum. We also prove that the eigenvalue $\lambda = 0$ is a simple zero of the Evans function. For simplicity of notation we shall skip the dependence on ε throughout this section.

Linearisation around a travelling wave solution leads for $z = u - \phi$ to the linear operator

$$Lz := (I - \varepsilon^2 \frac{d}{d\eta^2})^{-1}(z_{\eta\eta} + 2(\phi z)_\eta) - z_\eta, \tag{4.12}$$

Here, as in previous chapters, ϕ denotes the travelling wave solution of (4.1).

By adopting the notation (2.11)-(2.12) of Chapter 2, the linearised operator reads

$$Lz = A_\varepsilon z + 2B_\varepsilon(\phi z) - z_\eta.$$

With this formulation the next proposition follows easily.

Proposition 4.1. *The operator L generates a C_0-semigroup in $X = H^1(\mathbb{R}), L^2(\mathbb{R})$.*

Proof. First observe that A_ε is a bounded operator in X (see proofs in Chapter 2 of Lemma 2.3 and Lemma 2.4). Therefore it generates a C_0-semigroup, that we call $S(t)$. If T_a denotes the translation operator

$$T_a(u(\eta)) = u(\eta - a),$$

then for all $t \in \mathbb{R}$, T_t is the group generated by the operator $z \to -z_\eta$. An easy computation shows that $A_\varepsilon - \frac{d}{d\eta}$ generates the semigroup $T_t S(t)$. This is a C_0-semigroup of contractions, since $\| T_a \|_X = 1$.

The operator $z \to 2B_\varepsilon z$ is bounded in X (see Chapter 2 Lemmas 2.3 and 2.4). This and the fact that ϕ and ϕ' are bounded imply that the operator $z \to 2B_\varepsilon \phi z$ is bounded in X as well. Hence the operator $z \to Lz = A_\varepsilon z - z_\eta + 2B_\varepsilon \phi z$ generates a C_0-semigroup in X, see [51]. □

Assuming now that the spectral mapping theorem holds for L, we now study its spectrum. The eigenvalue problem of L reads

$$0 = L\zeta - \lambda\zeta = (I - \varepsilon^2 \frac{d^2}{d\eta^2})^{-1}(\zeta'' + 2(\phi\zeta)') - \zeta' - \lambda\zeta$$

or inverting the operator $(I - \varepsilon^2 \frac{d^2}{d\eta^2})^{-1}$ in (4.12):

$$\lambda\zeta = \zeta'' + (2\phi - 1)\zeta' + 2\phi'\zeta + \varepsilon^2(\zeta''' + \lambda\zeta''). \tag{4.13}$$

We transform the eigenvalue equation to a first order linear system of ODE's. Let $Y := (\zeta, \zeta', \zeta'')$, then (4.13) reads

$$\frac{dY}{d\eta} = A(\eta, \lambda)Y \tag{4.14}$$

where

$$A(\eta, \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\lambda - 2\phi'(\eta)}{\varepsilon^2} & \frac{1 - 2\phi(\eta)}{\varepsilon^2} & -\lambda - \frac{1}{\varepsilon^2} \end{pmatrix}. \tag{4.15}$$

The following properties are satisfied by the matrix $A(\eta, \lambda)$:

(i) $A(\eta, \lambda)$ is analytic with respect to λ for every $\eta \in \mathbb{R}$ and $\varepsilon > 0$ fixed.

(ii) The asymptotic matrices $A^{\pm}(\lambda) := \lim_{\eta \to \pm\infty} A(\eta, \lambda)$ exist for every λ in $\mathbb{C}$. The convergence is uniform in λ.

The characteristic polynomials corresponding to the asymptotic matrices read

$$P^{+}(\mu) = \varepsilon^2\mu^3 + (1+\varepsilon^2\lambda)\mu^2 + \mu - \lambda \quad \text{for } A^{+}, \tag{4.16}$$
$$P^{-}(\mu) = \varepsilon^2\mu^3 + (1+\varepsilon^2\lambda)\mu^2 - \mu - \lambda \quad \text{for } A^{-}. \tag{4.17}$$

In order to study the spectrum of the operator, we have to understand in which regions of $\mathbb{C}$ it is possible to construct bounded solutions of (4.10), i.e. for which values of λ a matching of solutions decaying to 0 at $-\infty$ with solutions that decay to 0 at $+\infty$ is possible. We look at the roots of the characteristic polynomials $P^{\pm}$. First we introduce the necessary notation.

Each polynomial $P^{\pm}$ has three complex roots (counting multiplicity) for every fixed λ in $\mathbb{C}$. $A^{\pm}(\lambda)$ being analytic in λ implies that the number of eigenvalues (counting multiplicity) of $A^{\pm}(\lambda)$ having negative (resp. positive) real part is constant as λ varies inside any of the connected components of the sets $\mathbb{C} - S^{\pm}$, where

$$S^{\pm} := \{\lambda \in \mathbb{C} : \ A^{\pm}(\lambda) \text{ has purely imaginary eigenvalues}\}.$$

Those sets for $A^{\pm}$ are

$$S^{+} = \{\lambda \in \mathbb{C} : \ \mathrm{Re}(\lambda) = \frac{-\tau^2}{1+\varepsilon^2\tau^2}, \ \mathrm{Im}(\lambda) = \frac{\tau(1-\varepsilon^2\tau^2)}{1+\varepsilon^2\tau^2}, \ \tau \in \mathbb{R}\}, \text{ and}$$
$$S^{-} = \{\lambda \in \mathbb{C} : \ \mathrm{Re}(\lambda) = \frac{-\tau^2}{1+\varepsilon^2\tau^2}, \ \mathrm{Im}(\lambda) = -\tau, \ \tau \in \mathbb{R}\},$$

see also Figure 4.1. Ω_r^{+} is the component containing the interval $(0, +\infty)$, Ω_c^{+} is the component containing the interval $(-\frac{1}{2\varepsilon^2}, 0)$, and Ω_l^{+} is the component containing the interval $(-\infty, -\frac{1}{2\varepsilon^2})$. The simply connected components of $\mathbb{C} - S^{-}$ are: Ω_r^{-} is the component containing the interval $(0, +\infty)$, and Ω_l^{-} is the component containing the interval $(-\infty, 0)$.

In the sequel $\mu_i^{\pm}(\lambda)$ for $i \in \{1, 2, 3\}$ will denote the eigenvalues of $A^{\pm}(\lambda)$, $v_i^{\pm}$ will denote the right eigenvectors of $A^{\pm}$ given by

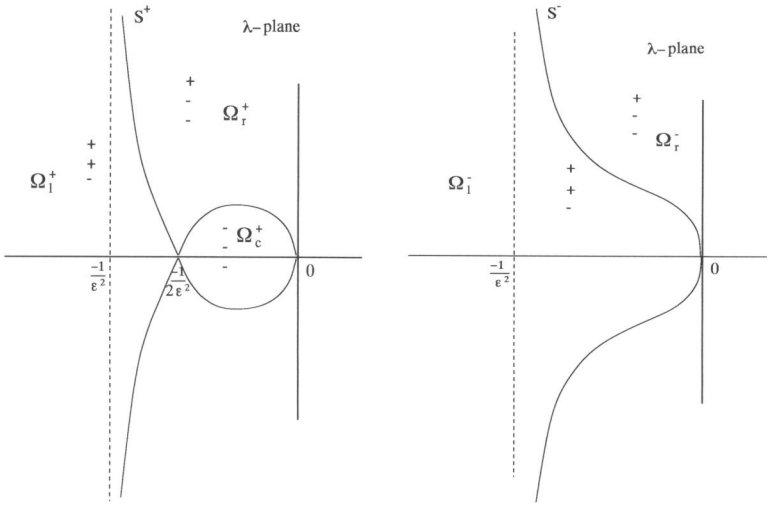
$$v_i^{\pm} = \left(1 \ , \ \mu_i^{\pm} \ , \ (\mu_i^{\pm})^2\right)^t,$$

and $w_i^{\pm}$ denote the left eigenvectors of $A^{\pm}$ such that $v_i^{\pm} w_i^{\pm} = 1$. Thus $w_i^{\pm}$ are given by

$$w_i^{\pm} = \left(\mu_i^{\pm}(\mu_i^{\pm} + \lambda + \frac{1}{\varepsilon^2}) \pm \frac{1}{\varepsilon^2} \, , \, \mu_i^{\pm} + \lambda + \frac{1}{\varepsilon^2} \, , \, 1\right) \frac{\varepsilon^2}{P_{\mu}^{\pm}(\mu_i)}.$$

Here the subscript μ indicates derivative with respect to μ.

In the next lemma we study the sign of the real part of the roots of polynomials $P^{\pm}$ for λ in each of the above connected components.



(a) Sign of real part of μ^+'s.

(b) Sign of real part of μ^-'s.

Figure 4.1: Distribution of sign of eigenvalues of $P^{\pm}$.

Lemma 4.2. *For each $\varepsilon > 0$ fixed, the signs of the real part of the eigenvalues of $A^{\pm}$ change with λ as follows:*

$$\begin{aligned} &\textit{If} \quad \lambda \in \Omega_r^+, \quad \textit{then} \quad \mathrm{Re}(\mu_1^+(\lambda)) > 0 > \mathrm{Re}(\mu_2^+(\lambda)) \geq \mathrm{Re}(\mu_3^+(\lambda)), \\ &\textit{if} \quad \lambda \in \Omega_c^+, \quad \textit{then} \quad 0 > \mathrm{Re}(\mu_1^+(\lambda)) \geq \mathrm{Re}(\mu_2^+(\lambda)) \geq \mathrm{Re}(\mu_3^+(\lambda)) \\ &\textit{if} \quad \lambda \in \Omega_l^+, \quad \textit{then} \quad \mathrm{Re}(\mu_1^+(\lambda)) \geq \mathrm{Re}(\mu_2^+(\lambda)) > 0 > \mathrm{Re}(\mu_3^+(\lambda)). \end{aligned}$$

Finally,

$$\begin{aligned} &\textit{if } \lambda \in \Omega_r^-, \quad \textit{then} \quad \mathrm{Re}(\mu_1^-(\lambda)) > 0 > \mathrm{Re}(\mu_2^-(\lambda)) \geq \mathrm{Re}(\mu_3^-(\lambda)), \\ &\textit{if } \lambda \in \Omega_l^-, \quad \textit{then} \quad \mathrm{Re}(\mu_1^-(\lambda)) \geq \mathrm{Re}(\mu_2^-(\lambda)) > 0 > \mathrm{Re}(\mu_3^-(\lambda)). \end{aligned}$$

Proof. The result holds by application of the Routh-Hurwitz criterion, see for instance [17]. We sketch the criterion for a third order polynomial in Appendix B.1. The method allows to count the number of roots with positive real part of a polynomial with real coefficients. Since in each component of $\mathbb{C} - S^{\pm}$ the number of roots with positive (resp. negative) real part does not change, we can apply this criterion by assuming that λ is real. Therefore we apply the criterion to P^+ separately for λ in $(-\infty, -\frac{1}{2\varepsilon^2})$, λ in $(-\frac{1}{2\varepsilon^2}, 0)$ and λ in $(0, -\infty)$, and to P^- separately for λ in $(-\infty, 0)$ and λ in $(0, \infty)$. □

The sets $\Omega_{r,l,c}^{\pm}$ are depicted in Figure 4.1, as well as the sign of the real part of the roots of the polynomials $P^{\pm}$.

We are now ready to define the Evans function. We follow the idea in [54]. Let $\Omega := \Omega_r^+ \cap \Omega_r^-$. For all $\lambda \in \Omega$ there is a unique solution Y^- of (4.13), which behaves

as $\exp(\mu_1^- \eta) v_1^-$ as $\eta \to -\infty$. Thus for all $\lambda \in \Omega$, Y^- decays to 0 as $\eta \to -\infty$ by Lemma 4.2. We take the Evans function $D(\lambda)$ to be the transmission coefficient such that

$$Y^-(x, \lambda) \sim D(\lambda) e^{\mu_1^+(\lambda)\eta} v_1^+ \quad \text{as } \eta \to +\infty. \tag{4.18}$$

for all $\lambda \in \Omega$. Clearly by Lemma 4.2, if $D(\lambda) = 0$, then Y^- decays to 0 as $\eta \to \pm\infty$, therefore λ is an eigenvalue and Y^- its corresponding eigenfunction. Conversely if Y is an eigenfunction for some $\lambda \in \Omega$, then Y must be a multiple of Y^-. This is because other independent solutions of (4.13) do not decay to 0 as $\eta \to -\infty$ (they behave as $\exp(\mu_2^- \eta) v_2^-$ and/or $\exp(\mu_3^- \eta) v_3^-$ when $\eta \to -\infty$).

$D(\lambda)$ can be expressed as the scalar product

$$D(\lambda) = Z^+ Y^-, \tag{4.19}$$

see [54], where Z^+, a row function, is the solution of the adjoint system $\frac{dZ^+}{d\eta} = -Z^+ A$, such that

$$Z^+ \sim e^{-\mu_1^+(\lambda)\eta} w_1^+ \quad \text{as } \eta \to +\infty.$$

The following properties are satisfied by the Evans function.

Lemma 4.3. *(i) $D(\lambda)$ does not depend on η. (ii) $D(\lambda)$ is analytic in the domain $\Omega := \Omega_r^+ \cap \Omega_r^-$. (iii) Also $\overline{D(\lambda)} = D(\bar{\lambda})$.*

The first statement is immediate from the formulation (4.19). Analyticity holds by standard arguments for ODE's and (4.19). The last statement holds by using the symmetry $A(\eta, \bar{\lambda}) = \overline{A(\eta, \lambda)}$.

Remark 4.4. *A second Evans function can be defined as a transmission coefficient on the set $\Omega_l^+ \cup \Omega_l^-$. In this case the Evans function is obtained as the coefficient $\widetilde{D}(\lambda)$ that makes*

$$Y^+ \sim \widetilde{D}(\lambda) \exp(\mu_3^- \eta) v_3^- \quad \text{as } \eta \to -\infty,$$

here Y^+ is a solution that behaves like $\exp(\mu_3^+) v_3^+$ as $\eta \to \infty$ (it decays to 0 as $\eta \to \infty$). $\widetilde{D}(\lambda)$ satisfies Lemma 4.3 on $\Omega_l^+ \cup \Omega_l^-$.

In order to locate the essential spectrum we observe that the operator $L - \lambda I$ can only be inverted in regions of the λ-plane that allow consistent splitting of roots, i.e regions where P^- and P^+ have the same number of roots with positive (resp. negative) real part. This is because a Green's function for the operator $L - \lambda I$ can only be constructed by matching linearly independent solutions of (4.13) that decay to zero as $\eta \to \infty$ to linearly independent solutions that decay to 0 as $\eta \to -\infty$. If for some λ these solutions are linearly dependent, λ is an eigenvalue. Observe that in the regions of consistent splitting the Evans function is analytic and its zeros are eigenvalues of L. Thus eigenvalues must be isolated in regions of consistent splitting, whereas the rest of points are in the resolvent set.

In our case Ω and $\Omega_l^- \cap \Omega_l^+$ are the regions of consistent splitting. We then have the following proposition

Proposition 4.5. *The essential spectrum of L is the set*

$$\mathbb{C} \setminus (\Omega \cup (\Omega_l^- \cap \Omega_l^+)) = \overline{(\Omega_l^- \cap \Omega_r^+) \cup (\Omega_l^- \cap \Omega_c^+)}.$$

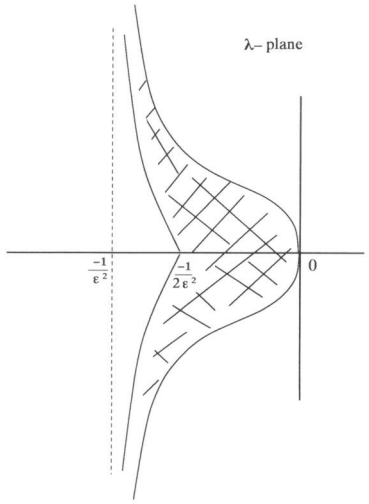


Figure 4.2: Essential spectrum

The essential spectrum is drawn in Figure 4.2. Clearly L is not sectorial.

As mentioned in the introduction, $\lambda = 0$ is always an eigenvalue of L, with eigenfunction ϕ', but $0 \notin \Omega$. In fact the function $D(\lambda)$ can be extended through the essential spectrum to a neighbourhood of $\lambda = 0$. Moreover it can be extended through the essential spectrum to a neighbourhood of the set S^-, as it is shown in the next lemma

Lemma 4.6. *There exists an open set Ω^* in $\mathbb{C}$ that contains the sets S^- and Ω, such that for all $\varepsilon > 0$, $D(\lambda)$ is well-defined and analytic in Ω^*.*

Proof. Observe that $D(\lambda)$ is well-defined for all λ such that

$$\mathrm{Re}(\mu_1^+(\lambda)) > \mathrm{Re}(\mu_2^+(\lambda)),\ \mathrm{Re}(\mu_3^+(\lambda)), \tag{4.20}$$

$$\mathrm{Re}(\mu_1^-(\lambda)) > \mathrm{Re}(\mu_2^-(\lambda)), \mathrm{Re}(\mu_3^-(\lambda)). \tag{4.21}$$

These conditions are satisfied for $\lambda \in S^-$. Indeed, if $\lambda \in S^-$, then $\lambda = \frac{-\tau^2}{1+\varepsilon^2\tau^2} - i\tau$ for $\tau \in \mathbb{R}$, and the eigenvalues of P^- are

$$\mu_1^-(\lambda) = \frac{-1+\Delta^{\frac{1}{2}}}{2\varepsilon^2},\ \mu_2^-(\lambda) = \tau i,\ \mu_3^-(\lambda) = \frac{-1-\Delta^{\frac{1}{2}}}{2\varepsilon^2},$$

where $\Delta = 1+4\varepsilon^2(\varepsilon^2\tau^2+1+\varepsilon^4\tau^2) - 4i\tau\varepsilon^2(1+\tau^2\varepsilon^2)$, with $\mathrm{Re}(\Delta^{\frac{1}{2}}) > 1$, this implies (4.21). Also if $\lambda \in S^-\backslash\{0\}$, then $\lambda \in \Omega_r^+$, and (4.20) holds by Lemma 4.2. Finally if $\lambda = 0$,

$$\mu_1^+(0) = 0,\ \mu_2^+(0) = \frac{-1+(1-4\varepsilon^2)^{\frac{1}{2}}}{2\varepsilon^2},\ \mu_3^+(0) = \frac{-1-(1-4\varepsilon^2)^{\frac{1}{2}}}{2\varepsilon^2},$$

hence (4.20) also holds. Since $\mu_1^\pm$ are analytic in $\lambda \in \overline{\Omega}$, there exists a neighbourhood U of S^-, in which (4.20) and (4.21) hold, and hence $D(\lambda)$ is well-defined. □

Then $D(0)$ is well-defined and clearly $D(0) = 0$. In the next lemma we compute $D'(0)$.

