

# On Logical Depth and the Running Time of Shortest Programs

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## Abstract

The logical depth with significance  $b$  of a finite binary string  $x$  is the shortest running time of a binary program for  $x$  that can be compressed by at most  $b$  bits. There is another definition of logical depth. We give two theorems about the quantitative relation between these versions: the first theorem concerns a variation of a known fact with a new proof, the second theorem and its proof are new. We select the above version of logical depth and show the following. There is an infinite sequence of strings of increasing length such that for each  $j$  there is a  $b$  such that the logical depth of the  $j$ th string as a function of  $j$  is incomputable (it rises faster than any computable function) but with  $b$  replaced by  $b+1$  the resulting function is computable. Hence the maximal gap between the logical depths resulting from incrementing appropriate  $b$ 's by 1 rises faster than any computable function. All functions mentioned are upper bounded by the Busy Beaver function. Since for every string its logical depth is nonincreasing in  $b$ , the minimal computation time of the shortest programs for the sequence of strings as a function of  $j$  rises faster than any computable function but not so fast as the Busy Beaver function.

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# 1 Introduction

The logical depth is related to complexity with bounded resources. Computing a string  $x$  from one of its shortest programs may take a very long time. However, computing the same string from a simple ‘print( $x$ )’ program of length about  $|x|$  bits takes very little time.

A program for  $x$  of larger length than a given program for  $x$  may decrease the computation time but in general does not increase it. Exceptions are, for example, cases where unnecessary steps are considered. Generally we associate the longest computation time with a shortest program for  $x$ . There arises the question how much time can be saved by computing a given string from a longer program.

## 1.1 Related Work

The minimum time to compute a string by a  $b$ -incompressible program was first considered in [4] Definition 1. The minimum time was called the *logical depth at significance  $b$*  of the string concerned. Definitions, variations, discussion and early results can be found in the given reference. A more formal treatment, as well as an intuitive approach, was given in the textbook [10], Section 7.7. In [1] the notion of *computational depth* is defined as  $K^d(x) - K(x)$ . This would or would not equal the negative logarithm of the expression  $Q^d(x)/Q(x)$  in Definition 2 as follows. In [9] L.A. Levin proved, in the so-called Coding Theorem

$$-\log Q(x) = K(x) + O(1) \quad (1)$$

(see also [10] Theorem 4.3.3). It remains to prove or disprove  $-\log Q^d(x) = K^d(x)$  up to a small additive term: a major open problem in Kolmogorov complexity theory, see [10] Exercises 7.6.3 and 7.6.4. For Kolmogorov complexity notions see Section 2.2, and for  $Q$  and  $Q^d$  see (3).

## 1.2 Results

There are two versions of logical depth, Definition 2 and Definition 3. The two versions are related. The version of Definition 3 almost implies that of Definition 2 (Theorem 1), but vice versa there is possible uncertainty (Theorem 2). We use Definition 3, that is,  $\text{depth}_b^{(2)}(x)$ . There is an infinite sequence of strings  $x_1, x_2, \dots$  with  $|x_{j+1}| = |x_j| + 1$  and an infinite sequence of positive integers  $b_1, b_2, \dots$ , which satisfy the following. For every  $j > 0$  the string  $x_j$  is computed by two programs that can be compressed by at

most  $b_j, b_j + 1$ -bits and take least computation time among programs of their lengths, respectively. Let these computation times be  $d_1^j, d_2^j$  steps. Then the function  $h(j) = d_1^j - d_2^j$  rises faster than any computable function but not as fast as the Busy Beaver function, the first incomputable function [11] (Theorem 3 and Corollary 1). For the associated shortest programs  $x_1^*, x_2^*, \dots$  of  $x_1, x_2, \dots$  the function  $s^*(j)$  defined as the minimum number of steps in the computation of  $x_j^*$  to  $x_j$  ( $j > 0$ ). Then the function  $s^*$  rises faster than any computable function but again not so fast as the Busy Beaver function (Corollary 2).

