Nearest-neighbour Markov point processes on graphs with Euclidean edges

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Abstract
We define nearest-neighbour point processes on graphs with Euclidean edges and linear networks. They can be seen as the analogues of renewal processes on the real line. We show that the Delaunay neighbourhood relation on a tree satisfies the Baddeley–Møller consistency conditions and provide a characterisation of Markov functions with respect to this relation. We show that a modified relation defined in terms of the local geometry of the graph satisfies the consistency conditions for all graphs with Euclidean edges.

Keywords & Phrases: Delaunay neighbours; Graph with Euclidean edges; Linear network; Markov point process; Nearest-neighbour interaction; Renewal process.

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1 Introduction

In recent years, a theory of point processes on linear networks has been emerging so as to be able to analyse, for example, the prevalence of accidents on motorways, the occurrence of street crimes and other data described in the first chapter of the pioneering monograph by Okabe and Sugihara [10]. Although there exists a mature theoretical framework for point processes on Euclidean spaces [6], the development of a similar theory on linear networks is complicated by the geometry inherent in the network. In particular, it is not possible to define strictly stationary models, as the network may not be closed under translations. For this reason, most attention has focussed on the development of second order summary statistics [11].

Little attention has been paid to model building with a few notable exceptions. The first serious work in this direction seems to be that by Baddeley et al. [4], who construct certain types of Cox processes as well as a Switzer-type and a cell process. The authors conclude that familiar procedures for constructing models tend not to produce processes on a linear network that are pseudostationary with respect to the shortest path distance, except when the network is a tree – an unrealistic assumption for a road network. Another important contribution is the work by Anderes et al. [1] who expand the modelling framework in various directions. They relax the assumption of [4, 10, 11] that a linear network consists of a finite union of straight line segments that intersect only in vertices, in the sense that the segments are replaced by parametrised rectifiable curves that may or may not overlap. The parametrisations have the additional advantage of naturally defining a weighted shortest path distance. In the motivating example where the linear network represents a road network, such a generalisation allows for bridges or tunnels and for distance to be measured in travel time where appropriate. Additionally, Anderes et al. [1] construct log-Gaussian Cox processes in terms of a Gaussian process on the network specified by an isotropic covariance function.

All models discussed so far are clustered in nature, that is, exhibit a positive association between the points. In this paper, our aim is to develop appropriate analogues of renewal processes, exploiting the one-dimensional nature of a linear network. Recall that renewal processes exhibit the property that the probability (in an infinitesimal sense) of an event at a given location conditional on the realisation of the process elsewhere depends only on the two nearest points, regardless of how far away they may be. Such models are known as nearest-neighbour Markov point processes [2] and may incorporate both inhibition and clustering [9]. In contrast to Cox models, the second order summary statistics may not be available in closed form, but the conditional intensities and likelihood are. The latter can be expressed as a product of interaction functions, which may be chosen to be isotropic.

The plan of this paper is as follows. In Section 2, we recall the definitions of Anderes et al. [1] regarding graphs with Euclidean edges, the weighted shortest path metric thereon and Poisson process defined on them. In Section 3, we extend the notion of a Markov point process with respect to the Delaunay nearest-neighbour relation [2] to graphs with Euclidean edges and state our main results. More specifically, we show that the Delaunay relation on a tree satisfies the Baddeley–Møller consistency conditions and we provide a characterisation of the Markov functions with respect to this relation. We then use the graph structure to define a modified Delaunay relation and show that it satisfies the consistency conditions on any graph with Euclidean edges. The proofs are given in Sections 4–5.

2 Preliminaries

2.1 Graphs with Euclidean edges

In their pioneering monograph on the subject, Okabe and Sugihara [10, page 31] define a network as a finite union

$$L = \bigcup_{i=1}^{n} l_i, \quad n \in \mathbb{N},$$

of straight line segments l_i in $\mathbb{R}^2$ or $\mathbb{R}^3$ that intersect only at their endpoints in such a way that L is connected. The representation is not unique since a line segment may arbitrarily be split in two pieces without affecting the union L.

A more general definition is given by Anderes, Møller and Rasmussen [1]. They replace the straight line segments by curves that are parametrised by bijections. In order to define a weighted shortest path distance on the graph, we impose the further condition that these parametrisations are diffeomorphisms and follow [1] to define the weight of an edge as the 'length' induced by the parametrisation.

Definition 1 *A graph with Euclidean edges in $\mathbb{R}^2$ is a triple $(V, E = (e_i)_i, \Phi = (\phi_i)_i)$ such that*

- *(V, E) is a finite, simple connected graph, i.e. has neither loops nor multiple edges;*

- *every edge* $e_i = \{v_i^1, v_i^2\} \in E$, $v_i^1, v_i^2 \in V$, *is parametrised by the inverse of a bijection* ϕ_i, *that is,* $\phi_i^{-1} : J_i \to \mathbb{R}^2$ *for a non-empty open interval* $J_i \subset \mathbb{R}$ *with endpoints* $\phi_i(v_i^j)$, $j = 1, 2$.

In other words, an edge is associated with a set $\phi_i^{-1}(J_i) \subset \mathbb{R}^2$ that does not contain the endpoints.

A graph with Euclidean edges gives rise to the space

$$L = (\{0\} \times V) \cup \bigcup_{i=1}^{n(E)} (\{i\} \times \phi_i^{-1}(J_i)),$$

where $n(E) < \infty$ is the cardinality of E. The labels i serve to identify the edges and will prevent paths from 'jumping from one edge to another' in case their interiors overlap. For instance, if L represents a road network, overlap is typically present due to tunnels and bridges [1].

As an aside, if there is no overlap between the edge interiors, one may drop the labels and simply consider the disjoint union

$$V \cup \bigcup_{i=1}^{n(E)} \phi_i^{-1}(J_i),$$

which in turn reduces to the classic linear networks if all edges $\phi_i^{-1}(J_i)$ are straight line segments. From now on, we will work with the general space L including the labels.

2.2 Weighted shortest path metric

The family Φ of parametrisations that is part of the definition of a graph with Euclidean edges can be used to define concepts of length and distance [1].

Definition 2 *Let* (V, E, Φ) *be a graph with Euclidean edges. For every* $i = 1, \dots, n(E)$, *define the* ϕ_i^{-1}*-induced length measure on the* σ*-algebra* $\{\phi_i^{-1}(B) : B \subset \bar{J}_i \text{ Borel}\}$ *on* $\phi_i^{-1}(\bar{J}_i)$ *generated by the functions* ϕ_i *as follows. The induced length of the set* $\phi_i^{-1}(B)$ *is equal to the Euclidean length of* B *in the closed interval* $\bar{J}_i$. *In particular, the edge* $e_i = \{v_i^1, v_i^2\}$ *has length* $|\phi(v_i^1) - \phi(v_i^2)|$ *equal to the Euclidean length of* J_i.

If we use the arc length parametrisation, the induced length of Definition 2 corresponds to the usual Euclidean length. A different set of parametrisations may induce a different length measure.

Definition 3 *Let* (V, E, Φ) *be a graph with Euclidean edges. A walk between two elements* (i, x) *and* (j, y) *of* L *travels alternatingly from* (i, x) *to* (j, y) *along a finite number of nodes and edges such that the nodes are endpoints of the edges. If* $i \neq 0$, *the first bit of the walk travels from* x *along* $\phi_i^{-1}(J_i)$; *similarly for* $j \neq 0$, *the last bit of the walk is along* $\phi_j^{-1}(J_j)$ *to* y. *In particular, a walk between two points along the same curve* $\phi_i^{-1}(J_i)$ *does not reverse its tracks.*

Definition 4 *Let (V, E, Φ) be a graph with Euclidean edges. A path between two different elements (i, x) and (j, y) of L is a walk in which all edge segments and all vertices are different. The weight of the path is the sum of the lengths of the edge segments in it, and the shortest path distance $d_G((i, x), (j, y))$ between two elements (i, x) and (j, y) of L is the smallest weight carried by any path between them.*

Note that it is possible that the shortest weighted path between two vertices joined by an edge is not along that edge! The assumption that the parametrisations are diffeomorphisms ensures that the edge lengths are finite. Moreover, the weighted shortest path distance on a graph with Euclidean edges defines a metric. For further details, see [1].

