Tensor rank is not multiplicative under the tensor product

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A B S T R A C T

The tensor rank of a tensor t is the smallest number r such that t can be decomposed as a sum of r simple tensors. Let s be a k-tensor and let t be an ℓ-tensor. The tensor product of s and t is a $(k+\ell)$-tensor. Tensor rank is sub-multiplicative under the tensor product. We revisit the connection between restrictions and degenerations. A result of our study is that tensor rank is not in general multiplicative under the tensor product. This answers a question of Draisma and Saptharishi. Specifically, if a tensor t has border rank strictly smaller than its rank, then the tensor rank of t is not multiplicative under taking a sufficiently hight tensor product power. The "tensor Kronecker product" from algebraic complexity theory is related to our tensor product but different, namely it multiplies two k-tensors to get a k-tensor. Nonmultiplicativity of the tensor Kronecker product has been known since the work of Strassen.

It remains an open question whether border rank and asymptotic rank are multiplicative under the tensor product. Interestingly, lower bounds on border rank obtained from general-

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ized flattenings (including Young flattenings) multiply under the tensor product.

1. Introduction

Let U_i, V_i be finite-dimensional vector spaces over a field $\mathbb{F}$. Let t be a k-tensor in $U_1 \otimes \cdots \otimes U_k$. The *tensor rank* of t is the smallest number r such that t can be written as a sum of r simple tensors $u_1 \otimes \cdots \otimes u_k$ in $U_1 \otimes \cdots \otimes U_k$, and is denoted by $\mathrm{R}(t)$. Letting $\mathbb{F}$ be the complex numbers $\mathbb{C}$, the *border rank* of t is the smallest number r such that t is a limit point (in the Euclidean topology) of a sequence of tensors in $U_1 \otimes \cdots \otimes U_k$ of rank at most r, and is denoted by $\underline{\mathrm{R}}(t)$.

Let $t \in U_1 \otimes \cdots \otimes U_k$ and $s \in V_1 \otimes \cdots \otimes V_\ell$ be a k-tensor and an ℓ-tensor respectively. Define the *tensor product* of t and s as the $(k+\ell)$-tensor

$$t \otimes s \ \in \ U_1 \otimes \cdots \otimes U_k \otimes V_1 \otimes \cdots \otimes V_\ell.$$

If $k = \ell$, then define the *tensor Kronecker product* of t and s as the k-tensor

$$t \boxtimes s \ \in \ (U_1 \otimes V_1) \otimes \cdots \otimes (U_k \otimes V_k)$$

obtained from $t \otimes s$ by grouping U_i and V_i together for each i. In algebraic complexity theory, the tensor Kronecker product is usually just denoted by '$\otimes$'. Using the tensor Kronecker product one defines the *asymptotic rank* of t as the limit $\lim_{n \to \infty} \mathrm{R}(t^{\boxtimes n})^{1/n}$. (This limit exists and equals the infimum $\inf_n \mathrm{R}(t^{\boxtimes n})^{1/n}$, see for example Lemma 1.1 in [1].) Asymptotic rank is denoted by $\underline{\mathrm{R}}(t)$.

This paper is about the relationship between tensor rank and the tensor product. It follows from the definition that rank is sub-multiplicative under the tensor product.

Proposition 1. *Let t, s be any tensors. Then, $\mathrm{R}(t \otimes s) \leq \mathrm{R}(t)\mathrm{R}(s)$.*

The result of this paper is that the above inequality can be strict.

Theorem. *Tensor rank is not in general multiplicative under tensor product. Specifically, if a tensor t has border rank strictly smaller than its tensor rank, then the tensor rank of t is not multiplicative under a taking a sufficiently high tensor power.*

The theorem answers a question posed in the lecture notes of Jan Draisma [2, Chapter 6] and a question of Ramprasad Saptharishi (personal communication, related to an earlier version of the survey [3]). The theorem was stated as a fact in [4, page 1097], referring to [5] for the proof; however, [5] studies only the tensor Kronecker product $\boxtimes$.

It has been known since the work of Strassen that tensor rank is not multiplicative under the tensor Kronecker product $\boxtimes$, see Example 3.

We construct three instances of this phenomenon (Proposition 13, Proposition 17 and Proposition 18) to prove the theorem. Explicitly, one of our examples is the following strict inequality (Proposition 14).

Example 2. Let b_1, b_2 be the standard basis of $\mathbb{C}^2$. Define the 3-tensor W_3 as $b_2 \otimes b_1 \otimes b_1 + b_1 \otimes b_2 \otimes b_1 + b_1 \otimes b_1 \otimes b_2 \in (\mathbb{C}^2)^{\otimes 3}$. Then we have the strict inequality $\mathrm{R}(W_3^{\otimes 2}) \leq 8 < 9 = \mathrm{R}(W_3)^2$.

In Section 5 we will prove that Example 2 is essentially minimal over the complex numbers, in the sense that if $s \in \mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and $t \in \mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^m$, then one has $\mathrm{R}(s \boxtimes t) = \mathrm{R}(s \otimes t) = \mathrm{R}(s)\mathrm{R}(t)$. This we prove using the theory of canonical forms of matrix pencils and a formula for their tensor rank.

Our general approach is to study approximate decompositions (or border rank decompositions) of tensors. It turns out that a border rank decomposition of a tensor t can be transformed into a tensor rank decomposition of tensor powers of t with a penalty that depends on the so-called error degree of the approximation. More precisely, the notion of border rank $\underline{\mathrm{R}}(t)$ has a more precise variant $\mathrm{R}^e(t)$ that allows only approximations with error degree at most e (see Section 2 for definitions). This variant goes back to [6] and [7]. We prove in Corollary 11(1) that

$$\mathrm{R}(s^{\otimes n}) \leq (ne+1)\mathrm{R}^e(s)^n, \tag{1}$$

which we use to construct nonmultiplicativity examples. In particular, we see that as soon as $\mathrm{R}^e(s) < \mathrm{R}(s)$, the quantity $\mathrm{R}(s)^n$ grows faster than the right-hand side of (1) and thus leads to nonmultiplicativity examples for large enough n.

