On the rate of decrease in logical depth

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The logical depth with significance b of a string x is the shortest running time of a program for x that can be compressed by at most b bits. Another definition is based on algorithmic probability. We give a simple new proof for the known relation between the two definitions. We also prove the following: Given a string we can consider the maximal decrease in logical depth when the significance parameter increases by 1. There exists a sequence of strings of lengths n = 1, 2, ..., such that this maximal decrease as a function of n rises faster than any computable function but not as fast as the Busy Beaver function. This holds also for the computation times of the shortest programs of these strings.

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1. Introduction

The logical depth is related to complexity with bounded resources and measures the tradeoff between program sizes and running times. Computing a string x from one of its shortest programs may take a very long time, but computing the same string from a simple “print(x)” program of length about |x| bits takes very little time.

A program p for x, i.e. such that U(p) = x where U is a universal Turing machine, of larger length than a given program q for x may require less computation time than q does. This need not always be the case, as a longer program might simply perform some pointless redundant steps.

In general we associate longer computation times with shorter programs for x. As a consequence, one may raise the question of how much time can be saved by computing a given string from a longer program.

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1. Previous work

The above question was first considered by C. Bennett in [1]. The minimum time to compute a string by a $b$-incompressible program was called the logical depth at significance $b$ of the string considered. Bennett also provided variations of this definition and studied their basic properties and relations. A more formal treatment, as well as an intuitive approach, was given in [4], Section 7.7.

1.2. Results

Section 2 introduces notations, definitions and results needed in the remainder. Section 3 presents two versions of logical depth and gives a simple new proof of quantitative relations between them. Section 4 shows that slight variations on the significance level may cause drastic variations of logical depth. Section 5 presents conclusions.

2. Preliminaries

We use string for a finite binary string. Strings are denoted by lower case letters, such as $x$, $y$, and $z$. The length of a string $x$ (the number of occurrences of bits in it) is denoted by $|x|$, and the empty string by $\epsilon$. Thus, $|\epsilon| = 0$. The $n$th string in the lexicographic length-increasing order is viewed also as the natural number $n$ and vice versa.

Computability, resource-bounded computation time, self-delimiting strings, big-O notation, and (prefix) Kolmogorov complexity are well-known and the properties, notations, are treated in [4]. Originally Kolmogorov complexity was introduced in 1965 in [2] and the prefix version in 1974 in [3]. The length of a self-delimiting version of a string of length $n$ will be $n + 2\log n + 1$ where log denotes the logarithm base 2. That is, the self-delimiting version of $x$ is $x' = 1^{|x|}0|x|$, where $||x||$ denotes the length of $|x|$. Restricting the computation time resource is indicated by a superscript giving the allowed number of steps, usually denoted by $d$.

We choose a reference optimal universal prefix Turing machine and call it $U$. Let $x$, $y$ be strings. The prefix Kolmogorov complexity $K(x|y)$ of $x$ with auxiliary $y$ is defined by

$$K(x|y) = \min_p |p : U(p, y) = x|,$$

If $x$ is a string of length $n$ then $K(x|y) \leq n + O(\log n)$. The notation $U^d(p, y) = x$ means that $U(p, y) = x$ within $d$ steps. The $d$-time-bounded prefix Kolmogorov complexity $K^d(x|y)$ is defined by

$$K^d(x|y) = \min_p |p : U^d(p, y) = x|.$$

If the auxiliary string $y$ is the empty string $\epsilon$, then we usually drop it. Similarly, we write $U(p)$ for $U(p, \epsilon)$. The string $x^*$ is a shortest program for $x$ if $U(x^*) = x$ and $K(x) = |x^*|$. A string $x$ is $c$-incompressible if $|x^*| \geq |x| - c$ and it is $c$-compressible if $|x^*| \leq |x| - c$.

Define $Q(x) = \sum_{p : U(p) = x} 2^{-|p|}$. In [3] L.A. Levin proved in the Coding Theorem that (see [4] Theorem 4.3.3 for details):

$$- \log Q(x) = K(x) + O(1)$$

3. Different versions of logical depth

The logical depth as defined in [1] for a string (the finite case) comes in two versions: one based on the compressibility of programs of prefix Turing machines and the other using the ratio between algorithmic probabilities with and without time limits.

The algorithmic probability version is based on the so-called a priori probability [4] and its time-bounded version:

$$Q(x) = \sum_{U(p) = x} 2^{-|p|}, \quad Q^d(x) = \sum_{U^d(p) = x} 2^{-|p|}.$$
Definition 1. Let \( x \) be a string and \( b \) a nonnegative integer. The logical depth, version 1, of \( x \) at significance level \( \epsilon = 2^{-b} \) is

\[
\text{depth}^{(1)}_{b}(x) = \min \left\{ d : \frac{Q^d(x)}{Q(x)} \geq \epsilon \right\}.
\]

The definition of logical depth based on the prefix Kolmogorov complexity version is based on the minimal time the reference optimal prefix Turing machine needs to compute \( x \) from a program which is \( b \)-incompressible.