2.3 Poisson process on graphs with Euclidean edges

Our goal in this paper is to define point process on graphs with Euclidean edges in terms of a density with respect to a unit rate Poisson process [9]. To do so, we recall Anderes et al.'s definition of a Poisson process on a graph with Euclidean edges [1].

Definition 5 *Let $(V, E = (e_i)_i, \Phi = (\phi_i)_i)$ be a graph with Euclidean edges and L the corresponding network. The Lebesgue–Stieltjes measure λ_G is defined as follows. For every i and every set B_i in the σ-algebra generated by ϕ_i, set*

$$\lambda_G^i(B_i) = \int_{J_i} 1\{\phi_i^{-1}(t) \in B_i\} \left| \frac{d}{dt}\phi_i^{-1}(t) \right| dt,$$

where $|\cdot|$ denotes the norm of the gradient. Then

$$\lambda_G \left(\cup_{i=1}^{n(E)} (\{i\} \times B_i) \right) = \sum_{i=1}^{n(E)} \lambda_G^i(B_i)$$

is a measure on L equipped with the finite disjoint union σ-algebra in which a set is measurable if and only if it can be written as $\cup_{i=0}^{n(E)}(\{i\} \times B_i)$ with B_i in the σ-algebra generated by ϕ_i for $i > 0$ and the power set of V for $i = 0$. By default, the set $\{0\} \times V$ has Lebesgue–Stieltjes measure zero.

Note that Definition 5 does not depend on the parametrisations.

Definition 6 *Let $(V, E = (e_i)_i, \Phi = (\phi_i)_i)$ be a graph with Euclidean edges and L the corresponding network. The unit rate Poisson process on L is defined as follows. For every i and every set B_i in the σ-algebra generated by ϕ_i,*

- *the number of points in $\{i\} \times B_i$ is Poisson distributed with expectation $\lambda_G(\{i\} \times B_i)$;*
- *given that the number of points falling in $\{i\} \times B_i$ is n_i, these n_i points are independent and identically distributed according to the probability density function $1/\lambda_G(\{i\} \times B_i)$.*

In words, the unit rate Poisson process scatters a Poisson number of points independently and uniformly on every edge, and the average number of such points is equal to the arc length of the edge. The definition does not depend on the parametrisation.

The integral of a measurable function $f : \mathbb{R}^2 \to \mathbb{R}^+$ with respect to the Lebesgue–Stieltjes measure λ_G is defined as the sum of the line integrals of f along the rectifiable curves parametrised by the diffeomorphism $\phi_i^{-1} : J_i \to \mathbb{R}^2$:

$$\int_L f d\lambda_G = \sum_{i=1}^{n(E)} \int_{J_i} f(\phi_i^{-1}(t)) \left| \frac{d}{dt} \phi_i^{-1}(t) \right| dt.$$

Higher order integrals are defined analogously. Note that the definition does not depend on the parametrisation.

Example 1 *As an example, consider the functions f_G^i defined on $\phi_i^{-1}(J_i)$ by*

$$f_G^i(x) = \left| (\frac{d}{dt} \phi_i^{-1})(\phi_i(x)) \right|^{-1}.$$

Then, for B_i in the σ-algebra generated by ϕ_i,

$$\begin{aligned} \int_L 1\{x \in B_i\} \, f_G^i(x) \, d\lambda_G(x) &= \int_{J_i} 1\{\phi_i^{-1}(t) \in B_i\} \left| (\frac{d}{dt} \phi_i^{-1})(\phi_i(\phi_i^{-1}(t)) \right|^{-1} \left| \frac{d}{dt} \phi_i^{-1}(t) \right| dt \\ &= |\phi_i(B_i)|, \end{aligned}$$

the Euclidean length of $\phi_i(B_i)$. In other words, the functions f_G^i, $i = 1, \ldots, n(E)$, define the weighted shortest path distance d_G on L.

A point process X on L is said to have probability density p with respect to the unit rate Poisson process if

$$P(X \in F) = \sum_{n=0}^{\infty} \frac{e^{-\lambda_G(L)}}{n!} \int_L \cdots \int_L 1\{\{x_1, \ldots, x_n\} \in F\} \, p(\{x_1, \ldots, x_n\}) \, d\lambda_G(x_1) \cdots d\lambda_G(x_n) \tag{1}$$

for all F in the usual σ-algebra on locally finite point configurations in L generated by the counts $X(B)$ with B in the finite disjoint union σ-algebra on L [6]. For convenience's sake, we will use the notation $\mathbf{x} = \{x_1, \ldots, x_n\}$ for configurations of finitely many distinct points in L. For further details on graphs with Euclidean edges, the reader may consult Anderes et al. [1].

3 Nearest-neighbour Markov point processes

The purpose of this section is to define appropriate analogues of the well-known class of renewal processes on the real line. Intuitively speaking, we are looking for a class of point processes in which the conditional behaviour at a given point depends on the remainder of the configuration only through its 'nearest neighbours'. We shall show that in the case that the network L is a tree, the shortest path distance may be used to define which points are each other's nearest neighbours. In the general case, we will use the geometry of the network to define a local neighbourhood relation.

3.1 The Delaunay relation

In this section, we adapt Baddeley and Møller's definition of configuration dependent neighbourhoods and cliques in Euclidean spaces [2] to our context.

Definition 7 *Let (V, E, Φ) be a graph with Euclidean edges and L the corresponding network. Let $\mathbf{x} \subseteq L$ be a finite configuration of distinct points and define the Delaunay relation $\sim_{\mathbf{x}}$ as the symmetric, reflexive relation on $\mathbf{x}$ given by*

$$x_1 \sim_{\mathbf{x}} x_2 \Leftrightarrow C(x_1|\mathbf{x}) \cap C(x_2|\mathbf{x}) \neq \emptyset$$

where, for $x_j \in \mathbf{x}$, $j = 1, 2$,

$$C(x_j|\mathbf{x}) = \{(i, y) \in L : d_G((i, y), x_j) \leq d_G((i, y), x) \text{ for all } x \in \mathbf{x}\}$$

is the Voronoi cell of x_j in $\mathbf{x}$. The $\mathbf{x}$–neighbourhood of a subset $\mathbf{z} \subseteq \mathbf{x}$ is defined as

$$N(\mathbf{z}|\mathbf{x}) = \{x \in \mathbf{x} : x \sim_{\mathbf{x}} z \text{ for some } z \in \mathbf{z}\}.$$

The configuration $\mathbf{z}$ is an $\mathbf{x}$-clique if for each $z_1, z_2 \in \mathbf{z}$, $z_1 \sim_{\mathbf{x}} z_2$. By convention, the empty set and singletons are cliques too.

A few remarks are in order. Although the space L can be seen as one dimensional, the Voronoi cells are not necessarily line segments and the number of neighbours a point can have is not restricted to two. Some further elementary properties of the relation are collected in the next lemma.