It follows from the definitions that also border rank and asymptotic rank are submultiplicative under the tensor product: $\underline{\mathrm{R}}(t \otimes s) \leq \underline{\mathrm{R}}(t)\underline{\mathrm{R}}(s)$, and $\underset{\sim}{\mathrm{R}}(t \otimes s) \leq \underset{\sim}{\mathrm{R}}(t)\underset{\sim}{\mathrm{R}}(s)$. We leave it as an open question whether these inequalities can be strict. In Section 4 we will see that lower bounds on border rank obtained from generalized flattenings (including Young flattenings) are in fact multiplicative under the tensor product.

It follows from $\mathrm{R}(t \boxtimes s) \leq \mathrm{R}(t \otimes s)$ that tensor rank, border rank and asymptotic rank are submultiplicative under the tensor Kronecker product: $\mathrm{R}(t \boxtimes s) \leq \mathrm{R}(t)\mathrm{R}(s)$, $\underline{\mathrm{R}}(t \boxtimes s) \leq \underline{\mathrm{R}}(t)\underline{\mathrm{R}}(s)$, and $\underset{\sim}{\mathrm{R}}(t \boxtimes s) \leq \underset{\sim}{\mathrm{R}}(t)\underset{\sim}{\mathrm{R}}(s)$. If t and s are 2-tensors (matrices), then tensor rank, border rank and asymptotic rank are equal and multiplicative under the tensor Kronecker product. However, for $k \geq 3$, it is well-known that each of the three inequalities can be strict, see the following example.

Example 3. Consider the following tensors

$$\mathrm{T}\left(\begin{array}{c} \bullet \\ \quad \searrow \bullet \\ \bullet \end{array} \right) = \sum_{i \in \{1,2\}} b_i \otimes b_i \otimes 1 \in \mathbb{F}^2 \otimes \mathbb{F}^2 \otimes \mathbb{F},$$

$$\mathrm{T}\left(\;\Big|\quad\bullet\;\right) = \sum_{i\in\{1,2\}} b_i \otimes 1 \otimes b_i \;\in\; \mathbb{F}^2 \otimes \mathbb{F} \otimes \mathbb{F}^2,$$

$$\mathrm{T}\left(\;\overset{\bullet}{\diagdown}\;\bullet\;\right) = \sum_{i\in\{1,2\}} 1 \otimes b_i \otimes b_i \;\in\; \mathbb{F} \otimes \mathbb{F}^2 \otimes \mathbb{F}^2.$$

(This graphical notation is borrowed from [8].) Each tensor has rank, border rank and asymptotic rank equal to 2, since they are essentially identity matrices. However the tensor Kronecker product is the 2×2 matrix multiplication tensor

$$\langle 2,2,2 \rangle = \mathrm{T}\left(\;\triangleright\;\bullet\;\right) = \sum_{i,j,k\in\{1,2\}} (b_i \otimes b_j) \otimes (b_j \otimes b_k) \otimes (b_k \otimes b_i)$$

whose tensor rank and border rank is at most 7 [9] and whose asymptotic rank is thus at most 7, which is strictly less that $2^3 = 8$. (The tensor rank of $\langle 2,2,2 \rangle$ equals 7 over any field [10] and the border rank of $\langle 2,2,2 \rangle$ equals 7 over the complex numbers $\mathbb{C}$ [11]. Both statements are in fact true for any tensor with the same support as $\langle 2,2,2 \rangle$ [12].)

2. Degeneration and restriction

We revisit the theory of degenerations and restrictions of tensors and how to transform degenerations into restrictions. Our non-multiplicativity results rely on these ideas. Let $t \in U_1 \otimes \cdots \otimes U_k$ and $s \in V_1 \otimes \cdots \otimes V_k$ be k-tensors. We say t *restricts* to s, written $t \geq s$, if there are linear maps $A_i : U_i \to V_i$ such that $(A_1 \otimes \cdots \otimes A_k)t = s$. Let $d, e \in \mathbb{N}$. We say t *degenerates to* s with approximation degree d and error degree e, written $t \unrhd_d^e s$, if there are linear maps $A_i(\varepsilon) : U_i \to V_i$ depending polynomially on ε such that $(A_1(\varepsilon) \otimes \cdots \otimes A_k(\varepsilon))t = \varepsilon^d s + \varepsilon^{d+1} s_1 + \cdots + \varepsilon^{d+e} s_e$ for some tensors $s_1, \ldots, s_e$. Naturally, $t \unrhd^e s$ means $\exists d : t \unrhd_d^e s$, and $t \unrhd_d s$ means $\exists e : t \unrhd_d^e s$, and $t \unrhd s$ means $\exists d \, \exists e : t \unrhd_d^e s$. (We note that our notation $t \unrhd_d s$ corresponds to $t \unrhd_{d+1} s$ in [5].) Clearly, degeneration is multiplicative in the following sense.

Proposition 4. *Let t_1, t_2, s_1, s_2 be tensors. If $t_1 \unrhd_{d_1}^{e_1} s_1$ and $t_2 \unrhd_{d_2}^{e_2} s_2$, then $t_1 \otimes t_2 \unrhd_{d_1+d_2}^{e_1+e_2}$ $s_1 \otimes s_2$ and $t_1 \boxtimes t_2 \unrhd_{d_1+d_2}^{e_1+e_2} s_1 \boxtimes s_2$.*

The error degree e is upper bounded by the approximation degree d in the following way.

