A Stochastic Resource-Sharing Network for Electric Vehicle Charging

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***Abstract*—We consider a distribution grid used to charge electric vehicles subject to voltage stability and various other constraints. We model this as a class of resource-sharing networks known as *bandwidth-sharing networks* in the communication network literature. Such networks have proved themselves to be an effective flow-level model of data traffic in wired and wireless networks. We focus on resource sharing networks that are driven by a class of greedy control rules that can be implemented in a decentralized fashion. For a large number of such control rules, we can characterize the performance of the system, subject to voltage stability constraints, by a fluid approximation. This leads to a set of dynamic equations that take into account the stochastic behavior of cars. We show that the invariant point of these equations is unique and can be computed by solving a specific ACOPF problem, which admits an exact convex relaxation. For the class of weighted proportional fairness control, we show additional appealing properties under the linearized Distflow model, such as fairness, and a product form property of the stochastic model.**

***Index Terms*—Electric vehicle charging, distribution network, AC power flow model, linearized Distflow, queueing theory, stochastic processes, fluid approximation.**

I. Introduction

The rise of electric vehicles (EVs) is unstoppable due to factors, such as the decreasing cost of batteries and various policy decisions [1]. These vehicles need to be charged and will therefore cause congestion in distribution grids in the (very near) future. This paper proposes to model and analyze such congestion by the use of a class of resource-sharing networks, which in the queueing network community is known as *bandwidth-sharing networks*. A bandwidth-sharing network is a specific class of queueing networks, where customers (in our context: cars) need to be served simultaneously. Their service requires the usage of multiple "servers" (in our case: all upstream lines between the location of the car and the feeder of the distribution grid). Determining how fast to charge each car taking into account network stability and the randomness of future arriving cars is one of the key problems in the analysis of distribution grids, leading to challenging mathematical problems.

Similar questions though have been successfully answered in communication networks, where the set of feasible schedules is determined by the maximum amount of data a communication channel can use per time unit, leading to the powerful concept of bandwidth-sharing networks [2]. Bandwidth-sharing networks couple the important fields of network utility maximization with stochastic process dynamics [3, 4]. Apart from yielding various qualitative insights, they have been instrumental in the comparison and optimization of various data network protocols, and even to new protocols [5]. The stochastic analysis of bandwidth-sharing networks was initially only restricted to some specific examples [6, 7]. By now, fluid and diffusion approximations are available, which are computationally tractable [8, 9, 10, 11, 12, 13] and hold for a large class of networks.

From a mathematical viewpoint, the present paper is influenced by [11]. However, in the setting of charging electrical vehicles, an important constraint that needs to be satisfied is voltage stability, making the bandwidth-sharing network proposed in this paper different from the above-mentioned works. This also causes new technical issues, as the capacity set can be non-polyhedral or even non-convex. The first paper to suggest the class of bandwidth-sharing networks in the content of EV charging is [14], where simulation studies were conducted to explore stability properties, assuming that the arrival times are exponentially distributed. Our work is a significant extension of [14], both in terms of models and results: we allow for load limits, finitely many parking spaces, deadlines (associated with parking times), and do not make any assumption on the joint distribution of the parking time and the demand for electricity. More importantly, despite our assumptions leading to an intricate class of measure-valued processes, we obtain a number of mathematical results that are computationally tractable and in some cases explicit.

We develop a fluid approximation for the number of uncharged EVs (for the single-node Markovian case see [15]), allowing the dynamics of the stochastic model to be approximated with a deterministic model. This model is still quite rich, as it depends on the joint distribution of the charging requirements and parking times. We show that the invariant point of this dynamical system is unique and can be characterized in a computationally friendly manner by formulating a nontrivial AC optimal-power-flow problem (ACOPF), which is tractable as its convex relaxation is exact in many cases. When we replace the AC load flow model with the simpler linearized Distflow [16], we obtain more explicit results, as the capacity set becomes polyhedral. For each load flow model, we compare the fluid approximation with its original stochastic model and also compare across load flow models. Our computational study, though certainly not exhaustive, indicates that our fluid models can accurately predict the performance of a specific control rule, with relative errors of no more than 10% in most cases.

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For the class of weighted proportional fairness controls where the weights are chosen as function of the line resistances, we derive a fairness property. In this case, all cars are charged at the same rate, independent of the location in the network, while keeping voltages stable. When the weights in are instead chosen equally, we can even derive an explicit formula for the invariant distribution of the original stochastic process. Specifically, we show that under certain assumptions, our network behaves like a multiclass processor sharing queue. Such properties have been proven quite fruitful in other areas of engineering, particularly in the analysis of computer systems [17], communication networks [6], and wireless networks [18].

Electric vehicles can be charged in several ways. Our setup can be seen as an example of slow charging, in which drivers typically park their car and are not physically present during charging (but are busy shopping, working, sleeping,. . .) For *queueing* models focusing on fast charging, we refer to [19, 20]. Both papers consider a gradient scheduler to control delays, but do not consider physical load flow models as is done here. [21] presents a queueing model for battery swapping. An early paper on a queueing analysis of EV charging, focusing on designing safe (in terms of voltage stability) control rules with minimal communication overhead, is [22].

We can only provide a small additional (i.e., non-queueing) sample of the already vast but still emerging literature on EV charging. The focus of [23] is on a specific parking lot and presents an algorithm for optimally managing a large number of plug-in EVs. Algorithms to minimize the impact of plug-in EV charging on the distribution grid are proposed in [24]. In [25], the overall charging demand of plug-in EVs is considered. Mathematical models where vehicles communicate beforehand with the grid to convey information about their charging status are studied in [26]. In [27], cars are the central object and a dynamic program is formulated that prescribes how cars should charge their battery using price signals.

The present paper aims to illustrate how state-of-the-art methods from the applied probability and queueing communities can contribute to the analysis of the interplay between EV charging and the analysis of congestion in distribution networks. Our analysis does not take into account other important features in distribution networks touched upon in some of the above-cited works. In particular, it would be useful to incorporate smart appliances/buildings/meters, rooftop solar panels, and other sources of electricity demand. We think that the tractability and generality of our formulation and the established connection with an OPF problem makes our framework promising towards a comprehensive stochastic network model of a distribution grid.

The paper is organized as follows. In Section II, we provide a detailed model description — in particular we introduce our stochastic model, the class of charging controls, and the load flow models. The section concludes with a fairness property of a specific class of controls. Section III shows how the proportional fairness control mechanism leads to a product-form property for the stochastic model. In Section IV, we consider our model in full generality. We present a fluid model of our system and show that the associated dynamic equations have a unique invariant point, which is shown to be stable under an additional assumption. Numerical validations are presented in Section V. All proofs are gathered in the Appendix.

