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ARTICLE

Quasi-graphic matroids*

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Abstract

Frame matroids and lifted-graphic matroids are two interesting generalizations of graphic matroids. Here, we introduce a new generalization, *quasi-graphic matroids*, that unifies these two existing classes. Unlike frame matroids and lifted-graphic matroids, it is easy to certify that a 3-connected matroid is quasi-graphic. The main result is that every 3-connected representable quasi-graphic matroid is either a lifted-graphic matroid or a frame matroid.

KEYWORDS

frame matroids, graphic matroids, matroids, representation

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1 | INTRODUCTION

Let G be a graph and let M be a matroid. For a vertex v of G we let $\text{loops}_G(v)$ denote the set of loop-edges of G at the vertex v. We say that G is a *framework* for M if

(1) $E(G) = E(M)$,

(2) $r_M(E(H)) \leq |V(H)|$ for each component H of G,

(3) for each vertex v of G we have $\text{cl}_M(E(G-v)) \subseteq E(G-v) \cup \text{loops}_G(v)$, and

(4) for each circuit C of M, the subgraph $G[C]$ has at most two components.

An earlier version of this article had a serious flaw that was pointed out to us by Daryl Funk. In order to overcome that issue we added condition (4) to the definition.

This definition is motivated by the following theorem that follows from a result due to Seymour [3].

Theorem 1.1. *Let G be a graph with c components and let M be a matroid. Then M is the cycle matroid of G if and only if G is a framework for M and $r(M) \leq |V(G)| - c$.*

We will call a matroid *quasi-graphic* if it has a framework. Next we will consider two classes of quasi-graphic matroids; namely "lifted-graphic matroids" and "frame matroids."

We say that a matroid M is a *lifted-graphic matroid* if there is a matroid M' and an element $e \in E(M')$ such that $M' \setminus e = M$ and M'/e is graphic. The following result is proved in Section 6.

Theorem 1.2. *Every lifted-graphic matroid is quasi-graphic.*

A *framed matroid* is a pair (M, V) such that M is a matroid, V is a basis of M, and each element of M is spanned by a subset of V with at most two elements. A matroid M is a *frame matroid* if there is a framed matroid (M', V) such that $M = M'$. The following result is proved in Section 5.

Theorem 1.3. *Every frame matroid is quasi-graphic.*

Our main result is that for matroids that are both 3-connected and representable, there are no quasi-graphic matroids other than those described above.

Theorem 1.4. *Let M be a 3-connected representable matroid. If M is quasi-graphic, then M is either a frame matroid or a lifted-graphic matroid.*

The representability condition in Theorem 1.4 is necessary; the Vámos matroid, for example, is quasi-graphic but it is neither a frame matroid nor a lifted-graphic matroid. However, for frameworks with loop-edges, we do not require representability.

Theorem 1.5. *Let G be a framework for a 3-connected matroid M. If G has a loop-edge, then M is either a frame matroid or a lifted-graphic matroid.*

Our proof of Theorem 1.5 uses results of Zaslavsky [4] who characterized frame matroids and lifted-graphic matroids using "biased graphs"; we review those results in Sections 6 and 5.

Given a graph G and a matroid M, via its rank oracle, one can efficiently check the conditions (1), (2), and (3). However, since M may have exponentially many circuits, it is not clear how one might efficiently check (4). The following result shows that one can easily certify that a 3-connected matroid is quasi-graphic.

Theorem 1.6. *A 3-connected matroid M is quasi-graphic if and only if there exists a graph G such that*

- **(i)** $E(G) = E(M)$,
- **(ii)** *G is connected,*
- **(iii)** $r(M) \leq |V(G)|$, *and*
- **(iv)** *for each vertex v of G we have* $\mathrm{cl}_M(E(G - v)) \subseteq E(G - v) \cup \mathrm{loops}_G(v)$.

We conjecture that the problem of recognizing 3-connected quasi-graphic matroids is tractable.

Conjecture 1.7. *There is a polynomial-time algorithm that given a 3-connected matroid M, via its rank-oracle, decides whether or not M is quasi-graphic.*

This contrasts with results of Rong and Whittle [1] who prove that there is no efficient algorithm for either frame matroid recognition or lifted-graphic matroid recognition. In fact, Rong and Whittle's results show that to certify that a matroid is either a frame matroid or a lifted graphic matroid requires an exponential number of rank evaluations in the worst case; those results were conjectured in an earlier version of this article.

We will use the notation and terminology of Oxley [2], except we denote $|E(M)|$ by $|M|$ and we define a graph G to be *k-connected* when $G - X$ is connected for each set $X \subseteq V(G)$ with $|X| < k$ (we do not require that $|V(G)| > k$); moreover, we consider that the graph with no vertices is connected.

2 | WEAK FRAMEWORKS AND MINORS

For the aid of readers familiar with the earlier flawed version of this article, we will keep the following two sections essentially unchanged except for correcting the transgression that was the root cause for the mistakes. To this end, we need to introduce a version of frameworks without the condition (4).

Let M be a matroid and G be a graph. We call G a *weak framework* for M if

(1) $E(G) = E(M)$,

(2) $r_M(E(H)) \leq |V(H)|$ for each component H of G, and

(3) for each vertex v of G we have $\mathrm{cl}_M(E(G - v)) \subseteq E(G - v) \cup \mathrm{loops}_G(v)$.