Proposition 5. *Let t, s be k-tensors. If $t \unrhd_d s$, then $t \unrhd_d^{kd-d} s$.*

Proof. Suppose $(A_1(\varepsilon) \otimes \cdots \otimes A_k(\varepsilon))t = \varepsilon^d s + \varepsilon^{d+1} s_1 + \cdots + \varepsilon^{d+e} s_e$. For every i let $B_i(\varepsilon)$ be the matrix obtained from $A_i(\varepsilon)$ by truncating each entry in $A_i(\varepsilon)$ to degree at most d. Then $(B_1(\varepsilon) \otimes \cdots \otimes B_k(\varepsilon))t = \varepsilon^d s + \varepsilon^{d+1} u_1 + \cdots + \varepsilon^{kd} u_{kd}$ for some k-tensors $u_1, \ldots, u_{kd}$. $\square$

For any $r \in \mathbb{N}$, let $b_1, \ldots b_r$ denote the standard basis of $\mathbb{F}^r$. Let $r, k \in \mathbb{N}$ and let

$$\mathrm{T}_r(k) := \sum_{i=1}^{r} (b_i)^{\otimes k} \ \in \ (\mathbb{F}^r)^{\otimes k}$$

be the rank-r order-k *unit tensor*. Let $s \in V_1 \otimes \cdots \otimes V_k$. The tensor rank of s is the smallest number r such that $\mathrm{T}_r(k) \geq s$, and is denoted by $\mathrm{R}(s)$. This definition of tensor rank is easily seen to be equivalent to the definition given in the introduction. The border rank of s is the smallest number r such that $\mathrm{T}_r(k) \trianglerighteq s$, and is denoted by $\underline{\mathrm{R}}(s)$. Note that this definition works over any field $\mathbb{F}$. When $\mathbb{F}$ equals $\mathbb{C}$, this definition of border rank is equivalent to the definition given in the introduction [13–15,5]. Define

$$\mathrm{R}_d^e(s) := \min\{r \in \mathbb{N} \mid \mathrm{T}_r(k) \trianglerighteq_d^e s\}$$

$$\mathrm{R}_d(s) := \min\{r \in \mathbb{N} \mid \mathrm{T}_r(k) \trianglerighteq_d s\}$$

$$\mathrm{R}^e(s) := \min\{r \in \mathbb{N} \mid \mathrm{T}_r(k) \trianglerighteq^e s\}.$$

(Our notation $\mathrm{R}_d(s)$ corresponds to $\mathrm{R}_{d+1}(s)$ in [5].) Error degree in the context of border rank was already studied in [6] and [7]. The following propositions follow directly from Proposition 4 and Proposition 5.

Proposition 6. $\mathrm{R}_{d_1+d_2}^{e_1+e_1}(s_1 \otimes s_2) \leq \mathrm{R}_{d_1}^{e_1}(s_1)\mathrm{R}_{d_2}^{e_2}(s_2).$

Proposition 7. *Let s be a k-tensor. Then $\mathrm{R}_d(s) = \mathrm{R}_d^{kd-d}(s).$*

The following theorem is our main technical result on which the rest of the paper rests. We note that for the tensor Kronecker product the statement is well-known in the context of algebraic complexity theory [6,7,16,17,8].

Theorem 8. *Let t, s be k-tensors. If $t \trianglerighteq^e s$ and $|\mathbb{F}| \geq e + 2$, then we have $t \boxtimes \mathrm{T}_{e+1}(k) \geq s$.*

Proof. By assumption there are matrices $A_i(\varepsilon)$ with entries polynomial in ε such that

$$\big(A_1(\varepsilon) \otimes \cdots \otimes A_k(\varepsilon)\big)t = \varepsilon^d s + \varepsilon^{d+1}s_1 + \cdots + \varepsilon^{d+e}s_e$$

for some tensors $s_1, \ldots, s_e$. Multiply both sides by ε^{-d} and call the right-hand side $q(\varepsilon)$,

$$\big(\varepsilon^{-d}A_1(\varepsilon) \otimes \cdots \otimes A_k(\varepsilon)\big)t = s + \varepsilon s_1 + \cdots + \varepsilon^e s_e =: q(\varepsilon).$$

Let $\alpha_0, \ldots, \alpha_e$ be distinct nonzero elements of the ground field $\mathbb{F}$ (by assumption our ground field is large enough to do this). View $q(\varepsilon)$ as a polynomial in ε. Write $q(\varepsilon)$ as follows (Lagrange interpolation):

$$q(\varepsilon) = \sum_{j=0}^{e} q(\alpha_j) \prod_{\substack{0 \le m \le e: \\ m \ne j}} \frac{\varepsilon - \alpha_m}{\alpha_j - \alpha_m}.$$

We now see how to write $q(0)$ as a linear combination of the $q(\alpha_j)$, namely

$$q(0) = \sum_{j=0}^{e} q(\alpha_j) \prod_{\substack{0 \le m \le e: \\ m \ne j}} \frac{\alpha_m}{\alpha_m - \alpha_j},$$

that is,

$$q(0) = \sum_{j=0}^{e} \beta_j \, q(\alpha_j) \quad \text{with} \quad \beta_j := \prod_{\substack{0 \le m \le e: \\ m \ne j}} \frac{\alpha_m}{\alpha_m - \alpha_j}.$$

