Stabilization and Blow-up for some Multidimensional Nonlinear PDE's

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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de
Technische Universiteit Eindhoven, op gezag van de
Rector Magnificus, prof.dr. R.A. van Santen, voor een
commissie aangewezen door het College voor
Promoties in het openbaar te verdedigen
op woensdag 15 januari 2003 om 16.00 uur

door

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geboren te Santiago, Chili

Dit proefschrift is goedgekeurd door de promotoren:

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Stabilization and blow-up for some multidimensional nonlinear PDE's /by
Ignacio A. Guerra Benavente.
Eindhoven : Technische Universiteit Eindhoven, 2003.
Proefschrift. - ISBN 90-386-0622-2
NUR 919
Subject headings: parabolic partial differential equations / elliptic partial differential equations / differential equations; asymptotics
2000 Mathematics Subject Classification : 35A07, 35B40, 35K60, 35K65, 35J55, 82C21

Druk: Universiteitsdrukkerij TU Eindhoven

Cover: Tadpole Galaxy : NASA, H. Ford (JHU), G. Illingworth (USCS/LO), M. Clampin (STScI), G. Hartig (STScI), and the ACS Science Team

This research was financially supported and carried out at the Centrum voor Wiskunde en Informatica (CWI), Amsterdam.

To my parents and
of course
to Pilar

Contents

Chapter 1

Introduction

In this thesis we study the time evaluation of solutions of certain partial differential equations (PDE's). Though these equations have a different physical background they describe -or can be thought to describe- mass balance. Consequently the involved PDE's have a divergence structure. They are special cases of the general form

$$u_t + \mathrm{div}\mathbf{B} = g \qquad \text{in} \quad \Omega, \quad t > 0, \tag{1.1}$$

where u denotes the density or concentration of a substance which evolves for $t > 0$ in a given domain $\Omega \subset \mathbb{R}^N$ ($N \geq 1$). The vector $\mathbf{B}$ denotes the mass current density or flux and the scalar g models the creation or depletion of substance by chemical reactions or by the presence of sources and sinks.

The total amount of substance in Ω at time t is represented by $\int\limits_\Omega u(x,t)\,dx$. Integrating (1.1) in Ω and applying the Divergence Theorem, we find for u the balance law

$$\frac{d}{dt}\int\limits_\Omega u(x,t)\,dx + \int\limits_{\partial\Omega} \mathbf{B}\cdot\vec{\nu}dS = \int\limits_\Omega g\,dx, \tag{1.2}$$

where $\partial\Omega$ denotes the boundary of Ω with outer unit normal vector $\vec{\nu}$. The second integral in the left hand side of (1.2) expresses the mass flow rate across $\partial\Omega$ and the integral in the right hand side can be interpreted as the rate of mass production in Ω at time t. For the special case $g \equiv 0$, and $\mathbf{B}\cdot\nu = 0$ on $\partial\Omega$, relation (1.2) implies that $\int\limits_\Omega u(x,t)\,dx$ is constant in time expressing mass conservation.

The actual form of the flux $\mathbf{B}$ and the reaction term g depends on the model or the constitutive law under consideration. For the moment, we will set $g \equiv 0$ and consider for $\mathbf{B}$ the general expression

$$\mathbf{B} = \mathbf{B}(x,t,u,\nabla u) \equiv -D(u,\nabla u)\nabla u + \boldsymbol{q}(x,t,u). \tag{1.3}$$

In others words, $\mathbf{B}$ has a diffusive part which is proportional to the vector $\nabla u = (\frac{\partial u}{dx_1},\ldots,\frac{\partial u}{dx_N})$, the gradient of u, and a lower order convective part. The diffusion term is characterized by the diffusion coefficient $D\colon \mathbb{R} \times \mathbb{R}^N \mapsto [0,\infty)$. In many physical situations we find $D(u, \nabla u) \geq D_0 > 0$ and in this case we say that the problem (1.1) is uniformly parabolic. In this thesis, we will also consider some cases of degenerate diffusion, allowing $D = 0$ in parts of the domain.

In this thesis one finds the following expressions for the diffusive part of $\mathbf{B}$:

(i) *Fick's Law.* This is the most commonly used form of the diffusive part of the flux. It states that

$$\mathbf{B} = -D\nabla u$$

where $D > 0$ is a particular constant. It measures how fast u diffuses from high to low values. This is a widely used expression, which holds for chemicals, particles, temperature, energy, velocity, and as well as populations. This law can be derived from the probabilistic study of Brownian motion, see Murray [62, p. 232-236]. If $D = 1$, we obtain the well-known *heat equation*,

$$u_t = \operatorname{div}(\nabla u) = \Delta u, \tag{1.4}$$

where we have introduced the notation $\Delta u := \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ the *Laplacian* of u. Equation (1.4) and its solutions are extensively discussed in the literature, see for instance Cannon [24], John [56, Chapter 7], Friedman [38].

(ii) *Concentration dependent diffusion.* The evolution of the density u of a gas which flows in a porous media, can be modelled by a flux that contains a density dependent diffusion:

$$\mathbf{B}(u, \nabla u) = -pu^{p-1}\nabla u \quad \text{with} \quad p > 1.$$

This results in the so-called porous media equation (PME), given by

$$u_t = \Delta u^p \quad \text{with} \quad p > 1. \tag{1.5}$$

This equation can be derived using Darcy's Law (conservation of momentum in a porous medium) combined with an equation of state for the gas. Properties of solutions and others applications of the PME are given, for instance, in the surveys by Aronson [2], Kalashnikov [57], Peletier [66], and Vazquez [74].

Equation (1.5), can be set in the form

$$\beta(v)_t = \Delta v \tag{1.6}$$

where we have used the change of variables $v = u^p$ and defined $\beta(v) := v^{1/p}$. The function β is called capacity. Some models may contain a function β which is non invertible or in a more general context a maximal monotone graph (Stefan problem). See Sacks [70] for regularity results for this formulation.

(iii) *Gradient concentration dependent diffusion.* In hydrodynamics the velocity u of a fluid can satisfy an equation of the type (1.4). These are the so-called

Newtonian fluids leading to linear viscous terms. Non-Newtonian fluids have nonlinear viscous terms, which are often modelled by the expression

$$\mathbf{B}(\nabla u) = -|\nabla u|^{p-2}\nabla u \quad \text{with} \quad p > 1,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$ and where p depends of the rheology of the fluid. This nonlinear flux produces the parabolic $p-$*Laplacian* equation:

$$u_t + \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0. \tag{1.7}$$

Fluids with $p > 2$ are called *dilatant* and those with $p < 2$ are called *pseudo-plastics*. Equation (1.7) also appears in other applications, see Diaz [29, p. 6] for some examples.

To explain the essential differences between these three types of diffusion we consider the equations (1.4), (1.5), and (1.7) in $\Omega = \mathbb{R}^N$, for $t > 0$, subject to the initial condition

$$u(x, 0) = u_0(x) \quad x \in \mathbb{R}^N.$$

Here $u_0 \geq 0$ ($u_0 \not\equiv 0$) in $\mathbb{R}^N$, $u_0 \in C(\mathbb{R}^N)$, and u_0 has a bounded support in $\mathbb{R}^N$, meaning that $\{x \in \mathbb{R}^N \mid u_0(x) > 0\}$ is bounded.

Solving the heat equation yields a solution $u = u(x, t)$ satisfying

(a) $u(\cdot, t) > 0$ for any $t \in \mathbb{R}^+ := (0, \infty)$, describing infinite speed of propagation of disturbances;

(b) $u \in C^\infty(\mathbb{R}^N \times \mathbb{R}^+)$, i.e. the solution is infinitely many times differentiable in x and t.

Solving the porous media equation yields

(a) the support of u remains bounded for all $t > 0$, describing finite speed of propagation of disturbances;

(b) $u \in C^\infty(\{u > 0\})$, i.e. only in the open set where $u > 0$ the solution is smooth.

The difference in behaviour is due to the presence of degenerate diffusion when $u = 0$.

Finally solving the parabolic p-Laplacian (1.7) with $p > 2$ we can also have degenerate diffusion. In fact, at a point where $|\nabla u| = 0$ equation (1.7) is degenerate if $p > 2$. Here we find that the finite speed of propagation property holds, however we do not obtain smoothing of solutions whenever $u(x, t) > 0$.

For the convective part of $\mathbf{B}$, we use the following expressions:

(i) *Linear convection.* Combining Fick's law with linear convection gives

$$\mathbf{B} = -D\nabla u + \boldsymbol{q}u,$$

where the vector $\boldsymbol{q}$ represents a flow field which transports u, In Chapter 5, we consider the radial flow out of a point source in $\mathbb{R}^3$, injecting a concentration u of substance into the surrounding medium.

(ii) *Non-local convection.* One of the main topics of this thesis is the study of a model of gravitational interaction. There the convection or flow is driven by the gradient of the gravitational potential, that is $\boldsymbol{q}(u) = -\nabla\phi$, where ϕ solves the boundary value problem

$$\Delta\phi = u \quad \text{in} \quad \Omega, \qquad \phi = 0 \quad \text{on} \quad \partial\Omega.$$

Similar models models are used to describe the movement of bacteria by chemotaxis, where now ϕ denotes the chemical potential. We observe that Fick's law combined with this type of nonlocal convection yields for the flux the expression

$$\mathbf{B} = -D\nabla u - u\nabla\phi$$

implying the equation

$$u_t = \text{div}(D\nabla u + u\nabla\phi).$$

If we evaluate the divergence in the right hand side, then we obtain

$$u_t = D\Delta u + \nabla\phi \cdot \nabla u + u^2, \tag{1.8}$$

an equation not in divergence form but with diffusion, convection, and reaction u^2. The interplay between these terms crucially influences the temporal behaviour of solutions. This is explained in Section 1.1.

So far we have only discussed properties of the flux $\mathbf{B}$. Reaction terms appears in equations such as (1.8) or in Section 6, where we consider the stationary solutions of (1.1) for the p-laplacian with a power like reaction term:, i.e.

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = u^q, \tag{1.9}$$

Here, existence of solutions depends on the competition between the effect of blow-up from the term u^q and spreading from the term $\text{div}(|\nabla u|^{p-2}\nabla u)$. In Section 6, we study a generalization of (1.9) to systems considering more general nonlinearities.

1.1 Stabilization and blow-up

The first question that comes to mind when dealing with equations (1.1) and (1.3) is the well-posedness of the corresponding initial-boundary value problem . That is to ask, given suitably defined initial and boundary data, whether the following hold:

(i) the problem has a solution;

(ii) this solution is unique; and

(iii) the solution depends continuously on the data of the problem.

In a parabolic problem like (1.1) and (1.3), we understand well-posedness in a finite time interval, i.e. $t \in (0, T]$ for some fixed time $T > 0$. In such case, we say that the problem is locally solvable and has a local solution.

Once local well-posedness is established, we ask whether this solution can be continued for all times $t > 0$, i.e is global. The answer to this question is strongly linked to the properties of the stationary problem

$$\mathrm{div}\mathbf{B} = g \qquad \text{in} \quad \Omega, \tag{1.10}$$

with the corresponding boundary conditions.

To illustrate the importance of (1.10), we consider the initial-boundary value problem:

$$\begin{cases} u_t - \Delta u = u^p & \text{in} \quad \Omega, \qquad t > 0, \\ u = 0 & \text{on} \quad \partial\Omega, \qquad t > 0, \\ u(0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.11}$$

and the associated stationary problem

$$\begin{cases} -\Delta u = u^p & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.12}$$

The relationship between these two problems has been studied in detail by Brezis et al. [22]. The main results are:

(i) Assume $p > 1$. If there exists a global classical solution of (1.11) for some $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, then there exists a weak solution of (1.12);

(ii) Assume $p > 1$. If there exists no weak solution of (1.12), then for any initial value $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ the solution of (1.11) blows up in finite time; and

(iii) If there exists a weak solution w of (1.12) then for any $u_0 \in L^\infty(\Omega)$, with $0 \leq u_0 \leq w$ the solution u of (1.11) with $u(0) = u_0$ is global.

Note that (ii) is a corollary of (i) and (iii) is a converse for (i). Here a classical solution is smooth and satisfies the equation in a pointwise manner throughout the domain Ω and for all $t > 0$. A weak solution is less regular and satisfies the equation in an integrated sense. Blow-up means that the supremum becomes infinite in finite time.

Having established these results about global existence or local blow-up, it is natural to ask for a detailed description of the solution near a blow-up point or near a stationary solution. In others words, we want to know how we can determine the blow-up rate and the decay rate for a specific problem.

We illustrate this for the special case of (1.11) when $p = 2$:

$$\begin{cases} u_t - \Delta u = u^2 & \text{in} \quad \Omega, \qquad t > 0 \\ u = 0 & \text{on} \quad \partial\Omega, \qquad t > 0 \\ u(0) = u_0 \geq 0 & \text{in} \quad \Omega, \end{cases} \tag{1.13}$$

where Ω is a bounded, smooth domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$.

Suppose $u = u(x,t)$ solves the equation in (1.13). Due to scale invariance , the function $\lambda^2 u(\lambda x, \lambda^2 t)$ solves again the equation for any $\lambda > 0$. This suggests that the quotient $\frac{|x|^2}{t}$ plays an special role in the dynamics described by the equation. The scale invariance also suggests to look for self-similar solutions of the form $u(x,t) = t^{-1}v(x/\sqrt{t})$. Note that such solutions are singular at $t = 0$ along any parabola $|x| = \lambda\sqrt{t}$ where $v(\lambda) \neq 0$. Using this observation, the local behaviour of blow-up at a point $x = a$ and $t = T$ can be described asymptotically by the self-similar transformation

$$u(x,t) = (T-t)^{-1}\bar{v}(\frac{x-a}{\sqrt{T-t}}, \tau) \quad \text{and} \quad \tau = -\log(T-t). \tag{1.14}$$

Substituting (1.14) in (1.13), gives

$$\bar{v}_\tau + \Delta\bar{v} - \frac{1}{2}y \cdot \nabla\bar{v} + \bar{v} - \bar{v}^2 = 0 \tag{1.15}$$

where $y = \frac{x-a}{\sqrt{T-t}}$. It was shown by Giga & Kohn [48, 49] that if $(x,t) = (a,T)$ is a point of blow-up, and if $1 \leq N < 6$ then

$$\lim_{t \to T}(T-t)u(a + y\sqrt{T-t}, t) = 1 \tag{1.16}$$

uniformly for $|y| < C$, with $C > 0$ arbitrary. This theorem results as a consequence of the stabilization of solutions of (1.15), as $\tau \to \infty$, towards the stationary solution $\bar{v} = 1$. Note that in this case $\bar{v} = 1$ is the only positive stationary solution of (1.15). Consequently $u(x,t) = (T-t)^{-1}$ is the only (backward)self-similar solution of (1.13). Therefore blow-up takes place at a rate $(T-t)^{-1}$ and with a sufficiently flat profile at $x = a$.

Global existence for (1.13), with $\Omega = \mathbb{R}^N$, can be established (provided $N > 3$) by constructing super solutions of the form $u(x,t) = (T+t)^{-1}v(x/\sqrt{T+t})$, where now the function v must satisfy

$$\Delta v + \frac{1}{2}y \cdot \nabla v + v + v^2 \leq 0.$$

Here super solutions v are of the form Ae^{-By^2} with $A, B > 0$. They give the decay rate $(T+t)^{-1}$ towards the stationary solution $u \equiv 0$ in $\mathbb{R}^N$, see Samarskii et al. [72, Chap. IV].

One of the main issues of this thesis is to study the temporal behaviour of solutions of an equation of the type

$$u_t = D\Delta u + \nabla\phi \cdot \nabla u + u^2 \quad \text{in} \quad \Omega, \quad t > 0 \tag{1.17a}$$

where

$$\Delta\phi = u \quad \text{in} \quad \Omega, \quad t > 0, \qquad \phi = 0 \quad \text{on} \quad \partial\Omega, \quad t > 0. \tag{1.17b}$$

Is it possible to arrive at similar conclusions for (1.17) as for (1.13)?. The answer appears to be much more complicated. Due to the presence of the nonlocal convection term, is not clear if the existence of stationary solutions characterizes the global behaviour of (1.17). In fact, here the large time behaviour not only depends on the existence of stationary solutions but also on D, on the shape of the initial data and on the spatial dimension N. For instance, when $N = 3$, we construct radially symmetric solutions with quite different temporal behaviour. This is shown in Figures 1.1–1.3. In Figure 1.1 we observe convergence to a stationary solution as $t \to \infty$. Decreasing the value of the diffusion coefficient D and starting from the same density we now obtain blow-up at the origin as $t \to T < \infty$, see Figure 1.2. Finally, keeping D fixed and modifying the initial value to a bump shaped form, gives a shock-like solution as $t \to T < \infty$, as is showed in Figure 1.3.

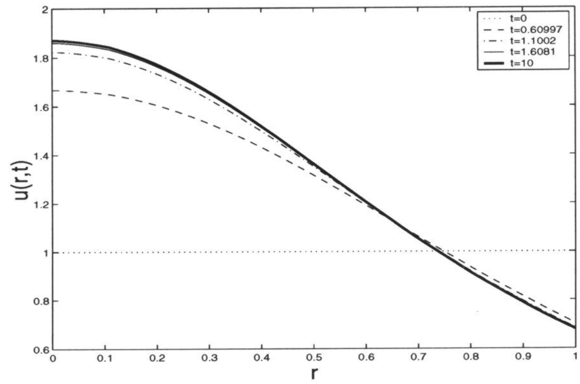

Figure 1.1: The solution $u(r,t)$ of (1.17) converges to an stationary solution.

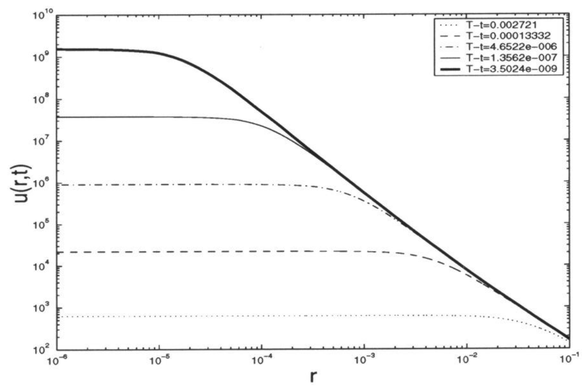

Figure 1.2: The solution $u(r,t)$ of (1.17) shows the onset of a singularity at the origin.

To arrive to such description of problem (1.17), we first considered possible stationary solutions for different values of the diffusion parameter D. In case of existence

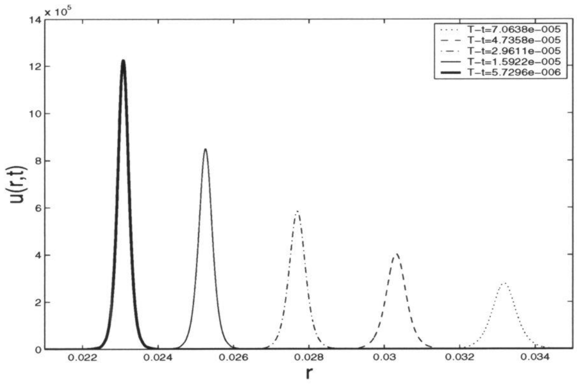

Figure 1.3: The solution $u(r,t)$ of (1.17) shows a smoothed imploding shock moving towards the origin.

of a stationary state, we look for conditions to ensure global existence, finding estimates on the solutions. On the other hand, for values of D for which there exists no stationary state, we ask whether the solutions cease to exists in finite of infinite time. In the case of finite time existence of solutions, by looking to the scale invariance of the problem, we can find self-similar solutions (as was explained above) which can describe blow-up patterns. These patterns can observed numerically in Figure 1.2–1.3.

In the remainder of the introduction, we introduce several models which are special cases of (1.1). We will consider the ideas and concepts explained above to analyse these problems. The models are N-dimensional with $N \geq 3$, and we will bring special attention to radially symmetric solutions.

1.2 Gravitational interaction

Chapters 2, 3, and 4 are devoted to the mathematical analysis of a model that arises in statistical mechanics and describes the gravitational interaction of particles. A detailed derivation and discussion on the physical assumptions was studied by Biler, Nadzieja and collaborators in [10, 13, 12, 15], Chavanis et al. [25, 26] and Wolansky [79]. Below we present a brief summary.

1.2.1 Model derivation

Consider a cluster of particles moving around in a bounded region $\Omega \subset \mathbb{R}^3$. Assuming that the particles move following a Brownian motion with a gravitational induced drift, the spatial particle density n satisfies the mass balance equation

$$n_t = \mathrm{div}\Big\{\frac{1}{\beta}(k\Theta\nabla n + n\nabla\phi)\Big\} \quad \text{in} \quad \Omega\times\mathbb{R}^+. \tag{1.18}$$

Here $\beta > 0$ is a friction coefficient, k the Boltzmann constant, Θ the temperature of the system and ϕ the gravitational potential, satisfying

$$\Delta\phi = 4\pi G n \quad \text{in} \quad \Omega\times\mathbb{R}^+, \tag{1.19}$$

where G is the gravitational constant. Equation (1.18) is called the Smoluchowski-Poisson equation. For constant temperature, in [79] this equation was derived in the context of gravitational interaction.

To ensure that the cluster of particles preserves mass we impose zero mass flux along the boundary: i.e.

$$(k\Theta\nabla n + n\nabla\phi)\cdot\vec{\nu} = 0 \quad \text{on} \quad \partial\Omega\times\mathbb{R}^+, \tag{1.20}$$

where $\vec{\nu}$ is the exterior normal at the boundary $\partial\Omega$. This implies

$$\int_\Omega n(x,t)\,dx = M \quad \text{for all} \quad t\in\mathbb{R}^+,$$

where M represents the total particle mass of the system, which is specified by the initial condition. Setting $n = 0$ outside the domain Ω, equation (1.19) gives what we call the "free" condition

$$\phi(x) = \int_\Omega \frac{G}{|x-y|} n(y,t)\,dy \quad \text{for} \quad x\in\Omega, \quad t\in\mathbb{R}^+$$

for the gravitational potential. However it appears to be convenient to work with a constant Dirichlet condition on ϕ along the boundary $\partial\Omega$. Following [15], we set

$$\phi = -\frac{GM}{R} \quad \text{on} \quad \partial\Omega\times\mathbb{R}^+, \tag{1.21}$$

with $R := \max_{x\in\Omega}|x|$. Note that this boundary condition corresponds to the gravitational potential of a mass M centered at the origin of a ball of radius R. Both definitions for the potential, the "free" and with Dirichlet boundary condition, coincide when the domain is a ball.

In general the temperature Θ in (1.18) varies in space and time. It satisfies a temperature(energy) balance equation containing thermal diffusion, heat convection and a term due to gravitational effects (cf. [13, Eq. (1.4)]):

$$\begin{aligned}\Big(\frac{\kappa}{2}n\Theta\Big)_t = \mathrm{div}(\lambda\nabla\Theta) + \mathrm{div}\left(\frac{\kappa}{2}\Theta\left\{\frac{1}{\beta}(k\Theta\nabla n + n\nabla\phi)\right\}\right)&\\ +\nabla\phi\cdot\left\{\frac{1}{\beta}(k\Theta\nabla n + n\nabla\phi)\right\} \qquad \text{in} \quad \Omega\times\mathbb{R}^+.&\end{aligned}$$

The combination of this equation and (1.18) results in the so-called Streater model. If we assume zero boundary flux on the boundary, the integrated form of the energy balance does not contain the thermal diffusivity λ,

$$E = \frac{\kappa}{2} \int\limits_{\Omega} n\Theta(x,t)\,dx + \frac{1}{2}\int_{\Omega} n\phi\,dx \quad \text{in} \quad \mathbb{R}^+,$$

where E denotes the total energy of the system and κ the specific heat of the particles. If the cluster resembles an ideal gas we have $\kappa = 3k$. Furthermore we expect that a large thermal diffusivity will result in a temperature which is nearly constant in space, i.e. $\Theta = \Theta(t)$. Taking this limit in the integrated energy balance, one finds

$$E = \frac{\kappa M}{2}\Theta(t) + \frac{1}{2}\int_{\Omega} n\phi dx \quad \text{in} \quad \mathbb{R}^+. \tag{1.22}$$

There are two ways to invoke the initial data for the system (1.18)-(1.22). If the energy E is given, it suffices to specify only the initial density

$$n(x,0) = n_0(x) \geq 0 \quad \text{for} \quad x \in \Omega. \tag{1.23}$$

Equivalently one can specify both initial density and temperature

$$\Theta(0) = \Theta_0 > 0.$$

Now E is fixed by (1.22) at $t = 0$.

Problem (1.18)–(1.23) was first derived for collisionless systems such as galaxies. The underlying argument is that rapid fluctuations of the gravitational field during the early stage of *violent relaxation* plays the same role as collisions, although the time scales involved for collisionless systems are smaller than for collisional systems (Brownian motion). The process of violent relaxation is considered in [26], and further results and interpretations can be found in [25].

Now, we turn to the analysis of problem (1.18)–(1.23). For that we assume that the domain Ω is a bounded open set, with boundary $\partial\Omega \in C^{1+\alpha}(\alpha > 0)$. As in Section 2, we introduce the dimensionless version of the system (1.18)-(1.23):

$$\begin{aligned}
n_t &= \operatorname{div}\{\Theta(t)\nabla n + n\nabla\phi\} && \text{in} \quad \Omega \times \mathbb{R}^+, && (1.24a)\\
\Delta\phi &= n && \text{in} \quad \Omega \times \mathbb{R}^+, && (1.24b)\\
0 &= (\Theta(t)\nabla n + n\nabla\phi)\cdot\vec{\nu} && \text{on} \quad \partial\Omega \times \mathbb{R}^+, && (1.24c)\\
\phi &= 0 && \text{in} \quad \partial\Omega \times \mathbb{R}^+, && (1.24d)\\
n(x,0) &= n_0(x) && \text{in} \quad \Omega, && (1.24e)
\end{aligned}$$

with the normalization

$$\int_{\Omega} n_0\,dx = 1, \quad \text{and} \quad n_0(x) \geq 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \sup_{x\in\Omega}|x| = 1, \tag{1.24f}$$

and with the energy balance

$$E = \kappa\Theta(t) + \int_{\Omega} n\phi dx \qquad \text{in} \quad \mathbb{R}^+. \tag{1.25}$$

Instead of studying problem (1.24)–(1.25) directly, we first analyze a simpler model, in which many of the principal features of the full problem are captured. This is the so-called isothermal model.

1.2.2 The isothermal model

We shall refer as the *Isothermal model*, if we drop the energy balance (1.25), and we complement (1.24) with a prescribed constant temperature; i.e.

$$\Theta(t) = \Theta^* = const. > 0 \quad \text{for} \quad t \in \mathbb{R}^+. \tag{1.26}$$

The isothermal model appears in various applications. For instance, it arises in the context of polytropic stars and the in biological phenomena of chemotaxis. The corresponding mathematical problem has received considerable attention in the past years because of its rich structure. As we discussed at the beginning of the introduction, different forms of blow-up can occur, as well as global existence. To our knowledge, there is no full description of these phenomena in $\mathbb{R}^3$. The reason is that in contrast to the two dimensional case, global existence in $\mathbb{R}^3$ not only depends on the given temperature, but also on the shape of the initial density profile.

In the following, we briefly review the principal features of the isothermal model. We start by recalling a result on radially symmetric stationary solutions.

Proposition 1.1 ([12]) *If $\Omega = B_1(0)$ is the unit ball centered at 0 in $\mathbb{R}^3$, then there exists $\bar{\Theta} > 0$ such that for $\Theta^* \geq \bar{\Theta}$ there exists a bounded positive stationary solution of (1.24) and (1.26), and there are no bounded positive stationary solutions if $\Theta^* < \bar{\Theta}$.*

As shown by Biler & Nadzieja in [15], local existence requires $n_0 \in L^p(\Omega)$ for $p > \frac{3}{2}$, implying that $n \in L^\infty_{loc}((0,T]; L^\infty(\Omega))$ for some $T \in (0,\infty)$. Hence we can allow for certain singular initial data, which result in solutions that are locally bounded in $(0,T]$. Let

$$T^* = \sup\{\ T > 0 \mid \text{Problem (1.24) and (1.26) has a solution in } (0,T]\ \}.$$

If $T^* = \infty$, the solution is defined globally and if $T^* < \infty$ we have at least

$$\lim_{t \to T^*} \|n(t)\|_{L^q(\Omega)} = \infty \quad \text{for each} \quad q > \frac{3}{2}. \tag{1.27}$$

For Problem (1.24) and (1.26) the optimal $L^p(\Omega)$ space for local existence seems to be $p = \frac{3}{2}$, since there exists a singular stationary solution in the radial case belonging to $L^q(\Omega) \setminus L^{3/2}(\Omega)$ with $q < \frac{3}{2}$. It is given by

$$U(x) = \frac{1}{4\pi|x|^2} \quad \text{provided} \quad \Theta^* = \frac{1}{8\pi}. \tag{1.28}$$

In the literature, this is refered to as the Chandrasekhar solution. Uniqueness for problem (1.24) and (1.26) is proven in [15] for $n_0 \in L^p(\Omega)$ with $p \geq 2$.

In [15], $T^* = \infty$ (global existence) was proved for large Θ^*. Using a different argument, in [16] global existence was obtained if there exists $B > 0$ such that $\Theta^* \geq \frac{1}{8\pi}(1+B)$ and $\|n_0\|_{L^1(B_r(0))} \leq (1+B)\frac{r^3}{r^2+B}$ for $r \in [0,1]$.

There are several conditions in the literature to ensure that $T^* < \infty$. We mention in particular the following result.

Theorem 1.2 ([12]) *Let $\Omega = B_1(0)$. If $\Theta^* < \frac{1}{24\pi}$, then $T^* < \infty$.*

The proof of this theorem does not give insight in the blow-up process. However, we know the principal types of blow-up for radially symmetric solutions for problem (1.24)–(1.26) from references [50, 51, 21]. First Herrero et al [50, 51] studied the problem of blow-up using careful matched asymptotic expansions. Later Brenner et al. [21] carried out an accurate numerical analysis and obtained various analytical results, such as existence and linear stability of self-similar profiles. The two principal types of blow-up found in the above references are (for $n(r,t) := n(x,t)$ with $r = |x|$):

- A solution $n(r,t)$ consists of an imploding smoothed-out shock wave which moves towards the origin. As $t \to T$, the bulk of such a wave is concentrated at distances $O((T-t)^{1/3})$ from the origin, has a width $O((T-t)^{2/3})$, and at its peak it reaches a height of order $O((T-t)^{-4/3})$. In this type of blow-up mass concentrates at the origin and at the blow-up time, i.e.

$$\lim_{r\to 0}\left[\lim_{t\to T}\int_0^r n(y,t)y^2\,dy\right] = C > 0. \tag{1.29}$$

  This situation is known as gravitational or chemotactic collapse and is depicted in Figure 1.4 (left), see also Figure 1.3. This type of blow-up produces $\lim_{t\to T^*}\|n(t)\|_{L^q(\Omega)} = \infty$ for all $q > 1$.

- A solution $n(r,t)$ has a self-similar blow-up in the explicit form

$$(T-t)n\left(\eta\sqrt{(T-t)\Theta^*}, t\right) \sim \Psi_1(\eta) := \frac{(6+\eta^2)}{(1+\frac{1}{2}\eta^2)^2} \quad \text{as} \quad t \to T.$$

  Note that this implies that n satisfies (1.29) with $C = 0$. Therefore no concentration of mass at the origin occurs at the blow-up time. This blow-up behaviour is depicted in Figure 1.4 (right), see also Figure 1.2.

We will return to the question of blow-up in Section 1.2.4 of this introduction.

First we consider Problem (1.24)–(1.25).

1.2.3 Non-isothermal model: main results

Stationary solutions on different kinds of domains were studied in [10]. We recall in particular:

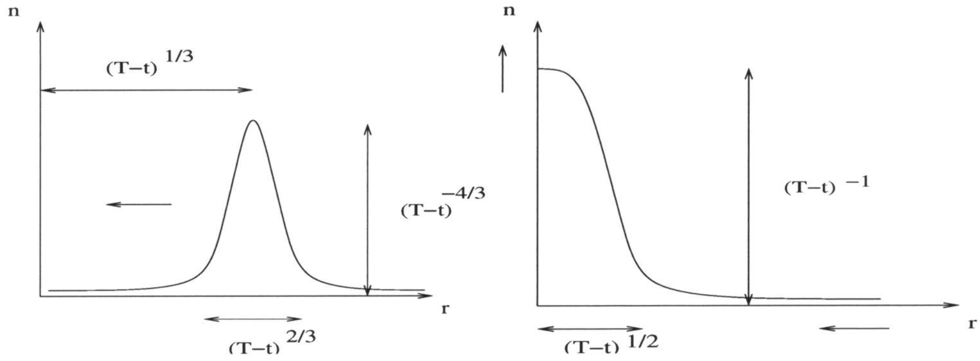

Figure 1.4: The profile $n(r,t)$ for blow-up with(left) and without(right) concentration of mass.

Proposition 1.3 ([10]) *Let $\Omega = B_1(0)$. Then for any $\kappa > 0$ there exists $E_\kappa \in \mathbb{R}$ such that for $E > E_\kappa$ there exists a bounded positive stationary solution of (1.24)–(1.25), and there are no bounded positive stationary solutions if $E < E_\kappa$.*

Local existence

Local existence is shown in Chapter 2. The proof uses a fixed-point argument and a careful construction of an invariant set to avoid degenerate diffusion in (1.24a). As in the isothermal case, local existence requires $n_0 \in L^p(\Omega)$ for $p > \frac{3}{2}$, and $\Theta(0) > 0$, implying that $n \in L^\infty_{loc}((0,T]; L^\infty(\Omega))$ for some $T \in (0,\infty)$. This existence result is in the framework of bounded solutions. However, unbounded solutions do exists. For instance the singular function U defined in (1.28), is a solution provided that $E = \frac{\kappa-2}{8\pi}$ and $\Theta(t) = const = \frac{1}{8\pi}$. Uniqueness is proven for $n_0 \in L^p(\Omega)$ with $p \geq 2$.

Conditions for global existence and blow-up

Let

$$T^* = \sup\{\ T > 0 \mid \text{Problem (1.24)–(1.25) has a solution in } (0,T]\ \}.$$

If $T^* = \infty$, the solution is defined globally and if $T^* < \infty$ we have blow up and as before we have that at least the norms satisfy (1.27).

We first give the following result about blow-up:

Theorem 1.4 *Let $\Omega = B_1(0)$. If $\kappa > 6$ and $E < \frac{1}{4\pi}$, then $T^* < \infty$.*

Unfortunately the proof of this theorem does not give insight in the structure of the blow-up. We will return to this question in Section 1.2.4. We now present a theorem which gives conditions for global existence.

Theorem 1.5 *Let $\Omega = B_1(0)$ and assume that solutions of Problem (1.24)–(1.25) are radially symmetric. If the pair $\langle n_0, \Theta_0\rangle$ satisfies one of the following conditions*

(i) $n_0 \in L^\infty(\Omega)$ *and* Θ_0 *is sufficiently large;*

(ii) $n_0 \log n_0 \in L^1(\Omega)$, *and there exists* $B > 0$, *such that*

$$\|n_0\|_{L^1(B_r(0))} \leq (1+B)\frac{r^3}{r^2+B} \quad \text{for} \quad r \in [0,1],$$

and

$$\Theta_0 \geq \frac{1}{8\pi}\frac{(1+B)(1+3B)}{(1+5B)} \exp\left(\frac{2}{\kappa}\left[\int\limits_\Omega n_0 \log n_0 \, dx - \log\left(\frac{3}{4\pi}\right)\right]\right);$$

(iii) $n_0 \equiv \frac{3}{4\pi}$, *and* $\Theta_0 \geq \frac{1}{8\pi}\gamma$ *with* $\gamma = 0.9519\ldots$,

then $T^* = \infty$ *(global existence).*

Remark 1.6 (i) *Due to the parabolic regularity,* $n_0 \in L^\infty(\Omega)$ *is not so restrictive.*

(ii) *Condition (iii) is an special case of (ii) and* $\gamma = \min\limits_{B>0} \frac{(1+B)(1+3B)}{1+5B} = 0.9519\ldots.$

(iii) *The condition on* n_0 *in (ii) implies a bound on the Morrey norm of exponent* $3/2$, *since* $\|n_0\|_{M^{3/2}(\Omega)} = \sup\limits_{x\in\mathbb{R}^3,\, 0\leq r\leq 1} r^{-1}\|n_0\|_{L^1(\Omega\cap B_r(x))}$. *In* [9], *the space* $M^{3/2}(\Omega)$ *was suggested as the natural space to prove existence.*

The proof of Theorem 1.5 contains two essential steps. To extend the local solution we first need a uniform bound from below on Θ. To achieve this we use a Lyapunov functional associated with (1.24)–(1.25), the so-called Boltzmann entropy (cf. [69]),

$$W(t) = \int\limits_\Omega n \log n \, dx - \frac{\kappa}{2} \log\left(E - \int\limits_\Omega n\phi \, dx\right).$$

This functional provides a uniform lower bound on Θ, which only depends on the initial data and κ. If Θ_0 is positive, then Θ remains positive in the whole existence interval, including the blow-up time.

In the second step we construct a control on n. Here we use the radial symmetry which allows us to transform equations (1.24a) and (1.24b) into a single equation, still containing Θ as unknown. It has the crucial property that an ordered pair of given Θ's results in an ordered pair of solutions. As a comparison function we now use the solution of (1.24a)-(1.24b) with a suitably chosen fixed Θ. Under certain hypotheses this auxiliary problem has a global solution which provides the control on n. The different conditions in Theorem 1.5 are closely related to global existence conditions for the auxiliary problem.

Convergence to stationary states

In Chapter 3 we give sufficient conditions for the convergence of global radially symmetric solutions to a stationary solution. The idea is the following. Assume that only one stationary solution exists. Then find a class of initial data which provide a uniform bound in time on $\|n(\cdot,t)\|_{L^p(\Omega)}$ for some $p>1$. Finally construct a lower bound on the temperature. These conditions together with the existence of a Lyapunov functional, provide the convergence to the unique stationary solution. We present here separate results for different classes of initial data, which provide solutions in $L^p(\Omega)$ for $p=\infty$ and $p=2$.

Theorem 1.7 *Let $\Omega = B_1(0)$ and assume that problem (1.24)–(1.25) has a radially symmetric solution. If there exists $B>0$ such that*

(i) $r^3 n_0(r) \leq \frac{3}{4\pi}\|n_0\|_{L^1(B_r(0))}$ *and* $\|n_0\|_{L^1(B_r(0))} \leq (1+B)\frac{r^3}{r^2+B}$ *for* $r=|x| \in [0,1]$;

(ii)
$$\Theta_0 \geq \frac{1}{8\pi}\frac{(1+B)(1+3B)}{(1+5B)} \exp\left(\frac{2}{\kappa}\left[\int_\Omega n_0 \log n_0 \, dx - \log\left(\frac{3}{4\pi}\right)\right]\right);$$

(iii) *$E \in \mathbb{R}$ and $\kappa > 0$ are such that Problem (1.24)–(1.25) has a unique stationary solution.*

Then, Problem (1.24)–(1.25) has a global solution in $L^\infty(\Omega)$ which converges to the stationary solution.

Remark 1.8 *If n_0 is constant in space, the normalization (1.24f) implies $n_0 \equiv \frac{3}{4\pi}$. Then (i) is satisfies and (ii) becomes $\Theta_0 \geq \frac{1}{8\pi}\gamma$, with γ as in Theorem 1.5.*

For a $L^2(\Omega)$ uniform bound, we have the following theorem.

Theorem 1.9 *Let $\Omega = B_1(0)$ and assume that problem (1.24)–(1.25) has a radially symmetric solution. If*

(i) $n_0 \in L^\infty(\Omega)$ *and*
$$\lambda\Theta_0 \geq \alpha \quad \text{and} \quad \|n_0\|^2_{L^2(\Omega)} \leq C\left((\lambda\Theta_0)^2 - \lambda\Theta_0\right),$$
where
$$\lambda = \exp\left(-\frac{2}{\kappa}\left[\int_\Omega n_0 \log n_0 \, dx - \log\left(\frac{3}{4\pi}\right)\right]\right);$$

(ii) *$E \in \mathbb{R}$ and $\kappa > 0$ are such that Problem (1.24)–(1.25) has a unique stationary solution.*

Then Problem (1.24)–(1.25) has a global solution in $L^2(\Omega)$ which converges to the stationary solution.

Remark 1.10 *Note that condition*

$(i)'$ $\quad n_0 \in L^\infty(\Omega)$ *and* Θ_0 *large,*

implies condition (i). In fact for a given n_0 we can choose Θ_0 sufficiently large so that condition (i) is satisfied.

We complement these theorems with a result which provides uniqueness for stationary solutions.

Proposition 1.11 *Let $\Omega = B_1(0)$. For a given $\kappa > 0$, there exists a sufficiently large energy $E(\kappa)$ such that for $E > E(\kappa)$ there exists a unique radially symmetric stationary solution for Problem (1.24)–(1.25).*

1.2.4 Asymptotic self-similar blow-up

In Chapter 4 we study blow-up for the isothermal and non-isothermal models using ideas of self-similarity introduced in Section 1.1. We complement the results of [50, 51, 21] on blow-up.for the isothermal case. In particular we provide a class of initial data which rules out the shock-like behaviour as showed in Figure 1.4 (left). We also present a blow-up description for the non-isothermal case, which is different from the one given in [26].

We studied blow-up for $\Omega = B_1(0)$ with radially symmetric initial data, which gives radial solutions. For this solutions the average density function $b(r,t)$ is defined by

$$b(r,t) := \frac{1}{r^3} \int\limits_{B_0(r)} n(x,t)\,dx, \tag{1.30}$$

This variable turns out to be most convenient when analyzing this system. In fact, it has the same scale invariance as $n(r,t)$, and it has in addition the advantage that solutions are smoothen. For example, if $n(r,t)$ is a delta function at the origin with unit mass, then $b(r,t) = r^{-3}$.

Let $D = (0,1)$ and set $D_T = D \times (0,T)$ for some time $T > 0$. Transformation (1.30) puts system (1.24), in the form

$$b_t = \frac{4\pi\Theta(t)}{3}\left(b_{rr} + \frac{4}{r}b_r\right) + \frac{1}{3}rbb_r + b^2 \qquad \text{in} \quad D_T \tag{1.31a}$$

$$b_r(0,t) = 0, \quad b(1,t) = 1, \qquad \text{for} \quad t \in [0,T), \tag{1.31b}$$

$$b(0,r) = b_0(r) \qquad \text{for} \quad r \in D. \tag{1.31c}$$

Here we have redefined $t := \frac{3}{4\pi}t$. Regarding the initial condition, we assume

$$b_0 \in C^2(\bar{D}), \quad \text{and} \quad \frac{r}{3}(b_0)_r + b_0 \geq 0 \quad \text{for} \quad r \in D, \tag{1.31d}$$

where the second condition is equivalent to $n_0 \geq 0$ in D. Note that conservation of mass is represented by $b(1,t) = 1$ for $t \in [0,T)$. In the isothermal case we consider (cf. 1.26)

$$\Theta(t) \equiv \Theta^* = \text{constant} > 0 \quad \text{for all} \quad t > 0. \tag{1.32}$$

In the non-isothermal case, condition (1.25) takes the form

$$E = \kappa\Theta(t) - \frac{1}{4\pi}\int_0^1 b(y,t)^2 y^4 \, dy \quad \text{in} \quad [0,T), \tag{1.33a}$$

where $E \in \mathbb{R}$ and $\kappa > 0$ are constants satisfying

$$\Theta(0) = \Theta_0 > 0. \tag{1.33b}$$

We denote by $b = b(r,t)$ the solution of problem (1.31)–(1.32) and by the pair $\langle b = b(r,t), \Theta = \Theta(t)\rangle$ the solution of problem (1.31),(1.33).

As was done for problems (1.24)–(1.26) and (1.24)–(1.25), we can now define the maximal interval of existence in terms of the average density $b(r,t)$. If $T^* < \infty$, then we must have

$$\lim_{t \to \bar{T}^*} \sup_{[0,1]} b(r,t) = \infty.$$

where $\bar{T}^* = \frac{3}{4\pi}T^*$. For the non-isothermal model the behaviour of $\Theta(t)$ near $t = T$ will de discussed later. Since problem (1.24) conserves mass, one finds for b

$$b(r,t) \leq \frac{1}{r^3} \quad \text{for} \quad r \in \overline{D},\, t > 0,$$

which implies a single point blow-up for $b(r,t)$ at the point $r = 0$.

Next we will state and motivate the results for the isothermal and non-isothermal case.

Isothermal case

The aim is to characterize the asymptotic behaviour near blow-up of the solution $b(r,t)$ of problem (1.31)–(1.32). We prove under certain hypotheses on the initial condition that if a solution of (1.31)–(1.32), blows up at time $T > 0$ and at the point $r = 0$, then it has the asymptotic form given by

$$b_*(r,t) = (T-t)^{-1}\varphi\left(\frac{r}{\sqrt{4\pi\Theta^*(T-t)/3}}\right)$$

where φ satisfies the boundary value problem

$$\begin{cases} \varphi_{\eta\eta} + \frac{4}{\eta}\varphi_\eta + \frac{1}{3}\eta\varphi\varphi_\eta - \frac{1}{2}\eta\varphi_\eta + \varphi^2 - \varphi = 0, & \text{for} \quad \eta > 0, \\ \varphi(0) \geq 1 \quad \varphi_\eta(0) = 0. \end{cases} \tag{1.34}$$

This problem has many solutions [21, 51]. We classify them by counting the number of times they cross the singular solution $\varphi_S(\eta) := \frac{6}{\eta^2}$. For that purpose, we introduce the set

$$\mathcal{S}_k = \{\varphi\colon \varphi \text{ is solution of (1.34), having } k \text{ intersections with } \varphi_S\}.$$

We will show that $\mathcal{S}_1$ is the relevant subset of solutions of (1.34) for the characterization of blow-up.

The hypotheses on the initial condition are

$$(b_0)_r \leq 0 \quad \text{for} \quad r \in D, \tag{1.35}$$

and

$$\frac{4\pi\Theta^*}{3}\left((b_0)_{rr} + \frac{4}{r}(b_0)_r\right) + \frac{1}{3}rb_0\,(b_0)_r + b_0^2 \geq 0 \quad \text{for} \quad r \in D. \tag{1.36}$$

We will show that this imply that $b_r \leq 0$ in D_T and $b_t \geq 0$ in D_T.

Next we introduce self-similarity in (1.31)–(1.32). Using scale invariance, we introduce

$$\tau = \log\left(\frac{T}{T-t}\right) \quad \text{and} \quad \eta = \frac{r}{(4\pi\Theta^*(T-t)/3)^{1/2}}; \tag{1.37}$$

and write

$$B(\eta,\tau) = (T-t)b(r,t). \tag{1.38}$$

The rectangle D_T transforms into the set

$$\Pi = \{(\eta,\tau) \mid \tau > 0,\ 0 < \eta < (4\pi\Theta^* T/3)^{-1/2}e^{\tau/2}\}. \tag{1.39}$$

The initial-boundary value problem (1.31)–(1.32) now becomes

$$B_\tau + B + \frac{1}{2}\eta B_\eta = B_{\eta\eta} + \frac{4}{\eta}B_\eta + \frac{1}{3}\eta BB_\eta + B^2 \qquad \text{in} \quad \Pi, \tag{1.40a}$$

$$B_\eta(0,\tau) = 0, \quad B\left((4\pi\Theta^* T/3)^{-1/2}e^{\tau/2}, \tau\right) = e^{-\tau}T \qquad \text{for} \quad \tau \in \mathbb{R}^+, \tag{1.40b}$$

$$B(\eta,0) = B_0(\eta) := Tb_0\left(\eta(4\pi\Theta^* T/3)^{1/2}\right), \qquad \text{for} \quad \eta \in \Pi(0) \tag{1.40c}$$

where $\Pi(0) = (0, (4\pi\Theta^* T/3)^{-1/2})$. We introduce for (1.40) the $\omega-$limit set

$$\begin{aligned} \omega \quad = \quad & \{\phi \in L^\infty(\mathbb{R}^+) \mid \exists \tau_j \to \infty \quad \text{such that} \\ & B(\cdot,\tau_j) \to \phi(\cdot) \quad \text{as} \quad \tau_j \to \infty \quad \text{uniformly on compact subsets of} \quad \mathbb{R}\} \end{aligned}$$

Note that a solution of (1.34) is a time independent solution of (1.40). Therefore the study of the blow-up behaviour amounts to analyze the large time behaviour of solutions of of $b(r,t)$ is reduced to (1.40) and in particular stabilization towards the stationary solutions of (1.34).

Now we have the elements to state our main result.

Theorem 1.12 *Let $\Theta^* < \frac{1}{8\pi}$. Let conditions (1.35), and (1.36) hold. Further, let $b(r,t)$ be a solution of problem (1.31)–(1.32) that blows up at $r = 0$ and at $t = T$. If b satisfies the growth condition*

$$b(0,t) \leq M(T-t)^{-1} \quad \textit{for} \quad t \in (0,T) \tag{1.41}$$

with $M > 0$, then

$$\omega \subset \mathcal{S}_1.$$

Remark 1.13 *The growth condition (1.41) has been observed numerically in [21]. There are analitical proofs of this condition for related equations, which we believe can be adapted for this case [72, 77].*

Remark 1.14 *There is numerical evidence [21] that shows that the set $\mathcal{S}_1$ contains only two elements. These elements are the profiles*

$$\varphi^* \equiv 1, \quad \textit{and} \quad \varphi_1(\eta) := \frac{6}{(1+\frac{\eta^2}{2})}.$$

Remark 1.15 *If we can prove that $\omega = \{\varphi\} \subset S_1$, then*

$$\lim_{t\to T}(T-t)b\left(\eta\sqrt{4\pi\Theta^*(T-t)/3}\right) = \varphi(\eta) \tag{1.42}$$

uniformly for $0 \leq \eta \leq C$ for some arbitrary $C > 0$. Numerical results in [21] show that for an open set of initial data, the convergence in (1.42) holds for $\varphi = \varphi_1$. This self-similar behaviour is depicted in Figure 1.4 (right), replacing $n(r,t)$ by $b(r,t)$. In constrast we know of no numerical evidence for the convergence in (1.42) with $\varphi \equiv 1$.

Remark 1.16 *Assumption (1.35) on the initial data gives in terms of n_0*

$$r^3 n_0(r) \leq \frac{3}{4\pi}\|n_0\|_{L^1(B_r(0))}. \tag{1.43}$$

Note that this condition was also used in Theorem 1.7.

The proof of Theorem 1.12 uses the observation that equation (1.40), without the term $\frac{1}{3}\eta B B_\eta$, is the same equation that arises in the study of self-similar blow-up for

$$\bar{b}_t = \Delta_N \bar{b} + \bar{b}^2. \tag{1.44}$$

with $N = 5$. Here Δ_N denotes the Laplacian in $\mathbb{R}^N$, see [48, 49]. Therefore we adapt the methods used for the analysis of (1.44) to prove Theorem 1.12. Note that Theorem 1.12 is very similar to the supercritical case ($N > 6$) for equation (1.44) when two different kinds of self-similar behaviour coexist Matos [60]. They are given by

$$\lim_{t \to T} (T-t)\bar{b}(\eta\sqrt{T-t}, t) = \bar{\varphi}(\eta)$$

uniformly for $|\eta| < C$ for some $C > 0$, with $\bar{\varphi} \equiv 1$ or $\bar{\varphi}$ is such that $\bar{\varphi}(\eta) = O(|\eta|^{-2})$, as $|\eta| \to \infty$.

Non-isothermal case

The blow-up behaviour of the solution $\langle b, \Theta \rangle$ of (1.31),(1.33), was studied by Chavanis et al. in [25]. There it was claimed that $\Theta(t)$ and $b(0,t)$ blow up at the same instant of time $T > 0$. To support this assertion, the authors assumed that $\Theta(t) \sim (T-t)^{-a}$ with $a > 0$, and derived a equation for the corresponding self-similar form. Since $a > 0$ is a-priori unknown, this results in a family of blow-up equations indexed by $a > 0$:

$$\begin{cases} \varphi_{\eta\eta} + \frac{4}{\eta}\varphi_\eta + \frac{1}{3}\eta\varphi\varphi_\eta - \frac{1}{2}(1-a)\eta\varphi_\eta + \varphi^2 - \varphi = 0, & \text{for} \quad \eta > 0, \\ \varphi(0) = \alpha \geq 0 \quad \varphi_\eta(0) = 0. \end{cases} \tag{1.45}$$

Note that (1.34) corresponds to (1.45) with $a = 0$. Guided by numerical evidence, it was argued in [25] that

$$\begin{cases} (T-t)b\left(\eta\sqrt{4\pi(T-t)\Theta(t)/3}, t\right) \to \varphi_a(\eta) \\ \\ \Theta(t)(T-t)^a \to Constant \end{cases} \quad \text{as} \quad t \to T \tag{1.46}$$

where φ_a is a solution of (1.45) with $a \sim 0.1$, which is bounded, decreasing and satisfies $\varphi_a(\eta) = O(\eta^{-2(1+a)})$ for large η. Recently, in [18], it was proved that for $a > 0$ such solutions cannot exist. Therefore the convergence (1.46) cannot hold and so the question of blow-up in this case remaind open.

In Chapter 4 we partly address this issue. As was pointed out to us by J. King, the energy relation (1.33) does not exclude the combination of a singular solution b and a finite temperature Θ. For example in the isothermal case at time of blow-up $b(r,T) = \frac{6}{r^2}$ near $r = 0$, which is unbounded and produces a finite temperature

$$\kappa\Theta(T) = E + \frac{1}{4\pi}\int_0^1 b(r,T)^2 r^4\, dr < \infty$$

This possibility was not addressed in [25].

To conclude we give a numerical result showing a generic blow-up behaviour with bounded temperature and singular density.

In our simulations we set $n_0 \equiv \frac{3}{4\pi}$, that is $b_0(r) \equiv 1$ for $r \in D$. Selecting E and κ so that blow-up occurs, we find that

(i) $\Theta_t > 0$ on $(0, T)$;

(ii) $\Theta(t) \to \bar{\Theta}$ where $\bar{\Theta}$ is a positive finite constant;

(iii) $(T-t)b\left(\eta\sqrt{4\pi\Theta(t)(T-t)/3}, t\right) \sim \varphi_1(y)$ as $t \to T$, where $\varphi_1(\eta) = \frac{6}{(1+\eta^2/2)^2}$.

We illustrate these findings in Figure 1.5, for a particular choice of E and κ.

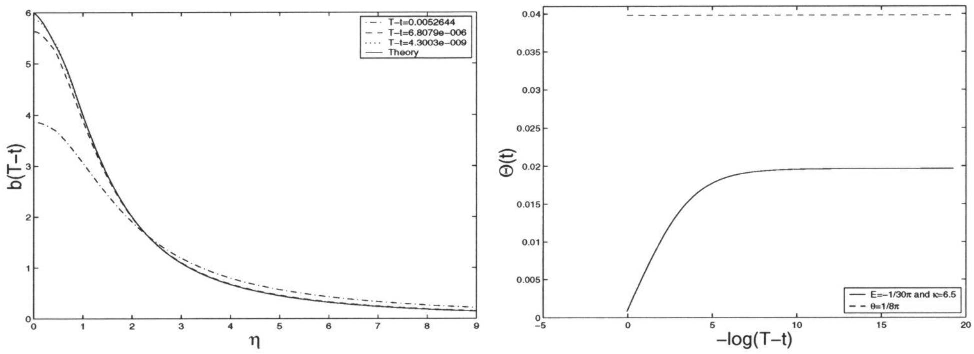

Figure 1.5: Convergence in self-similar variable to the profile φ_1(left) and the associated temperature behaviour(right), $E = -1/30\pi$, $\kappa = 6.5$, and $b_0 \equiv 1$.

1.3 Injection of reactive solutes from a three-dimensional well

Chapter 5 is devoted to the mathematical analysis of a problem arising in reactive solute transport in a homogeneous porous medium. We refer to van Duijn & Knabner [33] and Bear [3, Chapter 10] for the underlying physical assumptions, derivations and related problems.