Lemma 4.7. *For all $\varepsilon > 0$, $\lambda = 0$ is a simple zero of $D(\lambda)$, with*

$$D'(0) = \frac{2\varepsilon^2}{(-1+\sqrt{1+4\varepsilon^2})} > 0.$$

Proof. For the purposes of this lemma we adopt the notation $Y^- = (y_1^-, y_2^-, y_3^-)$ and $Z^+ = (z_1^+, z_2^+, z_3^+)$, for the components of the vector functions Y^- and Z^+.

The first derivative of $D(\lambda)$ is computed similarly as in [54], it reads

$$D'(\lambda) = \int_{-\infty}^{\infty} \left(Z^+(\lambda, s) \frac{dA(\lambda, s)}{d\lambda} Y^-(\lambda, s) \right) ds + D(\lambda) \left(\frac{d}{d\lambda}\{\mu_1^+ - \mu_1^-\} + \frac{dw_+(\lambda)}{d\lambda} v_+(\lambda) + w_-(\lambda) \frac{dv_-(\lambda)}{d\lambda} \right).$$

see Appendix B.2. Clearly if λ is an eigenvalue the derivative reduces to $D'(\lambda) = \int_{-\infty}^{\infty} Z^+ \frac{dA}{d\lambda} Y^-$, hence

$$D'(0) = \int_{-\infty}^{+\infty} z_3^+ \left(\frac{1}{\varepsilon^2} y_1^- - y_3^- \right). \tag{4.22}$$

An eigenfunction for $\lambda = 0$ is ϕ', and $\phi' \sim \mu_1^-(0) \exp(\mu_1^-(0)\eta)$ as $\eta \to -\infty$, then $y_1^- = \frac{1}{\mu_1^-(0)} \phi'$.

From the adjoint equation we see that z_3^+ satisfies

$$(z_3^+)''' \varepsilon^2 = (z_3^+)'(1 - 2\phi) + (z_3^+)'',$$

and from the choice of w_3^+, z_3^+ must behave as $e^{-\mu_1^+(0)\eta} \frac{\varepsilon^2}{P_\mu^+(\mu_1^+(0))}$ as $\eta \to +\infty$. Using that $\mu_1^+(0) = 0$ and $P_\mu^+(\mu_1^+(0)) = 1$ implies $z_3^+ \equiv \varepsilon^2$. Finally substitution of $y_1^- = (\mu^-)^{-1}\phi' = \frac{2\varepsilon^2}{-1+\sqrt{1+4\lambda}}\phi'$ and $z_3^+ = \varepsilon^2$ into (4.22), and computing this integral finishes the proof. □

4.3 Behaviour of D as $\varepsilon \to 0$

In this section, we prove that the Evans function of (4.1) converges to the Evans function of the diffusive Burgers' equation as $\varepsilon \to 0$.

To avoid confusion we will write the Evans function related to (4.1) as $D_\varepsilon(\lambda)$, and its domain Ω^* as Ω_ε^*. The polynomials $P^\pm$ will be written as $P_\varepsilon^\pm$, and their roots will be written as $\mu_{i,\varepsilon}^\pm$ for $i = 1, 2, 3$. We shall also write the travelling wave solutions of (4.1) as ϕ_ε. No subscript or the subscript 0 will mean that the set, function, polynomial, root, or travelling wave solution corresponds to Burgers' equation ($\varepsilon = 0$).

Let us recall briefly the linear stability problem corresponding to equation (4.1) with $\varepsilon = 0$. We consider travelling wave solutions of Burgers' equation that connect

0 at $-\infty$ to 1 at $+\infty$, hence the wave speed is 1, and the travelling wave coordinate is $\eta = x + t$, as for the $\varepsilon > 0$ problem. The eigenvalue problem reads

$$\lambda\zeta(\eta) = \zeta''(\eta) + (2\phi(\eta) - 1)\zeta'(\eta) + 2\phi'(\eta)\zeta(\eta), \tag{4.23}$$

or written as a first order system of ODE's

$$Y' = A_0(\eta, \lambda)Y = \begin{pmatrix} 0 & 1 \\ \lambda - 2\phi'(\eta) & 1 - 2\phi(\eta) \end{pmatrix} Y. \tag{4.24}$$

with $Y = (\zeta, \zeta')$. The characteristic polynomials of the limit matrices $A_0^{\pm}$ are $P_0^+(\mu) = \mu^2 + \mu - \lambda = 0$ with roots $\mu_{1,0}^+ = \frac{-1+\sqrt{1+4\lambda}}{2}$, $\mu_{2,0}^+ = \frac{-1-\sqrt{1+4\lambda}}{2}$; and $P_0^-(\mu) = \mu^2 - \mu - \lambda = 0$ with roots $\mu_{1,0}^- = \frac{1+\sqrt{1+4\lambda}}{2}$, $\mu_{2,0}^- = \frac{1-\sqrt{1+4\lambda}}{2}$. Clearly $\mathrm{Re}(\mu_{2,0}^{\pm}) < 0 < \mathrm{Re}(\mu_{1,0}^{\pm})$, for all λ such that $\mathrm{Re}(\lambda) > -\frac{1}{4}$.

The Evans function is then defined by

$$D_0(\lambda) = Y^- Z^+, \quad \text{for all } \lambda \text{ such that } \mathrm{Re}(\lambda) > -\frac{1}{4},$$

where Y^- is a solution of (4.24) that satisfies $Y^- \sim \exp(\mu_{1,0}^- \eta)\, v_{1,0}^-$ as $\eta \to -\infty$, with $v_{1,0}^- = (1, \mu_{1,0}^-)$. And Z^+ is a solution of the adjoint system to (4.24) that satisfies $Z^- \sim \exp(-\mu_{1,0}^+ \eta)\, w_{1,0}^+$ as $\eta \to +\infty$, with $w_{1,0}^+$ the left eigenvector to $\mu_{1,0}^+$ such that $w_{1,0}^+(1, \mu_{1,0}^+) = 1$.

We will need the behaviour of the eigenvalues of $P_\varepsilon^{\pm}$ for ε small, this is done in the next lemma.

Lemma 4.8. *If $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) \geq 0$, then*

$$\mu_{1,\varepsilon}^{\pm}(\lambda) = \mu_{1,0}^{\pm}(\lambda) + O(\varepsilon^2),\ \mu_{2,\varepsilon}^{\pm}(\lambda) = \mu_{2,0}^{\pm}(\lambda) + O(\varepsilon^2),$$
$$\text{and} \quad \mu_{3,\varepsilon}^{\pm}(\lambda) = -\frac{1}{\varepsilon^2} - (1 + \lambda) + O(\varepsilon^2) \text{ as } \varepsilon \to 0.$$

Proof. The roots of $P_0^{\pm}(\mu)$ are simple if λ has positive real part. By the implicit function theorem there are two roots that approach those of P_0 uniformly in $\lambda \in \Omega_{\varepsilon^*}^*$ for $\varepsilon^* > 0$ sufficiently small.

By using asymptotic expansions in ε of μ, we get the behaviour

$$\mu_{i,\varepsilon}^{\pm}(\lambda) = \mu_{i,0}^{\pm}(\lambda) + \varepsilon^2 \frac{-\mu_{i,0}^{\pm}(\lambda)(\mu_{i,0}^{\pm}(\lambda) + \lambda)}{\pm 1 + 2\mu_{i,0}^{\pm}(\lambda)} + O(\varepsilon^4), \quad \text{for } i = 1, 2 \quad \text{as } \varepsilon \to \infty.$$

We employ the scaling $\mu = \frac{1}{\varepsilon^2}\xi$ in $P_\varepsilon^{\pm}$, and apply the same argument as above to the scaled polynomial, for which -1 is a simple root. This gives the third root,

$$\mu_{3,\varepsilon}^{\pm} = -\frac{1}{\varepsilon^2} - (1 + \lambda) + O(\varepsilon^2) \quad \text{as } \varepsilon^2 \to 0.$$

□

Theorem 4.9. *For every $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) \geq 0$, $D_\varepsilon(\lambda) \to D_0(\lambda)$ as $\varepsilon \to 0$.*

Proof. We first observe that the solutions Z_ε of the adjoint system of (4.14) are of the form $Z_\varepsilon = (z_\varepsilon'' - (1+\varepsilon^2\lambda)z_\varepsilon'/\varepsilon^2 + (2\phi-1)z_\varepsilon/\varepsilon^2\,,\ -z_\varepsilon' + (1+\varepsilon^2\lambda)z_\varepsilon/\varepsilon^2\,,\ z_\varepsilon)$, where z_ε satisfies the equation $\varepsilon^2 z_\varepsilon''' = -\lambda z_\varepsilon + (1-2\phi)z_\varepsilon' + (1+\varepsilon^2\lambda)z_\varepsilon''$. Also a solution Z of the adjoint system is of the form $Z = (-z' - (1-2\phi)z, z)$, where z satisfies $z'(1-2\phi) + z'' = \lambda z$. Then computing the inner product (4.19) we get the Evans functions for $\varepsilon = 0$

$$D_0(\lambda) = Z^+Y^- = \zeta\,(-z' + (2\phi-1)z) + z\zeta'$$

and the Evans function for $\varepsilon > 0$

$$D_\varepsilon(\lambda) = Z_\varepsilon^+ Y_\varepsilon^- = \varepsilon^2\left(\frac{Z_\varepsilon^+}{\varepsilon^2}\right)Y_\varepsilon^- =$$

$$\zeta_\varepsilon\left(\frac{z_\varepsilon''}{\varepsilon^2}\varepsilon^2 - (1+\varepsilon^2\lambda)\frac{z_\varepsilon}{\varepsilon^2} + (2\phi_\varepsilon - 1)\frac{z_\varepsilon}{\varepsilon^2}\right) + \zeta_\varepsilon'\left(-\frac{z_\varepsilon'}{\varepsilon^2}\varepsilon^2 + (1+\varepsilon^2\lambda)\frac{z_\varepsilon}{\varepsilon^2}\right) + \zeta_\varepsilon''\frac{z_\varepsilon}{\varepsilon^2}\varepsilon^2.$$

We have divided z_ε, z_ε' and z_ε'' by ε^2 in D_ε so that the limit $\varepsilon \to 0$ makes sense. In fact if for some $\eta_0 \in \mathbb{R}$, $(\zeta_\varepsilon, \zeta_\varepsilon') \to (\zeta, \zeta')$ uniformly in some interval $(-\infty, \eta_0)$, $(z_\varepsilon/\varepsilon^2, z_\varepsilon'/\varepsilon^2) \to (z, z')$ uniformly on (η_0, ∞), and $\phi_\varepsilon \to \phi$ uniformly on $\mathbb{R}$ as $\varepsilon \to 0$, the result holds. The limit for ϕ_ε can be analysed separately. The proof is similar to the proof of Proposition 1.7 in Chapter 1. Next we prove the limit for ζ_ε. This is a singular limit. We shall apply geometric singular perturbation theory (GSPT), see [27] and [40]. For that we need to augment system (4.10), so that it becomes autonomous. We simply add the travelling wave equation. The complete system reads

$$(S_\varepsilon)\begin{cases} u' = v \\ v' = w \\ \varepsilon^2 w' = (\lambda - 2\psi)u + (1-2\phi)v - (1+\varepsilon^2\lambda)w \\ \phi' = \psi \\ \varepsilon^2\psi' = \phi - \phi^2 - \psi, \end{cases}$$

here $u = \zeta$. In the context of GSPT this is the slow system. The fast system is the system reformulated in the variable $\xi = \frac{1}{\varepsilon^2}\eta$, in this case

$$(F_\varepsilon)\begin{cases} \dot{u} = \varepsilon^2 v \\ \dot{v} = \varepsilon^2 w \\ \dot{w} = (\lambda - 2\psi)u + (1-2\phi)v - (1+\varepsilon^2\lambda)w \\ \dot{\phi} = \varepsilon^2\psi \\ \dot{\psi} = \phi - \phi^2 - \psi, \end{cases}$$

where the dots on top stand for $\frac{d}{d\xi}$. The limit $\varepsilon \to 0$ for (S_ε) is not well defined. The formal limit equations (taking $\varepsilon = 0$ in (S_ε)) can be seen as a flow on $\mathbb{R}^5$ restricted to a 3-dimensional manifold M_0, which is given by the singular equations, i.e.

$$M_0 = \{(u, v, w, \phi, \psi) \in \mathbb{R}^5 :\ w = (\lambda - 2\psi)u + (1-2\phi)v,\ \psi = \phi - \phi^2\}.$$

The aim of using SGPT is to restrict system (S_ε) to a 3-dimensional manifold which is close to M_0 for ε sufficiently small, and in which the limit $\varepsilon \to 0$ is regular. Part of the information to find this manifold is provided by the fast system (F_ε). M_0

coincides with the manifold of critical points of (F_0). Under the condition that M_0 is normally hyperbolic for (F_0), the first Fenichel's invariant manifold theorem, see [26], gives existence of a manifold M_ε, which is a C^k ε-perturbation of M_0, and that is locally invariant under the flow (F_ε).

The eigenvalues of the linearisation of (F_0) are $\{0, 0, -1, 0, -1\}$. Only the -1 eigenvalues (the eigenvalues with non trivial real part) have eigenvectors that are transversal to M_0, thus M_0 is normally hyperbolic. We use the Fenichel's invariant manifold theorem reformulated in [40], which is suitable for manifolds given by a graph on $\mathbb{R}^n$. Then for all $R > 0$, and for all $k \in \mathbb{N}$, there exists an $\varepsilon_0 > 0$, depending on R and k, such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a locally invariant manifold of system (F_ε), given by

$$M_\varepsilon = \{(u, v) \in B_R(0),\ \phi \in (-R, 1+R) :\ w = \Phi(u, v, \phi, \varepsilon, \lambda),\ \psi = \Psi(\phi, \varepsilon)\}$$

with Φ in $C^k(B_R(0) \times (-R, 1+R) \times [0, \varepsilon_0])$, and Ψ in $C^k((-R, 1+R) \times [0, \varepsilon_0])$. Also $\Phi(u, v, \phi, 0, \lambda) = (\lambda - 2(\phi - \phi^2))u + (1 - 2\phi)v$, and $\Psi(\phi, 0) = \phi - \phi^2$.

M_ε is in turn also locally invariant under the flow (S_ε); we can consider (S_ε) restricted to M_ε, this reads

$$(S'_\varepsilon) \begin{cases} u' = v \\ v' = \Phi(u, v, \phi, \varepsilon, \lambda) \\ \phi' = \Psi(\phi, \varepsilon), \end{cases}$$

and is now a regular perturbation of the system

$$(S_0) \begin{cases} u' = v \\ v' = \Phi(u, v, \phi, 0, \lambda) = (\lambda - 2(\phi - \phi^2))u + (1 - 2\phi)v \\ \phi' = \Psi(\phi, 0) = \phi - \phi^2. \end{cases}$$

The critical points of (S_0) are $(0, 0, 0)$ and $(0, 0, 1)$ if $\lambda \neq 0$. From the implicit function theorem it follows that for $\varepsilon > 0$ small there are two critical points of (S'_ε), thus they correspond to the only critical points $(0, 0, 0, 0, 0)$ and $(0, 0, 0, 1, 0)$ of (S_ε), hence $(0, 0, 0)$ and $(0, 0, 1)$ are the critical points of (S'_ε). Observe that there exists a 2-dimensional unstable manifold coming out of $(0, 0, 0)$ for system (S_0), which at $(0, 0, 0)$ is generated by the eigenvectors $\{(1, \mu^-_{0,1}, 0), (0, 0, 1)\}$. Next we see how this manifold perturbs to an unstable manifold of system (S_ε).

First we observe that the linearisation of (S_ε) around $(0, 0, 0, 0, 0)$ gives the 2-block matrix

$$\begin{pmatrix} A^-(\lambda, \varepsilon) & \\ & B(\varepsilon) \end{pmatrix} \quad \text{with} \quad B(\varepsilon) = \begin{pmatrix} 0 & 1 \\ 1/\varepsilon^2 & -1/\varepsilon^2 \end{pmatrix},$$

$B(\varepsilon)$ results from linearisation of the travelling wave system around $(0, 0)$, and $A^-(\lambda, \varepsilon) = \lim_{\eta \to -\infty} A(\eta, \lambda, \varepsilon)$. We distinguish between fast and slow eigenvalues. The fast eigenvalues are those which in the fast scaling approach -1 as $\varepsilon \to 0$, i.e. they approach the eigenvalues of (F_0) which have eigenvectors transversal to M_0.