The rest of the paper is organized as follows. Section 2 introduces notation, definitions and basic results needed for the paper. Section 3 defines two versions of logical depth and proves quantitative relations between them. In Section 4, we prove the other results mentioned.

## 2 Preliminaries

We use *string* or *program* to mean a finite binary string. Strings are denoted by the letters  $x, y$  and  $z$ . The *length* of a string  $x$  (the number of occurrences of bits in it) is denoted by  $|x|$ , and the *empty* string by  $\epsilon$ . Thus,  $|\epsilon| = 0$ . The notation “log” means the binary logarithm. Given two functions  $f$  and  $g$ , we say that  $f \in O(g)$  if there is a constant  $c > 0$ , such that  $f(n) \leq c \cdot g(n)$ , for all but finitely many natural numbers  $n$ . Restricting the computation time resource is indicated by a superscript giving the allowed number of steps, usually using  $d$ .

### 2.1 Computability

A pair of nonnegative integers, such as  $(p, q)$  can be interpreted as the rational  $p/q$ . We assume the notion of a computable function with rational arguments and values. A function  $f(x)$  with  $x$  rational is *semicomputable from below* if it is defined by a rational-valued total computable function  $\phi(x, k)$  with  $x$  a rational number and  $k$  a nonnegative integer such that  $\phi(x, k+1) \geq \phi(x, k)$  for every  $k$  and  $\lim_{k \rightarrow \infty} \phi(x, k) = f(x)$ . This means that  $f$  (with possibly real values) can be computed in the limit from below (see [10], p. 35). A function  $f$  is *semicomputable from above* if  $-f$  is semicomputable from below. If a function is both semicomputable from below and semicomputable from above then it is *computable*.

## 2.2 Kolmogorov Complexity

We refer the reader to the textbook [10] for details, notions, and history. We use Turing machines with a read-only one-way input tape, one or more (a finite number) of two-way work tapes at which the computation takes place, and a one-way write-only output tape. All tapes are semi-infinite divided into squares, and each square can contain a symbol. Initially, the input tape is inscribed with a semi-infinite sequence of 0's and 1's. The other tapes are empty (contain only blanks). At the start, all tape heads scan the leftmost squares on their tapes. If the machine halts for a certain input then the contents of the scanned segment of input tape is called the *program* or *input*, and the contents of the output tape is called the *output*. The machine thus described is a *prefix Turing machine*. Denote it by  $T$ . If  $T$  terminates with program  $p$  then the output is  $T(p)$ . The set  $\mathcal{P} = \{p : T(p) < \infty\}$  is *prefix-free* (no element of the set is a proper prefix of another element). By the ubiquitous Kraft inequality [8] we have

$$\sum_{p \in \mathcal{P}} 2^{-|p|} \leq 1. \quad (2)$$

We extend the prefix Turing machine with an extra read-only tape called the *auxiliary* or *conditional*. Initially it contains the auxiliary information consisting of a string  $y$ . We write  $T(p, y)$  and the set  $\mathcal{P}_y = \{p : T(p, y) < \infty\}$  is also prefix-free. The relation (2) holds also with  $\mathcal{P}_y$  substituted for  $\mathcal{P}$  and  $y$  is fixed auxiliary information. The unconditional case corresponds to the case where the conditional is  $\epsilon$ .

If  $T_1, T_2, \dots$  is a standard enumeration of prefix Turing machines, then certain of those are called universal. *Universal* prefix Turing machines are those that can simulate any other machine in the enumeration. Among the universal prefix Turing machines we consider a special subclass called *optimal*, see Definition 2.0.1 in [10]. To illustrate this concept let  $T_1, T_2, \dots$  be a standard enumeration of prefix Turing machines, and let  $U_1$  be one of them. If  $U_1(i, pp) = T_i(p)$  for every index  $i$  and program  $p$  and outputs 0 for inputs that are not of the form  $pp$  (doubling of  $p$ ), then  $U_1$  is also universal. However,  $U_1$  can not be used to define Kolmogorov complexity. For that we need a machine  $U_2$  with  $U_2(i, p) = T_i(p)$  for every  $i, p$ . The machine  $U_2$  is called an *optimal* prefix Turing machine. Optimal prefix Turing machines are a strict subclass of universal prefix Turing machines. The above example illustrates the strictness. The term ‘optimal’ comes from the founding paper [7].

