Some Stability Results for the Hopscotch Difference Method when Applied to Convection-Diffusion Equations[1]

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0. Introduction

The hopscotch method is a time stepping scheme applicable to wide classes of spatially discretized, multi-space dimensional, time-dependent partial differential equations (PDEs). In this contribution attention is focussed on the simple odd-even hopscotch method (OEH). Our aim is to present some interesting stability results of this method for convection-diffusion equations where the space discretization is carried out by standard symmetrical and/or one-sided finite differences. Our contribution is split up in three parts. First, in Section 1, we give a general formulation of the time-stepping scheme and outline its main computational features. Section 2 is devoted to the second part. Here we discuss linear stability properties of the method in the multi-dimensional case. We present explicit expressions for the crittical time step based on the von Neumann condition. We show that in certain cases an increase of diffusion may render the process unstable, an observation which is in clear contrast to the common practice. This strange phenomenon can occur only in the higher dimensional case. The third part is contained in Section 3. Here we discuss the spectral condition, but only in one space dimension. In accordance with results for the explicit Euler rule, we conclude that the spectral condition is misleading in the sense that it does not prevent large error growth.

1. The odd-even hopscotch scheme: some general aspects

In this section we briefly recall the OEH method which was first suggested by Gordon [3]. For an extensive discussion we refer to the work of Gourlay who invented the name hopscotch and made a thorough study of various techniques. Here we adopt his formulation [4-7].

Let the general form $u_t = Lu$ represent an evolutionary problem for a system of PDEs in d space dimensions. Boundary conditions will not be specified here as we do not discuss their influence. So, unless mentioned otherwise, our study of the OEH method will be carried out as if we were studying the pure initial value problem. We let L_h be a finite difference replacement of the space operator L. Then, at the gridpoint x_j, where j represents a multi-index $(j_1, \ldots, j_d)$, $u_t = Lu$ is replaced by the continuous time ordinary differential equation

$$\dot{U}_j = L_h U_j \,. \tag{1.1}$$

In what follows it is supposed that L is of at most second order and that in each co-ordinate direction L_h is, at most, 3-point coupled on the, possibly nonuniform, grid. Consequently, with regard to convection-diffusion problems, we in fact suppose standard central or one-sided spatial differencing.

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1. The lecture is based on the joint papers Hundsdorfer & Verwer [10] and Ten Thije Boonkkamp & Verwer [14]. I hereby gratefully acknowledge my colleagues for the pleasant co-operation.

According to GOURLAY [4-7] the OEH scheme for problem (1.1) is given by

$$U_j^{n+1} = U_j^n + \tau\phi_j^n L_h U_j^n + \tau\phi_j^{n+1} L_h U_j^{n+1} , \tag{1.2}$$

where $\tau = t_{n+1} - t_n$, U_j^n approximates $U_j(t)$ at $t = t_n$, and

$$\phi_j^n = \begin{cases} 1 \text{ if } (n + \sum_i j_i) \text{ is odd} , \\ 0 \text{ if } (n + \sum_i j_i) \text{ is even} . \end{cases} \tag{1.3}$$

From the computational point of view the following points are of interest:

1°. By writing down two successive steps of scheme (1.2),

$$U_j^{n+1} = U_j^n + \tau\phi_j^n L_h U_j^n + \tau\phi_j^{n+1} L_h U_j^{n+1} , \tag{1.4a}$$

$$U_j^{n+2} = U_j^{n+1} + \tau\phi_j^{n+1} L_h U_j^{n+1} + \tau\phi_j^{n+2} L_h U_j^{n+2} , \tag{1.4b}$$

its connection to the Peaceman-Rachford method [4-7] is shown. In particular, given a space grid, (1.4) may be interpreted as a 2-nd order integration formula using stepsize 2τ for the ODE system defined by (1.1) (see also [9]).