Now we want to write s as a restriction of $t \boxtimes \mathrm{T}_{e+1}(k)$. Define the linear maps $B_1 := \sum_{j=0}^{e} \beta_j \, \alpha_j^{-d} A_1(\alpha_j) \otimes b_j^*$ and $B_i := \sum_{j=0}^{e} \beta_j \, A_i(\alpha_j) \otimes b_j^*$ for $i \in \{2, \ldots, k\}$. Then $t \boxtimes \mathrm{T}_{e+1}(k) \ge s$ because

$$(B_1 \otimes \cdots \otimes B_k)(t \boxtimes \mathrm{T}_{e+1}(k)) = \sum_{j=0}^{e} \beta_j \left(\alpha_j^{-d} A_1(\alpha_j) \otimes \cdots \otimes A_k(\alpha_j) \right) t$$

$$= \sum_{j=0}^{e} \beta_j \, q(\alpha_j) = q(0) = s.$$

This finishes the proof. $\square$

Remark 9. In the statement of Theorem 8 we assume that $|\mathbb{F}|$ is large enough. For small fields one can do the following. For $k, d \in \mathbb{N}$, let $[0..d]$ denote the set $\{0, 1, 2, \ldots, d\}$ and define the k-tensor

$$\chi_d(k) := \sum_{\substack{a \in [0..d]^k: \\ a_1 + \cdots + a_k = d}} b_{a_1} \otimes \cdots \otimes b_{a_k} \in (\mathbb{F}^d)^{\otimes k}.$$

Let t, s be k-tensors. It is not hard to show that, if $t \trianglerighteq_d s$, then $t \boxtimes \chi_d(k) \ge s$. By definition of $\chi_d(k)$ we have $\mathrm{R}(\chi_d(k)) \le \binom{k+d-1}{k-1}$. We may thus conclude that $t \boxtimes \mathrm{T}_{\binom{k+d-1}{k-1}}(k) \ge s$.

We collect several almost immediate corollaries.

Corollary 10. *Let t_i, s_i be k_i-tensors for $i \in [n]$. Assume $\mathbb{F}$ is large enough.*

1. *If $\forall i\colon t_i \trianglerighteq^{e_i} s_i$, then $(t_1 \otimes \cdots \otimes t_n) \boxtimes \mathrm{T}_{\sum_i e_i + 1}(\sum_i k_i) \ge s_1 \otimes \cdots \otimes s_n$.*
2. *If $\forall i\colon t_i \trianglerighteq_{d_i} s_i$, then $(t_1 \otimes \cdots \otimes t_n) \boxtimes \mathrm{T}_{\sum_i (k_i - 1)d_i + 1}(\sum_i k_i) \ge s_1 \otimes \cdots \otimes s_n$.*

Proof. To prove the first statement, apply [Proposition 4](#) to obtain the degeneration $t_1 \otimes \cdots \otimes t_n \trianglerighteq^{\sum_i e_i} s_1 \otimes \cdots \otimes s_n$. [Theorem 8](#) yields the result. To prove the second statement, [Proposition 5](#) gives $t_i \trianglerighteq^{k_i d_i - d_i} s_i$. By [Proposition 4](#), $t_1 \otimes \cdots \otimes t_n \trianglerighteq^{\sum_i k_i d_i - d_i} s_1 \otimes \cdots \otimes s_n$. [Theorem 8](#) proves the statement. □

Corollary 11. *Let s be a k-tensor. Assume $\mathbb{F}$ is large enough.*

1. $\mathrm{R}(s^{\otimes n}) \leq (ne + 1)\mathrm{R}^e(s)^n$.
2. $\mathrm{R}(s^{\otimes n}) \leq ((k - 1)nd + 1)\mathrm{R}_d(s)^n$.

Proof. This follows from [Corollary 10](#). □

Corollary 12. *Let s be a k-tensor.*

1. $\lim_{n \to \infty} \mathrm{R}(s^{\otimes n})^{1/n} \leq \underline{\mathrm{R}}(s)$.
2. $\lim_{n \to \infty} \mathrm{R}(s^{\otimes n})^{1/n} = \lim_{n \to \infty} \underline{\mathrm{R}}(s^{\otimes n})^{1/n}$.
3. *If $\underline{\mathrm{R}}(s) < \mathrm{R}(s)$, then for some $n \in \mathbb{N}$, $\mathrm{R}(s^{\otimes n}) < \mathrm{R}(s)^n$.*

3. Tensor rank is not multiplicative under the tensor product

Because of [Corollary 11](#), in order to find nonmultiplicativity examples, it is enough to find a tensor t for which $\mathrm{R}^e(t) < \mathrm{R}(t)$. We will give three families of examples of nonmultiplicativity. For $k \geq 3$, define the k-tensor

$$W_k := \sum_{\substack{i \in \{1,2\}^k: \\ \mathrm{type}(i) = (k-1,1)}} b_{i_1} \otimes \cdots \otimes b_{i_k} \in (\mathbb{F}^2)^{\otimes k},$$

where $\mathrm{type}(i) = (k - 1, 1)$ means that i is a permutation of $(1, 1, \ldots, 1, 2)$.

Proposition 13. *Let $|\mathbb{F}|$ be large enough. Let $k \geq 3$. For n large enough, we have a strict inequality $\mathrm{R}(W_k^{\otimes n}) < \mathrm{R}(W_k)^n$. For example, $\mathrm{R}(W_3^{\otimes 7}) < \mathrm{R}(W_3)^7$ and $\mathrm{R}(W_8^{\otimes 2}) < \mathrm{R}(W_8)^2$.*

Proof. The rank of W_k equals k. This can be shown with the substitution method as explained in for example [18]. However, $\mathrm{R}^{k-1}(W_k) \leq 2$, namely

$$\left(\left(\begin{smallmatrix} 1 & 1 \\ \varepsilon & 0 \end{smallmatrix} \right) \otimes \cdots \otimes \left(\begin{smallmatrix} 1 & 1 \\ \varepsilon & 0 \end{smallmatrix} \right) \otimes \left(\begin{smallmatrix} 1 & -1 \\ \varepsilon & 0 \end{smallmatrix} \right) \right) T_2(k) = \varepsilon W_k + \varepsilon^2(\cdots) + \cdots + \varepsilon^k (b_2 \otimes \cdots \otimes b_2).$$

