Seminarium Algebra en Meetkunde

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BUEKENHOUT-TITS GEOMETRIES

We intend to discuss the following preprints:

-M. Aschbacher, Flag structures in Tits geometries, 18pp.

-             , Presheaves on Tits geometries, 43pp.

-             & St. D. Smith, Tits geometries over F_2 defined by groups over F_3, 7pp.

-F. Buekenhout, Diagram geometries for sporadic groups, 29pp.

-W.M. Kantor, Some exceptional 2-adic buildings, 29pp.

-F. Timmesfeld, Tits geometries and parabolic systems in finite groups, 113pp.

1.BASIC NOTIONS.

Throughout this section, I is an index set.

1.1 DEFINITION.

A *geometry* over I is

a triple Gamma =(V,*,t), where V is a set, * a symmetric and reflexive relation on V, called *incidence* and t: V --> I a map, called the *type map*, such that ((t(x)=t(y) and x*y)=> x=y) for all x,y in Gamma.

Thus, V is the disjoint union of subsets $V_i = t^{-1}(i)$ for i in I,

called the *parts* of Gamma. The *graph of* Gamma is the tuple (V,~) with ~ the adjacency relation defined by x~y if and only if x*y and x/=y for x,y in V. Notions such as *connectedness* and *cliques* will often be applied to Gamma when in fact they are meant for its graph.
If W is a subset of V and J a subset of I containing t(W),
we call the geometry (W, *|WxW , t|W) over J
the *(full) subgeometry* of Gamma over J induced on W. A *flag*

of Gamma is a subset F of V such that x*y for all x,y in F. Note that the restriction of t to a flag is injective. Let F be a flag. Then

the <u>type</u> of F, type(F), is the set t(F),

the <u>cotype</u> of F, cotype(F), is the set I\t(F),

the <u>rank</u> of F, rank(F), is the cardinality of type(F), and

the <u>corank</u> of F, corank(F), is the cardinality of cotype(F).

We shall write V^F={x in V | x*F}, where x*F means x*y for each y in F, and $V_F = V^F \setminus F$.

The <u>link</u> of F (in Gamma) is the full subgeometry over I induced on V^F, notation Gamma^F, and the <u>residue</u> of F is the full subgeometry over I\t(F) induced on V_F, notation Gamma_F or Res(F) when Gamma is clear.

The <u>radical</u> of Gamma, notation Rad(Gamma), is the set

{x in V | x*y for all y in V with t(y)/=t(x)}.

The <u>rank</u> of Gamma is the cardinality of I. A <u>chamber</u> is a flag of corank Ø.

The <u>transversality property</u> is said to hold for Gamma if every flag is contained in a chamber. Gamma is said to be <u>firm</u> if each flag which is not a chamber is contained in at least two chambers, and <u>thick</u> if each flag which is not a chamber is contained in at least three chambers. Gamma is said to be <u>thin</u> if the transversality property holds and each flag of corank 1 is in exactly two chambers.

Buekenhout defines Gamma to be <u>strongly connected</u> if for any two distinct i,j in I, and any flag F whose type is contained in I\{i,j} the subgraph of (V,~) on the points of

$[(t^{-1}(i) \cup t^{-1}(j))$ meet Res F] is connected.

Tits [A local approach...] defines Gamma to be <ins>residually connected</ins>

if for each flag of corank at least 2 its residue is connected and for each flag of corank at least 1 its residue is nonempty.

1.2 PROPOSITION.

Let I be finite and let Gamma be a geometry over I. Then Gamma is residually connected if and only if Gamma is strongly connected and satisfies the transversality property.

PROOF.

Clearly, strong connectedness and the transversality property imply residual connectedness.

Assume that Gamma is residually connected. If I has cardinality at most 2, then there is nothing to show. We apply induction on the rank of Gamma. Thus, for each nonempty flag its residue is residually connected and hence strongly connected. It remains to establish that for each x in V_i and y in V_j there is a path R from x to y with $t(R)$ contained in $\{i,j\}$.

By residual connectedness of Gamma there is a path R_1:

$x=x_0, x_1, \dots, x_m=y$.

If $t(R_1)$ is contained in $\{i,j\}$, we are done, so suppose there is k in I with $t(x_r)=k$ for some r $(1 \leq r \leq m)$. Now for any r with $t(x_r)=k$ proceed as follows.