Note that a graph G satisfies conditions $(i) - (iv)$ of Theorem 1.6 if and only if it is a connected weak framework. In this section, we will see that weak frameworks behave nicely under minors.

Lemma 2.1. *Let G be a weak framework for a matroid M. If H is a component of G, then H is a weak framework for $M|E(H)$.*

Proof. Note that conditions (1) and (2) are immediate. Condition (3) follows from the fact that for each flat F of M, the set $F \cap E(H)$ is a flat of $M|E(H)$. ■

The following result is very easy, but it is used repeatedly.

Lemma 2.2. *Let G be a weak framework for a matroid M. If v is a vertex of G that is incident with at least one nonloop-edge, then $r_M(E(G - v)) < r(M)$. Moreover, if v has degree one, then $r_M(E(G - v)) = r(M) - 1$.*

Proof. This follows directly from (3). ■

Lemma 2.3. *Let G be a connected weak framework for a matroid M and let H be a nonempty subgraph of G. Then $|V(H)| - r(M|E(H)) \geq |V(G)| - r(M)$.*

Proof. We can extend H to a spanning subgraph H^+ of G, adding one vertex and one edge at a time, with $|E(H^+)| - |E(H)| = |V(G)| - |V(H)|$. Clearly $|V(H^+)| - r(E(H^+)) \geq |V(G)| - r(M)$. If $H \neq H^+$, then there is a vertex $v \in V(H^+) - V(H)$ that has degree one in H^+. By Lemma 2.2, $r(E(H^+ - v)) = r(E(H)) - 1$ and, hence, $|V(H^+ - v)| - r(E(H^+ - v)) \geq |V(G)| - r(M)$. Now we obtain the result by repeatedly deleting vertices in $V(H^+) - V(H)$ in this way. ■

If X is a set of edges in a graph G, then $G[X]$ is the subgraph of G with edge-set X and with no isolated vertices; moreover, we will denote $V(G[X])$ by $V(X)$.

Lemma 2.4. *If G is a weak framework for a matroid M and $X \subseteq E(M)$, then $G[X]$ is a weak framework for $M|X$.*

Proof. Condition (1) is clearly satisfied. Condition (2) follows from Lemmas 2.1 and 2.3. Condition (3) follows from the fact that for each flat F of M, the set $F \cap E(H)$ is a flat of $M|E(H)$. Thus $G[X]$ is a weak framework for $M|X$. ■

The following two results give sufficient conditions for independence and dependence, respectively, for a set in a quasi-graphic matroid.

Lemma 2.5. *Let G be a weak framework for a matroid M. If F is a forest of G, then $E(F)$ is an independent set of M.*

Proof. We may assume that $E(F)$ is nonempty and, hence, that F has a degree-one vertex v. By Lemma 2.2, $r_M(E(F)) = r_M(E(F-v)) + 1$. Now the result follows inductively. ■

Lemma 2.6. *Let G be a weak framework for a matroid M. If H is a subgraph of G and $|E(H)| > |V(H)|$, then $E(H)$ is a dependent set of M.*

Proof. By Lemma 2.4 and (2), we have $r_M(E(H)) \leq |V(H)|$. So, if $|E(H)| > |V(H)|$, then $E(H)$ is a dependent set of M. ■

We can now prove Theorem 1.1. The "only if" direction is routine and left to the reader. For the "if" direction we prove the following stronger result in which we have replaced "framework" with "weak framework." This result is tantamount to the main theorem of [3], but we include the proof since it is short and the result is central to this article.

Theorem 2.7. *Let G be a graph with c components and let M be a matroid. If G is a weak framework for M and $r(M) \leq |V(G)| - c$, then M is the cycle matroid of G.*

Proof. By Lemma 2.5 and the fact that $r(M) \leq |V(G)| - c$, we have $r(E(H)) = |V(H)| - 1$ for each component H of G. Hence we may assume that G is connected. By Lemma 2.5, the edge-set of each forest of G is independent in M. Therefore, it suffices to prove, for each cycle C of G, that $E(C)$ is dependent in M. By Lemma 2.3, $|V(C)| - r(E(C)) \geq |V(G)| - r(E(G)) = 1$. So $r(E(C)) < |V(C)| = |E(C)|$ and, hence, $E(C)$ is dependent as required. ■

To consider the effect of contraction on weak frameworks, we consider two cases depending on whether or not we are contracting a loop-edge.

Lemma 2.8. *Let G be a weak framework for a matroid M and let e be a nonloop-edge of G. Then G/e is a weak framework for M/e.*

Proof. Conditions (1) and (2) are clearly satisfied. Let u and v be the ends of e in G, and let f be an edge of G that is incident with u but not with v. To prove (3) it suffices to prove that that there exists a cocircuit C in M such that $f \in C$, $e \notin C$, and C contains only edges incident with either u or v.