Lemma 1 *Let (V, E, Φ) be a graph with Euclidean edges and L the corresponding network. The Delaunay relation on L satisfies the following properties.*

- *If $\chi(\mathbf{y}|\mathbf{x}) = 1$ then also $\chi(\mathbf{y}|\mathbf{z}) = 1$ for all $\mathbf{y} \subseteq \mathbf{z} \subseteq \mathbf{x}$.*
- *If $\chi(\mathbf{y}|\mathbf{z}) = 0$ then also $\chi(\mathbf{y}|\mathbf{x}) = 0$ for all $\mathbf{y} \subseteq \mathbf{z} \subseteq \mathbf{x}$.*

Here $\chi(\mathbf{y}|\mathbf{x}) = 1$ if $\mathbf{y}$ is an $\mathbf{x}$-clique and zero otherwise.

The proof can be found in Section 4.

3.2 The Delaunay relation on trees

Recall that (V, E) is said to be a tree if it has no cycles, that is, there is no closed path $(v_0, v_1, \ldots, v_{p-1}, v_0)$, $v_i \in V$, of positive length ($p > 0$). It is well-known that a graph is a tree if and only if there is exactly one path between any two vertices [5].

A graph with Euclidean vertices (V, E, Φ) is said to be a tree if (V, E) is.

Lemma 2 *A graph with Euclidean edges is a tree if and only if there is exactly one path between any two points (i, x) and (j, y) in L.*

We are particularly interested in the geometrical arrangement of the paths between three points.

Lemma 3 *Let (V, E, Φ) be a graph with Euclidean edges that is a tree and L its associated network. Consider a triple $\mathbf{y} = \{y_1, y_2, y_3\} \subseteq L$. Then there exist unique paths between the elements of $\mathbf{y}$ that*

- *either form a wheel with three stokes of strictly positive length emanating from a vertex $(0, v) \in L$,*
- *or combine into a single path.*

The proofs of Lemma 2 and 3 will be given in Section 4.

A configuration $\mathbf{y} \subseteq L$ is said to be in general position if no three points lie on the boundary of the same d_G-ball. In other words, by Lemma 3, a configuration in general position cannot form a wheel with three or more equally long spokes. Such a configuration would lead to Delaunay cliques with more than two elements. Restriction to configurations in general position ensures that the size of cliques is at most two. Clearly, the class of all configurations in general position is hereditary.

Lemma 4 *Let (V, E, Φ) be a graph with Euclidean edges that is a tree and L its associated network. Then the clique sizes with respect to the Delaunay relation are at most two on the class of configurations in general position. Moreover, for all $\mathbf{y} = \{y_1, y_2\} \subseteq \mathbf{x}$ with $\mathbf{x}$ in general position, $\chi(\mathbf{y}|\mathbf{x}) = 1$ if and only if the midpoint of $\mathbf{y}$ with respect to the weighted shortest path metric along the unique path between y_1 and y_2 lies in $C(y_1|\mathbf{x}) \cap C(y_2|\mathbf{x})$.*

Since the clique structure depends on the configuration, consistency conditions must be imposed on the family of neighbourhood relations [2].

Definition 8 *Let (V, E, Φ) be a graph with Euclidean edges and L its associated network. Consider finite point configurations $\mathbf{y} \subseteq \mathbf{z} \subset L$ and points $u, v \in L$ such that $u, v \notin \mathbf{z}$. Then the Baddeley–Møller consistency conditions read as follows:*

(C1) $\chi(\mathbf{y}|\mathbf{z}) \neq \chi(\mathbf{y}|\mathbf{z} \cup \{u\})$ implies $\mathbf{y} \subseteq N(\{u\}|\mathbf{z} \cup \{u\})$;

(C2) if $u \not\sim_{\mathbf{x}} v$ for $\mathbf{x} = \mathbf{z} \cup \{u, v\}$, then

$$\chi(\mathbf{y}|\mathbf{z} \cup \{u\}) + \chi(\mathbf{y}|\mathbf{z} \cup \{v\}) = \chi(\mathbf{y}|\mathbf{z}) + \chi(\mathbf{y}|\mathbf{x})$$

where χ is the clique indicator function, i.e. $\chi(\mathbf{y}|\mathbf{x}) = 1\{\mathbf{y}$ is an $\mathbf{x}$-clique$\}$.

In general, the consistency conditions of Definition 8 do not hold, as illustrated by the following counterexample.

Example 2 *Take* $V = \{(-1,0),(1,0),(0,1)\} \subseteq \mathbb{R}^2$ *and connect all three vertices by straight line segments to form the complete graph* (V,E) *and the linear network* L*. Let* $\mathbf{z}$ *be a configuration containing a single point on each edge. Then* $\chi(\mathbf{z}|\mathbf{z}) = 1$*. However, any additional point* $u \notin \mathbf{z}$ *would split up two of the points in* $\mathbf{z}$*. Hence* $\chi(\mathbf{z}|\mathbf{z} \cup \{u\}) = 0$*, although the third point of* $\mathbf{z}$ *is no Delaunay neighbour of* u *in* $\mathbf{z} \cup \{u\}$*. Consequently, (C1) does not hold. Upon the addition of a second point* v *that is not a Delaunay neighbour of* u *in the resulting five point configuration,* $\chi(\mathbf{z}|\mathbf{z} \cup \{u,v\}) = \chi(\mathbf{z}|\mathbf{z} \cup \{u\}) = \chi(\mathbf{z}|\mathbf{z} \cup \{v\}) = 0$ *although* $\chi(\mathbf{z}|\mathbf{z}) = 1$*, in violation with (C2).*

The main theorem of this section is the following. Its proof will be given in Section 5.

Theorem 1 *Let* (V,E,Φ) *be a graph with Euclidean edges that is a tree and* L *its associated network. Then the Delaunay relation satisfies (C1)–(C2) on the family of configurations in general position.*

3.3 Markov functions

We are now ready to define Markov functions on graphs with Euclidean edges, in analogy with the spatial models of [2].

Definition 9 *Let* (V,E,Φ) *be a graph with Euclidean edges and* L *the corresponding network. Let* $\sim_{\mathbf{x}}$ *be a family of reflexive, symmetric relations on finite point configurations* $\mathbf{x}$ *in* L*. Then a function* p *from the set of finite point configurations into* $[0,\infty)$ *is a* Markov function *with respect to* $\sim_{\mathbf{x}}$ *if for all* $\mathbf{x}$ *such that* $p(\mathbf{x}) > 0$*,*

(a) $p(\mathbf{y}) > 0$ *for all* $\mathbf{y} \subseteq \mathbf{x}$*;*

(b) *for all* $u \in L$*,* $p(\mathbf{x} \cup \{u\})/p(\mathbf{x})$ *depends only on* u*, on* $N(\{u\}|\mathbf{x} \cup \{u\}) \cap \mathbf{x} = \{x \in \mathbf{x} : x \sim_{\mathbf{x}\cup\{u\}} u\}$ *and on the relations* $\sim_{\mathbf{x}}, \sim_{\mathbf{x}\cup\{u\}}$ *restricted to* $N(\{u\}|\mathbf{x} \cup \{u\})$*.*

The next theorem provides a Hammersley–Clifford factorisation. Similar results for spatial point processes in Euclidean spaces can be found in [2], [3], [8] or [12]. Recall that a function γ from the space of finite point configurations into $[0,\infty)$ is an interaction function [2] if the following properties hold. If $\gamma(\mathbf{x}) > 0$ then also (a) $\gamma(\mathbf{y}) > 0$ for all $\mathbf{y} \subseteq \mathbf{x}$ and (b) if additionally $\gamma(N(\{u\}|\mathbf{x} \cup \{u\})) > 0$ then $\gamma(\mathbf{x} \cup \{u\}) > 0$.