The matrix of the linearised system around $(0, 0, 0)$ of (S'_ε) is

$$\begin{pmatrix} 0 & 1 & 0 \\ \Phi_u(0, 0, 0, \varepsilon, \lambda) & \Phi_v(0, 0, 0, \varepsilon, \lambda) & \Phi_\phi(0, 0, 0, \varepsilon, \lambda) \\ 0 & 0 & \Psi_\phi(0, \varepsilon) \end{pmatrix}. \tag{4.25}$$

Since M_ε is locally invariant under the slow flow (S_ε), the eigenvalues of (4.25) must be eigenvalues of the matrix resulting from linearisation around $(0,0,0,0,0)$ in (S_ε). In particular the eigenvalue $\Psi_\phi(0,\varepsilon)$ is the positive eigenvalue of the travelling wave problem at $(0,0)$, i.e. $\omega_1 := \frac{-1+\sqrt{1+4\varepsilon^2}}{2\varepsilon^e} = 1-\varepsilon^2+O(\varepsilon^4)$ as $\varepsilon \to 0$. The negative eigenvalue of $B(\varepsilon)$ is a fast eigenvalue since $\omega_2 = \frac{-1+\sqrt{1+4\varepsilon^2}}{2\varepsilon^e} = \frac{-1}{\varepsilon^2} - 1 + \varepsilon^2 + O(\varepsilon^4)$ as $\varepsilon \to 0$.

Also the eigenvalues given by the 2×2 minor on the top left of (4.25) must correspond to eigenvalues of $A^-(\lambda,\varepsilon)$. These are the *slow* eigenvalues $\mu^-_{1,\varepsilon}$ and $\mu^-_{2,\varepsilon}$, since by Lemma 4.8, the remaining is a fast eigenvalue.

By the stable manifold theorem and continuation of solutions in parameters, there exist a 2-dimensional unstable manifold $W^U_\varepsilon(0)$ of (S'_ε) at $(0,0,0)$, which is a C^k perturbation of the unstable manifold at $(0,0,0)$ of (S_0). Locally the manifold $W^U_\varepsilon(0)$ is generated by the eigenvectors $\{(1,\mu^-_{\varepsilon,1},0),(0,0,1)\}$. Hence solutions of S_0 on the orbits of $W^U_0(0)$ behave as a linear combination of $\exp(\mu^-_{0,1}\eta)(1,\mu^-_{0,1},0)$ and $\exp(\eta)(0,0,1)$ as $\eta \to -\infty$. And solutions of S_ε on the orbits of $W^U_\varepsilon(0)$ behave as a linear combination of $\exp(\mu^-_{\varepsilon,1})(1,\mu^-_{\varepsilon,1},0)$ and $\exp(\omega_1\eta)(0,0,\omega_1)$ as $\eta \to -\infty$. Then $(u_\varepsilon(\eta),v_\varepsilon(\eta),\phi_\varepsilon(\eta)) \to (u(\eta),v(\eta),\phi(\eta))$ as $\varepsilon \to 0$ uniformly on $(-\infty,\eta_0)$. Since ϕ and ϕ_ε are travelling wave solutions, and the solutions belong to the unstable manifolds we get $(\zeta_\varepsilon,\zeta'_\varepsilon) \to (\zeta,\zeta')$ on $(-\infty,\eta_0)$.

The argument for the adjoint problem goes similarly. But first we have to introduce $\bar{w} = \frac{z_\xi}{\varepsilon^2}$, and hence study the system

$$\begin{cases} u' = -(\lambda - 2\psi)\bar{w} \\ v' = -u - (1-2\phi)\bar{w} \\ \varepsilon^2\bar{w}' = -v + (1+\varepsilon^2\lambda)\bar{w} \\ \phi' = \psi \\ \varepsilon^2\psi = \phi - \phi^2 - \psi. \end{cases}$$

Similar arguments apply here, the limit being $(u_\varepsilon,\bar{w}_\varepsilon,\phi_\varepsilon)$ to (u,v,ϕ) as $\varepsilon \to 0$, with $\bar{w}_\varepsilon \sim \exp(-\mu^+_{\varepsilon,1}\eta)$ and $v \sim \exp(-\mu^+_{0,1}\eta)$ as $\eta \to \infty$, where (u,v,ϕ) solve the system resulting from setting $\varepsilon = 0$ in the above sets of equations. □

4.4 Behaviour of $D(\lambda)$ as $|\lambda| \to \infty$

In this section we prove that $D(\lambda) \to 1$ as $|\lambda| \to \infty$ for $\lambda \in \Omega$. We first consider the limit for $\varepsilon > 0$ fixed.

We follow the approach by Pego and Weinstein in [53] and [54], and repeat the arguments for completeness. The proof is completed by verifying the sufficient conditions in our particular case. The second case turns out to be easier, the idea, explained below, appears for instance, in [75].

In order to prove the limit we shall write the Evans function as $D(\lambda) = Z^+V^-W^-Y^-$, where V^- is the matrix of right eigenvectors of A^-, and W^- the matrix of left eigenvectors of A^- such that $W^-V^- = 1$, i.e.

$$V^- = (v^-_1, v^-_2, v^-_3),$$

and

$$W^- = (w_1^-, w_2^-, w_3^-)^t.$$

Similarly, V^+ and W^+ will be the corresponding eigenvector matrices of A^+. We will prove the limit in Proposition 4.12. Namely, we prove that $W^-(\lambda)Y^-(\lambda, 0) \sim e_1$ as $|\lambda| \to \infty$, and that $Z^+(\lambda, 0)V^+(\lambda) \sim e_1^t$ as $|\lambda| \to \infty$, where $e_1 = (1, 0, 0)^t$. This requires a transformation of the system (4.14), and estimates on the matrix function

$$F(\eta, \lambda) = \begin{cases} F^+(\eta, \lambda) := W^+(\lambda)\left(A(\eta, \lambda) - A^+(\lambda)\right)V^+(\lambda) & \text{if } \eta > 0 \\ F^-(\eta, \lambda) := W^-(\lambda)\left(A(\eta, \lambda) - A^-(\lambda)\right)V^-(\lambda) & \text{if } \eta < 0. \end{cases} \tag{4.26}$$

Observe that if $X = W^-Y$, then $X' = F^-X + \operatorname{diag}(\mu_1^-, \mu_2^-, \mu_3^-)X$. Then if $F \to 0$ as $|\lambda| \to \infty$, $X^- = W^-Y^- \sim \exp(\mu_1^-\eta)e_1$ for all $\eta \in \mathbb{R}$, as $|\lambda| \to \infty$. Similarly, if $X = V^+Z$ then $X' = -ZF^- - \operatorname{diag}(\mu_1^+, \mu_2^+, \mu_3^+)X$, so that if $F \to 0$ as $|\lambda| \to \infty$ then $X^+ = V^+Z^+ \sim \exp(-\mu_1^+\eta)e_1^t$. If in addition $\mu_1^+ \sim \mu_1^-$ as $|\lambda| \to \infty$ then $D(\lambda) \to 1$ as $|\lambda| \to \infty$. Unfortunately if $\varepsilon > 0$, $F \to 0$ as $|\lambda| \to \infty$ does not hold, and the estimates presented in Lemma 4.11 below are needed to conclude the limit.

We start with a lemma that accounts for the behaviour of eigenvalues of the matrices $A^\pm$ as $|\lambda| \to \infty$.

Lemma 4.10. *Let $\mu_i^\pm$, $i = 1, 2, 3$ be the roots of the polynomials $P^\pm$ for $\lambda \in \Omega$, then if $\varepsilon > 0$ is fixed*

$$\mu_i^- \sim \mu_i^+ \quad \text{as } |\lambda| \to \infty \quad \text{for } i = 1, 2, 3.$$

Moreover

$$\begin{aligned} \mu_1^\pm &= \frac{1}{\varepsilon} + O(|\lambda|^{-1}) \quad \text{as } |\lambda| \to \infty, \\ \mu_2^\pm &= -\frac{1}{\varepsilon} + O(|\lambda|^{-1}) \quad \text{as } |\lambda| \to \infty, \\ \mu_3^\pm &= -\lambda - \frac{1}{\varepsilon^2} + O(|\lambda|^{-1}) \quad \text{as } |\lambda| \to \infty. \end{aligned}$$

Proof. Let $\delta = |\lambda|^{-1}$, and $\theta = \arg\lambda$ so that $\lambda = \frac{e^{i\theta}}{\delta}$. Reasoning as in Lemma 4.8, we get the asymptotic expansions

$$\mu_1^\pm = \frac{1}{\varepsilon} + \left(\mp 1 - 1 - \frac{1}{\varepsilon}\right)\frac{\delta}{2e^{i\theta}\varepsilon^2} + O(\delta^2) \quad \text{as } \delta \to 0$$

and

$$\mu_2^\pm = -\frac{1}{\varepsilon} + \left(\mp 1 + 1 - \frac{1}{\varepsilon}\right)\frac{\delta}{2e^{i\theta}\varepsilon^2} + O(\delta^2) \quad \text{as } \delta \to 0.$$

In order to capture the third eigenvalue, the one with least real part, we scale the polynomials $P^\pm$ by setting $\tilde{\mu} = \delta^\alpha \mu$, with $\alpha > 0$, then

$$\delta^{3\alpha} P^\pm\left(\frac{\tilde{\mu}}{\delta^\alpha}\right) = \varepsilon^2\tilde{\mu}^3 + \left(1 + \varepsilon^2\frac{e^{i\theta}}{\delta}\right)\tilde{\mu}^2\,\delta^\alpha \pm \tilde{\mu}\,\delta^{2\alpha} - e^{i\theta}\,\delta^{3\alpha-1}. \tag{4.27}$$

For ε fixed, the dominating terms are the third order term and the second order term, thus we take $\alpha = 1$. this gives

$$\tilde{\mu} = -e^{i\theta} - \frac{1}{\varepsilon^2}\delta + O(\delta^2), \quad \text{hence} \quad \mu_3^{\pm} = -\lambda - \frac{1}{\varepsilon^2} + O(\delta).$$

□

The following lemma states the crucial estimates on F.

Lemma 4.11. *F satisfies the following properties*

(i) If e_1 denotes the vector $(1,0,0)^t$, then

$$\int_{-\infty}^{\infty} |F(s,\lambda)e_1| ds \to 0 \text{ as } |\lambda| \to \infty.$$

(ii) There exists $C > 0$ independent of λ such that

$$\int_{-\infty}^{\infty} |F(s,\lambda)| ds \leq C \quad \text{as } |\lambda| \to \infty.$$

(iii)

$$\int_{|s|>\eta_0} |F(\lambda,s)|\, ds \to 0 \quad \text{as} \quad \eta_0 \to -\infty \quad \text{uniformly in} \quad \lambda \in \Omega.$$

Proof. Let us write

$$F^{\pm} = W^{\pm}(A - A^{\pm})V^{\pm} = \begin{pmatrix} r_{11}^{\pm} & r_{12}^{\pm} & r_{13}^{\pm} \\ r_{21}^{\pm} & r_{22}^{\pm} & r_{23}^{\pm} \\ r_{31}^{\pm} & r_{32}^{\pm} & r_{33}^{\pm} \end{pmatrix} \quad \text{where}$$

$$r_{ij}^{\pm} := \frac{2\phi'}{P_{\mu}^{\pm}(\mu_i^{\pm})}\varepsilon^2 + \mu_j^{\pm}\frac{2\phi - 1 \mp 1}{P_{\mu}^{\pm}(\mu_i^{\pm})}\varepsilon^2. \tag{4.28}$$

Then it is enough to prove that

$$\int_{-\infty}^{0} |r_{1j}^{-}(\lambda,s)|\, ds + \int_{0}^{\infty} |r_{1j}^{+}(\lambda,s)|\, ds \to 0 \quad \text{as } |\lambda| \to \infty \text{ for } j = 1,2,3 \tag{4.29}$$

$$\int_{-\infty}^{0} |r_{ij}(s)|\, ds + \int_{0}^{\infty} |r_{ij}(s)|\, ds < C \quad \text{uniformly in } \lambda \text{ for } i,j = 1,2,3 \tag{4.30}$$

$$\int_{-\infty}^{-\eta_0} |r_{ij}(s)|\, ds + \int_{\eta_0}^{\infty} |r_{ij}(s)|\, ds \to 0 \quad \text{as } \eta_0 \to \infty \text{ uniformly in } \lambda, \quad \text{for } i,j = 1,2,3. \tag{4.31}$$

Next we prove the necessary ingredients to get these estimates. Observe that since ϕ tends to zero exponentially as $\eta \to -\infty$, and $\phi - 1$ tends to zero exponentially as $\eta \to \infty$, then

$$\int_{0}^{\infty} |\phi'(s)|\, ds < \infty, \quad \int_{-\infty}^{0} |\phi'(s)|\, ds < \infty, \tag{4.32}$$

also

$$\int_0^\infty |\phi(s) - 1|\, ds < \infty, \quad \int_{-\infty}^0 |\phi(s)|\, ds < \infty. \tag{4.33}$$

From this, it remains to show that the constant coefficients of the elements $r_i^\pm$ of F in (4.28) do not spoil the limits (4.29)-(4.31). Using Lemma 4.10, it is not difficult to prove that

$$\frac{1}{P_\mu(\mu_i^\pm)} \to 0 \quad \text{as } |\lambda| \to \infty \text{ for } i = 1, 2, 3,$$

also

$$\frac{\mu_j^\pm}{P_\mu(\mu_i^\pm)} \to 0 \quad \text{as} \quad |\lambda| \to \infty \quad \text{for} \quad j = 1, 2,\ i = 1, 2, 3,$$

and

$$\frac{\mu_3^\pm}{P_\mu(\mu_3^\pm)} \to 0 \quad \text{as} \quad |\lambda| \to \infty, \quad \frac{\mu_3^\pm}{P_\mu(\mu_i^\pm)} \to c \quad \text{as} \quad |\lambda| \to \infty \quad \text{for } i = 1, 2.$$

These estimates together with (4.33) and (4.32) imply (4.29)-(4.31). □

With these two lemmas the proof of the limit for $\varepsilon > 0$ fixed is now analogous to the proofs of the limits in [54] and [53]. We have the following proposition.

Proposition 4.12. *Let $\varepsilon > 0$ and $\lambda \in \Omega$, then the Evans function, defined by (4.18) satisfies*

$$D(\lambda) \to 1 \quad \textit{as} \quad |\lambda| \to \infty.$$

Proof. Let us write $D(\lambda)$ as $D(\lambda) = Z^+(\lambda, 0)V^-(\lambda)W^-(\lambda)Y^-(\lambda, 0)$. Next we prove that $W^-(\lambda)Y^-(\lambda, 0) \sim e_1$ as $|\lambda| \to \infty$.

Let $v = -e_1 + e^{-\mu_1 \eta} W^- Y^-$ and $v(-\infty) = 0$, then v satisfies the equation

$$\frac{d}{d\eta} v = F^-(v + e_1) + W^-(A^- V^- - \mu_1^- I)v + W^-(A^- V^- - \mu_1^- I)e_1.$$

Since $(A^- V^- - \mu_1^- I)e_1 = 0$, using the variation of constants formula we get for any $\eta \le 0$,

$$v(\eta) = \int_{-\infty}^{\eta} \exp B(\lambda)(\eta - s) F^-(\lambda, s)(v(s) + e_1)\, ds$$

where $B(\lambda) = W^-(A^- V^- - \mu_1^- I)$. Observe that $B(\lambda)$ is a diagonal matrix, which entries are all non-positive, hence

$$|v(\eta)| \le \sup_{s \le \eta} |v(s)| \int_{-\infty}^{\eta} |F^-(\lambda, s)|\, ds + \int_{-\infty}^{\eta} |F^-(\lambda, s)|\, ds.$$

Then Lemma 4.11 *(iii)* implies that there exists η_0 sufficiently small such that $\int_{-\infty}^{\eta_0} |F^-(\lambda, s)|\, ds < \frac{1}{2}$ for all $\lambda \in \Omega$, then

$$\sup_{s \le \eta_0} |v(s)| \le \int_{-\infty}^{\eta_0} |F(\lambda, s)e_1|\, ds \to 0 \quad \text{as} \quad |\lambda| \to \infty. \tag{4.34}$$

Using again the variations of constants formula we get at η such that $\eta_0 < \eta \leq 0$

$$v(\eta) = v(\eta_0) \exp B(\lambda)(\eta - \eta_0) + \int_{\eta_0}^{\eta} \exp\left(B(\lambda)(\eta - s)\right) F^-(\lambda, s)(v(s) + e_1)\, ds,$$

hence

$$|v(\eta)| \leq C|v(\eta_0)| + C\int_{\eta_0}^{\eta} |F^-(\lambda, s)e_1|\, ds + C\int_{\eta_0}^{\eta} |F^-(\lambda, s)||v(s)|\, ds,$$

where $C = |\exp(-B(\lambda)\eta_0)|$. By Gronwall's lemma, Lemma 4.11 *(i)* and (4.34), we get

$$|v(\eta)| \leq C \exp\left(C\int_{\eta_0}^{\eta} |F^-(\lambda, s)|\, |v(s)|\, ds\right)\left(|v(\eta_0)| + \int_{\eta_0}^{\eta} |F^-(\lambda, s)e_1|\, ds\right) = o(1)$$
$$\text{as} \quad |\lambda| \to \infty.$$

This proves $W^-Y^-(\lambda, 0) \sim e_1$ as $|\lambda| \to \infty$. Similarly one proves that $Z^+V^+ \sim e_1^t$ as $|\lambda| \to \infty$. Lemma 4.10 implies $V^- \sim V^+$ as $|\lambda| \to \infty$, and hence $Z^+(\lambda, 0)V^- \sim e_1^t$ as $|\lambda| \to \infty$. □

To prove that $D_\varepsilon(\lambda)$ is continuous in $\varepsilon = 0$, one follows the same steps as for the case $\varepsilon > 0$, but taking the limit $\varepsilon \to 0$, so that the conditions on F as in Lemma 4.11 are not spoiled in the limit. Checking this is arduous and we shall not write it here. We state the result in the following proposition.

Proposition 4.13. *$D_\varepsilon(\lambda)$ is continuous in $\varepsilon \geq 0$ for all $\lambda \in \Omega \cup \infty$.*

4.5 Small ε

In this section we prove that there are no eigenvalues of L with positive real part if $\varepsilon^2 \leq \frac{1}{4}$.