Let $\epsilon > 0$ and let

$$\Omega_\epsilon = \{x \in \mathbb{R}^3 \colon |x| > \epsilon\}$$

denote the domain occupied by the porous medium. The hole $\{x \colon |x| < \epsilon\}$ represents a well through which liquid (water) is injected in the porous medium. The corresponding groundwater flow is characterized by the water flux vector $\boldsymbol{q}$ and the water content Θ, satisfying the balance equation

$$\partial_t \Theta = -\nabla \boldsymbol{q} \quad \text{for} \quad (x,t) \in \Omega_\epsilon \times \mathbb{R}^+.$$

At $t = 0$ a reactive chemical substance is mixed with the water in the well. Let C_e be the concentration of the injected solute. This solute is transported by the moving groundwater. The total flux of solute is given by the sum of convective flux and diffusive/dispersive flux

$$J = \boldsymbol{q}C - \Theta D \nabla C \quad \text{for} \quad (x,t) \in \Omega_\epsilon \times \mathbb{R}^+,$$

where $C = C(x,t)$ denotes the dissolved solute concentration and $D > 0$ the sum of molecular diffusion and mechanical dispersion. At the boundary $\partial\Omega_\epsilon$ with exterior normal $\vec{\nu}$, the flux J is given by

$$J \cdot \vec{\nu} = \boldsymbol{q} \cdot \vec{\nu} C_e \quad \text{for} \quad (x,t) \in \partial\Omega_\epsilon \times \mathbb{R}^+$$

where $\boldsymbol{q}$ is the prescribed water flux at the well. Assuming that adsorption is the principal reaction mechanism that takes place in the medium, we introduce $S = S(x,t)$ as the adsorbed concentration on $\Omega_\epsilon \times \mathbb{R}^+$. Consequently the mass balance equation reads

$$\partial_t(\Theta C + S) + \nabla \cdot J = 0 \quad \text{for} \quad (x,t) \in \Omega_\epsilon \times \mathbb{R}^+.$$

If the adsorption reaction is fast compared with the water flow, we can assume that the adsorbed concentration is an algebraic expression of the dissolved concentration, that is

$$S = \Psi(C).$$

The function Ψ is called the adsorption isotherm (see for instance [34]). In general Ψ is determined experimentally. Typical examples are

$$\Psi(C) = \frac{k_1 C}{1 + k_2 C}, \quad k_1, k_2 > 0, \qquad \text{(Langmuir isotherm)} \tag{1.47a}$$

or

$$\Psi(C) = kC^p, \quad k > 0, \quad p \in (0,1) \qquad \text{(Freundlich isotherm).} \tag{1.47b}$$

We consider the case of a saturated homogeneous medium, which implies that the water content θ is constant. This means that assuming a uniform normal water flux at $\partial\Omega_\epsilon$, the groundwater velocity $\mathbf{q}$ has the radial form

$$\mathbf{q}(x,t) = \frac{\Lambda(t)}{4\pi|x|^2}\mathbf{e}_r \quad \text{for} \quad (x,t) \in \Omega_\epsilon \times \mathbb{R}^+$$

where Λ represents the injected water rate at the well and $\mathbf{e}_r$ is the unit vector in the radial direction. Here we will consider the case $\Lambda(t) = \Lambda = const$, hence a prescribed constant flow rate at the well.

In a dimensionless setting, we find for the scaled solute concentration $u \colon \Omega_\epsilon \times [0,\infty) \mapsto [0,\infty)$ the following nonlinear initial-boundary value problem:

$$\beta(u)_t + \operatorname{div} F = 0 \quad \text{in} \quad \Omega_\epsilon, \quad t > 0, \tag{1.48a}$$
$$F \cdot \mathbf{e}_r = \boldsymbol{q} \cdot \mathbf{e}_r \quad \text{on} \quad \partial\Omega_\epsilon, \quad t > 0, \tag{1.48b}$$
$$u(\cdot,0) = u_0(\cdot) \quad \text{in} \quad \Omega_\epsilon, \tag{1.48c}$$

where

$$\beta(u) = u + \psi(u), \quad \text{with } \psi \text{ the scaled isotherm,} \tag{1.49}$$

describes the adsorption process. In this introduction we confine ourselves to examples (1.47), that is

$$\psi(u) = \frac{k_1 u}{1 + k_2 u}, \quad k_1, k_2 > 0, \qquad \text{(Langmuir isotherm)} \tag{1.50a}$$

or

$$\psi(u) = ku^p, \quad k > 0, \quad p \in (0,1) \qquad \text{(Freundlich isotherm).} \tag{1.50b}$$

The general case is treated in Chapter 5. The flux in (1.48) is given by

$$F = \boldsymbol{q}\, u - \nabla u$$

with $\boldsymbol{q} = \frac{1}{|x|^2}\mathbf{e}_r$. Note that all the constants are absorbed in ϵ and β.

Further note that equation (1.48a) with ψ from (1.50a) and (1.50b) behaves quite differently. In the case of the Langmuir isotherm $\psi \in C^1([0,\infty))$, implying that equation (1.48a) is uniformly parabolic. If ψ is given by the Freundlich isotherm (1.50b) then $\psi'(s) \to +\infty$ as $s \downarrow 0$. This means that equation (1.48a) is degenerate parabolic.

In a two-dimensional setting, Problem (1.48) was studied in [33] and [35]. In [33] the authors derived a radially symmetric self-similar solution of equation (1.48a) of the form $u(r,t) = f(r/\sqrt{t})$. This solution is defined on all $\mathbb{R}^2$ but does not satisfy boundary condition (1.48b). In [35] it was demonstrated that this solution describes the large-time behaviour for general two-dimensional radially symmetric solutions of (1.48a)–(1.48b) and rates of convergence were given.

The existence of self-similar solutions in two dimensions requires the well injection rate to be constant in time. In three spatial dimensions, self-similar solutions still exist but require the injection rate be such that $\Lambda(t) = \sqrt{t}$. From a practical point of view this is an unsatisfactory setup. Here we investigate the large-time behaviour of solutions under a constant injection rate.

Two natural questions arise from Problem (1.48): the behaviour as $\epsilon \downarrow 0$ and the behaviour as $t \to \infty$.

We first consider $\epsilon \downarrow 0$. Taking the formal limit in the combination (1.48a)–(1.48b) yields the equation

$$\beta(u)_t + \operatorname{div}(F) = \delta_{x=0} \quad \text{in} \quad \mathbb{R}^3, \quad t > 0 \tag{1.51a}$$

where $\delta_{x=0}$ denotes the Dirac distribution at the origin. Thus the boundary condition at the well appears as a source term in the equation. We complement equation (1.51a) with the initial condition

$$u(\cdot, 0) = u_0(\cdot) \qquad \text{in} \quad \mathbb{R}^3. \tag{1.51b}$$

Regarding the initial conditions (1.48c) and (1.51b), we take (1.48c) as the restriction of (1.51b) to Ω_ϵ, and assume

$$(H_{u_0}) \quad u_0 \in L^\infty(\mathbb{R}^3);\ u_0 \geq 0 \text{ in } \mathbb{R}^3;\ \lim_{|x|\to\infty} u_0(x) = 0;\ \int_{\mathbb{R}^3} \beta(u_0)\,dx < \infty.$$

Note that we allow non-radial initial data.

Our first theorem makes the stabilization as $\epsilon \downarrow 0$ precise.

Theorem 1.17 *Let* (H_{u_0}) *and* (1.49) − −(1.50) *be satisfied. Further, let* u^ϵ *be the unique weak solution of (1.48). Then*

$$u^\epsilon \to u \quad \textit{as} \quad \epsilon \to 0, \quad \textit{uniformly in compact subsets of } (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^+,$$

where u *is a weak solution of (1.51).*

The definition of weak solutions as well as the proof of Theorem 1.17 are given in Chapter 5. Weak solutions are introduced in this context to allow for solutions with non smooth behaviour at points that separates the regions $\{u = 0\}$ and $\{u > 0\}$.

Next we consider the large-time behaviour. We expect that different small well radii (ϵ) lead to the same large-time behaviour. This was shown rigorously in [35] for the two-dimensional case. With this in mind we consider only the large-time behaviour of (1.51) and for technical reasons we limit ourselves to radially symmetric solutions. Before we state the convergence result, we provide some motivation.

The radial form of equation (1.51a) is:

$$\beta(u)_t + \frac{1-2r}{r^2}u_r - u_{rr} = 0 \quad \text{in} \quad 0 < r < \infty, \quad t > 0, \tag{1.52}$$

and, as shown in Proposition 5.15 Chapter 5, its solutions satisfy the boundary condition

$$u(0,t) = 1 \quad \text{for all} \quad t > 0. \tag{1.53}$$

Equation (1.52) admits a nontrivial stationary solution $w = w(r)$, satisfying $w(0) = 1$ and $w(\infty) = 0$. It is given by

$$w(r) = 1 - e^{-1/r}, \tag{1.54}$$

and under the conditions of Theorem 1.18 below the solution u converges to this stationary state.

The appearance of (1.54) is quite different from the two-dimensional case. There the only bounded stationary solution satisfying $w(0) = 1$ is the constant state $w \equiv 1$. In [35] it was shown that the solution attains this state in a self similar way, namely

$$u(r,t) \sim f(r/\sqrt{t}) \quad \text{as} \quad t \to \infty$$

where $f(0) = 1$.

Here we assume an analogous behaviour with respect to (1.54), i.e.

$$\frac{u(r,t)}{w(r)} \sim f(r/t^\alpha) \quad \text{as} \quad t \to \infty \tag{1.55}$$

for some $\alpha > 0$, where $f(0) = 1$. To this end we set

$$\tilde{z}(r,t) := \frac{u(r.t)}{w(r)}$$

and introduce the coordinate transformation

$$\eta = r/t^\alpha, \qquad \tau = \log t.$$

Then $z(\eta, \tau) = \tilde{z}(r,t)$ satisfies:

$$e^{(2\alpha-1)\tau}\beta'(zw)z_\tau - \alpha e^{(2\alpha-1)\tau}\eta\beta'(zw)z_\eta - z_{\eta\eta} + \frac{1}{\eta}A\left(\frac{1}{\eta e^{\alpha\tau}}\right)z_\eta = 0, \tag{1.56}$$

where $A(s) := \dfrac{2s}{e^s - 1} + s - 2$ with $\lim_{s\to 0} A(s) = 0$.

To find the appropriate balance in (1.56), we observe that for fixed $\eta > 0$, $\tau \to \infty$ implies $r \to \infty$. Since $u(r,t) \to 0$ as $r \to \infty$, the behaviour of β near 0 is critical. Let us therefore assume

$$\beta(s) \sim s^p \quad (0 < p \le 1) \quad \text{as } s \downarrow 0. \tag{1.57}$$

Using this and $w(r) \to 1/r$, as $r \to \infty$, we find that the second and third term in (1.56) balance if and only if $\alpha = 1/(3-p)$.

The resulting equation is

$$\alpha\eta^{3-p}(f^p)_\eta + (\eta f_\eta - f)_\eta = 0 \text{ for } 0 < \eta < \infty, \tag{1.58}$$

where $f(\eta) := \lim_{\tau\to\infty} z(\eta,\tau)$.

Below we use the notation $[\cdot]_+ := \max\{\cdot, 0\}$, $\varphi_+ := [\varphi]_+$, and $\varphi_- := [-\varphi]_+$.

Theorem 1.18 *Let hypotheses (1.57) and* (H_{u_0}) *be satisfied, and let* u *be a weak solution of Problem* P. *Then we have the following estimates:*

$$0 \le e^{p\alpha\tau}\int_0^\infty [u^p - f^p w^p]_+ \eta^2\, d\eta \le L_1 e^{-\alpha\tau} + L\|\varphi_-\|_{L^\infty} e^{-\alpha\gamma\tau} \tag{1.59}$$

for all $\tau \in \mathbb{R}$, *and*

$$0 \le e^{p\alpha\tau}\int_0^\infty [f^p w^p - u^p]_+ \eta^2\, d\eta \le L_2 e^{-\alpha\tau} + L\|\varphi_+\|_{L^\infty} e^{-\alpha\gamma\tau} \tag{1.60}$$

for all $\tau \in \mathbb{R}$. Here L_1, L_2, and L are positive constants and $\alpha = 1/(3-p)$.

The function f is the unique solution of

$$\text{(S)} \quad \begin{cases} \alpha\eta^{2-p}\beta_p(f)_\eta + f_{\eta\eta} = 0 & \text{for } 0 < \eta < \infty, \\ f(0) = 1, \quad f(\infty) = 0. \end{cases}$$

Remark 1.19 *The mass of the system increases linearly in time. The scaling used in (1.59) (and (1.60)) is chosen to normalize the increase of mass:*

$$\frac{1}{t}\int_0^\infty [u^p - f^p w^p]_+ r^2\, dr = e^{p\alpha\tau}\int_0^\infty [u^p - f^p w^p]_+ \eta^2\, d\eta.$$

In this scaled metric the solutions u and fw converge. In the unscaled (original) metric the distance increases without bound.

Figure 1.6 shows the limit function $r \mapsto w(r)f(r/\sqrt{t})$ for different t, in the case $p = 1$.

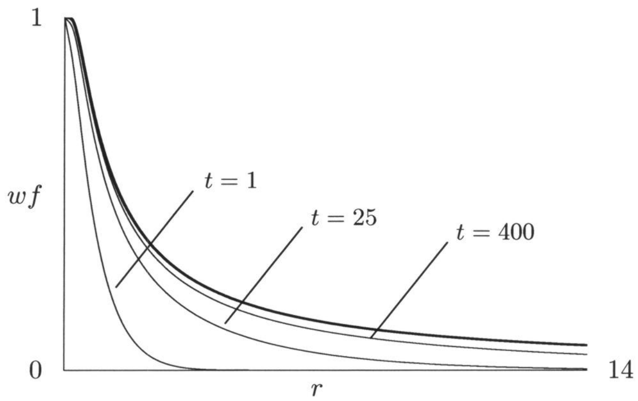

Figure 1.6: The function $r \mapsto w(r)f(r/\sqrt{t})$; $t = 1, 25, 400$.

1.4 A quasilinear elliptic system

In Chapter 6, we study the existence of positive radial solutions of a quasilinear elliptic system. Physical assumptions and model derivations for this type of equations can be found in DiBenedetto [28] and Diaz [29].

We motivate our main result by observing the following. Let Ω be a ball of radius $R > 0$ and let u, v be a solution of the weakly coupled system

$$\begin{aligned} -\Delta u &= v^p \quad \text{in } \Omega \\ -\Delta v &= u^q \quad \text{in } \Omega \\ u = v &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.61}$$

Solutions of this problem are necessarily radially symmetric. The existence of bounded radial positive solutions was studied by Peletier & van der Vorst in [67]. Their result asserts that for $N > 3$

(i) if $p, q \geq 1$ are such that

$$\frac{1}{p+1} + \frac{1}{1+q} > \frac{N-2}{N} \tag{1.62}$$

then (1.61) has a unique positive solution;

(ii) if $p, q \geq 1$ are such that

$$\frac{1}{p+1} + \frac{1}{1+q} \leq \frac{N-2}{N} \tag{1.63}$$

then (1.61) has no (positive) solution.

Our aim is to obtain a result of the type (i), for a nonlinear gradient dependent diffusion, that is, find conditions for existence of bounded positive radial solutions of a system of the form

$$\begin{cases} \operatorname{div}(a_i(|\nabla u_i|)\nabla u_i) + f_i(u_{i+1}(|x|)) = 0, & \text{for} \quad x \in \Omega \\ & \qquad\quad i = 1, \dots, n, \\ u_i(|x|) = 0, & \text{for} \quad x \in \partial\Omega, \end{cases} \tag{1.64}$$

where it is understood that $u_{n+1} = u_1$. Here for $i = 1, \dots, n$, the functions $\phi_i := s a_i(|s|)$, $s \in \mathbb{R}$, are odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, and the functions $f_i : \mathbb{R} \mapsto \mathbb{R}$ are odd, continuous, and such that $s f_i(s) > 0$ for $s \neq 0$.

Furthermore, concerning the functions ϕ_i, f_i, $i = 1, \dots, n$, we will assume that they belong to the class of asymptotically homogeneous functions (AH for short). We say that $h : \mathbb{R} \mapsto \mathbb{R}$ is AH at $+\infty$ of exponent $\delta > 0$ if for any $\sigma > 0$

$$\lim_{s \to +\infty} \frac{h(\sigma s)}{h(s)} = \sigma^{\delta}. \tag{1.65}$$

By replacing $+\infty$ by 0 in (1.65), we obtain a similar equivalent definition for a function h to be AH of exponent δ at zero.

In [27], the existence of solutions with positive components for a system of the form (1.64) with $n = 2$ and with the functions ϕ_i and f_i having the particular form $\phi_i(s) = |s|^{p_i - 2}s$, $\phi_i(0) = 0$, $p_i > 1$, $f_i(s) = |s|^{\delta_i - 1}s$, $f_i(0) = 0$, $\delta_i > 0$, $i = 1, 2$, was done. In [44], within the scope of the AH functions, the case of a single equation was considered. In both situations the central idea to obtain a-priori bounds was the blow-up method of Gidas & Spruck [45].

Next we develop some preliminaries in order to state our main theorem. For $i = 1, \dots, n$, let $\delta_i, \bar{\delta}_i$ be positive real numbers and $p_i, \bar{p}_i$ real numbers greater than one, and assume that the functions ϕ_i, f_i, $i = 1, \dots, n$ satisfy

$$(H_1) \qquad \lim_{s \to +\infty} \frac{\phi_i(\sigma s)}{\phi_i(s)} = \sigma^{p_i - 1}, \quad \lim_{s \to +\infty} \frac{f_i(\sigma s)}{f_i(s)} = \sigma^{\delta_i},$$

for all $\sigma > 0$,

$$(H_2) \qquad \prod_{i=1}^{n} \frac{\delta_i}{(p_i - 1)} > 1.$$

To the exponents p_i, δ_i, let us associate the system

$$(AS) \quad \begin{cases} (p_i - 1)E_i - \delta_i E_{i+1} = -p_i, & i = 1, \dots, n, \\ E_{n+1} = E_1. \end{cases}$$

From (H_2), it turns out that (AS) has a unique solution $(E_1, \dots, E_N)$, such that $E_i > 0$ for each $i = 1, \dots, n$. An explicit form for these solutions is given in the Appendix at the end of the Chapter 6.

Now we can establish our main existence theorem.

Theorem 1.20 *For $i = 1, \dots, n$, let ϕ_i be odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and $f_i : \mathbb{R} \mapsto \mathbb{R}$ odd continuous functions with $xf_i(x) > 0$ for $x \neq 0$, which satisfy (H_1), (H_2), and*

$$(H_3) \qquad \lim_{s \to 0} \frac{\phi_i(\sigma s)}{\phi_i(s)} = \sigma^{\bar{p}_i - 1}, \quad \lim_{s \to 0} \frac{f_i(\sigma s)}{f_i(s)} = \sigma^{\bar{\delta}_i},$$

for any $\sigma > 0$. Additionally, for $i = 1, \dots, n$, let us assume that

$$(H_4) \qquad \prod_{i=1}^{n} \frac{\bar{\delta}_i}{(\bar{p}_i - 1)} > 1,$$

$$(H_5) \qquad p_i < N, \ i = 1, \dots, n, \qquad \max_{i=1,\dots,n} \{E_i - \theta_i\} \geq 0,$$

where $\theta_i = \frac{N - p_i}{p_i - 1}$ and the E_i's are the solutions to (AS). Then problem (1.64) has a solution $(u_1, \dots, u_n)$ such that $u_i(r) > 0$, $r \in [0, R)$, for each $i = 1, \dots, n$.

Note that this theorem applied to Problem (1.61), implies that the exponents satisfy

$$1 < pq < 1 + \frac{2}{N-2}(\max\{p, q\} + 1). \tag{1.66}$$

Following the proof of Theorem 1.20, we find that condition (1.66) appears as a result of a non-existence of ground states for the system

$$\begin{aligned} -\Delta u &= v^p && \text{in } \mathbb{R}^N, \\ -\Delta v &= u^q && \text{in } \mathbb{R}^N, \\ u \geq 0 \quad v &\geq 0 && \text{in } \mathbb{R}^N. \end{aligned} \tag{1.67}$$

However in [73], condition (1.66) was improved for this particular system. They proved non existence of ground states for (1.67) provided (1.62) holds. Clearly this assumption is sharper than (1.66).

Having this in mind, as a corollary of Theorem 1.20, we can proof a result of the type

Theorem 1.21 *Let $f,\ g : \mathbb{R} \mapsto \mathbb{R}$ be odd continuous functions such that f is AH at $+\infty$ of exponent $\delta > 0$ and AH at 0 of exponent $\overline{\delta} > 0$, g is AH at $+\infty$ of exponent $\mu > 0$ and AH at 0 of exponent $\overline{\mu} > 0$ with $\mu\delta > 1$. Let also $\overline{p},\ \overline{q} > -1$ be such that $\overline{\delta}\overline{\mu} > (\overline{p}+1)(\overline{q}+1)$. Then, if*

$$\frac{N}{\delta+1} + \frac{N}{\mu+1} > N-2, \tag{1.68}$$

the problem

$$(DL) \quad \begin{cases} -\mathrm{div}((\log(1+|\nabla u|))^{\overline{p}}\nabla u) &= f(v),\ x \in \Omega \\ -\mathrm{div}((\log(1+|\nabla u|))^{\overline{q}}\nabla v) &= g(u),\ x \in \Omega \\ u(x) = v(x) &= 0 \quad x \in \partial\Omega, \end{cases}$$

has a non trivial radially symmetric solution (u, v) such that $u(x) > 0$ and $v(x) > 0$ for all $x \in \Omega$.

Apart from Theorem 1.21, we can also apply our existence result, Theorem 1.20, to a system that contains operators of the form $(-\Delta_p)^n$, $(-\Delta_q)^m$, where for $t > 1$ $\Delta_t u := \mathrm{div}(|\nabla u|^{t-2}\nabla u)$.

Finally, concerning extensions for the system (1.61) to evolution parabolic problems, we find that little is known. To illustrate this type of results, we consider the system

$$\begin{aligned} u_t - \Delta u = v^p \quad &\text{in} \quad \Omega \times (0,T), \\ v_t - \Delta v = u^q \quad &\text{in} \quad \Omega \times (0,T), \\ u(0) = u_0 \geq 0 \quad v(0) = v_0 \geq 0 \quad &\text{in} \quad \Omega. \end{aligned} \tag{1.69}$$

It was shown by Escobedo & Herrero in [36] that if $pq \leq 1$ then (1.69) has global existence. If $pq > 1$ then blow-up and global existence coexists. Global existence is ensured provided that the initial data is not so large.

Chapter 2

Global existence conditions for a non-local problem arising in statistical mechanics

2.1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set satisfying $\sup_{x\in\Omega}|x| = 1$. In Ω we consider the parabolic -elliptic system

$$
\begin{aligned}
n_t &= \operatorname{div}\{\Theta(t)\nabla n + n\nabla\phi\} \quad \text{in} \quad \Omega\times\mathbb{R}^+, &(2.1)\\
\Delta\phi &= n \quad \text{in} \quad \Omega\times\mathbb{R}^+, &(2.2)
\end{aligned}
$$

combined with the energy relation

$$
E = \kappa\Theta(t) + \int_\Omega n\phi dx \quad \text{in} \quad \mathbb{R}^+, \tag{2.3}
$$

where $E \in \mathbb{R}$ and $\kappa > 0$ are given parameters. At the boundary $\partial\Omega \in C^{1+\alpha}\,(\alpha > 0)$ we prescribe

$$
\begin{aligned}
(\Theta(t)\nabla n + n\nabla\phi)\cdot\vec{\nu} = 0 \quad &\text{on} \quad \partial\Omega\times\mathbb{R}^+, &(2.4)\\
\phi = 0 \quad &\text{in} \quad \partial\Omega\times\mathbb{R}^+, &(2.5)
\end{aligned}
$$

This chapter has been submitted to Advances in Differential Equations [32] where $\vec{\nu}$ denotes the exterior normal vector on $\partial\Omega$. At $t = 0$ we have the initial condition

$$n(x,0) = n_0(x) \quad \text{in} \quad \Omega, \tag{2.6}$$

satisfying

$$\int_\Omega n_0 \, dx = 1, \quad \text{and} \quad n_0(x) \geq 0 \quad \text{in} \quad \Omega. \tag{2.7}$$

This set of equations defines Problem **P** for the unknowns n, ϕ and Θ. The underlying model is discussed in Section 2, as well as some known properties of the system.

The purpose of this chapter is to demonstrate local existence for Problem **P** and to give sufficient conditions on E, κ, and n_0 for global existence. Local existence is shown in Section 3. The proof uses a Schauder fixed-point theorem and a careful construction of an invariant set to avoid degenerate diffusion in (2.1). It requires $n_0 \in L^p(\Omega)$ for $p > \frac{3}{2}$, implying that $n \in L^\infty_{loc}((0,T]; L^\infty(\Omega))$ for some $T \in (0,\infty)$. Hence we can allow for certain singular initial data which result in solutions that are locally bounded in $(0,T]$. Let

$$T^* = \sup\{\, T > 0 \mid \text{Problem } \mathbf{P} \text{ has a solution in } (0,T] \,\}.$$

If $T^* < \infty$ we have at least $\lim_{t \to T^*} \|n(t)\|_{L^q(\Omega)} = \infty$ for each $q > \frac{3}{2}$ and if $T^* = \infty$, the solution is defined globally. For Problem **P** the optimal $L^p(\Omega)$ space seems to be $p = \frac{3}{2}$, since there exists a singular stationary solution in the radial case belonging to $L^q(\Omega) \setminus L^{3/2}(\Omega)$, with $q < \frac{3}{2}$. This solution is given in Section 2.2.3. Uniqueness is proven for $n_0 \in L^p(\Omega)$ with $p \geq 2$.

Problem **P** was recently studied in [69]: local existence and uniqueness were obtained for $p > 3$. Although the result in [69] is proved only for $\kappa = 3$, the method seems applicable to any $\kappa > 0$.

In Section 2.4 we consider an auxiliary problem in which we drop the energy relation (2.3) and treat $\Theta(t)$ as a given function. This provides insight and bounds which we need in order to prove our main result about global existence. In Section 2.5, we first give the following result about blow-up:

Theorem 2.1 *Let $\Omega = B_1(0)$ be the unit ball in $\mathbb{R}^3$. If $\kappa > 6$ and $E < \frac{1}{4\pi}$, then $T^* < \infty$.*

Unfortunately the proof of Theorem 2.1 does not give any insight into the structure of the blow-up. This issue will be considered in a future publication.

Before stating the global existence result we note from (2.3) at $t = 0$, that instead of prescribing E and n_0 one could equivalently prescribe $\Theta_0 := \Theta(0)$ and n_0. In fact it seems more natural to consider Θ_0 and n_0 as initial values. In view of the physical interpretation of the model we consider $\Theta_0 > 0$. With this in mind we have

Theorem 2.2 *Let $\Omega = B_1(0)$ and assume that solutions of Problem* **P** *are radially symmetric. If the pair $\langle n_0, \Theta_0\rangle$ satisfies one of the following conditions*

(i) *$n_0 \in L^\infty(\Omega)$ and Θ_0 is sufficiently large;*

(ii) *$n_0 \log n_0 \in L^1(\Omega)$, and there exists $B > 0$, such that*

$$\|n_0\|_{L^1(B_r(0))} \le (1+B)\frac{r^3}{r^2+B} \quad \text{for} \quad r \in [0,1],$$

and

$$\Theta_0 \ge \frac{1}{8\pi}\frac{(1+B)(1+3B)}{(1+5B)} \exp\left(\frac{2}{\kappa}\left[\int_\Omega n_0 \log n_0\, dx - \log\left(\frac{3}{4\pi}\right)\right]\right);$$

(iii) *$n_0 \equiv \frac{3}{4\pi}$, and $\Theta_0 \ge \frac{1}{8\pi}\gamma$, with $\gamma = 0.9519\ldots$,*

then $T^* = \infty$ (global existence).

Remark 2.3 (i) *Due to the parabolic regularity, $n_0 \in L^\infty(\Omega)$ is not so restrictive.*

(ii) *Condition (iii) is an special case of (ii) and $\gamma = \min\limits_{B>0} \frac{(1+B)(1+3B)}{1+5B} = 0.9519\ldots.$*

(iii) *The condition on n_0 in (ii) implies a bound on the Morrey norm of exponent $3/2$, since $\|n_0\|_{M^{3/2}(\Omega)} = \sup\limits_{x\in\mathbb{R}^3,\, 0\le r\le 1} r^{-1}\|n_0\|_{L^1(\Omega\cap B_r(x))}$. In [9], the space $M^{3/2}(\Omega)$ was suggested as the natural space to prove existence.*

The proof of Theorem 2.2 contains two essential steps. To extend the local solution we first need a uniform bound from below on Θ. To achieve this we use a Lyapunov functional associated with Problem **P**, the so-called Boltzmann entropy (2.15). This functional provides a uniform lower bound on Θ, which only depends on the initial data and κ. If Θ_0 is positive, then Θ remains positive in the whole existence interval, including the blow-up time.

In the second step we construct a control on n. Here we use the radial symmetry which allows us to transform equations (2.1) and (2.2) into a single equation, still containing Θ as unknown. It has the crucial property that an ordered pair of given Θ's results in an ordered pair of solutions. As a comparison function we now use the solution of (2.1)-(2.2) with a suitably chosen fixed Θ. Under certain hypotheses this auxiliary problem has a global solution which provides the control on n. The different conditions in Theorem 2.2 are closely related to global existence conditions for the auxiliary problem.

2.2 Preliminaries

2.2.1 Model issues

Problem **P** was first derived for collisionless systems such as galaxies. The underlying argument is that rapid fluctuations of the gravitational field during the early stage of *violent relaxation* plays the same role as collisions, although the time scales involved for collisionless systems are smaller than for collisional systems (Brownian motion). The process of violent relaxation is considered in [26], and further results and interpretations can be found in [25].

Problem **P** also describes the evolution of density and temperature of a self-attracting cluster of Brownian particles in a bounded three-dimensional region. During the evolution mass and energy are conserved. A detailed derivation and discussion on the physical assumptions can be found in [10, 13, 79] and the references therein. Below we present a brief summary.

Suppose a cluster of particles is contained in a bounded region $\Omega \subset \mathbb{R}^3$. The spatial particle density n satisfies the mass balance equation

$$n_t = \operatorname{div}\Big\{\frac{1}{\beta}(k\Theta\nabla n + n\nabla\phi)\Big\} \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{2.8}$$

where $\beta > 0$ is the friction coefficient, k the Boltzmann constant and Θ the temperature of the system. To ensure that the cluster of particles preserves mass we impose zero mass flux along the boundary: i.e.

$$(k\Theta\nabla n + n\nabla\phi)\cdot\vec{\nu} = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+. \tag{2.9}$$

This implies

$$\int_\Omega n(x,t)\,dx = constant = M \quad \text{for all} \quad t > 0,$$

where M is the total particle mass of the system, specified by the initial condition.

The function ϕ in (2.8) is the gravitational potential. It satisfies

$$\Delta\phi = 4\pi G n \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{2.10}$$

with

$$\phi = -\frac{GM}{R} \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+. \tag{2.11}$$

Here G is the gravitational constant and $R := \max_{x\in\Omega}|x|$. Note that we have chosen as boundary condition the gravitational potential of a mass M centered at the origin of a ball of radius R, as was introduced in [25]. An alternative way to define the potential, physically more relevant, is to consider the convolution of the fundamental solution of the Laplacian in $\mathbb{R}^3$ with the density, see [15]. Both definitions coincide when the domain is a ball. For general domains the Dirichlet condition (2.11) is somewhat artificial, but we hope that it will provide insight for the study of more realistic boundary conditions, see discussion in [10].

In general the temperature varies in space and time. It satisfies an energy balance equation containing thermal diffusion, heat convection and a term due to gravitational effects [13, Eq. (1.4)]. This results in the so-called Streater model. However, the integrated energy balance does not contain the thermal diffusivity [13, Eq. (2.1)]. Furthermore we expect that a large thermal diffusivity will result in a temperature which is nearly constant in space. Taking this limit in the integrated energy balance, one finds

$$E = \frac{\kappa M}{2}\Theta(t) + \frac{1}{2}\int_{\Omega} n\phi dx \quad \text{in} \quad \mathbb{R}^+, \tag{2.12}$$

where E denotes the total energy of the system and κ the specific heat of the particles. If the cluster resembles an ideal gas we have $\kappa = 3k$.

Regarding the initial data for the system (2.8)-(2.12) there are two ways to proceed. If the energy E is given, it suffices to specify only the initial density

$$n(x,0) = n_0(x) \geq 0 \quad \text{for} \quad x \in \Omega. \tag{2.13}$$

Equivalently we can specify both initial density and temperature

$$\Theta(0) = \Theta_0 > 0.$$

Now E is fixed by (2.12) at $t = 0$.

If the temperature is constant in time as well we drop the energy balance (2.12) and obtain the isothermal model. This model also arises in the context of polytropic stars and the biological phenomena of chemotaxis. The corresponding mathematical problem has received considerable attention in the past years because of its rich structure. Blow-up in the form of singular solutions and gravitational collapse can occur, as well as global existence. To our knowledge there is no full description of these phenomena in $\mathbb{R}^3$. The reason is that in contrast to the two dimensional case, global existence in $\mathbb{R}^3$ not only depends on the parameters of the problem, but also on the shape of the initial density profile. A detailed discussion and references are given in [21], [50], and [51]. The isothermal model, however, plays a crucial role in the analysis presented in this paper.

Since Problem **P** has an additional equation, one expects that conservation of energy will act as a selection principle to favor global existence. This has been demonstrated in [78] for the two-dimensional case: the energy balance implies that temperature increases whenever density concentrates near a point. This in turn has a smoothing effect (through (2.8)) on the density profile, preventing blow-up from happening. Theorem 2.1 tells us that this general observation is not true in $\mathbb{R}^3$.

2.2.2 Non-dimensionalization

We put equations (2.8)-(2.12) in dimensionless form by setting

$$\tilde{x} = \frac{1}{R}x, \quad \tilde{n} = \frac{R^3}{M}n, \quad \tilde{\phi} = \frac{R}{4\pi GM}(\phi + \frac{GM}{R}) \tag{2.14}$$

and

$$\tilde{\Theta} = \frac{kR}{4\pi GM}\Theta, \quad \tilde{t} = \frac{4\pi GM}{\beta R^3}t.$$

Introducing $\tilde{E} = \frac{R}{2\pi GM^2}\left(E + \frac{1}{2}\frac{GM^2}{R}\right)$ and $\tilde{\kappa} = \frac{1}{k}\kappa$, and dropping the tildes, results in Problem **P**.

2.2.3 Lyapunov functional and stationary solutions

If a triple $\langle n, \phi, \Theta\rangle$ solves Problem **P**, then it is easy to check that

$$W(t) = \int_\Omega n \log n \, dx - \frac{\kappa}{2}\log\left(E - \int_\Omega n\phi \, dx\right) \quad \text{on} \quad \mathbb{R}^+ \tag{2.15}$$

satisfies

$$\frac{d}{dt}W(t) = -\int_\Omega \frac{|\Theta(t)\nabla n + n\nabla\phi|^2}{\Theta(t)n}dx, \quad \text{for all} \quad t > 0. \tag{2.16}$$

Hence W is a Lyapunov functional for Problem **P**, sometimes called the Boltzmann entropy, see [69] and [26] for the original reference.

One consequence of (2.16) is the following. Let $\langle n_s, \phi_s, \Theta_s\rangle$ denote a stationary solution of Problem **P**. Then (2.16) implies

$$\Theta_s \nabla n_s + n_s \nabla \phi_s \equiv 0 \quad \text{in} \quad \Omega.$$

Introducing the scaled potential $\psi := \dfrac{\phi_s}{\Theta_s}$ we observe that $n_s \equiv \dfrac{e^{-\psi}}{\int_\Omega e^{-\psi}\, dx}$, where ψ satisfies

$$(S) \quad \begin{cases} \Delta\psi = \dfrac{1}{\Theta_s}\dfrac{e^{-\psi}}{\int_\Omega e^{-\psi}\, dx} & \text{in} \quad \Omega, \\ \psi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The corresponding energy relation takes the form

$$E\frac{1}{(\Theta_s)^2} = \kappa\frac{1}{\Theta_s} - \int_\Omega |\nabla\psi|^2 dx. \tag{2.17}$$

Problem **S** has only one singular radially symmetric solution [19], the Chandrasekhar solution

$$n_s = U := \frac{1}{4\pi}\frac{1}{|x|^2}, \tag{2.18}$$

provided $\Theta_s = \frac{1}{8\pi}$. It is satisfies (2.17) for $E = \frac{(\kappa-2)}{8\pi}$. Observe that $U \in L^q(\Omega) \setminus L^{3/2}(\Omega)$, with $q < \frac{3}{2}$. If this solution is attained by Problem **P** for $t \uparrow T^* < \infty$, we have a blow-up without concentration of mass at the origin.

For completeness we recall a result [10, Proposition 5.6.] for bounded radially symmetric solutions of Problem **S** and (2.17).

Theorem 2.4 *Let $\Omega = B_1(0)$. For any $\kappa > 0$, there exists $E_\kappa \in \mathbb{R}$ such that:*

(i) *If $E > E_\kappa$ there exist bounded negative solutions;*

(ii) *If $E < E_\kappa$ there are no nontrivial bounded negative solutions.*

This observation is originally due to Antonov [1] as a result of a computational approach. He also showed that stationary solutions are local maxima or saddle points of an entropy and there is no global entropy maximum.

Theorem 2.4 is still open for general domains [10]. This is related to the non-trivial nature of the set of singular solutions [19].

2.2.4 Radially symmetric solutions

Our main theorem about global existence is stated in terms of radially symmetric solutions. Radial symmetry in Problem **P** not only reduces the spatial dimension, it also allows us to combine equations (2.1) and (2.2) into a single equation for the accumulated mass

$$Q(r,t) := \int_{B_r(0)} n(x,t)dx \quad \text{for} \quad r \in (0,1] \quad \text{and} \quad t \in \mathbb{R}^+.$$

This is shown in [12]. Introducing $t := \frac{3}{4\pi}t$ and $\vartheta := 12\pi\Theta$, we obtain in terms of $Q(y,t) := Q(r,t)$, with $y = r^3$, the equation

$$Q_t = y^{4/3}\vartheta(t)Q_{yy} + QQ_y \quad \text{for} \quad y \in (0,1) \quad \text{and} \quad t \in \mathbb{R}^+. \tag{2.19}$$

To transform the energy relation (2.3), we first note that (2.2) and (2.5) give $\int n\phi dx = -\int |\nabla\phi|^2 dx$. Further, radial symmetry and (2.2) imply $4\pi r^2 \partial_r \phi = Q(r,t)$. Finally we introduce $\mathcal{E} := 12\pi E$, to get in terms of $Q(y,t)$

$$\mathcal{E} = \kappa\vartheta(t) - \int_0^1 \frac{Q^2}{y^{4/3}} dy \quad \text{for} \quad t \in \mathbb{R}^+. \tag{2.20}$$

The boundary conditions for Q are

$$Q(0,t) = 0, \quad Q(1,t) = 1, \quad \text{for} \quad t \in \mathbb{R}^+, \tag{2.21}$$

and the initial condition becomes

$$Q(y,0) = Q_0(y) := \frac{4\pi}{3} \int_0^y n_0(y^{1/3}) dy \quad \text{for} \quad 0 \leq y \leq 1. \tag{2.22}$$

Equations (2.19)-(2.22) define Problem **Q**.

Note that

$$\vartheta(t) = constant = \frac{3}{2} \quad \text{and} \quad Q(y,t) = y^{1/3}$$

satisfy equation (2.19) and boundary conditions (2.21). The energy relation (2.20) is satisfied for $\mathcal{E} = \frac{3}{2}(\kappa - 2)$. This is the transformed Chandrasekhar solution (2.18).

2.3 Well-posedness for Problem P

Before we give a formal solution definition for Problem **P** we observe that ϕ is known in terms of n by the boundary value problem (2.2) and (2.5). Therefore we denote a solution by $\langle n, \Theta\rangle$ instead of the triple $\langle n, \phi, \Theta\rangle$.

We call $\langle n, \Theta\rangle$ a weak solution of Problem **P** if for some $T > 0$:

(i) $n \in L^2(0,T; H^1(\Omega))$ and $n_t \in L^2(0,T; (H^1(\Omega))')$;

(ii) $\Theta \in C([0,T])$ and $\Theta(t) > 0$ for $t \in [0,T]$;

(iii) the triple $\langle n, \phi, \Theta\rangle$, where $\phi \in C([0,T]; H_0^1(\Omega))$ solves the boundary value problem (2.2) and (2.5), satisfies (2.1) in the weak sense and (2.3) for all $t \in [0,T]$;

(iv) $n(\cdot, 0) = n_0 \geq 0$ a.e in Ω.

Remark 2.5 *The regularity in (i) implies $n \in C([0,T]; L^2(\Omega))$* [80, p. 260]. *Therefore Θ and ϕ are continuous in time in the sense of (ii) and (iii) respectively and the initial value of n can be prescribed.*

2.3.1 Local existence

Let $R_T := \Omega \times (0,T]$ for arbitrarily chosen $T > 0$.

The first result asserts local existence for Problem **P**.

Theorem 2.6 *Let $E \in \mathbb{R}$, $\kappa > 0$, and let $n_0 \in L^2(\Omega)$ be such that $\Theta(0) = \Theta_0 > 0$. Then there exists a weak solution $\langle n, \Theta\rangle$ of Problem* **P** *with $T = T(\|n_0\|_{L^2(\Omega)}, \Omega, \Theta_0) > 0$. It satisfies $n \geq 0$ in R_T and $n \in L^\infty_{loc}((0,T]; L^\infty(\Omega))$.*

Proof: The proof uses a Schauder fixed-point theorem [81, Corollary 9.7]. For any fixed $T > 0$, let

$$X = \{v \in L^2(0,T; H^1(\Omega)) \quad \text{with} \quad v_t \in L^2(0,T; (H^1(\Omega))')\}$$

and let $F\colon X \to C([0,T])$ be defined by

$$F(v)(t) = \frac{1}{\kappa}\left[\|v(t)\|^2_{H^{-1}(\Omega)} + E\right] \quad \text{for any} \quad t \in [0,T], \tag{2.23}$$

and for all $v \in X$. This map is clearly well-defined: observe that v and v_t belong to $L^2(0,T;H^{-1}(\Omega))$. Note that $F(n)(t)$ is the temperature $\Theta(t)$ whenever n is the solution of Problem **P**.

Next define $N\colon X \to X$, with $u = N(v)$ satisfying

$$\left.\begin{array}{l} u_t = \operatorname{div}(F(v)(t)\nabla u + u\nabla\phi) \\ \Delta\phi = u \end{array}\right\} \quad \text{in} \quad R_T \tag{2.24}$$

$$\left.\begin{array}{l} \phi = 0 \\ (F(v)(t)\nabla u + u\nabla\phi)\cdot\vec{\nu} = 0 \end{array}\right\} \quad \text{on} \quad \partial\Omega\times[0,T], \tag{2.25}$$

$$u(x,0) = n_0(x) \quad \text{for} \quad x\in\Omega. \tag{2.26}$$

For given $v \in X$, this problem is essentially Problem **P** with prescribed temperature. As we point out in Remark 2.13, we have local existence and uniqueness provided F remains positively bounded from below. Under this condition the operator N is well-defined.

To apply the fixed point theorem, we need to prove that there exists $\mathcal{C} \subset X$, with $\mathcal{C}$ convex, bounded and closed in $(X, \|\cdot\|)$, such that:

(i) $N(\mathcal{C}) \subset \mathcal{C}$;

(ii) N is weakly-weakly sequentially continuous in X.

For any $v \in \mathcal{C}$, the operator N has to be well defined. Thus in addition to (i) and (ii) we need

(iii) there exists $F_0 = F_0(\mathcal{C})$ such that $F(v)(t) \geq F_0 > 0$ for all $t \in [0,T]$ and for all $v \in \mathcal{C}$.

We show below that

$$\mathcal{C} = \big\{v \in X \mid v(0) = n_0, \quad \|v\|_{L^2(0,T;L^2(\Omega))} \leq RT^{1/2}, \quad \|\nabla v\|_{L^2(0,T;L^2(\Omega))} \leq R', \quad \text{and} \quad \|v_t\|_{L^2(0,T;(H^1(\Omega))')} \leq R''\big\},$$

for suitably chosen constants R, R', R'' and for T sufficiently small. In fact $R = 2\|n_0\|_{L^2(\Omega)}$, $R' = 2\|n_0\|_{L^2(\Omega)}/\Theta_0^{1/2}$ and $R'' = 2\|n_0\|_{L^2(\Omega)}\Theta_0^{1/2} + 4C\|n_0\|^2_{L^2(\Omega)}/\Theta_0^{1/2}$, where $C = C(\Omega)$ is a positive constant. Clearly $\mathcal{C}$ is convex, bounded, and closed in X. Note that $\mathcal{C}$ is not empty: the solution of the heat equation with initial value n_0 and diffusion coefficient $\Theta_0/4$ satisfies $\|\nabla n\|_{L^2(0,T;L^2(\Omega))} \leq \sqrt{2}\|n_0\|_{L^2(\Omega)}/\Theta_0^{1/2}$ and $\|n_t\|_{L^2(0,T;(H^1(\Omega))')} \leq \frac{1}{2\sqrt{2}}\|n_0\|_{L^2(\Omega)}\Theta_0^{1/2}$. Hence $n \in \mathcal{C}$ for $T > 0$.

We first show (iii). Differentiating expression (2.23), applying Cauchy-Schwartz and the continuous injections $(H^1(\Omega))' \hookrightarrow H^{-1}(\Omega)$ and $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, yields the estimate

$$\begin{aligned} \kappa|F(v)_t(t)| &\leq 2\|v(t)\|_{H^{-1}(\Omega)}\|v_t(t)\|_{H^{-1}(\Omega)} \\ &\leq C\|v(t)\|_{L^2(\Omega)}\|v_t(t)\|_{(H^1(\Omega))'} \quad \text{a.e. in} \quad [0,T], \end{aligned} \tag{2.27}$$

where $C = C(\Omega) > 0$. Integration now gives

$$|F(v)(t) - F(n_0)| \leq \frac{C}{\kappa} R'' R T^{1/2} \quad \text{for all} \quad t \in [0, T]. \tag{2.28}$$

Hence $F(v)(t) \geq F(n_0) - \frac{C}{\kappa} R'' R T^{1/2} = \Theta_0 - \frac{C}{\kappa} R'' R T^{1/2}$ for $0 < t \leq T$. If we now choose $F_0 = \Theta_0/2$ and T^* such that $\frac{C}{\kappa} R'' R (T^*)^{1/2} = \Theta_0/2$, we have established (iii) for all $0 < t \leq T \leq T^*$.

Next we verify (i) for a suitable $T \leq T^*$. Starting point is inequality (2.53) with $\Theta = F(v)(t)$ and $v \in \mathcal{C}$. It follows that the solution of (2.24)-(2.26) satisfies

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + F(v)(t)\|\nabla u\|^2_{L^2(\Omega)} \leq \|u\|_{L^3(\Omega)}\|\nabla \phi\|_{L^6(\Omega)}\|\nabla u\|_{L^2(\Omega)}.$$

Since (2.54) holds for any $\Theta > 0$, we use it with $\Theta = F_0$ to find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2_{L^2(\Omega)}) + (F(v)(t) - \frac{F_0}{2})\|\nabla u\|^2_{L^2(\Omega)} \\ &\qquad \leq \frac{C}{F_0^3}(\|u(t)\|^2_{L^2(\Omega)})^3 + \frac{F_0}{2}\|u(t)\|^2_{L^2(\Omega)} \end{aligned} \tag{2.29}$$

for $0 \leq t \leq T$ and for some $C = C(\Omega) > 0$.

Since $v \in \mathcal{C}$ and consequently $F(v)(t) \geq F_0$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2_{L^2(\Omega)}) + \frac{F_0}{2}\|\nabla u\|^2_{L^2(\Omega)} \leq \frac{C}{F_0^3}(\|u(t)\|^2_{L^2(\Omega)})^3 + \frac{F_0}{2}\|u(t)\|^2_{L^2(\Omega)} \tag{2.30}$$

in $[0, T]$. This inequality implies some useful bounds. Disregarding the gradient in the left-hand side of (2.30) gives a differential inequality in terms of $\|u(t)\|^2_{L^2(\Omega)}$. It follows that there exists $T_0 = T_0(\Theta_0, \Omega, \|n_0\|_{L^2(\Omega)})$ such that u is well defined in R_{T_0} and satisfies

$$\sup_{t \in [0, T_0]} \|u(t)\|^2_{L^2(\Omega)} \leq (2\|n_0\|_{L^2(\Omega)})^2, \quad \text{and thus} \tag{2.31a}$$

$$\|u\|_{L^2(0,T_0;L^2(\Omega))} \leq 2\|n_0\|_{L^2(\Omega)} T_0^{1/2}. \tag{2.31b}$$

Integrating (2.30) and using (2.31) gives

$$\begin{aligned} \|\nabla u\|_{L^2(0,T_0;L^2(\Omega))} &\leq C T_0^{1/2} + \|n_0\|_{L^2(\Omega)}/(F_0)^{1/2} = \\ &= C T_0^{1/2} + \sqrt{2}\|n_0\|_{L^2(\Omega)}/\Theta_0^{1/2}. \end{aligned} \tag{2.32}$$

for a positive constant $C = C(\Theta_0, \Omega, \|n_0\|_{L^2(\Omega)})$. Note that (2.31) and (2.32) imply

$$u \in L^\infty(0, T_0; L^2(\Omega)) \quad \text{and} \quad u \in L^2(0, T_0; H^1(\Omega)).$$

To show that $u \in \mathcal{C}$, for sufficiently small T, it remains to prove the bound on u_t. With $\xi \in L^2(0, T; H^1(\Omega))$, we have from (2.24)

$$\int_0^T \langle u_t, \xi \rangle \, dt = -\int_0^T F(v)(t) \int_\Omega \nabla u \nabla \xi \, dx dt + \int_0^T \int_\Omega u \nabla \phi \nabla \xi \, dx dt, \tag{2.33}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $(H^1(\Omega))'$, and $H^1(\Omega)$. To estimate the right hand side we first note that $F(v) \in L^\infty(0, T^*)$. Indeed, from (2.28) we deduce

$$F_0 < F(v)(t) \leq \frac{C}{\kappa} RR'' T^{1/2} + \Theta_0, \quad \text{for} \quad 0 \leq t \leq T \leq T^*. \tag{2.34}$$

Next we use (2.72) and interpolation inequality (2.71) from the appendix. This gives

$$\begin{aligned} \left| \int_\Omega u \nabla \phi \nabla \xi \, dx \right| &\leq \|u\|_{L^3(\Omega)} \|\nabla \phi\|_{L^6(\Omega)} \|\nabla \xi\|_{L^2(\Omega)} \\ &\leq C_s^{1/2} C_I \|u\|_{H^1(\Omega)} \|u\|_{L^2(\Omega)} \|\nabla \xi\|_{L^2(\Omega)}. \end{aligned}$$

Finally we combine this expression with (2.31), (2.32), and (2.34), and obtain after some manip ulation

$$\begin{aligned} \int_0^T |\langle u_t, \xi \rangle| \, dt \leq \{ C(T^{1/2} + T) + \sqrt{2} \|n_0\|_{L^2(\Omega)} \Theta_0^{1/2} \} \|\xi\|_{L^2(0,T,H^1(\Omega))} \\ + 2\sqrt{2} C_s^{1/2} C_I \|n_0\|_{L^2}^2 / \Theta_0^{1/2} \|\xi\|_{L^2(0,T,H^1(\Omega))} \end{aligned}$$

for some $C = C(\Omega, \|n_0\|_{L^2(\Omega)}, \Theta_0)$. Taking now $T_1 < T_0 < T^*$ sufficiently small we obtain that $u \in \mathcal{C}$ for $0 \leq T \leq T_1$ and consequently $N(\mathcal{C}) \subset \mathcal{C}$.

Next we show (ii): i.e. we claim that $v_k \in \mathcal{C}$, $v_k \rightharpoonup v$ in X implies $N(v_k) \rightharpoonup N(v)$ in X. For any such sequence v_k, define $u_k := N(v_k) \in \mathcal{C}$. Using the weak compactness of $\mathcal{C}$ we extract a subsequence $u_{k'} \in \mathcal{C}$ such that $u_{k'} \rightharpoonup u^*$ in X. We show below that $u^* = N(v)$, which proves the assertion. Since

$$v_{k'} \rightharpoonup v \quad \text{and} \quad u_{k'} \rightharpoonup u^* \quad \text{in} \quad \mathcal{C},$$

we obtain by Aubin's Lemma [59, pag. 58] for a subsequence, denoted again by k',

$$v_{k'} \to v \quad \text{and} \quad u_{k'} \to u^* \quad \text{in} \quad L^2\big(0, T; L^2(\Omega)\big).$$

We use this in (2.33) for $u_{k'}, v_{k'}$, and $\phi_{k'}$. Since $\Delta \phi_{k'} = u_{k'}$, we have $\phi_{k'} \to \phi^*$ in $L^2\big(0, T; H^2(\Omega)\big)$ satisfying $\Delta \phi^* = u^*$. Moreover, as $k' \to \infty$,

$$\begin{aligned} u_{k'_t} &\rightharpoonup u_t^* \quad \text{in} \quad L^2\big(0, T, (H^1(\Omega))'\big), \\ \nabla \phi_{k'} &\to \nabla \phi^* \quad \text{in} \quad L^2\big(0, T, H^1(\Omega)\big), \\ \nabla u_{k'} &\rightharpoonup \nabla u^* \quad \text{in} \quad L^2\big(0, T; L^2(\Omega)\big). \end{aligned}$$

Now suppose $F(v_{k'}) \to F(v)$ in $C([0, T])$. Then letting $k' \to \infty$ in (2.33) we obtain a solution u^* of problem (2.24)-(2.26) for the temperature $F(v)(t)$. By uniqueness we have $u^* = N(v)$.

It remains to show that $F(v_{k'}) \to F(v)$ in $C([0, T])$. In view of the continuous injection $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, we find from (2.23)

$$\kappa |F(v_{k'})(t) - F(v)(t)| \leq C(\Omega) \left[\|v_{k'}(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)} \right] \|v_{k'}(t) - v(t)\|_{L^2(\Omega)}.$$

This implies directly $F(v_{k'}) \to F(v)$ in $L^1([0,T])$. Writing (2.27) for the difference $F(v_{k'}) - F(v)$, using the continuous injection $(H^1(\Omega))' \hookrightarrow H^{-1}(\Omega)$, and integrating the result gives

$$\begin{aligned}\kappa \int_0^T |F(v_{k'})_t(t) - F(v)_t(t)|dt \quad &\leq \quad \int_0^T |(v(t), v_{k'_t}(t) - v_t(t))_{H^{-1}}|\, dt \\ &+ \quad \|v - v_{k'}\|_{L^2(0,T;L^2(\Omega))} \|(v_{k'})_t\|_{L^2(0,T;(H^1(\Omega))')}.\end{aligned}$$

Since $v_{k'_t} \rightharpoonup v_t$, in $L^2\big(0,T,H^{-1}(\Omega)\big)$, we obtain $F(v_{k'}) \to F(v)$ in $W^{1,1}([0,T])$. This concludes the proof of (ii) and establishes local existence for Problem $\mathbf{P}$.

The boundedness of n follows from [11, Theorem 2] and $n \geq 0$ a.e. in R_T is essentially demonstrated in [14].

Remark 2.7 *Let $n_0 \in L^p(\Omega)$ with $p > 3$ and let $\Theta(0) = \Theta_0 > 0$. Then Problem* $\mathbf{P}$ *has a local solution satisfying $n \in L^\infty\big(0,T;L^p(\Omega)\big)$ and $n^{p/2} \in L^2\big(0,T;H^1(\Omega)\big)$. The proof is almost identical to the proof of Theorem 2.6.*

2.3.2 Uniqueness

Uniqueness is stated for an equivalent formulation of Problem $\mathbf{P}$ in which we replace t by $\tau = \int_0^t \Theta(t)dt$. This transformation only affects equation (2.1), which now becomes

$$n_\tau = \operatorname{div}\Big\{\nabla n + \frac{n}{\Theta(\tau)}\nabla\phi\Big\} \quad \text{in} \quad R_{\hat{T}} \tag{2.35}$$

where $\hat{T} = \int_0^T \Theta(t)dt$. The problem stated in terms of x and τ is denoted by Problem $\mathbf{P}_e$. Without proof we remark that $\langle n = n(x,t), \Theta = \Theta(t)\rangle$ solves Problem $\mathbf{P}$ if and only if $\langle n = n(x,\tau), \Theta = \Theta(\tau)\rangle$ solves Problem $\mathbf{P}_e$. This is due to the strict positivity of Θ in the existence interval.

Theorem 2.8 *If $n_0 \in L^2(\Omega)$ and $\Theta_0 > 0$, then Problem $\mathbf{P}_e$ has at most one solution $\langle n, \Theta\rangle$.*

Proof: We use a uniqueness result of Biler & Nadzieja [14], who considered the problem

$$n_\tau = \operatorname{div}\big(\nabla n + nX(n)\big) \qquad \text{in} \qquad R_T, \tag{2.36}$$

$$\big(\nabla n + nX(n)\big)\cdot\vec{\nu} = 0 \qquad \text{on} \qquad \partial\Omega \times [0,T], \tag{2.37}$$

$$n(\cdot,0) = n_0 \qquad \text{in} \qquad \Omega, \tag{2.38}$$

where X is a general non-local vector field operator in $\mathbb{R}^3$. For this problem uniqueness in $L^2(\Omega)$ was proved in [14, Theorem 1 (i)] under the following condition: there exists $C > 0$ such that

$$(U) \qquad \|X(u) - X(v)\|_{L^6(\Omega)} \leq C\|u - v\|_{L^2(\Omega)}$$

for all $u, v \in L^2(\Omega)$.