By (3), there exist cocircuits C_e and C_f such that $e \in C_e$, that C_e contains only edges incident with v, that $f \in C_f$, and that C_f contains only edges incident with u. We may assume that $e \in C_f$ since otherwise we could take $C = C_f$. Since f is not incident with v, we have $f \notin C_e$. Then, by the strong circuit exchange axiom, there is a cocircuit C of M with $f \in C \subseteq (C_1 \cup C_2) - \{e\}$, as required. ■

Let e be a loop-edge of a graph G and let v be the vertex incident with v. We denote by $G \circ e$ the graph obtained from $G - v$ by first, for each nonloop edge $f = vw$ incident with v adding f as a loop-edge at w, and then for each loop-edge f of $G - e$ at v adding f as a loop-edge on an arbitrary vertex. The graph $G \circ e$ is well-defined unless there are multiple loop-edges at v.

Lemma 2.9. *Let G be a weak framework for M and let e be a loop-edge of G. If e is not a loop of M, then $G \circ e$ is a weak framework for M/e.*

Proof. Let v be the vertex incident with e. Condition (1) is clearly satisfied and condition (2) is also routine. By Lemma 2.4 and (2), we have $r_M(\text{loops}_G(v)) = 1$, so each element of $\text{loops}_G(v) - \{e\}$ is a loop in M/e. Each vertex $w \in V(G) - \{v\}$ is incident with the same edges in G as it is in H except for the elements in $\text{loops}_G(v)$. Moreover, $\text{cl}_M(E(G - w)) = \text{cl}_{M/e}(E(H - w)) \cup \{e\}$. Therefore (3) follows. ■

3 | BALANCED CYCLES

Let G be a weak framework for a matroid M and let C be a cycle of G. By Lemmas 2.3 and 2.5, $E(C)$ is either independent in M or $E(C)$ is a circuit in M. If $E(C)$ is a circuit of M, then we say that C is a *balanced* cycle of (M, G); when the matroid M is clear from the context, we will say that C is a *balanced* cycle of G. We recall that $M(G)$ denotes the cycle matroid of a graph G.

Lemma 3.1. *Let G be a weak framework for a matroid M. Then $M = M(G)$ if and only if each cycle of G is balanced.*

Proof. If $M = M(G)$, then each cycle of G is balanced. Conversely, suppose that each cycle of G is balanced. Let F be a maximal forest in G. Since each cycle is balanced, $E(F)$ is a basis of M. Then, by Theorem 1.1, $M = M(G)$. ■

A *theta* is a 2-connected graph that has exactly two vertices of degree 3 and all other vertices have degree 2. Observe that there are exactly three cycles in a theta.

Lemma 3.2. *Let G be a weak framework for a matroid M and let H be a theta-subgraph of G. If two of the cycles in H are balanced, then so too is the third.*

Proof. If there are two balanced cycles in H then $r_M(E(H)) \leq |E(H)| - 2 = |V(H)| - 1$. So, by Theorem 1.1, $M|E(H) = M(H)$ and, by Lemma 3.1, all cycles of H are balanced. ■

The following result describes the circuits of a matroid in terms of its weak framework; first we will give an unusual example to demonstrate one of the outcomes. If M consists of a single circuit and G is a graph with $E(G) = E(M)$ whose components are cycles, then G is a weak framework for M.

Lemma 3.3. *Let G be a weak framework for a matroid M. If C is a circuit of M, then either*

- *$G[C]$ is a balanced cycle,*
- *$G[C]$ is a connected graph with minimum degree at least two, $|C| = |V(C)| + 1$, and $G[C]$ has no balanced cycles, or*
- *$G[C]$ is a collection of vertex-disjoint nonbalanced cycles.*

Proof. We may assume that $G[C]$ is not a balanced cycle, and, hence, that $G[C]$ contains no balanced cycle. Next suppose that $|C| \geq |V(C)| + 1$. By Lemma 2.6, C is minimal with this property. Hence $G[C]$ is connected, the minimum degree of $G[C]$ is two, and $|C| = |V(C)| + 1$. Now suppose that $|C| \leq |V(C)|$ and consider a component H of $G[C]$; it suffices to show that H is a cycle. By Lemma 2.6 and the argument above, we may assume that $|E(H)| \leq |V(H)|$. If H is not a cycle there is a degree-one vertex v of H. Moreover, the edge e that is incident with v is not a loop-edge. Then, by (3), the element e is a coloop of $M|C$, which contradicts the fact that C is a circuit. ■

For a set X of elements in a matroid M we let

$$\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M).$$

Lemma 3.4. *Let G be a weak framework for M. If H is a component of G, then $\lambda_M(E(H)) \leq 1$.*

Proof. By Lemma 2.2, $r(E(M) - E(H)) \leq r(M) - (|V(H)| - 1)$. Hence $\lambda_M(E(H)) = r_M(E(H)) + r_M(E(M) - E(H)) - r(M) \leq |V(H)| + (r(M) - (|V(H)| - 1)) - r(M) = 1$. ■

A *loop-component* of a graph G is a component consisting of exactly one vertex and exactly one edge. The mistake in the earlier version of this article was that the third outcome of the following lemma was overlooked.