2°. The process consists of a componentwise application of the forward and backward Euler rule, in an alternately manner. For example, for odd points in the time-space mesh, (1.4a) is just the forward Euler scheme, while for the even points the backward Euler scheme is recovered. This suggests to first apply (1.4a), for a given n, on all the odd points, and then on all the even points.
Due to the 3-point coupling the great practical advantage is then that for the j-th grid point the only implicitely defined unknown is U_j^{n+1} itself. In other words, the computation is only diagonaly implicit and, specifically for convection-diffusion problems, often only scalar divisions are required. This is the most characteristic feature of the method and can also be exploited on a staggered grid [15]. Further, the near-expliciteness of the scheme is advantageous for vectorizing it for use on a super-computer. Eric de Goede and Jan ten Thije Boonkkamp, of our department, are currently building a (portable) CDC 205 code of the OEH pressure correction scheme published in [15]. They will develop a similar code for an ADI type pressure correction scheme.

3°. By implementing (1.4) in the so-called fast form

$$\begin{array}{lll} n := 0 & \text{odd:} & U_j^{n+1} = U_j^n + \tau L_h U_j^n \\ \text{label} & \text{even:} & U_j^{n+1} = U_j^n + \tau L_h U_j^{n+1} \\ & \text{even:} & U_j^{n+2} = 2U_j^{n+1} - U_j^n \\ & \text{odd:} & U_j^{n+2} = U_j^{n+1} + \tau L_h U_j^{n+2} \\ n := n+2 & \text{odd:} & U_j^{n+1} = 2U_j^n - U_j^{n-1}, \text{ goto label} \end{array} \tag{1.5}$$

it follows that one step of (1.4) involves only *one evaluation* of L_h. Hence, if we leave out of consideration the eventual solution of the nonlinear equations, with respect to computational costs per step this fast form is as cheap as the simple forward Euler rule.

4°. The fast form can be implemented using only one array of storage. It shall be evident that this may be of considerable interest to multi-dimensional problems.

5°. The above points are all positive for the method. Accepting the restriction to 3-point coupled systems, two drawbacks should be mentioned. First, for parabolic problems, without first-order terms, the method is unconditionally (linearly)stable. This, however, is deceptive as we have only conditionally convergence due to the Du Fort-Frankel accuracy deficiency which implies that convergence takes place for a smaller set of rules for refinement of the time-space mesh than allowed by stability. This deficiency manifests itself by large error constants which increase if the space mesh is refined. On a fixed space mesh the resulting loss of accuracy can be easily overcome by standard (global) Richardson extrapolation.[2] Unfortunately, it can be proved that the extrapolation procedure does not remove the DFF deficiency itself [1] (the conclusion made in [14] on this point is not correct). Second, for problems containing first-order terms, the method is conditionally stable. The critical values for the admissible time steps are close to that for the leap frog scheme (see section 2).

2. The multidimensional problem and the von Neumann condition

In this section we derive the critical time step for von Neumann stability of the OEH scheme for the above model when combined with central and one-sided differences for the spatial discretization. We first consider the central difference scheme [14].

We now turn to stability properties of the OEH method, more precisely, to linear stability properties. Nonlinear phenomena, like e.g. discussed in [2], Ch.10, are very difficult to study for methods of the explicit-implicit type. We do not know of any interesting result in that direction. Our linear model is the d-space dimensional convection-diffusion equation

$$u_t + (q \cdot \nabla)u = \epsilon \Delta u,\ t > 0,\ x \in \mathbb{R}^d$$

where $u(x,t) \in \mathbb{R}$ represents the convected and diffused variable, the vector $q = (q_1, \ldots, q_d)$ the (constant) velocity, and $\epsilon > 0$ a diffusion parameter.

Let h_k be a constant mesh width in the k-th direction and H_k and δ_k^2 the corresponding central finite difference operators for the first and second derivative, respectively. Then L_h can be written as

$$L_h U_j = \sum_k \left[-\frac{q_k}{2h_k} H_k + \frac{\epsilon}{h_k^2} \delta_k^2 \right] U_j \,. \quad \text{(summation from 1 to } d\text{)}$$