Note that the constant C in (U) does not depend on the choice of $u, v \in L^2(\Omega)$. In our case $X(n) = \frac{\nabla\phi}{\Theta(\tau)}$. Below, in Lemma 2.9, we show that again (U) holds but with C depending on both norms $\|u\|_{L^2(\Omega)}$ and $\|v\|_{L^2(\Omega)}$. Now suppose that Problem $\mathbf{P}_e$ admits two solutions $\langle n_1, \Theta_1\rangle$ and $\langle n_2, \Theta_2\rangle$ in some interval $[0, T]$. From the solution definition we know that both $\|n_1(t)\|_{L^2(\Omega)}$ and $\|n_2(t)\|_{L^2(\Omega)}$ are uniformly bounded in $[0, T]$. Therefore (U) is satisfied for the two solutions $n_1(t)$ and $n_2(t)$, with $0 \le t \le T$, for an appropriately chosen constant C. As a consequence we can apply the result of [14]. This proves the theorem.

Lemma 2.9 *Suppose there exist $\delta > 0$ and $u, v \in L^2(\Omega)$ such that*

$$\min\left\{\Theta_u := E + \int_\Omega |\nabla\phi_u|^2\,dx,\ \Theta_v := E + \int_\Omega |\nabla\phi_v|^2\,dx\right\} \ge \delta > 0,$$

where

$$\Delta\phi_u = u, \qquad \Delta\phi_v = v \qquad in \qquad \Omega, \tag{2.39}$$

$$\phi_u = \phi_v = 0 \qquad on \qquad \partial\Omega. \tag{2.40}$$

Then

$$\left\|\nabla\left(\frac{\phi_u}{\Theta_u} - \frac{\phi_v}{\Theta_v}\right)\right\|_{L^6(\Omega)} \le C\|u - v\|_{L^2(\Omega)}$$

where $C = C(\delta, \|u\|_{L^2(\Omega)}, \|v\|_{L^2(\Omega)})$.

Proof: Using

$$\|\nabla(\phi_u - \phi_v)\|_{L^6(\Omega)} \le C_I\|u - v\|_{L^2(\Omega)}$$

we estimate

$$\begin{aligned}
\left\|\nabla\left(\frac{\phi_u}{\Theta_u} - \frac{\phi_v}{\Theta_v}\right)\right\|_{L^6(\Omega)} &= \left\|\nabla\left(\frac{\phi_u}{\Theta_u} - \frac{\phi_u}{\Theta_v} + \frac{\phi_u}{\Theta_v} - \frac{\phi_v}{\Theta_v}\right)\right\|_{L^6(\Omega)} \\
&\le \frac{1}{\Theta_u\Theta_v}\|\nabla\phi_u\|_{L^6(\Omega)}|\Theta_u - \Theta_v| + \frac{C_I}{\Theta_v}\|u - v\|_{L^2(\Omega)} \\
&\le \frac{C_I}{\Theta_v}\left\{\frac{\|u\|_{L^2(\Omega)}}{\Theta_u}\left|\int_\Omega \left(|\nabla\phi_u|^2 - |\nabla\phi_v|^2\right)dx\right| + \|u - v\|_{L^2(\Omega)}\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
\left|\int_\Omega \left(|\nabla\phi_u|^2 - |\nabla\phi_v|^2\right)dx\right| &\le \|\nabla(\phi_u + \phi_v)\|_{L^2(\Omega)}\|\nabla(\phi_u - \phi_v)\|_{L^2(\Omega)}, \\
&\le C\big(\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\big)\|u - v\|_{L^2(\Omega)}
\end{aligned}$$

for some $C > 0$, we obtain the assertion. ■

■

Remark 2.10 *Note that for $n_0 \in L^p(\Omega)$ with $p > 2$, we can apply the above theorem and obtain uniqueness of solutions for such initial data. Thus the local solution given in Remark 2.7 of Theorem 2.6 for $n_0 \in L^p(\Omega)$ with $p > 3$ is unique.*

Remark 2.11 *A slight modification of the above argument directly gives uniqueness for $n_0 \in L^p(\Omega)$ with $p > 3$ and $\Theta_0 > 0$. Again following* [14, Theorem 1 (ii)], *we need to show*

$$(U') \qquad \|X(u) - X(v)\|_{L^\infty(\Omega)} \le C\|u - v\|_{L^p(\Omega)} \quad (p > 3).$$

With $X(n) = \frac{\nabla\phi}{\Theta(\tau)}$, inequality (U') results from inequality (2.72).

The next theorem extends the local existence result for $n_0 \in L^p(\Omega)$ with $p > \frac{3}{2}$. To do this we modify our definition of weak solution for $1 < p < 2$. We replace (i) by $n \in L^\infty(0,T;L^p(\Omega))$ and $n_t \in L^p\big(0,T;(W^{1,p'}(\Omega))'\big)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. We show that $n \in C\big([0,T];L^p(\Omega)\big)$, implying that n_0 can be prescribed. In fact as $n \in L^\infty\big(0,T;L^p(\Omega)\big)$ and $n \in C\big([0,T],(W^{1,p'}(\Omega))'\big)$, (since $n \in L^p\big(0,T;(W^{1,p'}(\Omega))'\big)$ and $n_t \in L^p\big(0,T;(W^{1,p'}(\Omega))'\big)$), we use the injection $L^p(\Omega) \hookrightarrow (W^{1,p'}(\Omega))'$, and follow the argument in [59, p. 23] to conclude.

Theorem 2.12 *Let $n_0 \in L^p(\Omega)$ with $p > 3/2$, and $\Theta(0) = \Theta_0 > 0$. Then there exists $T = T(\Omega, \|n_0\|_{L^p(\Omega)}, \Theta_0) > 0$, and a weak solution $\langle n, \Theta\rangle$ of Problem* **P**. *It satisfies $n^{p/2} \in L^2\big(0,T;H^1(\Omega)\big)$ and furthermore $n \in L^\infty_{loc}\big(0,T];L^\infty(\Omega)\big)$.*

Proof: Due to Remarks 2.7 and 2.10, we only need to demonstrate local existence for $n_0 \in L^p(\Omega)$ with $p \in (3/2, 3)$. We follow the proof of [14, Theorem 1 (iii)] and approximate $n_0 \in L^p(\Omega)$ by functions in $L^{p_*}(\Omega)$ with $p_* > 3$. Thus let $\{n_{0\epsilon}\} \subset L^{p_*}(\Omega)$ satisfy $\|n_{0\epsilon} - n_0\|_{L^p(\Omega)} \to 0$ as $\epsilon \to 0$.

For each $\epsilon > 0$ we consider Problem **P** with initial data $\langle n_{0\epsilon}, \Theta_0\rangle$. By Remark 2.10 there exists a solution $\langle n_\epsilon, \Theta_\epsilon\rangle$, with $\Theta_\epsilon \ge \Theta_0/2$, in some interval $[0, T_\epsilon]$. Next we use the estimate [14, Eq. (10)]

$$\|n_\epsilon(t)\|^p_{L^p(\Omega)} + \int_0^t \big|\nabla|n_\epsilon|^{p/2}\big|^2\, d\tau \le \exp\left(C\int_0^t \left\|\frac{\nabla\phi_\epsilon}{\Theta_\epsilon}\right\|^{\frac{2q}{q-3}}_{L^q(\Omega)} d\tau\right)\|n_{0\epsilon}\|^p_{L^p(\Omega)} \tag{2.41}$$

for almost every $t \in [0, T_\epsilon]$. Here $p \in (3/2, 3)$, $1/q = 1/p - 1/3$ and $C = C(\Omega, p)$. Note that $q > 3$. Further, using $\|\nabla\phi_\epsilon\|_{L^q(\Omega)} \le C\|n_\epsilon\|_{L^p(\Omega)}$ and the uniform lower bound on Θ_ϵ, we obtain

$$\|n_\epsilon(t)\|^p_{L^p(\Omega)} \le C\exp\left(\frac{2}{\Theta_0}\int_0^t \|n_\epsilon(\tau)\|^{\frac{2q}{q-3}}_{L^p(\Omega)}\, d\tau\right) \quad \text{for almost every} \quad t \in [0, T_\epsilon],$$

where $C = C(\bar\epsilon) > 0$ and $0 < \epsilon \le \bar\epsilon$. Since C does not depends on ϵ we have that $T_\epsilon = T$ and

$$\|n_\epsilon(t)\|_{L^p(\Omega)} \le C \quad \text{and} \quad \Theta_\epsilon(t) \ge \Theta_0/2 \quad \text{for almost every} \quad t \in [0, T] \tag{2.42}$$

and for all $0 < \epsilon \leq \bar{\epsilon}$. Using this and (2.41), we deduce

$$\|n_\epsilon(t)^{p/2}\|_{H^1(\Omega)} \leq C \quad \text{and} \quad \Theta_\epsilon(t) \geq \Theta_0/2 \quad \text{for almost every} \quad t \in [0,T]. \tag{2.43}$$

and for all $0 < \epsilon \leq \bar{\epsilon}$.

Next we separate the demonstration into two cases: $p < 2$ and $p > 2$.

We begin with $p < 2$. Under this condition, we have

$$\|\nabla n\|_{L^p(\Omega)} \leq C(\Omega, p)\|\nabla n^{p/2}\|_{L^2(\Omega)}\|n\|_{L^p(\Omega)}^{(2-p)/2} \quad \text{for} \quad n^{p/2} \in H^1(\Omega).$$

Combining this with (2.42) and (2.43), since $p < 2$, we obtain

$$\|\nabla n_\epsilon\|_{L^p(0,T;L^p(\Omega))} \leq C \quad \text{for all} \quad 0 < \epsilon \leq \bar{\epsilon}. \tag{2.44}$$

Consequently, we can check that $\|n_{\epsilon_t}\|_{L^p\left(0,T,(W^{1,p'}(\Omega))'\right)} \leq C$ for all $0 < \epsilon \leq \bar{\epsilon}$. Now using a compactness theorem [59, p. 141], with $L^p(\Omega) \hookrightarrow (W^{1,p'}(\Omega))'$, we find for a subsequence $\epsilon \to 0$,

$$n_\epsilon \to n \quad \text{in} \quad L^p\big(0,T;L^p(\Omega)\big). \tag{2.45}$$

Now using standard arguments and above estimates, we get as $\epsilon \to 0$

$$n_{\epsilon_t} \rightharpoonup n_t \quad \text{in} \quad L^p\big(0,T,(W^{1,p'}(\Omega))'\big), \tag{2.46}$$

$$\nabla\phi_\epsilon \to \nabla\phi \quad \text{in} \quad L^p\big(0,T,W^{1,p}(\Omega)\big), \tag{2.47}$$

$$\nabla n_\epsilon \rightharpoonup \nabla n \quad \text{in} \quad L^p\big(0,T;L^p(\Omega)\big). \tag{2.48}$$

To conclude, it suffices to prove $\Theta_\epsilon \to \Theta$ in $C([0,T])$. This follows from showing that

$$\kappa\int_0^T |\Theta_{\epsilon_t} - \Theta_t|\,dt \;\leq\; 2\int_0^T\left|\int_\Omega n_\epsilon\phi_{\epsilon_t} - n\phi_t\,dx\right|dt \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{2.49}$$

We obtain this using [81, Proposition 23.9 (d)], combining (i) strong convergence of $n_\epsilon \to n$ in $L^{p'}\big(0,T;L^p(\Omega)\big)$ with (ii) weak convergence of $\phi_{\epsilon_t} \rightharpoonup \phi_t$ in $L^p\big(0,T;L^{p'}(\Omega)\big)$. In fact (i) is consequence of (2.42) and (2.45); and (ii) yields using (2.46) and the estimate $\|\phi_{\epsilon_t}\|_{p'} \leq C\|\phi_{\epsilon_t}\|_{W^{1,p}(\Omega)} \leq C\|n_{\epsilon_t}\|_{(W^{1,p'}(\Omega))'}$, where we have used $p > \frac{3}{2}$.

Now we take the limit $\epsilon \to 0$ to conclude that n satisfies Problem **P**.

For $p > 2$, we use that $n_{0\epsilon} \in L^2(\Omega)$ and in particular (2.42) implies

$$\|n_\epsilon(t)\|_{L^2(\Omega)} \leq C \quad \text{and} \quad \Theta_\epsilon(t) \geq \Theta_0/2 \quad \text{for all} \quad t \in [0,T].$$

We follow the proof of Theorem 2.6, to find $\|n_{\epsilon_t}\|_{L^2\left(0,T,(H^1(\Omega))'\right)} \leq C$. With this we may apply again the compactness theorem [59, p. 141], now with $L^p(\Omega) \hookrightarrow (H^1(\Omega))'$, since $p > 2$, and obtain $n_\epsilon \to n$ in $L^p\big(0,T,L^p(\Omega)\big)$. Finally, we show (2.49) using $p = p' = 2$ and obtain $\Theta_\epsilon \to \Theta$ in $C([0,T])$, which concludes the proof of the theorem.
■

2.3.3 Radially symmetric solutions

In Section 2.2.4 we introduced Problem **Q** describing radially symmetric solutions of Problem **P** in the unit ball. In this paper we do not prove existence for Problem **Q**. Instead we shall assume that if $\Omega = B_1(0)$ and if n_0 is radially symmetric, then the corresponding weak solution is radially symmetric. By standard regularity theory weak solutions of Problem **P** satisfy equations (2.1)-(2.3) and boundary conditions (2.4)-(2.5) in a classical sense. With this in mind we introduce for Problem **Q** the following solution definition.

Let $D_T = (0,1) \times (0,T]$. A pair $\langle Q, \vartheta \rangle$ solves Problem **Q**, if for some $T > 0$:

(i) $Q \in C^{2,1}(D_T) \cap C(\overline{D_T})$, and $\vartheta \in C([0,T])$;

(ii) (Q, ϑ) satisfies equations (2.19)-(2.22);

(iii) $Q_y \geq 0$ in D_T and $\vartheta > 0$ in $[0,T]$.

Clearly radial solutions of Problem **P** with $n_0 \in L^p(B_1(0))$, $p > \frac{3}{2}$, satisfy this definition. This follows directly from the identity

$$Q_y(y,t) = \frac{4\pi}{3} n(y^{1/3}, t) \quad \text{for} \quad (y,t) \in D_T. \tag{2.50}$$

2.4 Prescribed temperature problem

In this section we study Problem **P** with prescribed temperature $\Theta(t)$ satisfying

$$\Theta\colon [0,T] \to \mathbb{R} \quad \text{such that} \quad \Theta \in C([0,T]) \quad \text{and} \quad \Theta(t) > \delta > 0 \quad \text{for} \quad t \in [0,T]. \tag{2.51}$$

Thus we drop the energy relation (2.3) and assume that ϑ in (2.1) and (2.4) is given and satisfies (2.51). We denote this modified problem by $\mathbf{P}^*$. Clearly, if Problem $\mathbf{P}^*$ has a radially symmetric solution, then the corresponding formulation in terms of the accumulated mass Q, which we denote by $\mathbf{Q}^*$, has a classical solution according to the definition given in Section 2.3.3.

We first recall some recent results of Biler & Nadzieja [16, 14], related to local existence for Problem $\mathbf{P}^*$ and global existence for Problem $\mathbf{Q}^*$.

Remark 2.13

(i) *Let* $n_0 \in L^2(\Omega)$ *and let* Θ *satisfy (2.51). Then there exists a time* $T = T(\Omega, \|n_0\|_{L^2(\Omega)}, \delta) > 0$ *so that Problem* $\mathbf{P}^*$ *has a unique weak solution in* $[0,T]$ *which satisfies* $n \in L^\infty_{loc}\big((0,T], L^\infty(\Omega)\big)$. *Proof: see* [14, Theorem 1 (i)].

(ii) *Let* $n_0 \in L^p(\Omega)$ *with* $p > \frac{3}{2}$ *and let* Θ *satisfy (2.51). Then there exist* $T = T(\Omega, \|n_0\|_{L^p(\Omega)}, \delta) > 0$ *so that Problem* $\mathbf{P}^*$ *has a weak solution in* $[0,T]$ *satisfying* $n \in L^\infty\big(0,T,L^p(\Omega)\big)$ *and* $n^{p/2} \in L^2\big(0,T,H^1(\Omega)\big)$. *For* $p > 3$ *the solution is unique. Proof: see* [14, Theorem 1 (ii) and (iii)].

(iii) *If for some* $B > 0$, $Q_0(y) \leq y\frac{1+B}{y^{2/3}+B}$ *for* $0 \leq y \leq 1$ *and* $\vartheta(t) = constant \geq \frac{3}{2}\frac{(1+B)(1+3B)}{(1+5B)}$ *then Problem* $\mathbf{Q}^*$ *has a global classical solution satisfying* $Q(y,t) \leq y\frac{1+B}{y^{2/3}+B}$ *for* $(y,t) \in \overline{D_\infty}$. *Proof: see Appendix.*

In the remainder of this section we present some new results related to Problem $\mathbf{P}^*$ with *constant* temperature $\Theta > 0$. We first extend a global existence result of [15].

Theorem 2.14 *For a given domain* Ω, *there exist positive constants,* α_1, α_2, A, B, *and* C *with* $\alpha_2 \geq \alpha_1$ *so that if the constant temperature* Θ *and the initial condition* n_0 *satisfy*

$$\Theta \geq \alpha_1 \quad \textit{and} \quad \|n_0\|^2_{L^2(\Omega)} \leq A + \frac{B}{\Theta^4}$$

or

$$\Theta \geq \alpha_2 \quad \textit{and} \quad \|n_0\|^2_{L^2(\Omega)} \leq C(\Theta^2 - \Theta),$$

then Problem $\mathbf{P}^*$ *has a global (weak) solution for which the* $L^2(\Omega)$ *norm is uniformly bounded in time.*

Proof: Integrating (2.1) we obtain the expression

$$\frac{1}{2}\frac{d}{dt}\|n\|^2_{L^2(\Omega)} + \Theta\|\nabla n\|^2_{L^2(\Omega)} = -\int_\Omega n\nabla n\nabla\phi\, dx. \tag{2.52}$$

As in the proof of [15, Theorem 2 (iii)] we estimate

$$\frac{1}{2}\frac{d}{dt}\|n\|^2_{L^2(\Omega)} + \Theta\|\nabla n\|^2_{L^2(\Omega)} \leq \|n\|_{L^3(\Omega)}\|\nabla\phi\|_{L^6(\Omega)}\|\nabla n\|_{L^2(\Omega)}. \tag{2.53}$$

The aim is to obtain a differential inequality for $\|n(t)\|^2_{L^2(\Omega)}$. From the appendix we first use (2.71) and then (2.72) with $r = 6$ and $p = 2$. This gives

$$\|n\|_{L^3(\Omega)}\|\nabla\phi\|_{L^6(\Omega)}\|\nabla n\|_{L^2(\Omega)} \leq \frac{\Theta}{2}\|n\|^2_{H^1(\Omega)} + \frac{C_1}{\Theta^3}(\|n\|^2_{L^2(\Omega)})^3. \tag{2.54}$$

Further, we use (2.70) with $q = 2$ and $p = 1$ to obtain

$$2\Theta\|n\|^2_{L^2(\Omega)} \leq \frac{\Theta}{2}\|n\|^2_{H^1(\Omega)} + C_2\Theta\|n\|^2_{L^1(\Omega)}. \tag{2.55}$$

The combination (2.53)-(2.55) eliminates the gradient term. Since $\|n\|_{L^1(\Omega)} = 1$, we are left with an inequality of the form

$$\frac{d}{dt}w \leq p_\Theta(w) := \frac{C_1}{\Theta^3}w^3 - \Theta w + C_2\Theta \quad \text{for} \quad t > 0 \tag{2.56}$$

with $w(t) := \|n(t)\|^2_{L^2(\Omega)}$. Here C_1 and C_2 are positive constants only depending on Ω. The assertions of the theorem now follow from particular properties of (2.56).

First observe that if $\Theta > \alpha_1 := \frac{3^{3/4}}{2^{1/2}}(C_2 C_1^{1/2})^{1/2}$, then $p_\Theta(w) = 0$ has two positive real roots $w_* < w^*$ and $p_\Theta(w) < 0$ for $w_* < w < w^*$. If $\Theta = \alpha_1$, these roots coincide. A simple calculation shows that

$$w_0 = C_2 + \frac{C_2^3 C_1}{\Theta^4}$$

satisfies $0 < w_0 < w_*$ for all $\Theta \geq \alpha_1$. Since $p_\Theta(w) > 0$ for $0 \leq w < w_*$, we deduce that $w(t)$, with $w(0) \leq w_0$, satisfies $w(t) \leq w_*$ for all $t \geq 0$. This proves the first assertion.

Next consider

$$w^0 := C_1^{-1/2}(\Theta^2 - \Theta).$$

Then $p_\Theta(w^0) \leq 0$ provided

$$\Theta \geq \alpha_2 = \left(\left(\frac{3 + C_2 C_1^{1/2}}{4} \right)^2 - \frac{1}{2} \right)^{1/2} + \frac{3 + C_2 C_1^{1/2}}{4}.$$

Clearly $\alpha_2 \geq \alpha_1$, since $p_\Theta(w) > 0$ for $\Theta < \alpha_1$ and for all $w > 0$. As before we have that $w(t)$, with $w(0) \leq w^0$, satisfies $w(t) \leq w^0 \leq w^*$ for all $t \geq 0$. This proves the second assertion. ■

In a similar fashion global existence results are obtained in $L^p(\Omega)$ for $p > 3/2$. Instead of (2.56) one now finds

$$\frac{d}{dt} w \leq \frac{C_1}{\Theta^\beta} w^\beta - \Theta w + C_2 \Theta \quad \text{for} \quad t > 0 \tag{2.57}$$

with $w(t) := \|n(t)\|_{L^p(\Omega)}^p$. Here

$$\beta = \begin{cases} \frac{2p-1}{2p-3} & \text{for} \quad 3/2 < p < 3 \\ \frac{p+2}{p} & \text{for} \quad p > 3. \end{cases}$$

Inequality (2.57) implies the following result.

Theorem 2.15 *For a given domain Ω, there exist positive constants, $\beta_1, \beta_2, \bar{A}, \bar{B}$, and $\bar{C}$ with $\beta_2 \geq \beta_1$ so that if the constant temperature Θ and the initial condition n_0 satisfy*

$$\Theta \geq \beta_1 \quad \textit{and} \quad \|n_0\|_{L^p(\Omega)}^p \leq \bar{A} + \frac{\bar{B}}{\Theta^{\beta+1}}$$

or

$$\Theta \geq \beta_2 \quad \textit{and} \quad \|n_0\|_{L^p(\Omega)}^p \leq \bar{C}(\Theta^\gamma - \Theta), \quad \textit{with} \quad \gamma = \frac{\beta+1}{\beta-1},$$

then Problem $\mathbf{P}^*$ *has a global (weak) solution for which the $L^p(\Omega)$ norm is uniformly bounded in time.*

2.5 Global existence for Problem Q

In this section we study radially symmetric solutions of Problem **P**. We will use the classical formulation in terms of Problem **Q**. Before we prove the global existence results, we first demonstrate the blow-up result Theorem 2.1.

Theorem 2.16 *Let $\kappa > 6$ and $\mathcal{E} < 3$. Then $T^* < \infty$.*

Proof: Suppose Problem **Q** has a global solution $Q = Q(y,t)$ and $\vartheta = \vartheta(t) \in (0, \infty)$ for all $t > 0$. Setting

$$w_\epsilon(t) := \int_\epsilon^1 Q(y,t) y^{-1/3}\, dy \quad \text{for all} \quad t > 0,$$

we find, after differentiating and using equation (2.19),

$$\begin{aligned} \frac{dw_\epsilon(t)}{dt} &= \vartheta(t) Q_y(1,t) - \epsilon \vartheta(t) Q_y(\epsilon,t) - \vartheta(t) + \vartheta(t) Q(\epsilon,t) \\ &\quad - \frac{1}{2}\frac{Q(\epsilon,t)^2}{\epsilon^{1/3}} + \frac{1}{2} + \frac{\kappa}{6}\vartheta(t) - \frac{\mathcal{E}}{6} + \frac{1}{6}\int_0^\epsilon Q^2(y,t) y^{-4/3}\, dy \quad \text{for all} \quad t > 0. \end{aligned} \tag{2.58}$$

Since $\vartheta(t) < \infty$, we obtain from (2.20)

$$Q(y,t)^2 / y^{4/3} \in L^1(0,1) \quad \text{for all} \quad t > 0,$$

which implies that we can choose a sequence $\epsilon_n \downarrow 0$ along which $\lim\limits_{\epsilon_n \downarrow 0} \frac{Q(\epsilon_n,t)}{\epsilon_n^{1/6}} = 0$ for all $t > 0$. Using this and $Q_y(1,t) \geq 0$ in (2.58), we find in the limit

$$\frac{dw_0(t)}{dt} \geq \frac{1}{2} - \vartheta(t) + \frac{\kappa}{6}\vartheta(t) - \frac{\mathcal{E}}{6} \quad \text{for all} \quad t > 0.$$

The parameter choice implies that $\frac{dw_0(t)}{dt} \geq \delta > 0$ for all $t > 0$. This contradicts $w_0 \leq \frac{3}{2}$ (implied by $Q \leq 1$ in $\overline{D_\infty}$). ■

Next we turn to global existence. The proof uses a comparison principle for the Q-equation (2.19) with respect to given ordered temperatures, and the fact that temperature is positively bounded from below. The results are stated in terms of an equivalent formulation, as in (2.35), in which we replace t by τ: i.e.

$$\begin{aligned} Q_\tau &= y^{4/3} Q_{yy} + \frac{1}{\vartheta(\tau)} Q Q_y \quad \text{in} \quad D_T & (2.59) \\ Q(0,\tau) &= 0, \qquad Q(1,\tau) = 1 \qquad \text{for} \quad \tau \in [0,T], & (2.60) \\ Q(y,0) &= Q_0(y) \qquad\qquad\qquad \text{for} \quad y \in [0,1], & (2.61) \end{aligned}$$

and the energy relation

$$\mathcal{E} = \kappa\vartheta(\tau) - \int_0^1 \frac{Q^2}{y^{4/3}} dy \qquad \text{for} \quad \tau \in [0, T]. \tag{2.62}$$

We first consider (2.59)-(2.61) for given ordered temperatures and ordered initial data.

Proposition 2.17 *Let $i = 1, 2$. Suppose Q_i solves (2.59)-(2.61) in D_{T_i} subject to $Q_0 = Q_{0i}$, and given $\vartheta = \vartheta_i$, satisfying (2.51) in $[0, T_i]$. Let $T = \min\{T_1, T_2\}$. If*

$$\vartheta_1 \le \vartheta_2 \quad in \quad [0, T], \quad and \quad Q_{01} \ge Q_{02} \quad in \quad (0, 1)$$

and if there exists $K > 0$ such that either

$$0 \le Q_{1_y} \le K \quad or \quad 0 \le Q_{2_y} \le K \quad in \quad \overline{D_T},$$

then

$$Q_1 \ge Q_2 \quad in \quad D_T.$$

Proof: Suppose $0 \le Q_{2_y} \le K$. Since

$$Q_{2_\tau} = y^{4/3} Q_{2_{yy}} + \frac{1}{\vartheta_1(\tau)} Q_2 Q_{2_y} + \left(\frac{1}{\vartheta_2(\tau)} - \frac{1}{\vartheta_1(\tau)}\right) Q_2 Q_{2_y} \quad \text{in} \quad D_T.$$

It follows from $\vartheta_1 \le \vartheta_2$ and $Q_{2y} \ge 0$ that

$$Q_{2_\tau} \le y^{4/3} Q_{2_{yy}} + \frac{1}{\vartheta_1(\tau)} Q_2 Q_{2_y} \quad \text{in} \quad D_T.$$

This inequality and the boundedness of Q_{2_y} allows us to use [52, Theorem 3.2], which show that Q_2 is a subsolution for the Q-equation with ϑ_1. ■

Next we use the Boltzmann entropy (2.15) in terms of $Q = Q(y, \tau)$ to establish a positive lower bound on ϑ (see [69] for a similar estimate).

Proposition 2.18 *Let $\langle Q, \vartheta \rangle$ be a solution of Problem (2.59)-(2.62). Suppose $\vartheta(0) = \frac{1}{\kappa}\left(\mathcal{E} + \int_0^1 \frac{Q_0^2(y)}{y^{4/3}} dy\right) > 0$. Then*

$$\vartheta(\tau) \ge \lambda \vartheta(0) \quad for \quad \tau > 0, \quad with \quad \lambda = \exp\left(-\frac{2}{\kappa} \int_0^1 Q_{0_y} \log Q_{0_y} dy\right). \tag{2.63}$$

Proof: Rewriting (2.15) results in

$$W(\tau) := \int_0^1 Q_y \log Q_y \, dy - \frac{\kappa}{2} \log\left(E + \int_0^1 \frac{Q^2}{y^{4/3}} dy\right) \quad \text{for} \quad \tau > 0$$

and differentiation gives, see also (2.16),

$$\frac{dW(\tau)}{d\tau} = -\int_0^1 \frac{Q_\tau^2}{y^{4/3}Q_y}\,dy \leq 0 \quad \text{for} \quad \tau > 0.$$

Hence $W(\tau)$ is decreasing in τ. As a consequence

$$\begin{aligned} W(\tau) &= \int_0^1 Q_y \log Q_y\,dy - \frac{\kappa}{2}\log\left(\mathcal{E} + \int_0^1 \frac{Q^2}{y^{4/3}}\,dy\right) \leq \\ &\leq W(0) = \int_0^1 Q_{0_y} \log Q_{0_y}\,dy - \frac{\kappa}{2}\log\left(\mathcal{E} + \int_0^1 \frac{Q_0^2}{y^{4/3}}\,dy\right) \quad \text{for} \quad \tau > 0. \end{aligned}$$

Here we use Jensen's inequality to estimate

$$\int_0^1 Q_y \log Q_y\,dy \geq \left(\int_0^1 Q_y\,dy\right)\log\left(\int_0^1 Q_y\,dy\right) = 0,$$

from which lower bound (2.63) directly follows. ■

Note that whenever ϑ is bounded away from zero, blow-up in Problem (2.59)-(2.62) can only occur at the boundary $y = 0$. This is a direct consequence of classical regularity theory, which implies that Q is smooth away from $y = 0$. Blow-up manifests itself through singular behaviour of Q_y as (y, τ) approaches the point $(0, T^*)$. This corresponds to unbounded density at the origin of the radially symmetric solution of Problem **P**. Below we use the comparison argument (Proposition 2.17) to control the behaviour of $Q_y(0,\tau)$. We show that this implies a uniform bound on $\|Q_y(\tau)\|_{L^2(0,1)}$ and thus on $\|n(\tau)\|_{L^2(B_1(0))}$ for all $0 \leq \tau < T^*$. Global existence for $Q = Q(y,\tau)$ as a consequence of Theorem 2.6. The results translate in a straightforward manner to the assertions of Theorem 2.2.

Theorem 2.19 *Let* $Q_0\colon [0,1] \mapsto [0,1]$ *be nondecreasing,* $Q_{0_y} \in L^\infty(0,1)$ *and* $Q_0(0) = 0$, $Q(1) = 1$. *Let* $\vartheta(0) = \vartheta_0 > 0$. *If either*

(i) ϑ_0 *is sufficiently large;*

or

(ii) *there exists* $B > 0$ *such that*

$$\vartheta_0 \geq \frac{3}{2}\frac{(1+B)(1+3B)}{(1+5B)\lambda}, \quad \text{and} \quad Q_0(y) \leq \frac{y(1+B)}{y^{2/3}+B},$$

with $\lambda = \exp\left(-\frac{2}{\kappa}\int_0^1 Q_{0_y}\log Q_{0_y}\,dy\right)$.

Then Problem (2.59)-(2.62) has a global solution $\langle Q, \vartheta \rangle$ in the sense of Section 2.3.3. Moreover there exist constants $L > 0$ and $\vartheta^* > 0$ such that

$$\vartheta(\tau) \leq \vartheta^* \quad \textit{and} \quad \|Q_y(\tau)\|_{L^2(0,1)} \leq \|Q_{0_y}\|_{L^2(0,1)} \exp(L\tau) \quad \textit{for all} \quad \tau > 0.$$

If (ii) is satisfied we have in addition

$$Q(y,\tau) \leq \frac{y(1+B)}{y^{2/3}+B} \quad \textit{for all} \quad (y,\tau) \in \overline{D_\infty}.$$

Proof: First we consider the auxiliary problem

$$(\mathbf{AP}) \quad \begin{cases} \bar{Q}_\tau &= y^{4/3}\bar{Q}_{yy} + \frac{1}{A}\bar{Q}\bar{Q}_y \quad &\text{in} \quad D_\infty, \\ \bar{Q}(0,\tau) &= 0, \qquad \bar{Q}(1,\tau) = 1 \quad &\text{for} \quad \tau > 0, \\ \bar{Q}(y,0) &= Q_0(y) \quad &\text{for} \quad y \in [0,1], \end{cases} \tag{2.64}$$

where $A > 0$ and where Q_0 satisfies the conditions of the theorem.

By Theorem 2.14 and Remark 2.13 (iii), we have: if either

(H_1) $A \geq \alpha_2$ and $\|Q_{0_y}\|^2_{L^2(0,1)} \leq C(A^2 - A)$

or

(H_2) $A = \frac{3}{2}(1+B)$ and $Q_0(y) \leq \frac{y(1+B)}{y^{2/3}+B}$ for $0 \leq y \leq 1$, and for some $B > 0$,

then Problem **AP** has a global solution $\bar{Q}\colon \overline{D_\infty} \mapsto [0,1]$. Since $Q_{0_y} \in L^\infty(0,1)$, the regularity theory of [12, Theorem 2] gives

$$\|\bar{Q}_y\|_{L^\infty(0,1)} \in L^\infty_{loc}([0,\infty)). \tag{2.65}$$

The conditions on Q_0 and ϑ_0 guarantee that Problem (2.59)-(2.62) has a classical solution in D_T for some T. Now suppose

$$T^* = \sup\{T > 0 \mid \text{solution of Problem (2.59)-(2.62) exists in } D_T\} < \infty. \tag{2.66}$$

Fix $A > 0$ such that (H_1) is satisfied, and choose $\vartheta_0 \geq \dfrac{A}{\lambda}$. By (2.63), we have

$$\vartheta(\tau) \geq A \quad \text{for all} \quad \tau \in [0, T^*) \tag{2.67}$$

and by Proposition 2.17 and (2.65), we find

$$Q(y,\tau) \leq \bar{Q}(y,\tau) \leq Ky \quad \text{for} \quad (y,\tau) \in [0,1] \times [0,T^*) \tag{2.68}$$

for some $K > 0$. Below we show that this implies a uniform bound on $\|Q_y(\cdot)\|_{L^2(\Omega)}$ in $[0,T^*)$. Multiplying (2.59) by $Q_\tau / y^{4/3}$ gives

$$\frac{1}{y^{4/3}}Q_\tau^2 = Q_\tau Q_{yy} + \frac{1}{\vartheta(\tau) y^{4/3}} Q_\tau Q Q_y \quad \text{in} \quad [0,1] \times (0,T^*). \tag{2.69}$$

Using (2.67) and (2.68), the second term on the right can be estimated by

$$\begin{aligned}\frac{1}{\vartheta(\tau)y^{4/3}}Q_\tau Q Q_y &\le \frac{1}{y^{4/3}}Q_\tau^2 + \frac{1}{4\vartheta^2(\tau)y^{4/3}}Q^2Q_y^2 \\ &\le \frac{1}{y^{4/3}}Q_\tau^2 + \frac{K^2}{4A^2}Q^2Q_y^2.\end{aligned}$$

Using this in (2.69) and integrating the results gives

$$\frac{d}{d\tau}\|Q_y(\tau)\|^2_{L^2(0,1)} \le \frac{K^2}{2A^2}\|Q_y(\tau)\|^2_{L^2(0,1)} \quad \text{for all} \quad 0 \le \tau < T^*.$$

Hence

$$\|Q_y(\tau)\|_{L^2(0,1)} \le \|Q_{0_y}\|_{L^2(0,1)} \exp\left(\left(\frac{M}{2A}\right)^2 T^*\right)$$

and

$$A \le \vartheta(\tau) \le \frac{1}{\kappa}\left(E + \int_0^1 \frac{\bar{Q}^2(y,\tau)}{y^{4/3}}\,dy\right) \le \frac{1}{\kappa}\left(E + \frac{3}{5}K^2\right)$$

for all $0 \le \tau < T^*$. This allows us to use Theorem 2.6 at T^{*-}, which contradicts (2.66). The uniform upper bound in the temperature follows from the observation

$$\bar{Q}(y,\tau) = \int_0^y \bar{Q}_y(y,\tau)\,dy \le y^{1/2}\|\bar{Q}_y(\tau)\|_{L^2(0,1)},$$

implying

$$\int_0^1 \frac{\bar{Q}^2(y,\tau)}{y^{4/3}}\,dy \le \frac{3}{2}\|\bar{Q}_y(\tau)\|^2_{L^2(0,1)},$$

and thus last expression is uniformly bounded if A satisfies (H_1) (Theorem 2.14).

If (ii) holds, global existence follows in a identical way. Again (2.67) and (2.68) hold, yielding the same bounds on $\|Q_y(\tau)\|_{L^2(0,1)}$ and $\vartheta(\tau)$. The pointwise bound on Q in D_∞ results from the fact that $y(1+B)/(y^{2/3}+B)$ is a supersolution for Problem **AP** if A and Q_0 satisfy (H_2). Take for instance $K = \frac{B+1}{B}$ in (2.68). The corresponding temperature bound is a direct consequence. ■

As a special case of Theorem 2.19 (ii) we have

Corollary 2.20 *If $Q_0(y) = y$, and $\vartheta_0 > \frac{3}{2}$, then Problem (2.59)-(2.62) has a global $\langle Q, \vartheta\rangle$ solution and $\vartheta_0 \le \vartheta(\tau) < \vartheta_0 + \frac{12}{5\kappa}$ for all $\tau \ge 0$.*

Proof: Since $\lambda = 1$, we can select a sufficiently small $B > 0$ such that Theorem 2.19 (ii) holds. The pointwise bound on Q implies $Q(y,\tau) \le y^{1/3}$ for all $(y,\tau) \in \overline{D_\infty}$. Since

$$\vartheta(\tau) = \vartheta_0 + \int_0^1 \frac{Q^2(y,\tau) - Q_0^2(y)}{y^{4/3}}\,dy,$$

the upper bound is immediate. ■

2.6 Appendix

2.6.1 Inequalities

For completeness we give in this appendix some inequalities which are used at various places in the paper.

Let Ω be a bounded open subset of $\mathbb{R}^N$ with a $C^{1+\alpha}(\alpha > 0)$ boundary.

First interpolation inequality. Let $N > 2$, $r \leq \frac{2N}{N-2}$ and let $p \leq q \leq r$ satisfy $\frac{1}{q} = \frac{\alpha}{p} + \frac{(1-\alpha)}{r}$ for some $\alpha \in (0,1)$. Then

$$\|n\|_{L^q(\Omega)} \leq C_s^{1-\alpha} \|n\|^{1-\alpha}_{H^1(\Omega)} \|n\|^{\alpha}_{L^p(\Omega)} \quad \text{for all} \quad n \in H^1(\Omega) \cap L^p(\Omega). \tag{2.70}$$

Proof: Use the Sobolev inequality $\|n\|_{L^r(\Omega)} \leq C_s \|n\|_{H^1(\Omega)}$ for $N > 2$ and $r \leq \frac{2N}{N-2}$, and the interpolation inequality $\|n\|_{L^q(\Omega)} \leq \|n\|^{\alpha}_{L^p(\Omega)} \|n\|^{1-\alpha}_{L^r(\Omega)}$. ∎

Second interpolation inequality. Let $N = 3$. Then

$$\|n\|_{L^3(\Omega)} \leq C_s^{1/2} \|n\|^{1/2}_{H^1(\Omega)} \|n\|^{1/2}_{L^2(\Omega)} \quad \text{for all} \quad n \in H^1(\Omega). \tag{2.71}$$

Proof: Take $p = 2$, $q = 3$, $r = \frac{2N}{N-2} = 6$ and $\alpha = 1/2$ in (2.70). ∎

Poisson's equation and L^p-norms. Let $n \in L^p(\Omega)$, $p > \frac{N}{2}$, and let ϕ satisfy (2.2) and (2.5). Then

$$\begin{cases} \|\nabla \phi\|_{L^r(\Omega)} \leq C_I \|n\|_{L^p(\Omega)} & \text{for} \quad 1 < r \leq \frac{pN}{N-p} \quad \text{and} \quad \frac{N}{2} < p < N, \\ \|\nabla \phi\|_{L^\infty(\Omega)} \leq C_I \|n\|_{L^p(\Omega)} & \text{for} \quad p > N. \end{cases} \tag{2.72}$$

where the constant C_I depends on Ω and p.

Proof: Since ϕ satisfies (2.2) with (2.5), we use the representation by the Green's function to obtain $\|\phi\|_{L^p(\Omega)} \leq \|\Delta \phi\|_{L^p(\Omega)}$ for $E > 2$ and $p > N/2$. If $p < N$ we combine this with the Sobolev inequality $\|\nabla \phi\|_{L^r(\Omega)} \leq C(\|\Delta \phi\|_{L^p(\Omega)} + \|\phi\|_{L^p(\Omega)})$ for $r \leq pN/(N-p)$ to obtain the desired inequality. If $p > N$ we proceed similarly. ∎

2.6.2 A result on global existence for constant temperature

Theorem 2.21 *If there exists $B > 0$ such that $Q_0(y) \leq y \frac{1+B}{y^{2/3}+B}$ for $0 \leq y \leq 1$ and $\vartheta^* \geq \frac{3}{2} \frac{(1+B)(1+3B)}{(1+5B)}$, then Problem Q^* with $\vartheta(t) = \vartheta^*$ has a global classical solution satisfying $Q(y,t) \leq y \frac{1+B}{y^{2/2}+B}$ for $(y,t) \in \overline{D_\infty}$.*

Proof. We follow the proof of [16, Theorem 1 (iii)], for a similar result. We can check that the function

$$q(y,t) = \frac{Ay}{y^{2/3} + B}$$

satisfies
$$q_t - \vartheta^* \, y^{4/3} \, q_{yy} + q q_y = (-\frac{2\vartheta^*}{9} + \frac{A}{3}) y^{2/3} + (-\frac{10\vartheta^*}{9} + A) B.$$
Hence q is a super-solution provided that
$$(-\frac{2\vartheta^*}{9} + \frac{A}{3}) y^{2/3} + (-\frac{10\vartheta^*}{9} + A) B \leq 0.$$
This inequality holds if either $\vartheta^* \geq \frac{3A}{2}$ or $\vartheta^* \geq \frac{3A}{2} \frac{1+3B}{1+5B}$ depending on the sign in front of the term $y^{2/3}$. Clearly in the second case we have a sharper estimate. Finally, since $A \geq 1 + B$ we choose $A = 1 + B$ to conclude the result. ■

Chapter 3

Convergence to stationary states

3.1 Introduction

In this chapter we study the convergence of solutions to a stationary state of Problem **P** from Chapter 2. We recall that this problem consists of the parabolic-elliptic system

$$n_t = \operatorname{div}\{\Theta(t)\nabla n + n\nabla\phi\} \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{3.1a}$$

$$\Delta\phi = n \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{3.1b}$$

combined with the energy relation

$$E = \kappa\Theta(t) + \int_\Omega n\phi dx \quad \text{in} \quad \mathbb{R}^+, \tag{3.1c}$$

where $E \in \mathbb{R}$ and $\kappa > 0$ are given parameters and where $\Omega \subset \mathbb{R}^3$ is a bounded open set. At the boundary $\partial\Omega \in C^{1+\alpha}$ $(\alpha > 0)$ we prescribe

$$(\Theta(t)\nabla n + n\nabla\phi) \cdot \vec{\nu} = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \tag{3.1d}$$

$$\phi = 0 \quad \text{in} \quad \partial\Omega \times \mathbb{R}^+, \tag{3.1e}$$

where $\vec{\nu}$ denotes the exterior normal vector on $\partial\Omega$. At $t = 0$ we have the initial condition

$$n(x, 0) = n_0(x) \quad \text{in} \quad \Omega. \tag{3.1f}$$

This chapter will appear as a paper in collaboration with T. Nadzieja (Zielona Gora, Poland).

We assume for n_0

$$\int_\Omega n_0\, dx = 1, \quad \text{and} \quad n_0 \geq 0 \quad \text{in} \quad \Omega, \tag{3.1g}$$

while $\sup\limits_{x\in\Omega} |x| = 1$.

Throughout this chapter we shall refer to this set of equations as Problem $\mathbf{P}$ for the unknowns n, ϕ and Θ. Note that ϕ can be obtained from n by the boundary value problem (3.1b) and (3.1f). Therefore we denote by $\langle n, \Theta\rangle$ the solution of Problem $\mathbf{P}$.

The problem of existence and uniqueness was studied in Chapter 2. For $n_0 \in L^2(\Omega)$ and $\Theta(0) > 0$, local existence and uniqueness of solutions was proved. Moreover assuming Ω is a ball and n_0 is radially symmetric, an a-priori estimate of the $L^2(\Omega)$ norm of solutions was obtained. This estimate guarantees global existence.

The main result of this chapter is to show convergence of global radially symmetric solutions towards stationary solutions. This requires a uniform bound on $\|n\|_{L^p(\Omega)}$ for some $2 \leq p \leq \infty$, a positive lower bound on $\Theta(t)$, and uniqueness of the stationary problem. The precise statement is given in Theorem 3.11.

A delicate issue is to obtain conditions on the data of Problem $\mathbf{P}$, i.e. $n_0, \Theta(0)$, and κ, which ensure the uniform L^p bound on $n(t)$. The uniform lower bound on $\Theta(t)$ was already constructed in the global existence proof of Chapter 2. There it was shown for radially symmetric solutions:

Proposition 3.1 *Let $n_0 \in L^\infty(\Omega)$. Let $\langle n, \Theta\rangle$ be a solution of Problem $\mathbf{P}$. Suppose $\Theta(0) = \frac{1}{\kappa}\Big(E + \int\limits_\Omega n_0\phi_0\, dy\Big) > 0$. Then*

$$\Theta(t) \geq \lambda\Theta(0) \quad \textit{for} \quad t > 0, \tag{3.2}$$

with

$$\lambda = \exp\left(-\frac{2}{\kappa}\left[\int\limits_\Omega n_0 \log n_0\, dx - \log\left(\frac{1}{\mathrm{vol}(\Omega)}\right)\right]\right). \tag{3.3}$$

The function ϕ_0 is determined by the solution of the boundary value problem (3.1b) and (3.1e) replacing n by n_0.

Remark 3.2 *Note that, the integral $\int\limits_\Omega n_0 \log n_0\, dx$ is finite whenever $n_0 \in L^\infty(\Omega)$.*

Inequality (3.2) for general domains was obtained in [18].

Concerning the uniqueness of stationary solutions, we shall show that it is sufficient to assume a large energy E, which is accomplish by taking large $\Theta(0)$, with n_0 and κ fixed.

To highlight some of the convergence results we give here explicit conditions on the data of Problem $\mathbf{P}$ that leads to uniform $L^\infty(\Omega)$ and $L^2(\Omega)$ bounds on $n(t)$. Here, as in the case of global existence (Theorem 2.2), we consider $\Omega = B_1(0)$ the unitary ball in $\mathbb{R}^3$ and n_0 radially symmetric. Thus again we restrict ourselves to the study of radially symmetric solutions and to emphasize that we shall sometimes denote by $n(r,t) = n(x,t)$ with $|x| = r$.

The next result shows convergence to a stationary solution by providing a class of initial data such that the corresponding solution $n(t)$ is uniformly bounded in $L^\infty(\Omega)$.

Theorem 3.3 *Let $\Omega = B_1(0)$ and let $\lambda > 0$ given by (3.3). Assume that Problem* **P** *has a radially symmetric solution $\langle n, \Theta\rangle$. Suppose there exists $B > 0$ such that*

(i) *the initial data n_0 satisfies $r^3 n_0(r) \le \frac{3}{4\pi}\|n_0\|_{L^1(B_r(0))}$ and $\|n_0\|_{L^1(B_r(0))} \le (1+B)\frac{r^3}{r^2+B}$ for $r \in [0,1]$;*

(ii) *a lower bound on initial temperature*

$$\Theta(0) \ge \frac{1}{8\pi}\frac{(1+B)(1+3B)}{\lambda(1+5B)};$$

(iii) *$E \in \mathbb{R}$ and $\kappa > 0$ are such that there exists a unique stationary solution $\langle n_s, \Theta_s\rangle$ of Problem* **P**.

Then Problem **P** *has a global solution $\langle n(t), \Theta(t)\rangle$, with $n(t)$ uniformly bounded in $L^\infty(\Omega)$. This solution converges to the stationary solution $\langle n_s, \Theta_s\rangle$.*

Remark 3.4 *Note that $n_0 \equiv constant = \frac{3}{4\pi}$ satisfies (i) for any $B > 0$. Since in this case condition (ii) holds for any $B > 0$ and $\lambda = 1$, (ii) becomes $\Theta_0 \ge \frac{1}{8\pi}\gamma$ with $\gamma = 0.9519....$ Here the value of γ is computed as in Theorem 2.2.*

Remark 3.5 *The first inequality in (i) is new with respect to the assumptions in Theorem 2.2. This extra condition implies the uniform $L^\infty(\Omega)$ bound in time for the solution. Combining the two conditions in (i), we have*

$$n_0(r) \le \frac{3}{4\pi}\frac{1+B}{r^2+B} \quad \textit{for} \quad r \in (0,1).$$

We will show that this estimate implies the uniform bound in time:

$$n(r,t) \le \frac{3}{4\pi}\frac{1+B}{r^2+B} \quad \textit{for} \quad r \in (0,1),\ t > 0.$$

The next result concerns with the $L^2(\Omega)$ bound.

Theorem 3.6 *Let $\Omega = B_1(0)$ and let $\lambda > 0$ given by (3.3). Assume that Problem* **P** *has a radially symmetric solution $\langle n, \Theta\rangle$. Suppose there exists α and C positive constants such that, if*

(i) $n_0 \in L^\infty(\Omega)$ *and* $\Theta_0 > 0$ *are such that*

$$\lambda\Theta_0 \geq \alpha \quad \text{and} \quad \|n_0\|^2_{L^2(\Omega)} \leq C((\lambda\Theta_0)^2 - \lambda\Theta_0),$$

(ii) $E \in \mathbb{R}$ *and* $\kappa > 0$ *are such that there exists only one stationary solution* $\langle n_s, \Theta_s \rangle$ *of Problem* **P**.

Then Problem **P** *has a global solution* $\langle n(t), \Theta(t) \rangle$, *with* $n(t)$ *uniformly bounded in* $L^2(\Omega)$. *This solution converges to the stationary solution* $\langle n_s, \Theta_s \rangle$.

Remark 3.7 *Note that condition*

$$(i)' \qquad n_0 \in L^\infty(\Omega) \quad \text{and large} \quad \Theta_0,$$

implies condition (i). In fact for a given n_0 *we can choose* Θ_0 *sufficiently large so that condition (i) is satisfied.*

Remark 3.8 *The* L^2*-bound in* n_0 *implies a uniform bound for* n, *given by*

$$\|n(t)\|^2_{L^2(\Omega)} \leq C((\lambda\Theta_0)^2 - \lambda\Theta_0).$$

The remainder of this chapter is organized as follows. In Section 3.2, we carried out the transformation from Section 2.2.4 for radially symmetric solutions. In terms of this transformed variables we discuss the existence and uniqueness of stationary solutions. Next we construct the uniform $L^\infty(\Omega)$ bound for the particular initial data mentioned in Theorem 3.3 In Section 3.3, the main convergence theorem is proven and as a first consequence we completes the proof of Theorem 3.3. In Section 3.4, without assuming radial symmetry, we construct an uniform $L^2(\Omega)$ bound of a solution of Problem **P** and as a consequence we prove Theorem 3.6.

3.2 Radially symmetric solutions

Following transformation from Section 2.2.4 for radially symmetric solutions we obtain for

$$Q(r,t) := \int_{B_r(0)} n(x,t)\,dx \quad \text{for} \quad r \in (0,1] \quad \text{and} \quad t \in \mathbb{R}^+.$$

the problem (with redefined temperature ϑ and the energy $\mathcal{E}$)

$$Q_t = y^{4/3}\vartheta(t) Q_{yy} + QQ_y \quad \text{for} \quad y \in (0,1) \quad \text{and} \quad t \in \mathbb{R}^+, \tag{3.4a}$$

$$\mathcal{E} = \kappa\vartheta(t) - \int_0^1 \frac{Q^2}{y^{4/3}} dy \quad \text{for} \quad t \in \mathbb{R}^+, \tag{3.4b}$$

$$Q(0,t) = 0, \quad Q(1,t) = 1, \quad \text{for} \quad t \in \mathbb{R}^+, \tag{3.4c}$$

and

$$Q(y,0) = Q_0(y) := \frac{4\pi}{3}\int_0^y n_0(y^{1/3})dy \quad \text{for} \quad 0 \le y \le 1. \tag{3.4d}$$

Equations (3.4) define Problem **Q** for the unknowns $\langle Q, \vartheta\rangle$. We call the pair $\langle Q, \vartheta\rangle$ a solution for Problem **Q** if satisfies equations (3.4) in the classical sense. Since $n \ge 0$, it is clear we only look for solutions satisfying $Q_y \ge 0$.

3.2.1 Stationary solutions

Existence of stationary solutions of Problem **Q** is well known [1, 25, 12, 10] see Theorem 2.4. We present here a result which asserts uniqueness of stationary solutions for appropriated choices of E and κ.

Theorem 3.9 *For any $\kappa > 0$, there exists an energy $\mathcal{E}(\kappa)$, such that for $\mathcal{E} > \mathcal{E}(\kappa)$ there exists a unique stationary solution $\langle Q_s, \vartheta_s\rangle$ for Problem* **Q**.

Proof. We introduce the new function $\bar{Q} := Q_s/\vartheta_s$ which satisfies the equation

$$y^{4/3}\bar{Q}_{yy} + \bar{Q}\bar{Q}_y = 0 \quad \text{for} \quad y \in (0,1) \tag{3.5}$$

and the boundary condition

$$\bar{Q}(0) = 0, \ \bar{Q}(1) = 1/\vartheta_s. \tag{3.6}$$

Next we introduce the variables [12]

$$v = 9y^{\frac{2}{3}}\bar{Q}_y, \qquad w = 3y^{-\frac{1}{3}}\bar{Q}, \qquad y = e^{3s}.$$

A simple computation shows that v, w satisfy the system of equations

$$v' = (2-w)v, \quad w' = v - w, \tag{3.7}$$

where $'$ denotes $\frac{d}{ds}$. The boundary conditions translate into $w(-\infty) = 0$, $w(0) = \frac{3}{\vartheta_s}$. Note that system (3.7) has solutions for all $s \in \mathbb{R}$. In particular there exists a unique trajectory $\langle v(s), w(s)\rangle$ connecting the critical point $\langle v(-\infty), w(-\infty)\rangle = \langle 0, 0\rangle$ and $\langle v(\infty), w(\infty)\rangle = \langle 2, 2\rangle$. This trajectory is shown in Figure 3.1. The corresponding solution $w(s)$ is nonnegative and bounded. Using this boundedness into the energy relation

$$\mathcal{E}(\vartheta_s) = \kappa\vartheta_s - \vartheta_s^2 \int_{-\infty}^{0} w^2(s)e^s \, ds.$$

shows that a large energy implies a large temperature and hence a small value of $w(0)$. This ensures the uniqueness. ■

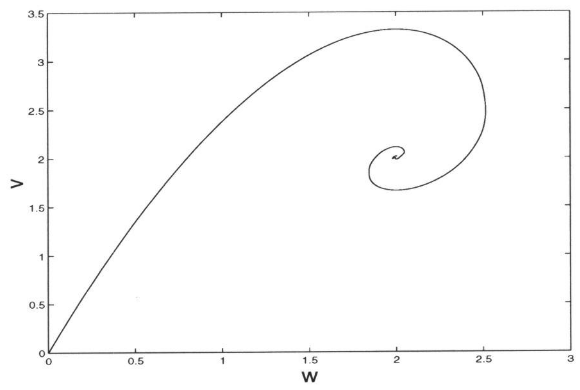

Figure 3.1: The solution $\langle v(s), w(s)\rangle$ of system (3.7) with limits $\langle 0,0\rangle$ as $s \to -\infty$ and $\langle 2,2\rangle$ as $s \to \infty$.

3.2.2 An uniform $L^\infty(\Omega)$ bound

In the next result we provide a class of initial data for Problem **Q** which give a uniform bound in time for Q_y .