Since x_{r-1}, x_{r+1} are in Res x_r, which is strongly connected by induction, there is a path $x_{r-1}, x_{r,1}, x_{r,2}, \dots, x_{r,m(r)}, x_{r+1}$ with $t(x_{r-1,a})$ in $\{i,j,t(x_{r-1}), t(x_{r+1})\}$ for each a $(1 \leq a \leq m(r))$.

The new path R_2 from x to y so obtained has type $t(R_2)$ contained in $(t(R_1) \cup \{i,j\}) \setminus \{k\}$.

Continuing with R_2 by repeating this process for a type k distinct from i,j occurring in $t(R_2)$, and so on, we find a path from x to y with types i,j only. This proves the proposition.

1.3 DEFINITION.

Let Gamma=(V,*,t) and Gamma'=(V',*',t') be geometries over I. A *morphism* from Gamma to Gamma' is a type and incidence preserving map f from V to V', i.e. t'(f(x))=t(x), and (x*y ==> f(x)*'f(y)) for all x,y in V. *Isomorphisms* and *automorphisms* of geometries over I are defined in the obvious way. Aut Gamma stands for the automorphism group of Gamma. A subgroup G of Aut Gamma is said to be *flag transitive* if for each subset J of I it is transitive on the flags of Gamma of type J. The geometry Gamma is called *flag transitive* whenever Aut Gamma is flag transitive. G is said to be *edge transitive* if for each subset J of I of cardinality at most 2, it is transitive on the flags of type J.

In the remainder of this section, Gamma=(V,*,t) *is a geometry over I.*

1.4 DEFINITION.

Let G be a group, and F = $(P_i)_{i \text{ in } I}$ a collection of subgroups of G indexed by I. Define the geometry Gamma=Gamma(G,F) as follows: Gamma=(V,*,t), with V the formal disjoint union of G/P_i over all i in I, where $t(xP_i)=i$, and xP_i*yP_j iff xP_i meets yP_j nontrivially. It is readily checked that Gamma is a geometry with edge transitive group G, acting on V by left multiplication. Such a geometry will be called a *coset geometry* on G with respect to F.

If J is a subset of I, we shall write P_J for the intersection of all P_j for j in J.

1.5 REMARK.

It is readily verified that

(i) G is transitive on the set of flags of type {r,s,t} if and only

if $P_r P_t$ meet $P_s P_t = P_{\{r,s\}} P_t$.

Furthermore, this is equivalent to

$(P_r$ meet $P_s)(P_r$ meet $P_t) = P_r$ meet $P_s P_t$.

(ii)G is flag transitive on Gamma if and only if for each subset J

of I and each i in I\J the equality $P_J P_i = \mathrm{meet}_{j \in J}(P_j P_i)$ holds.

1.6 THEOREM(ASCHBACHER).

Suppose Gamma'=(V',*',t') is a geometry over I with edge transitive

group G, and F is a chamber of Gamma'. For x in F,

let $P_{t(x)} = G_x$ be the stabilizer

of x in G under left multiplication on V',

set $F=(P_i)_{i \in I}$ and let Gamma=Gamma(G,F)=(V,*,t) be the correspon-

ding coset geometry. Then the map f : V' --> V given by gx --> $gP_{t(x)}$

for any g in G and x in F

is an isomorphism of geometries over I , commuting with the actions

of G on V and V'.

PROOF. Straightforward.

1.7 EXAMPLE(TIMMESFELD).

Let G be the alternating group Alt(7) on 7 letters.

Put a1=(2,3,5)(4,7,6), a2=(1,3,7)(2,5,4), a3=(1,6,2)(3,5,7),

I={1,2,3}, and $P_i = \langle aj,ak \rangle$ whenever {i,j,k}=I.

and $F=(P_i)_{i \in I}$. Then each P_i is a Frobenius group of order 21 and

Gamma(G,F) is a flag transitive residually connected geometry

such that the residue of any flag of rank 1 is isomorphic

to the incidence structure on the points and lines of the Fano plane.

Flag transitivity follows from verification of $P_1 P_3$ meet $P_2 P_3 = P_{\{1,2\}} P_3$,

see Remark 1.5. We shall see later, in 2.10, how residual connectedness

can be established.

1.8 DEFINITIONS.

A <u>Coxeter system</u> is a pair (W,R) consisting of a group W and a finite subset R of W for which there is a matrix $M=(m(r,s))_{r,s \text{ in } R}$ with $m(r,r)=1$ for all r in R , such that

$$\langle r \text{ in } R \mid (rs)^{m(r,s)}=1 \text{ for every } s \text{ in } R\rangle$$

is a presentation for W.