Theorem 2 *Let* (V,E,Φ) *be a graph with Euclidean edges such that* (V,E) *is a tree and* L *the corresponding network. Let* p *a measurable function from the set of finite point configurations into* $[0,\infty)$*. Then* p *is a Markov function on the family of configurations in general position with respect to the Delaunay relation if and only if*

$$p(\mathbf{x}) \propto 1\{\gamma(\mathbf{x}) > 0\} \prod_{x_i \in \mathbf{x}} \gamma(\{x_i\}) \prod_{i<j: x_i \sim_{\mathbf{x}} x_j} \gamma(\{x_i, x_j\})$$

for some measurable, non-negative interaction function γ*.*

If p can be normalised into a probability density, e.g. by assuming that $\gamma(\{x_i\})$ is bounded and $\gamma(\{x_i, x_j\}) \leq 1$, p is a Markov density and we can define a nearest-neighbour Markov point process by (1).

The characterisation allows some flexibility in the choice of γ.

Example 3 *Set $\gamma(\emptyset) = \alpha > 0$, $\gamma(\{x_i\}) \equiv \beta > 0$, and suppose that the interaction function for pairs depends only on the weighted shortest path distance between its elements:*

$$\gamma(\{x_i, x_j\}) = g(d_G(x_i, x_j))$$

for some function $g : [0, \infty) \to [0, \infty)$. For configurations $\mathbf{x}$ with $n(\mathbf{x}) > 2$, set

$$\gamma(\mathbf{x}) = 1\{g(d_G(x_i, x_j)) > 0 \text{ for all } x_i, x_j \in \mathbf{x}\}.$$

Clearly $\gamma(\mathbf{x}) > 0$ if and only if $g(d_G(x_i, x_j)) > 0$ for all $x_i, x_j \in \mathbf{x}$.

The function γ thus defined is an interaction function provided that if the conditions

- *$g(d_G(x_i, x_j)) > 0$ for all $x_i, x_j \in \mathbf{x}$ and*
- *$g(d_G(x_i, u)) > 0$ for all $x_i \in \mathbf{x}$ such that $x_i \sim_{\mathbf{x} \cup \{u\}} u$*

hold, then also $g(d_G(x_i, u)) > 0$ for all $x_i \in \mathbf{x}$. An alternative condition is [2, (G) in §5.2] in combination with the interaction function $\gamma(\mathbf{x}) = 1\{g(d_G(x_i, x_j)) > 0$ for all $x_i, x_j \in \mathbf{x}$ such that $x_i \sim_{\mathbf{x}} x_j\}$ for configurations $\mathbf{x}$ with $n(\mathbf{x}) > 2$. In either case, a sufficient condition is that the function g takes strictly positive values. Then the Papangelou conditional intensity reads

$$\lambda(u|\mathbf{x}) = \frac{p(\mathbf{x} \cup \{u\})}{p(\mathbf{x})} = \beta \, \frac{\prod_{x \sim_{\mathbf{x} \cup \{u\}} u} g(d_G(x, u))}{\prod_{x \sim_{\mathbf{x}} y, x \not\sim_{\mathbf{x} \cup \{u\}} y} g(d_G(x, y))},$$

in complete analogy with the conditional intensity of a renewal process on the real line.

3.4 The local Delaunay relation

As shown in Example 2, the Delaunay relation does not satisfy the consistency relations (C1)–(C2) if the graph (V, E) is not a tree. However, we may employ the neighbourhood relation implicit in the graph to define a local Delaunay relation. Such a procedure is similar to one employed in image analysis for edge detection and texture analysis [7].

Definition 10 *Let (V, E, Φ) be a graph with Euclidean edges and L its associated network. Define a symmetric and reflexive relation $\sim_E$ on L as follows:*

$$(i, x) \sim_E (j, y) \Leftrightarrow \begin{cases} \phi_i^{-1}(\partial J_i) \cap \phi_j^{-1}(\partial J_j) \neq \emptyset, & i, j \neq 0 \\ \phi_i^{-1}(\partial J_i) \cap \{y\} \neq \emptyset, & i \neq 0, j = 0 \\ \{x, y\} \in E, & i = j = 0. \end{cases}$$

Write, for $i, j \neq 0$, $\sim_{\mathbf{x}}^{i,j}$ for the Delaunay relation restricted to

$$L \cap \left((\{i, 0\} \times \phi_i^{-1}(\bar{J}_i)) \cup (\{j, 0\} \times \phi_j^{-1}(\bar{J}_j)) \right),$$

the restriction of L to at most two edges and their incident vertices, and define

$$(i,x) \sim^E_{\mathbf{z}} (j,y) \Leftrightarrow \begin{cases} (i,x) \sim^{i,j}_{\mathbf{z}\cap(\{i,0\}\times\phi_i^{-1}(\bar{J}_i))\cup(\{j,0\}\times\phi_j^{-1}(\bar{J}_j))} (j,y); (i,x) \sim_E (j,y), \\ \qquad i,j \neq 0 \\ (i,x) \sim^{i,i}_{\mathbf{z}\cap(\{i,0\}\times\phi_i^{-1}(\bar{J}_i))} (0,y); (i,x) \sim_E (0,y), \\ \qquad i \neq 0, j = 0 \\ (0,x) \sim^{k,k}_{\mathbf{z}\cap(\{k,0\}\times\phi_k^{-1}(\bar{J}_k))} (0,y); \{x,y\} = e_k \in E, \\ \qquad i = j = 0. \end{cases}$$

In words, a vertex is a $\sim_E$-neighbour of the edges it is incident with and edges are neighbours if they share a common vertex. Thus, the $\sim_E$-relation does not depend on the configuration. It does, however, crucially depend on the geometry of the graph – splitting an edge will result in a different relation. After combination with the Delaunay relation, the resulting relation $\sim^E_{\mathbf{z}}$ is configuration dependent and depends on the geometry of the graph.

The main result of this section is the following. Its proof will be given in Section 5.

Theorem 3 *Let (V, E, Φ) be a graph with Euclidean edges and L its associated network. Then the local Delaunay relation $\sim^E_{\mathbf{x}}$ satisfies (C1)–(C2).*

4 Proofs of Lemmata

Proof of Lemma 1:
We claim that $y_1 \sim_{\mathbf{x}} y_2$ implies that $y_1 \sim_{\mathbf{z}} y_2$ for any $\mathbf{z} \subseteq \mathbf{x}$ and $y_1, y_2 \in \mathbf{z}$. To see this, note that

$$\chi(\{y_1, y_2\}|\mathbf{x}) = 1 \Leftrightarrow \exists \xi \in L : d_G(\xi, x) \geq d_G(\xi, y_i) = d_G(\xi, y_2) \text{ for all } x \in \mathbf{x}.$$

A fortiori, for this ξ, $d_G(\xi, z) \geq d_G(\xi, y_i) = d_G(\xi, y_2)$ for all z in the smaller set $\mathbf{z} \subseteq \mathbf{x}$, so that $\chi(\{y_1, y_2\})|\mathbf{y}) = 1$, that is, $y_1 \sim_{\mathbf{z}} y_2$.

By convention, the clique indicator function takes the value one for singletons and the empty set regardless of the configuration. Hence we may focus on configurations $\mathbf{y}$ of cardinality at least two.

For the first statement, suppose that $\chi(\mathbf{y}|\mathbf{x}) = 1$. Pick any $y_1, y_2 \in \mathbf{y}$. Then $y_1 \sim_{\mathbf{x}} y_2$ and, by the above claim, $y_1 \sim_{\mathbf{z}} y_2$. Since y_1 and y_2 are chosen arbitrarily, $\chi(\mathbf{y}|\mathbf{z}) = 1$.