Applying [Corollary 11](#)(1) to this degeneration gives $\mathrm{R}(W_k^{\otimes n}) \leq (n(k-1) + 1)2^n$. Therefore, for n large enough, $\mathrm{R}(W_k^{\otimes n}) \leq 2^n(n(k-1) + 1) < k^n = \mathrm{R}(W_k)^n$. □

In fact, if $\mathrm{char}(\mathbb{F}) \neq 2$ and $\sqrt{2} \in \mathbb{F}$, then we can directly show a strict inequality for $n = 2$ and $k = 3$ as follows.

Proposition 14. $R(W_3^{\otimes 2}) \leq 8 < 9 = R(W_3)^2$ *if* $\operatorname{char}\mathbb{F} \neq 2$ *and* $\sqrt{2} \in \mathbb{F}$.

Proof. As mentioned in the proof of Proposition 13, $R(W_3) = 3$. If $c \in \mathbb{F} \setminus \{0\}$ such that $\sqrt{c} \in \mathbb{F}$, then $R(W_3 + c\, b_2 \otimes b_2 \otimes b_2) \leq 2$. Namely,

$$W_3 + c\, b_2 \otimes b_2 \otimes b_2 = \frac{1}{2\sqrt{c}}\big((b_1 + \sqrt{c}\, b_2)^{\otimes 3} - (b_1 - \sqrt{c}\, b_2)^{\otimes 3}\big).$$

(Over $\mathbb{C}$ this also follows from the fact that the *Cayley hyperdeterminant* evaluated at $W_3 + c\, b_2 \otimes b_2 \otimes b_2$ is a nonzero constant times c. One may also see this by noting that the image of $W_3 + c\, b_2 \otimes b_2 \otimes b_2$ under the *moment map* lies outside the image of the *moment polytope* associated to the orbit $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2 \cdot W$ [19,20].) We expand $W_3 \otimes W_3$ as

$$W_3 \otimes W_3 = \big(W_3 + b_2 \otimes b_2 \otimes b_2\big)^{\otimes 2} - \big(W_3 + \tfrac{1}{2}b_2 \otimes b_2 \otimes b_2\big) \otimes b_2 \otimes b_2 \otimes b_2$$
$$- b_2 \otimes b_2 \otimes b_2 \otimes \big(W_3 + \tfrac{1}{2}b_2 \otimes b_2 \otimes b_2\big).$$

By the above, we know that the rank of $W_3 + b_2 \otimes b_2 \otimes b_2$ and the rank of $W_3 + \tfrac{1}{2}b_2 \otimes b_2 \otimes b_2$ are at most 2. Therefore, the rank of $W_3 \otimes W_3$ is at most $2^2 + 2 + 2 = 8$. □

Remark 15. Let S_k be the symmetric group of order k. Clearly the tensor $W_3 \otimes W_3$ is invariant under the action of the subgroup $S_3 \times S_3 \subseteq S_6$ and under the action of the permutation $(14)(25)(36) \in S_6$ that swaps the two copies of W_3. Remarkably, the decomposition of $W_3 \otimes W_3$ given in the proof of Proposition 14 also has this symmetry, in the sense that the above actions leave the set of simple terms appearing in the decomposition invariant. The decomposition is said to be *partially symmetric*. In fact, each term is itself invariant under $S_3 \times S_3$.

Remark 16. It is stated in [21] that $R(W_3 \boxtimes W_3) = 7$, which implies that $R(W_3 \otimes W_3)$ equals 7 or 8. We obtained numerical evidence pointing to 8. After the first version of our manuscript appeared on the arXiv, Chen and Friedland delivered a proof that $R(W_3 \otimes W_3) \geq 8$ [22]. For the third power, it is known that $R(W_3 \boxtimes W_3 \boxtimes W_3) = 16$ [23]. A similar construction as in the proof of Proposition 14 gives $R(W_3 \otimes W_3 \otimes W_3) \leq 21$. This upper bound is improved to 20 in [22].

In Proposition 13, we took the nth power of a tensor in $(\mathbb{F}^2)^{\otimes k}$ with n large enough depending on k. In our next example, we take the square of a tensor in $(\mathbb{F}^d)^{\otimes k}$ with $d \geq 8$. For $k \geq 3$ and $q \geq 1$, define the tensor

$$\mathrm{Str}_q^k := \sum_{i=2}^{q+1} b_i \otimes b_i \otimes b_1 \otimes b_1^{\otimes k-3} + b_1 \otimes b_i \otimes b_i \otimes b_1^{\otimes k-3} \ \in (\mathbb{F}^{q+1})^{\otimes k}.$$

This tensor is named after Strassen, who used Str_q^3 to derive the upper bound $\omega \leq 2.48$ on the exponent of matrix multiplication [14,18].