II. Model description

In this section, we provide a detailed formulation of our model and explain various notational conventions that are used in the remainder of this work.

A. Network and infrastructure

Consider a radial distribution network described by a rooted tree graph $\mathcal{G} = (\mathcal{I}, \mathcal{E})$, where $\mathcal{I} = \{0, 1, \dots, I\}$, denotes its set of nodes (buses) and $\mathcal{E}$ is its set of directed edges, assuming that the node 0 is the root node (feeder). Further, we denote by $\epsilon_{ik} \in \mathcal{E}$ the edge that connects node i to node k, assuming that i is closer to the root node than k. Let $\mathcal{I}(k)$ and $\mathcal{E}(k)$ be the node and edge set of the subtree rooted in node $k \in \mathcal{I}$. The active and reactive power consumed by the subtree $(\mathcal{I}(k), \mathcal{E}(k))$ are $P_{\mathcal{I}(k)}$ and $Q_{\mathcal{I}(k)}$. The resistance, the reactance, and the real and active power losses along edge ϵ_{ik} are denoted by R_{ik}, X_{ik}, L^P_{ik}, and L^Q_{ik}, respectively. Moreover, V_i is the voltage at node i and V_0 is known. At any node, except for the root node, there is a charging station with $K_i > 0$, $i \in \mathcal{I} \backslash \{0\}$ parking spaces (each having an EV charger). Further, we assume there are $\mathcal{J} = \{1, \dots, J\}$ different types of EVs indexed by j.

B. Stochastic model for EVs

Type-j EVs arrive at node i according to a Poisson process with rate λ_{ij}. If all spaces are occupied, a newly arriving EV does not enter the system, but is assumed to leave immediately. Each EV has a random energy demand and a random parking time. These depend on the type of the car, but are independent between cars and are denoted by B_j and D_j, respectively, for type-j EVs. In queueing terminology, these quantities are respectively called *service requirements* and *deadlines*. The joint distribution of (B_j, D_j) is given by a bivariate probability distribution: $F_j(x, y) = P(B_j \leq x, D_j \leq y)$ for $x, y \geq 0$.

Our framework is general enough to distinguish between types. For example, we can classify types according to intervals of ratio of the charging requirement and parking time and/or according to the contract they have. An EV leaves the system after its parking time expires. It may be not fully charged. If an EV finishes its charge, it remains at its parking space without consuming power until its parking time expires. EVs that have finished their charge are called "*fully charged*".

C. State descriptor

We denote by $Q_{ij}(t) \in \{0, 1, \dots, K_i\}$ the total number of type-j EVs at node i at time $t \geq 0$, where $Q_{ij}(0)$ is the initial number of EVs. Thus, $Q_i(t) := \sum_{j=1}^{J} Q_{ij}(t)$ denotes the total number of EVs at node i. Further, we denote by $Z_{ij}(t) \in \{0, 1, \dots, Q_{ij}(t)\}$ the number of type-j EVs at node i with a not-fully-charged battery at time t and by $Z_{ij}(0)$ the number of vehicles initially at node i. Last, we write $Z_{ij}(\infty)$ or simlpy Z_{ij} to represent the process in steady-state.

For some fixed time, let $\boldsymbol{z} = (z_{ij} : i \in \mathcal{I}\backslash\{0\},\ j \in \mathcal{J}) \in \mathbb{R}_+^{I\times J}$ be the vector giving the number of uncharged EVs for all types and nodes. Note that although the vector that gives the number of uncharged EVs should have integer-valued coordinates, we allow real values in order to accommodate fluid analogues later. Moreover, we assume that EVs receive only active power during their charge and do not absorb reactive power; see [28] for a justification.

D. Charging control rule

An important part of our framework is the way we specify how the charging of EVs takes place. Given that the state of uncharged vehicles is equal to $\boldsymbol{z}$, we assume the existence of a function $\boldsymbol{p}(\boldsymbol{z}) = (p_{ij}(\boldsymbol{z}) : i \in \mathcal{I}\backslash\{0\},\ j \in \mathcal{J})$ that specifies the instantaneous rate of power each uncharged vehicle receives. Moreover, we assume that this function is obtained by optimizing a "global" function. Specifically, for a type-j EV at node i we associate a function $u_{ij}(\cdot)$, which is strictly increasing and concave in $\mathbb{R}_+$, twice differentiable in $(0,\infty)$ with $\lim_{x\to 0} u'_{ij}(x) = \infty$. The charging rate $\boldsymbol{p}(\boldsymbol{z})$ is then determined by

$$\max_{\boldsymbol{p}} \sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij}u_{ij}(p_{ij})$$

subject to a number of constraints, which take into account physical limits on the charging of the batteries, load limits, and most importantly voltage stability constraints.

Before we describe these constraints in detail, we first provide some comments on this charging protocol. An important example is the choice $u_{ij}(p_{ij}) = w_{ij}\log p_{ij}$, which is known as *weighted proportional fairness*. Note that this scheme assumes the existence of a virtual agent that is capable of optimizing the global function. In practice, this control may be achieved in a decentralized fashion, using mechanisms such as the *alternating direction* method of multipliers [29, 30]. It is even possible to come up with decentralized allocation schemes that achieve this control if the functions $u_{ij}(\cdot)$ are unknown, which dictates the use of proportional fairness with a specific choice of the weights w_{ij}. For background, we refer to [3, 5]. A shortcoming of our formulation is that it does not take into account the remaining time until the deadline expires and the remaining charging requirement. Our multiclass framework allows to at least partially overcome this, for example by letting the functions u_{ij} depend on the joint distribution of (B_j, D_j) and by allowing routing between classes.

We now turn to a discussion of the constraints. We assume that the highest power that the parking lot at node i can consume is $M_i > 0$ and that the maximum power rate which a type-j EV can be charged is $c_j^{\max}$. That is,

$$\sum_{j=1}^{J} z_{ij}p_{ij} \le M_i \quad \text{and} \quad 0 \le p_{ij} \le c_j^{max}. \tag{1}$$

In addition, we impose voltage stability constraints. These constraints rely on load flow models. Two of these models we consider in this paper are described next.