Proposition 4.14. *If $\varepsilon^2 \leq \frac{1}{4}$, then D_ε does not have zeros in the right half plane.*

Proof. First we assume that λ is an eigenvalue such that $Re(\lambda) > 0$, and that ζ is the corresponding eigenfunction. Hence by integrating the characteristic equation (4.13) we get

$$\lambda \int_{-\infty}^{\infty} \zeta\, d\eta = 0.$$

Let $\xi := \int_{-\infty}^{\eta} \zeta$, then ξ and its derivatives decay to zero as $\eta \to \pm\infty$. Also ξ satisfies the equation

$$\lambda\,(I - \varepsilon^2 \frac{d^2}{d\eta^2})\,\xi = (\frac{d}{d\eta} + (2\phi - 1) + \varepsilon^2 \frac{d^2}{d\eta^2})\xi_\eta. \tag{4.35}$$

Now following the hypothesis $\mathrm{Re}(\lambda) > 0$ we have

$$0 < \mathrm{Re}(\lambda) \int_{-\infty}^{\infty} (I - \varepsilon^2 \frac{d^2}{d\eta^2})\,\xi\,(I - \varepsilon^2 \frac{d^2}{d\eta^2})\,\xi^* =$$

$$\mathrm{Re}\left(\int_{-\infty}^{\infty} (\frac{d}{d\eta} + (2\phi - 1) + \varepsilon^2 \frac{d^2}{d\eta^2})\,\xi_\eta\,(I - \varepsilon^2 \frac{d^2}{d\eta^2})\,\xi^*\right) \tag{4.36}$$

where * denotes complex conjugate. Then

$$0 < \mathrm{Re}\Big(\int_{-\infty}^{\infty} \big(\xi_{\eta\eta}\xi^* + (2\phi - 1)\xi_\eta\xi^* + \varepsilon^2(\xi_{\eta\eta\eta}\xi^* - \xi_{\eta\eta}\xi^*_\eta - (2\phi - 1)\xi_\eta\xi^*_{\eta\eta} \\ -\varepsilon^2\xi_{\eta\eta\eta}\xi^*_{\eta\eta}\big)\, d\eta\Big) = -\int_{-\infty}^{+\infty} \big(|\xi_\eta|^2(1 - \varepsilon^2\phi') + \phi'|\xi|^2 + \varepsilon^2|\xi_{\eta\eta}|^2\big)\, d\eta. \quad (4.37)$$

We know from Chapter 2 Section 2.3 that if $\varepsilon^2 < \frac{1}{4}$ then $\phi' > 0$ and from (2.27) we get $(1-\varepsilon^2\phi') > 0$, which contradicts (4.37). Hence $\mathrm{Re}(\lambda) \leq 0$. Now if $\mathrm{Re}(\lambda) = 0$ but $\lambda \neq 0$ following the same steps as before we end up with

$$0 = -\int_{-\infty}^{+\infty} \big(|\xi_\eta|^2(1 - \varepsilon^2\phi') + \phi'|\xi|^2 + \varepsilon^2|\xi_{\eta\eta}|^2\big)\, d\eta,$$

and hence $\xi \equiv 0$. Notice that this does not give a contradiction for $\lambda = 0$.

□

Remark 4.15. *We can argue similarly if λ is an eigenvalue with real part smaller than $-\frac{1}{\varepsilon^2}$, and $\varepsilon^2 < \frac{1}{4}$. If we assume that λ is an eigenvalue, we call again by ζ its corresponding eigenfunction, and take $\xi := \int_{-\infty}^{x} \zeta$, we get*

$$0 > (\mathrm{Re}(\lambda) + \frac{1}{\varepsilon^2}) \int_{-\infty}^{+\infty} (I - \varepsilon^2\partial_{\eta\eta})\xi\, (I - \varepsilon^2\partial_{\eta\eta})\xi^* = \\ -\int_{-\infty}^{+\infty} \big(|\xi_\eta|^2(1 - \varepsilon^2\phi') + \phi'|\xi|^2 + \varepsilon^2|\xi_{\eta\eta}|^2\big)\, d\eta + \\ \frac{1}{\varepsilon^2} \int_{-\infty}^{+\infty} (I - \varepsilon^2\partial_{\eta\eta})\xi\, (I - \varepsilon^2\partial_{\eta\eta})\xi^* = \\ \int_{-\infty}^{\infty} (1 + \varepsilon^2\phi')|\xi_\eta|^2 + \int_{-\infty}^{+\infty} (\frac{1}{\varepsilon^2} - \phi')|\xi|^2 \geq 0,$$

a contradiction.

4.6 Large ε: numerical search for zeros

So far we have seen how the analysis for ε small confirms the stability results of Chapter 2. The aim of this section is to provide numerical evidence of linear stability of travelling waves for large values of ε. Before we present the numerical results we state the proposition which tells that zeros of D_ε can only emerge in pairs through the imaginary axis as ε increases.

Proposition 4.16. *Suppose that there exists $\varepsilon^* > \frac{1}{2}$ such that D_{ε^*} has a zero at some λ^* with positive real part. Then there exists $\bar{\varepsilon} \in (\frac{1}{2}, \varepsilon^*]$ such that $D_{\bar{\varepsilon}}$ has a pair of zeros in the imaginary axis aside from the origin.*

Proof. It is a consequence of the continuity of D_ε in ε, Proposition 4.12 and Rouche's theorem. Rouche's theorem says that if two analytic function are close to

each other in a simply connected domain, then they will have the same number of zeros (counting multiplicity) in that domain; since for $\varepsilon \leq \frac{1}{2}$ there are no eigenvalues with positive real part in the right hand plane, the only way zeros can enter this domain as ε increases is if they pop up from ∞ or cross the imaginary axis from the left half plane.The first possibility contradicts Proposition 4.12. Thus for $D_{\varepsilon^*}(\lambda)$ to have a zero in the right half plane, there must exist $\bar{\varepsilon} < \varepsilon^*$ and $\bar{\lambda}$ with $\mathrm{Re}(\bar{\lambda}) = 0$ such that $D_{\bar{\varepsilon}}(\bar{\lambda}) = 0$. $\bar{\lambda} \neq 0$ since by Lemma 4.7, $D'_{\varepsilon}(0) \neq 0$ for all ε. Then $\bar{\lambda} = \tau i$ for some $\tau \in \mathbb{R}\backslash\{0\}$, and Lemma 4.3-*(iii)* implies that $D_{\bar{\varepsilon}}(-\tau i) = 0$ as well. □

We compute numerically the Evans function along the imaginary axis. Since by Proposition 4.12, zeros of $D_{\varepsilon}(\lambda)$ only enter the right half plane through the imaginary axis, we take the imaginary axis as a wide contour around the right half plane. We look at the number of times that the graph of the curve $\tau \in \mathbb{R} \to D(\tau i)$ wraps around the origin (Winding number), this gives the number of zeros of $D(\lambda)$ in the right half plane, and hence, the number of isolated eigenvalues of the operator L.

In Figure 4.3 we have plotted the graphs of D_{ε} along the imaginary axis, for several values of ε^2, including $\varepsilon = 0$. As a guideline to interpret the results notice that if a new zero of D appears through the imaginary axis, the curve $D(\tau i)$ must intersect itself at the origin (since $D = 0$ for $\tau = 0$, for some $\tau = \tilde{\tau}$ and $-\tilde{\tau}$). However the numerical results do not exhibit this self-intersections. And the evolution of the graphs with respect to ε suggest that this is not going to happen at very large values of ε.

We have approximated the Evans function at each λ-value by first transforming Y^- by

$$\mathcal{V}^-(\eta, \lambda) = \exp(-\mu_1(\eta, \lambda)\eta) Y^-(\eta, \lambda),$$

where

$$\mu_1(\eta, \lambda) = \begin{cases} \mu_1^-(\lambda) & \text{if } \eta < \eta_m \\ \mu_1^+(\lambda) & \text{if } \eta \geq \eta_m, \end{cases}$$

for some $\eta_m \in \mathbb{R}$. Now $\mathcal{V}^-$ satisfies

$$\begin{aligned} \mathcal{V}^-(\eta, \lambda) &\sim v_-(\lambda) \quad \text{as } \eta \to -\infty \\ \mathcal{V}^-(\eta, \lambda) &\sim v_+(\lambda) D(\lambda) \quad \text{as } \eta \to +\infty, \end{aligned}$$

and the equation

$$\frac{d\mathcal{V}^-(\eta, \lambda)}{d\eta} = -\mu_1(\eta, \lambda)\mathcal{V}^-(\eta, \lambda) + A(\lambda, \eta)\mathcal{V}^-(\eta, \lambda). \tag{4.38}$$

We approximate numerically this equation on a finite interval $[\eta_0, \eta_f]$. First on the interval $[\eta_0, \eta_m]$ we solve the equation

$$\frac{d\mathcal{V}^-}{d\eta} = -\mu_1^-(\lambda)\mathcal{V}^- + A(\lambda, \eta)\mathcal{V}^-, \tag{4.39}$$

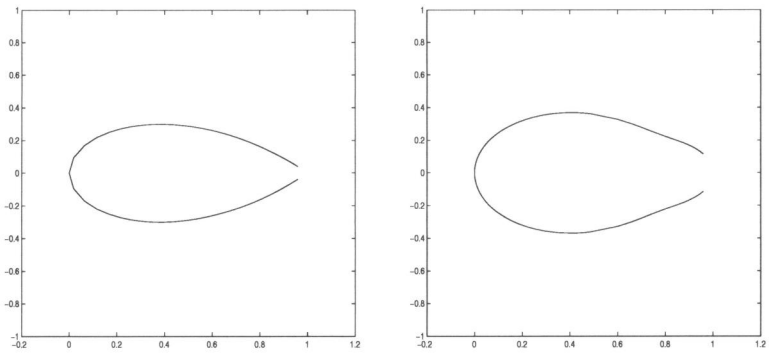
for $\eta_0 < 0$ sufficiently small, with initial condition the eigenvector $(1, \mu_1^-, (\mu_1^-)^2)$. On the interval $[\eta_m, \eta_f]$ we solve

$$\frac{d\mathcal{V}^-}{d\eta} = -\mu_1^+(\lambda)\mathcal{V}^- + A(\lambda, \eta)\mathcal{V}^-,$$

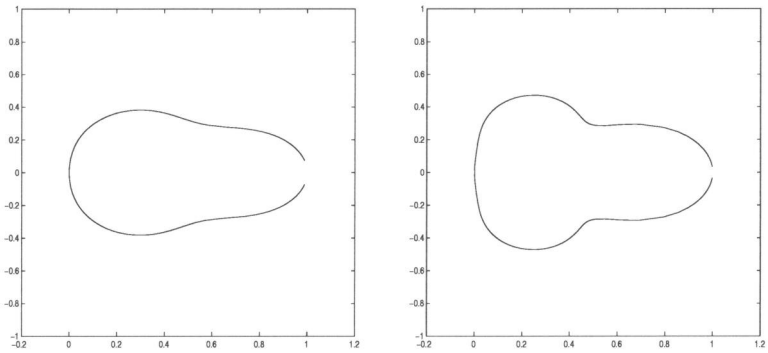
for $\eta_f > 0$ sufficiently large. The initial condition at η_m is taken to be the value of $\mathcal{V}^-$ at η_m obtained after solving (4.38). We capture the value of $\mathcal{V}^-$ at η_f and use it to approximate $D(\lambda)$. We then take

$$D(\lambda) = w^+ \mathcal{V}^-(\eta_f).$$

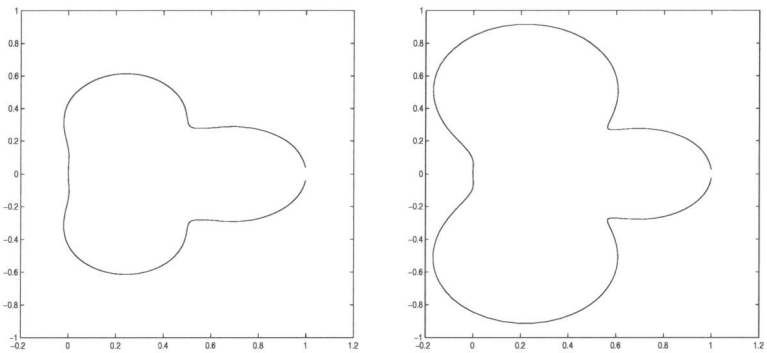
In practice this computation is repeated at each value $\lambda = \tau i$, where $\tau \in [-100, 100]$. We have only discretised the interval $[0, 100]$, since the symmetry of D gives $D(-\tau i)$ as $\overline{D(\tau i)}$. The partition of the τ-interval is not uniform. We use a finer grid near 0. From 0 to 1 we take 0.001 as τ-step size, and 0.5 for the rest of the interval. At each τ-step the systems (4.38) and (4.39) are solved simultaneously with the travelling wave equation (2.23). Here we have taken $\eta_0 = -50$, $\eta_m = 0$ and $\eta_f = 500$. We have used the Runge-Kutta solver implemented for matlab.



(a) $\varepsilon^2 = 0$ (b) $\varepsilon^2 = 0.2$



(c) $\varepsilon^2 = 1$ (d) $\varepsilon^2 = 5$



(e) $\varepsilon^2 = 10$ (f) $\varepsilon^2 = 20$

Figure 4.3: $\{D_\varepsilon(\tau i) : \tau \in \mathbb{R}\} \subset \mathbb{C}$ for different values of ε^2.

Chapter 5

Evans function for Burgers' equation

Preamble:
This chapter is devoted to the explicit computation of the Evans function of the diffusive generalised Burgers' equation, with convection term of the form $(u^p)_x$ for $p > 1$. We analyse the domain of analyticity of the Evans function. It turns out that for $p = 2$ the domain is a 2-sheeted Riemann surface, the cut given by the essential spectrum of the linearised operator. In this case, the Evans function has only one zero, $\lambda = 0$, which is simple. If $p \neq 2$ the Evans function can be extended through the essential spectrum, which has two branch points, hence it is a four valued function. In the first sheet $\lambda = 0$ is the only zero and is simple, here zeros of the Evans function correspond to eigenvalues of the linearised operator. However it has infinitely many poles and zeros in other of the sheets. Here zeros do not correspond to eigenvalues.

5.1 Introduction

In this section we construct the Evans function associated with the linear stability analysis of travelling wave solutions of the diffusive Burgers' equation

$$u_t = u_{xx} + (u^p)_x \quad \text{on } \mathbb{R} \times \mathbb{R}^+, \quad \text{with} \quad p > 1, \tag{5.1}$$

subject to bounded initial data satisfying

$$u(-\infty, 0) = u^- < u(+\infty, 0) = u^+. \tag{5.2}$$

Existence and uniqueness of equation (5.1) can be found in [49]. Equation (5.1) admits travelling wave solutions, i.e. solutions of the form $u(x,t) = \phi(\eta)$ with $\eta := x + ct$, connecting u^- at $\eta = -\infty$ to u^+ at $\eta = +\infty$, with wave speed given by $\frac{(u^+)^p - (u^-)^p}{u^+ - u^-}$. It is well-known that such solutions are (orbitally) stable, see for instance [39], [55], [57]. Our computation is however motivated by the following:

(i) We can discern from the explicit formula of the Evans function its domain of analyticity. In particular this confirms the gap lemma, see [28]. In fact

in this case the Evans function is analytic in a wider domain than just a small neighbourhood around the branch points of the essential spectrum. See Appendix B.3 for a version of the gap lemma in the two-dimensional case.

(ii) Equation (5.1) might appear as the reduced limit of a higher order equation, or of a system of equations. It is useful to know where the zeros of the Evans function of the reduced problem are, since they may turn into eigenvalues of the perturbed problem.

Our analysis is, in particular, motivated by the work of A. Doelman, R. Gardner and T. Kaper in [21]. In their paper the linear stability of N-pulses is studied for a coupled system of parabolic semilinear equations. This system is a singular perturbation of a single semilinear parabolic equation. In this way the spectral picture of the reduced problem allows the spectral set of the linearised operator of the perturbed problem to be analysed. The reduced eigenvalue problem of the reduced problem reads

$$\Psi'' + (p\phi^{p-1} - (1+\lambda))\Psi = 0.$$

This equation can be transformed into a hypergeometric equation, from which the Evans function can be computed explicitly, and a formula for the zeros, which are eigenvalues is given. It is this method of solving the eigenvalue equation that suggested to us the possibility of solving similarly the eigenvalue problem associated with equation (5.1).

This chapter is organised as follows. In Section 5.2 we give a framework for the problem of linearised stability. We comment briefly on travelling wave solutions and set up the problem of linearised stability in a weighted space, in which the linearised operator is self-adjoint. The essential spectrum is real and lies in the left half plane away from 0. Next we define the Evans function related to the eigenvalue problem in the weighted space, which up to a multiplying factor coincides with the Evans function of the operator in the original space. Finally in Section 5.3 we give an explicit formula for the Evans function. The formula allows one to prove that for $p > 1$ there are no eigenvalues away from the essential spectrum, except $\lambda = 0$ which is simple. However, we find zeros of the Evans function that do not correspond to eigenvalues. This, in particular, confirms that the Evans function is analytic on a 2-sheeted Riemann surface. Namely, we find that the Riemann surface consists of two complex planes cut and pasted through the essential spectrum, up to a countable number of real poles on the second complex plane. If $p = 2$ there are no poles or zeros in the second plane. If $p \neq 2$ we find infinitely many zeros and poles of the Evans function in the other planes, all of them real.