Proposition 17. *Assume that $\mathbb{F}$ is large enough. For $q \geq 7$ and any $k \geq 3$, we have a strict inequality $\mathrm{R}((\mathrm{Str}_q^k)^{\otimes 2}) < \mathrm{R}(\mathrm{Str}_q^k)^2$.*

Proof. The rank of Str_q^k equals $2q$, again by the substitution method. We have $\mathrm{R}^1(\mathrm{Str}_q^k) \leq q+1$, see the proof of Proposition 31 in [24]. Applying Corollary 11(1) to this degeneration gives $\mathrm{R}((\mathrm{Str}_q^k)^{\otimes n}) \leq (n+1)(q+1)^n$. Therefore, for $q \geq 7$ and $n = 2$, we have the strict inequality $\mathrm{R}((\mathrm{Str}_q^k)^{\otimes 2}) \leq 3(q+1)^2 < (2q)^2 = \mathrm{R}(\mathrm{Str}_q^k)^2$. $\quad\square$

Our third example uses matrix multiplication tensors. Let $n_1, n_2, n_3 \in \mathbb{N}$. Define the 3-tensor

$$\langle n_1, n_2, n_3 \rangle := \sum_{i \in [n_1] \times [n_2] \times [n_3]} (b_{i_1} \otimes b_{i_2}) \otimes (b_{i_2} \otimes b_{i_3}) \otimes (b_{i_3} \otimes b_{i_1})$$

$$\in (\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}) \otimes (\mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}) \otimes (\mathbb{F}^{n_3} \otimes \mathbb{F}^{n_1}).$$

Proposition 18. *Assume that $\mathbb{F}$ is large enough. For $n \geq 78$, we have a strict inequality $\mathrm{R}(\langle 2, 2, 4 \rangle^{\otimes n}) < \mathrm{R}(\langle 2, 2, 4 \rangle)^n$.*

Proof. The rank of $\langle 2, 2, 4 \rangle$ equals 14 over any field [25, Theorem 2]. On the other hand, $\mathrm{R}^4(\langle 2, 2, 4 \rangle) \leq 13$ over any field [26, Theorem 1]. Thus, when $\mathbb{F}$ is large enough Corollary 11(1) implies, for $n \geq 78$, the strict inequality $\mathrm{R}(\langle 2, 2, 4 \rangle^{\otimes n}) \leq 13^n(4n+1) < 14^n = \mathrm{R}(\langle 2, 2, 4 \rangle)^n$. $\quad\square$

In the language of graph tensors [8], Proposition 18 says that tensor rank is not multiplicative under taking disjoint unions of graphs.

4. Generalized flattenings are multiplicative

In the previous section we have seen that tensor rank can be strictly submultiplicative under the tensor product. We do not know whether the same is true for border rank. In fact, in this section we observe that lower bounds on border rank obtained from generalized flattenings are multiplicative. In this section we focus on 3-tensors for notational convenience. The ideas directly extend to k-tensors for any k.

Let t be a tensor in $V_1 \otimes V_2 \otimes V_3$. We can transform t into a matrix by grouping the tensor legs into two groups

$$V_1 \otimes V_2 \otimes V_3 \rightarrow V_1 \otimes (V_2 \otimes V_3)$$

$$v_1 \otimes v_2 \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3).$$

(There are three ways to do this for a 3-tensor.) This is called *flattening*. The rank of a flattening of t is a lower bound for the border rank of t. (Rank and border rank are equal for matrices.)

We now define generalized flattenings. Let t be a tensor in $V_1 \otimes V_2 \otimes V_3$. Instead of a basic flattening $V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$, we choose vector spaces V_1' and V_2' and apply some linear map $F : V_1 \otimes V_2 \otimes V_3 \to V_1' \otimes V_2'$ to t. To obtain a border rank lower bound using F we have to compensate for the fact that F possibly increases the border rank of a simple tensor. The following lemma describes the resulting lower bound.

Lemma 19. *Let* $t \in V_1 \otimes V_2 \otimes V_3$ *be a tensor. Let*

$$F : V_1 \otimes V_2 \otimes V_3 \to V_1' \otimes V_2'$$

be a linear map. The border rank of t is at least

$$\underline{R}(t) \geq \frac{R(F(t))}{\max R(F(v_1 \otimes v_2 \otimes v_3))}, \tag{2}$$

where the maximum is over all simple tensors $v_1 \otimes v_2 \otimes v_3$ *in* $V_1 \otimes V_2 \otimes V_3$.

Proof. Suppose $\underline{R}(t) = r$. Then there is a sequence of tensors t_i converging to t with $R(t_i) \leq r$ for each i. Each t_i thus has a decomposition into simple tensors $t_i = \sum_{j=1}^{r} t_{i,j}$. Since $F(t_i) \to F(t)$, there exists an i_0 such that for all $i \geq i_0$ we have $R(F(t_i)) \geq R(F(t))$. Moreover, we have the inequalities $R(F(t_i)) \leq \sum_{j=1}^{r} R(F(t_{i,j})) \leq r \cdot \max_s R(F(s))$, where the maximum is over all simple tensors s. We conclude that $\underline{R}(t) \geq R(F(t)) / \max_s R(F(s))$. $\square$

Note that the right-hand side of (2) might not be an integer. The lower bound in (2) is multiplicative under the tensor product in the following sense.

Proposition 20. *Let* $s \in V_1 \otimes V_2 \otimes V_3$ *and* $t \in W_1 \otimes W_2 \otimes W_3$ *be tensors. Let* $F_1 : V_1 \otimes V_2 \otimes V_3 \to V_1' \otimes V_2'$ *and* $F_2 : W_1 \otimes W_2 \otimes W_3 \to W_1' \otimes W_2'$ *be linear maps. The border rank of* $s \otimes t \in V_1 \otimes V_2 \otimes V_3 \otimes W_1 \otimes W_2 \otimes W_3$ *is at least*

$$\underline{R}(s \otimes t) \geq \frac{R(F_1(s))}{\max R(F_1(v_1 \otimes v_2 \otimes v_3))} \frac{R(F_2(t))}{\max R(F_2(w_1 \otimes w_2 \otimes w_3))}$$

where the maximizations are over simple tensors in $V_1 \otimes V_2 \otimes V_3$ *and in* $W_1 \otimes W_2 \otimes W_3$ *respectively.*