The stability analysis exploits the equivalence to the leapfrog-DFF scheme. This equivalence emerges by eliminating variables at the time level $n+1$ in equation (1.4b). For the odd points we then get the relation

$$U_j^{n+2} = U_j^n + \tau L_h U_j^n + \tau L_h U_j^{n+2}\,, \tag{2.1}$$

and for the even ones

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2. In fact, this simple, universal technique is attractive for any 2-nd order splitting method

$$\tilde{U} = U^n + \tfrac{1}{2}\tau F_1(t_n, U^n) + \tfrac{1}{2}\tau F_2(t_n + \tfrac{1}{2}\tau, \tilde{U})$$
$$U^{n+1} = \tilde{U} + \tfrac{1}{2}\tau F_1(t_{n+1}, U^{n+1}) + \tfrac{1}{2}\tau F_2(t_n + \tfrac{1}{2}\tau, \tilde{U})$$

for the numerical integration of nonlinear ODE systems

$$\dot{U} = F(t, U) \equiv F_1(t, U) + F_2(t, U),$$

as this method has an even error expansion in τ. This follows from symmetry properties. The proof follows readily from the material in section 2 of H.J. Stetter's paper '*Symmetric Two-Step Algorithms for Ordinary Differential Equations*', Computing 5, 267-280, 1970. The easiest way to see this, is to write the method as a 4-stage method containing only explicit and implicit Euler rules.

$$U_j^{n+2}=2U_j^{n+1}-U_j^n . \tag{2.2}$$

By evaluating the linear expression $L_h U_j^{n+2}$ at the odd points and inserting (2.2) at the occurring even ones. Relation (2.1) can be written as

$$(1+\sum_k 2\sigma_k)U_j^{n+2}=(1-\sum_k 2\sigma_k)U_j^n-\sum_k(c_k H_k-4\sigma_k\mu_k)U_j^{n+1} , \tag{2.3}$$

where μ_k is the standard averaging operator in the k-th direction and

$$\sigma_k=\epsilon\tau/h_k^2,\ c_k=q_k\tau/h_k . \tag{2.4}$$

Scheme (2.3) is the combined leapfrog Du Fort-Frankel scheme. From the 1D-formula (this case was studied earlier in [7])

$$U_j^{n+2}=U_j^n-2\tau q\left[\frac{U_{j+1}^{n+1}-U_{j-1}^{n+1}}{2h}\right]+2\tau\epsilon\left[\frac{U_{j-1}^{n+1}-[U_j^n+U_j^{n+2}]+U_{j+1}^{n+1}}{h^2}\right] \tag{2.3'}$$

the combination is readily recognized. Note that (2.3) contains only grid values at the uncoupled set of odd numbered points in space and time. Thus, if we ignore the start and completion of the OEH process and consider only the odd numbered points, we may proceed with (2.3) for the investigation of linear stability. Note that U_j^{n+1} in (2.2) is a grid value at an odd point. Hence, if the computation at the uncoupled set of odd numbered points is stable, we have also stability at all even points.

We shall now examine the stability of scheme (2.3). For this purpose we employ the classical method of VON NEUMANN [12]. So we introduce the Fourier mode

$$U_j^n=\xi^n e^{i\omega\circ x},\ \omega=(\omega_1,\ldots,\omega_d)^T\in\mathbf{R}^d,\ \xi\in\mathbf{C},\ i^2=-1 ,$$

and substitute into (2.3) to give

$$(1+\sigma)\xi^2+(\sum_k 2c_k i\sin\phi_k-4\sigma_k\cos\phi_k)\xi-(1-\sigma)=0 ,$$

where $\phi_k=\omega_k h_k$ and $\sigma=2(\sigma_1+\cdots+\sigma_d)$. In what follows we demand von Neumann stability in the strict sense, that is

$$|\xi|\leq 1,\ \text{all}\ |\phi_k|\leq\pi .$$

Bearing in mind that $\sigma>0$, as $\epsilon>0$, it then follows immediately from Th.6.1 in MILLER [11] that we have stability iff the complex number λ given by

$$\lambda=\sum_k r_k\cos\phi_k-c_k i\sin\phi_k,\ r_k=2\sigma_k/\sigma ,$$

satisfies $|\lambda|\leq 1$ for all $|\phi_k|\leq\pi$.

At this point we can make fruitful use of an interesting stability theorem due to HINDMARSH, GRESHO & GRIFFITHS [8] which they used in their stability analysis of the forward Euler-central difference scheme. As $\sum r_k\leq 1$, λ can be written as

$$\lambda = 1-i\sum_k c_k\sin\phi_k+\sum_k r_k(\cos\phi_k-1) .$$

Their stability theorem then says that $|\lambda|\leq 1$ for all $|\phi_k|\leq\pi$ iff $\sum r_k\leq 1$ and

$$\sum_k c_k^2/r_k\leq 1 . \tag{2.5}$$

Hence we can conclude immediately that this condition is sufficient and necessary for von Neumann stability (in the strict sense) of scheme (2.3).