Lemma 3.10 *If $(Q_0)_y \le Q_0/y$ for $y \in (0,1]$ then the solution $\langle Q, \vartheta\rangle$ of Problem* **Q** *satisfies*

$$Q_y < Q/y \quad in \quad (0,1] \times (0,\infty). \tag{3.8}$$

Proof. Let $T > 0$ be any finite time. We switch to the average density $b(r,t) = Q(y,t)/y$ with $y = r^3$. Using (3.4a) it follows that

$$b_t = \frac{\vartheta}{9} b_{rr} + \frac{4\vartheta}{9r} b_r + \frac{1}{3} r b b_r + b^2 \quad \text{in} \quad D_T := (0,1) \times (0,T). \tag{3.9}$$

Following the ideas of [39], we define the variable $w = w(r,t) := r^4 b_r$, which satisfies

$$w_t - \frac{\vartheta}{9}\left(w_{rr} - \frac{4}{r} w_r\right) - \frac{1}{3} r b w_r = \left(b + \frac{1}{3} r b_r\right) w \quad \text{in} \quad D_T.$$

Note that $w = 3yQ_y - 3Q$.

Assume for the moment the stronger assumption on the initial data:

$$(b_0)_r(r) < 0 \quad \text{for} \quad r \in (0,1) \quad \text{and} \quad (b_0)_{rr}(0) < 0. \tag{3.10}$$

This implies $w(0,r) = r^4 b_r(0,r) < 0$. Now if $b_0 = Q_0/y$ satisfies (3.10) then the function $b \equiv 1$ is a sub-solution of (3.9). Applying Hopf's Lemma to (3.9), we find that $w(1,t) = b_r(t,1) < 0$ for all $t > 0$. By global existence and regularity we have

$Q_y = b + \frac{rb_r}{3} < C(T)$ on $\overline{D_T}$ for each $T < \infty$. Since $T > 0$ was arbitrarily chosen, an application of the maximum principle gives

$$w = 3yQ_y - 3Q < 0 \quad \text{in} \quad D_\infty.$$

To conclude, we note that if b_0 is non-increasing, i.e. $(Q_0)_y \le Q_0/y$, then by the strong maximum principle, condition (3.10) is satisfied by a solution $b(r, \bar{t})$ in $(0, 1)$ for each $\bar{t} > 0$. This proves the result. ■

3.3 The convergence result and proof of Theorem 3.3

Now we state our main result.

Theorem 3.11 *Suppose the constants $\mathcal{E} \in \mathbb{R}$ and $\kappa > 0$ are such that the stationary solution $\langle Q_s, \vartheta_s \rangle$ of Problem* **Q** *is unique. Suppose that the initial data $\langle Q_0, \vartheta_0 \rangle$ are chosen such that the global solution $\langle Q, \vartheta \rangle$ of Problem* **Q** *satisfies $\|Q_y(t)\|_{L^p(0,1)} \le M$ for some $p \ge 2$ and $\vartheta(t) > c > 0$ for all $t > 0$. Then $\langle Q(t), \vartheta(t) \rangle \to \langle Q_s, \vartheta_s \rangle$ as $t \to \infty$, uniformly on $[0, 1]$.*

Proof. Denote by $\langle Q_s, \vartheta_s \rangle$, the unique stationary solution of Problem **Q**. Consider the functional

$$W(t) := \int_0^1 \left\{ Q_y(\log(\frac{Q_y}{Q_y^s} - 1) + Q_s \right\} dy - \log(\frac{\vartheta}{\vartheta_s}).$$

Below we show that W is a Lyapunov functional. Let

$$W'(t) = \int_0^1 (Q_t)_y(\log Q_y + 1)\, dy - \frac{\vartheta_t}{\vartheta}.$$

Integrating by parts, we obtain

$$W'(t) = -\int_0^1 \frac{Q_{yy}}{Q_y}\, dy - \frac{\vartheta_t}{\vartheta} = -\int_0^1 Q_t \left(\frac{Q_{yy}}{Q_y} + \frac{1}{\vartheta} Q y^{-\frac{4}{3}} \right) dy \tag{3.11}$$

$$= -\int_0^1 \frac{Q_t^2}{Q_y \vartheta} y^{-4/3}\, dy < 0. \tag{3.12}$$

It follows from the properties of the solution that W is bounded from below. Hence there exists a sequence $\{t_n\}$ such that $W'(t_n) \to 0$ as $n \to \infty$. This implies that for every $0 \le y \le 1$,

$$A(y, t_n) := \int_0^y Q_t(s, t_n)\, ds = \int_0^y \left\{ y^{4/3} \vartheta(t_n) Q_{yy}(s, t_n) + \frac{1}{2}(Q^2)_y(s, t_n) \right\} ds$$

tends to 0 as $t_n \to \infty$. Integrating by parts, we get

$$A(y,t_n) = \vartheta(t_n)\left[y^{4/3}Q_y(y,t_n) - \frac{4}{3}y^{1/3}Q(y,t_n) + \frac{4}{9}\int_0^y s^{-\frac{2}{3}}Q(s,t_n)\,ds\right] + \frac{1}{2}Q^2(y,t_n).$$

The uniform bound in time for $\|Q_y(t)\|$ and the lower bound on $\vartheta(t)$ imply that the family $Q(\cdot,t_n)$ is compact in C^0-topology and $\vartheta(t_n)$ is bounded. Therefore $Q(\cdot,t_n) \to \bar{Q}(\cdot)$ uniformly on $[0,1]$ and $\vartheta(t_n)$ converges to $\bar{\vartheta} > 0$. Moreover

$$\mathcal{E} = \kappa\bar{\vartheta} - \int_0^1 \frac{\bar{Q}^2(y)}{y^{4/3}}\,dy$$

Because $A(y,t_n) \to 0$ we conclude that $Q_y(\cdot,t_n)$ converges pointwise to $\bar{Q}_y$ on $(0,1]$, and $\bar{Q}$ satisfies

$$\bar{\vartheta}\left[y^{4/3}\bar{Q}_y(y) - \frac{4}{3}y^{1/3}\bar{Q}(y) + \frac{4}{9}\int_0^y s^{-\frac{2}{3}}\bar{Q}(s)\,ds\right] + \frac{1}{2}(\bar{Q})^2(y) = 0.$$

Differentiating above formula with respect to y we see that $y^{4/3}\bar{\vartheta}\bar{Q}_{yy} + \bar{Q}\bar{Q}_y = 0$, therefore $\bar{Q} = Q_s$, the unique stationary solution of Problem **Q**.

To conclude the theorem, let $\{s_n\}$ denote an arbitrary sequence satisfying $s_n \to \infty$ as $n \to \infty$. Because $W(t)$ is bounded there exists a sequence $\{t_n\}$ such that $|t_n - s_n| \downarrow 0$ and $W'(t_n) \to 0$. Clearly $Q(t_n) \to Q_s$ as $n \to \infty$. Now let $Q(s_n) \to Q_1$ as $n \to \infty$. We show that $Q_1 = Q_s$. From (3.11) we obtain

$$|W(t_n) - W(s_n)| = \int_{s_n}^{t_n}\int_0^1 \left(\frac{Q_t^2}{Q_y\vartheta}y^{-4/3}\,dy\right)dt \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that $\int_0^1\int_{s_n}^{t_n} |Q_t|\,dtdy \to 0$, Hence $\int_0^1 |Q(s_n) - Q(t_n)|\,dy \le \int_0^1\int_{s_n}^{t_n} |Q_t|\,dydt \to 0$, as $n \to \infty$ and therefore $Q_1 = Q_s$. ■

Proof of Theorem 3.3. Theorem 2.19 gives us

$$Q(y,t) \le \frac{y(1+B)}{y^{2/3}+B} \quad \text{for all} \quad (y,t) \in \overline{D_\infty}.$$

Together with Lemma 3.10, we find the $L^\infty(\Omega)$ bound on Q_y. Combining this with the uniform lower bound on temperature from Proposition 3.1, the result follows directly from Theorem 3.11. ■

3.4 An uniform $L^2(\Omega)$ bound and proof of Theorem 3.6

For the next result the condition that Ω is a ball in $\mathbb{R}^3$. We now assume that Ω is bounded in $\mathbb{R}^3$ with piecewise smooth boundary. We extend a global existence result of [15].

Theorem 3.12 *Suppose the initial temperature Θ_0 and the initial distribution n_0 satisfy*

$$\lambda\Theta_0 \geq \alpha \quad \textit{and} \quad \|n_0\|^2_{L^2(\Omega)} \leq C\left((\lambda\Theta_0)^2 - \lambda\Theta_0\right),$$

where α and C are specific constants depending only on the domain Ω and where $\lambda > 0$ is given by (3.3). Then Problem **P** *has a global (weak) solution for which the $L^2(\Omega)$ norm is uniformly bounded in time.*

Proof. As in Chapter 2, we manipulate equation (2.52) and obtain

$$\frac{1}{2}\frac{d}{dt}\|n\|^2_{L^2(\Omega)} + \Theta(t)\|\nabla n\|^2_{L^2(\Omega)} = -\int_\Omega n\nabla n\nabla\phi\,dx. \tag{3.13}$$

Using this, we obtain for $w(t) := \|n(t)\|^2_{L^2(\Omega)}$, the inequality (2.56)

$$\frac{d}{dt}w \leq p_{\Theta(t)}(w) := \frac{C_1}{\Theta(t)^3}w^3 - \Theta(t)w + C_2\Theta(t) \quad \text{for} \quad t > 0 \tag{3.14}$$

Here C_1 and C_2 are positive constants only depending on Ω. The assertion of the theorem now follow from particular properties of (3.14).

First observe that if $\Theta > \alpha_1 := \frac{3^{3/4}}{2^{1/2}}(C_2C_1^{1/2})^{1/2}$, then $p_{\Theta(t)}(w) = 0$ has two positive real roots $w_*(t) < w^*(t)$ and $p_{\Theta(t)}(w) < 0$ for $w_*(t) < w < w^*(t)$. If $\Theta \equiv \alpha_1$, these roots coincide.

Now consider

$$w^0(t) := C_1^{-1/2}(\Theta^2(t) - \Theta(t)).$$

Then $p_\Theta(w^0(t)) \leq 0$ provided

$$\Theta(t) \geq \alpha_2 = \left(\left(\frac{3 + C_2C_1^{1/2}}{4}\right)^2 - \frac{1}{2}\right)^{1/2} + \frac{3 + C_2C_1^{1/2}}{4}.$$

Clearly $\alpha_2 \geq \alpha_1$, since $p_{\Theta(t)}(w) > 0$ for $\Theta(t) < \alpha_1$ and for all $w > 0$. We now choose $w(0)$ such that $w(0) \leq \bar{w}^0 \leq w^0(t)$, where $\bar{w}^0 := C_1^{-1/2}\left((\lambda\Theta_0)^2 - \lambda\Theta_0\right)$. Note that $p_{\Theta(t)}(\bar{w}^0) < 0$ for all $t > 0$. Hence it follows $w(t) \leq w(0) \leq \bar{w}^0$. ■

We are now in a position to prove the second main theorem.

Proof of Theorem 3.3. Using the uniform bound on $\Theta(t)$ from Proposition 2.18 and employing the above L^2-bound of the solution n, i.e. the uniform bound in time of $\|Q_y\|_{L^2(0,1)}$, we apply Theorem 3.11 to obtain the result. ■

Chapter 4

Asymptotic self-similar blow-up for two models arising in statistical mechanics

4.1 Introduction

In this chapter we study the blow-up profiles for the parabolic-elliptic system

$$
\begin{aligned}
n_t &= \operatorname{div}\{\Theta(t)\nabla n + n\nabla\phi\} && \text{in} \quad \Omega \times \mathbb{R}^+, && (4.1a)\\
\Delta\phi &= n && \text{in} \quad \Omega \times \mathbb{R}^+, && (4.1b)\\
0 &= (\Theta(t)\nabla n + n\nabla\phi)\cdot\vec{\nu} && \text{on} \quad \partial\Omega \times \mathbb{R}^+, && (4.1c)\\
\phi &= 0 && \text{in} \quad \partial\Omega \times \mathbb{R}^+, && (4.1d)\\
n(x,0) &= n_0(x) && \text{in} \quad \Omega, && (4.1e)
\end{aligned}
$$

where $\Omega = B_1(0) = \{x \in \mathbb{R}^3\colon |x| \le 1\}$. The initial condition n_0 is chosen radially symmetric and such that

$$\int_\Omega n_0\, dx = 1, \quad \text{and} \quad n_0(x) \ge 0 \quad \text{in} \quad \Omega. \tag{4.1f}$$

Regarding temperature $\Theta = \Theta(t)$, we introduce two models:

This chapter is to appear as an article in collaboration with M.A. Peletier (CWI, Amsterdam) and J. Williams (Bath university, England) .

(i) Temperature is defined by:

$$\Theta(t) \equiv \Theta^* = \text{constant} > 0. \tag{4.2}$$

Equations (4.1)–(4.2) define a problem for the unknown density n and potential ϕ. However, we observe that since ϕ is known in terms of n by the boundary value problem (4.1b) and (4.1d), we denote a solution of (4.1)–(4.2) simply by n. Since the temperature remains constant we shall call this problem the isothermal model.

(ii) Temperature is given by the energy conservation relation

$$E = \kappa\Theta(t) + \int_\Omega n\phi\, dx \quad \text{in} \quad \mathbb{R}^+, \tag{4.3a}$$

where E and $\kappa > 0$ are given constants such that

$$\Theta(0) = \Theta_0 > 0. \tag{4.3b}$$

Equations (4.1),(4.3) define a problem for the unknown density n, potential ϕ, and temperature Θ. Arguing as in the previous problem, since ϕ can be expressed in terms of n, we denote a solution of problem (4.1),(4.3) by the pair $\langle n, \Theta\rangle$. We shall call this problem the non-isothermal model.

Problems (4.1)–(4.2) and (4.1),(4.3) are models for the evolution of a cluster of particles under gravitational interaction and Brownian motion. The isothermal model also arises in the study of the motion of bacteria by chemotaxis [6, 20]. The non-isothermal model was first introduced to describe galactic dynamics [26]. Note that both models are in dimensionless form. This was done in Chapter 2, where we have chosen mass and radius one. By the non-flux condition (4.1c), mass is preserved. In fact, condition (4.1f) gives

$$\int_\Omega n(x,t)\, dx = \int_\Omega n_0(x)\, dx = 1. \tag{4.4}$$

We know from Chapter 2 that problems (4.1)–(4.2), and (4.1)–(4.3) have a unique local solution if $n_0 \in L^2(\Omega)$. Moreover, this solution satisfies $n \in L^\infty(0,T; L^\infty(\Omega))$ for some $T > 0$. Now, since $\Omega = B_1(0)$ and n_0 is radially symmetric, by uniqueness this local solution is radially symmetric. For that reason, we restrict ourselves to the analysis of radial solutions, and to emphasize this we write $n(r,t) := n(x,t)$ with $r = |x| \in [0,1]$.

Since we are interested in the question when and how the isothermal and the non-isothermal model generate singularities, we define:

$$T_1 = \sup\{\, \tau > 0 \mid \text{Problem (4.1)–(4.2) has a solution in } (0,\tau]\,\},$$

and

$$T_2 = \sup\{\, \tau > 0 \mid \text{Problem (4.1),(4.3) has a solution in } (0,\tau]\,\}.$$

If $T_1 < \infty$, or $T_2 < \infty$ then we say that blow-up occurs for (4.1)–(4.2) and (4.1),(4.3) respectively. For $T_2 < \infty$ we find that energy conservation implies $\lim_{t \to T_2} \sup_{[0,1]} n(r,t) = \infty$. We shall discuss later the behaviour of $\Theta(t)$ near $t = T$.

There are various conditions in the literature which ensure $T_1 < \infty$ or $T_2 < \infty$. For example, to ensure $T_1 < \infty$, we can assume that $\Theta^* < \frac{1}{24\pi}$ [12] and to ensure $T_2 < \infty$ we can suppose that $E < \frac{1}{4\pi}$ and $\kappa > 6$ (Chapter 2). However, from the proof of these results we cannot infer how the blow-up occurs.

The first aim of this chapter is to characterize the asymptotic behaviour near blow-up of the solution $n(r,t)$ of problem (4.1)–(4.2). We prove that under certain conditions on the initial data, a solution $n = n(r,t)$ of (4.1)–(4.2), which blows up at time $T > 0$ and at the point $r = 0$, with the growth condition

$$\sup_{r \in [0,1]} n(r,t) \leq M(T-t)^{-1} \quad \text{for} \quad t \in (0,T),$$

has a structure near blow up given by

$$n_*(r,t) = (T-t)^{-1} \Psi \left(\frac{r}{\sqrt{\Theta^*(T-t)}} \right),$$

where the function Ψ belongs to a subset of solutions of a steady state problem; a subset that includes the functions

$$\Psi_1(\eta) := \frac{(6+\eta^2)}{(1+\frac{1}{2}\eta^2)^2} \quad \text{and} \quad \Psi^*(\eta) := 1 \quad \text{for} \quad \eta > 0,$$

(Theorem 4.1).

Our second goal is to provide a numerical description of blow-up in the non-isothermal case. We show that for constant initial data n_0, and for $E \in \mathbb{R}$ and $\kappa > 0$ ensuring blow-up, the corresponding solution $\langle n, \Theta \rangle$ of (4.1),(4.3), which blows up at time $T > 0$ and at the point $r = 0$, has a structure near blow up given by:

$$\begin{cases} n_*(r,t) = (T-t)^{-1} \Psi_1 \left(\frac{r}{\sqrt{\Theta(t)(T-t)}} \right), \\ \Theta_*(t) = \bar{\Theta} = \mathit{constant}. \end{cases}$$

See Figure 4.1 in the case $E = 0$ and $\kappa = 10$.

The form of blow-up for radially symmetric solutions for the isothermal model were already described in [50, 51, 21]. The problem of blow-up was first studied by Herrero et al. in [50, 51] using careful matched asymptotic expansions and later by Brenner et al. in [21], using an accurate numerical analysis and deriving various analytical results, such as existence and linear stability of self-similar profiles. Note however that no proof of convergence or characterization of blow-up in terms of initial data is given in these references The principal types of blow-up described in [50, 51, 21] are:

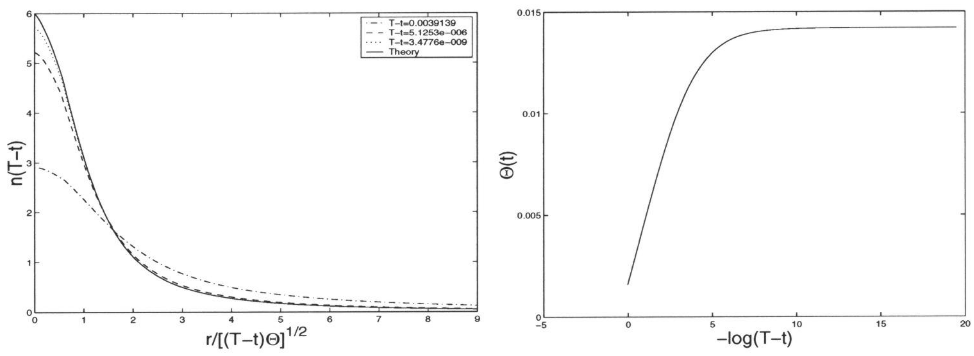

Figure 4.1: Convergence in self-similar variables to the profile Π_1(left) and the associated temperature behaviour(right), with $E = 0$, $\kappa = 10$, and $n_0 \equiv 3/4\pi$.

(a) A solution $n(r,t)$ consists of an imploding smoothed-out shock wave which moves towards the origin. As $t \to T$, the bulk of such a wave is concentrated at distances $O((T-t)^{1/3})$ from the origin, has a width $O((T-t)^{2/3})$, and at its peak it reaches a height of order $O((T-t)^{-4/3})$. This type of blow-up has the property of concentration of mass at the origin at the blow-up time, i.e.

$$\lim_{r\to 0}\left[\lim_{t\to T}\int_0^r n(y,t)y^2\,dy\right] = C > 0. \tag{4.5}$$

This situation is known as gravitational or chemotactic collapse and is depicted in Figure 4.2 (left). Note that the class of initial data considered in our study rules out this possibility.

(b) A solution $n(r,t)$ has a self-similar blow-up in the explicit form

$$(T-t)n\left(\eta\sqrt{(T-t)\Theta^*},t\right) \sim \Psi_1(\eta) \quad \text{as} \quad t \to T.$$

Note that this implies that n satisfies (4.5) with $C = 0$. Therefore no concentration of mass at the origin occurs at the blow-up time. This blow-up behaviour is depicted in Figure 4.2 (right). This is the type of behaviour found in our results.

4.2 Main results

For radial solutions, the average density function $b(r,t)$ is defined by

$$b(r,t) := \frac{3\omega_3}{r^3}\int_0^r n(y,t)y^2\,dy, \tag{4.6}$$

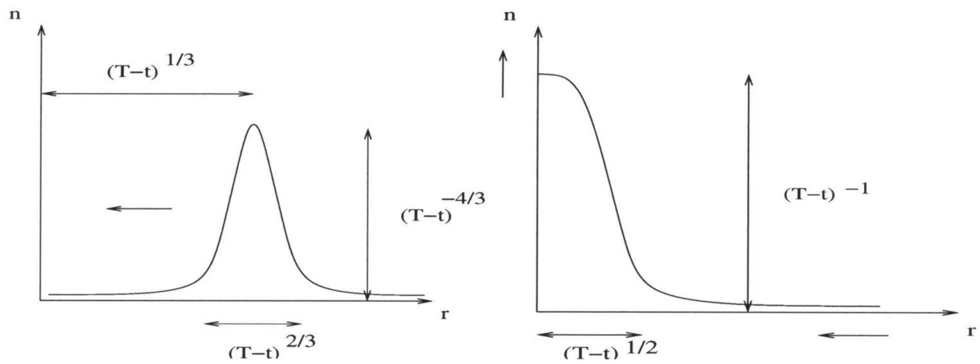

Figure 4.2: The profile $n(r,t)$ for blow-up with(left) and without(right) concentration of mass.

where $\omega_3 = \frac{4\pi}{3}$ is the volume of the unit ball in $\mathbb{R}^3$. This variable turns out to be most convenient when analyzing this system. In fact, it has the has the same scale invariance as $n(r,t)$, and it has in addition the advantage that solutions are smoothen. For example, if for some fixed $t > 0$ the density $n(r,t)$ is a delta function at the origin with unit mass, then $b(r,t) = r^{-3}$.

Let $D = (0,1)$ and set $D_T = D \times (0,T)$ for some time $T > 0$. Transformation (4.6) puts system (4.1) in the form

$$b_t = \omega_3 \Theta(t) \left(b_{rr} + \frac{4}{r} b_r \right) + \frac{1}{3} r b b_r + b^2 \quad \text{in} \quad D_T \tag{4.7a}$$

$$b_r(0,t) = 0, \quad b(1,t) = 1, \quad \text{for} \quad t \in [0,T), \tag{4.7b}$$

$$b(0,r) = b_0(r) \quad \text{for} \quad r \in D. \tag{4.7c}$$

Here we have redefined $t := \frac{1}{\omega_3} t$. Regarding the initial condition, we assume

$$b_0 \in C^2(\bar{D}), \quad \text{and} \quad \frac{r}{3}(b_0)_r + b_0 \geq 0 \quad \text{for} \quad r \in D, \tag{4.7d}$$

where the second condition is equivalent to $n_0 \geq 0$ in D. Note that the conservation of the mass (4.4) is represented by $b(1,t) = 1$ for $t \in [0.T)$. Concerning the function Θ, for the isothermal case condition (4.2) remains unchanged

$$\Theta(t) \equiv \Theta^* = \text{constant} > 0, \quad \text{for all} \quad t > 0. \tag{4.8}$$

In the non-isothermal case, condition (4.3a) takes the form

$$E = \kappa \Theta(t) - \frac{1}{3\omega_3} \int_0^1 b(y,t)^2 y^4 \, dy \quad \text{in} \quad [0,T), \tag{4.9a}$$

recalling that $E \in \mathbb{R}$ and $\kappa > 0$ are constants satisfying

$$\Theta(0) = \Theta_0 > 0. \tag{4.9b}$$

We denote the solution of (4.7)–(4.8) by $b = b(r,t)$, and the solution of (4.7),(4.9) by the pair $\langle b = b(r,t), \Theta = \Theta(t)\rangle$.

As was done for problems (4.1)–(4.2) and (4.1),(4.3), we can now define the maximal interval of existence in terms of the average density $b(r,t)$. If $T_1 < \infty$ or $T_2 < \infty$, then we must have

$$\lim_{t\to \bar{T}_i} \sup_{[0,1]} b(r,t) = \infty.$$

where $\bar{T}_i = T_i/\omega_3$ for $i = 1, 2$.

Since problem (4.1) conserves mass, one finds for b

$$b(r,t) \le \frac{1}{r^3} \quad \text{for} \quad r \in \overline{D},\ t > 0,$$

which implies a single point blow-up for $b(r,t)$ at the point $r = 0$.

Next we will state and motivate the results for the isothermal and non-isothermal case.

Isothermal case

To characterize the asymptotic behaviour near blow-up of the solution $b(r,t)$ of problem (4.7)–(4.8) it is important to study the solutions of an associated boundary-value problem

$$\begin{cases} \varphi_{\eta\eta} + \frac{4}{\eta}\varphi_\eta + \frac{1}{3}\eta\varphi\varphi_\eta - \frac{1}{2}\eta\varphi_\eta + \varphi^2 - \varphi = 0, & \text{for} \quad \eta > 0, \\ \varphi(0) \ge 1 \quad \varphi_\eta(0) = 0. \end{cases} \tag{4.10}$$

If b is a solution of (4.7)–(4.8), which blows up at time $T > 0$ and at the point $r = 0$, then we will show that it has the asymptotic form given by

$$b_*(r,t) = (T-t)^{-1}\varphi\left(\frac{r}{\sqrt{\omega_3\Theta^*(T-t)}}\right).$$

Equation (4.10) has multiple solutions [51, 21]. We classify them by counting the number of times they cross the singular solution $\varphi_S(\eta) := \frac{6}{\eta^2}$. For that purpose, we introduce the set

$$\mathcal{S}_k = \{\varphi\colon \varphi \text{ is solution of (4.10) having } k \text{ intersections with } \varphi_S\}.$$

We will show that $\mathcal{S}_1$ is the relevant subset of solutions of (4.10) for the characterization of blow-up.

For the initial condition, we assume

$$(b_0)_r \le 0 \quad \text{for} \quad r \in D, \tag{4.11}$$

and

$$\omega_3\Theta^*\left((b_0)_{rr} + \frac{4}{r}(b_0)_r\right) + \frac{1}{3}rb_0\,(b_0)_r + b_0^2 \ge 0 \quad \text{for} \quad r \in D. \tag{4.12}$$

We will show that this implies $b_r \le 0$ in D_T and $b_t \ge 0$ in D_T.

Next we introduce self-similarity in (4.7)–(4.8). Using scale invariance, we set:

$$\tau = \log\left(\frac{T}{T-t}\right) \quad \text{and} \quad \eta = \frac{r}{(\omega_3\Theta^*(T-t))^{1/2}}; \tag{4.13}$$

and for the unknown b we define

$$B(\eta, \tau) = (T-t)b(r,t). \tag{4.14}$$

The rectangle D_T transforms into

$$\Pi = \{(\eta,\tau) \mid \tau > 0,\ 0 < \eta < (\omega_3\Theta^*T)^{-1/2}e^{\tau/2}\}. \tag{4.15}$$

The initial-boundary problem (4.7)–(4.8) now becomes

$$B_\tau + B + \frac{1}{2}\eta B_\eta = B_{\eta\eta} + \frac{4}{\eta}B_\eta + \frac{1}{3}\eta B B_\eta + B^2 \quad \text{in} \quad \Pi, \tag{4.16a}$$

$$B_\eta(0,\tau) = 0, \quad B\left((\omega_3\Theta^*T)^{-1/2}e^{\tau/2}, \tau\right) = e^{-\tau}T \quad \text{for} \quad \tau \in \mathbb{R}^+, \tag{4.16b}$$

$$B(\eta, 0) = B_0(\eta) := Tb_0\left(\eta(\omega_3\Theta^*T)^{1/2}\right) \quad \text{for} \quad \eta \in \Pi(0), \tag{4.16c}$$

where $\Pi(0) = (0, (\omega_3\Theta^*T)^{-1/2})$.

Note that a solution of (4.10) is time independent solution of (4.16). Therefore the study of the blow-up behaviour of $b(r,t)$ is reduced to the analysis the large time behaviour of solutions $B(\eta,\tau)$ of (4.16), and in particular stabilization towards solutions φ of (4.10). For that, we introduce for (4.16) the ω-limit set

$$\omega = \{\phi \in L^\infty(\mathbb{R}^+) \mid \exists \tau_j \to \infty \quad \text{such that}$$
$$B(\cdot, \tau_j) \to \phi(\cdot) \quad \text{as} \quad \tau_j \to \infty \quad \text{uniformly on compact subsets of} \quad \mathbb{R}^+\}$$

Now we have the elements to state our main result.

Theorem 4.1 *Let $\Theta^* < \frac{1}{8\pi}$. Let b_0 be such that (4.11), and (4.12) hold, and let $b(r,t)$ be the corresponding solution of problem (4.7)–(4.8) that blows up at $r = 0$ and at $t = T$. If b satisfies the growth condition*

$$b(0,t) \le M(T-t)^{-1} \quad \textit{for} \quad t \in (0,T). \tag{4.17}$$

with $M > 0$, then

$$\omega \subset \mathcal{S}_1.$$

Remark 4.2 *The growth condition (4.17), has been observed numerically in [21]. There are analitical proofs of this condition for related equations, which we believe can be adapted for this case [77, 72].*

Remark 4.3 *There is numerical evidence [21] that shows that the set $\mathcal{S}_1$ contains only two elements. These elements are the profiles*

$$\varphi^* \equiv 1, \quad \text{and} \quad \varphi_1(\eta) := \frac{6}{(1+\frac{\eta^2}{2})}.$$

Remark 4.4 *If we can prove that $\omega = \{\varphi\} \subset S_1$, then*

$$\lim_{t\to T}(T-t)b\left(\eta\sqrt{\omega_3\Theta^*(T-t)}\right) = \varphi(\eta) \tag{4.18}$$

uniformly for $0 \le \eta \le C$ for some arbitrary $C > 0$. Numerical results in [21] show that for an open set of initial data the convergence in (4.18) holds for $\varphi = \varphi_1$. This self-similar behaviour is depicted in Figure 4.2 (right), replacing $n(r,t)$ by $b(r,t)$. In constrast, we know of no numerical evidence for the convergence towards φ^*.

Remark 4.5 *Assumption (4.11) on the initial data, gives in terms of n_0,*

$$r^3 n_0(r) \le \frac{1}{\omega_3}\|n_0\|_{L^1(B_r(0))}. \tag{4.19}$$

We remark also that there exists a family of b_0, which satisfy the conditions (4.7d), (4.12) and (4.11). These assumptions are satisfied, if for example $b_0(r) \equiv 1$, and also if

$$b_0(r) = K_1 + \frac{K_2}{r^3 + K_3}$$

for $K_i > 0$ for $i = 1 \dots 3$ with $K_1 + \frac{K_2}{1+K_3} = 1$ and $\Theta^* < K_2/24\pi$.

The proof of Theorem 4.1 uses the observation that equation (4.16), without the term $\frac{1}{3}\eta BB_\eta$, is the same equation that arises in the study of self-similar blow-up for the parabolic semilinear equation

$$\bar{b}_t = \Delta_N \bar{b} + \bar{b}^2, \tag{4.20}$$

with $N = 5$. Here Δ_N denotes the Laplacian in $\mathbb{R}^N$, see [48, 49]. Therefore we adapt the methods used for the analysis of this equation to prove Theorem 4.1. However here, due to the presence of the convection term, we need to construct a new Lyapunov functional, which is given in a implicit form as was done in [82]. Note that Theorem 4.1 is very similar to a result for the supercritical case ($N > 6$) for equation (4.20), when two different kinds of self-similar blow-up behaviour coexist [60].

Non-isothermal case

The blow-up behaviour of the solution $\langle b, \Theta\rangle$ of (4.7),(4.9), was studied by Chavanis et al. in [25]. There it was claimed that $\Theta(t)$ and $b(0,t)$ blow up at the same instant of time $T > 0$. To support this assertion, in [25] it was assumed that $\Theta(t) \sim (T-t)^{-a}$ with $a > 0$ and derived a corresponding self-similar equation. Since $a > 0$ is a-priori unknown, this results in a family of blow-up equations, indexed by $a > 0$:

$$\begin{cases} \varphi_{\eta\eta} + \frac{4}{\eta}\varphi_\eta + \frac{1}{3}\eta\varphi\varphi_\eta - \frac{1}{2}(1-a)\eta\varphi_\eta + \varphi^2 - \varphi = 0, & \text{for} \quad \eta > 0, \\ \varphi(0) \ge 1 \quad \varphi_\eta(0) = 0. \end{cases} \tag{4.21}$$

Note that (4.10) corresponds to (4.21) with $a = 0$. In [25], it was argued on the basis of numerical experience that

$$\begin{cases} (T-t)b\left(\eta\sqrt{\omega_3(T-t)\Theta(t)},t\right) \to \varphi^a(\eta) \\ \Theta(t)(T-t)^a \to Constant \end{cases} \quad \text{as} \quad t \to T \tag{4.22}$$

where φ^a is a solution of (4.21) with $a \sim 0.1$, which is bounded, decreasing and satisfies $\varphi^a(\eta) = O(\eta^{-2(1+a)})$ for large η. Recently, in [18], it was proved that for $a > 0$ such solutions cannot exists. Therefore the convergence (4.22) cannot hold and so the question of blow-up in this case remained open.

In this chapter we solve in part this issue. As was pointed out to us by J. King, the energy relation (4.9) does not exclude the combination of a singular solution b and a finite temperature Θ. For example, in the isothermal case at time of blow-up $b(r,T) = \frac{6}{r^2}$ near $r = 0$, which is unbounded and produces a finite temperature

$$\kappa\Theta(T) = E + \frac{1}{3\omega_3}\int_0^1 b(r,T)^2 r^4\, dr < \infty.$$

This possibility was not addressed in [25].

To conclude we give a numerical result showing generic blow-up behaviour with bounded temperature at singular density.

We set in our simulations $n_0 \equiv \frac{1}{\omega_3}$, that is $b_0(r) \equiv 1$ for $r \in D$. Selecting E and κ to ensure blow-up, we find that

(i) $\Theta_t > 0$ on $(0,T)$;

(ii) $\Theta(t) \to \bar{\Theta}$ where $\bar{\Theta}$ is a positive finite constant;

(iii) and recalling that $\varphi_1(\eta) = \frac{6}{(1+\eta^2/2)^2}$, we have the convergence

$$(T-t)b\left(\eta\sqrt{\omega_3\Theta(t)(T-t)},t\right) \sim \varphi_1(\eta) \quad \text{as} \quad t \to T.$$

We illustrate properties (i)–(iii) in Figure 4.3, for a particular choice of $E \in \mathbb{R}$ and $\kappa > 0$, for which we know blow-up is ensured.

Finally, from a view-point of analytical results, we prove that under certain initial condition, including the ones used in the numerical experiences, that the temperature must increase near $t = 0$. This agrees with conclusion (i).

The remainder of this chapter is organized as follows. Sections 4.4 and 4.5 are devoted to the proof of Theorem 4.1. In Section 4.3, we discuss the main results of this chapter comparing them with existing results from the literature. In Section 4.4, we study properties of solutions of problems (4.7)–(4.8) and (4.7),(4.9). In Section 4.5 we

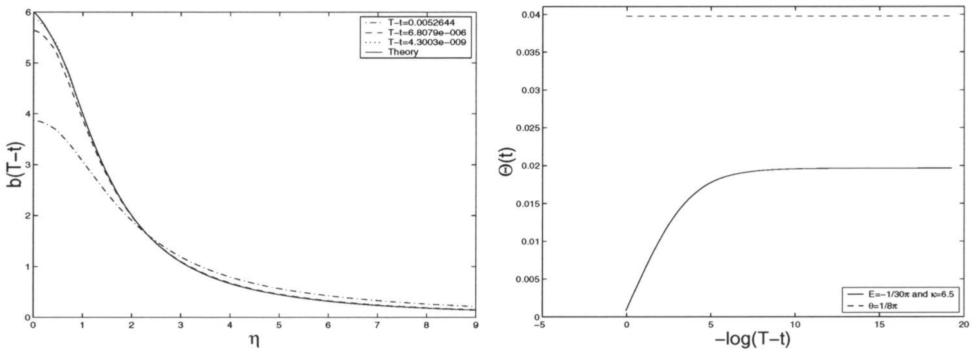

Figure 4.3: Convergence in self-similar variables to the profile φ_1(left) and the associated temperature behaviour(right), $E = -1/30\pi$, $k = 6.5$, and $b_0 \equiv 1$.

prove the convergence to a self-similar solutions. In Section 4.6, we introduce the self-similar problem for the non-isothermal case and we numerically study this problem. We conclude that section by proving that for certain initial data the temperature increases for short times.

4.3 Discussion

From our results we find that the non-isothermal and isothermal case present essentially the same features on blow-up behaviour for a constant initial density. This is explained by the fact that in the non-isothermal case the temperature remains bounded when a singularity appears in the density. Both cases are described by the same blow-up profile for the density, that is $n(r,t) = (T-t)^{-1}\Psi(\eta)$ near $t = T$, with

$$\Psi(\eta) = \Psi_1(\eta) = \frac{6+\eta^2}{(1+\frac{\eta^2}{2})^2} \quad \text{for} \quad \eta > 0. \tag{4.23}$$

for $\eta = \frac{r}{\Theta(t)(T-t)^{1/2}}$ containing the (variable)temperature.

This is in contrast with the findings of Chavanis et al. [26], where the authors concluded that the isothermal and non-isothermal model show different blow-up behaviour. For the non-isothermal model it was claimed that both temperature and density blow-up at the same time in what they called "gravothermal catastrophe". The arguments used there were numerical experiments as well as modelling considerations. The singularity in the temperature was fitted to the function $(T-t)^{-a}$, resulting in $a \sim 0.1$. As a result of the different scaling different blow-up profiles were found in the isothermal and non-isothermal cases: $n(r,t) = (T-t)^{-1}\Psi(\eta)$ near $t = T$, with

$$\Psi(\eta) = \Psi_1(\eta) \quad \text{(isothermal case) and } \Psi(\eta) = \Psi^a(\eta) \quad \text{(non-isothermal case),}$$

where Ψ^a is a bounded and decreasing function such that $\Psi^a(\eta) = O(\eta^{-2(1+a)})$ for large η.

In the numerical experiments [25] the maximum value of the density n $(n(0,t))$ that was obtained was of the order 10^4; at this point the calculation was discontinued. This corresponds in our simulations to calculations up to a time $-\log(T-t) \sim 6$,, indicated in Figure 4.4 by a circle. Note that we compute up to a time with $n(0,t)$ of the order 10^9.

We conjecture that the difference between our interpretation of our simulations (temperature remains bounded at blow-up) and those of [25] $(\Theta(t) = O((T-t)^{-a})$ as $t \to T)$ can be explained by the fact that our simulations continue further into the singularity. We hope to study this question in more detail in the future. We

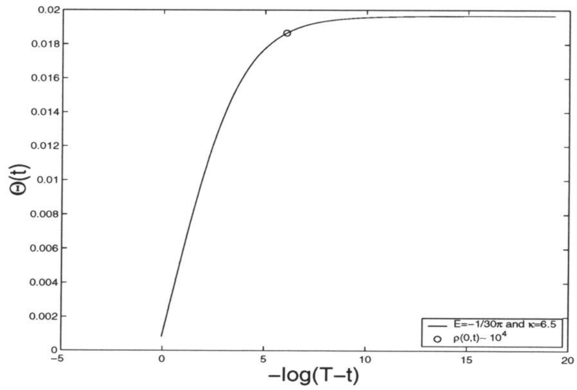

Figure 4.4: Temperature for $E = -1/30\pi$, $k = 6.5$, and $b_0 \equiv 1$: comparison of results with Chavanis et al. [25].

conjecture that a result similar to Theorem 4.1 can be proved for the non-isothermal model; the main difficulty lies in proving $b_t \geq 0$, which in turn implies $\Theta_t \geq 0$.

Now we describe a family of blow-up patterns which appears when we refine the asymptotic expansion for the profile $\varphi = \varphi^* \equiv 1$. This situation is closely related to the blow-up behaviour of (4.20), with $N < 6$. In fact, if a solution $\bar{b}$ blows up point at $x = 0$ and $t = T$, then

$$\lim_{t \to T} (T-t)\bar{b}(\eta\sqrt{T-t}, t) = 1$$

uniformly for $|\eta| < C$ for some arbitrary $C > 0$ [49]. Moreover it have been shown (cf. for instance [61, 76]), that a refined description gives the existence of two possible behaviours: either

$$\lim_{t \to T} (T-t)\bar{b}\left(\eta\sqrt{(T-t)|\log(T-t)|}, t\right) = \bar{\varphi}_1(\eta) \tag{4.24}$$

or

$$\lim_{t \to T} (T-t)\bar{b}\left(\eta(T-t)^{1/2m}, t\right) = \bar{\varphi}_m(\eta) \tag{4.25}$$

uniformly for $|\eta| < C$, for some $m \geq 2$ and where the family $\{\bar{\varphi}_i\}_{i\geq 1}$ is known explicitly.

For problem (4.7)–(4.8), it was shown in [51] that if blow-up is of the form $\varphi^* \equiv 1$, then there exists a the refined asymptotics given by either

$$(T-t)b\left(\eta\sqrt{(T-t)|\log(T-t)|^{1/3}}, t\right) \sim \tilde{\varphi}_1(\eta) \quad \text{as} \quad t \to T, \tag{4.26}$$

or

$$(T-t)b\left(\eta(T-t)^{\frac{1}{3}+\frac{1}{2(m+2)}}, t\right) \sim \tilde{\varphi}_m(\eta) \quad \text{as} \quad t \to T \tag{4.27}$$

for some $m \geq 2$. An implicit formula for the family $\{\tilde{\varphi}_m\}_{m\geq 1}$ is given in [21, eq. (43)]. The convergence towards these profiles remains an open problem.

In the asymptotic (4.27), we can take formally the limit $m \to \infty$ and find a non trivial scaling, that is

$$\lim_{t\to T}(T-t)b(\eta(T-t)^{1/3}, t) = \tilde{\varphi}_\infty(\eta). \tag{4.28}$$

This cannot be done from the semilinear equation where (4.25) holds. The convergence (4.28) represents the convection dominant behaviour of (4.7), which in terms of the density $n = n(r,t)$ describes an imploding wave moving towards the origin, as was shown in Figure 4.2 (left). Now the function $\tilde{\varphi}_\infty$ is discontinuous (cf. [50, Eq. (3.16)]),

$$\tilde{\varphi}_\infty(\eta) = \begin{cases} \frac{2C^3}{\eta^3} & \text{for} \quad \eta > C \\ 0 & \text{for} \quad \eta < C, \end{cases}$$

where $2C^3$ is the mass accumulated in the origin, which can be made arbitrary. In [50] this type of blow-up was studied using matched asymptotics. There it was suggested that this behaviour is stable and moreover it was expected that there exists an initial data such that (4.28) holds uniformly in subset away from the shock. A result of this type was proved in [37, Theorem 3] for a related situation.

4.4 Preliminaries

4.4.1 First estimates

In this section we develop some estimates for problems (4.7)–(4.8) and (4.7),(4.9), which in turn will imply bounds for the respective self-similar problems.

The following three lemmas hold for a solution $\langle b, \Theta\rangle$ of (4.7)–(4.8) or a solution b of (4.7),(4.9). We use indistinctly $T > 0$ as a blow-up time for both problems.

Lemma 4.6 *If b_0 satisfies (4.7d) then*

$$\frac{r}{3}b_r + b \geq 0 \quad \text{in} \quad D_T. \tag{4.29}$$

Proof. Note that the solution n of problem (4.1), satisfies the relation

$$n(r,t) = \frac{1}{\omega_3}[\frac{r}{3}b_r + b] \quad \text{in} \quad D_T. \tag{4.30}$$

Since $n_0 \geq 0$ in D, an application of the maximum principle to problem (4.1) shows that $n \geq 0$ in D_T. Using this and (4.30) the result follows. ∎

To prove the following results, we proceed as in [39], where similar estimates were found for the semilinear equation 4.20.

Lemma 4.7 *If b_0 satisfies (4.11) then*

$$b_r(r,t) < 0 \quad in \quad D_T. \tag{4.31}$$

Proof. Set $w(r,t) := r^4 b_r(r,t)$. Therefore differentiating (4.7a), we find

$$w_t - \omega_3\Theta\left(w_{rr} - \frac{4}{r}w_r\right) - \frac{1}{3}rbw_r = \left(b + \frac{1}{3}rb_r\right)w \tag{4.32}$$

Assume for the moment a stronger assumption on the initial data

$$(b_0)_r(r) < 0 \quad \text{for} \quad r \in (0,1) \quad \text{and} \quad (b_0)_{rr}(0) < 0. \tag{4.33}$$

This gives $w(0,r) = r^4 b_r(0,r) < 0$. Noting that under (4.33) the function $b \equiv 1$ is a sub-solution for (4.7), and by Hopf's Lemma, we find that $w(1,t) = b_r(t,1) < 0$ for all $t > 0$. Now choosing $\tilde{T} < T$ arbitrarily close to T, and noting that $\omega_3 n(r,t) = b + \frac{rb_r}{3} < C(\tilde{T})$ on $\overline{D_{\tilde{T}}}$, then by the maximum principle, we find $w < 0$ on D_T and whence $b_r < 0$ on D_T. To finish the proof, we note that by the strong maximum principle, if b_0 satisfies (4.11), then for each $t_1 \in (0,T)$, condition (4.33) holds for the function $b(r,t_1)$. This proves the result. ∎

Lemma 4.8 *If b_0 satisfies (4.11), then*

$$b(0,t) \geq (T-t)^{-1} \quad for \quad t \in [0,T), \tag{4.34}$$

Proof. Since the maximum of b in D is attained at $r = 0$ (by $b_r < 0$), therefore $\Delta b(0,t) \leq 0$. It follows from (4.7a) that $b_t(0,t) \leq b^2(0,t)$. Integrating this inequality on $(0,T)$ gives the result. ∎

The following results hold only for the isothermal model. Therefore in the sequel $b(r,t)$ is a solution to (4.7)–(4.8).

Lemma 4.9 *If b_0 satisfies (4.12) then $b_t \geq 0$ for all $t \in (0,T)$.*

Proof. Condition (4.12) implies that $b(r,\epsilon) \geq b(r,0)$ for ϵ small enough. Using the comparison theorem in [21, p. 1079], we find $b(r,t+\epsilon) \geq b(r,t)$ for $t \in (0, T-\epsilon)$. We conclude, using this and noting that $b_t(r,t) = \lim\limits_{\epsilon \to 0} \frac{b(r,t+\epsilon)-b(r,t)}{\epsilon}$. ∎

The next lemma gives a lower bound on b_r in D_T.

Lemma 4.10 *Let b_0 satisfy (4.11) and (4.12). Then*

$$\omega_3\Theta^* b_r^2(r,t) \leq \frac{2}{3}b(0,t)^3 \quad \textit{for} \quad (r,t) \in D_T. \tag{4.35}$$

Proof. Since $b_t \geq 0$ and $b_r \leq 0$, we multiply equation (4.7a) by b_r and obtain

$$0 \geq \omega_3\Theta^* \int_0^r b_r b_{rr} ds + \frac{1}{3}b^3(r,t) - \frac{1}{3}b^3(0,t).$$

Integrating by parts, we obtain

$$0 \geq \omega_3\Theta^* \left[b_r^2(r,t) - b_r^2(0,t) - \int_0^r b_r b_{rr}\, ds\right] + \frac{1}{3}b^3(r,t) - \frac{1}{3}b^3(0,t) \geq$$
$$\geq \omega_3\Theta^*[b_r^2(r,t) - b_r^2(0,t)] + \frac{2}{3}b^3(r,t) - \frac{2}{3}b^3(0,t).$$

Now, since $b_r^2(0,t) = 0$, we obtain the desired inequality. ■

To conclude this section we translate the properties of solutions derived above into estimates for the self-similar solutions associated with problem (4.1).

From hypothesis (4.17) and noting that $b \geq 1$ and $b_r \leq 0$ in D_T, we have the a priori bound

$$0 \leq B(\eta,\tau) \leq M \quad \text{for} \quad (\eta,\tau) \in \Pi. \tag{4.36}$$

Combining this with (4.35) and using (4.31), we obtain

$$0 \leq -B_\eta(\eta,\tau) \leq \bar{M} \quad \text{for} \quad (\eta,\tau) \in \Pi, \tag{4.37}$$

where $\bar{M}$ depends on M. Finally from (4.34), we get

$$1 \leq B(0,\tau) \quad \text{for} \quad \tau \in (0, (\omega_3\Theta^* T)^{-1/2}e^{\tau/2}). \tag{4.38}$$

4.4.2 The steady state equation (4.10)

We begin recalling the problem (4.10):

$$\varphi_{\eta\eta} + \frac{4}{\eta}\varphi_\eta + \frac{1}{3}\eta\varphi\varphi_\eta - \frac{1}{2}\eta\varphi_\eta + \varphi^2 - \varphi = 0 \quad \text{for} \quad \eta > 0, \tag{4.39a}$$

$$\varphi(0) \geq 1, \quad \varphi_\eta(0) = 0. \tag{4.39b}$$

Since $B(0,\tau) \geq 1$ therefore in the limit $\tau \to \infty$ we require that $\varphi(0) \geq 1$, and so condition (4.39b).

Equation (4.39a) has three special solutions:

$$\varphi_S(\eta) = \frac{6}{\eta^2}, \quad \varphi^*(\eta) = 1, \quad \text{and} \quad \varphi_*(\eta) = 0 \quad \text{for} \quad \eta > 0.$$

The singular solution φ_S and the constant φ^* will play an important role in the analysis. Note that φ_S satisfies

$$\varphi_S + \frac{1}{2}\eta(\varphi_S)_\eta = 0 \quad \text{and} \quad 0 = (\varphi_S)_{\eta\eta} + \frac{4}{\eta}(\varphi_S)_\eta + \frac{1}{3}\eta\varphi_S(\varphi_S)_\eta + (\varphi_S)^2. \tag{4.40}$$

For bounded non constant solutions we have the following theorem from [21] and [51].

Theorem 4.11 *There exists a countable set of solutions $\{\varphi_k\}_{k\in\mathbb{N}}$ of (4.39) such that $\varphi_k(0) > 1$ and $\varphi_k(0) \to \infty$ as $k \to \infty$, Moreover φ_k intersects k times the singular solution φ_S, and has the asymptotic behaviour $\varphi_k(\eta)\eta^2 = Const(k) > 0$.*

It was argued [21] that for each integer $k \geq 2$, the set

$$\mathcal{S}_k = \{\varphi\colon \varphi \text{ solution of (4.39) with } k \text{ intersections with } \varphi_S\}$$

is a singleton and that the set $\mathcal{S}_1$ contains only two elements. More precisely, $\mathcal{S}_1$ consists of the constant φ^* and surprisingly enough a function with an explicit form, given by

$$\varphi_1(\eta) = \frac{6}{1+\frac{\eta^2}{2}}.$$

If we relax condition (4.39b) to $\varphi(0) > 0$, we can find more solutions. In fact, a numerical solution φ_1^* of (4.39a) was found in [21, Figure 14] such that $\varphi_1^*(0) < 1$, $(\varphi_1^*)_\eta(0) = 0$ and intersects once with φ_S.

4.4.3 Comparison with the singular solution φ_S

From the preceding section, we recall that solutions φ of (4.39) are clasified by their intersections with φ_S. In this section we study the intersections of solutions B of (4.16) and φ_S. Our results are closely related to the ones found in reference [4], where the semilinear equation is studied.

First, we see that a solution B of (4.16) intersects at least once in $\bar{\Pi}(0)$ with the singular solution φ_S whenever $\Theta^* < \frac{1}{8\pi}$, since

$$\varphi_S(0) = \infty > B(0,0), \text{ and } \varphi_S\left((\omega_3\Theta^*T)^{-1/2}\right) < B\left((\omega_3\Theta T)^{-1/2}, 0\right) = T.$$

Therefore there exists $\eta_1 \in \Pi(0)$ such that $B(\eta_1, 0) = \varphi_S(\eta_1)$ and $B(\eta, 0) < \varphi_S(\eta)$ for $\eta < \eta_1$.

Lemma 4.12 *Under the assumptions (4.11), (4.12), and $\Theta^* < \frac{1}{8\pi}$, there exists a continuously differentiable function $\eta_1(\tau)$ with domain $[0, \infty)$ such that $\eta_1(0) = \eta_1$ and $B(\eta_1(\tau), \tau) = \varphi_S(\eta_1(\tau))$ for all $\tau \geq 0$.*

Proof. Define $H(\eta,\tau) := B(\eta,\tau) - \varphi_S(\eta)$. We first claim that H, H_η, and H_τ do not vanish simultaneously. Using Lemma 4.9 and the strong maximum principle, we find

$$b_t = (T-t)^{-1}(B_\tau + B + \frac{1}{2}\eta B_\eta) > 0 \quad \text{in} \quad D_T. \tag{4.41}$$

If there exists a point in Π where $H_\eta = H_\tau = H = 0$ then $H_\tau = 0$ implies $B_\tau = 0$, and using (4.40) the condition $H_\eta = 0$ combined with $H = 0$ implies

$$B + \frac{1}{2}\eta B_\eta = 0.$$

In turn this implies that $b_t = 0$ at some point of D_T, a contradiction with (4.41). Secondly, we claim that $H_\eta \neq 0$ at any point $(\bar\eta,\bar\tau) \in \Pi$ where $H(\bar\eta,\bar\tau) = 0$ and moreover $H(\eta,\bar\tau) < 0$ in a left neighborhood of $\bar\eta$. A proof of this is given in [4], and moreover it was deduced that $H_\eta(\bar\eta,\bar\tau) > 0$.

Now we prove that $H_{\eta_1}(\eta_1, 0) > 0$. This follows from the equation for $H(\eta, 0)$. On the left side of η_1, we find

$$\begin{aligned} H_{\eta\eta}(\eta,0) &+ \frac{4}{\eta}H_\eta(\eta,0) + \frac{1}{6}\eta H_\eta(\eta,0)(B(\eta,0)+\varphi_S) \\ &+ \frac{1}{6}\eta H(\eta,0)(B(\eta,0)+\varphi_S)_\eta \geq 0. \end{aligned}$$

Since $(B(\eta,0)+\varphi_S)_\eta \leq 0$ and $H(\eta_1,0) = 0$, we can apply Hopf's Lemma to conclude that $H_{\eta_1}(\eta_1,0) > 0$. The rest of the proof follows in the same manner as in [4] using the implicit function theorem. ■

Define the set

$$\Pi_1 = \{(\eta,\tau) \mid 0 < \eta < \eta_1(\tau)\}$$

and the value

$$\eta_2 = \sup\{\eta \in (\eta_1, (\omega_3\Theta^* T)^{-1/2}] \colon H(s,0) \geq 0 \text{ for } s \in [\eta_1,\eta]\}.$$

Since $H(\eta_1,0) = 0$ and $H_\eta(\eta_1,0) > 0$, the supremum is finite. We have $\eta_2 \leq (\omega_3\Theta^* T)^{-1/2}$. Define the set

$$\Pi_2 = \{(\eta,\tau) \mid \eta_1(\tau) < \eta < \eta_2(\tau) := \eta_2 e^{\tau/2}\}.$$

Let $F(\tau) = H(\eta_2(\tau),\tau)$. By definition of η_2, $F(\tau) \geq 0$. Also,

$$\frac{d}{d\tau}F(\tau) = H_\tau(\eta_2(\tau),\tau) + \frac{1}{2}\eta_2(\tau)H_\eta(\eta_2(\tau),\tau).$$

Using (4.41), we have

$$\frac{d}{d\tau}[e^\tau F(\tau)] \geq 0.$$

An integration yields $F(\tau) \geq 0$ for $\tau \geq 0$.

As was done in [4], applying the maximum principle, using Lemma 4.12, and noticing that $H(\eta_2(\tau),\tau) \geq 0$ for $\tau \geq 0$, we can prove the following lemma.

Lemma 4.13 *The function* $H(\eta,\tau) = B(\eta,\tau) - \varphi_S(\eta)$ *satisfies* $H < 0$ *in* Π_1 *and* $H > 0$ *in* Π_2.

And as a direct corollary, we find

Corollary 4.14 *Assume the conditions in Lemma 4.12. For each* $N > 0$ *there is* $\tau_N > 0$ *such that for* $\tau > \tau_N$, $B(\eta,\tau)$ *intersect* $\varphi_S(\eta)$ *at most once in* $\eta \in (0,N)$.

The reader can found the details of the proofs of these last two results in [4].

4.5 Convergence

In this section we prove the following convergence theorem:

Theorem 4.15 *Assume that* $\Theta^* < 1/8\pi$, *and let conditions (4.11) and (4.12) hold. Let* $B(\eta,\tau)$ *be a uniformly bounded global solution of (4.16). Then for every* $\tau_n \to \infty$ *there exists a subsequence relabelled* τ_n *such that the limit of* $B(\eta,\tau_n)$ *exists and equals* $\varphi \in \mathcal{S}_1$. *The convergence is uniform on every compact set of* $[0,\infty)$.