It is possible that two different optimal prefix Turing machines have

different computation times for the same input-output pairs or they have different sets of programs. To avoid these problems we fix a reference machine. Necessarily, the reference machine has a certain number of worktapes. A well-known result of [6] states that  $n$  steps of a  $k$ -worktape prefix Turing machine can be simulated in  $O(n \log n)$  steps of a two-worktape prefix Turing machine (the constant hidden in the big- $O$  notation depends only on  $k$ ). Thus, for such a simulating optimal Turing machine  $U$  we have  $U(i, p) = T_i(p)$  for all  $i, p$ ; if  $T_i(p)$  terminates in time  $t(n)$  then  $U(i, p)$  terminates in time  $O(t(n) \log t(n))$ . Altogether, we fix such a simulating optimal prefix Turing machine and call it the *reference optimal prefix Turing machine*  $U$ .

**DEFINITION 1** Let  $U$  be the reference optimal prefix Turing machine, and  $x, y$  be strings. The *prefix Kolmogorov complexity*  $K(x|y)$  of  $x$  given  $y$  is defined by

$$K(x|y) = \min\{|q| : U(q, y) = x\}.$$

(Earlier we wrote  $U(i, p)$  while we write here  $U(q, y)$ . The two are reconciled by writing  $i, p = i, r, y = q, y$ . That is,  $p = r, y$  for a program  $r$ , and  $q = i, r$ .)

The notation  $U^d(q, y) = x$  means that  $U(q, y) = x$  within  $d$  steps. The *d-time-bounded prefix Kolmogorov complexity*  $K^d(x|y)$  of  $x$  given  $y$  is defined by

$$K^d(x|y) = \min\{|q| : U^d(q, y) = x\}.$$

The default value for the auxiliary input  $y$  for the program  $q$ , is the empty string  $\epsilon$ . To avoid overloaded notation we usually drop this argument in case it is there. Let  $x$  be a string. Denote by  $x^*$  the first shortest program in standard enumeration such that  $U(x^*) = x$ . A string is *c-incompressible* if a shortest program for it is at most  $c$  bits shorter than the string itself.

### 3 Different Versions of Logical Depth

The logical depth [4] comes in two versions. One version is based on  $Q_U(x)$ , the so-called *a priori* probability [10] and its time-bounded version  $Q_U^d$ . Here  $U^d(p)$  means that  $U(p)$  terminates in at most  $d$  steps. For convenience we drop the subscript on  $Q_U$  and  $Q_U^d$  and consider  $U$  as understood.

$$Q(x) = \sum_{U(p)=x} 2^{-|p|}, \quad Q^d(x) = \sum_{U^d(p)=x} 2^{-|p|}. \quad (3)$$

**DEFINITION 2** Let  $x$  be a string,  $b$  a nonnegative integer. The logical depth, version 1, of  $x$  at significance level  $\varepsilon = 2^{-b}$  is

$$\text{depth}_\varepsilon^{(1)}(x) = \min \left\{ d : \frac{Q^d(x)}{Q(x)} \geq \varepsilon \right\}.$$

Using a program that is longer than another program for output  $x$  can shorten the computation time. The  $b$ -significant logical depth of an object  $x$  can also be defined as the minimal time the reference optimal prefix Turing machine needs to compute  $x$  from a program which is  $b$ -incompressible.