5.2 Travelling waves and linearised equation

In this section we derive the eigenvalue problem associated with the linear stability analysis of travelling wave solutions of equation (5.1). We set $u^- = 0$ and $u^+ = 1$ in (5.2) for simplicity. Let $\phi(\eta)$ with $\eta = x + ct$ be a travelling wave solution to (5.1). After substitution of $u(x,t) = \phi(x+ct)$ into (5.1) and integration we get that $\phi(\eta)$

must satisfy

$$c\phi(\eta) = \phi'(\eta) + \phi(\eta)^p \tag{5.3}$$

and the Rankine-Hugoniot condition gives $c = 1$. Equation (5.3) is of Bernoulli type; its family of solutions can be computed explicitly and reads

$$\phi(\eta) = \left(\frac{1}{1+\exp(-(p-1)\eta + k)}\right)^{\frac{1}{p-1}} \quad \text{for } k \in \mathbb{R}.$$

Observe that $\phi(\eta) \sim \exp(\eta)$ as $\eta \to -\infty$ and $\phi(\eta) - 1 \sim \exp(-\eta)$ as $\eta \to +\infty$.

Linearisation of equation (5.1) around a travelling wave solution gives, in the moving coordinate $\eta = x + t$, the equation

$$z_t = z_{\eta\eta} + pz(\phi^{p-1})_\eta + pz_\eta(\phi^{p-1}) - z_\eta =: Lz,$$

where $z = u - \phi$.

The eigenvalue problem for L consists of finding the values of $\lambda \in \mathbb{C}$ for which there are nontrivial solutions of

$$\zeta'' + (p\phi^{p-1} - 1)\zeta' + (p\phi^{p-1})'\zeta - \lambda\zeta = 0. \tag{5.4}$$

Observe that the essential spectrum of L is contained in the parabolic region $\mathcal{E} := \{\lambda \in \mathbb{C} : \ \mathrm{Im}(\lambda)^2 + \nu \mathrm{Re}(\lambda) \leq 0\}$ for $\nu = \max(1, (p-1)^2)$, see [34]. Also, if λ is an eigenvalue away from the essential spectrum its eigenfunctions ζ must behave as $\exp(\frac{-1+\sqrt{1+\lambda}}{2}\eta)$ when $\eta \to -\infty$, and as $\exp(\frac{-(p-1)-\sqrt{(p-1)^2+\lambda}}{2}\eta)$ when $\eta \to +\infty$.

Next we transform the operator L to a self-adjoint linear operator by introducing a weight function w. Let ψ be defined by $\zeta = w\psi$, where w satisfies $w' = aw$ for some function a. Then the eigenvalue problem in the weighted unknown ψ reads

$$\psi'' + (2a + p\phi^{p-1} - 1)\psi' + ((a^2 + a') + a(p\phi^{p-1} - 1) + p(p-1)\phi^{p-2}\phi' - \lambda)\psi = 0.$$

The choice $a = -\frac{p\phi^{p-1}-1}{2}$ in the above equation, gives $w = \phi(\eta)^{\frac{p}{2}} \exp(\frac{1-p}{2}\eta)$ and the eigenvalue problem

$$(\tilde{L} - \lambda I)\psi = \psi'' - \left(\frac{1}{4}(p\phi^{p-1} - 1)^2 - \frac{1}{2}p(p-1)\phi^{p-2}\phi' - \lambda\right)\psi = 0. \tag{5.5}$$

The linearised operator in the weighted space $\tilde{L}$ is self-adjoint, hence its spectrum is real. The limit equations of (5.5) at $\eta = +\infty$ and $\eta = -\infty$ give the following characteristic polynomials

$$P^-(\mu) = \mu^2 - \frac{1}{4} - \lambda \quad \text{at } -\infty \quad \text{and} \quad P^+(\mu) = \mu^2 - \frac{1}{4}(p-1)^2 - \lambda \quad \text{at } +\infty,$$

respectively. The essential spectrum of $\tilde{L}$ is then the set

$$\tilde{\mathcal{E}} := \{\lambda \in \mathbb{R} : \lambda \leq \max(-\frac{1}{4}, -\frac{(p-1)^2}{4})\},$$

see [34].

Let us consider the roots of $P^{\pm}$ as 2-valued functions. We have

$$\lambda \to \mu^-(\lambda) = \frac{1}{2}\sqrt{1+4\lambda} \quad \text{for} \quad P^- \quad \text{and}$$
$$\lambda \to \mu^+(\lambda) = \frac{1}{2}\sqrt{(p-1)^2+4\lambda} \quad \text{for} \quad P^+.$$

Let $\Omega_1^{\pm}$ denote the positive branch of $\mu^{\pm}$, and $\Omega_2^{\pm}$ the negative branch of $\mu^{\pm}$, i.e

$$\Omega_1^- = \{\lambda \in \mathbb{C} : \arg(\mu^-(\lambda)) = \frac{1}{2}\arg\left(\frac{1}{4}+\lambda\right) + 2k\pi,\ k \in \mathbb{Z}\}$$
$$\Omega_2^- = \{\lambda \in \mathbb{C} : \arg(\mu^-(\lambda)) = \frac{1}{2}\arg\left(\frac{1}{4}+\lambda\right) + (2k+1)\pi,\ k \in \mathbb{Z}\},$$

and similarly

$$\Omega_1^+ = \{\lambda \in \mathbb{C} : \arg(\mu^+(\lambda)) = \frac{1}{2}\arg\left(\frac{(p-1)^2}{4}+\lambda\right) + 2k\pi,\ k \in \mathbb{Z}\}$$
$$\Omega_2^+ = \{\lambda \in \mathbb{C} : \arg(\mu^+(\lambda)) = \frac{1}{2}\arg\left(\frac{(p-1)^2}{4}+\lambda\right) + (2k+1)\pi,\ k \in \mathbb{Z}\}.$$

and Ω^+ denote the domain of μ^+.

By standard results on asymptotic behaviour of ODE's there exists a solution of (5.5), ψ_1^-, such that if λ is away from $\{\lambda \in \mathbb{R} : \lambda \leq -\frac{1}{4}\}$, the cut of μ^-, then

$$\psi_1^- \sim \exp(\mu^- \eta) \quad \text{as } \eta \to -\infty. \tag{5.6}$$

ψ_1^- is well defined for λ in Ω_1^- away from the cut of μ^-. In particular there exist a coefficient D such that

$$\lim_{\eta\to\infty} \psi_1^- \exp(-\mu^+\eta) = D(\lambda) \quad \text{for } \lambda \in \Omega_1^- \cap \Omega_1^+. \tag{5.7}$$

Observe that $D(\lambda)$ is the Evans function as defined in Chapter 4.

Our claim is that $D(\lambda)$ is analytic in the Riemann surface defined by the sheets $\Omega_1^- \cap \Omega_1^+$, $\Omega_2^- \cap \Omega_1^+$, $\Omega_1^- \cap \Omega_2^+$ and $\Omega_2^- \cap \Omega_2^+$ cut and pasted through the essential spectrum $\mathcal{E}$ of $\tilde{L}$ and apart from a countable number of real poles that lay in $\Omega_2^- \cap \Omega_2^+$. This is seen in the next section after the explicit computation of $D(\lambda)$. A schematic picture of this Riemann surface for $p \neq 2$ is shown in Figure 5.1. Observe that in this case there are two branch points $-\frac{1}{4}$ and $-\frac{(p-1)^2}{4}$, and the essential spectrum contains two cuts of the domain, $(-\infty, \min\{-\frac{1}{4}, -\frac{(p-1)^2}{4}\})$ and $(\min\{-\frac{1}{4}, -\frac{(p-1)^2}{4}\}, \max\{-\frac{1}{4}, -\frac{(p-1)^2}{4}\})$. We denote $\Omega_1 := \Omega_1^- \cap \Omega_1^+$ and $\Omega_2 := \Omega_2^- \cap \Omega_2^+$ for simplicity of notation.

Finally we observe that the introduction of the weight function here is motivated by two reasons; the first reason is to push the spectrum away from the zero eigenvalue, and the second (that works for second order linear operators) is to make the operator self-adjoint. In view of the behaviour at $\eta = \pm\infty$ of ϕ, w and the eigenfunctions of L, it is not difficult to see that in the set $\mathbb{C}\backslash\mathcal{E}$ the spectrum of L and the spectrum of $\tilde{L}$ coincide. Moreover, the Evans functions are the same for both operators, provided that $\lambda \notin \tilde{\mathcal{E}}$.

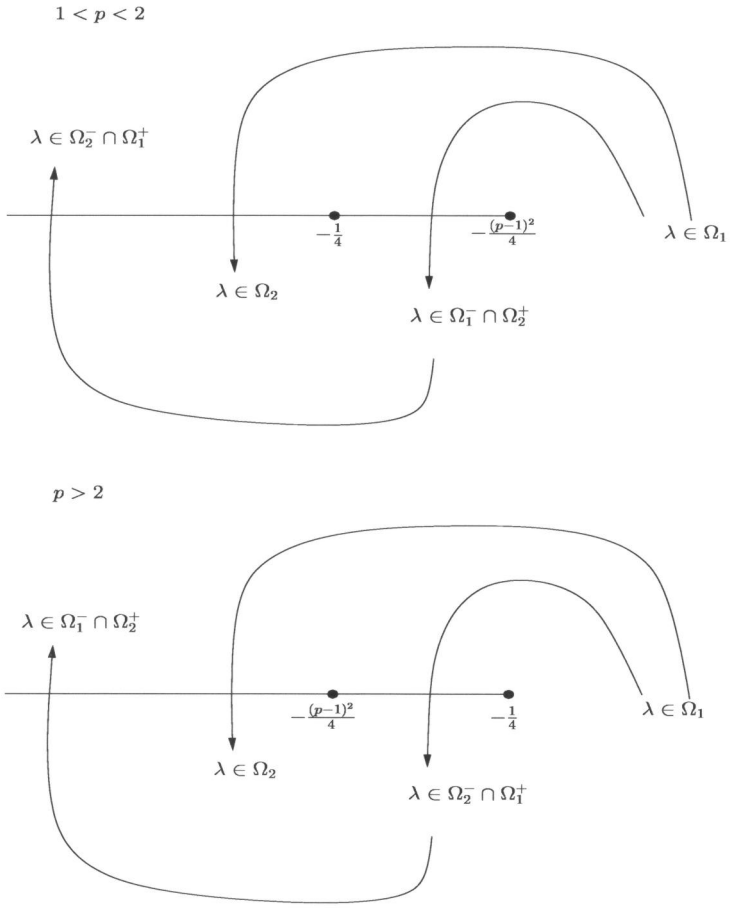


Figure 5.1: 4-sheeted Riemann surface

5.3 Explicit computation of the Evans function

In this section we give an explicit formula for the transmission coefficient $D(\lambda)$, and analyse its domain of analyticity in λ.

The idea is to transform equation (5.5) to an equation whose solutions can be better analysed. As we mentioned in the introduction, our approach is similar to that used in [21]. To be specific, we transform equation (5.5) to a hypergeometric equation. We first introduce a new unknown F, defined by setting $\psi = \phi^\alpha \exp(\beta\eta)\, F$, with α and β constants to be determined later. Finally a change of independent variable will be needed for technical reasons. In the forthcoming, Γ will denote the gamma function. The result is the following.

Theorem 5.1. *Let $D(\lambda)$ be the transmission coefficient defined by (5.7), then*

$$D(\lambda) = \frac{\Gamma\left(1+\frac{2\mu^-(\lambda)}{p-1}\right)\Gamma\left(\frac{2\mu^+(\lambda)}{p-1}\right)}{\Gamma\left(\frac{3p-2+2(\mu^+(\lambda)+\mu^-(\lambda))}{2(p-1)}\right)\Gamma\left(\frac{2(\mu^+(\lambda)+\mu^-(\lambda))-p}{2(p-1)}\right)}, \tag{5.8}$$

which domain lies in the Riemann surface defined by the positive and negative branches of μ^- and μ^+. Moreover

(i) For all $p > 1$, $\lambda = 0$ is the only zero of $D(\lambda)$ in Ω_1, and is simple.

(ii) For $p \neq 2$, there is a countable number of zeros of $D(\lambda)$. The zeros are given by

$$\lambda = \frac{(k-1)(k-2)(p-1)^2(kp-k-p)(kp-k-2p+1)}{(2-3p+2kp-2k)^2} \quad \text{for } k = 0, -1, \ldots,$$

which lie in Ω_2, and by

$$\lambda = \frac{k(k+1)(p-1)^2(kp-k+1)(kp-k+p)}{(2kp-2k+p)^2} \quad \text{for } k = -1, -2, \ldots$$

which for $p > \frac{2k}{2k+1}$ lie in Ω_2, and for $p < \frac{2k}{2k+1}$ lie in $\Omega_1^- \cap \Omega_2^+$.

(iii) There is also a countable number of poles of $D(\lambda)$ given by

$$\lambda = \frac{(k^2-1)(p-1)^2}{4}, \quad k = 0, -1, \ldots,$$

these λ's lie in Ω_2 and in $\Omega_1^- \cap \Omega_2^+$, and by

$$\lambda = \frac{(k(p-1)+1)(k(p-1)-1)}{4}, \quad k = -1, -2, \ldots,$$

which lie in Ω_2 and in $\Omega_2^- \cap \Omega_1^+$.

(iv) If $p = 2$ then, $D(\lambda)$ is analytic in $\Omega_1 \cup (\Omega_2 \backslash \{0\}) \cup \tilde{\mathcal{E}}$, and is given by

$$D(\lambda) = -\frac{1-\sqrt{1+4\lambda}}{1+\sqrt{1+4\lambda}}.$$

Proof. Throughout the proof we will need standard properties of hypergeometric functions and gamma functions, for which we refer for instance to [68].

Let F be defined by $\psi = \phi^\alpha \exp(\beta\eta)F$. Substitution of ψ into (5.5), and division by $\phi^\alpha \exp(\beta\eta)$, leads to the equation

$$F'' + 2(\alpha\frac{\phi'}{\phi} + \beta)F' + \left((\alpha^2-\alpha)\frac{\phi'^2}{\phi^2} + \alpha\frac{\phi''}{\phi} + 2\alpha\beta\frac{\phi'}{\phi} + \beta^2 - B\right)F = 0, \tag{5.9}$$

where $B := \left(\frac{3p^2}{4} - \frac{p}{2}\right)\phi^{2(p-1)} - \frac{1}{2}p^2\phi^{p-1} + \lambda + \frac{1}{4}$. Let $z = 1 - \frac{\phi'}{\phi}$ be the new independent variable, chosen such that the coefficient of F' is linear in z. Observe

also that $z = \phi^{p-1}$, by the travelling wave equation (5.3). With this observation, equation (5.9) in z, after dividing by $(p-1)^2 z(1-z)$, becomes

$$z(1-z)\ddot{F} + \left((1-2z) + \frac{2}{p-1}(\alpha(1-z)+\beta)\right)\dot{F} + AF = 0 \tag{5.10}$$

with

$$A := \frac{1}{(p-1)^2 z(1-z)}\Big((\alpha^2-\alpha)(1-z)^2 + \alpha(1-z)(1-pz) + 2\alpha\beta(1-z) + \beta^2 - (\frac{3p^2}{4} - \frac{p}{2})z^2 + \frac{1}{2}p^2 z - \lambda - \frac{1}{4}\Big).$$

The dots on top of F indicate derivatives with respect to z. For (5.10) to be a hypergeometric equation A must be a constant. This gives the conditions

$$(\alpha+\beta)^2 = \frac{1}{4} + \lambda, \tag{5.11}$$

and

$$\beta^2 = \frac{(p-1)^2}{4} + \lambda. \tag{5.12}$$

After setting the coefficients of $\dot{F}$ and F in (5.10) equal to $c-(a+b+1)z$ and $-ab$ respectively, F satisfies the hypergeometric equation

$$z(1-z)\ddot{F} + [c-(a+b+1)z]\dot{F} - ab\,F = 0 \tag{5.13}$$

with

$$a = \frac{3p-2+2\alpha}{2(p-1)}$$
$$b = \frac{2\alpha - p}{2(p-1)}$$
$$c = 1 + \frac{2(\alpha+\beta)}{p-1}.$$

Note also that $\alpha+\beta$ and β are 2-valued as functions of λ. Also note that the conditions (5.11) and (5.12) give us two possible choices: $\alpha+\beta = \mu^-$ or $-\mu^-$, and $\beta = \mu^+$ or $-\mu^+$. But in any of these cases $\alpha+\beta$ and β are 2-valued as functions of λ. We shall make a choice below.

To get the coefficient $D(\lambda)$, we have to look at the behaviour of ψ_1^- at $\eta = \pm\infty$ in terms of z, i.e. at $z=0$ and $z=1$ respectively. Since $\phi = z^{\frac{1}{p-1}}$, the explicit formula for ϕ gives $\exp(\eta) = (\frac{1-z}{z})^{-\frac{1}{p-1}}$ (up to translation in η). Hence, according to (5.6) and (5.7), for $\lambda \in \Omega_1^- \cap \Omega_1^+$, ψ_1^- behaves as

$$\psi_1^- \sim z^{\frac{\mu^-}{p-1}} \quad \text{for} \quad z \sim 0, \tag{5.14}$$

$$\psi_1^- \sim D(\lambda)(1-z)^{-\frac{\mu^+}{p-1}} \quad \text{for} \quad z \sim 1. \tag{5.15}$$

We look now at the solutions of (5.13). The space of solutions is spanned by the hypergeometric functions

$$F(a,b;c;z) \quad \text{and} \quad z^{1-c}F(a-c+1,b-c+1;2-c;z).$$

We take $\alpha+\beta=\mu^-$ so that $F(a,b;c;z)$ is regular at $z=0$. Then by (5.14), $\psi_1^- = z^{\frac{\alpha+\beta}{p-1}}(1-z)^{-\frac{\beta}{p-1}}F(a,b;c;z)$. In general $F(a,b;c;z)$ has a singularity in $z=1$, namely

$$\lim_{z\to 1}(1-z)^{-(c-a-b)}F(a,b;c;z) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \tag{5.16}$$

Since $c-a-b=\frac{2\beta}{p-1}$, we set $\beta=-\mu^+$, so that by (5.15) and (5.16) we get for $\lambda\in\Omega_1^-\cap\Omega_1^+$

$$D(\lambda)=\lim_{z\to 1}\psi_1^-(1-z)^{\frac{\mu^+}{p-1}}=\lim_{z\to 1}(1-z)^{\frac{2\mu^+}{p-1}}F(a,b;c;z)=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}.$$

This proves the first statement. Observe that the formula for $\psi_1^- = z^{\frac{\mu^-}{p-1}}(1-z)^{\frac{\mu^+}{p-1}}F(a,b;c;z)$ implies that the solution ψ_1^- is not only analytic for λ in Ω_1^- but for λ in the 4-sheeted Riemann surface R, which sheets are Ω_1, $\Omega_1^-\cap\Omega_2^+$, $\Omega_2^-\cap\Omega_1^+$ and Ω_2, see Figure 5.1. We also observe that we could have chosen $\alpha+\beta$ and β differently, for example to identity ψ_1^- with an expression involving $z^{1-c}F(a-c+1,b-c+1;2-c;z)$ instead of $F(a,b;c;z)$, or a linear combination of both. All possibilities, however, lead to the same expression for $D(\lambda)$. We have chosen the most transparent one.