Let us examine (2.5). We have

$$\sum_k c_k^2/r_k = \sum_k (\frac{\tau}{h_k})^2 \sum_k q_k^2 \leqslant 1 , \tag{2.6}$$

and observe that the diffusion parameter ϵ is absent in this condition. This is plausible because the DFF scheme is unconditionally stable for the pure diffusion problem $u_t = \epsilon \Delta u$. Next an interesting situation arises if we put $\epsilon = 0$. Then scheme (2.3) reduces to the leapfrog scheme for the pure convection problem which is known to be stable in the strict sense of von Neumann iff the CFL condition holds:

$$\sum_k \frac{\tau}{h_k} | q_k | \leqslant 1 . \tag{2.7}$$

Consequently, from Schwarz's inequality

$$(\sum_k \frac{\tau}{h_k} | q_k |)^2 \leqslant \sum_k (\frac{\tau}{h_k})^2 \sum_k q_k^2 , \tag{2.8}$$

it follows that the stability conditions are more restrictive for $\epsilon > 0$ than for $\epsilon = 0$. This implies that the critical time step, when considered as a function of $\epsilon \geqslant 0$, is discontinuous and takes on only two values, one for $\epsilon = 0$ and a smaller one for all $\epsilon > 0$. Hence if we add artificial diffusion to the OEH scheme for the pure convection problem we might destabilize the process. This observation is in clear contrast to the common practice which learns us that introducing artificial diffusion has a stabilizing effect.

Observe that we have equality in (2.8) iff $h_k | q_k |$ is independent of k, so that only in this case the restrictions on τ and h_k in (2.6), (2.7) are identical. Of course, this is trivially so for $d = 1$. If we put $h_k = h$, then (2.6), (2.7) lead to the time step restrictions

$$\tau^2 \leqslant h^2 / (d \sum_k q_k^2) , \tag{2.6'}$$

$$\tau^2 \leqslant h^2 / (\sum_k | q_k |)^2 . \tag{2.7'}$$

We see that when one of the velocities q_k dominates, the critical time step is approximately $\sqrt{d}$ times smaller than the critical time step imposed by the CFL condition.

We remark that the above pathological behaviour of the leapfrog-DFF scheme, and thus of the OEH scheme, has been observed earlier (see [13] and the references therein for numerical evidence). However, as far as we know, the expression for the critical time step implied by (2.6) is new and first given in [14].

For the sake of comparison we give the sufficient and necessary conditions for von Neumann stability of the forward Euler-central difference scheme for our convection-diffusion equation (see [8] for an extensive discussion)

$$\sum_k \frac{2\epsilon\tau}{h_k^2} \leqslant 1, \quad \sum_k \frac{q_k^2 \tau}{2\epsilon} \leqslant 1 .$$

The second of these is known as the convection-diffusion barrier. It shows that the forward-Euler central difference scheme becomes unconditionally unstable as $\epsilon \to 0$. In contrast, the OEH central difference scheme is stable for all $\epsilon \geqslant 0$ under condition (2.6).

We conclude this section with the result for the OEH one-sided difference scheme (standard upwind or downwind for convection terms and central differences for the diffusion term). In case of one space dimension, the equivalent DFF type scheme reads, for $q=q_1\geqslant 0$,

$$U_j^{n+2}=U_j^n-\frac{2\tau q}{h}\left[\frac{1}{2}[U_j^n+U_j^{n+2}]-U_{j-1}^{n+1}\right]+(\text{diff. term as in (2.3')}) \tag{2.8}$$

which is easily shown to be identical to (2.3') if we replace in (2.3') the diffusion parameter ϵ by $\epsilon+\frac{1}{2}hq$. The extra term $\frac{1}{2}hq$ is the common artificial diffusion coefficient.

Along the lines of the derivation for the central scheme, it can now be proved that in the multidimensional case the (strict) von Neumann stability condition is satisfied if and only if

$$\tau^2\sum_k\left[\frac{q_k^2}{\epsilon+\frac{1}{2}h_k|q_k|}\right]\sum_k\left[\frac{\epsilon+\frac{1}{2}h_k|q_k|}{h_k^2}\right]\leqslant 1\,. \tag{2.9}$$

If we remove $\frac{1}{2}h_k|q_k|$, condition (2.6) for the central scheme is recovered.