**DEFINITION 3** Let  $x$  be a string,  $b$  a nonnegative integer. The logical depth, version 2, of  $x$  at significance level  $b$ , is:

$$\text{depth}_b^{(2)}(x) = \min \{d : |p| \leq K(p) + b \wedge U^d(p) = x\}.$$

**REMARK 1** The program  $x^*$  is the first shortest program for  $x$  in enumeration order. It may not be the fastest shortest program for  $x$ . Therefore, if  $U^d(x^*) = x$  then  $d \geq \text{depth}_0^{(2)}(x)$ . For  $b > 0$  the value of  $\text{depth}_b^{(2)}(x)$  is monotonic nonincreasing until

$$\text{depth}_{|x|-K(x)+O(1)}^{(2)}(x) = O(|x| \log |x|),$$

where the  $O(1)$  term represents the length of a program to copy the literal representation of  $x$  in  $O(|x| \log |x|)$  steps. If  $x$  is random ( $|x| = n$  and  $K(x) \geq n$ ) then for  $b = O(\log n)$  we have  $\text{depth}_b^{(2)}(x) = O(n \log n)$ —we print a literal copy of  $x$ . These  $x$ 's, but not only these, are called *shallow*.

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For version (2) every program  $p$  of length at most  $K(p) + b$  must take at least  $d$  steps to compute  $x$ . Version (1) states that  $Q^d(x)/Q(x) \geq 2^{-b}$  and  $Q^{d-1}(x)/Q(x) < 2^{-b}$ . Statements similar to Theorem 1 and Remark 2 were shown in Theorem 7.7.1 and Exercise 7.7.1 in [10] and derive from [4] Lemma 3.

**THEOREM 1** If  $\text{depth}_b^{(2)}(x) = d$  then  $\text{depth}_{2^{-\beta}}^{(1)}(x) = d$  with  $b + 1 < \beta \leq b + K(b) + O(1)$ .

**PROOF.** The theorem states: if  $\text{depth}_b^{(2)}(x) = d$  then

$$\frac{1}{2^{b+K(b)+O(1)}} \leq \frac{Q^d(x)}{Q(x)} < \frac{1}{2^{b+1}}.$$

(Right  $<$ ) By way of contradiction  $Q^d(x) \geq 2^{-b-1}Q(x)$ . If for a nonnegative constant  $c$  all programs computing  $x$  within  $d$  steps are  $c$ -compressible, then the twice iterated reference optimal Turing machine (in its role as decompressor) computes  $x$  with probability  $2^c Q^d(x) \geq 2^{c-b-1}Q(x)$  from the  $c$ -compressed versions. But  $Q(x) \geq 2^{c-b-1}Q(x) + 2^{-b-1}Q(x)$ . Therefore  $c-b-1 < 0$  that is  $c < b+1$ . This implies that there is a program computing  $x$  within  $d$  steps that is  $(b+1)$ -incompressible. Then  $\text{depth}_{b+1}^{(2)}(x) = d$  contradicting the assumption that  $\text{depth}_b^{(2)}(x) = d$ . Hence  $Q^d(x) < 2^{-b-1}Q(x)$ .

(Left  $\leq$ ) By way of contradiction  $Q^d(x) < 2^{-B}Q(x)$  with  $B = b + K(b) + c$  and  $c$  is a large enough constant to derive the contradiction below. Consider the following lower semicomputable semiprobability (the total probability is less than 1): For every string  $x$  we enumerate all programs  $p$  that compute  $x$  in order of halting (time), and assign to each halting  $p$  the probability  $2^{-|p|+B}$  until the total probability would pass  $Q(x)$  with the next halting  $p$ . Since  $Q(x)$  is lower semicomputable we can postpone assigning probabilities. But eventually or never for a program  $p$  the total probability may pass  $Q(x)$  and this  $p$  and all subsequent halting  $p$ 's for  $x$  get assigned probability 0. Therefore, the total probability assigned to all halting programs for  $x$  is less than  $Q(x)$ . Since by contradictory assumption  $Q^d(x) < 2^{-B}Q(x)$  we have  $2^B Q^d(x) = \sum_{U^d(p)=x} 2^{-|p|+B} < Q(x)$ . Since  $Q(x) < 1$  we have  $Q^d(x) < 2^{-B}$  and therefore all programs that compute  $x$  in at most  $d$  steps are  $(B-O(1))$ -compressible given  $B$ , and therefore  $(B-K(B)-O(1))$ -compressible.