The next step is to find the zeros of $D(\lambda)$, i.e the poles of $\Gamma(a)$ and $\Gamma(b)$ that are not poles of $\Gamma(c)$ and $\Gamma(a+b-c)$.

We recall here that the poles of the gamma function are the negative integers. Hence setting $a=k$ for $k=0,-1\ldots$ we get that the poles of $\Gamma(a)$ are given by solving

$$2(\mu^-+\mu^+)=2k(p-1)+2-3p \quad \text{for } k=0,-1,\ldots \tag{5.17}$$

with respect to λ. This gives an explicit formula for λ, namely

$$\lambda=\frac{(k-1)(k-2)(p-1)^2(kp-k-p)(kp-k-2p+1)}{(2-3p+2kp-2k)^2} \quad k=0,-1,\ldots. \tag{5.18}$$

Similarly we get the poles of $\Gamma(b)$ by solving

$$2(\mu^-+\mu^+)=2k(p-1)+p, \quad \text{for } k=0,-1,\ldots \tag{5.19}$$

which gives

$$\lambda=\frac{k(k+1)(p-1)^2(kp-k+1)(kp-k+p)}{(2kp-2k+p)^2} \quad k=0,-1,\ldots. \tag{5.20}$$

It remains to study whether these explicit expressions for λ give zeros of $D(\lambda)$ or not, and if so, whether they are eigenvalues of the operator L ($\lambda\in\Omega_1$), or zeros on the other sheets. For that we look at the graph of $\lambda\to 2(\mu^++\mu^-)$ for $\lambda\in\mathbb{R}$ and the possibility of this graph to intersect the constant lines $2k(p-1)+2-3p$

and $2k(p-1)+p$. Observe that $\lambda \to 2(\mu^+ + \mu^-)$ is monotone for λ in each of the four sheets. Equation (5.19) is satisfied for $k=0$ and any $p>1$, this giving $\lambda = 0$. Clearly this solution is in Ω_1, hence $\lambda = 0$ is an eigenvalue. Also $\Gamma(a)$ does not have poles if $\lambda \in \Omega_1$, since for every $k = 0, -1, \ldots$ and $p > 1$, $2k(p-1)+2-3p < 0$. If $p > \frac{2k}{2k+1}$ for fixed $k \in \{-1, -2, \ldots\}$, the right hand side of (5.19) is negative, so $\Gamma(b)$ cannot have poles for $\lambda \in \Omega_1$. If $p \leq \frac{2k}{2k+1}$, the right hand side of (5.19) is positive, but only for $p=2$ and $k=-1$ does the equation (5.19) have a solution, which is $\lambda = -\frac{1}{4}$. This is because the graph of $\lambda \to 2(\mu^+ + \mu^-)$, which is increasing, does not intersect the lines $2k(p-1)+p$ when $p < \frac{2k}{2k+1}$ for $k = -1, -2, \ldots$. So the only zero of $D(\lambda)$ in Ω_1 is $\lambda = 0$.

Similar arguments give that the lines $2k(p-1)+2-3p$ $k = -1, -2, \ldots$ intersect the graph of $\lambda \to 2(\mu^+ + \mu^-)$ only if $\lambda \in \Omega_2$. This giving the zeros of $D(\lambda)$ expressed by the formula (5.17). If $p > 2$ then the lines $2k(p-1)+p$ cut the graph only if $\lambda \in \Omega_2$. If $p < 2$ and $p \leq \frac{2k}{2k+1}$ then the lines $2k(p-1)+p$ cut the graph only if $\lambda \in \Omega_1^- \cap \Omega_2^+$. Whereas if $p < 2$ and $p > \frac{2k}{2k+1}$ then the lines $2k(p-1)+p$ cut the graph of $2(\mu^+ + \mu^-)$ only if $\lambda \in \Omega_2^- \cap \Omega_1^+$.

In Figure 5.2 we have sketched all the possibilities for $p \neq 2$. We have drawn the graph of $\lambda \to 2(\mu^+ + \mu^-)$ for $\lambda \in \mathbb{R}$ in the different sets Ω_1, $\Omega_1^- \cap \Omega_2^+$, $\Omega_2^- \cap \Omega_1^+$ and Ω_2, and the lines $2k(p-1)+p$ and $2k(p-1)+2-3p$.

If $p=2$ the zeros of $D(\lambda)$ in Ω_2 are given by $\sqrt{1+4\lambda} = k-2$ for $k = 0, -1, \ldots$ and $\sqrt{1+4\lambda} = k+1$ with $k = 0, -1, \ldots$. In particular $\lambda = 0$ is a simple zero in Ω_2.

Next to give expressions of the poles of $D(\lambda)$, we have to find the poles of $\Gamma(c)$ and $\Gamma(a+b-c)$. Since $a+b-c = \frac{2\mu^+}{p-1}$, the poles of $\Gamma(a+b-c)$ are in $\Omega_2 \cap \mathbb{R}$ and in $\Omega_1^- \cap \Omega_2^+ \cap \mathbb{R}$ (the sheets that contain the negative branch of μ^+), and are given by

$$\lambda = \frac{(k^2-1)(p-1)^2}{4} \quad k = 0, -1, \ldots. \tag{5.21}$$

The poles of $\Gamma(c)$ are found by setting $\frac{2\mu^-}{p-1} = -1, -2, \ldots$, they are in $\Omega_2 \cap \mathbb{R}$ and $\Omega_2^- \cap \Omega_1^+ \cap \mathbb{R}$, and are given by

$$\lambda = \frac{(k(p-1)+1)(k(p-1)-1)}{4} \quad k = -1, -2, \ldots \tag{5.22}$$

If $p=2$ all poles of $\Gamma(a)$ and $\Gamma(b)$ in Ω_2 are removable, since they are given by

$$\lambda_a = \frac{(k-1)(k+1)}{4} \quad k = -2, -3, \ldots \text{ and } \lambda_b = \frac{(k-1)(k+1)}{4} \quad k = 0, -1, -2, \ldots.$$

This includes the zero $\lambda = -\frac{1}{4}$ that lies in $\tilde{\mathcal{E}}$. Observe that $\lambda = 0$ is a double pole and a simple zero in Ω_2 of $D(\lambda)$, so it is a simple pole of $D(\lambda)$ in Ω_2.

In fact only for $p=2$ and $p=0$, equations (5.18) and (5.20) can be written as second order polynomials in k, so there is no way these expression simplify out as (5.21) and (5.22), hence if $p \neq 2$ there are infinitely many poles and zeros of $D(\lambda)$ that are no removable.

The formula for $D(\lambda)$ when $p=2$ is obtained from the general formula (5.7), by using $\Gamma(z) = (z-1)\Gamma(z-1)$.

□

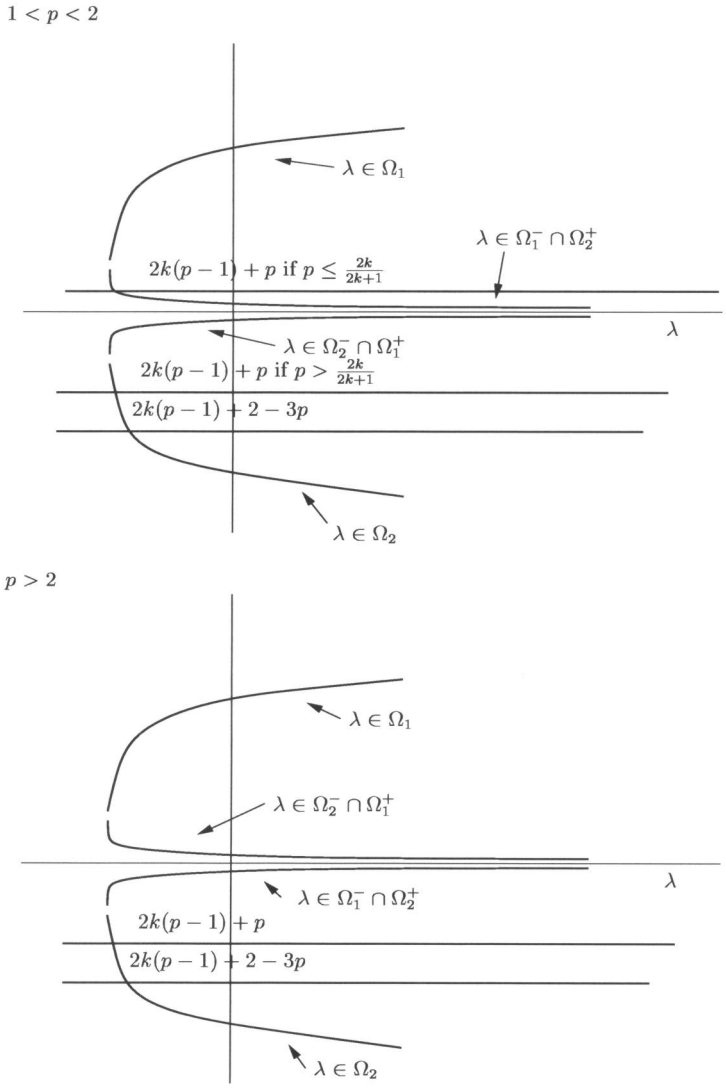


Figure 5.2: Graphs of $\lambda \in \mathbb{R} \to 2(\mu^+ + \mu^-)$ and the intersection with $2k(p-1)+p$ and $2k(p-1)+2-3p$.

Appendix A

Appendix to Chapter 3

Before going into the proof of Theorem 3.8, let us recall some aspects of the FME approach. We first define the space of finite elements V_h. Let $-l = x_0 < x_1 < \ldots < x_n = l$ be a uniform partition of $I = [-l, l]$, with $h = x_i - x_{i-1}$. Then we take V_h to be the space of continuous functions on I that are linear on the intervals (x_{i-1}, x_i). V_h is a subspace of H_0^1, moreover V_h satisfies the following approximation property see ([69]),

$$\inf_{\Psi \in V_h} \{\|\varphi - \Psi\| + h\|\varphi_x - \Psi_x\|\} \leq C_s \|\varphi\|_q h^q \quad 1 \leq q, \ \forall \ \varphi \in H^q(-l, l), \tag{A.1}$$

see ([69]). V_h is a Hilbert space. A base of V_h is given by the functions

$$\varphi_i(x_j) = \begin{cases} 1 & \text{if} \quad j = i \\ 0 & \text{if} \quad j \neq i. \end{cases}$$

Then a function of the space V_h has a unique representation as

$$u(x_j) = \sum_{i=0}^{n} u_i \varphi_i(x_j).$$

The bilinear form

$$(u, v)_h := \sum_{i,j}^{n} u_i v_j (\varphi_i, \varphi_j),$$

is an inner product on V_h. Here $(\cdot, \cdot)$ denotes the usual inner product on L^2. The inner products (φ_i, φ_j) are computed by extending the functions φ_i by the piecewise linear functions

$$\varphi_i(x) = \begin{cases} 0 \quad \text{if} \quad x \leq x_{i-1} \\ \frac{x - x_i}{h} + 1 \quad \text{if} \quad x_{i-1} \leq x \leq x_i \\ \frac{x_i - x}{h} + 1 \quad \text{if} \quad x_i \leq x \leq x_{i+1} \\ 0 \quad \text{if} \quad x \geq x_{i+1}. \end{cases}$$

A.1 Formal error estimates: proof of Theorem 3.8

To get error estimates of the full discrete equation, one usually consider a semi-discrete in space or time problem, the second step is then to compare the discrete

problem with the semi-discrete one. Here we have chosen to start with the semi-discrete in space equation, i.e. we first obtain Theorem 3.8, this theorem simplifies the proofs of Theorem 3.9 and Theorem 3.11. Let U be the solution of the semi-discrete problem (3.46), i.e. of

$$\begin{aligned}(U_t, \chi) + \varepsilon(U_{xt}, \chi_x) &= -(U_x, \chi_x) + ((U^2)_x, \chi) \\ (U(0), \chi) &= (u_0, \chi) \text{ for } \chi \in V\end{aligned}$$

and let be u the solution of the continuous problem.

$$\begin{aligned}(u_t, \varphi) + \varepsilon(u_{xt}, \varphi_x) &= -(u_x, \varphi_x) + ((u^2)_x, \varphi) \\ (u(0), \varphi) &= (u_0, \varphi) \text{ for } \varphi \in H_0^1.\end{aligned} \tag{A.2}$$

For simplicity we assume that $|U_0|_\infty \leq |u_0|_\infty$. Then Theorem 3.8 holds. The estimate (3.47) holds easily by testing equation (A.1) with U. Observe that this estimate implies that $|U|_\infty$ is uniformly bounded in $[0, T]$. This is later used to prove the error estimate. In fact the more general estimate

$$||U(t) - u(t)||_s \leq C(||u_0||_q)h^{q-s}\left(1 + \int_0^t (||u_t(\nu)||_q + ||u(\nu)||_q)d\nu\right), \tag{A.3}$$

holds if $u_0 \in H^q$, for $q \geq 1$ and $t > 0$. We prove it in the sequel. We define the error as $e := U(t) - u(t)$ for each $t \in [0, T]$. To simplify the analysis we further define $\eta := \tilde{u}(t) - u(t)$ and $\xi = U(t) - \tilde{u}(t)$, so that $e = \xi + \eta$. Here $\tilde{u}(t)$ is the unique solution of

$$(u_x - \tilde{u}_x, \chi) = 0 \quad \text{for } \chi \in V_h, \tag{A.4}$$

or the Ritz projection of $u(t)$ on V_h, see [69]. Using (A.1) one proves that η satisfies the estimates

$$||\tilde{u} - u|| = ||\eta|| \leq Ch^q||u||_q \quad \text{and} \quad ||\tilde{u} - u||_1 = ||\eta||_1 \leq Ch^{q-1}||u||_q, \tag{A.5}$$

see for instance [69]. With this estimates it only remains to show similar estimates for $||\xi||_1$ and $||\xi||$.

By (A.4), $(\eta_x, \chi_x) = 0$ and also $(\eta_{xt}, \chi_x) = 0$ for all $\chi \in V_h$. Hence subtraction of (3.46) and (A.2) gives

$$(\xi_t, \chi) + \varepsilon(\xi_{xt}, \chi_x) + (\xi_x, \chi_x) = B(U - u, \chi) - (\eta_t, \chi) \quad \text{for } \chi \in V_h, \tag{A.6}$$

where $B(\psi, \chi) := ((\psi U)_x, \chi) + ((\psi u)_x, \chi)$, and satisfies the following inequalities

$$|B(\psi, \chi)| \leq |(\psi u, \chi_x)| + |(\psi U, \chi_x)| \leq C(||u_0||_1)\,||\psi||\,||\chi||_1, \tag{A.7}$$

and

$$|B(\psi, \chi)| \leq C(||u_0||_1)\,(\psi, \chi_x) \leq C(||u_0||_1)\,||\psi||_{-1}\,||\chi||_2 \tag{A.8}$$

Here we have controlled $||U||_\infty$ and $||u||_\infty$ with the H^1-norm, and the stability estimates of Proposition 3.5 (for u) and (3.47) (for U). To control the inner product (ψ, χ_x) in (A.8) we use Poincaré-Friedrichs inequality, so that in general $(\psi, \chi) \leq ||\psi||_{-1}\,||\chi||_1$.

We first proceed to get an upper bound on $||\xi||_1$. To this end we set $\chi = \xi$ in (A.6), then using (A.7) we get

$$\frac{1}{2}\frac{d}{dt}||\xi(t)||_\varepsilon^2 + ||\xi_x||^2 \leq B(e,\xi) + ||\eta_t||\,||\xi|| \leq C||e||\,||\xi||_1 + ||\eta_t||\,||\xi||.$$

Arranging constants and dividing up by $||\xi||_\varepsilon$ in the latter, lead to

$$\frac{d}{dt}||\xi||_\varepsilon \leq C||e|| + ||\eta_t|| \leq C(||\xi||_\varepsilon + ||\eta||) + ||\eta_t||. \tag{A.9}$$

Finally application of Gronwall's lemma and (A.5) implies

$$||\xi(t)||_1 \leq C\int_0^t (||\eta_t(\nu)|| + ||\eta(\nu)||) \leq Ch^q \int_0^t (||u_t(\nu)||_q + ||u(\nu)||_q)\, d\nu,$$

which together with (A.5) proves (3.48) for $s = 1$. For $s = 0$, we use a duality argument. Let $\psi \in H^2$ be the solution of

$$-\varepsilon\psi_{xx} + \psi = \xi_t, \tag{A.10}$$

with homogeneous Dirichlet boundary conditions. In particular ψ satisfies $||\psi|| \leq ||\psi||_1 \leq ||\psi||_2 \leq ||\xi_t||$. With this observation, using equations (A.10) and (3.46), the estimate (A.8) and the Poincaré-Friedrichs inequality, we get

$$\begin{aligned} ||\xi_t||^2 = (\psi,\xi_t) + \varepsilon(\psi_x,\xi_{xt}) = -(\psi_x,\xi_x) + B(e,\psi) + |(\eta_t,\psi)| \leq \\ |(\psi_{xx},\xi)| + C||e||_{-1}\,||\psi||_2 + ||\eta_t||_{-1}\,||\psi||_1 \leq \\ ||\xi_t||\,(||\xi|| + C(||\xi|| + ||\eta||_{-1}) + ||\eta_t||_{-1})\,. \end{aligned}$$

This implies

$$||\xi_t|| \leq C(||\xi|| + ||\eta||_{-1} + ||\eta_t||_{-1}), \tag{A.11}$$

then by Gronwall's lemma and (A.5),

$$||\xi(t)|| \leq C\int_0^t (||\eta_t(\nu)||_{-1} + ||\eta(\nu)||_{-1})\, d\nu \leq Ch^{q+1}\int_0^t (||u_t(\nu)||_q + ||u(\nu)||_q)\, d\nu.$$

□

Remark A.1. *Observe that the estimates obtained for $\xi = \tilde{u} - U$ are of better order than for the error $u - U$. This is because the form of the equation, lets one obtain estimates on the derivative of the H^s of ξ, that depend on the H^{s-1} norm of η and not in the H^s-norm of η, see (A.9) and (A.11).*

The proofs of Theorem 3.9 and Theorem 3.11, are similar. At each time step one introduces the error $E^k = u^k - u(t_k)$. But $E^k = u^k - U(t_k) + e(t_k)$, thus only the estimates for $u^k - U(t_k)$ has to be deduced. For the implicit case boundedness of the solution U^k is achieved by the stability estimate (3.52). In the explicit discretisation it is first assumed that $|U^k|_\infty$ is uniformly bounded in $[0,T]$, since the error estimates imply convergence of the discrete solution to the solution of problem (3.2), then $|U^k|_\infty$ is uniformly bounded by the same bound as $|u(t)|_\infty$, if τ is sufficiently small.