Proof. Define $B^\sigma(\eta,\tau) := B(\eta,\sigma+\tau)$. We will first show that for any increasing unbounded given subsequence $\{n_j\}$, there exists a subsequence renamed $\{n_j\}$ such that B^{n_j} converges to a solution φ of (4.39) uniformly in compact subsets of $\mathbb{R}^+\times\mathbb{R}$.

Let $N \in \mathbb{N}$. We take i large enough such that the rectangle $\mathcal{Q}_{2N} = \{(\eta,\tau) \in \mathbb{R}^2 \colon 0 \le \eta \le 2N,\ |\tau| \le 2N\}$ lies in the domain of B^{n_i}. Let $\tilde{B}(\xi,\tau) = B^{n_i}(|\xi|,\tau)$ be a solution of

$$\tilde{B}_\tau = \Delta\tilde{B} - \frac{1}{2}\xi\cdot\nabla\tilde{B} + \frac{1}{3}(\xi\cdot\nabla\tilde{B})\tilde{B} + \tilde{B}^2 - \tilde{B}$$

on the cylinder given by

$$\Gamma_{2N} = \{(\xi,\tau) \colon \mathbb{R}^5\times\mathbb{R} \colon |\xi| \le 2N,\ |\tau| \le 2N\},$$

with $|\tilde{B}(\xi,\tau)|$ uniformly bounded in Γ_{2N} using (4.36).

By Schauder's interior estimates all partial derivatives of $\tilde{B}$ can be uniformly bounded on the subcylinder $\Gamma_N \subset \Gamma_{2N}$. Consequently B^{n_i}, $B^{n_i}_\tau$, $B^{n_i}_\eta$, and $B^{n_i}_{\eta\eta}$ are uniformly Lipschitz on $\mathcal{Q}_N \subset \mathcal{Q}_{2N}$. By Arzela-Ascoli, there is a subsequence $\{n_j\}_1^\infty$ and a function $\bar{B}$ such that B^{n_i}, $B^{n_i}_\tau$, $B^{n_i}_\eta$, and $B^{n_i}_{\eta\eta}$ converge to $\bar{B}$, $\bar{B}_\tau$, $\bar{B}_\eta$, and $\bar{B}_{\eta\eta}$, uniformly on $\mathcal{Q}_N$.

Repeating the construction for all N and taking a diagonal subsequence, we can conclude that

$$B^{n_j} \to \bar{B}, \quad B^{n_j}_\tau \to \bar{B}_\tau, \quad B^{n_j}_\eta \to \bar{B}_\eta, \quad \text{and} \quad B^{n_j}_{\eta\eta} \to \bar{B}_{\eta\eta}, \tag{4.42}$$

uniformly in every compact subset in $\mathbb{R}^+\times\mathbb{R}$. Clearly $\bar{B}$ satisfies (4.16a) and estimates (4.36) and (4.37). Assume for the moment that $\bar{B}$ is independent of τ. We postpone the proof of this assertion. This implies that $\bar{B}$ is a solution of (4.10). Moreover $\bar{B}$ intersects $\varphi_S(\eta)$ at most once. This follows from Corollary 4.14, which asserts that for every $N > 0$ the solution $B^{n_j}(\eta, \tau)$ intersects $\varphi_S(\eta)$ at most once on $[0, N]$ for each $\tau > \tau_N$. This concludes the proof.

Claim. The function $\bar{B}$ is independent of τ.

To prove this we use a non explicit Lyapunov functional in the spirit of Galaktionov [40] and Zelenyak [82].

We define the functional

$$E(\tau) = \int_0^z \Phi(\eta, B, B_\eta)\, d\eta,$$

where $z = (\omega_3 \Theta^* T)^{-1/2} e^{\tau/2}$ and $\Phi = \Phi(\eta, v, w)$ is a function to be determined. We introduce the function $\rho = \rho(\eta, v, w)$, such that the functional E has the form of a Lyapunov functional with a contribution on the boundary, that is

$$\frac{d}{d\tau} E(\tau) = - \int_0^z \rho(\eta, B, B_\eta)(B_\tau)^2\, d\eta + \Phi_w B_\tau|_0^z + \frac{1}{2} z \Phi(z, B(\tau, z), B_\eta(\tau, z)). \tag{4.43}$$

In the appendix we show that there exist functions $\Phi, \rho \colon \tilde{R} \subset \mathbb{R}^3 \mapsto \mathbb{R}$ such that (4.43) holds, where the domain $\tilde{R}$ is given by

$$\tilde{R} = R \cap \{0 \le v \le M,\ 0 \le -w \le \bar{M}\},$$

where $R = \{\eta > 0, v \ge 0, w \le 0\} \cup \{\eta = 0, v \ge 0, w = 0\}$, and the positive constants M and $\bar{M}$ are given by the estimates (4.36) and (4.37) respectively. Later, we show that it is sufficient to define Φ and ρ on the set $\tilde{R}$, to be allow to write $\rho(\eta, B(\eta, \tau), B_\eta(\eta, \tau))$ and $\Phi(\eta, B(\eta, \tau), B_\eta(\eta, \tau))$ for each $\eta, \tau > 0$ whenever B solves (4.16). Note that $\tilde{R}$ only depends on $M > 0$, remember that $\bar{M}$ can be expressed on M. To derive (4.43) and prove this convergence theorem, we construct ρ continuous in $R \setminus \{\eta = \bar{\eta},\ v > 1\}$ and such that

$$\frac{1}{C_0} \eta^4 e^{-C_0 \eta^2} \le \rho(\eta, v, w) \le \eta^4 e^{-\eta^2/12} \quad \text{for} \quad (\eta, v, w) \in \tilde{R}, \tag{4.44}$$

with $C_0 = C_0(M) > 0$ (Lemma 4.20). On the other hand, the function Φ has the form

$$\Phi(\eta, v, w) = \int_0^w (w - s)\rho(\eta, v, s)\, ds - \int_0^v \rho(\eta, \mu, 0)(\mu^2 - \mu)\, d\mu,$$

and therefore it is also continuous in $R \setminus \{\eta = \bar{\eta},\ v > 1\}$ with

$$\begin{cases} \Phi(\eta, v, w) \le C_1 \eta^4 e^{-\eta^2/12} \\ \Phi(\eta, v, w) \ge -C_1 \eta^4 e^{-\eta^2/12} \end{cases} \quad \text{for} \quad (\eta, v, w) \in \tilde{R} \tag{4.45}$$

for some positive constants $C_1(M) > 0$ (Lemma 4.21).

Now we show that it is sufficient to define Φ and ρ in $\tilde{R}$. In fact, if B is a solution of (4.16), then $(\eta, B(\eta,\tau), B_\eta(\eta,\tau)) \in \tilde{R}$ for each $\eta \geq 0$ and $\tau \geq 0$. To prove this, we first note that if $\eta \to 0$ then $B_\eta \to 0$, since B is bounded and solves (4.16) which contains the radially symmetric Laplacian in $\mathbb{R}^5$. Therefore for $\eta = 0$ corresponds the set $\{v \geq 0,\ w = 0\}$. On the other hand $B_\eta \leq 0$ and $B > 0$, this means that for $\eta > 0$, we consider $\{v \geq 0,\ w \leq 0\}$. Finally, we complete the assertion by noting that B and $|B_\eta|$ are uniformly bounded by M and $\bar{M}$ respectively.

Now having (4.43) and the corresponding properties of ρ an Φ, we proceed to prove the claim. An integration in the interval (a,b) of (4.43) gives

$$\int_a^b \int_0^z \rho B_\tau^2 \, d\eta d\tau = E(a) - E(b) + \psi(a,b) \tag{4.46}$$

where

$$\psi(a,b) := \int_a^b \frac{1}{2} z\Phi(z, B(z,\tau), B_\eta(z,\tau))\, d\tau + \\ + \int_a^b B_\tau(z,\tau) \left[\int_0^{B_\eta(z,\tau)} \rho(z, B(z,\tau), s)\, ds \right] d\tau. \tag{4.47}$$

Since $B_\tau(z,\tau) = -B(z,\tau) - \frac{1}{2} z B_\eta(z,\tau)$,

$$B_\tau(z,\tau) = -Te^{-\tau} - \frac{1}{2} b_r(1, T(1-e^\tau)).$$

Using Lemma 4.6 with $b = 1$, gives $|b_r(1, T(1-e^\tau))| \leq 3$ and consequently B_τ is uniformly bounded as $\tau \to \infty$. Employing this bound on B_τ and the uniform bounds on $B(z,\tau)$ and $|B_\eta(z,\tau)|$, the estimates (4.44) and (4.45) imply $|\Phi(z)| \leq Cz^4 e^{-z^2/12}$ and $\rho(z) \leq z^4 e^{-z^2/12}$ for some $C > 0$. Consequently

$$\lim_{a\to\infty} \{\sup_{b>a} \psi(a,b)\} = 0. \tag{4.48}$$

By (4.42), we have that there exists a sequence $n_j \to \infty$ such that $B^{n_j}(\eta,\tau)$ converge to $\bar{B}$ uniformly in compact subsets of $\mathbb{R}$. For any fixed N we will prove for a subsequence satisfying $\lim_{j\to\infty} (n_{j+1} - n_j) = \infty$ that

$$\int\int_{\mathcal{Q}_N} \rho \bar{B}_\tau^2 \, d\eta d\tau = \lim_{n_j\to\infty} \int\int_{\mathcal{Q}_N} \rho (B_\tau^{n_j})^2 \, d\eta d\tau = 0,$$

where we recall that $\mathcal{Q}_N = \{(\eta,\tau) \colon \mathbb{R}^2 \colon 0 \leq \eta \leq N,\ |\tau| \leq N\}$. For all j sufficiently large,

$$N \leq (\omega_3 \Theta^* T)^{-1/2} e^{\frac{1}{2}(n_j - N)} \quad \text{and} \quad n_{j+1} - n_j \geq 2N.$$

Consequently using (4.46), we find

$$\int_{-N}^{N}\int_{0}^{N} \rho(B_\tau^{n_j})^2\, d\eta d\tau \le \int_{-N}^{N+n_{j+1}-n_j} \int_0^{(\omega_3\Theta^* T)^{-1/2} e^{\frac{1}{2}(n_j-N)}} \rho(B_\tau^{n_j})^2\, d\eta d\tau$$
$$\le E(n_j-N) - E(n_{j+1}-N) + \psi(n_j-N, n_{j+1}-N).$$

Hence applying (4.48), we discover

$$\iint_{\mathcal{Q}_N} \rho(B_\tau^{n_j})^2\, d\eta d\tau \le \limsup_{j\to\infty}[E(n_j-N) - E(n_{j+1}-N)].$$

Next we divide the expression $E(n_j-N)-E(n_{j+1}-N)$ into three integrals, choosing K arbitrarily large

$$E(n_j-N) - E(n_{j+1}-N) =$$
$$= \int_0^K [\Phi(\eta, B^{n_j}(\eta,-N), B_\eta^{n_j}(\eta,-N)) - \Phi(\eta, B^{n_j}(\eta,-N), B_\eta^{n_j}(\eta,-N)]\, d\eta \tag{4.49}$$
$$+ \int_K^{\frac{1}{T^{1/2}}e^{\frac{n_j-N}{2}}} \Phi(\eta, B^{n_{j+1}}(\eta,-N), B_\eta^{n_{j+1}}(\eta,-N))\, d\eta \tag{4.50}$$
$$+ \int_K^{\frac{1}{T^{1/2}}e^{\frac{n_{j+1}-N}{2}}} \Phi(\eta, B^{n_j}(\eta,-N), B_\eta^{n_j}(\eta,-N))\, d\eta. \tag{4.51}$$

Integral (4.49) tends to zero as $j\to\infty$. In fact by the continuity of Φ in the second and third argument we obtain pointwise convergence and by the bounds (4.45) of Φ, we apply the Dominated Convergence Theorem to conclude. Expressions (4.50) and (4.51) can be made arbitrarily small since they can be bounded by

$$C\int_K^\infty \eta^4 e^{-\eta^2/12} d\eta,$$

where C is a positive, and K can be chosen arbitrary large.

Thus we have proved that $\iint_{\mathcal{Q}_N} \rho(B_\tau^{n_j})^2\, d\eta d\tau = 0$ for all N which in turn using (4.44) implies $\bar{B}_\tau = 0$. This proves the claim, and consequently the theorem. ■

4.6 Non-isothermal problem

In this section we provide a numerical description of blow-up for the non-isothermal model. Mimicking the isothermal model, we first introduce the corresponding self similar problem and then we show numerical results, whose methods are explained at the end of the section.

4.6.1 Self-similar problem

Let $T > 0$ be the blow-up time for problem (4.7),(4.9). Using the variable transformation

$$\tau = \log\left(\frac{T}{T-t}\right) \quad \text{and} \quad \eta = \frac{r}{(\omega_3(T-t)\Theta(t))^{1/2}}; \tag{4.52}$$

and for the unknown $\langle b(r,t), \Theta(t)\rangle$, we define

$$B(\eta,\tau) = (T-t)b(r,t), \quad \text{and} \quad \Theta(\tau) = \Theta(t). \tag{4.53}$$

The rectangle D_T transforms into the set

$$\bar{\Pi} = \{(\eta,\tau) \mid \tau > 0,\, 0 < \eta < (\omega_3 T\Theta(\tau))^{-1/2}e^{\tau/2}\}. \tag{4.54}$$

The self-similar problem for non-constant temperature reads

$$B_\tau + B + \frac{1}{2}(1-\bar{a})\eta B_\eta = B_{\eta\eta} + \frac{4}{\eta}B_\eta + \frac{1}{3}\eta B B_\eta + B^2 \quad \text{in} \quad \bar{\Pi} \tag{4.55a}$$

$$B_\eta(0,\tau) = 0, \quad B\left((\omega_3\Theta(\tau)T)^{-1/2}e^{\tau/2},\tau\right) = Te^{-\tau} \quad \text{for} \quad \tau \in \mathbb{R}^+ \tag{4.55b}$$

$$B(\eta,0) = B_0(\eta) := Tb_0\left((\eta\omega_3\Theta(0)T)^{1/2}\right) \quad \text{for} \quad \eta \in \Pi(0), \tag{4.55c}$$

where $\Pi(0) = (0, (\omega_3 T\Theta(0))^{-1/2})$ and $\bar{a} = \bar{a}(\tau)$ is a function given by

$$\bar{a}(\tau) = (T-t)\frac{\Theta_t}{\Theta}.$$

If for some $C > 0$, $\Theta(t) = C(T-t)^{-a}$ near $t = T$, then $\lim_{\tau\to\infty} \bar{a}(\tau) = a$. Moreover, since $\Theta(\tau) > c > 0$, then $a \geq 0$.

Using (4.38) and (4.31), problem (4.55) satisfies the bounds $B(0,\tau) \geq 1$ for $\tau \in (0,\infty)$, and $B_\eta(\eta,\tau) \leq 0$ for $(\eta,\tau) \in \bar{\Pi}$.

Now since the function $\bar{a} = \bar{a}(\tau)$ is a-priori unknown, and assuming $B(0,\tau) \leq M$, a steady state solution of (4.55), say $\varphi(\eta) = \lim_{\tau\to\infty} B(\eta,\tau)$, $a = \lim_{\tau\to\infty} \bar{a}(\tau)$ must be a bounded solutions φ of a family of equations indexed by $a > 0$:

$$\begin{cases} \varphi_{\eta\eta} + \frac{4}{\eta}\varphi_\eta + \frac{1}{3}\eta\varphi\varphi_\eta - \frac{1}{2}(1-a)\eta\varphi_\eta + \varphi^2 - \varphi = 0, & \text{for} \quad \eta > 0, \\ \varphi(0) \geq 1 \quad \varphi_\eta(0) = 0. \end{cases} \tag{4.56}$$

The condition $\varphi(0) \geq 1$ follows since $B(0,\tau) \geq 1$ for $\tau \in (0,\infty)$. Concerning bounded solutions of (4.56), when $a = 0$ this problem is equivalent to (4.10), which we know that has infinitely many bounded solutions (Theorem 4.11). On the other hand, an interesting result from [18] shows that the only non zero bounded solution of (4.56) is $\varphi^a \equiv 1$, if $a > 0$. In the next section, we show numerical results showing a generic convergence of $B(\eta,\tau)$ towards the profile φ_1 solution of (4.10). As a consequence $\Theta(t) \to \bar{\Theta}$ for $\bar{\Theta} < \infty$.

4.6.2 Numerical results

We fix for our simulations,

$$n_0 = 1/\omega_3 \quad \text{in} \quad D, \quad \text{that is} \quad b_0(r) \equiv 1 \quad \text{for} \quad r \in D. \tag{4.57}$$

Therefore, assuming (4.57) as initial condition and considering several values of E and κ such that $E < 1/4\pi$ and $\kappa > 6$, for which we know blow-up is assured, we conclude the following

(i) $\Theta_t > 0$ on $(0, T)$;

(ii) $\Theta(t) \to \bar{\Theta}$ where $\bar{\Theta}$ is a positive finite constant;

(iii) and recalling $\varphi_1(\eta) = \frac{6}{(1+\eta^2/2)}$, we have the convergence

$$(T-t)b(\eta\sqrt{\omega_3\Theta(t)(T-t)}, t) \sim \varphi_1(\eta) \quad \text{as} \quad t \to T.$$

The properties (i) and (ii) are shown in Figure 4.5. Here the line $\Theta = \frac{1}{8\pi}$, as a reference. We observe that all final temperatures showed here are such that $\bar{\Theta} < \frac{1}{8\pi}$. This coincides with the isothermal case where blow-up can occurs for $\Theta^* < 1/8\pi$.

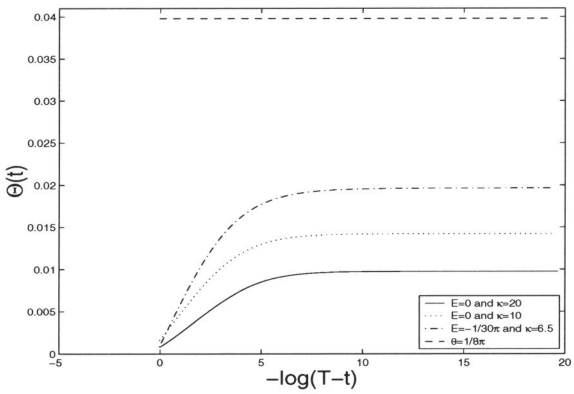

Figure 4.5: Convergence of temperatures

The convergence in (iii) is plotted in Figures 4.6, 4.7, and 4.9, for the corresponding profiles for the temperatures of Figure 4.5.

Finally since we use a moving mesh grid where the solution is calculated, in Figure 4.9 is shown how the points of the mesh accumulate near the singularity at $r = 0$.

4.6.3 Numerical method

For the numerical simulations we use an equation in the variables

$$Q(y,t) = b(r,t)r^3 \quad \text{and} \quad y = r^3.$$

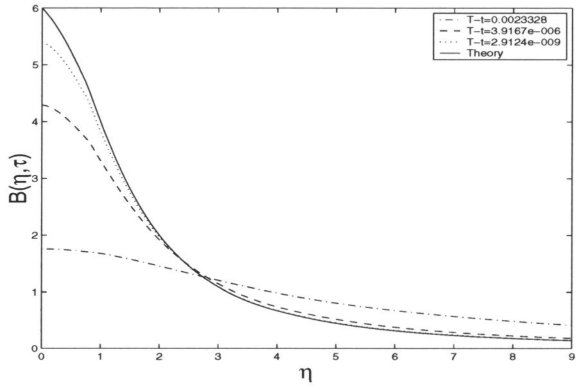

Figure 4.6: Convergence for $E = 0$ and $\kappa = 20$.

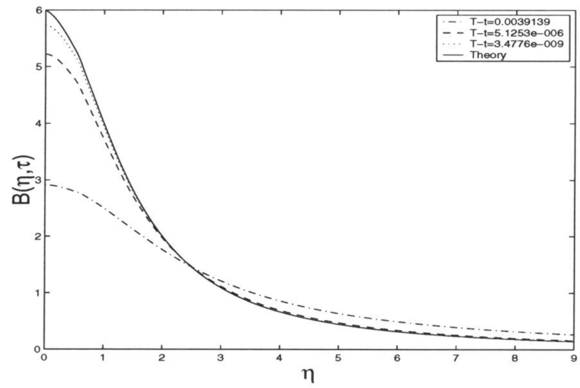

Figure 4.7: Convergence for $E = 0$ and $\kappa = 10$.

Therefore problem (4.7) becomes

$$Q_t = 9\omega_3\Theta(t)y^{4/3}Q_{yy} + QQ_y \tag{4.58}$$

with boundary conditions $Q(0,t) = 0 \quad Q(1,t) = 1$, and the particular initial data (4.57), that is $Q_0(y) = y$. This representation was first introduced in [12] and has been successfully used to study radially symmetric solutions for problem (4.1).

The variable Q represents the accumulated mass and so Q_y is proportional to the density n from problem (4.1), that is

$$n(r,t) = \frac{1}{\omega_3}Q_y(y,t).$$

Since density is positive, we search for monotone solutions ($Q_y \geq 0$). Blow-up for the solution n of (4.1), translates for a solution Q of (4.58) in the existence of a time $T > 0$ and sequences $y_n \to 0$ and $t_n \to T$ such that $Q_y(y_n, t_n) \to \infty$ as $t_n \to T$.

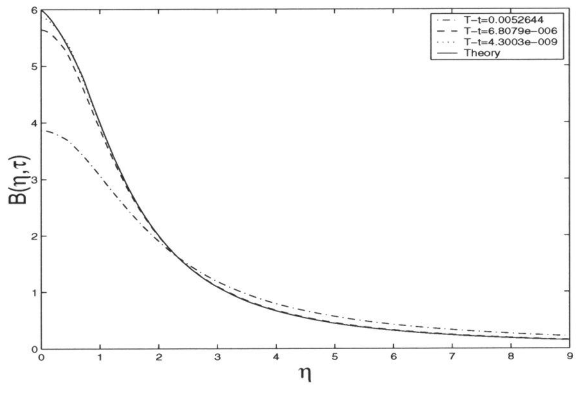

Figure 4.8: Convergence of $E = -1/30\pi$ and $\kappa = 6.5$.

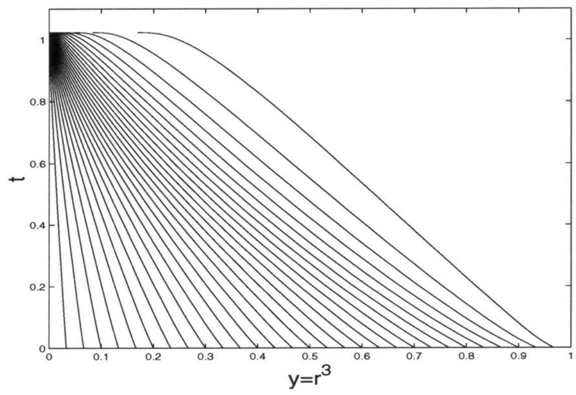

Figure 4.9: Moving mesh for $E = 0$ and $\kappa = 10$.

In the all simulations, the time of blow-up T was determined assuming that

$$Q_y(0,t)(T-t) \to C$$

for some $C > 0$ constant. This holds in the isothermal case with constant initial distribution [21].

All the pictures presented here were computed with a modified version of MOVCOL [55]. As written MOVCOL solves problems of the form

$$F(y,t,Q,Q_y) = \frac{d}{dy}G(y,t,Q,Q_y).$$

Assuming for the moment that temperature is fixed equal to Θ, this means solving

$$Q_t + 6\omega_3\Theta y^{1/3}Q_y = (9\omega_3\Theta y^{4/3}Q_y + Q^2/2)_y \text{ or } (Q_t - QQ_y)/y^{4/3} = 9\omega_3\Theta(Q_y)_y$$

We found however that neither of these forms produce monotone solutions. This has to do with where the right hand side G is evaluated. Therefore, we extended MOVCOL to solve problems of the form

$$F(y,t,Q,Q_y,Q_{yy}) = \frac{d}{dy}G(y,t,Q,Q_y)$$

and then we solved the problem in its original form (4.58).

MOVCOL uses Hermite collocation: the solution is represented by piecewise continuous cubic polynomials in the form

$$Q(y,t) = \sum_{j=1}^{N-1} \phi_1(s)U^j(t) + \phi_2(s)U_y^j(t) + \phi_3(s)U^{j+1}(t) + \phi_4(s)U_y^{j+1}(t) \tag{4.59}$$

where ϕ_i for $i = 1 \dots 4$ are the shape functions, cubic polynomials, and

$$s = (y - y_j)/(y_{j+1} - y_j) \quad \text{for} \quad j = 1 \dots N-1,$$

where N is the number of mesh points. The function $U^j(t)$ is an approximation of $Q(y_j,t)$ and $U_y^{j+1}(t)$ is an approximation of $Q_y(y_{j+1},t)$.

This gives a representation which is C^1 on $[0,1]$ and C^∞ on $[0,1] \setminus \{y_j\}$. Defining the vector $Y = [U, U_y]$, the goal is now to find Y, Y_t such that the differential equation is satisfied exactly at the four gauss points in each subinterval when applied to this representation of the solution. This is a method of lines approach since we have discretized in space to give a system solved as a system of coupled ODE's. Notice that there is no notion of upwinding with this form of discretization. Because of the implicitness of the resulting system and the grading of our mesh, we are not worried about that.

The mesh motion is based on the idea of equidistribution [53, 54]. See [23] for an application to a blow-up problem. We define a monitor function $M(y) > 0$ and we try to find the grid such that the integral of M is the same over every interval. If M is related to the error of the method then this can easily be shown to be the optimal grid.

Denoting y as the the computation variable, this means that

$$0 = x\int_0^1 M(s)ds - \int_0^{y(x)} M(s)ds$$

which is difficult to solve numerically as it is a DAE with no differential variables. Instead we set the right-hand side of this as the forcing term for the mesh motion

$$\tau y_t = x\int_0^1 M(s)ds - \int_0^{y(x)} M(s)ds$$

differentiating twice leads to the *moving mesh partial differential equation* MMPDE6 from [54, 23]

$$\tau(y_t)_{xx} = -(My_x)_x \tag{4.60}$$

which has the advantage that it can possess scaling invariance. Ignoring boundary conditions, the equation we are solving has scaling invariance, in fact if $Q(y,t)$ solves (4.58) then $\lambda^{-1}Q(\lambda^3 y, \lambda^2 t)$ for $\lambda > 0$ also does. We expect convergence to an asymptotically self-similar solution for which this scale invariance is very important, hence we would like to build it into the mesh equation as well as the physical PDE. This is done by choosing a monitor function which leads (4.60) to have the same scaling structure as (4.58). Setting $M(y) = Q_y^p$ implies $p = 1$. This means that in the blow-up limit, the mesh lines move on level sets of the similarity variables (excluding boundary effects which slowly pull the mesh points away).

For $p < 1$ the mesh motion will be too slow and will eventually 'freeze' as $Q_y(0,t) \to \infty$. For $p > 1$ the motion of the mesh is faster than the similarity variable causing additional stiffness and meaning that in the blow-up limit all mesh points will tend to zero in both the physical and similarity variable. However, because of the nature of the singularity in this problem, we found that $M = Q_y$ was not enough to resolve an evolving cube root and the error in approximating $Q \sim y^{1/3}$ required a smaller mesh. So, we use $M = Q_y^{3/2}$ for this initial data. Note that other initial condition could require another monitor function.

The parameter τ is determined by experience, too small and the system is too stiff, too large and the mesh does not move. The idea is that the moving mesh PDE has as its steady state the equidistributed grid and we want to relax onto that in a sufficiently short time relative to the natural dynamics of the PDE. In practice $10^{-2} \leq \tau \leq 10^{-4}$ is generally best but there is no hard and fast rule.

To include the change in temperature, we use an implicit/explicit method,

$$Q_t^n - Q^n Q_y^n - 9\omega_3 \Theta^{n-1} y^4 / 3 Q_{yy}^n = 0$$

where Θ^{n-1} was evaluated at the preceding time step. We recall here that temperature can be expressed by

$$E = \kappa\Theta - \frac{1}{9\omega_3}\int_0^1 \frac{Q^2}{y^{4/3}}\, dy$$

To compute this integral, we used two point Gauss integration

$$\int f \sim (f(g_1) + f(1-g_1))/2$$

over each interval smaller than a tolerance ($dy = .01$) and four point

$$\int f \sim (w_1 f(g_1) + w_2 f(g_2) + w_2 f(1-g_2) + w_1 f(1-g_1))$$

if the interval is large.

4.6.4 Increasing temperature near $t = 0$

We show that the temperature is increasing in small time interval for certain initial data.

Lemma 4.16 *Let $E < \frac{1}{40\pi}$ and $2\kappa - 3 - \int_0^1 r^4 (b_0)_r^2\, dr > 0$. If $\langle b, \Theta \rangle$ is a solution of (4.7), then there exists $t^* > 0$ such that $\Theta_t > 0$ on $(0, t^*)$.*

Proof. Multiplying (4.7a) by $b(r,t)$, gives

$$bb_t = \omega_3 \Theta \frac{1}{r^4}(r^4 b_r)_r b + \frac{1}{9} r(b^3)_r + b^3.$$

Integrating and using that $\frac{3}{2}\omega_3 \kappa \Theta_t = \int_0^1 bb_t r^4\, dr$, we find

$$\frac{3}{2}\omega_3 \kappa \Theta_t = \omega_3 \Theta \int_0^1 (r^4 b_r)_r b dr + \frac{1}{9} \int_0^1 r(b^3)_r r^4 dr + \int_0^1 b^3 r^4 dr$$

and then

$$\frac{3}{2}\omega_3 \kappa \Theta_t = \omega_3 \Theta \left[r^4 b_r b|_0^1 - \int_0^1 r^4 b_r^2 dr \right] + \frac{1}{9} \left[b^3 r^5 - 5 \int_0^1 b^3 r^4 dr \right] + \int_0^1 b^3 r^4 dr.$$

Simplifying, one finds

$$\frac{3}{2}\omega_3 \kappa \Theta_t = \omega_3 \Theta \left[b_r(1) - \int_0^1 r^4 b_r^2 dr \right] + \frac{1}{9} + \frac{4}{9} \int_0^1 b^3 r^4 dr.$$

Multiplying this by 9 and since $b_r(1,t) \geq -3$ (using (4.6)),

$$\frac{27}{2}\omega_3 \Theta_t \geq 9\omega_3 \Theta \left(-3 - \int_0^1 r^4 b_r^2 dr \right) + 1 + 4 \int_0^1 b^3 r^4 dr.$$

On the other hand we have $\int_0^1 b^2 r^4 dr \leq \frac{2}{3} \int_0^1 b^3 r^4 dr + 1/15$ and $\int_0^1 b^2 r^4\, dr = 3\omega_3 \kappa \Theta - 3\omega_3 E$, so

$$\frac{27}{2}\omega_3 \kappa \Theta_t \geq 9\omega_3 \left(2\kappa - 3 - \int_0^1 r^4 b_r^2 dr \right) \Theta + \frac{3}{5} - 18\omega_3 E.$$

This implies that $\Theta_t \geq 0$ for t near 0 if $E < 1/40\pi$ and $2\kappa - 3 - \int_0^1 r^4 (b_0)_r^2 dr > 0$. ■

4.7 Appendix

In this appendix we construct the Lyapunov functional E satisfying (4.43) and deduce the necessary properties of ρ and Φ. We start with a formal derivation for the Lyapunov functional. This construction requires that ρ solves a first order equation, and give an expression for Φ in terms of ρ. We solve the equation for ρ, with convenient data such that Φ and ρ have the needed properties. Finally we use smooth approximations of Φ to obtain a rigorous derivation of (4.43).

4.7.1 Formal derivation of a Lyapunov functional

To find the functions Φ and ρ satisfying (4.43), we first assume that they are regular enough and then compute

$$\frac{d}{d\tau}E(\tau) = \int_0^z \Phi_v B_\tau \, d\eta + \int_0^z \Phi_w B_{\tau\eta} \, d\eta + \frac{1}{2} z\Phi(z, B(\tau, z), B_\eta(\tau, z)). \tag{4.61}$$

Integrating by parts the second integral in (4.61) we obtain for this integral

$$\begin{aligned} \int_0^z \Phi_w B_{\tau\eta} \, d\eta &= -\int_0^z \frac{\partial}{\partial\eta}(\Phi_w) B_\tau \, d\eta + \Phi_w B_\tau|_0^z = \\ &- \int_0^z [\Phi_{\eta w} + \Phi_{vw} B_\eta + \Phi_{ww} B_{\eta\eta}] B_\tau \, d\eta + \Phi_w B_\tau|_0^z. \end{aligned} \tag{4.62}$$

Define the function $f = f(\eta, v, w)$ such that

$$f(\eta, v, w) = \frac{1}{\eta} w - \frac{\eta}{2} w + \frac{1}{3}\eta v w + v^2 - v.$$

Using this, equation (4.16a) takes the form $B_\tau = B_{\eta\eta} + f(\eta, B, B_\eta)$. Replacing this formula in (4.62), we substitute (4.62) into (4.61) and write

$$\frac{d}{d\tau}E(\tau) = \int_0^z \left\{[\Phi_v - \Phi_{\eta w} - \Phi_{vw} B_\eta + \Phi_{ww} f] B_\tau - \Phi_{ww}(B_\tau)^2\right\} d\eta \tag{4.63}$$

$$+ \Phi_w B_\tau|_0^z + \frac{1}{2} z\Phi(z, B(\tau, z), B_\eta(\tau, z)). \tag{4.64}$$

Now if we find functions $\rho = \rho(\eta, v, w) > 0$ and $\Phi = \Phi(\eta, v, w)$ solving the system

$$-\Phi_v + \Phi_{\eta w} + w\Phi_{vw} = \rho f \quad \text{and} \quad \Phi_{ww} = \rho, \tag{4.65}$$

then we find that formally E has the form of a Lyapunov functional with a contribution on the boundary, that is

$$\frac{d}{d\tau}E(\tau) = -\int_0^z \rho(\eta, B, B_\eta)(B_\tau)^2 \, d\eta + \Phi_w B_\tau|_0^z + \frac{1}{2} z\Phi(z, B(\tau, z), B_\eta(\tau, z)). \tag{4.66}$$

Therefore to obtain this expression, we solve the system (4.65). To do that, from (4.65) we deduce a first-order equation for ρ,

$$w\rho_v + \rho_\eta - f\rho_w = f_w \rho. \tag{4.67}$$

Employing the function ρ, we can construct Φ using (4.65) and find

$$\Phi(\eta, v, w) = \int_0^w (w - s)\rho(\eta, v, s) \, ds - \int_0^v \rho(\eta, \mu, 0) f(\eta, \mu, 0) \, d\mu. \tag{4.68}$$

where $f(\eta, \mu, 0) = -\mu + \mu^2$.

4.7.2 The first-order equation for ρ

We solve equation (4.67) using the method of characteristics. Since $\rho = \rho(\eta, v, w)$ is a function in three variables, let introduce a smooth curve $\mathbf{x} = \mathbf{x}(\eta) = (\eta, v^1(\eta), w^1(\eta))$ in $\mathbb{R}^3$. We shall obtain ρ by solving (4.67) along this curve; thus a solution ρ of (4.67) is given by

$$\frac{d}{d\eta}\rho = f_w \rho, \tag{4.69}$$

along the curves defined by

$$\frac{d}{d\eta}v^1 = w, \quad \frac{d}{d\eta}w^1 = -f. \tag{4.70}$$

In order to solve this system of ODE's, we select a vector $(\eta_0, v_0, w_0) \in \mathbb{R}^+ \times \mathbb{R}^2$ and introduce the solutions $\phi(\xi) = \phi(\xi, \eta_0, v_0, w_0)$ of the ODE:

$$\phi'' + f(\eta, \phi, \phi') = 0, \quad \text{with} \quad \phi|_{\eta=\eta_0} = v_0 \quad \text{and} \quad \phi'|_{\eta=\eta_0} = w_0, \tag{4.71}$$

where $' = \frac{\partial}{\partial \eta}$. Since the curve $\mathbf{x}$ satisfies equations (4.70), it can be expressed in terms of ϕ by setting

$$v^1(\eta) = \phi(\eta, \eta_0, v_0, w_0) \quad \text{and} \quad w^1(\eta) = \phi'(\eta, \eta_0, v_0, w_0). \tag{4.72}$$

where $v^1(\eta_0) = v_0$, and $w^1(\eta_0) = w_0$. Noting that $f_w = \frac{4}{\eta} - \frac{\eta}{2} + \frac{1}{3}\eta v$, by (4.69), we find

$$\rho(\eta, v, w) = \rho(\eta_0, v_0, w_0) \exp\left\{ \int_{\eta_0}^{s} \left[\frac{4}{\xi} - \frac{\xi}{2} + \frac{1}{3}\xi v^1(\xi) \right] d\xi \right\}.$$

From the proof of Theorem 4.15, we see that is only necessary to define ρ in the set $\tilde{R}$. For the moment we assume that the vector $(\eta, v, w) \in R$, remember that $R = \{\eta > 0, v \geq 0, w \leq 0\} \cup \{\eta = 0, v \geq 0, w = 0\}$. Now for each fixed $(\eta, v, w) \in R$, we calculate $\rho(\eta, v, w)$ by selecting a characteristic curve which connect this point with a reference point (η_0, v_0, w_0) for which we know the value of ρ. To select an appropriate curve and a reference point, we study some of the properties of solutions ϕ of (4.71) since they define the characteristic curves. It follows from standard theory of ODE's that solutions of (4.71) are locally smooth and continuous under perturbations. We observe however that in general we cannot extend the solution to the whole $\mathbb{R}^+$. In fact for each $(\eta, v, w) \in R$, there may exist a $\xi_1 \geq 0$ and/or a $\xi_2 \geq 0$ such that

$$\phi(\xi_1, \eta, v, w) = \infty \text{ with } \xi_1 < \eta \quad \text{and/or} \quad \phi(\xi_2, \eta, v, w) = -\infty \text{ with } \xi_2 > \eta.$$

In light of this, we choose to use forward solutions of (4.71) to define the characteristic curves. The next result show the possible behaviour of a forward solution ϕ of (4.71).

Lemma 4.17 *Select $(\eta, v, w) \in R$, and let $\phi(\xi) = \phi(\xi, \eta, v, w)$ be the solution of (4.71). For $\xi \geq \eta$, exactly one of the three alternatives holds:*

(i) $\phi \equiv 1$ *or* $\phi \equiv 0$*;*

(ii) *there exists* $\eta^* > \eta$ *such that* $\phi(\eta^*) = 0$ *and* $\phi(\xi) < 0$ *for* $\xi > \eta^*$*;*

(iii) $\phi(\xi) \to 0$ *as* $\xi \to \infty$ *and there exists a constant* $C > 0$ *such that* $\phi(\xi)\xi^2 \to C$ *as* $\xi \to \infty$.

Proof. Functions $\phi \equiv 1$ and $\phi \equiv 0$ are the only constant solutions of (4.71). To prove (ii)-(iii), we divide the proof in three steps and we assume that ϕ is not constant.

Step 1. We recall a property found in [51]. If $\phi(\xi) = \phi(\xi, \eta, v, w)$ solves (4.71) such that

$$\phi(\xi_0) > 1, \quad \phi'(\xi_0) = 0 \quad \text{for some large } \xi_0, \tag{4.73}$$

then $\phi(\xi)$ becomes negative at some $\xi_1 > \xi_0$.

Step 2. We next show that the solution $\phi(\xi)$ is bounded from above for $\xi > \eta$. Multiplying (4.71) by ϕ' in the interval (s, r) we obtain

$$\frac{\phi'(r)^2}{2} = \frac{\phi'(s)^2}{2} - \int_s^r \left[\frac{4}{\eta} - \frac{1}{2\eta} + \frac{\eta\phi}{3}\right] (\phi')^2 \, d\eta - G(\phi(r)) + G(\phi(s))$$

where $G(t) = \frac{t^3}{3} - \frac{t^2}{2}$. If $\phi(s) = 3/2$ and $\phi(t) \geq 3/2$ for all $t \in [s, r]$ then

$$|\phi'(r)| \leq |\phi'(s)|.$$

This implies that $\phi(\xi)$ is bounded from above for any finite ξ. Suppose now that ϕ is unbounded as $\xi \to \infty$. Then, for ξ large, and $\phi(\xi) > 3/2$, equation (4.71) asymptotically becomes:

$$\phi'' + \gamma\xi\phi' + \delta\phi = 0$$

for some positive γ and δ. This implies that there exists ξ_1 large such that $\phi'(\xi_1) = 0$ which is a contradiction with (4.73).

Step 3. We assert that if $\phi(\xi) \to C = \textit{constant}$ then $C = 0$. First observe that if $\phi(\xi) \to C$ as $\xi \to \infty$ then $C = 1$ or $C = 0$. Now, we claim that if ϕ is monotone near infinity, then $\phi(\xi) \not\to 1$ as $\xi \to \infty$. In fact, assuming $\phi < 1$, $\phi' > 0$ and $\phi'' < 0$ for ξ large, then ϕ cannot satisfy (4.71) since we find the inequation

$$\phi'' + \left(\frac{4}{\bar{\xi}} + \frac{1}{3}\bar{\xi}\phi - \frac{1}{2}\xi\right)\phi' + \phi(\phi - 1) < 0 \quad \text{for some large fixed } \xi \tag{4.74}$$

A similar situation occurs if we assume $1 < \phi(\xi) < 3/2$, $\phi' < 0$ and $\phi'' > 0$ for ξ large, now inequation (4.74) is satisfied with the opposite sign. The only remaining possibility is for ϕ to approach $C = 1$ in an oscillatory way. This is not possible since if ϕ has a maximum for ξ large then property (4.73) holds. This proves the assertion of this step.

We finish the proof using these three steps. Suppose that $\phi(\xi) \to 0$ as $\xi \to \infty$. Then equation (4.71) reduces for ξ large to

$$\phi'' - \frac{\xi}{2}\phi' - \phi = 0$$

and from here we deduce that $\phi(\xi)\xi^2 \sim C$ as $\xi \to \infty$. Combining this with the assertion in step 3, we obtain possibility (ii).

Combining steps 2, and 3 we find that the only possibility left is (ii), that is $\phi(\eta^*) = 0$ for a finite η^*. Note that once this occurs then $\phi(\xi) < 0$ for $\xi > \eta^*$ since ϕ cannot have a negative minimum. ■

Now with the properties of forward solutions of (4.71) at hand, we define a solution ρ in R, along the characteristics. From the previous lemma we can classify the vectors in R. We define $\{R_i\}$ for $i = 1 \dots 3$,

$$R_1 = \{(\eta, v, w) \in R \colon \phi(\xi, \eta, v, w) \text{ satisfies (i) in Lemma 4.17}\};$$

$$R_2 = \{(\eta, v, w) \in R \colon \phi(\xi, \eta, v, w) \text{ satisfies (ii) in Lemma 4.17}\};$$

$$R_3 = \{(\eta, v, w) \in R \colon \phi(\xi, \eta, v, w) \text{ satisfies (iii) in Lemma 4.17}\}.$$

Using these sets, we shall define ρ on each subset $R_i \subset R$ with $i = 1 \dots 3$. In the next result we introduce a parameter to give a definition of ρ with appropriate estimates.

Lemma 4.18 *There exists a large constant $\bar{\eta}$ such that for every $\eta_\epsilon > \bar{\eta}$, we have*

$$\phi(\xi, \eta_\epsilon, \epsilon, -\bar{\epsilon}) \leq 1 \quad \textit{for} \quad \xi \in (\bar{\eta}, \eta_\epsilon)$$

for all $0 < \epsilon \leq 1/2$, and $0 < \bar{\epsilon} \leq 1$.

Proof. Proving this lemma is equivalent to proving that each η_1 large enough satisfies the property

$$(L) \quad \begin{cases} \text{every solution } \phi \text{ of (4.71) with} \\ \phi(\eta_1) = 1 \text{ and } \phi'(\eta_1) < 0 \text{ satisfies: for all } \eta_2 > \eta_1 \\ \text{with } \phi(\eta_2) \in [0, 1/2],\ -\phi'(\eta_2) > 1. \end{cases}$$

To prove (L) for large η_1, we define the variable y such that $\xi = \eta_1 y$, and find the equation

$$0 = \frac{1}{\eta_1^2}\left(\ddot{\phi} + \frac{4}{y}\dot{\phi}\right) - \frac{y}{2}\dot{\phi} + \frac{1}{3}y\phi\dot{\phi} + \phi^2 - \phi, \quad \text{for} \quad y > 1 \tag{4.75}$$

$$\dot{\phi}(1) = -D\eta_1, \quad \phi(1) = 1, \tag{4.76}$$

where $\dot{} = \frac{d}{dy}$. Assuming η_1 large then $\frac{1}{2\eta_1^2} \ll y^2$, and the dominant terms of the above equation are

$$0 = \frac{1}{\eta_1^2}\ddot{\phi} - \frac{y}{2}\dot{\phi} + \frac{1}{3}y\phi\dot{\phi} + \phi^2 - \phi. \tag{4.77}$$

Note also that $\phi''(\eta_1) < 0$ for $\eta_1 > C$ where C is a large constant. The equation (4.77) near $y = 1$, becomes

$$0 = \frac{1}{\eta_1^2}\ddot{\phi} - \left(\frac{1}{2} - \frac{1}{3}\phi\right)\dot{\phi},$$

and integrating we have $\dot{\phi}(y) = -\frac{3}{2}\eta_1^2[(\frac{1}{2} - \frac{1}{3}\phi)^2 - \frac{1}{36}] - D\eta_1$. In the original variable ξ, we have $\phi'(\xi) = -\frac{3}{2}\eta_1[(\frac{1}{2} - \frac{1}{3}\phi)^2 - \frac{1}{36}] - D$. This shows that $\phi' \sim \eta_1$ and then larger than 1 for $\phi \leq 1/2$. To conclude, we set

$$\bar{\eta} = \min\{\eta_1 \colon \text{ with } \eta_1 \text{ satisfying property (L)}\} + 1. \blacksquare$$

Now we turn to calculate ρ in R. The idea is to use as a reference a point where $\eta_0 = \bar{\eta}$. In this way, owing to Lemma 4.18, we can obtain useful estimates for ρ. However it can happen that the function $\phi(\xi, \eta, v, w)$ is not defined at $\xi = \bar{\eta}$. In such situation, to define ρ, we introduce functions representing the intersection of $\phi(\cdot, \eta, v, w)$ with the line $\phi = 0$ for $\eta < \bar{\eta}$ and the line $\phi = 1$ for $\eta > \bar{\eta}$. We start, by computing $\rho(\eta, v, w)$ for $(\eta, v, w) \in R_3$ where $\phi(\bar{\eta}, \eta, v, w) \in (0, 1)$.

Case R_3. Fix a point $(\eta, v, w) \in R_3$. We choose $\eta_0 = \bar{\eta}$, $v_0 = \phi(\bar{\eta}, \eta, v, w)$ and $w_0 = \phi'(\bar{\eta}, \eta, v, w)$. Setting $\rho(\eta_0, v_0, w_0) = \eta_0^4 e^{-\eta_0^2/4}$, we find

$$\begin{aligned} \rho(\eta, v, w) &= \rho(\eta, v^1(\eta), w^1(\eta)) \\ &= \eta^4 e^{-\eta^2/4} \exp\left\{ \int_{\bar{\eta}}^{\eta} \frac{1}{3}\xi\phi(\xi, \bar{\eta}, \phi(\bar{\eta}, \eta, v, w), \phi'(\bar{\eta}, \eta, v, w))\, d\xi \right\}. \end{aligned} \tag{4.78}$$

Now we explain the reason why we choose $\eta_0 = \bar{\eta}$ as a reference point. This choice allows us to estimate the height of ϕ for $\xi > \bar{\eta}$, which in turn permits to control ρ for large η. In fact, since $\phi(\xi)\xi^2 \to C > 0$ as $\xi \to \infty$, there exists $\eta_\epsilon > \bar{\eta}$ such that $\phi(\eta_\epsilon, \eta, v, w) = \epsilon < 1/2$, and $-\phi'(\eta_\epsilon, \eta, v, w) = \bar{\epsilon} < 1$, with $\bar{\epsilon} \sim \frac{2\epsilon}{\eta_\epsilon}$. Then Lemma 4.18 implies $\phi(\xi) \leq 1$ for $\xi > \bar{\eta}$. Substituting this bound in (4.78) implies an exponential decay for ρ as $\eta \to \infty$.

Case R_1. Points in R_1 are of the form $(\eta, 1, 0)$ and $(\eta, 0, 0)$. Substituting $\phi \equiv 1$ and $\phi \equiv 0$ into formula (4.78) gives

$$\rho(\eta, 1, 0) = \eta^4 e^{-\frac{\eta^2}{12}} e^{-\frac{\bar{\eta}^2}{6}}, \quad \text{and} \quad \rho(\eta, 0, 0) = \eta^4 e^{-\frac{\eta^2}{4}}. \tag{4.79}$$

Case R_2. We provide a definition for ρ dividing R_2 in two disjoint subsets: $R_{2a} = R_2 \cup \{\eta \leq \bar{\eta}\}$ and $R_{2b} = R_2 \cup \{\eta > \bar{\eta}\}$.

Case R_{2a}. Fix a point $(\eta, v, w) \in R_{2a}$. Let η^* be given by Lemma 4.17 and define the function $L_0 \colon R_{2a} \mapsto \mathbb{R}^+$ such that $L_0(\eta, v, w) = \min\{\eta^*, \bar{\eta}\}$. Note that the function L_0 is continuous and represents the point η^* where $\phi(\eta^*, \eta, v, w)$ vanishes or equals $\bar{\eta}$ if $\phi(\bar{\eta}, \eta, v, w) \geq 0$. To find ρ, we choose $(\eta_0, v_0, w_0) = (\eta^*, 0, \phi'(\eta^*, \eta, v, w))$, and set $\rho(\eta_0, v_0, w_0) = \eta_0^4 e^{-\eta_0^2/4}$. This gives

$$\rho(\eta, v, w) = \rho(\eta, v^1(\eta), w^1(\eta)) = \eta^4 e^{-\eta^2/4} \times$$
$$\times \exp\left\{ \int_{L_0(\eta,v,w)}^{\eta} \frac{1}{3}\xi\phi\left(\xi, L_0(\eta, v, w), 0, \phi'(L_0(\eta, v, w), \eta, v, w)\right) d\xi \right\}. \tag{4.80}$$

Case R_{2b}. Here is convenient to define for any $(\eta, v, w) \in R_{2b}$ the function $L_1 \colon R_{2b} \mapsto \mathbb{R}^+$, by

$$L_1(\eta, v, w) = \begin{cases} \max\{\max\{\xi \in (0, \eta) \mid \phi(\eta, \xi, v, w) \geq 1\}, \bar{\eta}\} & \text{if} \quad v < 1, \\ \\ \min\{\xi \in (\eta, \infty) \mid \phi(\eta, \xi, v, w) \leq 1\} & \text{if} \quad v \geq 1. \end{cases}$$

The function L_1 is well defined for $v < 1$ since by contradiction if $\phi(\eta, \xi, v, w) = 0$ for $\xi > \bar{\eta}$ then ϕ has to attain a local maximum below the line 1, which is not possible. For $v \geq 1$, L_1 is well-defined by Lemma 4.17. Note that $\phi(L_1(\eta, v, w), \eta, v, w) \leq 1$. This function is continuous and represents the value η_* where $\phi(\eta_*, \eta, v, w) = 1$ or equals $\bar{\eta}$ if $\phi(\bar{\eta}, \eta, v, w) \in (0, 1)$ when $v < 1$. Fix a point $(\eta, v, w) \in R_{2b}$. we select $\eta_0 = L_1(\eta, v, w)$ and set $\rho(\eta_0, v_0, w_0) = \eta_0^4 e^{-\eta_0^2/12} e^{-\bar{\eta}^2/6}$, Then ρ is given by

$$\rho(\eta, v, w) = \rho(\eta, v^1(\eta), w^1(\eta)) = \eta^4 e^{-\eta^2/12} e^{-\bar{\eta}^2/6} \times$$
$$\times \exp\left\{ \int_{L_1(\eta,v,w)}^{\eta} \frac{1}{3}\xi[\phi\left(\xi, L_1(\eta, v, w), 1, \phi'(L_1(\eta, v, w), \eta, v, w)\right) - 1]\, d\xi \right\}. \tag{4.81}$$

4.7.3 Properties of ρ and Φ

In the previous section, we have find ρ solving (4.67). Here we show that this solution together with Φ, satisfies the required properties to prove Theorem 4.15.

We now prove a result which provide a lower bound for ρ in a subset of R.

Lemma 4.19 *Let M and $\bar{M}$ be the constants given by estimates (4.36) and (4.37) respectively. Then there exists a large constant $\bar{\eta}_0$ such that the function $G \colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ given by*

$$G(\eta) = \max\{L_1(\eta, a, b) \mid 1 \leq a \leq M \textit{ and } -\bar{M} \leq b \leq 0\}$$

satisfies $G(\eta) \leq C\eta$ for some constant $C = C(M) > 0$.

Proof. First note that $G(\eta) = L_1(M, 0, \eta)$ for $\eta > \bar{\eta}_0$, if we take a large enough $\bar{\eta}_0 > 0$. Therefore for $\eta > \bar{\eta}_0$, we define the variable $y \geq 1$ such that $\xi = \eta y$. As in (4.77), we find the problem

$$0 = \frac{1}{\eta^2}\ddot{\phi} - \frac{y}{2}\dot{\phi} + \frac{1}{3}y\phi\dot{\phi} + \phi^2 - \phi \quad \text{for} \quad y > 1, \ \dot{\phi}(1) = 0, \text{ and } \phi(1) = M. \tag{4.82}$$

We have two possibilities, either $M > 3/2$ or $M \leq 3/2$. Suppose first that $M \leq 3/2$. A solution of problem (4.82) has the form

$$\phi(y) = \frac{A}{B}(y-1) + C(1 - e^{B\eta_1^2(y-1)}) + M \tag{4.83}$$

for A, B, C positive constants. Then for some y close to 1, the solution ϕ crosses 1, which solves this part. Suppose now that $M > 3/2$, in this case we have for y near 1, a solution of the form (4.83) with $B < 0$. In this case for some y_1 near 1, the solution takes the linear form $\phi = \frac{A}{B}(y-1) + C + M$. Hence the equation in (4.82) with small second order term becomes

$$(\phi - \phi^2) + y\left(\frac{1}{2} - \frac{\phi}{3}\right)\dot{\phi} = 0$$

and is valid till ϕ is near $3/2$. From this equation we can estimate where is the y_2 such that $\phi(y_2) = 1$. Integrating above equation in the interval (y_1, y_2) estimating $\phi(y_1) = M$ we find

$$\frac{y_2}{y_1} = \frac{M^3(a-1)}{(M-1)a^3} \quad \text{with} \quad a = 3/2.$$

At the point $y = y_2$ the equation takes now the form $\frac{\ddot{\phi}}{\eta^2} = -A + y_2(\frac{1}{2} - \frac{\phi}{3})\dot{\phi}$ and integrating

$$\frac{\dot{\phi}}{\eta^2} = -Ay + C - \frac{3y_2}{2}(\frac{1}{2} - \frac{\phi}{3})^2$$

this shows that for y near y_2 the solution crosses the line 1. We conclude by noticing that $G(\eta) \sim y_2\eta$, and by the above analysis $y_2 \sim M^2$. This proves the lemma. ■

Now to obtain estimates for ρ and Φ, we narrow the set R and define

$$\tilde{R} = R \cap \{0 \leq v \leq M,\ 0 \leq -w \leq \bar{M}\}$$

where M and $\bar{M}$ are constants given by estimates (4.36) and (4.37). Also as a consequence of the above lemma, we redefine if necessary $\bar{\eta} = \max\{\bar{\eta}, \bar{\eta}_0\}$.

Lemma 4.20 *The function ρ is continuous in $R\backslash\{\eta = \bar{\eta}, v > 1\}$ and for $(\eta, v, w) \in R$, one finds*

$$\rho(\eta, v, w) \leq \eta^4 e^{-\eta^2/12}. \tag{4.84a}$$

In addition if $(\eta, v, w) \in \tilde{R}$, then

$$\rho(\eta, v, w) \geq \frac{1}{C_0}\eta^4 e^{-C_0\eta^2} \tag{4.84b}$$

for some constant $C_0 = C_0(M) > 0$.

Proof. We start by proving (4.84). We abuse notation and redefine $R_i = \tilde{R} \cap R_i$ for $i = 1..3$.