E. AC voltage model

We first consider a minor simplification of the full AC power flow equations. The angle between voltages in distribution networks are small and hence they can be chosen so that the phasors have zero imaginary components [31, Chapter 3]. Under this assumption, Kirchhoff's law [32, Eq. 1] for our model takes the form, for $\epsilon_{pk} \in \mathcal{E}$,

$$V_pV_k - V_kV_k - P_{\mathcal{I}(k)}R_{pk} - Q_{\mathcal{I}(k)}X_{pk} = 0, \tag{2}$$

where $p \in \mathcal{I}$ denotes the unique parent of node k. The previous equations are non-linear. Applying the transformation

$$\boldsymbol{W}(\epsilon_{pk}) = \begin{pmatrix} V_p^2 & V_pV_k \\ V_kV_p & V_k^2 \end{pmatrix} =: \begin{pmatrix} W_{pp} & W_{pk} \\ W_{kp} & W_{kk} \end{pmatrix}$$

leads to the following linear equations (in terms of $\boldsymbol{W}(\epsilon_{pk})$):

$$W_{pk} - W_{kk} - P_{\mathcal{I}(k)}R_{pk} - Q_{\mathcal{I}(k)}X_{pk} = 0,\ \epsilon_{pk} \in \mathcal{E}\,. \tag{3}$$

Note that $\boldsymbol{W}(\epsilon_{pk})$ are positive semidefinite matrices (denoted by $\boldsymbol{W}(\epsilon_{pk}) \succeq 0$) of rank one. The active and reactive power consumed by the subtree $(\mathcal{I}(k), \mathcal{E}(k))$ are given by

$$\begin{aligned} P_{\mathcal{I}(k)} &= \sum_{l\in\mathcal{I}(k)}\sum_{j=1}^{J} z_{lj}p_{lj} + \sum_{l\in\mathcal{I}(k)}\sum_{\epsilon_{ls}\in\mathcal{E}(k)} L_{ls}^P, \\ Q_{\mathcal{I}(k)} &= \sum_{l\in\mathcal{I}(k)}\sum_{\epsilon_{ls}\in\mathcal{E}(k)} L_{ls}^Q, \end{aligned} \tag{4}$$

where by [14, Appendix B],

$$\begin{aligned} L_{ls}^P &= (W_{ll} - 2W_{ls} + W_{ss})R_{ls}/(R_{ls}^2 + X_{ls}^2), \\ L_{ls}^Q &= (W_{ll} - 2W_{ls} + W_{ss})X_{ls}/(R_{ls}^2 + X_{ls}^2). \end{aligned}$$

Note that W_{kk} are dependent on vectors $\boldsymbol{p}$ and $\boldsymbol{z}$. We write $W_{kk}(\boldsymbol{p}, \boldsymbol{z})$, when we wish to emphasize this. If we use this model to describe voltages, the function $\boldsymbol{p}(\boldsymbol{z})$ is then given by

$$\begin{aligned} &\max_{\boldsymbol{p}} && \sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij}u_{ij}(p_{ij}) \\ &\text{subject to} && (1), (3),\ \underline{v}_i \le W_{ii} \le \overline{v}_i, \\ & && \boldsymbol{W}(\epsilon_{ik}) \succeq 0,\ \text{rank}(\boldsymbol{W}(\epsilon_{ik})) = 1,\ \epsilon_{ik} \in \mathcal{E}, \end{aligned} \tag{5}$$

for $z_{ij} > 0$. If $z_{ij} = 0$, then take $p_{ij} = 0$. In addition, $0 < \underline{v}_k \le W_{00} \le \overline{v}_k$ are the voltage limits. Observe that the optimization problem (OP) (5) is non-convex and is NP hard due to rank-one constraints. Removing the non-convex constraints yields a convex relaxation, which is a second-order cone program. Further, by Remark 1, the upper bound in the voltage constraint of OP (5) can be replaced by $W_{ii} < \infty$. Thus, by [33, Theorem 5], we obtain that the convex-relaxation OP is exact.

F. Linearized Distflow model

Though the previous voltage model is tractable enough for a convex relaxation to be exact, it is rather complicated. Assuming that the active and reactive power losses on edges are small relative to the power flows, but now allowing the voltages to be complex numbers, we derive a linear approximation of the previous model, called the *linearized (or simplified)*

Distflow model [16]. In this case, the voltage magnitudes $W_{kk}^{lin} := |V_k^{lin}|^2$ have an analytic expression [32, Lemma 12]:

$$W_{kk}^{lin}(\boldsymbol{p}, \boldsymbol{z}) = W_{00} - 2 \sum_{\epsilon_{ls} \in \mathcal{P}(k)} R_{ls} \sum_{m \in \mathcal{I}(s)} \sum_{j=1}^{J} z_{mj} p_{mj}, \quad (6)$$

where the $\mathcal{P}(k)$ is the unique path from the feeder to node k.

Remark 1. *Note that $W_{kk}^{lin} \leq W_{00}$ for all nodes k, as we assume that the nodes only consume power, and by [32, Lemma 12] we obtain $W_{kk}(\boldsymbol{p}, \boldsymbol{z}) \leq W_{kk}^{lin}(\boldsymbol{p}, \boldsymbol{z})$.*

To derive the representation of the power allocation mechanism $\boldsymbol{p}(\boldsymbol{z})$ in this setting, one replaces the constraints in (5) by (1) and $\underline{v}_k \leq W_{kk}^{lin}(\boldsymbol{p}, \boldsymbol{z}) \leq \overline{v}_k$.

When adding stochastic dynamics, the resulting model is still rather complicated. Even Markovian assumptions yield a multidimensional Markov chain of which the transition rates are governed by solutions of nonlinear programming problems. In the following proposition, we identify a special case of our setting in which $\boldsymbol{p}(\boldsymbol{z})$ is explicit and leads to a fair allocation of power to all users, in the sense that the charging rate of a battery does not depend on the location where it is parked. Specifically, if we remove the load constraints by setting $M_i = \infty$ and $c_j^{\max} = \infty$, we can not directly control the power each node receives after the functions u_{ij} have been set. As a consequence, this may lead to the situation that the nodes close to the feeder consume almost all the power.

As a result, an important question is under what assumptions all EVs in the system are charged at the same power rate. The following proposition give a partial positive answer to this question. Let $SR_i = \sum_{\epsilon_{ls} \in \mathcal{P}(i)} R_{ls}$ and $\delta = \frac{W_{00} - \underline{v}}{2}$.