A.2 Up-winding

In both our methods we discretized the convection term by one-side up-wind discretisation, that is to say that the derivative of u^2 at each finite element x_i is approximated by

$$(u^2)_x(x_i) \sim \frac{u_i^2 - u_{i-1}^2}{h}. \tag{A.12}$$

But if we compute the derivative of some $u \in V_h$ at a point x_i of $[-l, l]$, we have to compute

$$(u_x, \varphi_i)_h = -(u, \varphi_i')_h = -\sum_{j=0}^{n} u^j(\varphi_j, \varphi_i').$$

We need to determine the values of the integrals (φ_j, φ_i'), Then by inserting the continuous extensions of φ_i, and computing the integrals give

$$(\varphi_i, \varphi_j') = 0 \quad \text{for} \quad |i - j| > 1,$$

and

$$(\varphi_i, \varphi_{i-1}') = \frac{1}{2}, \ (\varphi_i, \varphi_i') = 0, \ (\varphi_i, \varphi_{i+1}') = -\frac{1}{2},$$

hence

$$(u_x, \varphi) = \frac{u_{i+1} - u_{i-1}}{2}. \tag{A.13}$$

In this way the variational formulation has to be modified, since up-winding does not correspond to the standard FEM approximation (A.13).

There are several ways to get up-wind by FEM. For example:

The artificial diffusion method: We approximate the terms (u_x, φ_i) by

$$\frac{h}{2}(u_{xx}, \varphi_i) + (u_x, \varphi_i).$$

The addition of a discrete laplacian of u to the usual approximation of u_x by FEM yields to the discretisation

$$\frac{h}{2}(-u_{i-1} + 2u_i - u_{i+1}) + \frac{1}{2}(u_{i+1} - u_{i-1})$$

which coincides with the up-wind discretisation (A.12). Observe that the diffusion term introduces an error of order $O(h)$. So it does not reduce the order of the spatial error, as obtained in Theorem 3.8.

Quadrature approximation: If we approximate integrals on the interval (x_i, x_{i+1}) for $i = 0 \ldots n$ by

$$\int_{x_i}^{x_{i+1}} f(x)dx \sim hf(x_i),$$

we can define a new inner product on the space V_h. This gives for the base $\{\varphi_i\}_i$

$$< \varphi_i, \varphi_{i-1}' >= 1, \ < \varphi_i, \varphi_i' >= -1, \ < \varphi_i, \varphi_{i+1}' >= 0,$$

hence

$$< u_x, \varphi >_h = u_i - u_{i-1}.$$

This inner product, $< \cdot, \cdot >_h$, is equivalent to $(\cdot, \cdot)_h$, so the error induced in H^s, for $s = 0, 1$ is of the same order as for $(\cdot, \cdot)_h$.

Remark A.2. *For the one-side method,*

$$u_x(x_i) \sim \frac{1}{h}(u_{i+1} - u_i),$$

then one may use

$$-\frac{h}{2}(u_{xx}, \varphi_i) + (u_x, \varphi_i),$$

for the artificial diffusion method, and the quadrature approximation consists now of taking the integrals as follows

$$\int_{x_i}^{x_{i+1}} f(x)dx \sim hf(x_{i+1}).$$

Appendix B

Appendix to Chapter 4 and Chapter 5

B.1 Sketch of Routh-Hurwitz Criterion

There are several ways to determine the sign of the real part of eigenvalues of matrices. This question is of relevance in the context of linear stability of ODEs. We shall give here a theorem without proof due to Routh and (independently) to Hurwitz that gives an algorithm to count the number (counting multiplicity) of positive roots of a polynomial with real coefficients, see [17] and references therein.

Before stating the result we need the following concepts and notation. Let $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ be a polynomial of degree n ($a_0 \neq 0$), with $a_i \in \mathbb{R}$. The following matrix, constructed from the coefficients of P, is called the Routh-Hurwitz matrix

$$\begin{pmatrix} a_1 & a_3 & a_5 & \cdots & a_{2n-1} \\ a_0 & a_2 & a_6 & \cdots & a_{2n-2} \\ 0 & a_1 & a_3 & \cdots & a_{2n-3} \\ 0 & a_0 & a_2 & \cdots & a_{2n-4} \\ 0 & 0 & a_1 & \cdots & a_{2n-5} \\ 0 & 0 & a_0 & \cdots & a_{2n-6} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}.$$

Let also H_k denote the principal minor of order k of the Routh-Hurwitz matrix, and $V[b_1, b_2, b_3, \ldots, b_m]$ denotes the variation of sign in the sequence $b_1, b_2, b_3, \ldots b_m$, $m \in \mathbb{N}$. Then the following theorem holds:

Theorem B.1. *If $H_k \neq 0$ for all $k = 0, 1, \ldots, n$, then $P(x)$ has no purely imaginary zeros and the number of its zeros with positive real part is equal to*

$$V[a_0, H_1, H_3, \ldots] + V[1, H_2, \ldots].$$

Proof of Lemma 4.2

We only need to apply Theorem B.1 to

$$P^+(\mu) = \varepsilon^2\mu^3 + (1 + \varepsilon^2\lambda)\mu^2 + \mu - \lambda = 0$$

assuming that $\lambda \in \mathbb{R}$. In this case the Routh-Hurwitz matrix reads

$$\begin{pmatrix} 1+\varepsilon^2\lambda & -\lambda & 0 \\ \varepsilon^2 & 1 & 0 \\ 0 & 1+\varepsilon^2\lambda & \lambda \end{pmatrix}.$$

Then we have

$$\begin{cases} a_0 = \varepsilon^2 \\ H_1 = 1+\varepsilon^2\lambda \\ H_2 = 1+2\varepsilon^2\lambda \\ H_3 = -\lambda(1+2\varepsilon^2\lambda) \end{cases}$$

If $\lambda > 0$, then $a_0 > 0$, $H_1 > 0$, $H_2 > 0$, $H_3 < 0$, hence $V[1, H_2] = 0$ and $V[a_0, H_1, H_3] = 1$ and P^+ has one root with positive real part and two roots with negative real part. This corresponds to $\lambda \in \Omega_r^+$. If $-\frac{1}{2\varepsilon^2} < \lambda < 0$, then $a_0 > 0$, $H_1 > 0$, $H_2 > 0$, $H_3 > 0$ and so $V[1, H_2] = 0$, and $V[a_0, H_1, H_3] = 0$, hence all roots of P^+ have negative real part. This corresponds to $\lambda \in \Omega_c^+$. If $-\frac{1}{\varepsilon^2} < \lambda < -\frac{1}{2\varepsilon^2}$, then $a_0 > 0$, $H_1 > 0$, $H_2 < 0$, $H_3 < 0$ hence $V[1, H_2] = 1$, and $V[a_0, H_1, H_3] = 1$, and P^+ has two roots with positive real part and one with negative real part. This corresponds to $\lambda \in \Omega_l^+$. Finally, if $\lambda < -\frac{1}{\varepsilon^2}$, then $a_0 > 0$, $H_1 < 0$, $H_2 < 0$, $H_3 < 0$ hence $V[1, H_2] = 1$, and $V[a_0, H_1, H_3] = 1$ and P^+ has two roots with positive real part and one with negative real part. Here also $\lambda \in \Omega_l^+$.

Applying the same criterion to

$$P^-(\mu) = \varepsilon^2\mu^3 + (1+\varepsilon^2\lambda)\mu^2 - \mu - \lambda = 0,$$

we obtain that for $\lambda > 0$ there is one root with positive real part and two with negative real part. And for $\lambda < 0$ there are two roots with positive real part and one with negative real part. This corresponds to $\lambda \in \Omega_r^-$ and $\lambda \in \Omega_l^-$ respectively.

B.2 Formula for $D'(\lambda)$

Let $Y' = A(\eta, \lambda)Y$ be a linear system, for $\lambda \in \mathbb{C}$, $\eta \in \mathbb{R}$, where the matrix $A(\eta, \lambda)$ tend to constants matrices $A^{\pm}(\lambda)$ as $t \to \pm\infty$. Assume that there exists $\Omega \subset \mathbb{C}$ such that for all $\lambda \in \Omega$, both, $A^{\pm}$ have one eigenvalue with positive real part, and the rest of eigenvalues have negative real part. As in Chapter 4, we denote these positive eigenvalues by $\mu_1^{\pm}$, and similarly we denote by $v_1^{\pm}$ the right eigenvector of $A^{\pm}$ associated to $\mu_1^{\pm}$, and $w_1^{\pm}$ the left eigenvector of $A^{\pm}$ associated to $\mu_1^{\pm}$, such that $v_1^{\pm}w_1^{\pm} = 1$. Then we can define the Evans function, as in (4.19), by $D(\lambda) = Z^+Y^-$, where Y^- and Z^+ are as in Chapter 4. Recall $D(\lambda)$ does not depend on η.
Next we prove that

$$D'(\lambda) = \int_{-\infty}^{\infty} \left(Z^+(\lambda, s) \frac{dA(\lambda, s)}{d\lambda} Y^-(\lambda, s) \right) ds + D(\lambda) \left(\frac{d}{d\lambda}\{\mu_1^+ - \mu_1^-\} + \frac{dw_+(\lambda)}{d\lambda} v_+(\lambda) + w_-(\lambda) \frac{dv_-(\lambda)}{d\lambda} \right). \tag{B.1}$$

First we observe that $D'(\lambda) = Z^+_\lambda Y^- + Z^+ Y^-_\lambda$, and does not depend on η, in particular $D'(\lambda) = Z^+_\lambda Y^-(0) + Z^+ Y^-_\lambda(0)$, (the subscript λ indicates derivative with respect to λ). Let us compute $Z^+_\lambda Y^-(0)$ and $Z^+ Y^-_\lambda(0)$. We first observe that

$$\frac{d}{d\eta} Z^+_\lambda = -Z^+ A_\lambda - Z^+_\lambda A,$$
$$\frac{d}{d\eta} Z^+_\lambda = A_\lambda Y^- + A Y^-_\lambda,$$

this easily implies that

$$\frac{d}{d\eta}(Z^+ Y^-_\lambda) = -\frac{d}{d\eta}(Z^+_\lambda Y^-) = Z^+ A_\lambda Y^-. \tag{B.2}$$

Integration of (B.2) then giving

$$Z^+_\lambda Y^-(0) = \lim_{s\to\infty} Z^+_\lambda Y^-(s) + \int_0^\infty Z^+ A_\lambda Y^-, \tag{B.3}$$

and

$$Z^+ Y^-_\lambda(0) = \lim_{s\to\infty} Z^+ Y^-_\lambda(-s) + \int_{-\infty}^0 Z^+ A_\lambda Y^-. \tag{B.4}$$

Using that $Y^- \to v_1^-$ as $\eta \to -\infty$, $Y^- \to D(\lambda) v_1^+$ as $\eta \to \infty$, and similarly that $Z^+ \to D(\lambda) w_1^+$ as $\eta \to -\infty$ and $Z^+ \to w_1^+$ as $\eta \to \infty$ we get

$$\lim_{s\to\infty} Z^+_\lambda Y^-(s) = \left\{ -\frac{d\mu_1^+}{d\lambda} + \frac{dw_1^+}{d\lambda} v_1^+ \right\} D(\lambda)$$

and

$$\lim_{s\to\infty} Z^+ Y^-_\lambda(-s) = \left\{ \frac{d\mu_1^-}{d\lambda} + \frac{dv_1^-}{d\lambda} w_1^- \right\} D(\lambda).$$

Substituting these expressions into (B.3) and (B.4) respectively and summing up the two, we get (B.1). □

B.3 Gap lemma for second order equations

Let

$$Y' = A(\lambda, x) Y \tag{B.5}$$

be a 2-dimensional linear system. Where $A(x, \lambda)$ is an 2×2 matrix. The following hypotheses on $A(x, \lambda)$ are assumed:

H1. $A(x, \lambda)$ is $C^k(\mathbb{R})$ in $x \in \mathbb{R}$, and analytic in $\lambda \in \mathbb{C}$.

H2. $A(x, \lambda)$ tends uniformly on $\mathbb{C}$ to constant matrices $A^+(\lambda)$ as $x \to \infty$, and to $A^-(\lambda)$ as $x \to -\infty$ at exponential rate, namely there exist constants $C_\pm > 0$ and $\alpha > 0$ such that

$$\|A(x, \lambda) - A^\pm\| \leq C_\pm \exp(-\alpha |x|) \text{ as } |x| \to \infty.$$

H3. There exists a domain $\Omega \subset \mathbb{C}$, that contains the semi-plane $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$, such that for all $\lambda \in \Omega$ both $A^+(\lambda)$ and $A^-(\lambda)$ have one eigenvalue with positive real part, and one eigenvalue with negative real part. Let μ^+ denote the eigenvalue of A^+ with negative real part, and μ^- the eigenvalue of A^- with positive real part if λ is in Ω.

H4. Finally, the set Ω^* is chosen so that $\lambda \in \Omega^*$ if the following conditions hold:

(i) If $\beta_+(\lambda) := \min\{\mu(\lambda) \in \sigma(A^+(\lambda))\} - \mu^+$, then for all $\lambda \in \Omega^*$, $\beta_+ < \alpha$.

(ii) Similarly if $\beta_-(\lambda) := \max\{\mu(\lambda) \in \sigma(A^-(\lambda))\} - \mu^-$, then for all $\lambda \in \Omega^*$, $\beta_- > -\alpha$.

Lemma B.2 (gap-lemma). *If hypotheses* **H1** - **H4** *are satisfied, then the Evans function associated to problem (B.5) can be analytically extended to* Ω^*.

Proof. If Y^- and Y^+ are the exponentially decaying solutions at $-\infty$ and ∞, which define the Evans function of (B.5) on Ω, we only have to prove that both Y^- and Y^+ extend analytically in λ to Ω^*. We prove it only for Y^+, the proof for Y^- is analogous.

We introduce the unknown

$$z = \exp(-\mu^+ x)Y,$$

hence z satisfies

$$z' = \left(A(x,\lambda) - \mu^+ \mathrm{I}\right) z = B(\lambda)z + R(x,\lambda)z, \tag{B.6}$$

where $B(\lambda) = (A^+(\lambda) - \mu^+(\lambda))$ and $R(x,\lambda) = A(x,\lambda) - A^+(\lambda)$.

We seek solutions of (B.6) that are uniformly bounded on some interval $(x_0, -\infty)$, and that are analytic in $\lambda \in \Omega^*$. Using the variation of constants formula we reduce the equation to the following integral equation

$$z(x) = \tilde{z}(x) + \int_x^\infty \exp(B(\lambda)(x-s))R(s,\lambda)z(s)\,ds = \tilde{z}(x) + \mathcal{F}\,z(x), \;\; x \in [x_0, \infty) \tag{B.7}$$

where $\tilde{z}(x)$ is a bounded solution of $\tilde{z}' = B(\lambda)\tilde{z}$ on $[x_0, \infty)$, and $x_0 > 0$ arbitrary. This is possible since for every λ, $B(\lambda)$ has at least one zero eigenvalue. Observe that the operator $\mathcal{F}$ maps $C([x_0, \infty])$ into itself. Moreover, by **H2.** and **H4.**

$$\|\exp(B(\lambda)(x-s))R(s,\lambda)\| \leq \exp\left((\beta_+ - \alpha)(s-x)\right) \text{ for all } s \geq x, \; x \geq 0,$$

and hence x_0 can be chosen sufficiently large so that

$$\sup_{x \geq x_0} \int_x^\infty |\exp(B(\lambda)(x-s))R(s,\lambda)|ds \leq 1$$

and hence $\mathcal{F}$ is a contraction on $C([x_0, \infty))$. By the fixed point theorem, for any given $\tilde{z}$ in $C([x_0, \infty))$ there exist a unique solution of (B.7) in $C([x_0, \infty))$. If $\tilde{z}$ is

a solution of $\tilde{z}' = B(\lambda)\tilde{z}$ then z is solution of (B.6) on $[x_0, \infty)$ and is analytic in $\lambda \in \Omega^*$. Thus $y_+ = z \exp(\mu^+(\lambda)x)$ on $[x_0, \infty)$ and can be extended to satisfy (B.5) on $\mathbb{R}$ by continuation of solutions. Analyticity follows from the analyticity of z and $\mu^+(\lambda)$. □

Bibliography

[1] J. Alexander, R. Gardner, and C. Jones. A topological invariant arising in the stability analysis of travelling waves. *J. Reine Angew. Math.*, 410:167–212, 1990.