If $(\eta, v, w) \in R_1$ the estimate (4.84) follows directly by definition. If $(\eta, v, w) \in R_{2a}$, using that $\phi > 0$ in $(\eta, L_0(\eta, v, w))$, then the integral in (4.80) is negative, and implies $\rho(\eta, v, w) \leq \eta^4 e^{-\eta^2/4}$. The continuity of ϕ in $(\eta, L_0(\eta, v, w))$, produces the lower bound $C(M)\eta^4 e^{-\eta^2/4} \leq \rho(\eta, v, w)$, where $C(M) < 1$.

$$C(M)\eta^4 e^{-\eta^2/4} \leq \rho(\eta, v, w) \leq \eta^4 e^{-\eta^2/4} \quad \text{for} \quad (\eta, v, w) \in R_{2a}, \tag{4.85}$$

for some $C(M) < 1$. From which (4.84) follows. For any $(\eta, v, w) \in R_{2b}$, we use (4.81) and find the upper bound $\rho(\eta, v, w) \leq \eta^4 e^{-\eta^2/12} e^{-\bar{\eta}^2/6}$. For a lower bound, we find

$$\begin{cases} \rho(\eta, v, w) \geq \eta^4 e^{-\eta^2/4} & \text{for} \quad v \leq 1, \\ \rho(\eta, v, w) \geq \eta^4 e^{-\eta^2/12} e^{-\bar{\eta}^2/6} e^{-C(M)\eta^2} & \text{for} \quad v > 1, \end{cases}$$

where $C(M) > 0$ and we have used Lemma 4.19 for $v > 1$. Finally for $(\eta, v, w) \in R_3$, we have two cases, if $\eta \leq \bar{\eta}$ then the estimate (4.85) holds and if $\eta > \bar{\eta}$ then the above estimate for R_{2b} with $v \leq 1$ holds.

Claim. ρ is continuous in $R \setminus \{\eta = \bar{\eta},\ v > 1\}$.

Before to prove this we note that R_2 is an open set and R_1 and R_3 are closed.

We first see that ρ is continuous within R_{2a} and R_{2b}, by continuity of L_0 and L_1. For the elements in R_1, the definition of ρ is as for R_2, therefore there is continuity of ρ between R_2 and R_1.

The delicate part is to proof continuity between R_3 and R_2. Taking a sequence $(\eta_n, v_n, w_n) \in R_2$, we associate a solution $\phi_n(\cdot, \eta_n, v_n, w_n)$. Suppose that $(\eta_n, v_n, w_n) \to (\eta, v, w) \in R_3$. Now if $\phi(\cdot, \eta, v, w)$ is the solution of (4.71) then $\phi_n \to \phi$ in compact subsets of $\mathbb{R}^+$. Therefore by Lemma 4.18, for $n \geq n_0 \in \mathbb{N}$, we find $\phi_n(\bar{\eta}) \in (0, 1)$. Then $(\eta_n, v_n, w_n) \in R_2$ for $n \geq n_0$, have the same definition of ρ as for $(\eta, v, w) \in R_3$.

Finally if $v \leq 1$ and $\eta = \bar{\eta}$, then ρ is continuous. If η close enough to $\bar{\eta}$ then we have that $\eta_0 = \bar{\eta}$. So the computation of ρ uses the same formula, independent of the subset of R to which (η, v, w) belongs. ■

For Φ we deduce the following lemma

Lemma 4.21 *The function Φ is continuous in $R \setminus \{\eta = \bar{\eta}, v > 1\}$ and if $(\eta, v, w) \in R$, then*

$$\Phi(\eta, v, w) \leq \{w^2 + \frac{v^2}{2}\}\eta^4 e^{-\eta^2/12}$$

and

$$\Phi(\eta, v, w) \geq -P(v)\eta^4 e^{-\eta^2/12}$$

where $P(v) = \frac{v^3}{3}$ for $v \geq 1$ and $P(v) = 0$ else.

Proof. Follows directly from the definition of Φ (4.68) and using the upper bound for ρ (4.84a). ■

4.7.4 Regularizing argument

In the beginning of this appendix, we formally constructed a Lyapunov functional $E(\tau)$ with Φ and ρ satisfying (4.66). In the previous section, we obtained a solution ρ of (4.67) and Φ given by (4.65). Moreover these functions satisfy the properties found in Lemmas 4.20 and 4.21. From these results we do not obtain enough regularity to derive (4.66). To do this, we need to introduce a regularization of Φ. This is done in the next result.

Lemma 4.22 *Let ρ be a solution of (4.67) obtained as in the previous section and Φ in terms of ρ by (4.65). Then for a given $\tau_2, \tau_1 > 0$ such that $\tau_2 > \tau_1$, we have*

$$E(\tau_2) - E(\tau_1) = -\int_{\tau_1}^{\tau_2}\int_0^z \rho(\eta, B, B_\eta)(B_\tau)^2\, d\eta d\tau + \psi(\tau_1, \tau_2), \tag{4.86}$$

where ψ is defined by (4.47).

Proof. First for the analysis that follows we extend ρ and Φ to $\mathbb{R}^3$ by setting it to zero outside R. Define $\varrho \in C^\infty(\mathbb{R}^2)$ by

$$\varrho(x) = \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } \ |x| \leq 1, \\ 0 & \text{if } \ |x| > 1, \end{cases}$$

and $C = (\int \varrho)^{-1}$. Set $\varrho_\epsilon(x) = C\epsilon^{-2}\varrho(x/\epsilon)$, and note that this family of functions satisfies $\int_{\mathbb{R}^2} \varrho_\epsilon\, dx = 1$ and $\text{supp}(\varrho_\epsilon) \subset B_0(\epsilon)$.

Let $\Psi\colon \mathbb{R}^3 \to \mathbb{R}$ be any locally integrable function. We define for $\epsilon > 0$ the translation $\Psi^\epsilon(\eta, v, w) = \Psi(\eta - \epsilon, v, w)$ and its mollification $\Psi_\epsilon = \varrho_\epsilon * \Psi^\epsilon$ in $\mathbb{R}^3$, that is

$$\begin{aligned} \Psi_\epsilon(\eta, v, w) &= \int_{\mathbb{R}^3} \varrho_\epsilon(\eta - \epsilon - y_1, v - y_2)\Psi(y_1, y_2, w)\, dy_1 y_2 \\ &= \int_{B_0(\epsilon)} \varrho_\epsilon(y_1, y_2)\Psi(\eta - \epsilon - y_1, v - y_2, w)\, dy_1 dy_2 \end{aligned}$$

for $(\eta, v, w) \in \mathbb{R}^3$. Note that this convolution is only in the variables (η, v).

For Φ it is enough to regularize only in the variables (η, v) since it has the required regularity in w. In fact, since ρ is continuous in $R\backslash\{\eta = \bar{\eta}\}$, $\Phi_{ww} = \rho$ is also continuous in $R \setminus \{\eta = \bar{\eta}\}$. Therefore $\Phi_\epsilon \in C^2(R)$ and we may write

$$\begin{aligned} E_\epsilon(\tau_2) - E_\epsilon(\tau_1) &= +\int_{\tau_1}^{\tau_2}\int_0^z \left\{L\Phi_\epsilon B_\tau - (\Phi_\epsilon)_{ww}(B_\tau)^2\right\} d\eta d\tau \\ &+ \int_{\tau_1}^{\tau_2} (\Phi_\epsilon)_w B_\tau|_0^z\, d\tau + \frac{1}{2}\int_{\tau_1}^{\tau_2} z\Phi_\epsilon(z, B(\tau, z), B_\eta(\tau, z))d\tau, \end{aligned} \tag{4.87}$$

where $E_\epsilon(\tau) = \int_0^z \Phi_\epsilon \, d\eta$ and we have introduced the linear operator $L\Phi_\epsilon = (\Phi_\epsilon)_v - (\Phi_\epsilon)_{\eta w} - w(\Phi_\epsilon)_{vw} + (\Phi_\epsilon)_{ww} f$. The idea is to obtain (4.86) from (4.87) letting $\epsilon \to 0$.

To prove (4.86), we first show that

$$\int_{\tau_1}^{\tau_2} \int_0^z (L\Phi_\epsilon)(\eta, B(\eta,\tau), B_\eta(\eta,\tau)) B_\tau \, d\eta \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{4.88}$$

To prove this we first examine the expression $(L\Phi_\epsilon)(\eta, v, w)$. Recalling that $\rho = \Phi_{ww}$, in the sequel we write $\rho_\epsilon = (\Phi_\epsilon)_{ww}$. Since the derivatives of Φ in the variable η and v are weakly defined, the equation $L\Phi = 0$ is understood in the weak sense. In particular, we have $0 = [L\Phi]_\epsilon$ and we find

$$\begin{aligned} L\Phi_\epsilon &= (\Phi_\epsilon)_v - (\Phi_\epsilon)_{\eta w} - w(\Phi_\epsilon)_{vw} + \rho_\epsilon f \\ &= [L\Phi]_\epsilon + \rho_\epsilon f - (\rho f)_\epsilon \\ &= \rho_\epsilon f - (\rho f)_\epsilon. \end{aligned}$$

Using this expression, we turn to prove (4.88). We proceed in two steps.

1. We assert that

$$[L\Phi_\epsilon](\eta, B, B_\eta) = [(\rho f)_\epsilon - (\rho_\epsilon) f](\eta, B, B_\eta) \to 0 \quad \text{for all} \quad (\eta, \tau) \in (\mathbb{R}^+)^2, \quad \eta \neq \bar{\eta}. \tag{4.89}$$

This gives pointwise convergence of $L\Phi_\epsilon(\eta, B, B_\eta) B_\tau \to 0$ a.e. in $(\mathbb{R}^2)^+$. This follows since B_τ is bounded on compact sets if $(\eta, B, B_\eta) \in \tilde{R}$. See beginning of proof of Theorem 4.15.

The convergence in (4.89), is a consequence of

$$(\rho f)_\epsilon - (\rho_\epsilon) f \to 0 \quad \text{for all} \quad (\eta, v, w) \in R, \quad \eta \neq \bar{\eta}. \tag{4.90}$$

To confirm this, let $V \subset R$ be open and such that $(\eta, v, w) \in V$ with $\eta \neq \bar{\eta}$. Then

$$\|(\rho f)_\epsilon - \rho f\|_{C^0(V)} \leq \|(\rho f)_\epsilon - (\rho f)^\epsilon\|_{C^0(V)} + \|(\rho f)^\epsilon - \rho f\|_{C^0(V)}. \tag{4.91}$$

The second term on the right hand side converges to 0 with ϵ, since ρf is uniformly continuous on bounded sets; and the first term also vanishes in the limit because $(\rho f)^\epsilon$ is continuous. On the other hand

$$\|\rho_\epsilon f - \rho f\|_{C^0(V)} \leq \|\rho_\epsilon f - \rho^\epsilon f\|_{C^0(V)} + \|\rho^\epsilon f - \rho f\|_{C^0(V)}. \tag{4.92}$$

By reasoning similar to (4.91), the right hand side goes to 0 with ϵ. Combining (4.91) and (4.92), the assertion (4.90) is proved.

2. We claim that there exists $g \in L^1(\mathbb{R}^+ \times (\tau_1, \tau_2))$ such that $|L\Phi_\epsilon(\eta, B, B_\eta)| \leq g(\eta, \tau)$. To find g, it is convenient to write

$$|L\Phi_\epsilon(\eta, B, B_\eta)| \leq |(\rho f)_\epsilon| + |\rho_\epsilon f|.$$

The first term can be written as

$$\left|(\rho f)_\epsilon\right| = \left| \int\limits_{B_0(\epsilon)} \varrho_\epsilon(y_1,y_2)[\rho f](\eta-\epsilon-y_1, B-y_2, B_\eta)\,dy_1 dy_2 \right|.$$

Then

$$\left|(\rho f)_\epsilon\right| \le \sup_{\substack{\eta-\epsilon\le s\le \eta+\epsilon\\ B-\epsilon\le t\le B+\epsilon}} [\rho f]^\epsilon(s,t,B_\eta).$$

Using the form of f and the estimates on ρ, we obtain

$$|[\rho f](\eta,v,w)| \le C\eta^4 e^{-\eta^2/12}(\frac{1}{\eta}+C\eta) \quad \text{for} \quad (\eta,v,w)\in\tilde{R} \tag{4.93}$$

for $C>0$ a constant depending on M. Using this estimate, we find

$$\left|(\rho f)_\epsilon\right| \le g_2(\eta) := C\min\{\eta^3, (\eta+2\epsilon)^5 e^{-(\eta+2\epsilon)^2/12}\}.$$

Similarly, using now the estimate $0 \le \rho(\eta,v,w) \le \eta^4 e^{-\eta^2/12}$ for $(\eta,v,w)\in R$, we obtain

$$|\rho_\epsilon f| \le \left| \sup_{\substack{\eta-\epsilon\le s\le \eta+\epsilon\\ B-\epsilon\le v\le B+\epsilon}} \rho^\epsilon(s) \right| |f| \le g_2(\eta) := C\min\{\eta^4, (\eta+2\epsilon)^4 e^{-(\eta+2\epsilon)^2/12}\}.$$

Taking $g = g_1 + g_2$, we have $g\in L^1(\mathbb{R}^+\times(\tau_1,\tau_2))$ which proves the convergence in (4.89).

Reasoning as in the proof of (4.88), we obtain

$$\int_{\tau_1}^{\tau_2}\int_0^z \rho_\epsilon B_\tau^2\,d\eta d\tau \to \int_{\tau_1}^{\tau_2}\int_0^z \rho B_\tau^2\,d\eta d\tau \quad \text{as} \quad \epsilon\to 0. \tag{4.94}$$

In fact using the argument to prove (4.89), we deduce

$$[\rho_\epsilon - \rho](\eta, B(\eta,\tau), B_\eta(\eta,\tau)) \to 0 \quad \text{for all} \quad (\eta,\tau)\in(\mathbb{R}^+)^2, \quad \eta\ne\bar{\eta}. \tag{4.95}$$

Moreover the estimate $0\le\rho(\eta,v,w)\le C\eta^4 e^{-\eta^2/12}$ for $(\eta,v,w)\in\tilde{R}$, yields

$$|\rho_\epsilon| \le \sup_{\substack{\eta-\epsilon\le s\le \eta+\epsilon\\ B-\epsilon\le t\le B+\epsilon}} [\rho](s,t,B_\eta) \le g_3(\eta) = C\min\{\eta^4, (\eta+2\epsilon)^4 e^{-(\eta+2\epsilon)^2/12}\}.$$

This combined with (4.95) concludes the proof of (4.94), since $g_3\in L^1(\mathbb{R}^+\times(\tau_1,\tau_2))$.

In the same manner, we can proof that $E_\epsilon(\tau)\to E(\tau)$ as $\epsilon\to 0$, as well as the convergence of the boundary term in (4.87). Combining these results with (4.88) and (4.94), we let $\epsilon\to 0$ in (4.87) to find (4.86). ■

Chapter 5

Asymptotic results for injection of reactive solutes from a three-dimensional well

5.1 Introduction

Suppose a homogeneous and saturated porous medium occupies the region

$$\Omega_\epsilon = \{x \in \mathbb{R}^3 \colon |x| > \epsilon\}.$$

Here $\epsilon > 0$ denotes the radius of an injection well, which induces a radially symmetric flow in Ω_ϵ. At a certain instance ($t = 0$), a reactive solute at tracer concentration is added to the fluid in the well and subsequently carried into the porous medium. Within the medium, the solute interacts with the porous matrix by means of equilibrium adsorption.

Following van Duijn & Knabner [33] or the introduction of this thesis, where a detailed derivation was presented, we find for the scaled solute concentration $u \colon \Omega_\epsilon \times [0, \infty) \mapsto [0, \infty)$ the following nonlinear initial-boundary value problem:

$$(\mathrm{P}_\epsilon) \quad \begin{cases} \beta(u)_t + \mathrm{div}\bar{F} = 0 & \text{in} \quad \Omega_\epsilon, \quad t > 0 & (5.2) \\ \bar{F} \cdot \mathbf{e}_r = u_e \boldsymbol{q} \cdot \mathbf{e}_r & \text{on} \quad \partial\Omega_\epsilon, \quad t > 0 & (5.3) \\ u(\cdot, 0) = u_0(\cdot) & \text{in} \quad \Omega_\epsilon. & (5.4) \end{cases}$$

This chapter has appeared in Journal of Mathematical Analysis and Applications [31]

Here $\bar{F} = \boldsymbol{q}\, u - \nabla u$ denotes the solute flux, $\boldsymbol{q} = \frac{\Lambda}{|x|^2}\mathbf{e}_r$ the induced flow field, and $\Lambda > 0$ the Peclet number of the problem, which combines the effects of flow rate and dispersion. In (5.3), u_e denotes the solute concentration in the injection well and $\mathbf{e}_r$ is the unit vector in radial direction. The adsorption mechanism is accounted for by the nonlinear term $\beta = \beta(u)$. Generally it takes the form

$$\beta(u) = u + \psi(u), \tag{5.4}$$

where ψ is called the adsorption isotherm (see for instance VAN DUIJN & KNABNER [34]). Typical examples are

$$\psi(u) = \frac{k_1 u}{1 + k_2 u}, \quad k_1, k_2 > 0, \qquad \text{(Langmuir isotherm)}$$

or

$$\psi(u) = k u^p, \quad k > 0, \quad p \in (0,1) \qquad \text{(Freundlich isotherm).}$$

In a two-dimensional setting, Problem P_ϵ was previously considered by van Duijn & Knabner [33] and van Duijn & Peletier [35]. In [33] the authors derived a radially symmetric self-similar solution of equation (5.2) of the form $u(r,t) = f(r/\sqrt{t})$. This solution is defined on all $\mathbb{R}^2$ but does not satisfy boundary condition (5.3). In [35] it was demonstrated that this solution describes the large-time behaviour for general two-dimensional radially symmetric solutions of (5.2)–(5.3) and rates of convergence were given.

The existence of self-similar solutions in two dimensions requires the well injection rate to be constant in time. In three spatial dimensions self-similar solutions still exist but require the injection rate and therefore Λ to grow as $\sqrt{t}$. From a practical point of view this is an unsatisfactory setup and the main goal of this paper is to investigate the large-time behaviour of solutions under a constant injection rate. We do this in the framework of a contamination event (see also [35]), i.e. assuming that far away from the well no solute (contaminant) is present.

Two natural questions arise form Problem P_ϵ: the behaviour as $\epsilon \downarrow 0$ and as $t \to \infty$. Since in [35] the authors were only concerned with radially symmetric solutions, their proofs of the limiting behaviour as $\epsilon \downarrow 0$ and as $t \to \infty$ follow essentially along the same lines. This is due to the scale invariance of the equation and the boundary condition. In this paper the proofs are quite different and are treated separately.

We first consider the behaviour as $\epsilon \downarrow 0$. Taking the formal limit in the combination (5.2)–(5.3) yields the equation

$$\beta(u)_t + \mathrm{div}(F) = u_e \delta_{x=0} \quad \text{in} \quad \mathbb{R}^3, \quad t > 0 \tag{5.5}$$

where $\delta_{x=0}$ denotes the Dirac distribution at the origin. Thus the boundary condition at the well appears as a source term in the equation. We refer to (5.5), together with the initial condition

$$u(\cdot, 0) = u_0(\cdot) \quad \text{in} \quad \mathbb{R}^3 \tag{5.6}$$

as Problem P_0 or (P_0).

Regarding the initial conditions (5.4) and (5.6), we take (5.4) as the restriction of (5.6) to Ω_ϵ, and assume

$$(H_{u_0}) \quad u_0 \in L^\infty(\mathbb{R}^3);\ u_0 \geq 0 \text{ in } \mathbb{R}^3;\ \lim_{|x|\to\infty} u_0(x) = 0;\ \int_{\mathbb{R}^3} \beta(u_0)\, dx < \infty.$$

Note that we allow non-radial initial data.

With respect to the nonlinear capacity term $\beta = \beta(u)$ we assume the regularity

$$(H_\beta 1) \qquad \beta \in C^\infty(0,\infty) \cap C([0,\infty)),$$

and the structural properties

$$(H_\beta 2) \qquad \beta(0) = 0,\ \beta'(s) > 0,\ \text{and } \beta''(s) \leq 0 \text{ for } s > 0.$$

Later, when we consider the large-time behaviour, we will add some additional hypotheses, essentially expressing that $\beta(u)$ behaves as u^p $(0 < p \leq 1)$ near $u = 0^+$.

Since equation (5.2) is scale invariant, we may set $\Lambda = 1$ after redefining $\epsilon := \epsilon/\Lambda$. By redefining $\beta(u) := \beta(u_e u)/u_e$ we may also set $u_e = 1$.

Our first theorem makes the stabilization as $\epsilon \downarrow 0$ precise.

Theorem 5.1 *Let* (H_{u_0}) *and* $(H_\beta 1\text{–}2)$ *be satisfied. Further, let* u^ϵ *be the unique weak solution of* (P_ϵ)*. Then*

$$u^\epsilon \to u \quad \text{as} \quad \epsilon \to 0, \quad \text{uniformly in compact subsets of } (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^+,$$

where u *is a weak solution of Problem* P_0*.*

The definition of weak solutions as well as the proof of Theorem 5.1 are given in Section 5.2.

Next we consider the large-time behaviour. We expect that different small well radii (ϵ) lead to the same large-time behaviour. This was shown rigorously [35] for the two-dimensional case. With this in mind we consider only the large-time behaviour for Problem P_0 and for technical reasons we limit ourselves to radially symmetric solutions. Before we state the convergence result, we provide some motivation.

The radial form of equation (5.5) is:

$$\beta(u)_t + \frac{1-2r}{r^2} u_r - u_{rr} = 0 \quad \text{in} \quad 0 < r < \infty, \quad t > 0, \tag{5.7}$$

and, as shown in Proposition 5.15, its solutions satisfy the boundary condition

$$u(0,t) = 1 \quad \text{for all} \quad t > 0. \tag{5.8}$$

The initial condition takes the form

$$u(r,0) = u_0(r) \quad \text{for} \quad 0 < r < \infty. \tag{5.9}$$

Equation (5.7) admits a nontrivial stationary solution $w = w(r)$, satisfying $w(0) = 1$ and $w(\infty) = 0$. It is given by

$$w(r) = 1 - e^{-1/r}, \tag{5.10}$$

and under the conditions of Theorem 5.3 below the solution u converges to this stationary state.

The appearance of (5.10) is quite different from the two-dimensional case. There the only bounded stationary solution satisfying $w(0) = 1$ is the constant state $w \equiv 1$. In [35] it was shown that the solution attains this state in a self similar way, namely

$$u(r,t) \sim f(r/\sqrt{t}) \quad \text{as} \quad t \to \infty$$

where $f(0) = 1$.

In this paper we assume an analogous behaviour with respect to (5.10), i.e.

$$\frac{u(r,t)}{w(r)} \sim f(r/t^\alpha) \quad \text{as} \quad t \to \infty \tag{5.11}$$

for some $\alpha > 0$, where $f(0) = 1$. To this end we set

$$\tilde{z}(r,t) := \frac{u(r.t)}{w(r)}$$

and introduce the coordinate transformation

$$\eta = r/t^\alpha, \qquad \tau = \log t.$$

Then $z(\eta,\tau) = \tilde{z}(r,t)$ satisfies:

$$e^{(2\alpha-1)\tau}[\beta(zw)_\tau - \alpha\eta\beta(zw)_\eta] + \frac{e^{-\alpha\tau} - 2\eta}{\eta^2}(zw)_\eta - (zw)_{\eta\eta} = 0. \tag{5.12}$$

To obtain the convergence (5.11), we study the large-τ behaviour of (5.12). In particular we need to select the exponent α so that the appropriated terms in (5.12) balance as $\tau \to \infty$. For this purpose we rewrite the equation as

$$e^{(2\alpha-1)\tau}\beta'(zw)z_\tau - \alpha e^{(2\alpha-1)\tau}\eta\beta'(zw)z_\eta - z_{\eta\eta} + \frac{1}{\eta}A\left(\frac{1}{\eta e^{\alpha\tau}}\right)z_\eta = 0, \tag{5.13}$$

where $A(s) := \dfrac{2s}{e^s - 1} + s - 2$ with $\lim\limits_{s\to 0} A(s) = 0$.

To find the appropriate balance, we observe that for fixed $\eta > 0$, $\tau \to \infty$ implies $r \to \infty$. Since $u(r,t) \to 0$ as $r \to \infty$, the behaviour of β near 0 is critical. Let us assume

$$\beta(s) \sim s^p \quad (0 < p \leq 1) \quad \text{as } s \downarrow 0. \tag{5.14}$$

Using this and $w(r) \to 1/r$, as $r \to \infty$, we find that the second and third term in (5.13) balance if and only if $\alpha = 1/(3-p)$.

The resulting equation is

$$\alpha\eta^{2-p}(f^p)_\eta + f_{\eta\eta} = 0 \quad \text{or} \quad \alpha\eta^{1-p}(f^p)_\eta + (\eta f_\eta - f)_\eta = 0 \text{ for } 0 < \eta < \infty, \tag{5.15}$$

where $f(\eta) := \lim_{\tau\to\infty} z(\eta,\tau)$. Note the resemblance between (5.15) and the limiting equation obtained in [33].

Before we state the main convergence theorem, we specify some additional hypotheses on β. Related to (5.14) we assume that there exists $0 < p \le 1$ such that

$$(H_\beta 3) \qquad \frac{\beta'(s)}{ps^{p-1}} = \ell + O(s^\gamma) \qquad \text{as} \quad s \downarrow 0,$$

for some $\ell > 0$ and $\gamma \in (0, 3-p)$. Furthermore we assume the lower bound

$$(H_\beta 4) \qquad \inf_{s\in[0,1]} \frac{\beta'(s)}{ps^{p-1}} = m > 0.$$

Let $\beta_p(s) := \ell s^p$ and $\varphi(s) := \dfrac{\beta'(s) - \beta_p'(s)}{ps^{p-1+\gamma}}$.

Remark 5.2 *The simplest function β that satisfies (H_β5–4) is*

$$\beta(s) = ks^p \quad p \in (0,1],$$

with $\ell = m = k$, $\varphi \equiv 0$, and for any $\gamma \in (0, 3-p)$. Hypotheses (H_β3–4) are also fulfilled by the examples given at the beginning of the introduction. In the case of the Freundlich isotherm,

$$\beta(s) = s + ks^p \quad p \in (0,1),$$

we have $\ell = m = k$, and $\gamma = 1-p$. Note that this choice implies $\varphi(s) = 1/p > 0$. In the Langmuir isotherm case,

$$\beta(s) = s + \frac{k_1 s}{k_2 s + 1} \qquad k_1, k_2 > 0,$$

we have $p = 1$, $\ell = k_1 + 1$, $m = 1 + \frac{k_1}{(k_2+1)^2}$, $\gamma = 1$, and $\varphi(s) = -k_1 k_2 \frac{k_2 s + 2}{(k_2 s+1)^2} \le 0$.

Below we use the notation $[\cdot]_+ := \max\{\cdot, 0\}$, $\varphi_+ := [\varphi]_+$, and $\varphi_- := [-\varphi]_+$.

Theorem 5.3 *Let hypotheses (H_β1–4) and (H_{u_0}) be satisfied, and let u be a weak solution of Problem P_0. Then we have the following estimates:*

$$0 \le e^{p\alpha\tau} \int_0^\infty [u^p - f^p w^p]_+ \eta^2 \, d\eta \le L_1 e^{-\alpha\tau} + L\|\varphi_-\|_{L^\infty} e^{-\alpha\gamma\tau} \tag{5.16}$$

for all $\tau \in \mathbb{R}$, *and*

$$0 \le e^{p\alpha\tau} \int_0^\infty [f^p w^p - u^p]_+ \eta^2 \, d\eta \le L_2 e^{-\alpha\tau} + L\|\varphi_+\|_{L^\infty} e^{-\alpha\gamma\tau} \tag{5.17}$$

for all $\tau \in \mathbb{R}$. *Here* L_1, L_2, *and* L *are positive constants and* $\alpha = 1/(3-p)$.

The function f *is the unique solution of*

$$\text{(S)} \quad \begin{cases} \alpha\eta^{2-p}\beta_p(f)_\eta + f_{\eta\eta} = 0 & \text{for } 0 < \eta < \infty, \\ f(0) = 1, \quad f(\infty) = 0. \end{cases}$$

Figure 5.1 shows the limit function $r \mapsto w(r)f(r/\sqrt{t})$ for different t, in the case $p = 1$.

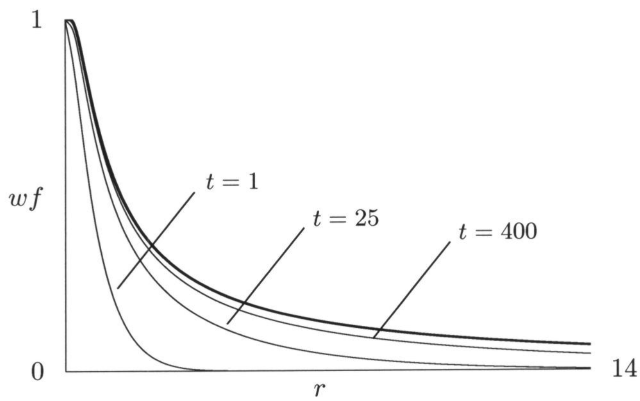

Figure 5.1: The function $r \mapsto w(r)f(r/\sqrt{t})$; $t = 1$, 25, 400.

Remark 5.4 *Note that the constants in the estimates of Theorem 5.3 depend on* p. *For instance, it follows from the proof that if* $p = 1$ *then* $L_2 = 0$. *An immediate consequence of this fact concerns functions* β *of the form*

$$\beta(s) = s + \frac{k_1 s}{k_2 s + 1} \qquad k_1, k_2 > 0.$$

Here $p = 1$ *and* $\varphi \le 0$ *(Remark 5.2), so that* $fw \le u$.

Remark 5.5 *The mass of the system increases linearly in time. The scaling used in (5.16) (and (5.17)) is chosen to normalize the increase of mass:*

$$\frac{1}{t} \int_0^\infty [u^p - f^p w^p]_+ r^2 \, dr = e^{p\alpha\tau} \int_0^\infty [u^p - f^p w^p]_+ \eta^2 \, d\eta.$$

In this scaled metric the solutions u *and* fw *converge. In the unscaled (original) metric the distance increases without bound.*

We conclude with a statement about the applicability of the results. Problem P_ϵ clearly describes an idealized flow model. Homogeneous porous media with spherical symmetry do not occur in any practical setting. Problem P_ϵ and related questions are studied mainly out of mathematical curiosity, in particular to understand the difference between the two and three dimensional case. Having said this, we see two reasons why our results could be of interest to applied researchers.

(i) Idealized models such as Problem P_ϵ can be used as benchmarks for complex contaminant transport codes.

(ii) In practical situations there is often a need to work with local solutions near singular points (such ar wells). Our stability result and error analysis can be used for that purpose. Replacing the contaminant concentration by the self-similar solution in a neighborhood of a well will considerably reduce the computational effort, since less grid refinement is required.

5.2 Convergence as $\epsilon \to 0$

5.2.1 Weak solutions of (P_ϵ)

Let T be a fixed positive number which eventually tends to infinity and let $E_T^\epsilon = \Omega_\epsilon \times (0, T]$. Note that we have rescaled the problem such that $u_e = \Lambda = 1$.

Definition 5.6 *A weak solution of Problem P_ϵ is a non-negative function u such that*

(i) $u \in C(E_T^\epsilon)$ *and* $\nabla u \in L^2(E_T^\epsilon)$,

(ii) *For every test function* $\phi \in L^2(0,T;H^1(\Omega_\epsilon)) \cap H^1(0,T,L^2(\Omega_\epsilon))$ *that vanishes for large* $|x|$ *and at* $t = T$,

$$\int_{E_T^\epsilon} \{\beta(u)\phi_t + (\boldsymbol{q}\, u - \nabla u)\nabla\phi\}\, dxdt + \int_{\Omega_\epsilon} \beta(u_0)\phi(0)\, dx + \\ + \frac{1}{\epsilon^2}\int_0^T \int_{\partial\Omega_\epsilon} \phi\, dSdt = 0. \tag{5.18}$$

If u satisfies (i) and (ii) with the equality replaced by $\geq$ ($\leq$) and with $\phi \geq 0$ in E_T^ϵ then we call u a sub(super)solution. Here and in the sequel, we use the obvious notation $\phi(0) = \phi(t = 0)$.

Theorem 5.7 (Existence for (P_ϵ)) *Let (H_{u_0}) and $(H_\beta 1$–$2)$ be satisfied. Then there exists a unique weak solution of (P_ϵ).*

The proof of existence will be given in Section 5.2.3, the uniqueness follows from Proposition 5.10 below.

Proposition 5.8 *Let u be the weak solution of Problem P_ϵ. For each $t > 0$,*

$$\int_{\Omega_\epsilon} \beta(u(t))\, dx = \int_{\Omega_\epsilon} \beta(u_0)\, dx + 4\pi t.$$

The proof of Proposition 5.8 follows along the same lines as in [47].

5.2.2 Uniqueness of (P_ϵ)

Throughout this section we denote $\Omega_\epsilon^m = \{x \in \mathbb{R}^3 \mid \epsilon < |x| < m\}$ and similarly $E_T^{\epsilon,m} = \Omega_\epsilon^m \times (0,T]$.

In order to prove the comparison result for Problem P_ϵ, we introduce as in [30] an equivalent definition of solution, which we call *generalized solution:*

Definition 5.9 *A generalized solution of Problem P_ϵ is a function u satisfying:*

(i) *u is bounded, nonnegative, and continuous on E_T^ϵ;*

(ii) *for any $t \in (0,T]$ and any bounded domain $\Omega'_\epsilon \subset \Omega_\epsilon$ with smooth boundary $\partial\Omega'_\epsilon := \Gamma_\epsilon \cup \Gamma$, such that $\Gamma_\epsilon \subset \partial B_\epsilon$ and $\Gamma \cap \partial B_\epsilon = \emptyset$,*

$$\int_{\Omega'_\epsilon} \beta(u(t))\phi(t)\, dx - \int_{\Omega'_\epsilon} \int_0^t \{\beta(u)\partial_t\phi + u\boldsymbol{q}\nabla\phi + u\Delta\phi\}\, dxdt +$$

$$+\frac{1}{\epsilon^2}\int_0^t \int_{\Gamma_\epsilon} \phi\, dSdt + \int_0^t \int_{\Gamma} u\partial_\nu\phi\, dSdt = \int_{\Omega'_\epsilon} \beta(u_0)\phi(0)\, dx \qquad (5.19)$$

for all $\phi \in C^{2,1}(\Omega'_\epsilon \times (0,t])$, $\phi \geq 0$ with $\partial\phi/\partial\nu = 0$ on $\Gamma_\epsilon \times (0,t)$ and $\phi = 0$ on $\Gamma \times (0,t]$.

We define a subsolution (supersolution) by (i) and (ii) with the equality replaced by $\leq$ ($\geq$).

For the proof of equivalence between generalized and weak solutions we refer to [30].

Proposition 5.10 *Let u^1 and u^2 be generalized sub- and supersolutions with initial data u_0^1 and u_0^2 respectively. Then for any $t \in [0,T]$, we have*

$$\int_{\Omega_\epsilon} [\beta(u^1(t)) - \beta(u^2(t))]_+\, dx \leq \int_{\Omega_\epsilon} [\beta(u_0^1) - \beta(u_0^2)]_+\, dx.$$

Proof. Let $\bar{u} = u^1 - u^2$ and $\bar{\beta} = \beta(u^1) - \beta(u^2)$. Subtracting equations (5.19) we find

$$\int_{\Omega'_\epsilon} \bar{\beta}(t)\phi(t)\,dx - \int_{\Omega'_\epsilon} \bar{\beta}(0)\phi(0)\,dx \le \int_0^t \int_{\Omega'_\epsilon} \{\bar{\beta}(t)\partial_t\phi + \bar{u}(\boldsymbol{q}\nabla\phi + \Delta\phi)\}\,dxdt$$
$$- \int_0^t \int_\Gamma \bar{u}\partial_\nu\phi\,dSdt. \qquad (5.20)$$

Following [5] we define a family of weight functions $\omega_\lambda\colon \mathbb{R}^3 \to \mathbb{R}^+$, for each $\lambda > 0$, by

$$\omega_\lambda(x) = \begin{cases} 1 & \text{if } |x| \in (\epsilon, 1), \\ e^{-\sqrt{\lambda}(|x|-1)} & \text{if } |x| \in (1, \infty). \end{cases}$$

Hypothesis $(H_\beta 2)$ implies that there exists $b_0 > 0$ such that $\beta'(s) \ge b_0$ for all $s \in \mathbb{R}$. We define $A\colon \Omega'_\epsilon \times \mathbb{R} \mapsto \mathbb{R}$ by:

$$A(x,t) = \begin{cases} \frac{\beta(u^1)-\beta(u^2)}{u^1-u^2} & \text{if } u^1 \ne u^2, \\ b_0 & \text{if } u^1 = u^2. \end{cases}$$

We choose $\xi \in C_c^\infty(\Omega_\epsilon)$ such that $0 \le \xi \le 1$, with $\partial\xi/\partial\nu = 0$ in Γ_ϵ. In addition let $\Omega'_\epsilon = \Omega_\epsilon^m$ where $m > 0$ is such that $\operatorname{supp}\xi \subset B_m$. We introduce smooth functions $A_m\colon \Omega_\epsilon^m \times (0,T) \mapsto \mathbb{R}$, satisfying

$$0 < b_0 \le A_m \le \|A\|_{L^\infty(E_T^\epsilon)} + \frac{1}{m}, \qquad \|\frac{A_m - A}{\sqrt{A_m}}\|_{L^2(E_T^{\epsilon,m})} \to 0. \qquad (5.21)$$

Consider for each C_m the problem

$$(\mathrm{PA_m})\begin{cases} A_m\partial_\tau\phi + \boldsymbol{q}\nabla\phi + \Delta\phi &= \lambda\phi \quad \text{in} \quad \Omega_\epsilon^m \times [0,t] \\ \partial_\nu\phi &= 0 \quad \text{on} \quad \partial B_\epsilon \times [0,t] \\ \phi &= 0 \quad \text{on} \quad \partial B_m \times [0,t] \\ \phi(x,t) &= \xi(x)\omega_\lambda(x) \quad \text{in} \quad \Omega_\epsilon^m. \end{cases}$$

This equation has a unique solution $\phi_m \in C^{2,1}(\overline{\Omega_\epsilon^m} \times [0,t])$, $\phi_m \ge 0$. Using ϕ_m as a test function, we find

$$\int_{\Omega_\epsilon^m} \bar{\beta}(t)\xi(x)\omega_\lambda(x)\,dx - \int_{\Omega_\epsilon^m} \bar{\beta}(0)\phi_m(x,0)\,dx \le \int_{E_t^\epsilon} \bar{u}(A - A_m)\partial_t\phi_m\,dxdt$$
$$+\lambda \int_{E_t^\epsilon} \bar{u}\phi_m\,dxdt - \int_0^t \int_{\partial\Gamma_m} \bar{u}\partial_\nu\phi_m\,dSdt. \qquad (5.22)$$

Lemma 5.11 *The functions ϕ_m satisfy the following properties:*

(i) $0 \le \phi_m \le \omega_\lambda$ *in* E_t^ϵ

(ii) $\displaystyle\int_{E_t^{\epsilon,m}} A_m|\partial_\tau \phi_m|^2 dxdt \le C;$

(iii) $\displaystyle\sup_{0\le\tau\le t} \int_{\Omega_\epsilon^m} |\nabla\phi_m(\tau)|^2 dx \le C;$

(iv) $0 \le -\phi_{m\nu} \le Ce^{-\sqrt{\lambda}m} \quad on\, \partial B_m \times [0,t].$

Proof. Part (i) is a consequence of the maximum principle. Parts (ii–iii) are standard estimates. To prove (iv), we follow the ideas of [5]. We fix $m_0 < m$ such that $\operatorname{supp}\xi \subset B_{m_0}$ and define $\tilde{\omega}_\lambda\colon B_m \to [0,1]$ separately on the two subsets B_{m_0} and $\Omega_{m_0}^m$. In B_{m_0} we set $\tilde{\omega}_\lambda = \omega_\lambda$, and in $\Omega_{m_0}^m$ we define $\tilde{\omega}_\lambda$ as the solution of

$$\begin{array}{rcll} \boldsymbol{q}\nabla\tilde{\omega}_\lambda + \Delta\tilde{\omega}_\lambda - \lambda\tilde{\omega}_\lambda &=& 0 & \text{in} \quad \Omega_{m_0}^m \\ \tilde{\omega}_\lambda &=& \omega_\lambda & \text{on} \quad \partial B_{m_0} \\ \tilde{\omega}_\lambda &=& 0 & \text{on} \quad \partial B_m. \end{array} \tag{5.23}$$

By (i) we have $0 \le \phi_m \le \tilde{\omega}_\lambda$ on $B_{m_0} \times (0,t]$; by an application of the comparison principle on $\Omega_{m_0}^m \times (0,t]$ it follows that $0 \le \phi_m \le \tilde{\omega}_\lambda$ on $\Omega_\epsilon^m \times (0,t]$. Therefore $0 \le -\phi_{m\nu} \le -\tilde{\omega}_{\lambda\nu}$ on ∂B_m. To estimate $\tilde{\omega}_{\lambda\nu}$ we introduce another auxiliary function $\bar{\omega}_\lambda$, defined by $\bar{\omega}_\lambda = \omega_\lambda$ in B_{m_0} and the solution of

$$\begin{array}{rcll} \Delta\bar{\omega}_\lambda - \lambda\bar{\omega}_\lambda &=& 0 & \text{in} \quad \Omega_{m_0}^m \\ \bar{\omega}_\lambda &=& \omega_\lambda & \text{on} \quad \partial B_{m_0} \\ \bar{\omega}_\lambda &=& 0 & \text{on} \quad \partial B_m, \end{array} \tag{5.24}$$

in $\Omega_{m_0}^m$. By a standard argument we have $\nabla\tilde{\omega}_\lambda \cdot \mathbf{e}_r < 0$ in $\Omega_{m_0}^m$. The function $\tilde{\omega}_\lambda$ is therefore subsolution for (5.24). Then

$$0 \le -\phi_{m\nu} \le -\tilde{\omega}_{\lambda\nu} \le -\bar{\omega}_{\lambda\nu} \quad \text{on} \quad \partial B_m$$

which proves (iv), because $\bar{\omega}_{\lambda\nu} \le c(\lambda, m_0)e^{-\sqrt{\lambda}m}$ on ∂B_m. ■

We continue the proof of Theorem 5.10. Using (5.21) and Lemma 5.11 the inequality (5.22) yields

$$\int_{\Omega_\epsilon^m} (\beta(u^1(t)) - \beta(u^2(t)))\xi\omega_\lambda\, dx \le \int_{\Omega_\epsilon^m} [\beta(u_0^1) - \beta(u_0^2)]_+ \omega_\lambda\, dx$$
$$+ \int_{E_t^{\epsilon,m}} \bar{u}(A - A_m)\partial_\tau\phi_m\, dxdt + \int_{E_t^{\epsilon,m}} \lambda(u^1 - u^2)\omega_\lambda\, dxdt + Cm^2 e^{-\sqrt{\lambda}m}.$$

With the estimate

$$\|\bar{u}(A - A_m)\partial_\tau\phi_m\|_{L^1(E_t^{\epsilon,m})} \le C\|\frac{A - A_m}{\sqrt{A_m}}\|_{L^2(E_t^{\epsilon,m})}\|\sqrt{A_m}\partial_t\phi_m\|_{L^2(E_t^{\epsilon,m})},$$

we find in the limit $m \to \infty$,

$$\int_{\Omega_\epsilon} (\beta(u^1(t)) - \beta(u^2(t)))\xi\omega_\lambda \, dx \leq \int_{\Omega_\epsilon} [\beta(u_0^1) - \beta(u_0^2)]_+ \omega_\lambda \, dx + \int_{E_t^\epsilon} \lambda(u^1 - u^2)\omega_\lambda \, dxdt. \tag{5.25}$$

In (5.25), we take a sequence $\{\xi_n\}$ that converges pointwise to $sgn(\bar{\beta}_+)$. We then let $\lambda \to 0$ to obtain the result; the convergence of the term $\int_{E_t^{\epsilon,m}} \lambda(u^1 - u^2)\omega_\lambda \, dxdt$ follows from the L^1-bound (Proposition 5.8) and $(H_\beta 2)$. ■

5.2.3 Existence for (P_ϵ)

Now we use solutions of a regularized problem to prove the existence of solutions for (P_ϵ). Let $\delta_n := 1/n$ and introduce the approximations $\{u_{0n}\}$ and $\{u_{ne}\}$,

$$\begin{array}{l} u_{0n} \in C^\infty(\mathbb{R}^3), \quad \text{with} \quad \|u_{0n}\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \delta_n; \\ u_{0n} \downarrow u_0 \quad \text{uniformly on compact subsets of } \Omega_\epsilon; \\ u_{0n}(x) = \delta_n \quad \text{for} \quad n - 1 \leq |x| \leq n; \\ \nabla u_{0n}(x) \cdot \mathbf{e}_r = 0 \quad \text{at} \quad |x| = \epsilon \end{array}$$

and

$$u_{ne}(x,t) := 1 - (1 - u_{0n}(x))e^{-nt} \quad \text{for} \quad |x| = \epsilon \quad \text{and} \quad 0 \leq t \leq T.$$

Then consider the regularized version of (P_ϵ),

$$(P_{n\epsilon}) \quad \left\{ \begin{array}{rcl} \beta(u)_t + \operatorname{div}(\bar{F}) &=& 0 \quad \text{in} \quad E_T^{\epsilon,n}, \\ \bar{F} \cdot \mathbf{e}_r &=& u_{ne}\boldsymbol{q} \cdot \mathbf{e}_r \quad \text{at} \quad |x| = \epsilon, \quad t > 0, \\ u &=& \delta_n \quad \text{at} \quad |x| = n, \quad t > 0, \\ u(x,0) &=& u_{0n}(x) \quad \text{in} \quad \Omega_\epsilon^n. \end{array} \right.$$

Let $u_n^\epsilon \in C^\infty(E_T^{\epsilon,n}) \cap C^{2+\alpha,1+\alpha/2}(\overline{E_T^{\epsilon,n}})$, be the unique solution of $(P_{n\epsilon})$ (see [58], Theorem 7.4), which satisfies

$$\delta_n \leq u_n^\epsilon(x,t) \leq \max\{\|u_0\|_{L^\infty}, 1\} + \delta_n,$$

and

$$\int_{E_T^{\epsilon,n}} |\nabla u_n^\epsilon|^2 \, dxdt \leq M, \tag{5.26}$$

where M is independent of n and ϵ (see [74], Theorem 4).

With the above estimates, we are ready to prove the existence for (P_ϵ).

Proof of Theorem 5.7. For this proof we fix $\epsilon > 0$. Using Bernšteĭn estimates as in [65], we find

$$\|\nabla u_n^\epsilon(x,t)\|_{L^\infty(\Omega^m_{\epsilon+1/m}\times[\frac{1}{m},T])} \leq C(m) \quad \text{for all} \quad n \geq m. \tag{5.27}$$

Using GILDING [46], we find that, for $n \geq m$,

$$|u_n^\epsilon(x,t_2) - u_n^\epsilon(x,t_1)| \leq C(m)|t_2 - t_1|^{\frac{1}{2}} \tag{5.28}$$

for all $1/m \leq t_1 \leq t_2 \leq T$ and $x \in \Omega^m_{\epsilon+1/m}$. By a standard argument we combine estimates (5.26), (5.27), and (5.28), to conclude the existence of a solution of (P_ϵ). ∎

5.2.4 Weak solutions of Problem P_0 and proof of Theorem 5.1

We now turn to Problem P_0. Let $E_T = \mathbb{R}^3 \times (0,T)$.

Definition 5.12 *A weak solution of Problem P is a non-negative function u such that*

(i) $u \in C(E_T)$ *and* $\nabla u \in L^2(E_T)$.

(ii) *For every test function* $\phi \in H^1(E_T)$ *with* $\int_{\mathbb{R}^3} |\boldsymbol{q}||\nabla\phi|^2\,dx < \infty$, *that vanishes for large* $|x|$ *and at* $t = T$,

$$\int_{E_T} [\beta(u)\phi_t + \{\boldsymbol{q}\,u - \nabla u\}\nabla\phi]\,dxdt + \int_{\mathbb{R}^3} \beta(u_0)\phi(0)\,dx + 4\pi\int_0^T \phi(0,t)\,dt = 0. \tag{5.29}$$

If u satisfies (5.18) with the equality replaced by $\geq$ ($\leq$) *and with* $\phi \geq 0$ *in* E_T *then we call u sub(super)solution.*

Remark 5.13 *Since* $|\boldsymbol{q}| \in L^1_{loc}(\mathbb{R}^3)$, *the integrals in (5.29) are well-defined,*

$$\left|\int_{\mathbb{R}^3} \boldsymbol{q}u\nabla\phi\,dx\right|^2 \leq \left(\int_{\operatorname{supp}\phi} |\boldsymbol{q}|u^2\,dx\right)\left(\int_{\operatorname{supp}\phi} |\boldsymbol{q}||\nabla\phi|^2\,dx\right) < \infty.$$

The existence of a weak solution of (P_0) is a consequence of Theorem 5.1. Uniqueness holds in the class of solutions of (P_0) that are obtained as limits of solutions of (P_ϵ), since the comparison principle (Proposition 5.10) carries over to the limit. However, due to the singularity of $\boldsymbol{q}$ at the origin, uniqueness in the class of all solutions of (P_0) remains an open question.

We have the following properties of the weak solution of (P_0).

Proposition 5.14 *Let u be a weak solution of Problem* P_0*. Then*

$$\int_{\mathbb{R}^3} \beta(u(t))\,dx = \int_{\mathbb{R}^3} \beta(u_0)\,dx + 4\pi t \quad \textit{for all} \quad t \geq 0.$$

The proof of this proposition is similar to the proof of Proposition 5.8.

The singularity of $\boldsymbol{q}$ at the origin creates a "pseudo-boundary condition":

Proposition 5.15 *For any weak solution u of Problem* P_0 *we have*

$$u(0,t) = 1 \quad \textit{for} \quad 0 < t \leq T.$$

Proof. Consider a fixed function $\rho \in C_c^\infty(0,T)$, and the functions $\eta_n : \mathbb{R}^3 \mapsto \mathbb{R}$ given by

$$\eta_n(r) = \begin{cases} 1 - nr & \text{if } 0 \leq r \leq \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < r. \end{cases}$$

Let $\phi_n(x,t) := \rho(t)\eta_n(|x|)$. We estimate $\int_{E_T} \boldsymbol{q} u \nabla \phi_n \, dx$ by

$$\int_0^T \inf_{x \in B_{\frac{1}{n}}} \{u(x,t)\}\rho(t)\,dt \leq -\frac{1}{4\pi} \int_{E_T} \boldsymbol{q} u \nabla \phi_n \, dxdt \leq \int_0^T \sup_{x \in B_{\frac{1}{n}}} \{u(x,t)\}\rho(t)\,dt,$$

therefore in the limit, $n \to \infty$, we find

$$\lim_{n \to \infty} \int_{E_T} \boldsymbol{q} u \nabla \phi_n \, dxdt = -4\pi \int_0^T u(0,t)\rho(t)\,dt.$$

Using the boundedness of $\int_{E_T} |\nabla u|^2 \, dxdt$,

$$\left| \int_{E_T} \nabla u \nabla \phi_n \, dxdt \right| \leq \left(\int_{E_T} |\nabla u|^2 \, dxdt \right)^{\frac{1}{2}} \left(\int_{E_T} |\nabla \eta_n|^2 \, dxdt \right)^{\frac{1}{2}} \to 0$$

as $n \to \infty$. As u is bounded near the origin $0 \leq |\int_{E_T} \beta(u)(\phi_n)_t \, dxdt| \leq C \int_{\mathbb{R}^3} \eta_n \, dx \to 0$ as $n \to \infty$, and with a similar argument $\int_{\mathbb{R}^3} \beta(u_0)\phi_n(0)\,dx \to 0$ as $n \to \infty$.

Using the above estimates and taking the limit in (5.29) as $n \to \infty$ we have

$$\int_0^T (u(0,t) - 1)\rho(t)\,dt = 0 \quad \text{for all} \quad \rho \in C_c^\infty(0,T),$$

which proves the lemma. ■

Finally we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. Using estimates (5.27) and (5.28), we have

$$\|u^\epsilon\|_{C^{0+1,0+\frac{1}{2}}(\Omega^m_{1/m}\times[\frac{1}{m},T])} \leq C(m) \quad \text{for all} \quad \epsilon < 1/m.$$

Extending u^ϵ by zero on B_ϵ, we extract a subsequence of u^ϵ that converges *a.e.* in E_T to a limit u.

Fix $\phi \in C_c^\infty([0,T)\times\mathbb{R}^3)$. Since u^ϵ is uniformly bounded, and $|\boldsymbol{q}| \in L^1_{loc}(\mathbb{R}^3)$, the pointwise convergence of u^ϵ implies

$$\lim_{\epsilon\to 0}\int_{E_T} \boldsymbol{q}u^\epsilon\nabla\phi\, dxdt = \int_{E_T} \boldsymbol{q}u\nabla\phi\, dxdt.$$

Using the bound $\int_{E_T^\epsilon} |\nabla u^\epsilon|^2\, dxdt \leq M$, we have (after extracting a subsequence),

$$\int_{E_T^\epsilon} \nabla u^\epsilon\nabla\phi\, dxdt \to \int_{E_T} \nabla u\nabla\phi\, dxdt \quad \text{as} \quad \epsilon\to 0.$$

Therefore

$$\lim_{\epsilon\to 0}\int_{E_T^\epsilon} \beta(u^\epsilon)\phi_t + (\boldsymbol{q}u^\epsilon - \nabla u^\epsilon)\nabla\phi\, dxdt = \int_{E_T} \beta(u)\phi_t + (\boldsymbol{q}u - \nabla u)\nabla\phi\, dxdt.$$

Furthermore by the continuity of ϕ we have

$$\int_{\Omega_\epsilon} \beta(u_0)\phi(x,0)\, dx + \int_0^T\int_{\partial\Omega_\epsilon} \frac{\phi}{\epsilon^2}\, dSdt \to \int_{\mathbb{R}^3} \beta(u_0)\phi(x,0)\, dx + 4\pi\int_0^T \phi(0,t)dt.$$

as $\epsilon\to 0$. Combining these results we conclude that u satisfies equation (5.29) for all $\phi \in C_c^\infty([0,T)\times\mathbb{R}^3)$. To extend this equation to all ϕ as mentioned in the definition we note that the set $C_c^\infty([0,T)\times\mathbb{R}^3)$ is dense in the set of all such ϕ with respect to the norm

$$\|\phi\|^2_{L^2(E_T)} + \|\phi_t\|^2_{L^2(E_T)} + \|(\sqrt{|\boldsymbol{q}|}+1)\nabla\phi\|^2_{L^2(E_T)}.$$

■

5.3 Asymptotic behaviour for a solution of (P_0)

5.3.1 Preliminaries

To study the long-term behaviour we consider an extension to Problem P_0:

$$(P_0')\quad \begin{cases} \beta(u)_t + \operatorname{div}(\bar{F}) = \delta_{x=0} + G(x,t) & \text{in} \quad \mathbb{R}^3, \quad t>0 \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^3. \end{cases}$$

Here $G \in L^1(0,T,L^1(\mathbb{R}^3))$.

The notion of weak solutions of (P_0') follows along the same lines as above. For (P_0') we can state a comparison principle:

Proposition 5.16 *Let u^1 be a subsolution and u^2 a supersolution of (P_0') with data u_0^1, G_1 and u_0^2, G_2. Then for each $t \in [0,T]$,*

$$\int_{\mathbb{R}^3} [\beta(u^1(t)) - \beta(u^2(t))]_+ \, dx \le \int_{\mathbb{R}^3} [\beta(u_0^1) - \beta(u_0^2)]_+ \, dx + \int_{E_t} [G_1 - G_2]_+ \, dxdt.$$

The proof of Proposition 5.16 is a direct extension of that of Proposition 5.10.

Lemma 5.17 *Let $G \equiv 0$. Then $w(r) = 1 - e^{-\frac{1}{r}}$ is a stationary solution of (P_0') satisfying*

(i) $0 \le w(r)r \le w(r)^p r^p \le 1$ *for all* $r \ge 0$;

(ii) $\dfrac{1}{1+r} \le w(r) \le \min\left\{\dfrac{2}{1+2r}, 1\right\}$ *for all* $r > 0$.

Proof. We only demonstrate (ii). The function $z(s) = w(1/s)$ satisfies $z' = 1 - z$. The function $y(s) = \frac{s}{s+1}$ satisfies $y' < 1 - y'$ this implies the first inequality. The second follows along the same lines. ■

To prepare the proof of Theorem 5.3 we derive some relevant properties of the solutions of (S).

Proposition 5.18 *Let f be a solution of* (S) *and consider the set $P_f = \{\eta > 0 \mid f(\eta) > 0\}$. Then*

(i) $f \in C^\infty(P_f)$;

(ii) $f' < 0$, $f'' > 0$ *on* P_f;

(iii) $f' \to 0$ *as* $\eta \to \infty$;

(iv) $\lim_{\eta \to 0^+} f'(\eta) = -K$ *with* $K \in (0,\infty)$;

(v) $\int_0^\infty \beta_p(f)\eta^{2-p} d\eta = 1$;

(vi) *If* $p = 1$, *then* $0 \le f(\eta) \le Ca^{-1}e^{-\ell\eta^2/4}$ *for* $\eta > a$; *if* $p < 1$, *then* $\sup P_f < \infty$.