Proposition 1 (Load balancing)**.** *Let $\underline{v}_i = \underline{v}$ and the network be a line. Moreover, assume that $M_i = \infty$ and $c_j^{max} = \infty$. Take $u_{ij}(p_{ij}) = w_i log(p_{ij})$ and the power flow model* (6)*. If $w_i = SR_i$, then we have that $p_{ij}(\boldsymbol{z}) = p(\boldsymbol{z}) > 0$. Moreover, $p(\boldsymbol{z}) = \delta \Big(\sum_{i=1}^{I} SR_i \sum_{j=1}^{J} z_{ij} \Big)^{-1}$.*

A similar result can be shown for more general trees, under the assumption that the root node has only one child.

III. An allocation mechanism with the product-form property

The dynamics of the high-dimensional non-Markovian stochastic process $(Q_{ij}(t), Z_{ij}(t))$, $t \geq 0$, are in general not tractable from a probabilistic viewpoint. To obtain a Markovian description, we would also have to keep track of all residual parking times and charging requirements, leading to a measure-valued process as in [11]. Therefore, we consider fluid approximations of $(Q_{ij}(t), Z_{ij}(t))$ in the next section, which are more tractable.

Despite the complexity of our stochastic model, we are able to identify a special case for which the entire network behaves like a multiclass processor sharing queue, of which the invariant distribution is explicit, and for which even time-dependent properties are known [4, 34].

We take $J = 1$ for convenience and drop all indices j from the notation in this section. For every node i, let $\rho_i = \lambda_i \mathbb{E}[B] SR_i / \delta$ and $\rho = \sum_{i=1}^{I} \rho_i$.

Theorem 1. *Assume $J = 1$ and $K_i = M_i = \infty$ for all i, and a line network. Assume also $c^{max} = \infty$. Furthermore, consider the power allocation rule $\boldsymbol{p}(\boldsymbol{z})$ under the linearized Distflow model* (6)*. If the power allocation rule is proportional fairness, i.e., $u_i(p_i) = \log p_i$, then, for every $\boldsymbol{n} \in \mathbb{N}_+^I$,*

$$\lim_{t \to \infty} \mathbb{P}\big(\boldsymbol{Z}(t) = \boldsymbol{n}\big) = (1 - \rho) (\sum_{i=1}^{I} n_i)! \prod_{i=1}^{I} \frac{\rho_i^{n_i}}{n_i!}, \quad (7)$$

provided $\rho < 1$.

Note that this result is valid for arbitrary distributions of the charging requirments and as such it provides an inensitivity property. This result is another exhibition of the appealing nature of proportional fairness, which has also shown to give similar nice properties in communication network models [4, 6, 7, 18].

The proof of this theorem follows from a similar argument as in [6], making a connection with the class of Whittle networks, by showing that a specific local balance property called *balanced fairness* is satisfied. We explain this procedure for the case of exponential charging times in the proofs section.

IV. Fluid Approximation

In this section, we develop a fluid approximation for the stochastic model defined in Section II, of which the invariant point is characterized through an OPF problem. To do so, we follow a similar approach as in [11] and [35].

The fluid approximation, which is deterministic, can be thought of as a formal law of large numbers approximation. More precisely, consider a family of models as defined in Section II, indexed by a scaling parameter $n \in \mathbb{N}$. The fluid scaling is given by $\frac{Z_{ij}^n(\cdot)}{n}$. To obtain a non-trivial fluid limit, we choose the following scaling for the node parameters in the n^{th} system. The maximum power that node i can consume is nM_i, the arrival rate is $n\lambda_{ij}$, the number of parking spaces is nK_i; all other parameters remain unchanged. A mathematically rigorous justification of this scaling is beyond the scope of this study, and will be pursued elsewhere. If the set of feasible power allocations is polyhedral, the methods from [11] can be applied directly to achieve this justification. Formal or rigorous, this scaling gives rise to the following definition of a fluid model.

Definition 1 (Fluid model)**.** *A nonnegative continuous vector-valued function $\boldsymbol{z}(\cdot)$ is a fluid-model solution if it satisfies the functional equations (for $i = 1, \ldots, I, j = 1, \ldots, J$)*

$$z_{ij}(t) = z_{ij}(0) \mathbb{P}\Big(B_j^0 > \int_0^t p_{ij}(\boldsymbol{z}(u)) du, D_j^0 \geq t\Big) + \int_0^t \gamma_{ij}(s) \mathbb{P}\Big(B_j > \int_s^t p_{ij}(\boldsymbol{z}(u)) du, D_j \geq t - s\Big) ds,$$

where $\gamma_{ij}(t) := \lambda_{ij} \mathbb{1}_{\{q_i(t) < K_i\}}$, and $q_i(t) = \sum_j q_{ij}(t)$, with

$$q_{ij}(t) = q_{ij}(0) \mathbb{P}\big(D_j^0 \geq t\big) + \int_0^t \gamma_{ij}(s) \mathbb{P}\big(D_j \geq t - s\big) ds.$$

Further, B_j^0 and D_j^0 are the energy demand and the parking time for the initial population in the system

The time dependent fluid-model solution can be used directly to approximate the evolution of the system at time t, e.g., one may take $\mathbb{E}[Z_{ij}(t)] \approx z_{ij}(t)$. This set of equations can be extended to time-varying arrival rates by replacing λ_{ij} by $\lambda_{ij}(t)$. Also, one can consider schemes in which blocked cars are not lost, but routed to adjacent parking lots, which lead to further modifications to γ_{ij}. The fluid-model equations, though still rather complicated, have an intuitive meaning, e.g. the term $\mathbb{P}\big(B_j > \int_s^t p_{ij}(\boldsymbol{z}(u))du, D_j \geq t-s\big)$ resembles the fraction of cars of type-j admitted to the system at time s at node i which are still in the system at time t (for this to happen, their deadline needs to exceed $t-s$ and their service requirement needs to be bigger than the service allocated, which is $\int_s^t p_{ij}(\boldsymbol{z}(u))du$). To solve the fluid model equations, one can proceed numerically by Picard iteration. In some cases, the set of equations can also be solved explicitly, as the next proposition illustrates.

Proposition 2. *Assume that $K_i = \infty$, $J = 1$, and the energy demands and parking times are exponential and independent. Under the assumptions of Proposition 1, we have that*

$$z_i(t) = z_i^* + (z_i(0) - z_i^*)e^{-t/\mathbb{E}[D]},$$

with

$$z_i^* = \mathbb{E}[D]\left(\gamma_i - \Lambda_i^*/\mathbb{E}[B]\right), \quad \Lambda_i^* = \frac{\gamma_i \delta}{\sum_{k=1}^{I} S R_k \gamma_k}.$$

The previous proposition continues holds for K_i big enough such that the parking lots are never full.