For the second statement, if $\chi(\mathbf{y}|\mathbf{z}) = 0$, there exist $y_1, y_2 \in \mathbf{z}$ such that $y_1 \not\sim_{\mathbf{z}} y_2$. By the claim, also $y_1 \not\sim_{\mathbf{x}} y_2$, hence $\chi(\mathbf{y}|\mathbf{x}) = 0$. □

Proof of Lemma 2:
Suppose that (V, E, Φ) is a tree. For vertices, since (V, E) is a tree, there is a unique path between each pair of vertices. Hence we may restrict ourselves to a pair of points (i, x) and (j, y) of which at least one belongs to the set $L \setminus (\{0\} \times V)$.

If $i = j$, one path runs along the edge. Since a walk does not reverse its tracks by definition, any other walk from (i, x) to (i, y) must visit at least one of the two end vertices of e_i before returning, hence use e_i twice.

If $i = 0$ and $j \neq 0$, note there is a unique path from x to v_j^1 in the graph (V, E) and hence a corresponding one in the labelled space L. If this path includes edge e_j, by deleting the j-labelled curve segment from y to v_j^1, one obtains a path from (i, x) to (j, y), otherwise such a path is found by extending the path to v_j^1 with this segment. Moreover, any other path would have to pass at least one of the vertices in e_j, that is, coincide with the construction up to there and hence entirely.

Finally, if $i, j \neq 0$, as seen in the previous case, there is a unique path from $(0, v_i^1)$ to (j, y). If this path includes edge e_i, this yields the path from (i, x) to (j, y). Otherwise, precede by the segment along e_i from (i, x) to $(0, v_i^1)$. Any other path would have to pass at least one of the vertices in e_i, that is, coincide with the construction from there and hence entirely.

Reversely, let (V, E, Φ) be such that there is a unique path between any pair of distinct points (i, x) and (j, y) in L (which exists since a graph with Euclidean edges is connected by definition). In particular, there is a unique path between any pair of 0-labelled vertices, and therefore (V, E) is a tree. □

Proof of Lemma 3:
Since L is a tree, by Lemma 2, there are unique paths from y_1 to y_2 and from y_2 to y_3, say with consecutive labelled vertices

$$y_1, (0, v_1), \ldots, (0, v_p), y_2, (0, v_{p+1}), \ldots, (0, v_{p+q}), y_3,$$

$p, q \geq 0$ and no vertex replication in $v_1, \ldots, v_p$ nor in $v_{p+1}, \ldots, v_{p+q}$.

Suppose that $v_k = v_l$ for some $1 \leq k \leq p$ and $p+1 \leq l \leq p+q$. Then the paths $((0, v_k), \ldots, (0, v_p), y_2)$ and $((0, v_l), \ldots, (0, v_{p+1}), y_2)$ both connect $(0, v_k) = (0, v_l)$ and y_2 in L, and must therefore coincide. Extending if possible, we may assume that $v_{k-1} \neq v_{l+1}$. If some earlier vertex v_i, $i < k-1$ would be identical to v_j for $l+1 < j \leq p+q$, a cycle would be created from v_i via $v_k = v_l$ to $v_j = v_i$. Hence the sequences $(v_1, \ldots, v_{k-1})$ and $(v_{l+1}, \ldots, v_{p+q})$ do not intersect and the paths $(y_1, (0, v_1), \ldots, (0, v_{k-1}))$, $(y_2, (0, v_p), \ldots, (0, v_{k+1}))$ and $(y_3, (0, v_{p+q}), \ldots, (0, v_{l+1}))$ are connected at $(0, v_k) = (0, v_l)$ and therefore form three stokes connected at a single hub provided the lengths are positive. It therefore remains to consider the cases $k = 1$, $k = p$ or $l = p + q$.

If $l = p + q$, y_3 may lie on the path from y_1 to y_2; if it does not, the curve segment from $(0, v_l)$ to y_3 forms a stoke of positive length. Similarly, if $k = p$, y_2 may lie on the path from y_1 to y_3; if it does not it, the curve segment from $(0, v_k)$ to y_2 forms a stoke of positive length. Also, if $k = 1$, y_1 may lie along the path between y_2 and y_3; if it does not, the curve segment from y_1 to $(0, v_k)$ forms a stoke of positive length.

Finally, if there is no $1 \leq k \leq p$ such that $v_k = v_l$ for some $p+1 \leq l \leq p+q$, the unique path from y_1 to y_3 runs via y_2. □

Proof of Lemma 4:
Suppose that $\chi(\mathbf{y}|\mathbf{x}) = 1$ for some $\mathbf{y} = \{y_1, y_2, y_3\} \subseteq \mathbf{x} \subseteq L$. By Lemma 1, $\chi(\mathbf{y}|\mathbf{y}) = 1$ hence one needs only consider $\mathbf{x} = \mathbf{y}$.

By Lemma 3, the elements of $\mathbf{y}$ either form a wheel or a path. First consider the case that $\mathbf{y}$ is a wheel with three stokes emanating from a hub $(0, v)$. Ordering the stokes according to their length, without loss of generality suppose that $a \leq b \leq c$ where $a = d_G(y_1, (0, v))$, $b = d_G(y_2, (0, v))$ and $c = d_G(y_3, (0, v))$. Since by assumption $\mathbf{y}$ is in general position, $a < b < c$. Therefore, as L is a tree, $C(y_2|\mathbf{y}) \cap C(y_3|\mathbf{y}) = \emptyset$ and $y_2 \not\sim_{\mathbf{y}} y_3$.

It remains to consider the case that all three elements of $\mathbf{y}$ lie along a path, without loss of generality from y_1 via y_2 to y_3. Then $C(y_1|\mathbf{y}) \cap C(y_3|\mathbf{y}) = \emptyset$, again using the tree property to ensure that there are no paths to connect y_1 and y_3 other than via y_2. Therefore $y_1 \not\sim_{\mathbf{y}} y_3$. In conclusion, there cannot be a clique of cardinality three or larger.

To prove the second assertion, let $\mathbf{y} = \{y_1, y_2\}$ and write ξ for the midpoint with respect to d_G along the path between y_1 and y_2. Since L is assumed to be a tree, by Lemma 2, the path is unique. Since all $\phi_i \in \Phi$ are bijections, also the midpoint is unique. Let $\mathbf{x} \supseteq \mathbf{y}$ be some configuration in general position.

If $\xi \in C(y_1|\mathbf{x}) \cap C(y_2|\mathbf{x})$, clearly $\chi(\mathbf{y}|\mathbf{x}) = 1$. Reversely, suppose that $\mathbf{y}$ is a clique in $\mathbf{x}$. Then there exists some $\eta \in L$ such that $\eta \in C(y_1|\mathbf{x}) \cap C(y_2|\mathbf{x})$. By Lemma 3, the triple $\{\eta, y_1, y_2\}$ either forms a path or a wheel. In the first case, the property of equidistance to y_1 and y_2 implies that $\eta = \xi$ and the proof is complete.

Next consider the case that η and $\mathbf{y}$ form three stokes of strictly positive length connected at some hub $(0, v)$. Since $d_G(\eta, y_j) = d_G(\eta, (0, v)) + d_G((0, v), y_j)$, $j = 1, 2$, by the uniqueness of the paths from η to y_j (cf. Lemma 2), $d_G((0, v), y_1) = d_G((0, v), y_2)$. Hence $\xi = (0, v)$, now using the uniqueness of the path between y_1 and y_2. It remains to show that no other point of $\mathbf{x}$ lies closer to ξ than the y_j, $j = 1, 2$. Since L is a tree, any such $x \in \mathbf{x}$ is connected either to the hub ξ or to exactly one of the tree stokes. Any $x \in \mathbf{x}$ connected to the end points y_1, y_2 of the stokes lies further from ξ than y_1 and y_2.