Definition 2. Let \( x \) be a string and \( b \) a nonnegative integer. The logical depth, version 2, of \( x \) at significance level \( b \), is

\[
\text{depth}^{(2)}_{b}(x) = \min \left\{ d : p \in [0, 1]^* \land U^d(p) = x \land |p| \leq K(p) + b \right\},
\]

the least number of steps to compute \( x \) by a \( b \)-incompressible program.

Remark 1. The relation of almost equality between those definitions is known [4, Theorem 7.7.1] based on the more informal [1, Lemma 3]. As far as we know that is the only existing proof. We give a simple new proof based on algorithmic probability.

Theorem 1. Let \( x \) be a string and \( b, d \) positive integers. Then

(i) \( \text{depth}^{(1)}_{b+d}(x) = d \) implies \( \text{depth}^{(2)}_{b-O(1)}(x) = d \), and

(ii) \( \text{depth}^{(2)}_{b}(x) = d \) implies \( \text{depth}^{(1)}_{2^{-b}c}(x) \leq d \) for some \( b' \leq b + K(b) + O(1) \).

Proof. (i): Let \( Q^d(x) \geq 2^{-b} Q(x) \) with \( d \) least. Let \( c \) be the greatest integer such that all programs computing \( x \) within \( d \) steps are \( c \)-compressible. Therefore every \( p \) such that \( U^d(p) = x \) satisfies \( U^c(p) = x \) for some \( r \). To compute string \( x \) by \( U \) using program \( r \) requires that \( r \) is concatenated with an extra program \( q \) of constant length \( c' \geq 0 \) to restart \( U \) after it has computed \( p \). Then \( U^d(U(tp)) = x \) and \( |r| \leq |p| - c - c' \). For every \( p \) there are possibly more than one such \( r \). Let \( P \) be the set of these \( p \)'s and \( R \) be the set of the \( r \)'s. Hence \( \sum_{r \in R} 2^{-|r|} > \sum_{p \in P} 2^{-|p|-c-c'} \). That is, the reference optimal universal Turing machine computes \( x \) with probability at least \( 2^{-c-c'} Q^d(x) \geq 2^{-c-c'} Q_2 Q(x) \) the programs in \( R \). Then \( Q(x) \geq 2^{c+c'-b} Q_2 Q(x) + 2^{-b} Q(x) \) (the left-hand side comprises all halting programs for \( x \) no matter what is the running time and the right-hand side comprises a lower bound on the contribution of the programs in \( R \) plus the contribution of the programs in \( P \)). This means that \( 2^{c+c'-b} < 1 \) and therefore \( c + c' - b < 0 \), that is \( c + c' < b \). Among the programs in \( P \) there is a program which is \( b \)-incompressible (otherwise \( c \) is not greatest as assumed above) and since \( c + c' < b \) it is \((b - c') = (b - O(1))\)-incompressible.

(ii): Assume that \( d \) is the least number of steps to compute \( x \) by a \( b \)-incompressible program. By way of contradiction let \( Q^d(x) < 2^{-b} Q(x) \) with \( B = b + K(b) + d > 0 \) for a large enough constant \( e \). Let an auxiliary distribution \( R \) be defined as follows. Computably enumerate the set \( P \) of all programs \( p \) such that \( U^d(p) = x \). Let \( R(x) = \sum_{p \in P} 2^{B-p} \). Then \( R(x) = 2^B Q^d(x) < Q(x) \). The function \( R \) given \( B \) be lower semicomputable. Since \( Q \) is a semimeasure and \( R(x) < Q(x) \) for all \( x \) the function \( R \) is a semimeasure. A program \( p \in P \) adds \( 2^{-|p|+B} \) probability to \( R(x) \) and \( p \) is therefore \((B - O(1))\)-compressible given \( B \), and hence \((B - K(B) - O(1))\)-compressible. It remains to bound the constant term. For sufficiently large \( e \),

\[
B - K(B) - O(1) = b + K(b) + e - K(b + K(b) + e) - O(1)
\]

\[
> b + K(b) + e - K(b + K(b)) - K(e) - O(1)
\]

\[
> b.
\]

The first inequality follows from \( K(u + v) \leq K(u) + K(v) + O(1) \), and the second inequality holds since \( K(b + K(b)) \leq K(b) + K(b) + O(1) \leq K(b) + O(1) \) and \( K(e) < e/2 \) for \( e \) large enough. Hence all programs that compute \( x \) in \( d \) steps are \((b + 1)\)-compressible contradicting the assumption. Therefore \( Q^d(x) \geq 2^{-b} Q(x) \).

Remark 2. With \( b, d \) as in the theorem a slight modification in the proof yields

\[
\frac{1}{2^{b+K(b)+O(1)}} \leq \frac{Q^d(x)}{Q(x)} \leq \frac{1}{2^{b-O(1)}},
\]

where (i) corresponds to the right inequality and (ii) to the left inequality.