We now turn to the behavior of our fluid model in equilibrium, i.e., for $t = \infty$. In this case, we obtain a computationally tractable characterization through a particular OPF problem. Before we state our main theorem, we introduce some notation. Let

$$\gamma_{ij} := \frac{\lambda_{ij}}{\lambda_i} \min\left\{\lambda_i, K_i \left(\sum_{j'=1}^{J} \frac{\lambda_{ij'}}{\lambda_i} E[D_{j'}]\right)^{-1}\right\},$$

where $\lambda_i := \sum_{j=1}^{J} \lambda_{ij}$. Furthermore, define $g_{ij}(x) := \gamma_{ij}\mathbb{E}[\min\{D_j x, B_j\}]$ and the node allocation (the power which type-j EVs consume at node i), $\Lambda_{ij}(\boldsymbol{z}) := z_{ij}p_{ij}(\boldsymbol{z})$. Also, for a random variable Y, denote by $\inf(Y)$ the leftmost point of its support.

Theorem 2 (Characterization of invariant point)**.** *(i) If z_{ij}^* is an invariant point for the fluid model, it is given by the solution of the fixed-point equation*

$$z_{ij}^* = \gamma_{ij}\mathbb{E}[\min\{D_j, \frac{B_j}{p_{ij}(\boldsymbol{z}^*)}\}]. \tag{8}$$

(ii) Let $\inf(D_j/B_j) \leq 1/c_j^{max}$. The solution $\boldsymbol{z}^*$ of (8) is unique and is given by $z_{ij}^* = \frac{\Lambda_{ij}^*}{g_{ij}^{-1}(\Lambda_{ij}^*)}$, where $\boldsymbol{\Lambda}^*$ is the unique solution of the optimization problem

$$\begin{aligned} \mathit{max}_{\boldsymbol{\Lambda}} \quad & \sum_{i=1}^{I}\sum_{j=1}^{J} G_{ij}(\Lambda_{ij}) \\ \mathit{subject\ to} \quad & W_{ik} - W_{kk} - P_{\mathcal{I}(k)}R_{ik} - Q_{\mathcal{I}(k)}X_{ik} = 0, \\ & \underline{v}_i \leq W_{ii} \leq \overline{v}_i,\ \Lambda_{ij} \leq M_i, \\ & 0 \leq \Lambda_{ij} \leq g_{ij}(c_j^{max}),\ \epsilon_{ik} \in \mathcal{E}\,. \end{aligned} \tag{9}$$

$G_{ij}(\cdot)$ is a strictly concave function such that $G_{ij}'(\cdot) = u_{ij}'(g_{ij}^{-1}(\cdot))$.

By (4), observe that W_{kk} depends on $\boldsymbol{z}$ through the products $z_{ij}p_{ij}(\boldsymbol{z})$. By the definition of $\boldsymbol{\Lambda}$, we have that W_{kk} depends only on $\boldsymbol{\Lambda}$. That is, the previous OP is indeed independent of the fixed point $\boldsymbol{z}^*$.

The intuition behind (8) follows from a result in queueing theory known as Little's law [17], which says that the expected number of customers equals the arrival rate times the expected sojourn time of a particular customer. The sojourn time of a customer of type-j at node i is the minimum of its deadline, and its potential service time. The latter quantity is approximated by the total service time divided by the service rate, which will be approximately constant in equilibrium for large systems (this is called the snapshot principle in queueing theory [36]). To arrive at (9) the essential idea is to add Little's law (8) to the set of KKT conditions that characterize $\boldsymbol{p}(\boldsymbol{z})$, and rewrite all equations in such a way that they form the KKT conditions for the problem (9). We refer to the Appendix for more details.

To show that a fluid model solution converges towards the equilibrium point as $t \to \infty$, we also assume that the power rate which an EV gets does not increase if a newly arriving EV enters the system:

Definition 2. *An allocation mechanism is called "monotone" if $0 < \boldsymbol{y} \leq \boldsymbol{z}$ implies that $p_{ij}(\boldsymbol{y}) \geq p_{ij}(\boldsymbol{z})$.*

By adapting techniques from [10], it can be shown that this assumption holds in the linearized Distflow model. We conjecture it is true for the more general AC allocation mechanism as well.

Proposition 3. *Let the allocation mechanism is monotone, the fixed point unique, $K_i = \infty$, we have that $z_{ij}(t) \to z_{ij}^*$, as $t \to \infty$.*

We conjecture this result to be true for general AC allocation mechanisms, without imposing the condition $K_i = \infty$, but the mathematical techniques required to establish this are beyond the scope of this paper.

V. Numerical evaluation

In this section, we give a numerical illustration of our results. We consider the case $J = 1$, and assume $I = 2$ nodes on a line. Fix $V_0 = 1$ in the p.u. system, take $R_{01} = X_{01} = 0.01$, $R_{12} = X_{12} = 0.005$, and $\underline{v}_1 = \underline{v}_2 = (0.9)^2$. We consider a weighted proportional fairness allocation as in Proposition 1. Further, the energy demand and the parking times are exponentially distributed with unit mean, and we take $K_1 = K_2 = K$ in all our computations.

As a first experiment we consider this model without load constraints, i.e., $M_i = \infty$ and $c^{\max} = \infty$. We focus on an overloaded case, i.e., $\lambda_i = 1.2K_i$, and compare $\mathbb{E}[Z_i]$ between the AC and Distflow allocation mechanisms, as well as their fluid approximations z_i^*.

First, we compare the two power flow models for the stochastic model computing the expected number of uncharged EVs at any node, Table I. We note that the expectations for the

AC model are higher as this model takes into account power losses leading to lower service rates.

TABLE I

K	$\mathbb{E}[Z_1], \mathbb{E}[Z_2]$ (Distflow)	$\mathbb{E}[Z_1], \mathbb{E}[Z_2]$ (AC)	Rel. error
10	$4.5336, 4.6179$	4.6801, 4.6924	3.13 %, 1.58 %
20	$14.0174, 14.0385$	$14.1725, 14.1948$	1.09 %, 1.10 %

Next, we evaluate the fluid approximation for the two load models, see Table II. Observe again that the number of uncharged EVs for the AC model is higher. The relative error between the two load flow models is similar to what we saw in Table I.