Proof. Combine F_1 and F_2 into a single linear map

$$F : V_1 \otimes V_2 \otimes V_3 \otimes W_1 \otimes W_2 \otimes W_3 \to (V_1' \otimes W_1') \otimes (V_2' \otimes W_2').$$

One then follows the proof of Lemma 19 and uses the fact that matrix rank is multiplicative under the tensor Kronecker product. $\square$

Young flattenings [27,28] are a special case of generalized flattenings. For completeness, we finish with a concise description of Young flattenings and the corresponding multiplicativity statement. We work over the complex numbers $\mathbb{C}$. Let $S_\lambda V$ be an irreducible GL_V-module of type λ. Consider the space $V \otimes S_\lambda V$ as a GL_V-module under the diagonal action. The *Pieri rule* says that we have a GL_V-decomposition

$$V \otimes S_\lambda V \cong \bigoplus_\mu S_\mu V,$$

where the direct sum is over partitions μ of length at most $\dim V$ obtained from λ by adding a box in the Young diagram of λ. This decomposition yields GL_V-equivariant embeddings $S_\mu V \hookrightarrow V \otimes S_\lambda V$, called *Pieri inclusions* or *partial polarization maps*. These maps are unique up to scaling. Such a Pieri inclusion corresponds to a GL_V-equivariant map $\phi_{\mu,\lambda} : V^* \to S_\mu V^* \otimes S_\lambda V$. Every element $\phi_{\mu,\lambda}(v)$ is called a *Pieri map*. The *Young flattening* $F_{\mu,\lambda}$ on $V_1 \otimes V_2^* \otimes V_3$ is obtained by first applying the map $\phi_{\mu,\lambda}$ to one tensor leg,

$$V_1 \otimes V_2^* \otimes V_3 \to V_1 \otimes S_\mu V_2^* \otimes S_\lambda V_2 \otimes V_3,$$

and then flattening into a matrix,

$$V_1 \otimes S_\mu V_2^* \otimes S_\lambda V_2 \otimes V_3 \to (V_1 \otimes S_\mu V_2^*) \otimes (S_\lambda V_2 \otimes V_3).$$

Note that for any simple tensor $v_1 \otimes v_2 \otimes v_3$, the rank of $F_{\mu,\lambda}(v_1 \otimes v_2 \otimes v_3)$ equals the rank of $\phi_{\mu,\lambda}(v_2)$. Proposition 20 thus specializes as follows.

Proposition 21. *Let $s \in V_1 \otimes V_2 \otimes V_3$ and $t \in W_1 \otimes W_2 \otimes W_3$. Let λ, μ and ν, κ be pairs of partitions as above. The border rank of $s \otimes t \in V_1 \otimes V_2 \otimes V_3 \otimes W_1 \otimes W_2 \otimes W_3$ is at least*

$$\underline{\mathrm{R}}(s \otimes t) \geq \frac{\mathrm{R}(F_{\mu,\lambda}(s))}{\max \mathrm{R}(\phi_{\mu,\lambda}(v_2))} \frac{\mathrm{R}(F_{\nu,\kappa}(t))}{\max \mathrm{R}(\phi_{\nu,\kappa}(w_2))}$$

where the maximizations are over $v_2 \in V_2$ and $w_2 \in W_2$ respectively.

We refer to [29] for an overview of the applications of Young flattenings.

5. Multiplicativity for complex matrix pencils and 2-tensors

In this section all vector spaces are over the complex numbers. The goal of this section is to prove the following proposition.

Proposition 22. *Let $s \in \mathbb{C} \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ and $t \in \mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^m$. Then*

$$R(t \boxtimes s) = R(t \otimes s) = R(t)R(s).$$

Remark 23. Proposition 22 shows that Example 2 is essentially minimal over the complex numbers. Namely, any example of non-multiplicativity of tensor rank under $\otimes$ must either be with a 5-tensor in $(\mathbb{C}^d \otimes \mathbb{C}^d) \otimes (\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})$ with $d_1, d_2, d_3 \geq 3$, $d \geq 2$ or in a tensor space of order 6 or more. Moreover, one can show using Proposition 22 and the well-known classification of the $\mathrm{GL}_2^{\times 3}$-orbits in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ that if $s, t \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathrm{R}(s \otimes t) < \mathrm{R}(s)\mathrm{R}(t)$, then s and t are both isomorphic to the tensor W_3.

The elements of $\mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^m$ are often called matrix pencils. The tensor rank of matrix pencils is completely understood, in the sense that every matrix pencil is equivalent under local isomorphisms to a pencil in canonical form, for which the rank is given by a simple formula. This formula will allow us to give a short proof of Proposition 22.

We begin with introducing the canonical form for matrix pencils. For a proof we refer to [30, Chapter XII]. Recall that the standard basis elements of $\mathbb{C}^n$ are denoted by $b_1, \ldots, b_n$.

Definition 24. Given $t_i \in U \otimes V_i \otimes W_i$, define $\mathrm{diag}_U(t_1, \ldots, t_n)$ as the image of $\bigoplus_{i=1}^{n} t_i$ under the natural inclusion $\bigoplus_i (U \otimes V_i \otimes W_i) \to U \otimes \left(\bigoplus_i V_i \right) \otimes \left(\bigoplus_i W_i \right)$. For $\varepsilon \in \mathbb{N}$ define the tensor $L_\varepsilon \in \mathbb{C}^2 \otimes \mathbb{C}^\varepsilon \otimes \mathbb{C}^{\varepsilon+1}$ by

$$
L_\varepsilon := b_1 \otimes \sum_{i=1}^{\varepsilon} b_i \otimes b_i \; + \; b_2 \otimes \sum_{i=1}^{\varepsilon} b_i \otimes b_{i+1}
$$