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\footnote{We assume that \( R(x) > 0 \). If \( R(x) = 0 \) then there is no program \( p \) such that \( U^d(p) = x \) and (ii) is vacuously correct.}

Remark 3. Notice that it is possible to use \(K(d)\) instead of \(K(b)\) in the above proof by changing the construction of the semi-probability as follows: knowing \(d\) generate all programs \(p\) computing \(x\) within \(d\) steps and let the semi-probabilities be proportional to \(2^{-|p|}\) and the sum be less than \(Q(x)\). In this way, \(K(b)\) in Theorem 1 can be improved to \(\min(K(b), K(d)) + O(1)\). ◇

4. The graph of logical depth

Slight changes of the significance level \(b\) can drastically change the value of the logical depth.

Lemma 1. Let \(\phi\) be defined by

\[
\phi(n) = \max_{|x|=n} \min_d U^d(x^*) = x.
\]

Then \(\phi\) is not computable and grows faster than any computable function.

Proof. If a function \(\phi\) as in the lemma were computable, then for an \(x\) of length \(n\) we could run \(U\) for \(\phi(n)\) steps on any program of length \(n + O(\log n)\). Among those programs that halt within \(\phi(n)\) steps, we could select the ones which output \(x\). Subsequently, we could select from that set a program of minimum length from which \(x\) can be computed. Such a program has length \(K(x)\). This would imply that \(K\) would be computable. But the function \(K\) is incompressible: contradiction. Therefore \(\phi\) cannot be computable. Since this holds for every function majoring \(\phi\), the function \(\phi\) must grow faster than any computable function. □

Definition 3. The busy beaver function \(BB : \mathbb{N} \to \mathbb{N}\) is defined by

\[
BB(n) = \max\{d : |p| \leq n \land U^d(p) < \infty\}.
\]

This is a variant in terms of number of steps from the original Busy Beaver function in [5]. The following result was mentioned informally in [1].

Lemma 2. The running time of a program \(p\) of length \(n\) that halts is at most \(BB(n)\). The running time of a shortest program (shortest in prefix Kolmogorov complexity) for a string \(x\) of length \(n\) is at most \(BB(n + O(\log n))\).

Proof. The first statement of the lemma follows from Definition 3. The second statement follows from the observation that \(K(x) \leq n + O(\log n)\) for every \(x\) of length \(n\). □

Theorem 2. The function

\[
f(n) = \max_{|x|=n, 0 \leq b \leq n} \{x : \text{depth}_b^{(2)}(x) - \text{depth}_{b+1}^{(2)}(x)\}
\]

grows faster than any computable function but not as fast as the Busy Beaver function.

Proof. Let \(\phi\) be the incompressible function of Lemma 1. For every \(n \geq 1\) select \(x_n\) of length \(n\) such that \(U^{(n)}(x_n) = x_n\). Since a shortest program can be compressed by at most \(O(1)\) bits we have \(\text{depth}_{O(1)}^{(2)}(x_n) = \phi(n)\). The optimal reference prefix Turing machine \(U\) can compute \(x\) from its self-delimiting version \(x'\) of length \(n + O(\log n)\) using an \(O(1)\)-bit program in time \(g(n)\) with \(g\) a computable function.

For every \(n\) we can consider the finite sequence \(L_n^0, L_n^1, \ldots, L_n^{n-K(n)+O(\log n)}\) with \(L_n^m = \text{depth}_m^{(2)}(x_n) - \text{depth}_{m+1}^{(2)}(x_n)\) \((0 \leq m \leq n - K(n) + O(\log n))\), and \(\phi(n) = \sum_{m=0}^{n-K(n)+O(\log n)} L_n^m\). Then, the average \((\phi(n) - g(n))/(n - K(n) + O(\log n))\) is incompressible (as it grows faster than any computable function), and therefore the function \(f\) defined by

\[
f(n) = \max_m\{L_n^m : 0 \leq m \leq n - K(x) + O(\log n)\}
\]

also grows faster than any computable function, since \(f(n) \geq (\phi(n) - g(n))/(n - K(n) + O(\log n))\). □

5. Conclusion

In [1] one version of logical depth is based on compression and one is based on algorithmic probability. These are closely related [1,4]. We gave a new proof of this relation based on incompressibility and algorithmic probability. For a string \(x\) the logical depth with significance \(b\) is the minimum number of steps \(d\) in a computation of \(x\) from a program \(p\) that is less than \(b\) bits longer than \(K(p)\). This \(b\) is called the significance. There is a sequence of strings such that for each string
changing the significance parameter by only 1 can change the depth by a large amount. As a function of the position of the string in the sequence this amount grows faster than any computable function but not as fast as the Busy Beaver function. The computation times of shortest programs can similarly rise faster than any computable function but not as fast as the Busy Beaver function.

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