Lemma 3.5. *If G is a weak framework for a 3-connected matroid M with $|M| \geq 4$ and G has no isolated vertices, then either*

- **(a)** *G is connected,*
- **(b)** *G has exactly two components one of which is a loop-component, or*
- **(c)** *every component of G is a loop-component.*

Proof. It follows from (3) and the fact that M has no coloops that each component of G is either a loop-component or has at least two edges. We may assume that G has a nonloop-component, say H, and we may further assume that $|E(G) - E(H)| \geq 2$. Then, by Lemma 3.4, $(E(H), E(G) - E(H))$ is a 2-separation. However this contradicts the fact that M is 3-connected. ■

That oversight turns out to be catastrophic. Let M be a matroid and let G be a graph, with $E(G) = E(M)$, whose components are all loop-components. Then G is a weak framework for M. So all matroids admit weak frameworks!

To conclude this section, we consider additional properties of the weak frameworks that satisfying outcomes (a) and (b) of Lemma 3.5. We will start by showing that, in case (b), M is a coextension of a graphic matroid.

Lemma 3.6. *Let G be a weak framework, without isolated vertices, for a connected matroid M. If G has exactly two components and e is an edge in a loop-component of G, then $M/e = M(G \setminus e)$. Moreover M has a connected weak framework.*

Proof. Let H denote the component of G that does not contain e. By Lemma 2.9, H is a connected weak framework for M/e. By Lemma 2.1, we have $r_M(E(H)) \leq |V(H)|$. Since M is connected, $e \in \mathrm{cl}_M(E(H))$. Therefore $r(M/e) \leq |V(H)| - 1$. Then, by Theorem 2.7, we have $M/e = M(H)$.

Let v be the vertex of G incident with e and let $w \in V(G)\{v\}$. Construct a graph G^+ from G by adding a new edge f with ends v and w and construct a new matroid M^+ by adding f as a coloop to M. Note that G^+ is a weak framework for M^+ and hence, by Lemma 2.8. G^+/f is a weak framework for M^+/f. Moreover, as f is a coloop of M^+, we have $M^+/f = M$. Therefore G^+/e is a connected weak framework for M. ■

The following result shows that connected weak frameworks are in fact frameworks; this implies the "if" direction of Theorem 1.6.

Lemma 3.7. *Let M be a matroid with at least four elements and let G be a connected weak framework for M. If C is a circuit of M, then $G[C]$ has at most two components.*

Proof. Suppose that $G[C]$ has more than two components. By Lemma 3.3, each component of $G[C]$ is a nonbalanced cycle. Let P be a shortest path connecting two components of $G[C]$; let these components be C_1 and C_2. Since C is a circuit, $E(C_1 \cup C_2)$ is independent. By Lemmas 2.6 and 3.3, $E(C_1 \cup C_2 \cup P)$ is a circuit of M.

Let $e \in E(P)$ and $f \in E(C_1)$. By the strong exchange property for circuits, there is a circuit C' of G with $e \in C' \subseteq (C \cup E(P)) - \{f\}$. However this is inconsistent with the outcomes of Lemma 3.3. ■

Finally, we show that every connected weak framework for a 3-connected matroid is necessarily 2-connected.

Lemma 3.8. *Let M be a 3-connected matroid with $|M| \geq 4$. If G is a connected weak framework for M, then G is 2-connected.*

Proof. Suppose otherwise. Then there is a pair (H_1, H_2) of subgraphs of G such that $G = H_1 \cup H_2$, $|V(H_1) \cap V(H_2)| = 1$, and $|V(H_1)|, |V(H_2)| \geq 2$. Note that H_1 and H_2 are both connected. Now $M(G)$ is not 3-connected, so, by Theorem 1.1, $r(M) = |V(G)|$. Therefore $\lambda_M(E(H_1)) \leq |V(H_1)| + |V(H_2)| - |V(G)| = 1$. Since M is 3-connected either $|E(H_1)| \leq 1$ or $|E(H_2)| \leq 1$; we may assume that $|E(H_1)| = 1$. Let $e \in E(H_1)$. Since H_1 is connected and $|V(H_1)| \geq 2$, the edge e is not a loop-edge. Therefore, by (3), e is a coloop of M. This contradicts the fact that M is 3-connected. ■

4 | QUASI-GRAPHIC MATROIDS

The following result shows that the class of quasi-graphic matroids is closed under taking minors.

Lemma 4.1. *Let G be a framework for a matroid M. For each $e \in E(M)$,*

- *$G - e$ is a framework for M,*
- *if e is not a loop-edge of G, then G/e is a framework for M/e, and*
- *if e is a loop-edge of G and e is not a loop of M, then $G \circ e$ is a framework for M/e.*

Proof. By Lemma 2.4, $G - e$ is a weak framework for M. Moreover, (4) is clearly preserved under deletion, so $G - e$ is a framework for M.

Suppose that e is a nonloop-edge of G. By Lemma 2.8, G/e is a weak framework for M/e. Consider a circuit C of M/e. Either C or $C \cup \{e\}$ is a circuit of M. Let $C' \in \{C, C \cup \{e\}\}$ be a circuit of M. Then $G[C']$ has at most two components, and, hence, $G/e[C]$ has at most two components. So G/e is a framework for M/e.

Finally suppose that e is a loop-edge of G and that e is not a loop of M. By Lemma 2.9, $G \circ e$ is a weak framework for M/e. Consider a circuit C of M/e. Either C or $C \cup \{e\}$ is a circuit of M. Let $C' \in \{C, C \cup \{e\}\}$ be a circuit of M. Then $G[C']$ has at most two components and, by Lemma 3.3, if $G[C']$ has two components then each of the components is 2-connected. Thus $G \circ e[C]$ has at most two components. So $G \circ e$ is a framework for M/e. ■

The following result is a strengthening of Lemma 3.5 for frameworks.