Any $x \in \mathbf{x}$ connected to the stoke of η satisfies, since η is assumed to lie in $C(y_1|\mathbf{x}) \cap C(y_2|\mathbf{x})$,

$$d_G(\eta, x) \geq d_G(\eta, y_j) = d_G(\eta, \xi) + d_G(\xi, y_j), \quad j = 1, 2,$$

again using the uniqueness of paths for the last equality. If the connection is at the endpoint η, then $d_G(\xi, x) = d_G(\xi, \eta) + d_G(\eta, x)$ hence, by the above equation, $d_G(\xi, x) \geq 2d_G(\xi, \eta) + d_G(\xi, y_j) \geq d_G(\xi, y_j)$. If the connection is at a vertex $(0, w)$, $w \in V$,

$$\begin{aligned} d_G(\xi, x) &= d_G(\xi, (0, w)) + d_G(x, (0, w)) \\ &= d_G(\xi, (0, w)) + d_G(\eta, x) - d_G(\eta, (0, w)) \\ &\geq d_G(\xi, (0, w)) + d_G(\eta, y_j) - d_G(\eta, (0, w)) \\ &= d_G(\xi, (0, w)) + d_G(y_j, (0, w)) \geq d_G(\xi, y_j). \end{aligned}$$

Finally, any $x \in \mathbf{x}$ connected to the stoke of a y_j at ξ or some other vertex $(0, w)$ satisfies, since η is assumed to lie in $C(y_1|\mathbf{x}) \cap C(y_2|\mathbf{x})$,

$$d_G(\eta, x) \geq d_G(\eta, y_j) = d_G(\eta, \xi) + d_G(\xi, y_j)$$

so that $d_G(\xi, x) = d_G(\eta, x) - d_G(\eta, \xi) \geq d_G(\xi, y_j)$ and the proof is complete. □

5 Proof of main theorems

Proof of Theorem 1:
First, note that for $\mathbf{y}$ of size zero or one, (C1) and (C2) are automatically satisfied. The assumption that L is a tree and the restriction to configurations in general position imply, by Lemma 4, that $\chi(\mathbf{y}|\mathbf{z}) = 0$ for all $\mathbf{z}$ when the cardinality of $\mathbf{y}$ is three or more. Hence (C1) and (C2) hold for $\mathbf{y}$ with $n(\mathbf{y}) \geq 3$ as well and it suffices to consider pairs $\mathbf{y} = \{y_1, y_2\}$.

Condition (C1) First, consider (C1). Take $y_1, y_2 \in \mathbf{z} \subseteq L$, $u \in L$ with $u \notin \mathbf{z}$, and suppose that $\chi(\{y_1, y_2\}|\mathbf{z})$ differs from $\chi(\{y_1, y_2\}|\mathbf{z} \cup \{u\})$. By Lemma 1, if $\chi(\{y_1, y_2\}|\mathbf{z} \cup \{u\}) = 1$, also $\chi(\{y_1, y_2\}|\mathbf{z}) = 1$, so it suffices to consider the case that $\chi(\{y_1, y_2\}|\mathbf{z} \cup \{u\}) = 0$ but $\chi(\{y_1, y_2\}|\mathbf{z}) = 1$.

Let $\tilde{y}_j$ be the point lying halfway between y_j and u along the unique path between them (cf. Lemma 2) and let ξ be the halfway point between y_1 and y_2. These points exist since the parametrisations are bijections. By Lemma 4, since $\chi(\mathbf{y}|\mathbf{z}) = 1$, $\xi \in C(y_1|\mathbf{z}) \cap C(y_2|\mathbf{z})$, so that for all $z \in \mathbf{z}$, $d_G(\xi, z) \geq d_G(\xi, y_j)$, $j = 1, 2$. Also by construction $d_G(\tilde{y}_j, y_j) = d_G(\tilde{y}_j, u)$. We shall show that, for $j = 1, 2$, $\tilde{y}_j \in C(y_j|\mathbf{z} \cup \{u\}) \cap C(u|\mathbf{z} \cup \{u\})$ and therefore $y_j \sim_{\mathbf{z} \cup \{u\}} u$. To do so, we must demonstrate that

$$d_G(\tilde{y}_j, z) \geq d_G(\tilde{y}_j, u) = d_G(\tilde{y}_j, y_j) \tag{2}$$

for $z \in \mathbf{z} \setminus \{y_j\}$.

By Lemma 3, since L is a tree, the paths between the three points u, y_1, y_2 form either a wheel with three stokes joined at some hub $\eta = (0, v)$ with $v \in V$, or the points lie on a path.

First, consider the case that u, y_1 and y_2 lie on a path. If u were an extremity of this path, say y_1 is on the path between u and y_2, then

$$d_G(\xi, u) = d_G(\xi, y_1) + d_G(y_1, u) > d_G(\xi, y_1).$$

By the assumption $\chi(\mathbf{y}|\mathbf{z}) = 1$ and Lemma 4, for any $z \in \mathbf{z}$, also $d_G(\xi, z) \geq d_G(\xi, y_1) = d_G(\xi, y_2)$ so that $\xi \in C(y_1|\mathbf{z} \cup \{u\}) \cap C(y_2|\mathbf{z} \cup \{u\})$, thus violating the assumption that $\chi(\mathbf{y}|\mathbf{z} \cup \{u\}) = 0$. We conclude that u has to lie on the path from y_1 to y_2. Suppose that (2) does not hold, i.e. that for some j and some $z \in \mathbf{z} \setminus \{y_j\}$, the distance $d_G(\tilde{y}_j, z) < d_G(\tilde{y}_j, y_j)$. Then

$$d_G(\xi, z) \leq d_G(\xi, \tilde{y}_j) + d_G(\tilde{y}_j, z) < d_G(\xi, \tilde{y}_j) + d_G(\tilde{y}_j, y_j) = d_G(\xi, y_j)$$

using uniqueness of paths (Lemma 2). However, if $d_G(\xi, z) < d_G(\xi, y_j)$, then $\xi \notin C(y_j|\mathbf{z})$ in contradiction with Lemma 4.

Next suppose that $\{u, y_1, y_2\}$ form a wheel and, without loss of generality, the path from y_1 to y_2 passes first ξ and then η. Since $\chi(\mathbf{y}|\mathbf{z} \cup \{u\}) = 0$, the intersection of the $C(y_j|\mathbf{z} \cup \{u\})$, $j = 1, 2$, is empty and in particular does not contain ξ. Hence

$$d_G(\xi, u) < d_G(\xi, y_1) = d_G(\xi, y_2).$$

Therefore,

$$d_G(\eta, u) < d_G(\eta, y_2) \leq d_G(\eta, y_1)$$

with equality only if $\eta = \xi$. Consequently, $\tilde{y}_j$ lies on the stoke of y_j for each $j = 1, 2$. Now, if $d_G(\tilde{y}_2, z) < d_G(\tilde{y}_2, y_2)$ for some $z \in \mathbf{z} \setminus \{y_2\}$, since $\tilde{y}_2$ lies on the path from ξ to y_2 via η, then

$$d_G(\xi, z) \le d_G(\xi, \tilde{y}_2) + d_G(\tilde{y}_2, z) < d_G(\xi, \tilde{y}_2) + d_G(\tilde{y}_2, y_2) = d_G(\xi, y_2)$$

in contradiction with the assumption that $\xi \in C(y_2|\mathbf{z})$. Similarly for y_1, recalling that $d_G(\xi, u) < d_G(\xi, y_1)$, the point $\tilde{y}_1$ lies on the path between ξ and y_1. Hence if $d_G(\tilde{y}_1, z) < d_G(\tilde{y}_1, y_1)$, then

$$d_G(\xi, z) \le d_G(\xi, \tilde{y}_1) + d_G(\tilde{y}_1, z) < d_G(\xi, \tilde{y}_1) + d_G(\tilde{y}_1, y_1) = d_G(\xi, y_1)$$

in contradiction with the assumption that $\xi \in C(y_1|\mathbf{z})$. Therefore (2) cannot be violated and (C1) holds.