$$
= b_1 \otimes \begin{pmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & \ddots & \vdots \\ & & & 1 \ 0 \end{pmatrix} + b_2 \otimes \begin{pmatrix} 0 \ 1 & & \\ 0 & 1 & \\ \vdots & & \ddots \\ 0 & & & 1 \end{pmatrix}
$$

and for $\eta \in \mathbb{N}$ define the tensor $N_\eta \in \mathbb{C}^2 \otimes \mathbb{C}^{\eta+1} \otimes \mathbb{C}^\eta$ by

$$
N_\eta := b_1 \otimes \sum_{i=1}^{\eta} b_i \otimes b_i \; + \; b_2 \otimes \sum_{i=1}^{\eta} b_{i+1} \otimes b_i
$$

$$
= b_1 \otimes \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \\ 0 \ 0 & \cdots & 0 \end{pmatrix} + b_2 \otimes \begin{pmatrix} 0 \ 0 & \cdots & 0 \\ 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}.
$$

Theorem 25 (*Canonical form*). *Let* $t \in \mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^m$. *There exist invertible linear maps* $A \in \mathrm{GL}_2$, $B \in \mathrm{GL}_n$ *and* $C \in \mathrm{GL}_m$ *and natural numbers* $\varepsilon_1, \ldots, \varepsilon_p, \eta_1, \ldots, \eta_q \in \mathbb{N}$ *and an* $\ell \times \ell$ *Jordan matrix* F *such that, with* $M = b_1 \otimes I_\ell + b_2 \otimes F$, *we have*

$$
(A \otimes B \otimes C)t = \mathrm{diag}_{\mathbb{C}^2}(0, L_{\varepsilon_1}, \ldots, L_{\varepsilon_p}, N_{\eta_1}, \ldots, N_{\eta_q}, M), \tag{3}
$$

where the 0 *stands for some* 0*-tensor of appropriate dimensions. The right-hand side of* (3) *is called the canonical form of* t.

Next we give a formula for the tensor rank of matrix pencils in canonical form (Theorem 27). Theorem 27 is due to Grigoriev [31], JáJá [32] and Teichert [33], see also [5, Theorem 19.4] or [29, Theorem 3.11.1.1].

Definition 26. Let F be a Jordan matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$. Let $d(\lambda_i)$ be the number of Jordan blocks in F of size at least two with eigenvalue λ_i. Define $m(F) := \max_i d(\lambda_i)$.

Theorem 27. *Let* $t = \operatorname{diag}_{\mathbb{C}^2}(0, L_{\varepsilon_1}, \ldots, L_{\varepsilon_p}, N_{\eta_1}, \ldots, N_{\eta_q}, b_1 \otimes I_\ell + b_2 \otimes F)$ *be a tensor in canonical form as in* (3). *The tensor rank of* t *equals*

$$\mathrm{R}(t) = \sum_{i=1}^{p} (\varepsilon_i + 1) + \sum_{i=1}^{q} (\eta_i + 1) + \ell + m(F).$$

Example 28. Let $W_3 = b_2 \otimes b_1 \otimes b_1 + b_1 \otimes b_2 \otimes b_1 + b_1 \otimes b_1 \otimes b_2 \in (\mathbb{C}^2)^{\otimes 3}$ as in Example 2. The canonical form of W_3 is

$$W_3 \cong b_1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b_2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so in the notation of Theorem 25 we have $p = q = 0$ and $F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We can thus apply Theorem 27 with $\ell = 2$ and $m(F) = 1$ to get $\mathrm{R}(W_3) = 2 + 1 = 3$.

We are now ready to give the short proof of Proposition 22.

Proof of Proposition 22. Let $s \in \mathbb{C} \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, $t \in \mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^m$. We may assume that $s = 1 \otimes \sum_{i=1}^{r} b_i \otimes b_i$ with $r = \mathrm{R}(s)$. By Theorem 25 we may assume that t is in canonical form, $t = \operatorname{diag}_{\mathbb{C}^2}(0, L_{\varepsilon_1}, \ldots, L_{\varepsilon_p}, N_{\eta_1}, \ldots, N_{\eta_q}, M)$. The tensor Kronecker product $t \boxtimes s$ is isomorphic to

$$t \boxtimes s \cong \operatorname{diag}_{\mathbb{C}^2}(\underbrace{t, \ldots, t}_{r}).$$

By an appropriate local basis transformation we put this in canonical form

$$t \boxtimes s \cong \operatorname{diag}_{\mathbb{C}^2}(L_{\varepsilon_1}^{\oplus r}, \ldots, L_{\varepsilon_p}^{\oplus r}, N_{\eta_1}^{\oplus r}, \ldots, N_{\eta_q}^{\oplus r}, M^{\oplus r}),$$

which by Theorem 27 has rank $r \cdot \mathrm{R}(t) = \mathrm{R}(s)\mathrm{R}(t)$. $\quad\square$

Remark 29. Proposition 22 is also true over the finite field $\mathbb{F}_q$ when $q \geq n, m$. To see this one may use the formula from [5, Section 19.5] for the rank of pencils over finite fields, which for $q \geq n, m$ is as follows:

$$\mathrm{R}(t) = \sum_{i=1}^{p}(\varepsilon_i + 1) + \sum_{i=1}^{q}(\eta_i + 1) + \ell + \delta(B).$$

Here B is the *regular part* of the pencil t and $\delta(B)$ is the number of *invariant divisors* of B that do not decompose into a product of *unassociated linear factors*. (We refer to [5] for definitions.) The invariant divisors of $\mathrm{diag}(B, \dots, B)$ are just the invariant divisors of B counted for each copy of B and so Proposition 22 follows.

We note that part of the results in this section have been independently obtained in Section 2 of [22].

Acknowledgements

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