Lemma 4.2. *If G is a framework for a 3-connected matroid M with $|M| \geq 4$ and G has no isolated vertices, then either*

(a) *G is connected, or*

(b) *G has exactly two components one of which is a loop-component.*

Moreover M has a connected framework.

Proof. By Lemma 3.5, if G does not satisfy (a) or (b), then each component of G is a loop-component. Since M is 3-connected, and therefore simple, M has a circuit of length at least three. However any such circuit violates (4). Hence G indeed satisfies (a) or (b).

If G is itself not connected, then G satisfies (b). Then, by Lemma 3.6, M has a connected weak framework G'. By Lemma 3.7, G' is a connected framework for M. ■

Note that Theorem 1.6 follows directly from Lemmas 4.2 and 3.7.

5 | FRAME MATROIDS

A *simple framed matroid* is a framed matroid (M, V) with M simple. The *support graph* of a simple framed matroid (M, V) is the graph $G = (V, E(M))$ such that, for each $v \in V$, the edge v is a loop-edge at the vertex v, and, for each $e \in E(M) - V$, the edge e has ends u and v where $\{e, u, v\}$ is the unique circuit of M in $V \cup \{e\}$.

Lemma 5.1. *If G is the support graph of a simple framed matroid (M, V), then G is a framework for M and for each circuit C of M, the subgraph $G[C]$ is connected.*

Proof. By construction $E(G) = E(M)$ and, since V is a basis of M, for each component H of G we have $r(E(H)) = |V(H)|$. Moreover, for each vertex v of G, the hyperplane of M spanned by $V - \{v\}$ is $E(G - v)$. Hence G is a weak framework for M. Finally, if H is a subgraph of G, then $r_M(E(H))$ is the sum, taken over all components H' of H, of $r_M(E(H'))$. Therefore, if C is a circuit of M, then $G[C]$ is connected. ■

We can now prove that every frame matroid is quasi-graphic.

Proof of Theorem 1.4. Let M be a frame matroid. Recall that the class of quasi-graphic matroids is closed under taking minors, so we may assume that M has a basis V such that (M, V) is a framed matroid. Moreover, M is quasi-graphic if and only if its simplification is, so we may assume that M is simple. Now it follows from Lemma 5.1 that M is quasi-graphic. ■

Next we characterize frame matroids using frameworks. These results are due to Zaslavsky [4,5] but we include proofs for completeness since they play a central role in this article.

Let G be a graph and let $\mathcal{B}$ be a subset of the cycles of G. We say that $\mathcal{B}$ satisfies the *theta-property* if there is no theta in G with exactly two of its three cycles in $\mathcal{B}$. The following result is contained in [4][Theorem 2.1].

Theorem 5.2. *Let G be a graph and let $\mathcal{B}$ be a collection of cycles in G that satisfy the theta-property. Now let $\mathcal{I}$ denote the collection of all sets $I \subseteq E(G)$ such that there is no $C \in \mathcal{B}$ with $E(C) \subseteq I$ and $|E(H)| \le |V(H)|$ for each component H of $G[I]$. Then $\mathcal{I}$ is the collection of independent sets of a matroid with ground set $E(G)$.*

Proof. To prove that $(E(G), \mathcal{I})$ is a matroid it suffices to check the following conditions, which are effectively a reformulation of the circuit axioms in terms of independent sets:

- **(a)** $\emptyset \in \mathcal{I}$,
- **(b)** for each $J \in \mathcal{I}$ and $I \subseteq J$, we have $I \in \mathcal{I}$, and
- **(c)** for each set $I \in \mathcal{I}$ and $e \in E(G) - I$ either $I \cup \{e\} \in \mathcal{I}$ or there is a unique minimal subset C of $I \cup \{e\}$ that is not in $\mathcal{I}$.

Conditions (a) and (b) follow from the construction.

We call the cycles of G in $\mathcal{B}$ *balanced*. Let $I \in \mathcal{I}$ and $e \in E(G) - I$ with $I \cup \{e\} \notin \mathcal{I}$. Let C_1 and C_2 be minimal subsets of $I \cup \{e\}$ that are not in $\mathcal{I}$. Suppose for a contradiction that $C_1 \neq C_2$. By definition, for each $i \in \{1, 2\}$, we have $G[C_i - \{e\}]$ is connected, $e \in C_i$, and either $G[C_i]$ is a balanced cycle or $|C_i| > |V(C_i)|$. Consider $J = (C_1 \cup C_2) - \{e\}$. Since $J \subseteq I$, we have $J \in \mathcal{I}$. Since $G[C_1 - \{e\}]$ and $G[C_2 - \{e\}]$ are connected, $G[J]$ is connected. Therefore $|J| \le |V(J)|$. It follows that $|C_1| \le |V(C_1)|$ and $|C_2| \le |V(C_2)|$. Hence $G[C_1]$ and $G[C_2]$ are balanced cycles. Now $G[J]$ is the union of two paths, each connecting the ends of e, and $|J| \le |V(J)|$, so $G[C_1 \cup C_2]$ is a theta. By the theta-property, $G[J]$ has a balanced cycle. However, this contradicts the fact that $J \in \mathcal{I}$. ■

We denote the matroid $(E(G), \mathcal{I})$ in Theorem 5.2 by $FM(G, \mathcal{B})$. The following result is an easy application of [4, Theorem 2.1].