Condition (C2) Next, consider (C2). Take $y_1, y_2 \in \mathbf{z} \subseteq L$ and $u, v \in L$ such that $u, v \notin \mathbf{z}$ with $u \not\sim_{\mathbf{x}} v$ where $\mathbf{x} = \mathbf{z} \cup \{u, v\}$. Write $\mathbf{y} = \{y_1, y_2\}$. Lemma 1 implies that if $\chi(\mathbf{y}|\mathbf{z}) = 0$, the same is true upon adding points to $\mathbf{z}$ and (C2) holds in this case. Therefore, it suffices to consider the case that $\chi(\{y_1, y_2\}|\mathbf{z}) = 1$. The same lemma implies that if $\chi(\{y_1, y_2\}|\mathbf{z} \cup \{u, v\}) = 1$, this remains true when deleting points from $\mathbf{z}$ and (C2) holds. The only case left to consider is that when $\chi(\{y_1, y_2\}|\mathbf{z}) = 1$ and $\chi(\{y_1, y_2\}|\mathbf{z} \cup \{u, v\}) = 0$. We must show that exactly one of $\chi(\{y_1, y_2\}|\mathbf{z} \cup \{u\})$ and $\chi(\{y_1, y_2\}|\mathbf{z} \cup \{v\})$ takes the value 1 and will do so by contradiction.

Let $\eta \in L$ be the point that lies halfway between u and v along the unique path between them (cf. Lemma 2) and write ξ for the halfway point between y_1 and y_2. These points exist since the parametrisations are bijections.

Suppose that $\chi(\mathbf{y}|\mathbf{z} \cup \{u\}) = 0 = \chi(\mathbf{y}|\mathbf{z} \cup \{v\})$. By the proof of condition (C1) above, the triple $\{u, y_1, y_2\}$ form a direct path with u on the path from y_1 to y_2, or a wheel with three stokes connecting in a hub. In either case

$$d_G(\xi, u) < d_G(\xi, y_1) = d_G(\xi, y_2).$$

The same is true for the triple $\{v, y_1, y_2\}$. Therefore the ensemble can be seen as a path from y_1 to y_2 that passes branches leading off to u and to v if we include the degenerate cases of a branch consisting of the single point u, respectively v. Without loss of generality, suppose the order is y_1, then the stokes of u and v, and finally y_2. We claim that such an arrangement would imply that $u \sim_{\mathbf{x}} v$ in contradiction with the assumption.

To prove the claim, we show that $\eta \in C(u|\mathbf{z} \cup \{u, v\}) \cap C(v|\mathbf{z} \cup \{u, v\})$, that is $d_G(\eta, u) = d_G(\eta, v) \le d_G(\eta, z)$ for all $z \in \mathbf{z}$. Suppose not. Then for some $z \in \mathbf{z}$, possibly y_1 or y_2, $d_G(\eta, z) < d_G(\eta, u) = d_G(\eta, v)$ and therefore

$$d_G(\xi, z) \le d_G(\xi, \eta) + d_G(\eta, z) < d_G(\xi, \eta) + d_G(\eta, u) = d_G(\xi, \eta) + d_G(\eta, v).$$

The right hand side is equal to either $d_G(\xi, u)$ or $d_G(\xi, v)$ and recalling that both are strictly smaller than $d_G(\xi, y_j)$ we obtain $d_G(\xi, z) < d_G(\xi, y_1) = d_G(\xi, y_2)$ in contradiction with the assumption that $\chi(\mathbf{y}|\mathbf{z}) = 1$, that is, $\xi \in C(y_j|\mathbf{z})$ for $j = 1, 2$ (Lemma 4). Hence $C(u|\mathbf{z} \cup \{u, v\}) \cap C(v|\mathbf{z} \cup \{u, v\})$ contains η, implying $u \sim_{\mathbf{x}} v$ in contradiction with the assumption.

Finally, suppose that $\chi(\mathbf{y}|\mathbf{z} \cup \{u\}) = 1 = \chi(\mathbf{y}|\mathbf{z} \cup \{v\})$. By Lemma 4, $\xi \in C(y_1|\mathbf{z} \cup \{u\}) \cap C(y_2|\mathbf{z} \cup \{u\})$ and therefore $d_G(\xi, u) \geq d_G(\xi, y_1) = d_G(\xi, y_2)$. Similarly, $d_G(\xi, v) \geq d_G(\xi, y_1) = d_G(\xi, y_2)$. By assumption, $\chi(\mathbf{y}|\mathbf{z} \cup \{u, v\}) = 0$, which means that $C(y_1|\mathbf{z} \cup \{u, v\}) \cap C(y_2|\mathbf{z} \cup \{u, v\}) = \emptyset$ and in particular does not contain ξ. Therefore, recalling the assumption $\chi(\mathbf{y}|\mathbf{z}) = 1$, we have that $\min(d_G(\xi, u), d_G(\xi, v)) < d_G(\xi, y_1) = d_G(\xi, y_2)$, a contradiction.

In conclusion, exactly one of $\chi(\mathbf{y}|\mathbf{z} \cup \{u\})$ and $\chi(\mathbf{y}|\mathbf{z} \cup \{v\})$ takes the value one. □

Proof of Theorem 2:
Suppose that p is a Markov function. By Theorem 1, the Delaunay relation satisfies (C1) and (C2) and therefore [2, Thm 4.13] implies that p can be factorised as

$$p(\mathbf{x}) = \prod_{\mathbf{y} \subseteq \mathbf{x}} \gamma(\mathbf{y})^{\chi(\mathbf{y}|\mathbf{x})}$$

for some interaction function γ under the convention that $0^0 = 0$. Since by Lemma 4 the cliques have cardinality at most two, the factorisation reduces to

$$p(\mathbf{x}) \propto \prod_i \gamma(\{x_i\}) \prod_{i<j} \gamma(\{x_i, x_j\})^{\chi(\{x_i, x_j\}|\mathbf{x})} \left(\prod_{\mathbf{y} \subseteq \mathbf{x}: n(\mathbf{y}) > 2} \gamma(\mathbf{y}) \right)^0 .$$

As γ is an interaction function, it is hereditary, so $\gamma(\mathbf{x}) > 0$ if and only if $\gamma(\mathbf{y}) > 0$ for all $\mathbf{y} \subseteq \mathbf{x}$, hence

$$\left(\prod_{\mathbf{y} \subseteq \mathbf{x}: n(\mathbf{y}) > 2} \gamma(\mathbf{y}) \right)^0 = 1\{\gamma(\mathbf{x}) > 0\}.$$

When $\gamma(\mathbf{x}) > 0$, also $\gamma(\{x_i\})$ and $\gamma(\{x_i, x_j\})$ are strictly positive, so

$$p(\mathbf{x}) = \gamma(\emptyset) \prod_i \gamma(\{x_i\}) \prod_{i<j, x_i \sim_{\mathbf{x}} x_j} \gamma(\{x_i, x_j\}).$$

Clearly $p(\mathbf{x}) = 0$ when $\gamma(\mathbf{x}) = 0$.