Since  $\text{depth}_b^{(2)}(x) = d$ , there exists a  $b$ -incompressible program from which  $x$  can be computed in  $d$  steps. Since  $K(b+K(b)) \leq K(b, K(b)) + O(1) \leq K(b) + O(1)$  by an easy argument [10] and  $K(c) = O(\log c) < c/2$  we have that  $B - K(B) - O(1) = b + K(b) + c - K(b+K(b)+c) - O(1) \geq b + K(b) + c - K(b+K(b)) - K(c) - O(1) > b$ . This gives the required contradiction. Hence  $Q^d(x) \geq 2^{-B}Q(x)$ .  $\square$

**REMARK 2** We can replace  $K(b)$  by  $K(d)$  by changing the construction of the semiprobability: knowing  $d$  we generate all programs that compute  $x$  within  $d$  steps and let the semiprobabilities be proportional to  $2^{-|p|}$  and the sum be at most  $Q(x)$ . In this way  $K(b)$  in Theorem 1 is substituted by  $\min\{K(b), K(d)\}$ .  $\diamond$

**REMARK 3** Possibly  $Q^d(x) = Q^{d+1}(x)$ . Moreover, while  $Q^d(x)/Q(x) \geq 2^{-b}$  for  $d$  least, possibly also  $Q^d(x)/Q(x) \geq 2^{-b+1}$ . Both events happen, for example, if  $p$  computes  $x$  in  $d$  steps but not in  $d-1$  steps, there is no program for  $x$  halting in  $d+1$  steps, and  $|p| < b-1$  while  $Q^{d-1} < 2^{-b}$ . In

the next theorem if we write  $\text{depth}_{2^{-b}}^{(2)}(x) = d$  then  $d, b$  are least integers for which this equality holds.  $\diamond$

**THEOREM 2** *Let  $U^d(p) = U^d(q) = U^d(r) = x$  and  $U^{d-1}(p), U^{d-1}(r) = \infty$  (the computation does not halt in  $d-1$  steps) with  $K(p)$  least,  $|q| \leq K(x) + b$ , and  $||r| - |q||$  (the absolute value of the difference in lengths) is minimal. If  $\text{depth}_{2^{-b}}^{(1)}(x) = d$  then  $\text{depth}_{\beta}^{(2)}(x) = d$  with  $b - K(p) + K(x) - O(1) \leq \beta \leq b + ||r| - |q|| - O(1)$ .*

**PROOF.** (Left  $\leq$ ) By assumption  $Q^d(x)/Q(x) \geq 2^{-b}$  and by Remark 3 we have  $Q^d(x)/Q(x) < 2^{-b+1}$ . Since it is easy to see that  $2^{-K(x)} < Q(x)$ , we have  $Q(x) = 2^{-K(x)+c}$  for a positive constant  $c$  by (1). Therefore

$$Q^d(x) = \sum_{U^d(p)=x} 2^{-|p|} < 2^{-(K(x)+b-1-c)}.$$

Hence every program  $p$  such that  $U^d(p) = x$  satisfies  $K(x) + b - 1 - c < |p|$ . Denote the set of these programs by  $D$ . Let  $p \in D$  be such that  $U^{d-1}(p) = \infty$ . Let  $(D - 1)$  be the set of such programs. Then  $(D - 1) \subseteq D$ . By the convention in Remark 3 we have  $(D - 1) \neq \emptyset$ . Since  $K(x) \leq K(p) + O(1)$  for all  $p$  satisfying  $U(p) = x$  (the  $O(1)$  term is a nonnegative constant independent of  $x$  and  $p$ ) we have  $K(p) + (b - f_x(d)) < |p|$  for all  $p \in (D - 1)$ , with  $f_x(d) = K(p) - K(x) + O(1) > 0$ . If  $\text{depth}_{\beta}^{(2)}(x) = d$  then  $|p| \leq K(p) + \beta$  for  $p \in (D - 1)$ . Hence  $\beta \geq b - f_x(d)$ .