Proof. Parts (i–iv) follow from Proposition 2.3 in [33]. Part (v) is a simple integration of the equation in (S). For part (vi), case $p < 1$ we refer to [33]. For the case $p = 1$, (S) has the explicit solution $f(\eta) = \mathrm{erfc}(\frac{\sqrt{\ell}}{2}\eta)$. This implies

$$f(\eta) \le -f'(0)2a^{-1}e^{-\ell\eta^2/4} \quad \text{for} \quad \eta > a.$$

■

5.3.2 Proof of Theorem 5.3

We consider Problem P_0 in the radially symmetric form. Let $S_T = \{(r,t) : 0 < r < \infty,\ 0 < t < T\}$.

Proof of Theorem 5.3. The proof is based on Proposition 5.16, applied to u and fw. We claim that the following estimates hold:

$$0 \le e^{p\alpha\tau} \int_0^\infty [\beta(u) - \beta(fw)]_+ \eta^2 d\eta \le L_1 e^{-\alpha\tau} + L_3 e^{-\tau} + L\|\varphi_-\|_{L^\infty} e^{-\alpha\gamma\tau} \tag{5.30}$$

for all $\tau \in \mathbb{R}$, and

$$0 \le e^{p\alpha\tau} \int_0^\infty [\beta(fw) - \beta(u)]_+ \eta^2 d\eta \le L_2 e^{-\alpha\tau} + L\|\varphi_+\|_{L^\infty} e^{-\alpha\gamma\tau} \tag{5.31}$$

for all $\tau \in \mathbb{R}$. By $(H_\beta 4)$ the function $\psi(s) := \beta(s) - ms^p$ is non decreasing. Therefore, if $a > b$ we have

$$\beta(a) - \beta(b) = \psi(a) - \psi(b) + m(a^p - b^p).$$

By this observation estimates (5.30–5.31) imply (5.16–5.17).

Let $h(r,t) := f(r/t^\alpha)$ for all $(r,t) \in (0,\infty) \times (0,\infty)$. Then h satisfies

$$r^{1-p}\beta_p(h)_t - h_{rr} = 0 \quad \text{in} \quad S_T \tag{5.32}$$

Using (5.32) and $\beta'(s) = \beta_p'(s) + \varphi(s)s^\gamma$, the function $g(r,t) := h(r,t)w(r)$ satisfies (P′) with

$$G(r,t) = \varphi(g)pg^{p-1+\gamma}g_t - \beta_p(h)_t w^p(r^{1-p}w^{1-p} - 1) + \frac{((1-rw)r)_r}{r^2}h_r.$$

Writing $G(r,t) := G_{1+}(r,t) + G_{1-}(r,t) + G_2(r,t) + G_3(r,t)$, with

$$G_{1\pm}(r,t) := \mp\frac{\alpha}{p+\gamma}\varphi_\pm(g)pw^{p+\gamma}(f^{p+\gamma})'\frac{r}{t^{\alpha+1}},$$

$$G_2(r,t) := \frac{\alpha\beta_p(f)'w^p r}{t^{\alpha+1}}(r^{1-p}w^{1-p} - 1), \text{ and } G_3(r,t) := \frac{((1-rw)r)_r}{r^2}h_r,$$

we note that $G_{1+} \ge 0$, $G_{1-} \le 0$, $G_2 \ge 0$, and $G_3 \le 0$. These inequalities follows directly from Lemma 5.17 and Proposition 5.18.

Now we compute estimates for the integrals associated with each part of G. For G_{1+}, we have

$$\begin{aligned}\int_0^\infty G_{1+}(r,t)r^2 dr &\le -\frac{C\|\varphi_+\|_{L^\infty}}{t^{\alpha\gamma}} \int_0^\infty (f^{p+\gamma})'\eta^{3-p-\gamma} d\eta \\ &\le t^{-\alpha\gamma}C\|\varphi_+\|_{L^\infty} \int_0^\infty f^{p+\gamma}\eta^{2-p-\gamma} d\eta = L\|\varphi_+\|_{L^\infty} t^{-\alpha\gamma}\end{aligned}$$

since $\gamma < 3 - p$. Hence L is a positive constant. We have a similar estimate for G_{1-} replacing $\|\varphi_+\|_{L^\infty}$ by $\|\varphi_-\|_{L^\infty}$.

For G_2 we have two cases. For $p < 1$ we use $1 - r^{1-p}w^{1-p} \le w \le 1/r$ to obtain

$$\int_0^\infty G_2(r,t)r^2 dr \le \frac{\alpha}{t^\alpha} \int_0^\infty (\beta_p(f))' \eta^{2-p} d\eta \le \frac{(2-p)\alpha}{t^\alpha} \int_0^\infty \beta_p(f)\eta^{1-p} d\eta = \frac{L_2}{t^\alpha}.$$

For $p = 1$, we use $1 - r^{1-p}w^{1-p} = 0$, so that $G_2 \equiv 0$.

Computing the integral of G_3, gives

$$-\int_0^\infty G_3(r,t)r^2 dr = \int_0^\infty (r(wr-1))_r h_r dr = \int_0^\infty r(1-wr)h_{rr} dr \le h_r|_0^\infty = \frac{K}{t^\alpha},$$

where K is defined in Proposition 5.18. Here we used Lemma 5.17 (ii). To complete the proof we use the sign of the functions $G_{1\pm}$, G_2, and G_3, and Proposition 5.16. ■

Chapter 6

A weakly coupled system

6.1 Introduction

In this chapter we will study existence of positive solutions to a system of the form

$$(D) \quad \begin{cases} -(r^{N-1}\phi_i(u_i'(r)))' = r^{N-1}f_i(u_{i+1}(r)) \\ & i = 1, \dots, n \\ u_i'(0) = 0 = u_i(R), \end{cases}$$

where it is understood that $u_{n+1} = u_1$. Here for $i = 1, \dots, n$, the functions ϕ_i are odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and the $f_i : \mathbb{R} \mapsto \mathbb{R}$ are odd continuous functions such that $sf_i(s) > 0$ for $s \neq 0$. Also $' = \frac{d}{dr}$.

System (D) is particularly important when the homeomorphisms ϕ_i take the form $\phi_i(s) = sa_i(|s|)$, $s \in \mathbb{R}$ since it is satisfied by the radial solutions of the system

$$\begin{cases} \operatorname{div}(a_i(|\nabla u_i|)\nabla u_i) + f_i(u_{i+1}(|x|)) = 0, \\ & i = 1, \dots, n \\ u_i(|x|) = 0, \quad x \in \partial\Omega, \end{cases}$$

where Ω denotes the ball in $\mathbb{R}^N$ centered at zero and with radius $R > 0$.

Furthermore, concerning the functions ϕ_i, f_i, $i = 1, \dots, n$, we will assume that they belong to the class of asymptotically homogeneous functions (AH for short). We say that $h : \mathbb{R} \mapsto \mathbb{R}$ is AH at $+\infty$ of exponent $\delta > 0$ if for any $\sigma > 0$

$$\lim_{s \to +\infty} \frac{h(\sigma s)}{h(s)} = \sigma^\delta. \tag{6.1}$$

This chapter has appeared in Abstract and Applied Analysis [41]

By replacing $+\infty$ by 0 in (6.1) we obtain a similar equivalent definition for a function h to be AH of exponent δ at zero. AH functions have been recently used in [44] and [42] in connection with quasilinear problems. They form an important class of non homogeneous functions which without being necessarily asymptotic to any power have the suitable homogeneous asymptotic behavior given by (6.1). In a very different context they have been used in applied probability and statistics where they are known as regularly varying functions, see for example [69], [71].

By a solution to (D) we understand a vector function $\mathbf{u} = (u_1, \ldots, u_n)$ such that $\mathbf{u} \in C^1([0,T], \mathbb{R}^n)$ and $\phi_i(u_i') \in C^1([0,T], \mathbb{R})$, $i = 1 \ldots, n$, which satisfies (D).

In [27], the existence of solutions with positive components for a system of the form (D) with $n = 2$ and with the functions ϕ_i and f_i having the particular form $\phi_i(s) = |s|^{p_i - 2}s$, $\phi_i(0) = 0$, $p_i > 1$, $f_i(s) = |s|^{\delta_i - 1}s$, $f_i(0) = 0$, $\delta_i > 0$, $i = 1, 2$, was done. In [44], within the scope of the AH functions, the case of a single equation was considered. In both situations the central idea to obtain a-priori bounds was the blow-up method of Gidas and Spruck, see [45]. As a consequence of our results in this chapter, those in [27] and [44] are greatly generalized.

Next we develop some preliminaries in order to state our main theorem. For $i = 1, \ldots, n$, let $\delta_i, \bar{\delta}_i$ be positive real numbers and $p_i, \bar{p}_i$ real numbers greater than one, and assume that the functions ϕ_i, f_i, $i = 1, \ldots, n$ satisfy

$$(H_1) \qquad \lim_{s \to +\infty} \frac{\phi_i(\sigma s)}{\phi_i(s)} = \sigma^{p_i - 1}, \quad \lim_{s \to +\infty} \frac{f_i(\sigma s)}{f_i(s)} = \sigma^{\delta_i},$$

for all $\sigma > 0$,

$$(H_2) \qquad \prod_{i=1}^{n} \frac{\delta_i}{(p_i - 1)} > 1.$$

To the exponents p_i, δ_i, let us associate the system

$$(AS) \quad \begin{cases} (p_i - 1)E_i - \delta_i E_{i+1} = -p_i, & i = 1, \ldots, n, \\ E_{n+1} = E_1. \end{cases}$$

From (H_2), it turns out that (AS) has a unique solution $(E_1, \ldots, E_N)$, such that $E_i > 0$ for each $i = 1, \ldots, n$. An explicit form for these solutions is given in the Appendix at the end of the chapter.

Now we can establish our main existence theorem.

Theorem 6.1 *For $i = 1, \ldots, n$, let ϕ_i be odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and $f_i : \mathbb{R} \mapsto \mathbb{R}$ odd continuous functions with $x f_i(x) > 0$ for $x \neq 0$, which satisfy (H_1), (H_2), and*

$$(H_3) \qquad \lim_{s \to 0} \frac{\phi_i(\sigma s)}{\phi_i(s)} = \sigma^{\bar{p}_i - 1}, \quad \lim_{s \to 0} \frac{f_i(\sigma s)}{f_i(s)} = \sigma^{\bar{\delta}_i},$$

for any $\sigma > 0$. Additionally, for $i = 1, \ldots, n$, let us assume that

$$(H_4) \qquad \prod_{i=1}^{n} \frac{\bar{\delta}_i}{(\bar{p}_i - 1)} > 1,$$

(H_5) $$p_i < N, \ i = 1, \dots, n, \qquad \max_{i=1,\dots,n} \{E_i - \theta_i\} \geq 0,$$

where $\theta_i = \frac{N-p_i}{p_i-1}$ *and the* E_i*'s are the solutions to (AS). Then problem* (D) *has a solution* $(u_1, \dots, u_n)$ *such that* $u_i(r) > 0$, $r \in [0, R)$, *for each* $i = 1, \dots, n$.

The plan of this chapter is as follows. We begin Section 6.2 by discussing some properties of the AH functions that will be used throughout the paper. Then we provide an abstract functional analysis setting for problem (D) so that finding solutions to that problem is equivalent to solving a fixed point problem. Section 6.3 is first devoted to the study of a-priori bounds for positive solutions to problem (D) and then to prove our main theorem by using Leray Schauder degree arguments. To show the a-priori bounds we argue by contradiction and thus by using some suitable *rescaling functions* we find that there must exist a vector solution $\mathbf{v} = (v_1, \dots, v_n)$ defined on $[0, +\infty)$ (vector ground state) to a system of the form

$$(D_p) \quad \begin{cases} -(r^{N-1}|v_i'(r)|^{p_i-2}v_i'(r))' = C_i r^{N-1}|v_{i+1}(r)|^{\delta_i - 1}v_{i+1}(r) \ r \in [0, +\infty) \\ \hfill i = 0, \dots, n, \\ v_i'(0) = 0, \quad v_i(r) \geq 0, \ r \in [0, +\infty), \end{cases}$$

where $v_{n+1} = v_1$ and C_i are positive constants, $i = 1, \dots, n$. We observe here the interesting fact that in this asymptotic system only properties of ϕ_i, f_i at $+\infty$ appear. We reach then a contradiction, and hence the existence of a-priori bounds, by using hypothesis (H_5) which prevents the existence of such a vector ground state.

In all of our previous argument the existence of suitable rescaling functions is crucial. The lemma for their existence (as well as some of their key properties) is stated without proof at the beginning of Section 6.3 and its proof (which is delicate and rather lengthy and technical is postponed to Section 6.4. In Section 6.5 we give some applications that illustrate our existence result. In particular, in Theorem 6.13 we apply our existence results to a system that contains operators of the form $(-\Delta_p)^n$, $(-\Delta_q)^m$, where for $t > 1$ $\Delta_t u := \mathrm{div}(|\nabla u|^{t-2}\nabla u)$. We end the chapter with an Appendix which contains some technical results.

We introduce now some notation. Throughout this section vectors in $\mathbb{R}^n$ will be written in boldface. $C_\#$ will denote the closed linear subspace of $C[0, R]$ defined by $C_\# = \{u \in C[0, R] \mid u(R) = 0\}$. We have that $C_\#$ is a Banach space with respect to the norm $\|\cdot\| := \|\cdot\|_\infty$. Also we will denote by $C_\#^n$, the Banach space of the $n-$tuples of elements of $C_\#$ endowed with the norm $||\mathbf{u}||_n := \sum_{i=1}^{n} ||u_i||$, where $\mathbf{u} = (u_1, \dots, u_n) \in C_\#^n$.

Finally we adopt the following conventions. By $\mathbb{R}_+$ and $\mathbb{R}^+$ we mean $[0, +\infty)$ and $(0, +\infty)$ respectively. For a function $H : \mathbb{R} \mapsto \mathbb{R}$ (with $\lim_{s \to 0} \frac{H(s)}{s} = 0$) we define $\hat{H}(s) := \frac{H(s)}{s}$, $s \neq 0$, $\hat{H}(0) = 0$, and we note that if H is AH of exponent p (at $+\infty$ or zero) then $\hat{H}$ is AH of exponent $p-1$. Also if γ_i, $i = 1, \dots, n$, are real numbers or functions, we define $\gamma_{n+i} = \gamma_i$ for all $i = 1, \dots, n$.

6.2 Preliminaries and abstract formulation

We begin this section with a proposition.

Proposition 6.2 *Let $h : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function with $h(0) = 0$, $th(t) > 0$ for $t \neq 0$, and let $H(t) := \int_0^t h(s)ds$ and $\hat{H} : \mathbb{R} \mapsto \mathbb{R}$ as defined in the Introduction.*

(i) *If h is AH of exponent $\rho > 0$ at $+\infty$, then there exists $t_0 > 0$ and positive constants d_1 and d_2 with $1 < d_1 \leq d_2$ such that*

$$d_1 \leq \frac{th(t)}{H(t)} \leq d_2, \quad \text{for all} \quad t \geq t_0, \tag{6.2}$$

$h(s) \to +\infty$ as $s \to +\infty$, $\hat{H}(t)$ is increasing for $t \geq t_0$ and

$$d_1 h(s) \leq d_2 h(t) \quad \text{for all } s,\ t \text{ such that} \quad t_0 \leq s \leq t. \tag{6.3}$$

(ii) *If h is AH of exponent $\rho > 0$ at 0 then there exists $t_0 > 0$ and positive constants d_1 and d_2, with $1 < d_1 \leq d_2$ such that*

$$d_1 \leq \frac{th(t)}{H(t)} \leq d_2, \quad \text{for all } |t| \leq t_0,$$

$\hat{H}(t)$ is increasing in $[-t_0, t_0]$, and

$$d_1|h(s)| \leq d_2|h(t)|$$

for all s, t with $|s| \leq |t| \leq t_0$.

Proof. We only prove (i), since (ii) is similar. From Karamata's theorem (see [69], p. 17, Theorem 0.6), it follows that for any $\sigma > 0$

$$\lim_{t\to+\infty} \frac{h(\sigma t)}{h(t)} = \sigma^\rho \quad \text{if and only if} \quad \lim_{t\to+\infty} \frac{H(t)}{th(t)} = \frac{1}{\rho+1}, \tag{6.4}$$

and thus, if h is AH of exponent $\rho > 0$, for $\varepsilon \geq 0$ (less than $\min\{\rho, 1\}$) there is a $t_0 > 0$, such that for all $t \geq t_0$,

$$\frac{\rho+1-\varepsilon}{t} \leq \frac{h(t)}{H(t)} \leq \frac{\rho+1+\varepsilon}{t}. \tag{6.5}$$

Setting $d_1 := \rho + 1 - \varepsilon > 1$ and $d_2 := \rho + 1 + \varepsilon$ we have that (6.2) holds. Now since $h(t) = H'(t)$, from (6.5) we obtain that

$$C_1 t^{d_1 - 1} \leq h(t) \leq C_2 t^{d_2 - 1} \quad \text{for all} \quad t \geq t_0, \tag{6.6}$$

for some positive constants C_1, C_2 and thus $h(t) \to +\infty$ as $s + \infty$.

We observe now that the function $\hat{H}$ is a C^1 function for $t > 0$, and that $\hat{H}'(t) = \frac{th(t)-H(t)}{t^2}$. Then from (6.5) and since $d_1 > 1$, we find that $\hat{H}'(t) > 0$ for $t \geq t_0$, i.e., $\hat{H}$ is ultimately increasing. Finally, and again from (6.5) for $t_0 \leq s \leq t$, we have that $d_1 h(s) \leq d_1 d_2 \hat{H}(s) \leq d_1 d_2 \hat{H}(t) \leq d_2 h(t)$, ending the proof of the proposition. ■

As a consequence of this proposition we have the following result, which will be used to prove our main result.

Proposition 6.3 *Let $h : \mathbb{R} \mapsto \mathbb{R}$ be continuous and asymptotically homogeneous at $+\infty$ (at 0) of exponent $\rho > 0$ satisfying $th(t) > 0$ for $t \neq 0$. Let $\{w_n\}$ and $\{t_n\} \subseteq \mathbb{R}^+$ be sequences such that $w_n \to w$ and $t_n \to +\infty$ ($t_n \to 0$) as $n \to \infty$. Then,*

$$\lim_{n\to\infty} \frac{h(t_n w_n)}{h(t_n)} = w^\rho. \tag{6.7}$$

Proof. We only prove the case when h is AH at $+\infty$, the other case being similar. Let $H(s) := \int_0^s h(t)dt$ and assume first $w \neq 0$. Then $t_n w_n \to +\infty$ and by writing

$$\frac{h(t_n w_n)}{h(t_n)} = \frac{t_n w_n h(t_n w_n)}{H(t_n w_n)} \frac{\hat{H}(t_n w_n)}{\hat{H}(t_n)} \frac{H(t_n)}{t_n h(t_n)} \tag{6.8}$$

we see from (6.4) that to obtain (6.7) it suffices to prove that

$$\lim_{n\to\infty} \frac{\hat{H}(t_n w_n)}{\hat{H}(t_n)} = w^\rho. \tag{6.9}$$

Since by Proposition 6.2, $\hat{H}$ is ultimately increasing, given $\varepsilon > 0$ sufficiently small, there exists $n_0 > 0$ such that for all $n \geq n_0$

$$\frac{\hat{H}(t_n(w-\varepsilon))}{\hat{H}(t_n)} \leq \frac{\hat{H}(t_n w_n)}{\hat{H}(t_n)} \leq \frac{\hat{H}(t_n(w+\varepsilon))}{\hat{H}(t_n)}$$

and thus (6.9) follows by using the fact that $\hat{H}$ is AH of exponent ρ and $\varepsilon > 0$ is arbitrarily small. Assume now that $w = 0$. We claim then that

$$\lim_{n\to\infty} \frac{h(t_n w_n)}{h(t_n)} = 0.$$

If not,

$$\frac{h(t_{n_k} w_{n_k})}{h(t_{n_k})} \geq \mu,$$

for some subsequences $\{t_{n_k}\}$, $\{w_{n_k}\}$, which implies that $t_{n_k} w_{n_k}$ must tend to $+\infty$. Let now $\varepsilon > 0$ be such that $\varepsilon < \mu^{1/\rho}$. Since $w_{n_k} \to 0$, there exists $k_0 > 0$ such that $w_{n_k} < \varepsilon$ and as $t_{n_k} w_{n_k} \to +\infty$, both $t_{n_k} w_{n_k}$ and $t_{n_k}\varepsilon$ belong to the range where $\hat{H}$ is increasing for $k \geq k_0$.. Hence,

$$0 \leq \frac{\hat{H}(t_{n_k} w_{n_k})}{\hat{H}(t_{n_k})} \leq \frac{\hat{H}(t_{n_k}\varepsilon)}{\hat{H}(t_{n_k})}.$$

Using now that $\hat{H}$ is AH of exponent ρ, by letting $k \to \infty$ we find that

$$\limsup_{k\to\infty} \frac{\hat{H}(t_{n_k} w_{n_k})}{\hat{H}(t_{n_k})} \le \varepsilon^\rho$$

and hence, by (6.8), $\mu \le \limsup_{k\to\infty} \frac{h(t_{n_k} w_{n_k})}{h(t_{n_k})} \le \varepsilon^\rho < \mu$, a contradiction. ■

Finally, regarding properties of AH (at ∞ or 0) that we will need later on, it is simple to see that if $\chi,\ \psi : \mathbb{R} \mapsto \mathbb{R}$ are AH functions of exponent p and q respectively, then $\chi \circ \psi$ is AH of exponent $r = pq$. Also, if ϕ is an increasing odd homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ which is AH of exponent $p-1$, then its inverse ϕ^{-1} is AH of exponent p^*-1, where $p^* = \frac{p}{p-1}$.

We now find a functional analysis setting for problem (D). A simple calculation shows that finding non trivial solutions with positive components to problem (D) is equivalent to finding non trivial solutions to the problem

$$(A) \qquad \begin{cases} -(r^{N-1}\phi_i(u_i'(r)))' = r^{N-1} f_i(|u_{i+1}(r)|) \\ \\ u_i'(0) = 0 = u_i(R). \end{cases} \qquad i = 1, \dots, n,$$

Let $(u_1(r), \dots, u_n(r))$ be a non trivial solution of (A). Then for each $i = 1, \dots, n$, we have that $u_i(r) \ge 0$ and is non increasing on $[0, R]$. By integrating the equations in (A), it follows that $u_i(r)$ satisfies

$$u_i = M_i(u_{i+1})$$

where $M_i : C_\# \mapsto C_\#$ is given by

$$M_i(v)(r) = \int_r^R \phi_i^{-1}[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} f_i(|v(\xi)|) d\xi] ds,$$

for each $i = 1, \dots, n$. Let us define $T_0 : C_\#^n \mapsto C_\#^n$ by

$$T_0(\boldsymbol{u}) := (M_1(u_2), ..., M_i(u_{i+1}), ..., M_n(u_1)),$$

where $\boldsymbol{u} = (u_1, \dots, u_n)$. Clearly T_0 is well defined and fixed points of T_0 will provide solutions of (A), and hence componentwise positive solutions of (D).

Define now the operator $T_h : C_\#^n \times [0,1] \mapsto C_\#^n$ by

$$T_h(\mathbf{u}, \lambda) := \left(\tilde{M}_1(u_2, \lambda), ..., M_i(u_{i+1}), ..., M_n(u_1)\right)$$

where $\tilde{M}_1 : C_\# \times [0,1] \mapsto C_\#$ is the operator defined by

$$\tilde{M}_1(v, \lambda)(r) : \int_r^R \phi_1^{-1}[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} (f_1(|v(\xi)|) + \lambda h) d\xi] ds$$

with $h > 0$ a constant to be fixed later. Define also $S : C^n_{\#} \times [0,1] \mapsto C^n_{\#}$ by

$$S(\mathbf{u},\lambda) = (N_1(u_2,\lambda), ..., N_i(u_{i+1},\lambda), ..., N_n(u_1,\lambda)) \tag{6.10}$$

where $N_i : C_{\#} \times [0,1] \mapsto C_{\#}$ is the operator defined by

$$N_i(v,\lambda)(r) = \int_r^R \phi_i^{-1}[\frac{\lambda}{s^{N-1}} \int_0^s \xi^{N-1} f_i(|v(\xi)|) d\xi] ds, \ i = 1, \dots, n. \tag{6.11}$$

It follows from Proposition 2.2 of [44] that all the operators $\tilde{M_1}, M_i, N_i$, $i = 1, \dots, n$, are completely continuous, hence the operators T_0, T_h and S are also completely continuous. We note that $T_h(\cdot, 0) = T_0 = S(\cdot, 1)$.

To prove existence of a fixed point of T_0 we use suitable a-priori estimates and degree theory. Indeed, we will show that there exists $R_1 > 0$, such that $deg_{LS}(I - T_0, B(0, R_1), 0) = 0$, and also that the index $i(T_0, 0, 0)$ is defined and it satisfies $i(T_0, 0, 0) = 1$, from where the existence of a fixed point of T_0 follows by the excision property of the degree.

Finally in this section, in our next lemma we will select the constant h that appears in the definition of the operator T_h, and hence fix this operator once for all.

Lemma 6.4 *For $i = 1, \dots, n$ let the homeomorphisms ϕ_i, and the functions f_i satisfy (H_1) and (H_2). Then there exists $h_0 > 0$ such that the problem*

$$\mathbf{u} = T_h(\mathbf{u}, 1) \tag{6.12}$$

has no solutions for $h \geq h_0$.

Proof. We argue by contradiction and thus we assume that there exists a sequence $\{h_k\}_{k \in \mathbb{N}}$, with $h_k \to +\infty$ as $k \to \infty$, such that the problem

$$\mathbf{u} = T_{h_k}(\mathbf{u}, 1)$$

has a solution $\mathbf{u}_k = (u_{1,k}, \dots, u_{n,k})$, for each $k \in \mathbb{N}$. Then $\boldsymbol{u}_k$ satisfies

$$u_{1,k}(r) = \int_r^R \phi_1^{-1}[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}(f_1(|u_{2,k}(\xi)|) + h_k) d\xi] ds \tag{6.13}$$

$$u_{i,k}(r) = \int_r^R \phi_i^{-1}[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} f_i(|u_{i+1,k}(\xi)|) d\xi] ds, \ i = 2, \dots, n, \tag{6.14}$$

for each $k \in \mathbb{N}$. Clearly $u_{i,k}(r) > 0$, $r \in [0, R)$, and is non increasing for $r \in [0, R]$, for all $k \in \mathbb{N}$, and all $i = 1, \dots, n$. From (6.13)

$$u_{1,k}(r) \geq (R - r)\phi_1^{-1}(\frac{r h_k}{N}), \quad \text{for all } r \in [0, R]$$

and thus for $r \in [0, \frac{7R}{8}]$, (we choose this interval for convenience, but any other interval of the form $[0,T] \subset [0,R)$ will work as well) we find that

$$u_{1,k}(r) \geq \frac{R}{8}\phi_1^{-1}(\frac{Rh_k}{4N}) \tag{6.15}$$

where we have used that $u_{1,k}(r) \geq u_{1,k}(R/4)$ for all $r \in [0, R/4]$. Then, using that $f_i(x) \to +\infty$ as $x \to +\infty$, from (6.14) and (6.15), by iteration, we conclude that for any $A > 0$, there exists $k_A > 0$ such that for all $r \in [0, \frac{3R}{4}]$

$$u_{i,k}(r) \geq A \quad \text{for all} \quad k \geq k_A \quad \text{and all} \quad i = 1, \dots, n. \tag{6.16}$$

Now, from the second of (H_1) and (i) of Proposition 6.2 there exist $t_0 > 0$, $1 < d_1 \leq d_2$ such that

$$d_1 f_i(\tau) \leq d_2 f_i(t) \tag{6.17}$$

for all $t \geq \tau \geq t_0$ and all $i = 1, \dots, n$. Hence, by (6.12), and by increasing A if necessary,

$$d_1 f_i(|u_{i+1,k}(r)|) \leq d_2 f_i(|u_{i+1,k}(\xi)|) \tag{6.18}$$

for all $\xi \in [0, r]$ with $r \in [0, 3R/4]$. Since from (6.13) and (6.14) we also have that

$$u_{i,k}(r) \geq \int_r^{\frac{3R}{4}} \phi_i^{-1}[\frac{1}{s^{N-1}} \int_0^r \xi^{N-1} f_i(|u_{i+1,k}(\xi)|) d\xi] ds, \ i = 1, \dots, n, \tag{6.19}$$

then, for $k \geq k_A$, from (6.18) and the monotonicity of ϕ_i^{-1} we have that

$$u_{i,k}(r) \geq \int_r^{\frac{3R}{4}} \phi_i^{-1}[d\ f_i(u_{i+1,k}(r)) d\xi] ds, \quad r \in [\frac{R}{4}, \frac{3R}{4}],$$

where $d = \frac{d_1 R}{4Nd_2 3^{N-1}}$. Thus for all $r \in [\frac{R}{4}, \frac{R}{2}]$, we find that

$$u_{i,k}(r) \geq \frac{R}{4}\phi_i^{-1}(d\ f_i(u_{i+1,k}(r))), \tag{6.20}$$

for all k large enough and for all $i = 1, \dots, n$. Next, setting

$$b_{i,k}(r) := \frac{R}{4}\frac{\phi_i^{-1}(d\ f_i(u_{i+1,k}(r)))}{\phi_i^{-1}(f_i(u_{i+1,k}(r)))}, \tag{6.21}$$

(6.20) becomes

$$u_{i,k}(r) \geq b_{i,k}(r)\phi_i^{-1}(f_i(u_{i+1,k}(r))), \quad r \in [\frac{R}{4}, \frac{R}{2}]. \tag{6.22}$$

Observing that by (6.16) and (H_1), $b_{i,k}(r) \to c_i$ as $k \to \infty$, uniformly in $[\frac{R}{4}, \frac{R}{2}]$, where c_i is a positive constant, we have that $b_{i,k}(r) \geq \tilde{C}$ for all $r \in [\frac{R}{4}, \frac{R}{2}]$, for all $i = 1, \dots, n$

and for all k sufficiently large and where $\tilde{C}$ is a positive constant. Hence, by (6.6) in the proof of Proposition 6.2, for $\varepsilon > 0$ small there is a $k_0 \in \mathbb{N}$ such that

$$u_{i,k}(r) \geq C u_{i+1,k}^{\frac{\delta_i}{p_i-1}-\varepsilon}(r), \quad r \in [\frac{R}{4}, \frac{R}{2}], \tag{6.23}$$

for all $k \geq k_0$ and all $i = 1, \ldots, n$, and where C is a positive constant. Now, by iterating in (6.23), we find that

$$u_{1,k}(r) \geq C(u_{1,k}(r))^{\prod_{i=1}^{n}\left(\frac{\delta_i}{p_i-1}-\varepsilon\right)}, \tag{6.24}$$

where C is a new positive constant. Since by (H_2) we may choose

$$0 < \varepsilon < \min\{\frac{\delta_i}{p_i - 1}, \; i = 1, \ldots, n\}$$

so that $\prod_{i=1}^{n}\left(\frac{\delta_i}{p_i-1} - \varepsilon\right) > 1$, from (6.24), we have

$$(u_{1,k}(r))^{\prod_{i=1}^{n}\left(\frac{\delta_i}{p_i-1}-\varepsilon\right)-1} \leq \frac{1}{C}, \quad \text{for any fixed } r \in [\frac{R}{4}, \frac{R}{2}],$$

which by (6.16) gives a contradiction for large k. This ends the proof of the lemma. ■

6.3 A-priori bounds and proof of the main result

In this section we will use the blow up method to find a priori bounds for the positive solutions of problem (D_h) and then prove Theorem 6.1. Let ϕ_i, f_i, $i = 1, \ldots, n$ be as in Theorem 6.1 and set

$$\Phi_i(s) = \int_0^s \phi_i(t)dt, \quad F_i(s) = \int_0^s f_i(t)dt, \quad i = 1, \ldots, n. \tag{6.25}$$

In extending the blow up method to our situation it turns out that a key step is to find a solution $(x_1, \ldots, x_n)$ in terms of s (for s near $+\infty$) to the system

$$F_i(x_{i+1})x_i = x_{i+1}\Phi_i(x_i s), \qquad i = 1, \ldots, n. \tag{6.26}$$

In this respect we can prove the following.

Lemma 6.5 *Assume that the homeomorphisms ϕ_i, and the functions f_i, $i = 1, \ldots, n$ satisfy (H_1), (H_2), (H_3), and (H_4). Then*

(i) *there exist positive numbers s_0, x_i^0, and increasing diffeomorphisms α_i defined from $[s_0, +\infty)$ onto $[x_i^0, +\infty)$, $i = 1, \ldots, n$, which satisfy*

$$F_i(\alpha_{i+1}(s))\alpha_i(s) = \alpha_{i+1}(s)\Phi_i(\alpha_i(s)s), \tag{6.27}$$

for all $s \in [s_0, +\infty)$.

(ii) *The functions* α_i *satisfy*

$$\lim_{s\to\infty} \frac{f_i(\alpha_{i+1}(s))}{s\phi_i(\alpha_i(s)s)} = \frac{\delta_i+1}{p_i}, \quad i=1,\ldots,n. \tag{6.28}$$

(iii) *The functions* α_i *satisfy*

$$\lim_{s\to+\infty} \frac{\alpha_i(\sigma s)}{\alpha_i(s)} = \sigma^{E_i} \quad \textit{for all} \quad \sigma \in (0,+\infty) \quad i=1,\ldots,n,$$

where the E_i*'s are the solutions to* (AS).

We call these α_i's functions *rescaling* variables for system (D).

The proof of this lemma is rather lengthy and delicate and thus in order not to deviate the attention of the reader we postpone it until section 4.

We next find a-priori bounds for positive solutions. To this end let h satisfy the conditions of Lemma 6.4 and consider the family of problems

$$(D_\lambda) \quad \begin{cases} [r^{N-1}\phi_1(u_1')]' + r^{N-1}(f_1(|u_2(r)|) + \lambda h) = 0, \\ [r^{N-1}\phi_i(u_i')]' + r^{N-1}f_i(|u_{i+1}(r)|) = 0, \quad \lambda \in [0,1],\ i=2,\ldots,n, \end{cases}$$

$$u_i'(0) = 0 = u_i(R) \quad \text{for } i=1,\ldots,n.$$

Clearly, a solution to (D_λ) is a fixed point of $T_h(\cdot,\lambda)$.

Theorem 6.6 *Under the conditions of Theorem 6.1, solutions to problem* (D_λ) *are a-priori bounded.*

Proof. We argue by contradiction and thus we assume that there exists a sequence $\{(\mathbf{u}_k,\lambda_k)\} \in C^n_\# \times [0,1]$, with $\mathbf{u}_k = (u_{1,k},\ldots,u_{n,k})$, such that $(\mathbf{u}_k,\lambda_k)$ satisfies (D_{λ_k}) and $\|\mathbf{u}_k\| = \sum_{i=1}^{n} \|u_{i,k}\| \to \infty$ as $k\to\infty$. It is not difficult to check by using the equations in (D_{λ_k}) that $\sum_{i=1}^{n}\|u_{i,k}\| \to \infty$ as $k\to\infty$ if and only if $\|u_{i,k}\|\to\infty$ as $k\to\infty$ for each $i=1,\ldots,n$. Hence, by redefining the sequence $(\mathbf{u}_k,\lambda_k)$ if necessary, we can assume that $\|u_{i,k}\| \geq s_0$ (s_0 as in Lemma 6.5) for all $i=1,\ldots,n$ and for all $k>0$. Let us set

$$\gamma_k = \sum_{i=1}^{n} \alpha_i^{-1}(\|u_{i,k}\|) \quad \text{and} \quad t_{i,k} = \alpha_i(\gamma_k). \tag{6.29}$$

Then, $\gamma_k\to\infty$ as $k\to\infty$, and $\|u_{i,k}\| \leq t_{i,k}$, for each $i=1,\ldots,n$. Also by (6.28)

$$\lim_{k\to\infty} \frac{f_i(t_{i+1,k})}{\gamma_k\phi_i(t_{i,k}\gamma_k)} = \frac{\delta_i+1}{p_i}. \tag{6.30}$$

Next we define the change of variables $y = \gamma_k r$, $w_{i,k}(y) = \frac{u_{i,k}(r)}{t_{i,k}}$ and set $\mathbf{w}_k := (w_{1,k}, \ldots, w_{n,k})$. Clearly we have $|w_{i,k}(y)| \leq 1$ for all $y \in [0, \gamma_k R]$. In terms of these new variables and since $(\mathbf{u}_k, \lambda_k)$ satisfies (D_{λ_k}), we obtain that $(\mathbf{w}_k, \lambda_k)$ satisfies

$$-(y^{N-1}\phi_1(t_{1,k}\gamma_k w'_{1,k}(y)))' = y^{N-1}[\frac{f_1(t_{2,k}|w_{2,k}(y)|)}{\gamma_k} + \frac{\lambda_k h}{\gamma_k}], \tag{6.31}$$

$$-(y^{N-1}\phi_i(t_{i,k}\gamma_k w'_{i,k}(y)))' = y^{N-1}\frac{f_i(t_{i+1,k}|w_{i+1,k}(y)|)}{\gamma_k}, \quad i = 2, \ldots, n, \tag{6.32}$$

$$w'_{i,k}(0) = 0 = w_{i,k}(\gamma_k R) \quad \text{for} \quad i = 1, \ldots, n, \tag{6.33}$$

where now $' = \frac{d}{dy}$. Let now $T > 0$ be fixed and assume, by passing to a subsequence if necessary, that $\gamma_k R > T$ for all $k \in \mathbb{N}$. We observe that by the usual argument, $w'_{i,k}(y) \leq 0$ and $w_{i,k}(y) \geq 0$ for all $i = 1, \ldots, n$, for all $k \in \mathbb{N}$, and for all $y \in [0, T]$.

Claim. The sequences $\{w'_{i,k}\}_k$, $i = 1, \ldots, n$, are bounded in $C[0, T]$. Indeed, assume by contradiction that for some $i = 1, \ldots, n$, $\{w'_{i,k}\}$ contains a subsequence, renamed the same, with $||w'_{i,k}||_{C[0,T]} \to \infty$ as $k \to \infty$. Then there exists a sequence $\{y_k\}$, $y_k \in [0, T]$, such that for any $A > 0$ there is n_0 such that $|w'_{i,k}(y_k)| > A$ for all $k > n_0$. Integrating (6.31) (respectively (6.32)) from 0 to y_k, we obtain

$$\phi_i(t_{i,k}\gamma_k|w'_{i,k}(y_k)|) = {y_k}^{1-N}\int_0^{y_k}\frac{s^{N-1}f_i(t_{i+1,k}w_{i+1,k}(s))}{\gamma_k}ds + \frac{\lambda_k h y_k}{N\gamma_k}. \tag{6.34}$$

Now let t_0, d_1, d_2 be as in Proposition 6.2 and set $M = \max_{i\in\{1,\ldots,n\}} \sup_{x\in[0,t_0]} f_i(x)$. Since $t_{i+1,k} \to +\infty$ as $k \to \infty$, by redefining the sequence if necessary, we may assume that $\frac{M}{f_i(t_{i+1,k})} \leq \frac{d_2}{d_1}$ for all $i = 1, \ldots, n$ and all $k \in \mathbb{N}$. Also, since $w_{i+1,k}(s) \leq 1$, if $t_{i+1,k}w_{i+1,k}(s) \geq t_0$, then by Proposition 6.2 we have that

$$\frac{f_i(t_{i+1,k}w_{i+1,k}(s))}{f_i(t_{i+1,k})} \leq \frac{d_2}{d_1}. \tag{6.35}$$

Since if $t_{i+1,k}w_{i+1,k}(s) \leq t_0$ (6.35) holds by the definition of M, we have that indeed (6.35) holds for all $i = 1, \ldots, n$, all $k \in \mathbb{N}$ and all $s \in [0, T]$. Hence from (6.34) and the monotonicity of ϕ_i we find that

$$\frac{\phi_i(t_{i,k}\gamma_k A)}{\phi_i(t_{i,k}\gamma_k)} \leq \frac{d_2}{d_1}\frac{f_i(t_{i+1,k})T}{\phi_i(t_{i,k}\gamma_k)\gamma_k N} + \frac{hT}{N\phi_i(t_{i,k}\gamma_k)\gamma_k}.$$

Thus, by (H_1) and (6.30), and by letting $k \to \infty$ in this last inequality we find that

$$A^{p_i-1} \leq \frac{d_2}{d_1}\frac{(\delta_i + 1)T}{p_i N},$$

which is a contradiction since A can be taken arbitrarily large and hence the claim follows.

From this claim and Arzela Ascoli Theorem, by passing to a subsequence if necessary, we have that $\mathbf{w}_k \to \mathbf{w} := (w_1, \ldots, w_n)$ in $C^n[0, T]$. Also, by (6.29),

$$1 = \sum_{i=1}^{n}\frac{\alpha_i^{-1}(t_{i,k}w_{i,k}(0))}{\gamma_k} = \sum_{i=1}^{n}\frac{\alpha_i^{-1}(t_{i,k}w_{i,k}(0))}{\alpha_i^{-1}(t_{i,k})},$$

and hence, by letting $k \to \infty$ and using (iii) of Lemma 6.5, we obtain

$$1 = \sum_{i=1}^{n} w_i^{\frac{1}{E_i}}(0),$$

which implies that w is not identically zero.

Now by integrating (6.31) (respectively (6.32)) from 0 to $y \in [0, T]$ and using (6.33), we obtain

$$-\phi_i(t_{i,k}\gamma_k w'_{i,k}(y)) = \tilde{f}_{i,k}(y)\frac{f_i(t_{i+1,k})}{\gamma_k}, \tag{6.36}$$

for $i = 1, \ldots, n$ and all $k \in \mathbb{N}$, where

$$\tilde{f}_{1,k}(y) = y^{1-N}\int_0^y s^{N-1}\frac{f_1(t_{2,k}w_{2,k}(s))}{f_1(t_{2,k})}ds + \frac{\lambda_k h y}{N f_1(t_{2,k})}, \tag{6.37}$$

and

$$\tilde{f}_{i,k}(y) = y^{1-N}\int_0^y s^{N-1}\frac{f_i(t_{i+1,k}w_{i+1,k}(s))}{f_i(t_{i+1,k})}ds, \quad i = 2, \ldots, n. \tag{6.38}$$

Using now Proposition 6.3, we have that $\frac{f_i(t_{i+1,k}w_{i+1,k}(s))}{f_i(t_{i+1,k})} \to (w_{i+1}(s))^{\delta_i}$ for each $s \in [0, T]$ and $i = 1, \ldots, n$, and thus by (6.35) we may use the Dominated Convergence Theorem to conclude that

$$\lim_{k\to\infty} \tilde{f}_{i,k}(y) = y^{1-N}\int_0^y s^{N-1}w_{i+1}^{\delta_i}(s)ds := \tilde{f}_i(y) \tag{6.39}$$

for each $y \in (0, T]$. From (6.36),

$$-w'_{i,k}(y) = \frac{\phi_i^{-1}(\tilde{g}_{i,k}(y)\mu_k)}{\phi_i^{-1}(\mu_k)}, \tag{6.40}$$

where

$$\tilde{g}_{i,k}(y) = \frac{\tilde{f}_{i,k}(y)f_i(\alpha_{i+1}(\gamma_k))}{\gamma_k\phi_i(\gamma_k\alpha_i(\gamma_k))} \quad \text{and} \quad \mu_k = \phi_i(\gamma_k t_{i,k}).$$

Then $\mu_k \to +\infty$ as $k \to \infty$ and by (6.39) and (ii) of Lemma 6.5,

$$\tilde{g}_{i,k}(y) \to \frac{\delta_i + 1}{p_i}\tilde{f}_i(y) \quad \text{as} \quad k \to \infty \quad \text{for each } y \in [0, T]. \tag{6.41}$$

Integrating (6.40) over $[0, y]$, we obtain

$$w_{i,k}(0) - w_{i,k}(y) = \int_0^y \frac{\phi_i^{-1}(\tilde{g}_{i,k}(s)\mu_k)}{\phi_i^{-1}(\mu_k)}\,ds. \tag{6.42}$$

Then, since by (6.39) there exists $A > 0$ such that $|\tilde{f}_i(y)| \leq A$ for all $i = 1, \ldots, n$ and all $y \in [0, T]$, using (6.41) and the monotonicity of ϕ^{-1}, by another application of the Dominated Convergence Theorem, we find that

$$w_i(0) - w_i(y) = (\frac{\delta_i + 1}{p_i})^{\frac{1}{p_i-1}}\int_0^y \Big(s^{1-N}\int_0^s t^{N-1}w_{i+1}^{\delta_i}(t)dt\Big)^{\frac{1}{p_i-1}}\,ds,$$

and hence that w_i satisfies

$$(D_p)_T \quad \begin{cases} -(y^{N-1}|w_i'(y)|^{p_i-2}w_i'(y))' = (\frac{\delta_i+1}{p_i})y^{N-1}w_{i+1}^{\delta_i}(y) & y \in (0,T] \\ \\ w_i'(0) = 0, \quad w_i(y) \geq 0 \text{ for all } y \in [0,T]. \end{cases}$$

We observe next that each component $w_i(y)$ is decreasing on $[0,T]$. Thus if for some i, $w_i(0) = 0$, then necessarily $w_i(y) = 0$ for all $y \in [0,T]$. But from $(D_p)_T$ it follows that $w_{i+1}(y) = 0$ for all $y \in [0,T]$ and hence by iterating, that $\mathbf{w} \equiv \mathbf{0}$ on $[0,T]$, which cannot be. Now, for the purpose of our next argument let us call $\{\mathbf{w}_k^T\}$ the final subsequence, solution to (6.31), (6.32) and (6.33), which by the limiting process provided us with the non trivial solution $\mathbf{w}$ to $(D_p)_T$ defined in $[0,T]$. We also set $\mathbf{w}^T \equiv \mathbf{w}$. Let us choose next $T_1 > T$. By repeating the limiting process following (6.33), this time starting from the sequence $\{\mathbf{w}_k^T\}$, we will find a subsequence $\{\mathbf{w}_k^{T_1}\}$, which as $k \to \infty$ will provide us with a non trivial solution $\mathbf{w}^{T_1}$ to $(D_p)_{T_1}$. Clearly $\mathbf{w}^{T_1}$ is an extension of $\mathbf{w}^T$ to the interval $[0,T_1]$, which satisfies $w_i^{T_1}(y) \geq 0$, $i = 1,\ldots,n$. It is then clear that by this argument we can obtain a non trivial solution (called again $\mathbf{w}$) to (D_p), i.e. $\mathbf{w}$ satisfies

$$(D_p) \quad \begin{cases} -(y^{N-1}|w_i'(y)|^{p_i-2}w_i'(y))' = (\frac{\delta_i+1}{p_i})y^{N-1}w_{i+1}^{\delta_i}(y), & y \in (0,+\infty) \\ \\ w_i'(0) = 0, \quad w_i(y) \geq 0 \text{ for all } y \in [0,+\infty). \end{cases}$$

We claim now that under the hypotheses of Theorem 6.1 such a non trivial solution cannot exist. The proof of this claim is entirely similar to Lemma 2.1 in [27] so we just sketch it. An integration of the equations of (D_p) over $[0,r]$, $r \in (0,+\infty)$, shows that $w_i'(r) \leq 0$, for all $r > 0$, and that

$$-r^{N-1}|w_i'(r)|^{p_i-2}w_i'(r) \geq (\frac{\delta_i+1}{p_i})^{\frac{1}{p_i-1}}\frac{r^N}{N}w_{i+1}^{\delta_i}(r), \quad \text{for all } r > 0, \tag{6.43}$$

Also it must be that $w_i(r) > 0$ for all $r > 0$ and all $i = 1,\ldots,n$. Now by Proposition 2.1 and Lemma 2.1 in [27], see also [63] for related results, we have that for all $i = 1,\ldots,n$, $w_i \in C^2(0,+\infty)$ and that

$$rw_i'(r) + \theta_i w_i(r) \geq 0, \qquad \text{for all } r > 0. \tag{6.44}$$

Hence, from (6.43)

$$\frac{\theta_i w_i(r)}{r} \geq -w_i'(r) \geq Cr^{\frac{1}{p_i-1}}w_{i+1}^{\frac{\delta_i}{p_i-1}}(r) \quad \text{for all } r > 0,$$

where C is a positive constant. (In the rest of this argument C will denote a positive constant that may change from one step to the other). Multiplying this inequality by r^{E_i+1}, using (6.44) and system (AS), we obtain

$$r^{E_i}w_i(r) \geq C(r^{E_{i+1}}w_{i+1}(r))^{\frac{\delta_i}{p_i-1}}, \quad \text{for all } r > 0 \text{ and } i = 1,\ldots,n. \tag{6.45}$$

Iterating this expression $n-1$ times, we find first that

$$r^{E_i}w_i(r) \geq C(r^{E_i}w_i(r))^{\prod_{j=1}^{n}\frac{\delta_j}{p_j-1}} \quad \text{for each} \quad i=1,\ldots,n, \tag{6.46}$$

and thus by hypothesis (H_2),

$$w_i(r) \leq Cr^{-E_i} \quad \text{for each} \quad i=1,\ldots,n. \tag{6.47}$$

By (6.44), $r^{\theta_i}w_i(r)$ is non decreasing, and thus combining with (6.46),

$$C_i \equiv w_i(r_0)r_0^{\theta_i} \leq w_i(r)r^{\theta_i} \leq Cr^{-E_i}r^{\theta_i} = Cr^{-(E_i-\theta_i)}, \tag{6.48}$$

for all $r > r_0 > 0$, for all $i=1,\ldots,n$ and where the C_i's are positive constants. If the strict inequality holds in hypothesis (H_5), we obtain a contradiction by letting $r \to +\infty$ in (6.48) and the claim follows in this case.

Next, let us assume that for some $j \in \{1,\ldots,n\}$ we have that $E_j = \theta_j$. Integrating the j-th equation of (R_p) on (r_0, r), $r_0 > 0$, using (6.45) and iterating $n-2$ times, we obtain

$$r^{N-1}|w_j'(r)|^{p_j-1} \geq C\int_{r_0}^{r} s^{N-1-E_{j+1}\delta_j}(s^{E_j}w_j(s))^{P_j'}ds$$

where $P_j' := \prod_{i=1,i\neq j}^{n}\frac{\delta_i}{(p_i-1)}$. Hence, since $w_j(r)r^{\theta_j}$ is non decreasing, and using that $E_j = \theta_j$,

$$r^{N-1}|w_j'(r)|^{p_j-1} \geq C\int_{r_0}^{r} s^{-1}ds \quad \text{for all} \quad r > r_0,$$

where C is a positive constant. Hence by (6.44), we find that

$$r^{\theta_1}w_j(r) \geq C(\log(\frac{r}{r_0}))^{\frac{1}{p_j-1}} \quad \text{for all} \quad r > r_0, \tag{6.49}$$

which combined with (6.47) $(i=j)$ and using that $E_j = \theta_j$, yields again a contradiction and thus the claim follows.

In this form we have concluded the proof that solutions to (D_λ) are a priori bounded. ■

To prove Theorem 6.1 we need a last lemma. Let S be as defined by (6.10) and $B(0,\rho)$ denote the open ball centered at 0 and having radius ρ in $C_{\#}^n$.

Lemma 6.7 *Under the assumptions of Theorem 6.1, there exists $\rho_0 > 0$ such that the equation*

$$\mathbf{u} = S(\mathbf{u}, \lambda) \tag{6.50}$$

has no solutions $(\mathbf{u},\lambda) \in \left(\overline{B(0,\rho)} \setminus \{0\}\right) \times [0,1]$ for all $0 \leq \rho \leq \rho_0$. In particular, the index $i(S(\cdot,1),0,0) \equiv i(T_0,0,0)$ is defined and $i(T_0,0,0) = 1$.

Proof. We argue by contradiction and thus we assume that there exist sequences $\{\mathbf{u}_k\}$ in $C^n_\#$, $\{\rho_k\}$, $\rho_k > 0$ such that $||\boldsymbol{u}_k|| = \rho_k \to 0$, and a sequence $\{\lambda_k\}$, $\lambda_k \in [0,1]$, such that

$$\mathbf{u}_k = S(\mathbf{u}_k, \lambda_k). \tag{6.51}$$

Let $\mathbf{u}_k = (u_{1,k}, \ldots, u_{n,k})$. Since $u_{i,k}(s) \leq ||\boldsymbol{u}_k||$ for all $s \in [0,R]$, by (ii) of Proposition 6.2 we find that there exist $k_0 > 0$, $d_1 > 1$, $d_2 \geq d_1$, such that $f_i(u_{i,k}(s)) \leq \frac{d_2}{d_1} f_i(||\boldsymbol{u}_k||)$, for all $s \in [0,R]$, for all $k \geq k_0$, and for all $i = 1, \ldots, n$.

Then, from (6.51) we obtain that $\boldsymbol{u}_k$, $k \geq k_0$, satisfies

$$\|u_{i,k}\| \leq \phi_i^{-1}[\frac{\lambda_k R d_2 f_i(\|u_{i+1,k}\|)}{N d_1}]R, \quad 1 = 1, \ldots, n,$$

and hence

$$\phi_i\Big(\frac{\|u_{i,k}\|}{R}\Big) \leq \frac{Rd_2}{Nd_1} f_i(\|u_{i+1,k}\|). \tag{6.52}$$

Using the fact that the functions ϕ_i and f_i, $i = 1, \ldots, n$, are AH near zero, we have that given $\varepsilon > 0$ small, there are $s_0 > 0$ and positive constants C, $\tilde{C}$, such that

$$Cs^{\overline{\delta}_i+\varepsilon} \leq f_i(s) \leq \tilde{C}s^{\overline{\delta}_i-\varepsilon} \quad \text{for all} \quad 0 \leq s \leq s_0, \tag{6.53}$$

and

$$Cs^{\overline{p}_i-1+\varepsilon} \leq \phi_i(s) \leq \tilde{C}s^{\overline{p}_i-1-\varepsilon} \quad \text{for all} \quad 0 \leq s \leq s_0. \tag{6.54}$$

Hence by combining (6.52), (6.53), and (6.54), we obtain $||u_{i,k}|| \leq C||u_{i+1,k}||^{\frac{\overline{\delta}_i-\varepsilon}{\overline{p}_i-1+\varepsilon}}$ for all $i = 1, \ldots, n$ and $k \in \mathbb{N}$ sufficiently large. Then by iteration, there is a positive constant C such that $||u_{i,k}||^{1-\prod_{j=1}^{n}\frac{\overline{\delta}_j-\varepsilon}{\overline{p}_j-1+\varepsilon}} \leq C$ for each $i = 1, \ldots, n$. But this is not possible for $\varepsilon > 0$ small since $||u_{i,k}|| \to 0$ as $k \to \infty$, and since by (H_4) $1 - \prod_{j=1}^{n} \frac{\overline{\delta}_j-\varepsilon}{\overline{p}_j-1+\varepsilon} < 0$ for $\varepsilon > 0$ small enough. That the index $i(S(\cdot,1),0,0) = i(T_0,0,0)$ is defined and that $i(T_0,0,0) = 1$ is elementary. ■

Proof of Theorem 6.1. It follows from Theorem 6.6 that if $(\mathbf{u}, \lambda)$ is a solution to the equation

$$\mathbf{u} = T_h(\mathbf{u}, \lambda), \quad \lambda \in [0,1],$$

$\mathbf{u} = (u_1, \ldots, u_n)$, then there is a positive constant R_1 such that $\sum_{i=1}^{n} \|u_i\| \leq R_1$ for all $\lambda \in [0,1]$ and where we may assume $R_1 > \rho_0$. Thus if $B(0,R_1)$ denotes the ball centered at 0 in $C^n_\#$ with radius $R_1 > C$, we have that the Leray-Schauder degree of the operator

$$I - T_h(\cdot,\lambda) : \overline{B(0,R_1)} \mapsto C^n_\#$$

is well defined and constant with $\lambda \in [0,1]$. Then, by Lemma 6.4

$$\begin{aligned} deg_{LS}(I - T_0, B(0,R_1), 0) &= deg_{LS}(I - T_h(\cdot,0), B(0,R_1), 0) \\ &= deg_{LS}(I - T_h(\cdot,1), B(0,R_1), 0) \\ &= 0. \end{aligned} \tag{6.55}$$

Thus by Lemma 6.7, the excision property of the Leray Schauder degree, and (6.55), we conclude that there must be a solution of the equation

$$\mathbf{u} = T_0(\mathbf{u})$$

with $\mathbf{u} \in B(0,R_1) \setminus \overline{B(0,\varepsilon_0)}$, for $\varepsilon_0 > 0$ small enough. ■

Remark 6.8 *We point out that condition (H_5) in our main Theorem 6.1 is only used to conclude that problem (D_p) has no non trivial solutions on $[0,+\infty)$. Thus it can be replaced by any other condition which ensures this property and enlarges the set of parameters $\{\delta_i, p_i\}$ $i = 1,\dots,n$, for which the conclusion of Theorem 6.1 remains true. This remark will be illustrated in Theorem 6.15.*

6.4 Proof of Lemma 6.5

Throughout this section we will use freely the definition $\hat{H}$ for a function H that we gave in the Introduction.