Theorem 5.3. *If G is a graph and $\mathcal{B}$ is a collection of cycles in G that satisfies the theta-property, then $FM(G, \mathcal{B})$ is a frame matroid.*

Proof. Let G^+ be obtained from G by adding a loop-edge e_v at each vertex of v. Since we only added loop-edges, the pair $(G^+, \mathcal{B})$ still satisfies the theta-property. Let $M^+ = FM(G^+, \mathcal{B}^+)$ and $V = \{e_v : v \in V(G)\}$. By the definition of $FM(G^+, \mathcal{B}^+)$, the set V is a basis of M^+. For each nonloop edge e of G with ends u and v, the set $\{e_u, e, e_v\}$ is a circuit of M^+ and for each loop-edge e of G at v, the set $\{e, e_v\}$ is a circuit of M^+. Therefore M^+ is a framed matroid and hence $FM(G, \mathcal{B})$ is a frame matroid. ■

The next result follows directly from Lemma 3.3 and the definition of $FM(G, \mathcal{B})$.

Lemma 5.4. *Let G be a framework for a matroid M and let $\mathcal{B}$ denote the set of nonbalanced cycles of (M, G). Then $M = FM(G, \mathcal{B})$ if and only if for each circuit C of M the subgraph $G[C]$ is connected.*

The following result is the main theorem in [5].

Theorem 5.5. *A matroid M is a frame matroid if and only if there is a graph G and a collection $\mathcal{B}$ of cycles of G satisfying the theta-property such that $M = FM(G, \mathcal{B})$.*

Proof. The "if" direction of the result follows from Theorem 5.3. For the converse, since it is straightforward to add loops and parallel elements, we may assume that M is simple, and that (M, V) is a framed matroid for some basis V of M. Let G be the support graph of (M, V) and let $\mathcal{B}$ denote the set of cycles C of G such that $E(C)$ is dependent in M. By Lemma 5.1, G is a framework for M and, for each circuit C' of M, the subgraph $G[C']$ is connected. Now, by Lemma 5.4, we have $M = FM(G, \mathcal{B})$. ■

6 | LIFTED-GRAPHIC MATROIDS

We say that a matroid M is a *lift* of a matroid N if there is a matroid M' and an element $e \in E(M')$ such that $M' \setminus e = M$ and $M'/e = N$. The following result implies Theorem 1.2.

Theorem 6.1. *If G is a graph and M is a lift of $M(G)$, then G is a framework for M. Moreover, if C_1 and C_2 are disjoint cycles in G, then $E(C_1 \cup C_2)$ is dependent in M.*

Proof. Let e be an element of a matroid M' such that $M' = M$ and $M'/e = M(G)$. Thus $E(M) = E(G)$. For each component H of G, $r_{M'/e}(E(H)) = |V(H)| - 1$ so $r_M(E(H)) = r_{M'}(E(H)) \leq r_{M'/e}(E(H)) + 1 = |V(H)|$. For a vertex v of G, we have $\mathrm{cl}_M(E(G - v)) \subseteq \mathrm{cl}_{M'}(E(G - v) \cup \{e\}) - \{e\} = \mathrm{cl}_{M'/e}(E(G - v)) \subseteq E(G - v) \cup \mathrm{loops}_G(v)$. So G is a weak framework for M.

Now consider two disjoint cycles C_1 and C_2 of G and let $X = G[C_1 \cup C_2]$. So $r_M(X) = r_{M'}(X) \leq r_{M'/e}(X) + 1 = r_{M(G)}(X) + 1 = |X| - 1$ and, hence, X is dependent. Condition (4) follows. ■

Next, we will give an alternate characterization of lifted-graphic matroids using frameworks; again, these results are due to Zaslavsky [4,6], but the proofs are included here for completeness.

Theorem 6.2. *Let G be a graph and let $\mathcal{B}$ be a collection of cycles in G that satisfy the theta-property. Now let $\mathcal{I}$ denote the collection of all sets $I \subseteq E(G)$ such that there is no $C \in \mathcal{B}$ with $E(C) \subseteq I$ and $G[I]$ contains at most one cycle. Then $\mathcal{I}$ is the set of independent sets of a matroid on $E(G)$.*

Proof. As noted in the proof of Theorem 5.2, to prove that $(E(G), \mathcal{I})$ is a matroid it suffices to check the following conditions:

(a) $\emptyset \in \mathcal{I}$,

(b) for each $J \in \mathcal{I}$ and $I \subseteq J$, we have $I \in \mathcal{I}$, and

(c) for each set $I \in \mathcal{I}$ and $e \in E(G) - I$ either $I \cup \{e\} \in \mathcal{I}$ or there is a unique minimal subset C of $I \cup \{e\}$ that is not in $\mathcal{I}$.

Conditions (a) and (b) follow from the construction.