(Right  $\leq$ ): By assumption  $Q^d(x) \geq 2^{-b}Q(x)$ . Let  $P$  be the set of programs  $p$  such that  $U(p) = x$ , the set  $Q$  consist of programs  $q \in P$  such that  $|q| \geq |p| + b$  with  $|q|$  least and  $p \in P$ , while the sets  $D, (D - 1)$  are defined above. Then  $D, Q \subseteq P$ ,  $(D - 1) \subseteq D$ , and

$$\begin{aligned} Q^d(x) &= \sum_{U^d(p)=x} 2^{-|p|} \geq 2^{-b} \sum_{U(p)=x} 2^{-|p|} \\ &= \sum_{p \in P} 2^{-|p|-b} \geq \sum_{q \in Q} 2^{-|q|}. \end{aligned}$$

The last sum is at most the first sum and the programs of  $Q$  constitute all the programs in  $P$  that have length at least  $K(x) + b$  (a shortest program in  $P$  trivially having length  $K(x)$  and therefore a shortest program in  $Q$  has length  $K(x) + b$ ). Since  $D \subseteq P$  either  $D = Q$  or  $D \cap Q \neq \emptyset$ . It follows that there exist programs  $q \in D$  at least as short as the shortest program of  $Q$ . Since a shortest program in  $Q$  has length  $K(x) + b$  therefore  $|q| \leq K(x) + b$ .

By the convention of Remark 3  $(D - 1) \neq \emptyset$ . Choose  $r \in (D - 1)$  and  $q \in D$  with  $|q| \leq K(x) + b$  such that  $||r| - |q||$  is minimal. Additionally,  $K(r) \geq K(x) + O(1)$  (since  $U(r) = x$  and an  $O(1)$  term independent of  $r$  and  $x$ ). Therefore  $|r| \leq K(r) + b + g_x(d)$  with  $g_x(d) = ||r| - |q|| - O(1)$ . It follows that if  $\text{depth}_\beta^{(2)}(x) = d$  then  $\beta \leq b + g_x(d)$ .  $\square$

According to Theorems 1, 2 version 2 implies version 1 with the same depth  $d$  and almost the same parameter  $b$ , while version 1 implies version 2 with the same depth  $d$  but more uncertainty in the parameter  $b$ . We choose version 2 as our final definition of logical depth.

## 4 The graph of logical depth

Even slight changes of the significance level  $b$  can cause large changes in logical depth.

**LEMMA 1** *Every function  $\phi$  such that  $\phi(x) \geq \min\{d : U^d(p) = x, |p| = K(x)\}$  is incomputable and rises faster than any computable function.*

**PROOF.** By [5] we have  $K(K(x)|x) \geq \log n - 2 \log \log n - O(1)$ . (This was improved to the optimal  $K(K(x)|x) \geq \log n - O(1)$  recently in [3].) Hence there is no computable function  $\phi(x) \geq \min\{d : U^d(p) = x, |p| = K(x)\}$ . If there were, then we could run  $U$  for  $d$  steps on any program of length  $n + O(\log n)$ . Among the programs that halt within  $d$  steps we select the ones which output  $x$ . Subsequently, we select from this set a program of minimum length. This is a shortest program for  $x$  of length  $K(x)$ . Therefore, the assumption that  $\phi$  is computable implies that  $K(K(x)|x) = O(1)$  and hence a contradiction.  $\square$

**DEFINITION 4** The *Busy Beaver function*  $BB : \mathbb{N} \rightarrow \mathbb{N}$  is defined by

$$BB(n) = \max\{d : |p| \leq n \wedge U^d(p) < \infty\}$$

The following result was mentioned informally in [4].