TABLE II

K	z_1^*, z_2^* (Distflow)	z_1^*, z_2^* (AC)	Rel. error
10	$4.5769, 4.5769$	$4.7356, 4.7513$	$3.35\%, 3.67\%$
20	$14.0300, 14.0300$	$14.1849, 14.2069$	$1.09\%, 1.25\%$
30	$23.6820, 23.6820$	$23.8357, 23.8597$	$0.64\%, 0.74\%$
40	$33.4293, 33.4293$	$33.5823, 33.6073$	$0.45\%, 0.53\%$
50	$43.2330, 43.2330$	$43.3857, 43.4112$	$0.35\%, 0.41\%$

Not surprisingly, the original stochastic model is not numerically tractable for high values of K. Our results though show that this is not a problem, as our fluid approximation performs very well:

TABLE III

Relative error of fluid approximations		
K	Distflow	AC
10	$0.95\%, 0.86\%$	$1.18\%, 1.25\%$
20	$0.09\%, 0.06\%$	$0.08\%, 0.08\%$

Now, we move to the case that $M_i < \infty$ and $c^{\max} < \infty$. In this case, we fix $c^{\max} = 1$ and we plot the relative error between the fluid approximation and the stochastic model for the fraction of EVs that get successfully charged. Using arguments from queueing theory, it can be shown that this fraction equals $1 - \mathbb{E}[Z_i]/\mathbb{E}[Q_i]$. For the fluid approximation, we replace the numerator with z_i^*, while $\mathbb{E}[Q_i]$ can be computed explicitly, using the Erlang Loss formula [4]. Figures 1 and 2 show the results for all possible values of M_i and for both load models. Though the quality of the fluid approximation deteriorates, the relative error is generally below 10% and for reasonably high values of M_i, it is even smaller. We also expect these results to improve for bigger K. For higher values of K one needs to solve millions of OPF problems of type (5) to obtain the steady-state behavior of the stochastic model, while the fluid model only requires the solution of a single OPF problem.

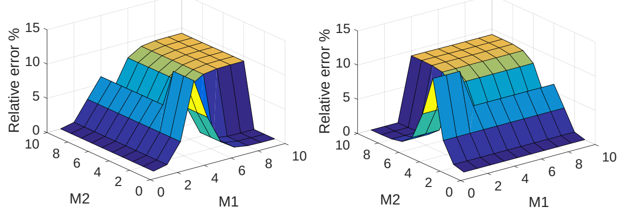

Fig. 1. Distflow in case $\boldsymbol{K} = (10, 10)$ and $\boldsymbol{\lambda} = (12, 12)$

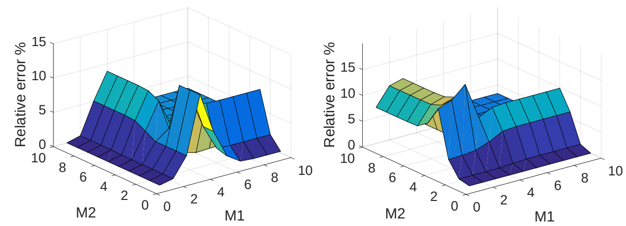

Fig. 2. AC in case $\boldsymbol{K} = (10, 10)$ and $\boldsymbol{\lambda} = (12, 12)$

VI. Concluding remarks

This paper has proposed a queueing network model for electric vehicle charging. The main result is a fluid approximation of the number of uncharged jobs in the system, which is derived by combining key ideas from network utility maximization (using class of utility maximizing scheduling disciplines), queueing theory (Little's law and the snapshot principle), and load flow models (AC and linearized Distflow) from power systems engineering. Our fluid approximation explicitly captures the interaction between these three three elements, as well as physical network parameters, and can be computed using convex programming techniques. Our approach can easily be extended to impose other reliability constraints, such as line limits, and our fluid approach can be extended to deal with superchargers, by allowing additional queues at each station.

We focused on the specific class of weighted proportional fairness protocols. Our optimization framework allows for a further comparison between charging protocols, which is a natural next step for further research. As mentioned in the Introduction, another important problem is to extend our model to include other features such as smart appliances/buildings/meters, rooftop solar panels, and other sources of electricity demand, and to allow for batteries to de-charge. This naturally lead to various questions about pricing schemes. Again, we think that our characterization of the performance of the system for a fixed control in terms of the OPF problem (9) can be a useful starting point to design economic mechanisms.

A full mathematical examination of our system is beyond the scope of the present work. In a follow-up work we will rigorously show that z_{ij}^* is a good approximation of Z_{ij} by developing a fluid limit theorem using the framework of measure-valued processes, extending the framework of [11].

Appendix

Proof of Proposition 1

Under the assumptions of Proposition 1, and observing that W_{ii}^{lin} is decreasing in i, (5) takes the following form:

$$\max_{\boldsymbol{p}} \quad \sum_{i=1}^{I} \sum_{j=1}^{J} z_{ij} w_i \log(p_{ij})$$

$$\text{subject to} \quad \sum_{k=1}^{I} R_{k-1k} \sum_{m=k}^{I} \sum_{j=1}^{J} z_{mj} p_{mj} \geq \delta.$$

By the Karush-Kuhn-Tucker (KKT) conditions, there exists $h \in \mathbb{R}_+$ such that for any $i \in \mathcal{I}\setminus\{0\}$ and $j \in \mathcal{E}$,

$$\frac{z_{ij} w_i}{p_{ij}} = h z_{ij} SR_i. \tag{10}$$

and

$$h\Big(\sum_{k=1}^{I} R_{k-1k} \sum_{m=k}^{I}\sum_{j=1}^{J} z_{mj}p_{mj} - \delta\Big) = 0 \tag{11}$$

Note that h cannot be zero due to (10). Again, by (10), we have that $\frac{w_i}{p_{ij}SR_i}$ should be constant in i, j. In particular

$$\frac{w_i}{p_{ij}SR_i} = \frac{w_1}{p_{11}R_1}. \tag{12}$$

Choosing $w_i = SR_i$, we see that $p_{ij} = p_{11}$, for all possible i, j. Combining (11) and (12), we derive the expression for $p(\boldsymbol{z})$.

Proof of Theorem 1

Using similar analysis as in the proof of Proposition 1, it can be shown that

$$p_i(\boldsymbol{z}) = \frac{\delta}{SR_i \sum_{l=1}^{I} z_l}.$$

Next, observe that the so-called balance property [37] holds: for $i, k \in \mathcal{I}\setminus\{0\}$,

$$p_i(\boldsymbol{z}+\boldsymbol{e}_k)p_k(\boldsymbol{z}) = p_i(\boldsymbol{z})p_k(\boldsymbol{z}+\boldsymbol{e}_i),$$

where $\boldsymbol{e}_i$, $i = 1,\ldots,I$ denote unit vectors.