The matrix M is uniquely determined by the system (W,R), in fact, m(r,s) is the order of rs in W for any two distinct r,s in R, see [Bourbaki].

In this situation, W is called the <u>Coxeter group</u> of (W,R) and the pair (R,M) is called the <u>Coxeter diagram</u> of (W,R).

The Coxeter diagram can be viewed as a labeled graph by letting {r,s} for r,s in R be an edge whenever m(r,s) > 2.

If J is a subset of R, denote by W_J the subgroup of W generated by J, and by V_J the subgroup $W_{\neg I\backslash J}$.

If w is in W, denote by $l_R(w)$, or just l(w), the length of a shortest expression of w as a product of elements in R.

1.9 THEOREM.

Let (W,R) be a Coxeter system and let I,J,K be subsets of R. Then

(i) The pair (W_I, I) is a Coxeter system with $(W_I$ meet $R) = I$.

(ii) $(W_I$ meet $W_J W_K) = W_{I \text{ meet } J} W_{I \text{ meet } K}$.

(iii) $W=V_{\neg}V_j$ for i,j, in I if and only if i,j are in distinct connected components of the Coxeter diagram.

1.10 COROLLARY.

Let $F=(V_r)_{r \text{ in } R}$. Then Gamma(W,F) is a thin flag transitive geometry over R.

PROOF.

Flag transitivity is an immediate consequence of REMARK 1.5 and the above theorem.

Since the transversality property holds for any flag transitive geometry with a chamber, we need only show that a flag of cotype $\{r\}$ is contained in precisely two chambers. Let F' be the unique flag in F of this cotype, then $xV_r * F'$ implies x in $\langle r \rangle$ so the only chamber distinct from F containing F' is $(F' \cup rV_r)$. Hence Gamma is thin.

1.11 DEFINITIONS AND NOTATION.

The Coxeter diagrams whose Coxeter groups are finite have a finite number of connected components, each of which is isomorphic to exactly one of A_n $(n \geq 1)$, $B_n = C_n$ $(n \geq 2)$, D_n $(n \geq 4)$, E_n $(n=6,7,8)$, F_4, G_2,

H_n $(n=2,3,4)$, I_2^m $(m \geq 7)$, in the notation of [Bourbaki].

If the diagram is connected, the Coxeter group and system are called *irreducible*, otherwise they are called *reducible*. Note that due to the above theorem reducible Coxeter groups are direct products of irreducible ones, the components in the direct product decomposition corresponding to the connected components of the diagram.

A finite Coxeter group is also called *spherical*.

1.12 DEFINITION.

Let G be a group. A *Tits system* in G is a 4-tuple (B,N,W,R) consisting of two subgroups B,N of G, a group W and a subset R of W such that

(1) G is generated by B and N.

(2) The intersection H of B and N is a normal subgroup of N. with $W=N/H$.

(3) For any r in R and w in W we have

(3') rBw is contained in $BwB \cup BrwB$, and

(3'') rBr^{-1} is not contained in B.

Note that the notation rBw makes sense as $rBm=rBm'$ for any two representatives m, m' of w in N.

The group W is called the <u>Coxeter group</u> of the system.

Tits systems are studied in [Tits, Buildings of ...,SLN 386]. and [Bourbaki, Groupes et algebres de Lie, Chap. 4,5,6].

1.13 THEOREM.

Let G be a group with Tits system (B,N,W,R). Then

(i) The pair (W,R) is a Coxeter system.

(ii) For every subset J of R the set $G_J = BW_J B$ is a subgroup of G. Moreover, for any subgroup of G containing B there is a subset I of R such that this subgroup is G_I.

(iii) Let I,J be subsets of R. The map $W_I w W_J \dashrightarrow G_I w G_J$ is a bijection from $W_I\backslash W/W_J$ onto $G_I\backslash G/G_J$.

(iv) If U,V are subsets of W closed under taking products of subexpressions of expressions of elements w of W as products of l(w) elements of R, then BUBVB = BUVB.

(v) $G_I G_K$ meet $G_J G_K = G_{I \text{ meet } J}\, G_K$.

(vi) For any subset J of R the subgroup G has Tits system (B,N_J,W_J,J), where N_J is the inverse image of W_J in N under the natural map from N to W.

PROOF. See [Tits, Bourbaki].