It is of interest to note that, as contrasted with the derivation for the central scheme, the derivation of (2.9) holds for all $\epsilon\geqslant 0$ (the quantity σ introduced above is always positive due to the artificial diffusion). This implies that here the critical time step depends continuously on $\epsilon\geqslant 0$. For $\epsilon=0$ we thus recover again the CFL condition (2.7). Using Schwarz's inequality, it follows that

$$\left[\sum_k\frac{\tau|q_k|}{h_k}\right]^2=\left[\sum_k\frac{\tau|q_k|(\epsilon+\frac{1}{2}h_k|q_k|)^{\frac{1}{2}}}{h_k(\epsilon+\frac{1}{2}h_k|q_k|)^{\frac{1}{2}}}\right]^2\leqslant \text{ left hand side of (2.9)}$$

with equality iff h_kq_k is independent of k. Consequently, like for the central scheme we have the unexpected result that without a diffusion term the one-sided scheme usually allows a larger time-step than with diffusion, provided $d>1$. For $d=1$ we get, uniformly in $\epsilon\geqslant 0$, the CFL condition $\tau|q|\leqslant h$, like we found for the central scheme.

For the sake of comparison, figure 1 shows the two maximal allowable stepsizes as a (linearly interpolated) function of ϵ for the following case: $d=2$, $q_1=1$, $q_2=.1$, $h_1=h_2=1/100$. Despite the unconditional stability of the scheme for the purely parabolic problem, we observe that as ϵ increases the maximal time step of the one-sided scheme continuously decreases to a limit value, being the maximal value ot the central scheme for $\epsilon>0$. This last value is approximately a factor $\sqrt{2}$ smaller than the maximal stepsize imposed by the CFL condition.

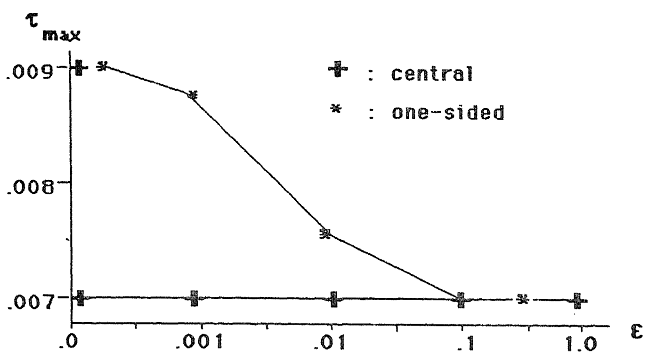

FIGURE 1. Illustration of maximal stepsizes.

3. The one-dimensional problem and the spectral condition

In *Method of Lines* papers the spectral condition is often employed to study the effect for large n of an error in the initial value if τ, h are fixed. While being perfectly allright for the pure diffusion problem, for convection-diffusion equations this condition is known to lead to misleading results, unfortunately. In particular the simple forward Euler scheme has been used to illustrate this (see e.g. [2], sections 10.5, 10.6 and the references therein).

Following [10], we now briefly discuss the spectral condition property for the OEH method. Because for the hopscotch scheme the derivations are highly complicated, our discussion is confined to the homogeneous, one-space dimensional problem

$$u_t + qu_x = \epsilon u_{xx}, \quad 0<x<1, \quad t>0, \tag{3.1}$$

$$u(x,0) \text{ given}, \quad 0 \leqslant x \leqslant 1, \quad u(0,t)=u(1,t)=0, \quad t \geqslant 0,$$

where $q \geqslant 0$, $\epsilon > 0$.