If B is an exponential random variable, the process $\boldsymbol{Z}(t),\ t \geq 0$ is Markov. Let at time $t \geq 0$ the process be at state $\boldsymbol{Z}(t) = \boldsymbol{z}$. The process can move to the state $\boldsymbol{z}+\boldsymbol{e}_i$ with rate λ_i and to the state $\boldsymbol{z}-\boldsymbol{e}_i$ with rate $z_i p_i(\boldsymbol{z})/\mathbb{E}[B]$, if $z_i > 0$. Setting $\mu_i = \frac{SR_i\mathbb{E}[B]}{\delta}$, the transition rates become the same as a that of a multiclass Markovian processor sharing queue with I classes of customers and mean service times $1/\mu_i$. By [38], the stationary distribution of a processor sharing queue is given by (7).

Further, it is shown in [37], that the stationary distribution of a processor sharing queue is insensitive to the distribution of service times if the balanced property satisfied. That is, (7) holds for general charging requirement B. The construction carried out in [37] can also be carried out in our setting.

Proof of Proposition 2

We denote $\mathbb{E}[B] = 1/\mu$ and $\mathbb{E}[D] = 1/\nu$, and for simplicity take $z_i(0) = 0$. By the assumption that $K_i = \infty$, it follows $\gamma_i(t) \equiv \gamma_i \equiv \lambda_i$. Before we move to the main part of the proof, we show some helpful relations.

Under the assumptions of Proposition 1, we can solve (9) explicitly. To see this, note that $G_i'(\Lambda_i) = w_i\frac{\gamma_i - \mu\Lambda_i}{\nu\Lambda_i}$. Writing the KKT conditions for OP (9), we have that there exists $h \in R_+$, such that for any $i \in \mathcal{I}\setminus\{0\}$, the optimal solution $\boldsymbol{\Lambda}^*$ satisfies

$$w_i\frac{\gamma_i - \mu\Lambda_i^*}{\nu\Lambda_i^*} = hSR_i,$$

and

$$\sum_{k=1}^{I} SR_k\Lambda_k^* = \delta. \tag{13}$$

The first equation (taking into account $w_i = SR_i$) implies that

$$\frac{\gamma_i - \mu\Lambda_i^*}{\nu\Lambda_i^*} = \frac{\gamma_1 - \mu\Lambda_1^*}{\nu\Lambda_1^*}.$$

Finally, we get the following expressions for Λ_i^*,

$$\Lambda_i^* = \frac{\gamma_i}{\gamma_1}\Lambda_1^*, \tag{14}$$

and by (13),

$$\Lambda_i^* = \frac{\gamma_i\delta}{\sum_{k=1}^{I} SR_k\gamma_k}. \tag{15}$$

Next, by (8), we compute the fixed point, which is given by

$$z_i^* = \frac{\gamma_i - \mu\Lambda_i^*}{\nu}. \tag{16}$$

It is helpful to note the following relation for the fixed points. Combining the last equation and (14), we get

$$z_i^* = \frac{\gamma_i - \mu(\gamma_i/\gamma_1)\Lambda_1^*}{\nu} = \frac{\gamma_i}{\gamma_1}\frac{\gamma_1 - \mu\Lambda_1^*}{\nu} = \frac{\gamma_i}{\gamma_1}z_1^*. \tag{17}$$

Now, we move to the main part of the proof. Under the Markovian assumptions, $z_i(\cdot)$ is given (alternatively) by the following ODE:

$$z_i'(t) = \gamma_i - \nu z_i(t) - \mu z_i(t)\frac{\delta}{\sum_{k=1}^{I} SR_k z_k(t)}.$$

The last ODE has a unique solution $\boldsymbol{z}(\cdot)$ for given initial point. So, it is enough to show that the function $z_i(t) = z_i^*(1-e^{-\nu})$ satisfies the previous ODE. Plugging in it into the ODE, we have that

$$\nu z_i^* e^{-\nu t} = \gamma_i - \nu z_i^*(1-e^{-\nu t}) - \frac{\mu z_i^*(1-e^{-\nu t})\delta}{\sum_{k=1}^{I} SR_k z_k^*(1-e^{-\nu t})},$$

which can be simplified to

$$\gamma_i = \nu z_i^* + \frac{\mu z_i^*\delta}{\sum_{k=1}^{I} SR_k z_k^*}.$$

By (16), we derive

$$\Lambda_i^* = \frac{z_i^*\delta}{\sum_{k=1}^{I} SR_k z_k^*}.$$

Now, we apply (17) to get

$$\Lambda_i^* = \frac{\gamma_i\delta}{\sum_{k=1}^{I} SR_k\gamma_k},$$

which holds by (15).

Proof of Theorem 2

As a preliminary, we observe some properties of the function $g_{ij}(\cdot)$. Define $a_{ij} = \inf\{x \geq 0 : g_{ij}(x) = \gamma_{ij}/\mathbb{E}[D_j]\}$. By [11, Lemma 6], we have that $g_{ij}(\cdot)$ is continuous, strictly increasing in $[0, a_{ij}]$, and constant in $(a_{ij}, \infty]$. Further, by [11, Lemma 7], we obtain that if $a_{ij} < \infty$ then $\inf(D_j/B_j) = 1/a_{ij}$, and if $a_{ij} = 0$ then $\inf(D_j/B_j) = 0$.

Now, we move to the main part of the proof. First, we note that for $i \in \mathcal{I}\setminus\{0\}$, we have that

$$W'_{kk}(\boldsymbol{z}, \boldsymbol{p}) = z_{ij} W'_k(\boldsymbol{\Lambda}), \tag{18}$$

where the prime denotes the derivative with respect to p_{ij} and Λ_{ij}, respectively. To see this, observe that by (2) and the definition of the bandwidth allocation $\boldsymbol{\Lambda}$, as $V_k(\boldsymbol{z}, \boldsymbol{p})$ depends on vector $\boldsymbol{z}$ and $\boldsymbol{p}$ only through the product $z_{ij}p_{ij}$, we get

$$V_k(\boldsymbol{z}, \boldsymbol{p}) = V_k(\boldsymbol{\Lambda}).$$

Applying the transformation $\Lambda_{ij}(\boldsymbol{z}, \boldsymbol{p}) = z_{ij}p_{ij}$, this yields

$$\frac{\partial}{\partial p_{ij}} V_k(\boldsymbol{z}, \boldsymbol{p}) = \frac{\partial}{\partial \Lambda_{ij}} V_k(\boldsymbol{\Lambda}) \frac{\partial}{\partial p_{ij}} \Lambda_{ij} = z_{ij} V'_k(\boldsymbol{\Lambda}).$$