To prove Lemma 6.5 we need some preliminary propositions. We begin by noting that the functions $\hat{\Phi}_i$, $\hat{F}_i$, $i = 1,\cdots,n$ defined in (6.25) are C^1 functions from $\mathbb{R}^+$ onto $\mathbb{R}^+$. Also $\hat{\Phi}_i$ is AH of exponent $p_i - 1 > 0$ at $+\infty$ and of exponent $\overline{p}_i - 1 > 0$ at zero and $\hat{F}_i$ is AH of exponent $\delta_i > 0$ at $+\infty$ and of exponent $\overline{\delta}_i > 0$ at zero. Furthermore $\hat{F}_i$ is strictly increasing in some interval of the form $(-t_1, t_1)$, $t_1 > 0$, and in some interval of the form $(t_2, +\infty)$, $t_2 \geq t_1$.

For $\boldsymbol{x} = (x_1,\dots,x_n) \in (\mathbb{R}^+)^n$, we have that solving (6.26) is equivalent to solving

$$\hat{F}_i(x_{i+1}) - \hat{\Phi}_i(x_i s)s = 0, \quad i = 1,\dots,n. \tag{6.56}$$

Proposition 6.9 *For each fixed $s > 0$ there exists a solution $\boldsymbol{x} \in (\mathbb{R}^+)^n$ of the system (6.56).*

Proof. Let us fix $s > 0$ and suppose that $\boldsymbol{x} \in (\mathbb{R}^+)^n$ is a solution to (6.56). We have that (6.56) is in turn equivalent to the system

$$x_i = \Psi_i(x_{i+1}) \quad i = 1,\dots,n, \tag{6.57}$$

where $\Psi_i(t) := \frac{1}{s}\hat{\Phi}_i^{-1}(\frac{\hat{F}_i(t)}{s})$ for $t \geq 0$, and $i = 1,\dots,n$. Here and in what follows, for simplicity of the notation we will not show the dependence on s. Hence the component x_1 of the solution satisfies $l(x_1) = 0$, where

$$l(t) = t - (\Psi_1 \circ \Psi_2 \circ \cdots \circ \Psi_n)(t), \tag{6.58}$$

Conversely, if x_1 satisfies $l(x_1) = 0$,, then by recursively defining $x_i = \Psi_i(x_{i+1})$, $i = 2, \ldots, n$, we find that $\boldsymbol{x} = (x_1, \ldots, x_n)$ satisfies (6.56). Thus we are led to study the zeros of the function l. Since the function Ψ_i is AH of exponent $\frac{\delta_i}{p_i - 1}$ at $+\infty$ and of exponent $\frac{\overline{\delta}_i}{\overline{p}_i - 1}$ at zero, $i = 1, \ldots, n$, we obtain that the function $\Psi_1 \circ \Psi_2 \ldots \circ \Psi_n$ is AH at $+\infty$ of exponent $\prod_{i=1}^{n} \frac{\delta_i}{p_i - 1}$ and at zero of exponent $\prod_{i=1}^{n} \frac{\overline{\delta}_i}{\overline{p}_i - 1}$. Thus for a given $\varepsilon > 0$ there are $t_2 > t_1 > 0$ and two positive constants $C_1 \equiv C_1(s)$ and $C_2 \equiv C_2(s)$ such that

$$\frac{l(t)}{t} \leq 1 - C_2 t^{\prod_{i=1}^{n} (\frac{\delta_i}{p_i - 1} - \varepsilon) - 1} \quad \text{for all} \quad t \geq t_2, \tag{6.59}$$

and

$$\frac{l(t)}{t} \geq 1 - C_1 t^{\prod_{i=1}^{n} (\frac{\overline{\delta}_i}{\overline{p}_i - 1} - \varepsilon) - 1} \quad \text{for all} \quad 0 < t \leq t_1. \tag{6.60}$$

Since by (H_2) and (H_4) we may choose $\varepsilon > 0$ such that

$$\prod_{i=1}^{n} \Big(\frac{\delta_i}{p_i - 1} - \varepsilon \Big) - 1 > 0 \quad \text{and} \quad \prod_{i=1}^{n} \Big(\frac{\overline{\delta}_i}{\overline{p}_i - 1} - \varepsilon \Big) - 1 > 0,$$

we have by (6.59) that $l(t) < 0$ for all large t and by (6.60) that $l(t) > 0$ for all small positive t. Thus the equation $l(t) = 0$ has at least one solution, which is what we wanted to prove. ■

We note that for each $s > 0$ the set of solutions of $l(t) = 0$ is bounded (the bound depending on s) but may not be a singleton. For $s > 0$ let us define $\beta_1(s) := \min\{t \mid l(t) = 0\}$ and $\gamma_1(s) := \max\{t \mid l(t) = 0\}$ (these max and min are reached), and define recursively $\beta_2(s)$ to $\beta_n(s)$ by $\beta_i(s) = \Psi_i(\beta_{i+1}(s))$, $\gamma_2(s)$ to $\gamma_n(s)$ by $\gamma_i(s) = \Psi_i(\gamma_{i+1}(s))$. Then, $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_n) : \mathbb{R}^+ \mapsto (\mathbb{R}^+)^n$, $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_n) : \mathbb{R}^+ \mapsto (\mathbb{R}^+)^n$ and $\boldsymbol{\beta}(s)$, $\boldsymbol{\gamma}(s)$ are solutions to (6.56) for each $s > 0$.

Proposition 6.10 *We have that*

(i) $\beta_i(s) \to +\infty$ *as* $s \to +\infty$ *for each* $i = 1, \ldots, n$.

(ii) *For each* $m > 0$ *there is a constant* $M = M(m)$ *such that for all* $0 < s \leq m$, *it follows that* $||\boldsymbol{\gamma}(s)|| \leq M$.

Proof. (i) We will show first that

$$\liminf_{s \to +\infty} \beta_i(s) > C \quad \text{for all} \quad i = 1, \ldots, n, \tag{6.61}$$

where C is a positive constant. Suppose by contradiction that for some $j \in \{1, \ldots, n\}$ there is a sequence $\{s_k\} \to +\infty$ such that $\beta_j(s_k) \to 0$. By (6.56),

$$\hat{\Phi}_i^{-1}(\hat{F}_i(\beta_{i+1}(s_k))) \geq \beta_i(s_k) \quad i = 1, \ldots, n, \tag{6.62}$$

for $s_k \geq 1$ and thus by an iteration process starting with $i = j-1$ we conclude that $\beta_i(s_k) \to 0$ for all $i = 1, \ldots, n$. Using now that the function $\hat{\Phi}_i^{-1} \circ \hat{F}_i$ is AH at 0 of exponent $\frac{\overline{\delta}_i}{\overline{p}_i - 1}$, we obtain from (6.62) that given $\varepsilon > 0$ small enough, there is a positive constant C such that

$$\Big(\beta_{i+1}(s_k)\Big)^{\frac{\overline{\delta}_i}{\overline{p}_i-1}-\varepsilon} \geq C\beta_i(s_k) \quad i = 1, \ldots, n,$$

and hence

$$\Big(\beta_j(s_k)\Big)^{\prod\limits_{i=1}^{n}(\frac{\overline{\delta}_i}{\overline{p}_i-1}-\varepsilon)-1} \geq C$$

for some other positive constant C. Since by (H_4) we may choose $\varepsilon > 0$ so that $\prod\limits_{i=1}^{n}(\frac{\overline{\delta}_i}{\overline{p}_i-1} - \varepsilon) > 1$, this is a contradiction and thus (6.61) holds. We conclude then that there are a positive constant C_1 and $s_0 > 0$ such that $\beta_i(s) \geq C_1$ for all $s \geq s_0$ and all $i = 1, \ldots, n$. Hence, $\hat{F}_i(\beta_{i+1}(s)) \geq s\hat{\Phi}_i(sC_1)$ for all $s \geq s_0$, which implies that $\lim\limits_{s\to+\infty} \beta_{i+1}(s) = +\infty$ and (i) is proved.

To show (ii) we assume there is an $m_1 > 0$ and a sequence $\{s_k\} \subset [0, m_1]$ such that $\gamma_j(s_k) \to +\infty$ as $k \to \infty$ for some component γ_j of $\boldsymbol{\gamma}(s) = (\gamma_1(s), \ldots, \gamma_n(s))$. Since by (6.56)

$$\frac{1}{m_1}\hat{\Phi}_i^{-1}\Big(\frac{1}{m_1}\Big(\hat{F}_i(\gamma_{i+1}(s_k))\Big)\Big) \leq \gamma_i(s_k) \quad i = 1, \ldots, n, \tag{6.63}$$

by iteration (starting with $i = j-1$) we find that $\gamma_i(s_k) \to +\infty$ for all $i = 1, \ldots, n$. Using now that $\hat{\Phi}_i^{-1}$ and $\hat{F}_i$ are AH at $+\infty$ of exponents $\frac{1}{p_i-1}$ and δ_i respectively, for $\varepsilon > 0$ small enough we obtain from (6.63) that $(\gamma_{j+1}(s_k))^{\frac{\delta_j}{p_j-1}-\varepsilon} \leq \tilde{C}\gamma_j(s_k)$, and thus by iterating, we conclude that $\Big(\gamma_{j+1}(s_k)\Big)^{\prod\limits_{j=1}^{n}(\frac{\delta_j}{p_j-1}-\varepsilon)-1} \leq C$ for k large, where $\tilde{C}$ and C are positive constants. Since by (H_2) we may choose $\varepsilon > 0$ such that $\prod\limits_{j=1}^{n}(\frac{\delta_j}{p_j-1} - \varepsilon) > 1$, we have reached a contradiction. Hence (ii) is proved and the proposition follows. ■

We begin now the proof of Lemma 6.5.

Proof of (i) of Lemma 6.5. Suppose first that there is a function $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) : \mathbb{R}^+ \mapsto (\mathbb{R}^+)^n$ such that $\boldsymbol{\alpha}(s)$ is a solution of class C^1 to (6.56) for s in some subinterval I of $\mathbb{R}^+$. Then, for $s \in I$, $\boldsymbol{\alpha}(s)$ satisfies

$$\hat{F}_i(\alpha_{i+1}(s)) - \hat{\Phi}_i(\alpha_i(s)s)s = 0, \quad i = 1, \ldots, n. \tag{6.64}$$

By differentiating with respect to s, we find that $\boldsymbol{\alpha}$ is a solution to the system of differential equations

$$a_i(s, \boldsymbol{\alpha}(s))\alpha_i'(s) - b_i(s, \boldsymbol{\alpha}(s))\alpha_{i+1}'(s) = -c_i(s, \boldsymbol{\alpha}(s)), \ i = 1, \ldots, n, \tag{6.65}$$

where $' = \frac{d}{ds}$, and

$$a_i(s, \boldsymbol{\alpha}) = s^2 \hat{\Phi}_i'(s\alpha_i), \; b_i(s, \boldsymbol{\alpha}) = \hat{F}_i'(\alpha_{i+1}) \; \text{ and } \; c_i(s, \boldsymbol{\alpha}) = \phi(s\alpha_i). \tag{6.66}$$

Conversely if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is a solution to (6.65) in I, then $\boldsymbol{\alpha}(s)$ satisfies

$$\hat{F}_i(\alpha_{i+1}(s)) - \hat{\Phi}_i(\alpha_i(s)s)s = C_i, \; i = 1, \ldots, n. \; \text{ for all } \; s \in I. \tag{6.67}$$

Hence if for some $s_0 \in I$ $(s_0, \boldsymbol{\alpha}(s_0))$ satisfies (6.56), then $(s, \boldsymbol{\alpha}(s))$ satisfies (6.56) for all $s \in I$. At this point the proof of (i) of Lemma 6.5 consists in showing that indeed the initial value problem

$$(IV) \quad \begin{cases} a_i(s, \boldsymbol{x}) x_i'(s) - b_i(s, \boldsymbol{x}) x_{i+1}' = -c_i(s, \boldsymbol{x}), & i = 1, \ldots, n, \\ x(s_0) = \boldsymbol{x}_0 \end{cases}$$

has a solution defined for all $s \geq s_0$, for some initial condition $(s_0, \boldsymbol{x}_0)$ which satisfies (6.56). Thanks to Proposition 6.9 we know that we can choose a pair $(s_0, \boldsymbol{x}_0)$ satisfying (6.56) for any $s_0 > 0$.

Observing that the system in (IV) has the form (6.76) in the Appendix, with $x_i'(s)$ in the place of X_i, we have that we can solve for the $x_i'(s)$ in any subset of $\mathbb{R}^+ \times (\mathbb{R}^+)^n$ where $\prod_{i=1}^n \frac{b_i(s,\boldsymbol{x})}{a_i(s,\boldsymbol{x})} \neq 1$ is satisfied. We will find next a point $(s_0, \boldsymbol{x}_0)$ and hence by continuity a neighborhood of this point where $\prod_{i=1}^n \frac{b_i(s,\boldsymbol{x})}{a_i(s,\boldsymbol{x})} \neq 1$ holds. To this end, let us define the lower and upper envelopes of $\hat{F}_i$ by

$$\hat{F}_i^-(x) = \inf_{s \in [0,x]} \hat{F}_i(s), \quad \hat{F}_i^+(x) = \sup_{s \in [0,x]} \hat{F}_i(s).$$

Then, v, $\hat{F}_i^-$, and $\hat{F}_i^+$ are nondecreasing and since $\hat{F}_i$ is ultimately increasing, there exists $\overline{m_1} > 0$ such that $\hat{F}_i^-(x) = \hat{F}_i^+(x) = \hat{F}_i(x)$ for all $x \geq \overline{m_1}$ and all $i = 1, \ldots, n$.

After computing the derivatives in (6.66), we find that

$$\prod_{i=1}^n \frac{b_i(s, \boldsymbol{x})}{a_i(s, \boldsymbol{x})} = D(s, \boldsymbol{x}) \prod_{i=1}^n \frac{F_i(x_{i+1})}{\Phi_i(sx_i)}, \tag{6.68}$$

where $D(s, \boldsymbol{x}) = \prod_{i=1}^n (\frac{x_{i+1} f_i(x_{i+1})}{F_i(x_{i+1})} - 1)(\frac{sx_i \phi_i(sx_i)}{\Phi_i(sx_i)} - 1)^{-1}$. Now, since the functions f_i, ϕ_i are respectively AH at $+\infty$ of exponent δ_i and $p_i - 1$, from (6.4) in the proof of Proposition 6.2, we have that for $\varepsilon > 0$ small there is an $m_1 \geq \overline{m_1}$ such that for all $\boldsymbol{x} = (x_1, \ldots, x_n)$ with $x_i \geq m_1$ and $s \geq 1$ it holds that

$$0 < \delta_i - \varepsilon \leq \frac{x_{i+1} f_i(x_{i+1})}{F_i(x_{i+1})} - 1 \leq \delta_i + \varepsilon$$

and

$$0 < p_i - 1 - \varepsilon \leq \frac{sx_i \phi_i(sx_i)}{\Phi_i(sx_i)} - 1 \leq p_i - 1 + \varepsilon,$$

for all $i = 1, \ldots, n$. Thus $D(s, \boldsymbol{x}) \geq \prod_{i=1}^{n} \frac{\delta_i - \varepsilon}{p_i - 1 + \varepsilon}$ for all $(s, \boldsymbol{x})$ in the set

$$\mathcal{S} := \{(s, \boldsymbol{x}) \mid s \geq 1, x_i \geq m_1, \quad i = 1, \ldots, n\}.$$

Since by (H_2) we may choose ε small enough so that $\prod_{i=1}^{n} \frac{\delta_i - \varepsilon}{p_i - 1 + \varepsilon} > 1$, we have that $D(s, \boldsymbol{x}) > 1$ for all $(s, \boldsymbol{x}) \in \mathcal{S}$. Then from (6.68), we find that

$$\prod_{i=1}^{n} \frac{b_i(s, \boldsymbol{x})}{a_i(s, \boldsymbol{x})} > \prod_{i=1}^{n} \frac{F_i(x_{i+1})}{\Phi_i(sx_i)} = \prod_{i=1}^{n} \frac{\hat{F}_i(x_{i+1})}{s\hat{\Phi}_i(sx_i)},$$

for all $(s, \boldsymbol{x}) \in \mathcal{S}$.

By using (i) of Proposition 6.10 we choose now $s_0 > 1$ such that $\beta_i(s) > m_1$ for all $i = 1, \ldots, n$ and all $s \geq s_0$ and set $\boldsymbol{x}_0 = \boldsymbol{\beta}(s_0)$. Then $(s_0, \boldsymbol{x}_0)$ satisfies (6.56) and $(s_0, \boldsymbol{x}_0) \in \mathrm{Int}\mathcal{S}$, implying that $\prod_{i=1}^{n} \frac{b_i(s_0, \boldsymbol{x}_0)}{a_i(s_0, \boldsymbol{x}_0)} > 1$. By continuity the same is true for $(s, \boldsymbol{x}) \in \Omega_0 := (s_0 - \mu_0, s_0 + \mu_0) \times B(\boldsymbol{x}_0, \varepsilon_0)$ for some small $\mu_0 > 0$ and $\varepsilon_0 > 0$ and where $B(\boldsymbol{x}_0, \varepsilon_0)$ is the ball in $\mathbb{R}^n$ centered at $\boldsymbol{x}_0$ and with radius ε_0. By using (6.76) in the Appendix we can solve for the derivatives x_i' in (IV) in terms of $(s, \boldsymbol{x}) \in \Omega_0$ to obtain the equivalent initial value problem

$$(IV_e) \quad \begin{cases} x_i' = \dfrac{\dfrac{c_i(s, \boldsymbol{x})}{a_i(s, \boldsymbol{x})} + \displaystyle\sum_{k=1}^{n-1} \left[\frac{c_{i+k}(s, \boldsymbol{x})}{a_{i+k}(s, \boldsymbol{x})} \prod_{\ell=0}^{k-1} \frac{b_{i+\ell}(s, \boldsymbol{x})}{a_{i+\ell}(s, \boldsymbol{x})} \right]}{\displaystyle\prod_{j=1}^{n} \frac{b_j(s, \boldsymbol{x})}{a_j(s, \boldsymbol{x})} - 1}, & i = 1, \ldots, n, \\ \boldsymbol{x}(s_0) = \boldsymbol{x}_0. \end{cases}$$

Since the right hand in the system in (IV_e) is continuous in Ω_0, by the theory of ordinary differential equation problem (IV_e) has a solution $\boldsymbol{\alpha} = (\alpha_1(s), \ldots, \alpha_n(s))$ defined in an interval $(s_0 - \gamma_0, s_0 + \gamma_0)$, with $\gamma_0 \leq \mu_0$ which can be extended to the right as a solution of (IV_e) (this extension is also denoted by $\boldsymbol{\alpha}$) to a maximal interval of existence of the form $[s_0, w)$.

We claim that $w = +\infty$. We argue by contradiction and so we assume $w < +\infty$. Indeed, since $\boldsymbol{\alpha}(s)$ satisfies (6.56), by the definition of the vector function $\boldsymbol{\beta}$ we have that

$$\begin{aligned} \hat{F}_1^+(\alpha_2(s)) \geq \hat{F}_1(\alpha_2(s)) &= s\hat{\Phi}_1(s\alpha_1(s)) \\ &\geq s\hat{\Phi}_1(s\beta_1(s)) \\ &= \hat{F}_1(\beta_2(s)) = \hat{F}_1^+(\beta_2(s)), \end{aligned}$$

and thus $\alpha_2(s) \geq \beta_2(s) > m_1$ for all $s \in [s_0, w)$. By iteration we conclude that $\alpha_i(s) > \beta_i(s) > m_1$ for all $i = 1, \ldots, n$ and all $s \in [s_0, w)$. Hence, $(s, \boldsymbol{\alpha}(s)) \in \mathrm{Int}\mathcal{S}$ for all $s \in [s_0, w)$. On the other hand by the choice of m_1 the function $\hat{F}_i(y)$ is strictly increasing for $y \in [m_1, +\infty)$, and thus it holds that $\hat{F}_i'(y) > 0$ for all $y \geq m_1$. Then from the definition of a_i, b_i, and c_i in (6.66) and the fact that $\hat{\Phi}_i'(x) > 0$ for all $x > 0$, we see that the numerator on the right hand side of the equations in (IV_e) is positive for all $(s, \boldsymbol{x}) \in \mathcal{S}$ and thus $\alpha_i'(s) > 0$ for all $s \in [s_0, w)$. Also, it can be easily verified that $\alpha_i(s) \leq \gamma_i(s)$ for all $s \in [s_0, w)$ and all $i = 1, \ldots, n$. Indeed, by the definition of $\hat{F}_i^-$,

$$\begin{aligned} \hat{F}_1^-(\alpha_2(s)) \leq \hat{F}_1(\alpha_2(s)) &= s\hat{\Phi}_1(s\alpha_1(s)) \\ &\leq s\hat{\Phi}_1(s\gamma_1(s)) \\ &= \hat{F}_1(\gamma_2(s)) = \hat{F}_1^-(\gamma_2(s)), \end{aligned}$$

and therefore by the monotonicity of $\hat{F}_1^-$, we have that $\alpha_2(s) \leq \gamma_2(s)$ for all $s \in [s_0, w)$, and thus iterating, we find that $\alpha_i(s) \leq \gamma_i(s)$ for all $s \in [s_0, w)$ and all $i = 1, \ldots, n$. Hence, by (ii) of Proposition 6.10, we obtain that $\alpha_i(s)$ is bounded in $[s_0, w)$ and then $\lim\limits_{s \to w^-} \boldsymbol{\alpha}(s) = \boldsymbol{d} = (d_1, \ldots, d_n)$ and $(w, \boldsymbol{d}) \in \mathcal{S}$. But from the continuity of the α_i's, and the fact that $\boldsymbol{\alpha}(s)$ satisfies (6.56), we obtain that $\prod\limits_{i=1}^{n} \frac{\hat{F}_i(d_{i+1})}{w\hat{\Phi}_i(wd_i)} = 1$ which implies $\prod\limits_{i=1}^{n} \frac{b_i(w,\boldsymbol{d})}{a_i(w,\boldsymbol{d})} > 1$. Hence we conclude that $\boldsymbol{\alpha}(s)$ can be extended to the right of w, a contradiction and our claim is proved.

Thus the domain of the solution $\boldsymbol{\alpha}$ to $(IV)_e$ is $[s_0, +\infty)$. Now for $i = 1, \ldots, n$, $\alpha_i'(s) > 0$, and $\alpha_i(s) \geq \beta_i(s)$, for all $s \geq s_0$. Hence by (i) of Proposition 6.10 $\alpha_i(s) \to +\infty$ as $s \to +\infty$. Then $\alpha_i : [s_0, +\infty) \to [\beta_i(s_0), +\infty)$, is a diffeomorphism onto $[\beta_i(s_0), +\infty)$, for each $i = 1, \ldots, n$. Also $(s, \boldsymbol{\alpha}(s))$ satisfies (6.56) for each $s \in [s_0, +\infty)$. This concludes the proof of (i) of Lemma 6.5. ■

Proof of (ii) of Lemma 6.5 By (6.56), for each $i = 1, \ldots, n$, we have that

$$\frac{F_i(\alpha_{i+1}(s))\alpha_i(s)}{\alpha_{i+1}(s)\Phi_i(\alpha_i(s)s)} = 1, \quad \text{for all} \quad s > s_0.$$

Since we can write

$$\frac{f_i(\alpha_{i+1}(s))}{s\phi_i(\alpha_i(s)s)} = \frac{f_i(\alpha_{i+1}(s))\alpha_{i+1}(s)}{F_i(\alpha_{i+1}(s))} \frac{F_i(\alpha_{i+1}(s))\alpha_i(s)}{\Phi_i(\alpha_i(s)s)\alpha_{i+1}(s)} \frac{\Phi_i(\alpha_i(s)s)}{s\phi_i(\alpha_i(s)s)\alpha_i(s)},$$

and

$$\lim_{s \to \infty} \frac{f_i(\alpha_{i+1}(s))\alpha_{i+1}(s)}{F_i(\alpha_{i+1}(s))} = \delta_i + 1 , \qquad \lim_{s \to \infty} \frac{\Phi_i(\alpha_i(s)s)}{\phi(\alpha_i(s)s)\alpha_i(s)s} = \frac{1}{p_i},$$

we have that (6.28) follows immediately. ■

Proof of (iii) of Lemma 6.5. We begin the proof by observing that if $h : \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable then by an obvious modification in Karamata's theorem, (see [69], page 17, Theorem 0.6), we have that

$$\lim_{s \to +\infty} \frac{sh'(s)}{h(s)} = E > 0, \quad \text{if and only if} \quad \lim_{s \to +\infty} \frac{h'(\sigma s)}{h'(s)} = \sigma^{E-1}, \tag{6.69}$$

for all $\sigma > 0$. Then, by L'Hôpital's rule, we find that

$$\lim_{s \to +\infty} \frac{h(\sigma s)}{h(s)} = \sigma^E, \quad \text{for all} \quad \sigma > 0. \tag{6.70}$$

From this observation the rest of the proof consists in showing that

$$\frac{s\alpha_i'(s)}{\alpha_i(s)} \to E_i, \quad \text{as} \quad s \to +\infty, \quad \text{for each } i = 1, \dots, n. \tag{6.71}$$

Since $\boldsymbol{\alpha}$ satisfies (6.65) and (6.66) for s large, by computing the derivatives of the coefficient functions in (6.65), we find that $\boldsymbol{\alpha}$ satisfies

$$A_i(s)\frac{s\alpha_i'(s)}{\alpha_i(s)} - B_i(s)\frac{s\alpha_{i+1}'(s)}{\alpha_{i+1}(s)} = -1, \quad \text{for } i = 1, \dots, n, \tag{6.72}$$

where

$$A_i(s) = \Big[1 - \frac{\Phi_i(s\alpha_i(s))}{s\alpha_i(s)\phi_i(s\alpha_i(s))}\Big], \quad B_i(s) = \frac{f_i(\alpha_{i+1}(s))}{s\phi_i(s\alpha_i(s))}\Big[1 - \frac{F_i(\alpha_{i+1}(s))}{\alpha_{i+1}(s)f_i(\alpha_{i+1}(s))}\Big],$$

for $i = 1, \dots, n$. We note that for each fixed s this system has the form (6.77) and thus it can be solved for $\frac{s\alpha_i'(s)}{\alpha_i(s)}$ if $\prod_1^n \frac{B_i(s)}{A_i(s)} \neq 1$. Furthermore using the AH properties of the ϕ_i's and f_i's functions it can be seen that there exists $s_0 > 0$ such that $\prod_1^n \frac{B_i(s)}{A_i(s)} \neq 1$, for all $s \geq s_0$ (we leave these calculations to the interested reader). Then, since $\lim_{s\to+\infty} A_i(s) = \frac{p_i - 1}{p_i}$ and $\lim_{s\to+\infty} B_i(s) = \frac{\delta_i}{p_i}$, by letting $s \to +\infty$ in (6.72), it follows that (6.71) holds true, concluding the proof of (iii) of Lemma 6.5. This in turn ends the proof of Lemma 6.5. ∎

6.5 Applications

In this section we wish to show by means of simple examples the applicability of our main theorem. We will denote by Ω the open ball, centered at 0 with radius $R > 0$ in $\mathbb{R}^N$.

Theorem 6.11 *Let ϕ, $\psi : \mathbb{R} \mapsto \mathbb{R}$ be defined by*

$$\phi(s) = |s|^{p_2-2}s + s\theta_1(s) + a|s|^{p_1-2}s, \quad p_2 > p_1 > 1,$$

$$\psi(s) = |s|^{q_2-2}s + s\theta_2(s) + b|s|^{q_1-2}s, \quad q_2 > q_1 > 1,$$

where a, b are positive constants, and for $i = 1, 2$, $\theta_i : \mathbb{R} \mapsto \mathbb{R}$, are even continuous functions with $0 \leq \theta_i(s)$, $s\theta_i(s)$ non decreasing for all $s > 0$, $\lim_{s\to 0} s\theta_i(s) = 0$, and such that

$$\lim_{s\to+\infty} \frac{s\theta_1(s)}{|s|^{p_2-2}s} = 0 \quad \textit{and} \quad \lim_{s\to 0} \frac{s\theta_1(s)}{|s|^{p_1-2}s} = 0,$$

$$\lim_{s\to+\infty} \frac{s\theta_2(s)}{|s|^{q_2-2}s} = 0 \quad \textit{and} \quad \lim_{s\to 0} \frac{s\theta_2(s)}{|s|^{q_1-2}s} = 0.$$

Let also $f, g : \mathbb{R} \mapsto \mathbb{R}$ *by odd continuous functions defined by*

$$f(s) = |s|^{\delta_2 - 1}s + \xi_1(s) + c|s|^{\delta_1 - 1}s, \quad \delta_2 > \delta_1 > 0,$$
$$g(s) = |s|^{\mu_2 - 1}s + \xi_2(s) + d|s|^{\mu_1 - 1}s, \quad \mu_2 > \mu_1 > 0,$$

where c, d *are positive constants and for* $i = 1, 2$, $\xi_i : \mathbb{R} \mapsto \mathbb{R}$, *are odd continuous (non necessarily increasing) functions such that* $0 \leq \xi_i(s)$, *for all* $s > 0$, *and*

$$\lim_{s \to +\infty} \frac{\xi_1(s)}{|s|^{\delta_2 - 1}s} = 0 \;\; \textit{and} \;\; \lim_{s \to 0} \frac{\xi_1(s)}{|s|^{\delta_1 - 1}s} = 0,$$

$$\lim_{s \to +\infty} \frac{\xi_2(s)}{|s|^{\mu_2 - 1}s} = 0 \;\; \textit{and} \;\; \lim_{s \to 0} \frac{\xi_2(s)}{|s|^{\mu_1 - 1}s} = 0.$$

Then, if $\max\{p_2, q_2\} < N$, $\frac{\delta_2 \mu_2}{(p_2 - 1)(q_2 - 1)} > 1$, $\frac{\delta_1 \mu_1}{(p_1 - 1)(q_1 - 1)} > 1$, *and*

$$\max\Big\{ \frac{p_2(q_2 - 1) + \delta_2 q_2}{\delta_2 \mu_2 - (p_2 - 1)(q_2 - 1)} - \frac{N - p_2}{p_2 - 1}, \quad \frac{q_2(p_2 - 1) + \mu_2 p_2}{\delta_2 \mu_2 - (p_2 - 1)(q_2 - 1)} - \frac{N - q_2}{q_2 - 1} \Big\} \geq 0, \tag{6.73}$$

the problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p_2 - 2}\nabla u) - \mathrm{div}(\theta_2(|\nabla u|)\nabla u) - a\mathrm{div}(|\nabla u|^{p_1 - 2}\nabla u) \\ \qquad = |v(x)|^{\delta_2 - 1}v(x) + \xi_1(v(x)) + c|v(x)|^{\delta_1 - 1}v(x), \quad x \textit{ in } \Omega \\ -\mathrm{div}(|\nabla v|^{q_2 - 2}\nabla v) - \mathrm{div}(|\theta_2(|\nabla v|)\nabla v) - b\mathrm{div}(|\nabla u|^{q_1 - 2}\nabla v) \\ \qquad = |u(x)|^{\mu - 1}u(x) + \xi_2(u(x)) + d|u(x)|^{\mu_1 - 1}u(x), \quad x \textit{ in } \Omega \\ \qquad u(x) = v(x) = 0, \quad x \in \partial\Omega, \end{cases}$$

has a componentwise positive radial solution (u, v) *of class* C^1.

Proof. It can be easily shown that the function ϕ is AH of exponent $p_2 - 1$ at $+\infty$ and of exponent $p_1 - 1$ at zero, while ψ is AH of exponent $q_2 - 1$ at $+\infty$ and of exponent $q_1 - 1$ at zero. Also, the function f is AH of exponent δ_2 at $+\infty$ and of exponent δ_1 at zero, while g is AH of exponent μ_2 at $+\infty$ and of exponent μ_1 at zero. It only remains to show that condition (H_5) is fulfilled. Indeed, in this case system (AS) is given by

$$\begin{aligned} (p_2 - 1)E_1 - \delta_2 E_2 &= -p_2 \\ -\mu_2 E_1 + (q_2 - 1)E_2 &= -q_2, \end{aligned}$$

and thus

$$E_1 = \frac{p_2(q_2 - 1) + \delta_2 q_2}{\delta_2 \mu_2 - (p_2 - 1)(q_2 - 1)} \quad \text{and} \quad E_2 = \frac{q_2(p_2 - 1) + \mu_2 p_2}{\delta_2 \mu_2 - (p_2 - 1)(q_2 - 1)}.$$

Also,

$$\theta_1 = \frac{N-p_2}{p_2-1} \quad \text{and} \quad \frac{N-q_2}{q_2-1},$$

and thus (H_5) is given by (6.73) and the result follows directly from Theorem 6.1. ■

Remark 6.12 *A particular but illustrative case for the functions θ_i is given by*

$$\theta_1(s) = \sum_{j=1}^{m_1} b_j|s|^{\alpha_j-2}, \qquad \theta_2(s) = \sum_{j=1}^{m_2} c_j|s|^{\beta_j-2},$$

where $b_j \geq 0,\ \alpha_j \in (p_1, p_2),\ j = 1, \ldots, m_1$ and $c_j \geq 0,\ \beta_j \in (q_1, q_2),\ j = 1, \ldots, m_2$. Thus

$$\phi(s) = |s|^{p_2-2}s + \sum_{j=1}^{m_1} b_j|s|^{\alpha_j-2}s + a|s|^{p_1-2}s$$

and

$$\psi(s) = |s|^{q_2-2}s + \sum_{j=1}^{m_2} c_j|s|^{\beta_j-2}s + b|s|^{q_1-2}s.$$

In the next example we show that our method allows us to find existence of positive solutions to some Δ_p, Δ_q systems.

For $p,\ q > 1,\ n,\ m \in \mathbb{N}$ and $\mu,\ \delta$ positive real numbers such that $\mu\delta \neq (p-1)^n(q-1)^m$, define

$$A := \frac{\delta q Q_m + p(q-1)^m P_n}{\mu\delta - (p-1)^n(q-1)^m}, \quad B := \frac{\mu p P_n + q(p-1)^n Q_m}{\mu\delta - (p-1)^n(q-1)^m},$$

where for any $k \in \mathbb{N}$, $P_k = \sum_{i=1}^{k}(p-1)^{i-1}$ and $Q_k = \sum_{i=1}^{k}(q-1)^{i-1}$. We have the following existence result.

Theorem 6.13 *Let $f,\ g : \mathbb{R} \mapsto \mathbb{R}$ be odd continuous functions such that f is AH at $+\infty$ of exponent $\delta > 0$ and AH at 0 of exponent $\overline{\delta} > 0$, g is AH at $+\infty$ of exponent $\mu > 0$ and AH at 0 of exponent $\overline{\mu} > 0$. Assume also that $\min\{\mu\delta,\ \overline{\mu}\overline{\delta}\} > (p-1)^n(q-1)^m$. Then, if $N > \max\{p,\ q\}$ and*

$$\max\Big\{A - \frac{N-p}{p-1},\ (p-1)^{n-1}A + pP_{n-1} - \frac{N-p}{p-1},$$
$$B - \frac{N-q}{q-1},\ (q-1)^{m-1}B + qQ_{m-1} - \frac{N-q}{q-1}\Big\} \geq 0, \qquad (6.74)$$

the problem

$$(S) \quad \begin{cases} (-\Delta_p)^n u = f(v); \quad (-\Delta_q)^m v = g(u), & \text{in } \Omega \\ ((\Delta_p)^i u)(x) = ((\Delta_q)^j v)(x) = 0,\ i = 0, 1, \ldots, n-1, \\ \qquad j = 0, 1, \ldots, m-1,\ x \in \partial\Omega, \end{cases}$$

has a nontrivial radially symmetric solution $(u(x), v(x))$ *such that* $u(x) > 0$ *and* $v(x) > 0$ *for all* $x \in \Omega$.

Proof. We apply the result in Theorem 6.1 to the problem

$$(SS)\quad \begin{cases} -\Delta_p u_i = u_{i+1},\ i = 1,\dots,n-1; \quad -\Delta_p u_n = f(u_{n+1}), & \text{in } \Omega \\ -\Delta_q u_{n+j} = u_{n+j+1},\ j = 1,\dots,m-1, & \text{in } \Omega \\ -\Delta_q u_{n+m} = g(u_1) \quad \text{in } \Omega \\ u_i(x) = 0,\ i = 1,\dots,n+m, \quad x \in \partial\Omega. \end{cases}$$

By a solution to (SS) we mean a vector function $(u_1(x), \dots, u_{n+m}(x))$, $x \in \overline{\Omega}$, that satisfies (SS). Indeed, in this case the functions ϕ_i, f_i defined by

$$\phi_i(t) = \begin{cases} |t|^{p-2}t & \text{for } i = 1.\dots,n \\ |t|^{q-2}t & \text{for } i = n+1,\dots,n+m \end{cases}$$

$$f_i(t) = \begin{cases} t & \text{for } i = 1.\dots,n-1, n+1,\dots,n+m-1 \\ f(t) & \text{for } i = n \\ g(t) & \text{for } i = n+m, \end{cases}$$

satisfy the hypotheses of the theorem with $\delta_i = 1$ for $i = 1,\dots,n-1, n+1,\dots,n+m-1$, $\delta_n = \delta$, $\delta_{n+m} = \mu$, $\overline{\delta}_i = 1$ for $i = 1,\dots,n-1,n+1,\dots,n+m-1$, $\overline{\delta}_n = \overline{\delta}$, $\overline{\delta}_{n+m} = \overline{\mu}$, $\overline{p}_i = p_i = p$ for $i = 1,\dots,n$ and $\overline{q}_i = q_i = q$ for $i = n+1,\dots,n+m$. Furthermore, system (AS) for this problem is given by

$$(p-1)E_i - E_{i+1} = -p,\ i = 1,\dots,n-1,\ (p-1)E_n - \delta E_{n+1} = -p$$
$$(q-1)E_{n+i} - E_{n+i+1} = -q,\ i = 1,\dots,m-1,\ (q-1)E_{n+m} - \mu E_1 = -q$$

which has the unique solution $(E_1,\dots,E_{n+m})$ given by $E_1 = A$, $E_{n+1} = B$, $E_i = (p-1)^{i-1}A + pP_{i-1}$, $i = 2,\dots,n$, $E_{n+i} = (q-1)^{i-1}B + qQ_{i-1}$, $i = 2,\dots,m$. Also, $\theta_i = \frac{N-p}{p-1}$ for $i = 1,\dots,n$ and $\theta_i = \frac{N-q}{q-1}$ for $i = n+1,\dots,n+m$. Since, as it can be checked, either $E_1 \le \dots \le E_n$ or $E_1 \ge \dots \ge E_n$, and $E_{n+1} \le \dots \le E_{n+m}$ or $E_{n+1} \ge \dots \ge E_{n+m}$, we see that hypothesis (6.74) corresponds to hypothesis (H_5) in Theorem 6.1. Hence, according to that theorem, for $N > \max\{p,q\}$, (SS) will have a radial solution which is positive componentwise in Ω. The result follows now by setting $u(x) = u_1(x)$ and $v(x) = u_{n+1}(x)$. ■

Remark 6.14 *It is interesting to note that if*

$$pP_{n-1} - \frac{N-p}{p-1} \ge 0 \quad \text{or} \quad qQ_{m-1} - \frac{N-q}{q-1} \ge 0$$

then (6.74) is automatically satisfied. Thus for instance, if $p = q = 2$, n, $m > 1$, *then* $P_{n-1} = n-1$ *and* $Q_{m-1} = m-1$, *and we have that the problem*

$$\begin{cases} (-\Delta)^n u = f(v); \quad (-\Delta)^m v = g(u), \quad \text{in } \Omega \\ u(x) = ((\Delta)^i u)(x) = v(x) = ((\Delta)^j v)(x) = 0 \quad x \in \partial\Omega, \\ i = 1,\dots,n-1, \quad j = 1,\dots,m-1, \end{cases}$$

has a non trivial radial positive componentwise solution (u, v) *whenever*

$$\max\{2n,\ 2m\} \geq N > 2$$

for any choice of $\mu, \overline{\mu}, \delta, \overline{\delta}$ *satisfying* $\mu\delta > 1$ *and* $\overline{\mu}\overline{\delta} > 1$.

Our last application illustrates the Remark 3.1 following the proof of Theorem 6.1. It is known from [73], Theorem 1.1, that the problem

$$(DD) \quad \begin{cases} -\Delta u = |v|^{\delta-1}v; \quad -\Delta v = |u|^{\mu-1}u, \quad \text{in } \mathbb{R}^N \\ \quad u(x) \geq 0, \quad v(x) \geq 0, \quad x \in \mathbb{R}^N, \end{cases}$$

where $\delta > 0$, $\mu > 0$ does not possess non trivial positive radially symmetric solution if

$$\frac{N}{\delta+1} + \frac{N}{\mu+1} > N - 2,$$

and thus we have the following existence result.

Theorem 6.15 *Let* $f,\ g : \mathbb{R} \mapsto \mathbb{R}$ *be odd continuous functions such that* f *is AH at* $+\infty$ *of exponent* $\delta > 0$ *and AH at 0 of exponent* $\overline{\delta} > 0$, g *is AH at* $+\infty$ *of exponent* $\mu > 0$ *and AH at 0 of exponent* $\overline{\mu} > 0$ *with* $\mu\delta > 1$. *Let also* $\overline{p},\ \overline{q} > -1$ *be such that* $\overline{\delta}\overline{\mu} > (\overline{p}+1)(\overline{q}+1)$. *Then, if*

$$\frac{N}{\delta+1} + \frac{N}{\mu+1} > N - 2, \tag{6.75}$$

the problem

$$(DL) \quad \begin{cases} -\mathrm{div}((\log(1+|\nabla u|))^{\overline{p}}\nabla u) = f(v), \ x \in \Omega \\ \\ -\mathrm{div}((\log(1+|\nabla u|))^{\overline{q}}\nabla v) = g(u), \ x \in \Omega \\ \\ u(x) = v(x) = 0 \quad x \in \partial\Omega, \end{cases}$$

has a non trivial radially symmetric solution (u, v) *such that* $u(x) > 0$ *and* $v(x) > 0$ *for all* $x \in \Omega$.

Proof. For this problem we have that $\phi_1(s) = (\log(1+|s|))^{\overline{p}}s$ and $\phi_2(s) = (\log(1+|s|))^{\overline{q}}s$ are AH at $+\infty$ of exponent 1 and AH at 0 of exponents $\overline{p}+1$ and $\overline{q}+1$ respectively. Moreover, the limiting problem at infinity is (DD) and thus the result follows. ■

Remark 6.16 *We observe that for* (DL) *condition* (H_5) *of Theorem 6.1 becomes*

$$\max\Big\{\frac{\delta+1}{\delta\mu-1}, \frac{\mu+1}{\delta\mu-1}\Big\} \geq N - 2.$$

Thus condition (6.75) above improves the set of δ, μ *values for which existence of positive solutions is guaranteed by Theorem 6.1.*

Finally, for related existence results of positive solutions for the case $\overline{p} = \overline{q} = 0$, *in* (DL), *see [67] and [75].*

6.6 Appendix

Here we briefly consider the solutions to the system (AS), which for convenience of the reader we repeat here.

$$(AS) \quad (p_i - 1)E_i - \delta_{i+1}E_{i+1} = -p_i \quad \text{for} \quad i = 1, \dots, n.$$

This system is a particular case of the system

$$a_i X_i - b_i X_{i+1} = -c_i, \quad \text{for} \tag{6.76}$$

where a_i, b_i, c_i are constants and which has played an important role in this paper.

It can be easily verified that if $a_i \neq 0$, $i = 1, \dots, n$ and $\prod_{i=1}^{n} \frac{b_i}{a_i} \neq 1$, then (6.76) has the unique solution $\mathbf{X}= (X_1, \dots, X_n)$

$$X_i = \frac{\dfrac{c_i}{a_i} + \displaystyle\sum_{k=1}^{n-1}\Big[\frac{c_{i+k}}{a_{i+k}} \prod_{\ell=0}^{k-1} \frac{b_{i+\ell}}{a_{i+\ell}}\Big]}{\displaystyle\prod_{j=1}^{n} \frac{b_j}{a_j} - 1}, \qquad i = 1, \dots, n, \tag{6.77}$$

with the usual convention that $a_{n+k} = a_k$, $b_{n+k} = b_k$ and $c_{n+k} = c_k$ for $k = 1, \dots, n$. Clearly if $a_i > 0, b_i > 0, c_i > 0$, $i = 1, \dots, n$, then $X_i > 0$ for all $i = 1, \dots, n$. Hence, if $p_i > 1$, $\delta_i > 0$ and $\prod_{i=1}^{n} \delta_i > \prod_{i=1}^{n} (p_i - 1)$ then (AS) has the unique solution $\mathbf{E}$ $= (E_1, \dots, E_n)$

$$E_i = \frac{\dfrac{p_i}{p_i - 1} + \displaystyle\sum_{k=1}^{n-1}\Big[\frac{p_{i+k}}{p_{i+k} - 1} \prod_{\ell=0}^{k-1} \frac{\delta_{i+\ell}}{p_{i+\ell} - 1}\Big]}{\displaystyle\prod_{j=1}^{n} \frac{\delta_j}{p_j - 1} - 1}, \qquad i = 1, \dots, n, \tag{6.78}$$

such that $E_i > 0$, $i = 1, \dots, n$.

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Summary

In this thesis we consider three mathematical problems. These problems have their origins in different physical problems yielding certain classes of multidimensional nonlinear elliptic and parabolic PDE's. The three-dimensional setting is the main topic of this research and radial symmetry plays a special role.

Chapter 1 contains model derivations and gives an overview of the results presented in Chapters 2–6, contrasting them with the existing theory.

Chapters 2–4 concern a model of gravitational interaction and diffusion of particles, confined to a three-dimensional region. The system conserves mass and energy and has the density of particles and temperature as unknowns. We call this the non-isothermal model. To study this model, the fixed temperature (isothermal) model is revisited.

In Chapter 2 we give conditions for the non-isothermal problem to have a unique solution. In addition conditions for global existence and finite time existence are given.

In Chapter 3 we present criteria on the data of the non-isothermal problem which ensure the convergence of global solutions towards stationary states.

Chapter 4 deals with finite time solutions for the isothermal and the non-isothermal model. We see that for certain initial data the behaviour of the solutions of the isothermal model is given by self-similar solutions. Numerical results are obtained to describe the blow-up behaviour in the non-isothermal case.

In Chapter 5 we study a problem that arises from the injection of water with a reactive solute from a well into the ground. We consider a radial flow profile and see that the large time behaviour of solutions is given by a self-similar solution. We also study the behaviour as the well size goes to zero.

Chapter 6 studies a quasilinear elliptic system. We prove that this problem has at least one nontrivial radially symmetric solution. As a corollary, this result gives existence of radial solutions for higher-order nonlinear partial differential equations.

Chapters 2–6 are transcriptions of articles.

Chapter 2: Global existence conditions for a non-local problem arising in statisti-

cal mechanics, submitted to Advances in Differential Equations, with C. J. van Duijn (TU/e) and M. A. Peletier (CWI and TU/e);

Chapter 3: Convergence to a stationary solution for a model arising in statistical mechanics, in preparation, with T. Nadzieja (Zielona Góra, Poland);

Chapter 4: Asymptotic self-similar blow-up for two models arising in statistical mechanics, in preparation, with M.A. Peletier and J. Williams (Bath University, England);

Chapter 5: Asymptotic results for injection of reactive solutes from a three-dimensional well, Journal of Mathematical Analysis and Applications **280** (2000), 367–385, with C. J. van Duijn and M. A. Peletier;

Chapter 6: Existence of positive radial solutions for a weakly coupled system via blow up, Abstract and Applied Analysis **3**(1–2) (1998), 105–131, with R. Manásevich (Universidad de Chile) and M. García-Huidobro (P. Universidad Católica, Chile).

Samenvatting

In dit proefschrift beschouwen we drie problemen die behoren tot een klasse van meerdimensionale niet-lineaire elliptische en parabolische partiële differentiaalvergelijkingen. Deze vinden hun oorsprong in diverse natuurkundige problemen. Het hoofdonderwerp van dit onderzoek is het drie-dimensionale geval waarin radiële symmetrie een bijzondere rol speelt.

In Hoofdstuk 1 worden de modellen geïntroduceerd en wordt een overzicht van de resultaten zoals die naar voren komen in de overige hoofdstukken gegeven. We vergelijken de resultaten met bestaande theorie.

Hoofdstukken 2–4 hebben betrekking op een model van zwaartekrachtinteractie en diffusie van deeltjes in een drie-dimensionaal gebied. Het systeem behoudt massa en energie en de dichtheid van deeltjes en de temperatuur zijn de onbekenden. We noemen dit het niet-isotherme model. Om dit model te bestuderen, bekijken wij het probleem met constante temperatuur (het isotherme model) opnieuw.

In Hoofdstuk 2 leiden we voorwaarden af voor het niet-isotherme probleem die garanderen dat dit een unieke oplossing heeft. Voldoende voorwaarden worden afgeleid voor oplossingen die voor alle tijd bestaan en voor oplossingen die in eindige tijd opblazen (finite time blow-up).

In Hoofdstuk 3 gebruiken we dit resultaat om globale oplossingen en de convergentie naar stationaire oplossingen te bestuderen.

Hoofdstuk 4 behandelt blow-up-gedrag van oplossingen van het isotherme en het niet-isotherme model. We laten zien dat dit gedrag wordt gegeven door een gelijkvormigheidoplossing. In het isotherme geval geven we een analytisch bewijs en voor het niet-isotherme geval een numeriek resultat.

Hoofdstuk 5 bestudeert een probleem dat ontstaat bij het injecteren van water met reactieve stoffen in de grond via een bron. Hierbij ontstaat een radieel stromingsprofiel. We laten zien dat hier het lange-termijn gedrag van de oplossing wordt gegeven door een gelijkvormigheidoplossing. Ook bestuderen we de gedrag als de maat van de bron naar nul gaat.

In Hoofdstuk 6 bestuderen we een quasi-lineair elliptisch systeem. We bewijzen dat dit probleem een niet-triviale radieel-symmetrische oplossing heeft. Een gevolg van dit resultaat is het bestaan van radiële oplossingen voor hogere-orde niet-lineaire partiële differentiaalvergelijkingen.

Hoofdstukken 2–6 zijn transcripties van artikelen:

Hoofdstuk 2: Global existence conditions for a non-local problem arising in statistical mechanics, ter publicatie aangeboden in Advances in Differential Equations, met C. J. van Duijn (TU/e) en M. A. Peletier (CWI and TU/e);

Hoofdstuk 3: Convergence to stationary solutions in a model of self-graviting systems, in voorbereiding, met T. Nadzieja (Zielona Góra, Poland);

Hoofdstuk 4: Asymptotic self-similar blow-up for two models arising in statistical mechanics, in voorbereiding, met M.A. Peletier en J. Williams (Bath university, England);

Hoofdstuk 5: Asymptotic results for injection of reactive solutes from a three-dimensional well, Journal of Mathematical Analysis and Applications **280** (2000), 367–385, met C. J. van Duijn en M. A. Peletier;

Hoofdstuk 6: Existence of positive radial solutions for a weakly coupled system via blow up, Abstract and Applied Analysis **3**(1–2) (1998), 105–131, met R. Manásevich (Universidad de Chile) en M. García-Huidobro (P. Universidad Católica, Chile).

Acknowledgements

I would like to thank all the people who contribute to the realization of this thesis. In first place my thanks to the Centrum voor Wiskunde en Informatica, for giving the financial support for this doctorate research. I also thank the department of Mathematical Engineering of University of Chile and Eindhoven University of Technology.

I would like to express my deepest gratitude to my promotor prof.dr.ir C.J. van Duijn for his patience, guidance, thoughtful suggestions and continuous encouragement. My co-promotor dr. M. A. Peletier has been very supportive and I thank him for the many useful discussions that we had along these years and for his brilliant suggestions and contributions to this work. Special thanks to prof.dr. J. Hulshof and prof.dr.ir. L.A. Peletier for their helpful comments.

My warm thanks go to Raúl Manásevich and Marta García-Huidobro for their contribution to this work, continuous support and guidance at the beginning of my career. I am indebted to the group of Nonlinear Analysis from the Mathematical Engineering department at University of Chile, who provided me with the tools to carry out this research.

I am grateful to Piotr Biler, Tadeusz Nadzieja, and Grzegorz Karch for useful discussions and their contribution to the understanding of the last part of this work. I am very grateful to them for the time I spent in Wroclaw. I hope that this thesis can provide the start for a fruitful cooperation.

I gratefully acknowledge Jeff Williams, who has contributed in great deal to the numerical work presented in this thesis. I also thank Profs. J. King and D. Hilhorst for their hospitality during my visit to their respective universities.

I thank my parents, my brother and sisters, for their constant support. Finally, my special thanks to my wife Pilar, for her support, patience, and understanding throughout the years of this research.

Curriculum Vitae

Ignacio Guerra was born on August 12, 1974 in Santiago, Chile. After completing the secondary school in Caracas (Venezuela) and Santiago (Chile) from March 1992 until May 1998 he studied Mathematical Engineering at the University of Chile in Santiago. He graduated with a thesis in Nonlinear Partial Differential Equations under the supervision of prof. Raúl Manásevich.

From June 1998 until January 2003 he carried out a doctoral research at the Centrum voor Wiskunde en Informatica (CWI) in Amsterdam. The research project was supported by CWI and was entitled "Partial Differential Equations in Porous Media Research". In this function he carried out this research under the supervision of prof. dr. ir. C.J. van Duijn.

Stellingen

bij het proefschrift

Stabilization and Blow-up for some Multidimensional Nonlinear PDE's

door

Ignacio Antonio Guerra Benavente

I

While studying parabolic equations it is useful to introduce and make use of several function spaces.

Olga Aleksandrovna Ladyženskaja (1985)

See results on local existence in Chapters 2 and 5.

II

The time evolution of a cluster of self-gravitating particles following Brownian motion -starting from a three-dimensional constant distribution- has a very similar behaviour, whether we consider the constant temperature model or the energy conservation model. See results in Chapters 2–4.

III

For nonlinear equations, such as the Navier-Stokes equations, it is known that a regular solution for the nonstationary problem need not to exist for all times $t \geq 0$. At some finite time the solution may go to infinity or loose its regularity. Even if the solution exists for all $t \geq 0$, it need not converge towards the solution of the stationary problem as $t \longrightarrow +\infty$, when the boundary conditions and the forces converge towards a stationary situation.

Olga Aleksandrovna Ladyženskaja (1970)

IV

Blow-up is a property that cannot be expected to take place for the magnitudes that describe the behaviour of biological or physical systems. Usually, in many problems in applied mathematics blow-up occurs only for some approximation of the real problem, and it indicates the presence not of a real singularity, but rather of a change in the orders of magnitude of the values of some quantity that characterizes the state of a system. For instance, this is a well established fact in combustion theory, where blow-up just means that the values of the temperature and other physical magnitudes rise several orders of magnitude when ignition takes place (cf. [1]).

Extract from an article of J.J.L. Velázquez.

[1] A. Liñán and F.A. Williams, Fundamental aspects of combustion, Oxford University Press, (1993).

V

There exists a strong resemblance between the argument used to prove an a-priori bound of solutions for the system studied in Chapter 6 (cf. [2]), and the argument to prove the following result found in [3] (see also [4]). Let Ω be a bounded convex domain in $\mathbb{R}^N$, or $\Omega = \mathbb{R}^N$, let $p > 1$, and assume that u solves

$$\begin{aligned} u_t &= \Delta u + u^p \quad \text{in} \quad \Omega \times (0,T), \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(0) &= u_0 \geq 0 \quad \text{in} \quad \Omega, \end{aligned}$$

and blows up at $t = T$. Then

$$\sup_{\Omega\times[0,T)} u(x,t)(T-t)^{1/(p-1)} < \infty$$

provided that $p < \frac{N+2}{N-2}$ or $N \leq 2$.

[2] B. Gidas, and J. Spruck, A-priori bounds for positive solutions of nonlinear elliptic Equations, *Comm. in P.D.E.* **6** (1981), 883-901.
[3] Y. Giga and R. Kohn, Characterizing blowup using similarity variables, *Indiana Univ. Math. J.* **36** (1987), 1-40.
[4] F. B. Weissler, An L^∞ blow-up estimate for a nonlinear heat equation, *Comm. Pure Appl. Math.* **38** (1985), 291-296.

VI

In principle, the matter is a conscious matter, but it is required a very developed organic existence to be able to cross -the threshold beyond of which it can show itself as a consciousness.

Theilhard de Chardin

VII

The world is made of three infinities, the infinitely big, the infinitely small and the infinitely complex.

Theilhard de Chardin

VIII

We can have a non-consequent way of thinking, but we have to know exactly when we are not being consistent.

IX

Time travelling is unlikely to happen because if it were possible then someone from the future would have already travelled to the past.

X

The closest we can get to immortality is to pass our genes from generation to generation, which is guaranteed if we have at least two offspring.

XI

Science also suffers natural selection, some disciplines go extinct, and others evolve and new branches are created. It is very little what we can do to change the course of this natural selection.