1.14 DEFINITION.

The subgroups G_I of a group G with a Tits system (B,N,W,R) are called the <u>standard parabolics (parabolic subgroups)</u> of G with respect to the Tits system. The conjugates of standard parabolics are called <u>parabolics</u> of G.

A parabolic is called <u>maximal</u> if it is a proper subgroup of G not contained in any other proper subgroup of G. Thus maximal parabolics have the form $g\,G_{R\setminus\{r\}}\,g^{-1}$ for some g in G and r in R.

With any Tits system (B,N,W,R) in a group G we can associate a flag-transitive geometry Gamma (G,F) over R, where $F=(G_{R\setminus\{r\}})_{r \text{ in } R}$.

We shall call this geometry the <u>building associated with the Tits system</u>.

1.15 COROLLARY.

Let G be a group with Tits system. Then its building is a thick geometry over R on which G acts flag transitively.

PROOF.

As for Corollary 1.10. Note that the building is a thick geometry as B<r>P, where $P=G_{R\setminus\{r\}}$, is not contained in P U rP.

1.16 EXAMPLE.

Take G=GL(n,K), where K is a field, B the subgroup of G consisting of upper triangular matrices and N the subgroup of G of monomial matrices.

Then H=(B meet N) = the group of diagonal matrices and W=N/H is isomorphic to the symmetric group Sym(n) on n letters.

Identify W with the subgroup of G of permutation matrices in the obvious way and let R={(i,i+1) | i=1,...,n-1}. Then (B,N,W,R) is a Tits system. Its building can be identified with the geometry Gamma= (V,*,t) over I={0,...,n-2} in which V_i is the set of all proper nonempty subspaces of the projective space PG(n-1,K) over K of projective dimension i, two members x,y of V are incident (i.e., x*y) if x is contained in y or vice versa, and t(x) stands for the projective dimension of x in V. For, identify R and I by means of (i,i+1) <->i-1. Since G acts on Gamma flag transitively, we can apply Theorem 1.6 in order to identify the building and Gamma.

1.17 REMARK.

The finite buildings of rank at least two have been determined by Fong-Seitz for rank = 2 under an additional hypothesis

and Tits for rank > 2.

They are essentially the (twisted) Chevalley groups $A_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$, $E_n(q)$, $F_4(q)$, $G_2(q)$, ${}^2A_n(q)$, ${}^2D_n(q)$, ${}^3D_4(q)$, ${}^2E_6(q)$, ${}^2F_4(q)$ in the notation of [Gorenstein]. The associated buildings are denoted by $[\,{}^aX_n(q)\,]$, if $G = {}^aX_n(q)$. Here we use the convention a=1 for normal Chevalley groups. Thus, $[A_{n-1}(q)]$ is the building discussed in the above example, if K is the field of q elements.

The following theorem by Seitz extends a theorem by D.G. Higman, and determines all flag transitive groups on a building of rank >1 coming from the Tits system of a (twisted) Chevalley group.

1.18 THEOREM(SEITZ).

Let Gamma be a finite building of rank >1 and of irreducible type. and suppose L is a subgroup of G=Aut Gamma which is flag transitive on Gamma. Then either L contains the generalized Fitting subgroup of G or one of the following holds:

(i) Gamma=$[A_2(2)]$, G=PSL(3,2) and L has order 3.7.

(ii) Gamma=$[A_2(8)]$, G=PGammaL(3,8) and L has order 9.73.

(iii) Gamma=$[A_4(2)]$, G=PSL(4,2) and L is isomorphic to Alt(7).

(iv) Gamma=$[C_2(2)]$, G=Sp(4,2) and L is isomorphic to Alt(6).

(v) Gamma=$[G_2(2)]$, G=$G_2(2)$ and L = G'

(vi) Gamma=$[\,{}^2F_4(2)]$, G=${}^2F_4(2)$ and L=G'.

(vii) Gamma=$[C_2(3)]$, G=PSp(4,3) and L is a maximal parabolic of PSU(4,2) of order 2^6*3*5.

1.19 DEFINITIONS.

For m at least 2, a generalized m-gon is a firm connected geometry of rank 2 whose graph has diameter m and girth (minimal length of a cycle) 2m.

A diagram on I is a collection D of nonempty sets D_J , indexed by the subsets J of I of cardinality 2, consisting of geometries over J.

The graph of D is the has vertex set I; its vertices i,j are adjacent whenever some member of $D_{\{i,j\}}$ is not a generalized 2-gon.

Notions such as connectedness will be applied freely to D when in fact they are meant to apply to its graph.