For space discretization we take a uniform grid with mesh width $h = 1/(m+1)$, and we use either standard central or upwind differencing for the convection term qu_x and standard central differences for the diffusion term ϵu_{xx}. Let $U(t) \in \mathbb{R}^m$ represent the resulting semi-discrete approximation on the space grid. This continuous time vector function is the solution to the linear, constant coefficient ODE initial value system

$$\dot{U} = AU, \; t>0, \; U(0) = U^0, \; U^0 \text{ given}, \tag{3.2}$$

with A the finite difference matrix

$$A = -\frac{q}{2h}(-E^T + E) + \frac{\epsilon}{h^2}(E^T - 2I - E) \quad (\textit{for central diff.}) \tag{3.3a}$$

$$A = -\frac{q}{h}(-E^T + I) + \frac{\epsilon}{h^2}(E^T - 2I - E) \quad (\textit{for one-sided diff.}) \tag{3.3b}$$

E is the familiar shift operator in $\mathbb{R}^m$, i.e., all entries of the matrix E are zero except those of the first upper diagonal which are equal to one.

The 2-stage OEH scheme (2.4) now fits in the matrix-vector formulation

$$U^{n+1} = U^n + \tau I_o A U^n + \tau I_E A U^{n+1}, \tag{3.4a}$$

$$U^{n+2} = U^{n+1} + \tau I_o A U^{n+2} + \tau I_E A U^{n+1}, \tag{3.4b}$$

where I_o, I_E are diagonal $m \times m$ matrices with entries either 0 or 1. The j-th diagonal entry of I_o is 1 if j is odd and 0 if j is even, while $I_E = I - I_o$, I the unit matrix. Define $A_o = \tau I_o A$ and $A_E = \tau I_E A$. Then the OEH finite difference scheme can be presented in the matrix-vector recurrence form

$$U^{n+2} = TU^n, \quad n = 0, 2, \ldots, \tag{3.5}$$

$$T = (I - A_o)^{-1}(I + A_E)(I - A_E)^{-1}(I + A_o), \tag{3.6}$$

and the (amplification) matrix T governs the stability of the scheme.

For difference schemes like (3.5) a conventional definition of the celebrated Lax-Richtmyer stability reads: a constant $\gamma > 0$ exists such that for any initial function U^o

$$\|U^n\| \leqslant \gamma \|U^o\| \tag{3.7}$$

for all (even) $n \geqslant 0$, $0 \leqslant n\tau \leqslant t_e$ with t_e fixed, while assuming some specific relation between τ and h. Hereby, $\|\cdot\|$ is an appropriate vector norm on $\mathbb{R}^m$. The present definition is equivalent to requiring that $\|T^n\|$ be uniformly bounded, i.e.

$$\|T^n\| \leqslant \gamma \tag{3.8}$$

for all (even) $n \geqslant 0$, $0 \leqslant n\tau \leqslant t_e$, and a specific relation between τ and h. Here $\|T\|$ is the natural matrix norm induced by the vector norm $\|\cdot\|$.

Lax-Richtmyer stability guarantees that no error growth will occur if we fix time t and simultaneously refine the time and space mesh in a specified way. Unfortunately, finding necessary and/or sufficient conditions on τ,h establishing Lax-Richtmyer stability may turn out to be very difficult, even for simple scalar problems like (3.1) (see also [17] and the references therein). A necessary condition is the spectral radius condition $\sigma(T) \leqslant 1+O(\tau)$ and sufficient is $\|T\| \leqslant 1+O(\tau)$ (uniformly in τ,h under the specific relation mentioned in the definition).

If T is a normal matrix, then $\sigma(T)=\|T\|_2$, so that then the spectral radius condition $\sigma(T) \leqslant 1+O(\tau)$ is both sufficient and necessary for Lax-Richtmyer stability in the L_2-norm. Here the stability can be investigated by using Fourier transformations and the analysis is equivalent to von Neumann analysis. We emphasize that this analysis cannot be applied directly to the hopscotch scheme, because for this scheme eigenvectors of the operator A are not eigenvectors of T. However, as we have discussed in the previous section, von Neumann analysis is applicable by discharging half of the gridpoints.

If T is not normal or other norms than the L_2-norm are used, results are almost always hard to obtain. This is also the case for the present hopscotch scheme (3.5). For example, one can show [10] that if $\epsilon=0$ and one-sided differences are used (omitting the right boundary condition), we have $\|T\|_\infty \leqslant 1$ iff the CFL condition $\tau q \leqslant h$ is satisfied. However, to our experience, if $\epsilon > 0$ there is no feasible way to deriving results on Lax-Richtmyer stability. For this reason we now resort to another, but weaker form of stability.