Recalling that $W_{kk} = V_k V_k$, we have that

$$W'_{kk}(\boldsymbol{p}) = 2V'_k(\boldsymbol{p})V_k(\boldsymbol{p}) = 2z_{ij}V'_k(\boldsymbol{\Lambda})V_k(\boldsymbol{\Lambda}) = z_{ij}W'_{kk}(\boldsymbol{\Lambda}).$$

By the KKT conditions for (5) there exist $(h_i^1, h_i^2, h_{ij}^3, \bar{h}_{ij}^4) \in \mathbb{R}_+^{2I+2(I\times J)}$ such that for $i \in \mathcal{I}\setminus\{0\}$, $j \in J$,

$$z^*_{ij} u'_{ij}(p_{ij}(\boldsymbol{z}^*)) = \sum_{k=1}^{I} W'_{kk}(\boldsymbol{p}(\boldsymbol{z}^*))(h_k^1 - h_k^2) + h_{ij}^3 z^*_{ij} + \bar{h}_{ij}^4,$$

and

$$h_i^1(W_{ii}(\boldsymbol{p}(\boldsymbol{z}^*)) - \overline{v}_i) = 0, \quad h_i^2(W_{ii}(\boldsymbol{p}(\boldsymbol{z}^*)) - \underline{v}_i) = 0,$$
$$h_{ij}^3(z^*_{ij}p_{ij}(\boldsymbol{z}^*) - M_i) = 0, \quad \bar{h}_{ij}^4(p_{ij}(\boldsymbol{z}^*) - c_j^{max}) = 0.$$

Setting $h_{ij}^4 = \bar{h}_{ij}^4/z^*_{ij}$, the previous equations take the following (equivalent) form

$$u'_{ij}(p_{ij}(\boldsymbol{z}^*)) = \frac{1}{z^*_{ij}} \sum_{k=1}^{I} W'_{kk}(\boldsymbol{p}(\boldsymbol{z}^*))(h_k^1 - h_k^2) + h_{ij}^3 + h_{ij}^4,$$

and

$$h_i^1(W_{ii}(\boldsymbol{p}(\boldsymbol{z}^*)) - \overline{v}_i) = 0, \quad h_i^2(W_{ii}(\boldsymbol{p}(\boldsymbol{z}^*)) - \underline{v}_i)) = 0,$$
$$h_{ij}^3(z^*_{ij}p_{ij}(\boldsymbol{z}^*) - M_i) = 0, \quad h_{ij}^4(p_{ij}(\boldsymbol{z}^*) - c_j^{max}) = 0.$$

By definition of $g_{ij}(\cdot)$ and (8), we have that $\Lambda_{ij}(z^*) = g_{ij}(p_{ij}(\boldsymbol{z}^*))$. Moreover, by the assumption in Theorem 2, we have that $1/a_{ij} = \inf(D_j/B_j) \leq 1/c_j^{\max}$. That is, $c_j^{\max} \leq a_{ij}$. Thus, $g_{ij}(\cdot)$ is strictly increasing in $[0, c_j^{\max}]$. This implies $p_{ij}(\boldsymbol{z}^*) = g_{ij}^{-1}(\Lambda_{ij}(z^*))$, and we note that $(p_{ij}(\boldsymbol{z}^*) - c_j^{max}) = 0$ if and only if $(\Lambda_{ij}(\boldsymbol{z}^*) - g_{ij}(c_i^{max})) = 0$. Using the last observations and (18), the above equations can be rewritten as follows

$$u'_{ij}(g_{ij}^{-1}(\Lambda_{ij}(\boldsymbol{z}^*))) = \sum_{k=1}^{I} W'_{kk}(\boldsymbol{\Lambda})(h_k^1 - h_k^2) + h_{ij}^3 + h_{ij}^4,$$

and

$$h_i^1(W_{ii}(\boldsymbol{\Lambda}(\boldsymbol{z}^*)) - \overline{v}_i) = 0, \quad h_i^2(W_{ii}(\boldsymbol{\Lambda}(\boldsymbol{z}^*)) - \underline{v}_i) = 0,$$
$$h_{ij}^3(\Lambda_{ij}(\boldsymbol{z}^*) - M_i) = 0, \quad h_{ij}^4(\Lambda_{ij}(\boldsymbol{z}^*) - g_{ij}(c_i^{max})) = 0.$$

Now, we observe that the last equations are KKT conditions for the OP (9). To complete the proof, it remains to be shown that the function $G_{ij}(\cdot)$ is strictly concave. To this end, observe that $g_{ij}^{-1}(\cdot)$ is strictly increasing and $u'_{ij}(\cdot)$ is strictly decreasing since the last is strictly concave function. It follows that $G'_{ij}(\cdot)$ is strictly decreasing and hence $G_{ij}(\cdot)$ strictly concave function. The last implies that OP (9) has unique solution independent of the fixed point $\boldsymbol{z}^*$, say $\boldsymbol{\Lambda}^*(\boldsymbol{z}^*) = \boldsymbol{\Lambda}^*$. Further, the unique fixed point is given by

$$z^*_{ij} = \frac{\Lambda^*_{ij}}{p_{ij}(\boldsymbol{z}^*)} = \frac{\Lambda^*_{ij}}{g_{ij}^{-1}(\Lambda^*_{ij})}. \tag{19}$$

Proof of Proposition 3

We present the proof for the case that K_i is big enough such that $\mathbb{1}_{\{q_i(t)<K_i\}} = 1$. As there common parts with [11, Theorem 3, Corollary 1], we present only a short proof sketch. By following the same argument as in [11, Theorem 3] there exist $\boldsymbol{l}, \boldsymbol{h} \in (0, \infty)^{I\times J}$, such that

$$l_{ij} \leq \liminf_{t\to\infty} z_{ij}(t) \leq \limsup_{t\to\infty} z_{ij}(t) \leq h_{ij}. \tag{20}$$

Further, using the assumption of the monotone network, $\boldsymbol{l}$ and $\boldsymbol{h}$ satisfy the following relations

$$l_{ij} = \gamma_{ij}\mathbb{E}[\min\{D_j, \frac{B_j}{p_{ij(\boldsymbol{l})}}\}], \; h_{ij} = \gamma_{ij}\mathbb{E}[\min\{D_j, \frac{B_j}{p_{ij(\boldsymbol{h})}}\}].$$

By uniqueness of the fixed point, we have that $l_{ij} = h_{ij}$. Now, by (20), the result follows.

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