If for any two distinct i,j in I, there is an integer m(i,j) such that $D_{\{i,j\}}$ consists of all generalized m(i,j)-gons, then D is determined uniquely by the Coxeter diagram (I,M), where $M=(m(i,j))_{i,j \text{ in } I}$ with the convention that m(i,i)=1 for all i in I, cf. 1.8.

Conversely, any Coxeter diagram (R,M) determines a unique diagram over R with $D_{\{i,j\}}$ ={generalized m(i,j)-gons} for all distinct i,j in I.

By abuse of language, such a diagram D will also be referred to as a Coxeter diagram.

A geometry Gamma over I is said to belong to a diagram D if Gamma_T is isomorphic to a member of D_J whenever T is a flag of cotype J and corank 2.

If Gamma is a residually connected geometry belonging to a diagram D then G is said to be a geometry of type D. A geometry of type a Coxeter diagram will be called a Tits geometry. Thus, buildings are Tits geometries.

The minimal diagram D(Gamma) of a geometry Gamma is a diagram D such that D_J consists of a representative system of geometries for the isomorphism classes of Gamma_T as T varies over all flags of cotype J. Clearly, the minimal diagram of a geometry is unique up to "isomorphism".

If i is in I, then D\{i} denotes the restriction of D to I\{i}.

1.20 PROPOSITION.

Let Gamma be a residually connected geometry over I belonging to diagram D. Suppose J and K partition I and are unions of connected of D. Then x*y for any x,y in V with t(x) in J and t(y) in K.

PROOF.

If J or K is empty, there is nothing to prove. If J and K both have rank 1, then the result is clear from the definition of the graph of D. Without loss of generality, we assume that J has cardinality at least 2. By residual connectedness, there is a vertex of type in J which is incident to y. Hence, Proposition 1.1 yields a path $x=x_0, x_1, \ldots, x_n=y$ such that $t(x_i)$ is in J for all i<n.

If n=1, then we are done. Suppose n>1. Now x_{n-2}, y are both in the residue of x_{n-1}, and the latter geometry belongs to $D\setminus\{t(x_{n-1})\}$.

Since $t(x_{n-2})$ is in J and t(y) is in K, we get by induction on the rank of Gamma that $y*x_{n-2}$. Thus by induction on the length of a path from x to y, we obtain x*y as wanted.

1.21 EXERCISE.

Show that there is a bijective correspondence between Tits geometries Gamma=(V,*,t) of type A_3 and generalized projective spaces P of projective dimension 3 such that the members of V_1, V_2, V_3 correspond to points, lines , and planes of P respectively, and such that * corresponds to symmetrized inclusion. Hint: Use the axioms of Veblen and Young for projective spaces, see [TAMASCHKE],[TITS].

2. FLAG TRANSITIVITY.

In this section I *is a finite index set*, Gamma=(V,*,t)

is a geometry over I,

and Gamma(G,C) *is a coset geometry over* I *with* C=$(P_i)_{i \text{ in } I}$.

Our goal is to give an inductive criterium for flag transitivity and for residual connectedness in the case of coset geometries.

2.1 DEFINITION.

Let $\underline{\underline{T}}$ be a set of flags of Gamma.

A *flag structure* Sigma on Gamma over $\underline{\underline{T}}$ is

a collection of partial subgeometries $\text{Sigma}^T=(U^T,o^T,t|U^T)$

over I of Gamma indexed

by the members T of $\underline{\underline{T}}$, such that

the following properties hold:

(i) For any S,T in $\underline{\underline{T}}$ with S contained in T, the geometry

Sigma^T is a partial subgeometry of Sigma^S.

(ii) For any T in $\underline{\underline{T}}$, we have that T is contained in Rad (Sigma^T)

(iii) The null set is a member of $\underline{\underline{T}}$, and $\text{Sigma}^{\text{null}} = \text{Gamma}^{\text{null}}$

A flag structure is called *dense* if for each T in $\underline{\underline{T}}$ and each flag X

of Sigma^T with T of corank at most 2 in X, we have X in $\underline{\underline{T}}$, and

rigid if $\underline{\underline{T}}$ is the collection of all flags of Gamma.