We call cycles of G in $\mathcal{B}$ *balanced*. Let $I \in \mathcal{I}$ and $e \in E(G) - I$ with $I \cup \{e\} \notin \mathcal{I}$. Let C_1 and C_2 be minimal subsets of $I \cup \{e\}$ that are not in $\mathcal{I}$. Suppose for a contradiction that $C_1 \neq C_2$. By definition, for each $i \in \{1, 2\}$, either $G[C_i]$ is a balanced cycle, $G[C_i]$ is the union of two vertex disjoint nonbalanced cycles, or $G[C_i]$ is 2-edge-connected and $|C_i| = |V(C_i)| + 1$. Consider $J = (C_1 \cup C_2) - \{e\}$. Since $J \subseteq I$, we have $J \in \mathcal{I}$ so either $G[J]$ is a forest or $G[J]$ contains a unique cycle.

For each $i \in \{1, 2\}$, there is a cycle A_i of $G[C_i]$ that contains e. Since $G[J]$ contains at most one cycle, either $A_1 = A_2$ or $A_1 \cup A_2$ is a theta.

First suppose that $A_1 = A_2$. Since $C_1 \neq C_2$, the cycle A_1 is nonbalanced. Therefore, for each $i \in \{1, 2\}$, there is a nonbalanced cycle B_i in $G[C_i - e]$. Since $G[J]$ contains a unique cycle $B_1 = B_2$. But then $C_1 = E(A_1 \cup B_1)$ and $C_2 = E(A_2 \cup B_2)$, contradicting the fact that $C_1 \neq C_2$.

Now suppose that $A_1 \cup A_2$ is a theta, and let C be the cycle in $(A_1 \cup A_2) - e$. Since J is independent, C is not balanced. By the theta-property and symmetry, we may assume that A_1 is not balanced. Then there is a non-balanced cycle B_1 in $G[C_1 - \{e\}]$. Since $G[J]$ has at most one cycle $C = B_1$. Therefore $C_1 = E(A_1 \cup A_2)$ and, hence, A_2 is nonbalanced. Then there is a nonbalanced cycle B_2 in $G[C_2 - \{e\}]$. Since $G[J]$ has at most one cycle $C = B_2$, however, this contradicts the fact that $C_1 \neq C_2$. ■

We denote the matroid $(E(G), \mathcal{I})$ in Theorem 6.2 by $LM(G, \mathcal{B})$.

Theorem 6.3. *If G is a graph and $\mathcal{B}$ is a collection of cycles in G that satisfies the theta-property, then $LM(G, \mathcal{B})$ is a lift of $M(G)$ and, hence, G is a framework of $LM(G, \mathcal{B})$.*

Proof. Let G^+ be obtained from G by adding a loop-edge e at a vertex v. Note that $(G^+, \mathcal{B})$ satisfies the theta-property; let $M^+ = LM(G^+, \mathcal{B})$. By the definition of $LM(G^+, \mathcal{B})$, for each cycle C of G, $\{e\} \cup E(C)$ is dependent in M^+. Hence $E(C)$ is a dependent set M^+/e. Similarly, by the definition of $LM(G^+, \mathcal{B})$, for each forest F of G, the set $\{e\} \cup E(F)$ is independent in M^+ and, hence, $E(F)$ is independent in M^+/e. Thus $M^+/e = M(G)$ and, hence, M is a lift of $M(G)$. So, by Theorem 1.2, G is a framework for $LM(G, \mathcal{B})$. ■

The following result is a direct consequence of Lemma 3.3 and the definition of $LM(G, \mathcal{B})$.

Lemma 6.4. *Let G be a framework for a matroid M and let $\mathcal{B}$ denote the set of balanced cycles of (M, G). Then $M = LM(G, \mathcal{B})$ if and only if for each pair (C_1, C_2) of disjoint cycles of G, the set $E(C_1 \cup C_2)$ is dependent in M.*

The following result, which is a converse to Theorem 6.3, is proved in [6, Section 3].

Theorem 6.5. *If G is a graph, M is a lift of $M(G)$, and $\mathcal{B}$ is the set of balanced cycles of (M, G), then $M = LM(G, \mathcal{B})$.*

Proof. The result follows immediately from Lemmas 6.1 and 6.4. ■

7 | FRAMEWORKS WITH A LOOP-EDGE

In this section, we prove Theorem 1.5 that is an immediate consequence of the following two results.

Theorem 7.1. *Let G be a framework for a 3-connected matroid M, let $\mathcal{B}$ be the set of balanced cycles of G, and let e be a nonbalanced loop-edge at a vertex v. If $e \in \mathrm{cl}_M(E(G-v))$, then $M = LM(G, \mathcal{B})$.*

Proof. By Lemmas 4.2 and 3.6, we may assume that G is connected.

We will start by proving, for any nonbalanced cycle C of G, that $E(C) \cup \{e\}$ is a circuit of M. By Lemmas 2.6 and 3.3 we may assume that $v \notin V(C)$. Let P be a minimal path from $\{v\}$ to $V(C_2)$ and let $X = \{e\} \cup E(P \cup C)$. By Lemma 2.6, X is dependent. Let f be the edge of P that is incident with v. By (3) and the fact that $e \in \mathrm{cl}_M(E(G-v))$, there is a cocircuit C^* of M such that $C^* \cap X = \{f\}$. Therefore X is not a circuit of M. So, by Lemma 3.3, $\{e\} \cup E(C)$ is a circuit of M, as required.