In what follows we shall require inequality (3.8) for fixed h. This form of (fixed-h) stability deals with the effect of an error in the initial value U^o if we fix τ and h and let $n \to \infty$. It is usually more tractable to investigate than Lax-Richtmyer stability via the following result from linear algebra

$$\sup_{n \geqslant 0} \|T^n\| < \infty \Leftrightarrow \sigma(T) \leqslant 1 \text{ and any eigenvalue } \lambda \text{ of } T \text{ with } |\lambda| = 1 \text{ is not defect}\,. \tag{3.9}$$

We note that this result assumes that T is of fixed dimension, which is not the case in the Lax-Richtmyer definition. In literature, the condition of the right-hand side of (3.9) is often called the (fixed-h) spectral condition.

In [10] it is proved that the hopscotch scheme (3.5) satisfies this spectral condition, if and only if τ is restricted to

$$\tau^2(q^2 - 4\epsilon^2/h^2) \leqslant h^2 \quad (\textit{for central diff.}) \tag{3.10a}$$

$$\text{no restriction on } \tau \geqslant 0 \quad (\textit{for one-sided diff.}) \tag{3.10b}$$

The proof of this result is rather complicated and too long to reproduce here. The unconditional (fixed-h) stability of the one-sided scheme also holds for $\epsilon=0$, omitting the right boundary condition.

Comparing (3.10) with the (von Neumann) conditions (3.6), (3.9) for $d=1$, which yield the familiar CFL condition $\tau q \leqslant h$, we see that the spectral condition allows a much larger time step. The central scheme is even unconditionally (fixed-h) stable if $h|q| \leqslant 2\epsilon$, while this holds for the one-sided scheme for any h. This difference requires an explanation, the more so because as $\tau,h \to 0$, the CFL condition must be obeyed for getting convergence.

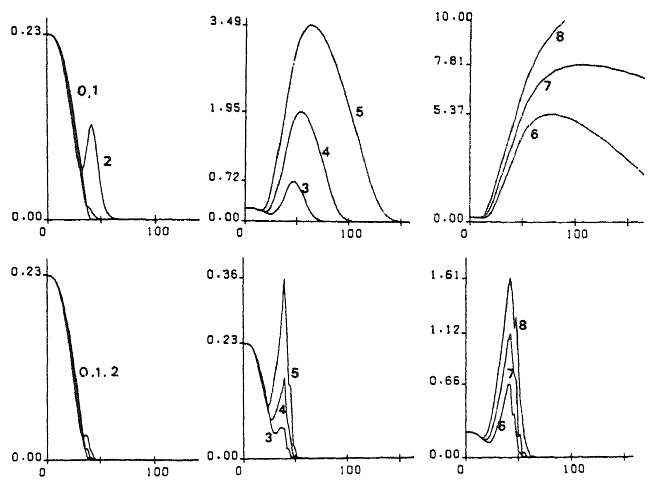

FIGURE 2. The plots represent $\log_{10}(1+\|U^{2n}\|_2)$ as a function of n. The three upper plots belong to the central scheme, and the three lower ones to the one-sided scheme. The integer numbers $i=0(1)8$ correspond to τ according to $\tau=1/(40-2i)$. Further, $h=1/40$, $q=1$, $\epsilon=0.01$, $u(x,0)=\sin(\pi x)$. The case $i=0$ corresponds to the CFL condition and $i=8$ yields the maximal τ-value allowed by condition (3.10a).

Assuming strict inequality, the spectral condition implies $\|T^n\|\to 0$ as $n\to\infty$. Unfortunately, it tells us nothing about the magnitude of $\|T^n\|$ for finite n. In fact, while (3.10) is satisfied, it may well occur that for finite n the solution first grows enormously before it decays to zero. The growth may be so large that from a computational point of view we can no longer call the computation stable. The plots of figure 2 (copied from [10]) serve to illustrate this. Of interest is that for the one-sided scheme, which is identical to the central scheme with ϵ replaced by $\epsilon+\frac{1}{2}h|q|$, the growth remains reasonably small. In this 1D-example diffusion helps for getting more stability. Note that the vertical (logarithmic) scaling differs per plot. We emphasize that the initial decay of the solutions, which happens for all stepsizes, is due to our very smooth initial function. With a less smooth $u(x,0)$, we get the same pictures, but without initial decay and with a stronger intermediate growth.

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