[2] C. Amick, J. Bona, and M. Schonbek. Decay of solutions of some nonlinear wave equations. *J. Differential Equations*, 81:1–19, 1989.

[3] D. N. Arnold, J. Douglas, Jr., and V. Thomée. Superconvergence of a finite element approximation to the solution of a Sobolev equation in a single space variable. *Math. Comp.*, 36(153):53–63, 1981.

[4] G. Barenblatt, V. Entov, and V. Ryzhik. *Theory of fluid flows through natural rocks.*, volume 3 of *Theory and Applications of Transport in Porous Media.* Kluwer, Dordrecht, 1990.

[5] G. I. Barenblatt, M. Bertsch, R. Dal Passo, V. M. Prostokishin, and M. Ughi. A mathematical model of turbulent heat and mass transfer in stably stratified shear flow. *J. Fluid Mech.*, 253:341–358, 1993.

[6] G. I. Barenblatt, M. Bertsch, R. Dal Passo, and M. Ughi. A degenerate pseudoparabolic regularization of a nonlinear forward-backward heat equation arising in the theory of heat and mass exchange in stably stratified turbulent shear flow. *SIAM J. Math. Anal.*, 24(6):1414–1439, 1993.

[7] G. I. Barenblatt, J. García-Azorero, A. De Pablo, and J. L. Vázquez. Mathematical model of the non-equilibrium water-oil displacement in porous strata. *Appl. Anal.*, 65(1-2):19–45, 1997.

[8] G. K. Batchelor. *An introduction to fluid dynamics.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, paperback edition, 1999.

[9] J. Bear. *Dynamics of fluids in porous media.* Elsevier, New York, 1972.

[10] J. Bear. *Hydraulics of groundwater.* McGraw-Hill: Series in Water Resources and Environmental Engineering, New York, 1979.

[11] P. Bedrikovetsky. *Mathematical theory of oil and gas recovery.* Kluwer, 1993.

[12] A. Y. Beliaev and R. J. Schotting. Analysis of a new model for unsaturated flow in porous media including hysteresis and dynamic effects. *Comput. Geosci.*, 5(4):345–368 (2002), 2001.

[13] T. B. Benjamin, J. L. Bona, and J. J. Mahony. Model equations for long waves in nonlinear dispersive systems. *Philos. Trans. Roy. Soc. London Ser. A*, 272(1220):47–78, 1972.

[14] A. Bourgeat and M. Panfilov. Effective two-phase flow through highly heterogeneous porous media: capillary nonequilibrium effects. *Comput. Geosci.*, 2(3):191–215, 1998.

[15] H. Brezis. *Analyse fonctionnelle : theorie et applications.* Masson, 1983.

[16] E. Coddington and N. Levinson. *Theory of ordinary differential equations.* McGraw-Hill, New York, 1955.

[17] W. A. Coppel. *Stability and asymptotic behavior of differential equations.* D. C. Heath and Co., Boston, Mass., 1965.

[18] C. Cuesta and J. Hulshof. A model problem for groundwater flow with dynamic capillary pressure: stability of travelling waves. To appear in Nonlinear Analysis TMA.

[19] C. Cuesta, C. J. van Duijn, and J. Hulshof. Infiltration in porous media with dynamic capillary pressure: travelling waves. *European J. Appl. Math.*, 11(4):381–397, 2000.

[20] E. DiBenedetto and M. Pierre. On the maximum principle for pseudoparabolic equations. *Indiana Univ. Math. J.*, 30(6):821–854, 1981.

[21] A. Doelman, R. A. Gardner, and T. J. Kaper. Large stable pulse solutions in reaction-diffusion equations. *Indiana Univ. Math. J.*, 50(1):443–507, 2001.

[22] J. W. Evans. Nerve axon equations. III. Stability of the nerve impulse. *Indiana Univ. Math. J.*, 22:577–593, 1972/73.

[23] J. W. Evans. Nerve axon equations. IV. The stable and the unstable impulse. *Indiana Univ. Math. J.*, 24(12):1169–1190, 1974/75.

[24] R. E. Ewing. The approximation of certain parabolic equations backward in time by Sobolev equations. *SIAM J. Math. Anal.*, 6:283–294, 1975.

[25] R. E. Ewing. Time-stepping Galerkin methods for nonlinear Sobolev partial differential equations. *SIAM J. Numer. Anal.*, 15(6):1125–1150, 1978.

[26] N. Fenichel. Persistence and smoothness of invariant manifolds for flows. *Indiana Univ. Math. J.*, 21:193–226, 1971/1972.

[27] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J. Differential Equations*, 31(1):53–98, 1979.

[28] R. A. Gardner and K. Zumbrun. The gap lemma and geometric criteria for instability of viscous shock profiles. *Comm. Pure Appl. Math.*, 51(7):797–855, 1998.

[29] B. H. Gilding. The occurrence of interfaces in nonlinear diffusion-advection processes. *Arch. Rational Mech. Anal.*, 100(3):243–263, 1988.

[30] S. M. Hassanizadeh. Dynamic effects in the capillary pressure saturation relationship. In *Proceedings of* 4^{th} *International Conference on Civil Engineering*, volume 4 of *Water Resources and Envirom. Eng.*, pages 141–149, Tehran., 1997. Sharif U. of Technology.

[31] S. M. Hassanizadeh, M. A. Celia, and H. K. Dahle. Dynamic effect in the capillary pressure-saturation relationship and their impacts on unsaturated flow. *Vadose Zone Hydrology*, 1, 2002.

[32] S. M. Hassanizadeh and W. G. Gray. Thermodynamic basis of capillary pressure in porous media. *Water Resour. Res.*, 29:3389–3405, 1993.

[33] S. M. Hassanizadeh, R. J. Schotting, and A. Y. Beliaev. A new capillary pressure-saturation relationship including hysteresis and dynamic effects. In B. et.al, editor, *Proceedings of the XIII International Conference on Computational Methods in Water Resources*, pages 245–251, Calgary, Canada, June 2000. Balkema.

[34] D. Henry. *Geometric theory of semilinear parabolic equations.* Springer-Verlag, Berlin, 1981.

[35] E. Hopf. The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Comm. Pure Appl. Math.*, 3:201–230, 1950.

[36] P. Howard and K. Zumbrun. Pointwise estimates and stability for dispersive-diffusive shock waves. *Arch. Ration. Mech. Anal.*, 155(2):85–169, 2000.

[37] J. Hulshof. An elliptic-parabolic free boundary problem: continuity of the interface. *Proc. Roy. Soc. Edinburgh Sect. A*, 106(3-4):327–339, 1987.

[38] J. Hulshof and J. R. King. Analysis of a Darcy flow model with a dynamic pressure saturation relation. *SIAM J. Appl. Math.*, 59(1):318–346 (electronic), 1999.

[39] A. M. Il′in and O. A. Oleĭnik. Asymptotic behavior of solutions of the Cauchy problem for some quasi-linear equations for large values of the time. *Mat. Sb. (N.S.)* English transl. in *Amer. Math. Soc. Translations.*, pages 19–23, 1964.

[40] C. K. R. T. Jones. Geometric singular perturbation theory. In *Dynamical systems (Montecatini Terme, 1994)*, pages 44–118. Springer, Berlin, 1995.

[41] T. Kapitula. The Evans function and generalized Melnikov integrals. *SIAM J. Math. Anal.*, 30(2):273–297 (electronic), 1999.

[42] T. Kato. A spectral mapping theorem for the exponential function, and some counterexamples. Technical report, University of Wisconsin-Madison, Mathematical research center, 1982.

[43] D. Korteweg and G. de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philos. Mag.*, 39(5):422, 1895.

[44] R. Lattès and J.-L. Lions. *The method of quasi-reversibility. Applications to partial differential equations.* Translated from the French edition and edited by Richard Bellman. Modern Analytic and Computational Methods in Science and Mathematics, No. 18. American Elsevier Publishing Co., Inc., New York, 1969.

[45] R. J. LeVeque. *Numerical methods for conservation laws.* Birkhäuser Verlag, Basel, second edition, 1992.

[46] T. P. Liu. Invariants and asymptotic behavior of solutions of a conservation law. *Proc. Amer. Math. Soc.*, 71(2):227–231, 1978.

[47] M. Mei and C. Schmeiser. Asymptotic profiles of solutions for the bbm-burgers equation. *Funkcialaj Ekvacioj*, 44:151–170, 2001.

[48] A. Novick-Cohen and R. L. Pego. Stable patterns in a viscous diffusion equation. *Trans. Amer. Math. Soc.*, 324(1):331–351, 1991.

[49] O. Oleĭnik. Discontinuous solutions of non-linear differential equations. In *Seminari 1962/63 Anal. Alg. Geom. e Topol., Vol. 1, Ist. Naz. Alta Mat.*, pages 404–436. Ediz. Cremonese, Rome, 1962/1963.

[50] V. Padrón. Sobolev regularization of a nonlinear ill-posed parabolic problem as a model for aggregating populations. *Comm. Partial Differential Equations*, 23(3-4):457–486, 1998.

[51] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Springer-Verlag, New York, 1983.

[52] R. L. Pego. Remarks on the stability of shock profiles for conservation laws with dissipation. *Trans. Amer. Math. Soc.*, 291(1):353–361, 1985.

[53] R. L. Pego, P. Smereka, and M. I. Weinstein. Oscillatory instability of traveling waves for a KdV-Burgers equation. *Phys. D*, 67(1-3):45–65, 1993.

[54] R. L. Pego and M. I. Weinstein. Eigenvalues, and instabilities of solitary waves. *Philos. Trans. Roy. Soc. London Ser. A*, 340(1656):47–94, 1992.

[55] L. A. Peletier. Asymptotic stability of travelling waves. In *Instability of continuous systems (IUTAM Sympos., Herrenalb, 1969)*, pages 418–422. Springer, Berlin, 1971.

[56] A. Quarteroni and A. Valli. *Numerical approximation of partial differential equations.* Springer-Verlag, Berlin, 1994.

[57] D. H. Sattinger. Stability of travelling waves of nonlinear parabolic equations. In *VII. Internationale Konferenz über Nichtlineare Schwingungen (Berlin, 1975), Band I, Teil 2*, pages 209–213. Abh. Akad. Wissensch. DDR, Abt. Math.–Naturwissensch.–Techn., No. 4N. Akademie-Verlag, Berlin, 1977.

[58] R. E. Showalter. The final value problem for evolution equations. *J. Math. Anal. Appl.*, 47:563–572, 1974.

[59] R. E. Showalter. *Hilbert space methods for partial differential equations.* Pitman, London, 1977. Monographs and Studies in Mathematics, Vol. 1.

[60] R. E. Showalter. Initial and final value problems for degenerate parabolic evolution systems. *Indiana Univ. Math. J.*, 28(6):883–893, 1979.

[61] R. E. Showalter. Local regularity, boundary values and maximum principles for pseudoparabolic equations. *Applicable Anal.*, 16(3):235–241, 1983.

[62] R. E. Showalter and T. W. Ting. Pseudoparabolic partial differential equations. *SIAM J. Math. Anal.*, 1:1–26, 1970.

[63] R. E. Showalter and T. W. Ting. Asymptotic behavior of solutions of pseudoparabolic partial differential equations. *Ann. Mat. Pura Appl. (4)*, 90:241–258, 1971.

[64] V. G. Smiles D. E. and V. M. A test of the uniqueness of the soil moisture characteristic during transient, non-hysteretic flow of water in rigid soil. *Soil Sci. Soc. Amer. Proc.*, 35:535–539, 1971.

[65] J. Smoller. *Shock waves and reaction-diffusion equations.* Springer, New York, 1983.

[66] F. Stauffer. Time dependence of the relations between capillary pressure, water content and conductivity during drainage of porous media. In *LAHR Symp. on Scale Effects in Porous Media*, Thessaloniki, Greece., Aug. 29 - Sep. 1 1978.

[67] M. Stecher and W. Rundell. Maximum principles for pseudoparabolic partial differential equations. *J. Math. Anal. Appl.*, 57(1):110–118, 1977.

[68] N. M. Temme. *Special functions. An introduction to the classical functions of mathematical physics.* New York, NY: Wiley. xii, 1996.

[69] V. Thomée. *Galerkin finite element methods for parabolic problems.* Springer-Verlag, Berlin, 1984.

[70] T. W. Ting. A cooling process according to two-temperature theory of heat conduction. *J. Math. Anal. Appl.*, 45:23–31, 1974.

[71] C. van Duijn, J. Molenaar, and M. de Neef. The effect of capillary forces on immisble two-phase flow in heterogeneous porous media. *Transp. Porous Media*, 21:71–93, 1995.

[72] C. J. van Duijn, A. Mikelič, and I. S. Pop. Effective equations for two-phase flow with trapping on the micro scale. *SIAM J. Appl. Math.*, 62:1531–1568, 2002.

[73] C. J. van Duyn and L. A. Peletier. Nonstationary filtration in partially saturated porous media. *Arch. Rational Mech. Anal.*, 78(2):173–198, 1982.

[74] J. Vázquez and E. Zuazua. Complexity of large time behaviour of evolution equations with bounded data. To appear.

[75] E. Yanagida. Stability of fast travelling pulse solutions of the FitzHugh-Nagumo equations. *J. Math. Biol.*, 22(1):81–104, 1985.

Samenvatting

De modellering van verticale grondwaterstroming met dynamische capillaire druk leidt tot een eigenaardige niet-lineaire diffusie-convectie vergelijking, een pseudo-parabolische partiële differentialvergelijking met een derde orde term (tweede orde in plaats en eerste orde in tijd).

In Hoofdstuk 1 bestuderen we de lopende golf oplossingen van deze vergelijking, met speciale aandacht voor het effect van de dynamisch term en de vorming van grenslagen.

In Hoofdstuk 2 beschouwen we een gereduceerde versie van de vergelijking, de zogenaamde Burgers' vergelijking uitgebreid met een lineaire derde orde term, die gemengde afgeleiden heeft. We bewijzen dat het Cauchy-probleem een unieke oplossing heeft en dat monotone lopende golven stabiel zijn.

In Hoofdstuk 3 bestuderen we het lange-termijn gedrag van oplossingen met numerieke technieken. De numerieke voorbeelden laten zien dat het lange-termijn gedrag door oplossingen van de Burgers' vergelijking gegeven wordt. Afhankelijk van de begintoestand gaat de oplossing naar een lopende golf, een 'rarefaction' golf of een gelijkvormingheidsoplossing.

In Hoofdstuk 4 keren we terug naar de kwestie van stabiliteit van lopende golven, nu door middel van een lineaire stabiliteitsanalyse: de vergelijking wordt gelineariseerd rond een lopende golf, wat leidt tot een eigenwaarde probleem dat via de Evans functie kan worden bestudeerd. De analyse van de Evans functie suggereert dat lopende golven (monotoon en niet-monotoon) stabiel zijn.

In Hoofdstuk 5 berekenen we expliciet de Evans functie voor de gegeneraliseerde Burgers' vergelijking, die formeel een gereduceerde singuliere limiet is van de pseudo-parabolische Burgers' vergelijking.

Summary

Modelling vertical non-steady groundwater flow with dynamic capillary pressure leads to a peculiar nonlinear diffusion-convection equation, namely, a degenerate second order diffusion equation extended with a third order term with two space derivatives and one time derivative.

A rigorous study of global travelling wave solutions of this equation is given in Chapter 1, with emphasis on the role played by the dynamic term and the appearance of fronts.

In Chapter 2 we consider a special case of the model equation, in which the underlying equation is the Burgers' equation and the third order term is linear. We investigate stability of travelling wave solutions of the resulting pseudo-parabolic Burgers' equation by energy methods, and establish stability of monotone travelling wave solutions.

This study is extended in Chapter 3 by a numerical analysis with emphasis on the large time behaviour of the Cauchy problem. Depending on the initial data the solution converges to a self-similar source type solution, to a rarefaction wave or to a travelling wave. In particular, the numerical evidence suggests that non-monotone travelling waves are stable as well.

In Chapter 4 we come back to the stability of travelling wave solutions, now by means of a linear stability analysis. This leads to the study of the Evans function corresponding to the eigenvalue problem of the linearised equation. The analysis again suggests that travelling waves (monotone and non-monotone) are stable.

Finally, in Chapter 5 we compute explicitly the Evans function arising in the linear stability analysis of the generalised Burgers' equation, which is, formally, a reduced singular limit of the pseudo-parabolic Burgers' equation.

Acknowledgements

Many people have contributed to the production of this thesis. In particular I want to thank my advisor Prof. Joost Hulshof, who has provided me with support and mathematical insight, and for his encouragement to go abroad. I also thank my co-advisor Prof. Hans van Duijn for giving me a position at CWI regardless of my background, and for encouraging me to go to conferences and meetings.

I would also like to thank all the members of the committee for their careful reading and fruitful comments on the thesis. Special thanks are dedicated to Sorin Pop, for his numerical insight, his collaboration in Chapter 3, and enjoyable discussions. And to Prof. John R. King and Jan Bouwe van den Berg for their demonstrated interest in the development of this manuscript.

I gratefully acknowledge the financial support provided by NWO and the RTN project Fronts-Singularities, and thank the people that have been behind this with their decisive signatures.

I should mention that life abroad has been much easier thanks to my colleagues at CWI, VU and University of Nottingham. Coffee breaks, lunches, small chats, dinners at Diemen, and going out from time to time, have been crucial in the course of my research. Particular thanks in this case go to Bob Planqué and Dorina Jibetean, my *paranimfen.*

En las últimas lineas quiero mostrar mi gratitud a las personas más cercanas. En primer lugar gracias papá, mamá, José Martín, Mamel y Ángel Andrés por apoyarme, cada uno a vuestra manera, desde casa. Y finalmente, Klaus, gracias por tu ayuda y por haber estado siempre ahí con tu sonrisa.