By Lemma 6.4, it suffices to prove that if C_1 and C_2 are vertex-disjoint cycles of G, then $E(C_1 \cup C_2)$ is dependent in M; we may assume that C_1 and C_2 are nonbalanced. By the preceding paragraph we may assume that neither C_1 nor C_2 is equal to $G[\{e\}]$ and both $E(C_1) \cup \{e\}$ and $E(C_2) \cup \{e\}$ are circuits of M. So, by the circuit-exchange property, $E(C_1 \cup C_2)$ is dependent, as required. ■

Theorem 7.2. *Let G be a framework for a 3-connected matroid M with $|M| \geq 4$, let $\mathcal{B}$ be the set of balanced cycles of G, and let e be a loop-edge at a vertex v. If $e \notin \mathrm{cl}_M(E(G-v))$, then $M = FM(G, \mathcal{B})$.*

Proof. First we consider the case that G is not connected. By Lemma 4.2, G has exactly two components one of which is a loop-component; let f be the edge in the loop-component. By Lemma 3.6, $M/f = M(G)$. Since M is 3-connected, M/f has no loops and, hence, $f = e$. However, $e \notin \mathrm{cl}_M(E(G-v))$ and hence e is a coloop of M, contradicting fact that M is 3-connected. Hence G is connected.

Suppose by way of contradiction that $M \neq FM(G, \mathcal{B})$. Then, by (4) and Lemmas 3.3 and 5.4, there exist disjoint nonbalanced cycles C_1 and C_2 of (M, G) such that $E(C_1 \cup C_2)$ is a circuit in M.

Since $e \notin \mathrm{cl}_M(E(G-v))$, neither C_1 nor C_2 is equal to $G[\{e\}]$. Since G is connected, there is a path from v to $V(C_1 \cup C_2)$ in G; let P be a minimal such path. We may assume that P has an end in $V(C_1)$. By Lemmas 2.6 and 3.3, $E(C_1 \cup P) \cup \{e\}$ is a circuit of M. Let $f \in E(C_1)$; by the circuit exchange property, there exists a circuit C in $(E(C_1 \cup C_2 \cup P) \cup \{e\}) - \{f\}$. By Lemma 3.3, $C = E(C_2) \cup \{e\}$. However this contradicts the fact that $e \notin \mathrm{cl}_M(E(G-v))$. ■

8 | REPRESENTABLE MATROIDS

A framework G for a matroid M is called *strong* if G is connected and $r_M(E(G-v)) = r(M) - 1$ for each vertex v of G.

Lemma 8.1. *If M is a 3-connected quasi-graphic matroid with $|M| \geq 4$, then M has a strong framework.*

Proof. By Lemma 4.2, M has a connected framework. Let G be a connected framework having as many loop-edges as possible. Suppose that G is not a strong framework and let $v \in V(G)$ such that $r_M(E(G) - v) < r(M) - 1$. Let C^* be a cocircuit of M with $C^* \cap E(G-v) = \emptyset$; if possible we choose C^* so that it contains a loop-edge of G. Since M is 3-connected, $|C^*| \geq 2$ and, by Lemma 2.6, there is at most one loop-edge at v. Therefore C^* contains at least one nonloop-edge. Let L denote the set of nonloop-edges of $G - C^*$ incident with v. By our choice of C^*, the set L is nonempty.

Let H be the graph obtained from G by replacing each edge $f = vw \in L$ with a loop-edge at w. By Lemma 3.8, H is connected. Note that H is framework for M. However, this contradicts our choice of G. ■

We can now prove our main theorem that, if M is a 3-connected representable quasi-graphic matroid, then M is either a frame matroid or a lifted-graphic matroid.

Proof of Theorem 1.4.. Let $M = M(A)$, where A is a matrix over a field $\mathbb{F}$ with linearly independent rows. We may assume that $|M| \geq 4$. Therefore, by Lemma 8.1, M has a strong framework G. ■

Claim. *There is a matrix $B \in \mathbb{F}^{V(G)\times E(G)}$ such that*

- *the row-space of B is contained in the row-space of A, and*
- *for each $v \in V(G)$ and nonloop edge e of G, we have $B[v, e] \neq 0$ if and only if v is incident with e.*

Proof of claim. Let $v \in V(G)$ and let $C^* = E(M) - \mathrm{cl}_M(E(G - v))$. By the definition of a strong framework, C^* is a cocircuit of M. Since $r(E(M) - C^*) < r(M)$, by applying row-operations to A we may assume that there is a row w of A whose support is contained in C^*. Since C^* is minimally codependent, the support of row-w is equal to C^*. Now we set the row-v of B equal to the row-w of A. ■

Note that $M(B)$ is a frame matroid and G is a framework for $M(B)$. We may assume that $r(M(A)) > r(M(B))$ since otherwise $M(A)$ is a frame matroid. Since G is a connected framework for both $M(A)$ and $M(B)$, it follows that $r(M(B)) = |V(G)| - 1$ and that $r(M(A)) = |V(G)|$. Up to row-operations we may assume that A is obtained from B by appending a single row. By Lemma 1.1, $M(B) = M(G)$. Hence M is a lift of $M(G)$.

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