**LEMMA 2** *The running time of a program  $p$  is at most  $BB(|p|)$ . The running time of a shortest program for a string  $x$  of length  $n$  is at most  $BB(n + O(\log n))$ .*

**PROOF.** The first statement of the lemma follows from Definition 4. For the second statement we use the notion of a simple prefix-code called a *self-delimiting* code. This is obtained by reserving one symbol, say 0, as a stop

sign and encoding a string  $x$  as  $1^x0$ . We can prefix an object with its length and iterate this idea to obtain ever shorter codes:  $\bar{x} = 1^{|x|}0x$  with length  $|\bar{x}| = 2|x| + 1$ , and  $x' = \overline{|x|}x$  of length  $|x| + 2||x|| + 1 = |x| + O(\log |x|)$  bits. From this code  $x$  is readily extracted. The second statement follows since  $K(x) \leq |x| + O(\log |x|)$ .  $\square$

**THEOREM 3** *There is an infinite sequence of strings  $x_1, x_2, \dots$  with  $|x_{j+1}| = |x_j| + 1$  ( $j \geq 1$ ) and an infinite sequence  $b_1, b_2, \dots$  of integers such that  $f(j) = \text{depth}_{b_j}^{(2)}(x_j)$  is incomputable (faster than any computable function) and  $g(j) = \text{depth}_{b_j+1}^{(2)}(x_j)$  is computable.*

**PROOF.** Let  $\phi(x) \geq \text{depth}_0^{(2)}(x)$  be an incomputable function as in Lemma 1. The function  $\psi$  defined by  $\psi(x) = \text{depth}_{n-K(x)+O(\log n)}^{(2)}(x) = O(n \log n)$  for  $|x| = n$  is computable. Namely, a self-delimiting encoding of  $x$  can be done in  $n + O(\log n)$  bits. Let  $q$  be such an encoding with  $q = 1^{\|x\|}0|x|x$  (where  $\|x\|$  is the length of  $|x|$ ). Let  $r$  be a self-delimiting program of  $O(1)$  bits which prints the encoded string. Consider the program  $rq$ . Since  $x$  can be compressed to length  $K(x)$ , the running time  $\text{depth}_{n-K(x)+O(\log n)}^{(2)}(x)$  is at most the running time of  $rq$  which is  $O(n \log n)$ .  $\square$

**COROLLARY 1** Define the function  $h$  by  $h(j) = f(j) - g(j)$ . Then  $h$  is a gap in the logical depths of which the significance differs by 1. The function  $h(j)$  rises faster than any computable function but not faster than  $BB(|x_j| + O(\log |x_j|))$  by Lemma 2.

**COROLLARY 2** Let  $s^*(j) = \text{depth}_{O(1)}^{(2)}(x_j)$  be the minimal time of a computation of a shortest program for  $x_j$ , and  $f$  be the function in statement of Theorem 3. Then  $f(j) \leq s^*(j) \leq BB(|x_j| + O(\log |x_j|))$ .

Namely, the logical depth function  $\text{depth}_b^{(2)}(x)$  is monotonic nonincreasing in the significance argument  $b$  for all strings  $x$  by its Definition 3. By Lemma 2 and Corollary 1 the Corollary 2 follows.

## 5 Conclusion

We resolve quantitative relations between the two versions of logical depth in the literature. One of these relations was known by another proof, the other

relation is new. We select one version that approximately implies the other, and study the behavior of the resulting logical depth function associated with a string  $x$  of length  $n$ . This function is monotonic nonincreasing. For argument 0 the logical depth is at least the minimum running time of the computation from a shortest program for  $x$ . The function decreases to  $O(n \log n)$  for the argument  $|x| - K(x) + O(\log |x|)$ . We show that there is an infinite sequence of strings such that maximum gap of logical depths resulting from consecutive significance levels rises faster than any computable function, that is, incomputably fast, but not more than the Busy Beaver function. This shows that logical depth can increase tremendously for only an incremental difference in significance. Moreover, the minimal computation times of associated shortest programs rises incomputably fast but not so fast as the Busy Beaver function.

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