Reversely, any function of the specified form is a Markov function. To see this, suppose $p(\mathbf{x}) > 0$. Then $\gamma(\mathbf{x}) > 0$ and for all subsets $\mathbf{y}$ of $\mathbf{x}$ also $\gamma(\mathbf{y}) > 0$. Hence $p(\mathbf{y}) > 0$. Furthermore, the ratio $p(\mathbf{x} \cup \xi)/p(\mathbf{x})$ can be written as

$$\gamma(\xi)\, 1\{\gamma(\mathbf{x} \cup \{\xi\}) > 0\} \prod_i \gamma(\{x_i, \xi\})^{\chi(\{x_i, \xi\}|\mathbf{x} \cup \{\xi\})} \times \prod_{i<j} \gamma(\{x_i, x_j\})^{\chi(\{x_i, x_j\}|\mathbf{x} \cup \{\xi\}) - \chi(\{x_i, x_j\}|\mathbf{x})}.$$

By Theorem 1, the Delaunay relation satisfies (C1), hence the product

$$\prod_{i<j} \gamma(\{x_i, x_j\})^{\chi(\{x_i, x_j\}|\mathbf{x} \cup \{\xi\}) - \chi(\{x_i, x_j\}|\mathbf{x})}$$

depends only on x_i and x_j in $N(\{\xi\}|\mathbf{x} \cup \{\xi\})$ and the relations $\sim_{\mathbf{x}}$ and $\sim_{\mathbf{x} \cup \{\xi\}}$ on this neighbourhood. Furthermore, recalling that γ is an interaction function and $\gamma(\mathbf{x}) > 0$,

$$1\{\gamma(\mathbf{x} \cup \{\xi\}) > 0\} \prod_i \gamma(\{x_i, \xi\})^{\chi(\{x_i, \xi\}|\mathbf{x} \cup \{\xi\})} = 1\{\gamma(N(\{\xi\}|\mathbf{x} \cup \{\xi\})) > 0\} \prod_{\xi \sim_{\mathbf{x} \cup \{\xi\}} x_i} \gamma(\{x_i, \xi\})$$

depends only on $N(\{\xi\}|\mathbf{x} \cup \{\xi\}))$ and the relation $\sim_{\mathbf{x}\cup\{\xi\}}$ on this neighbourhood. □

Proof of Theorem 3: Let $\mathbf{y} \subseteq \mathbf{z} \subseteq L$ and $u, v \in L$ be such that $u, v \notin \mathbf{z}$ and $u \not\sim_{\mathbf{z}\cup\{u,v\}} v$.

We first observe that cliques in $\sim_E$ consist of points lying on closed edges emanating from a single vertex. Thus, the size of a $\sim_E$-clique may be larger than two. Moreover, the Delaunay relation restricted to a pair of such edges coincides with the sequential neighbourhood relation with respect to an ordering defined by the parametrisations on the edges. In other words, consecutive points on a single edge are each other's nearest neighbours; also the point on one of the edges that is closest to the vertex that joins the two edges, if it exists, is a nearest neighbour of the point closest to that vertex on the other edge, if it exists, and no other points are nearest neighbours. In particular, each point has at most two nearest neighbours. Cliques in the combined relation $\sim_{\mathbf{z}}^E$ therefore are either empty, consist of a single point, of two consecutive points on a single edge, or of points on different edges that are closest on their edge to the central vertex from which all edges emanate. The clique size is therefore at most the degree of the central vertex.

Condition (C1) If $\chi(\mathbf{y}|\mathbf{z}) = 0$ there exists a pair $y_1 \neq y_2 \in \mathbf{y}$ for which $y_1 \not\sim_{\mathbf{z}}^E y_2$. Then either y_1 and y_2 lie on edges that are not adjacent, in which case $y_1 \not\sim_{\mathbf{z}\cup\{u\}}^E y_2$ and $\chi(\mathbf{y}|\mathbf{z} \cup \{u\}) = 0$, or y_1 and y_2 lie on adjacent edges but y_1 and y_2 are not sequential neighbours. The addition of u cannot make them sequential neighbours, so $\chi(\mathbf{y}|\mathbf{z} \cup \{u\}) = 0$.

Suppose therefore that $\chi(\mathbf{y}|\mathbf{z}) = 1$ but there exists a pair $y_1 \neq y_2 \in \mathbf{y}$ for which $y_1 \not\sim_{\mathbf{z}\cup\{u\}}^E$ y_2. Then y_1 and y_2 must lie on $\sim_E$-related edges (either a single one, or two adjacent ones), be consecutive in $\mathbf{z}$ but not in $\mathbf{z} \cup \{u\}$. This can only happen if u lies in between y_1 and y_2, making y_1 and y_2 both $\sim_{\mathbf{z}\cup\{u\}}^E$-neighbours of u.

Condition (C2) As shown when proving (C1), if $\chi(\mathbf{y}|\mathbf{z}) = 0$, also $\mathbf{y}$ cannot be a clique in configurations with more points. If $\chi(\mathbf{y}|\mathbf{z} \cup \{u, v\}) = 1$, all points in $\mathbf{y}$ lie on adjacent edges and are sequential neighbours in the set $\mathbf{z} \cup \{u, v\})$ restricted to their edges. The same remains true when u and v are removed, so that $\chi(\mathbf{y}|\mathbf{z}) = 1$. Hence it remains to consider the case that $\chi(\mathbf{y}|\mathbf{z}) = 1$ but $\chi(\mathbf{y}|\mathbf{z} \cup \{u, v\}) = 0$. In this case, the points of $\mathbf{y}$ must lie on a single edge or on a number of edges that emanate from a single vertex $(0, w) \in L$, $w \in V$.

Now, if u and v do not lie on any of these $\mathbf{y}$-edges, $\chi(\mathbf{y}|\mathbf{z} \cup \{u, v\}) = 1$, in contradiction with the assumptions.

If exactly one of u and v does not lie on any of the $\mathbf{y}$-edges, without loss of generality u, then $\chi(\mathbf{y}|\mathbf{z} \cup \{u\}) = \chi(\mathbf{y}|\mathbf{z}) = 1$. Moreover, since by assumption $\chi(\mathbf{y}|\mathbf{z} \cup \{u, v\}) = 0$, there is a pair $y_1, y_2 \in \mathbf{y}$ that are not consecutive in the configuration $\mathbf{z} \cup \{u, v\}$ restricted to the edge or edges on which y_1 and y_2 lie. Since they are adjacent in the configuration $\mathbf{z}$, v must lie in between y_1 and y_2, which implies that $\chi(\mathbf{y}|\mathbf{z} \cup \{v\}) = 0$ in accordance with (C2).

Finally, suppose both u and v lie on the $\mathbf{y}$-edges that emanate from $(0, w)$. If $\mathbf{y} = \{y_1, y_2\}$ consists of two consecutive points in $\mathbf{z}$, since y_1 and y_2 are no longer consecutive in the configuration $\mathbf{z} \cup \{u, v\}$, they must be separated by either u or v but not by both, since by assumption u and v are not sequential neighbours in $\mathbf{z} \cup \{u, v\}$. Hence (C2) holds. If $\mathbf{y}$ consists of more than two points $\mathbf{y} = \{y_1, \ldots, y_k\}$, the assumption $\chi(\mathbf{y}|\mathbf{z}) = 1$ implies that the y_j, $j = 1, \ldots, k$, must lie on different edges $e_1, \ldots, e_k$ emanating from w and no points of $\mathbf{z}$ lie between y_i and w. Since u and v are not sequential neighbours, they cannot both lie between some y_i and $(0, w)$; one of them, however, must, since the clique indicator function of $\mathbf{y}$ in $\mathbf{z} \cup \{u, v\}$ takes the value zero. Consequently, also in this case (C2) is seen to hold. $\square$

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