2.2 EXAMPLE.

Let Gamma=(G,C) be a coset geometry over I. Set

$\underline{\underline{T}}$={gF| g in $P_{t(F)}$ and F contained in C}, and for T in $\underline{\underline{T}}$ with J=t(T),

i,j in I, and x,y in gP_J, set

$$U^T = \{zP_k \mid z \text{ in } gP_J \text{ and } k \text{ in } I\}, \text{ and}$$

$$xP_i \; o^T \; yP_j \quad \text{whenever} \quad y^{-1}x \text{ is in } P_{J\cup\{j\}} P_{J\cup\{i\}}.$$

Then the collection $\Sigma=\Sigma(G,C)$ consisting of all

$\Sigma^T = (U^T, o^T\ t|U^T)$ for T in $\mathcal{T}$ is a dense flag structure

on Γ. It is rigid if and only if G is

flag transitive on Γ.

2.3 EXAMPLE.

A trivial example of a rigid flag structure Σ on Γ is provided

by $\Sigma^T = \Gamma^T$ for each T in $\mathcal{T}$={flags of Γ}. In fact,

due to Lemma 2.5 below, this is the only rigid example.

2.4 NOTATION.

For $\mathcal{T}$ a collection of flags and T in $\mathcal{T}$, we set

$$\mathcal{T}^T = \{S \text{ in } \mathcal{T} \mid S \text{ contains } T\} \text{ and } \mathcal{T}_T = \{S\setminus T \mid S \text{ contained in } \mathcal{T}\}.$$

In the remainder of this section, Σ *denotes a flag structure on* *Γ over* $\mathcal{T}$ *and* $\Sigma^T = (U^T, o^T, t|U^T)$ *for every* T in $\mathcal{T}$.

2.5 LEMMA.

Let Σ be a flag structure on Γ and let T be a member of the flag set $\mathcal{T}$ of Σ.

(i) The members of $\mathcal{T}^T$ are flags of Σ^T.

(ii) If Σ is rigid, then the geometry Σ^T is the full subgeometry Γ^T of Γ.

PROOF.

(i) Let S be a member of $\underline{\underline{T}}^{T}$.

By axiom (ii) of the definition of flag structure, S is a flag of Σ^{S}. Hence, by axiom (i), S is a flag of Σ^{T}.

(ii) Suppose that x,y in U^{T} satisfy $x*y$. We have to show that $x \circ^{T} y$. Write $R=T \cup \{x,y\}$. Then R is a flag of Γ, and hence by rigidity and (i) a flag of Σ^{T}. Hence $x \circ^{T} y$.

2.6 DEFINITION.

For Σ a flag structure on Γ and T in $\underline{\underline{T}}$, we denote by Σ_{T} the full induced subgeometry of Σ^{T} on V_{T}, and by $\Sigma(T)$ the collection of geometries

$$\Sigma(T)^{S} = (W, \circ^{(S \cup T)} | (W \times W), t|W) \text{ where } W=(U^{S})_{T},$$

indexed by S in $\underline{\underline{T}}_{T}$.

It is easy to check that $\Sigma(T)$ defines a flag structure on Σ_{T} over $\underline{\underline{T}}_{T}$.

2.7 LEMMA.

Let Σ be a dense flag structure on Γ over $\underline{\underline{T}}$. Then

(i) If T in $\underline{\underline{T}}$, then $\Sigma(T)$ is a dense flag structure on Σ_{T}

(ii) Every flag of Γ of rank at most 2 is in $\underline{\underline{T}}$.

PROOF. Easy.

2.8 LEMMA.

Let Σ be a dense flag structure on Γ and X a flag of Γ.

(i) If X has rank 3 then X is in $\underline{\underline{T}}$ if and only if it is a flag of Σ^{T} for some flag T of rank 1.

(ii) The flag X is in $\mathbb{T}$ if and only if it is a flag of Σ^{T} for every T contained in $(X \text{ meet } \mathbb{T})$.

PROOF.

If X, T are in $\mathbb{T}$ and X contains T, then X in $\mathbb{T}^{T}$, whence X is a flag of Σ^{T} by Lemma 2.5.

This proves an implication for each of (i) and (ii).

The reverse implication for (i) is a consequence of the definition of density and Lemma 2.7; the reverse implication of (ii) follows from (i) by induction on the rank of X.

2.9 DEFINITION.

A flag structure Σ on Γ is called *residually connected* if Σ^{T} is connected for each T in $\mathbb{T}$ of corank at least 2 and nonempty for each T in $\mathbb{T}$ of corank at least 1.

2.10 LEMMA.

Let $\Gamma(G,C)$ be a coset geometry.

(i) $\Sigma(G,C)$ is residually connected if and only if $P_J = \langle P_{J\cup\{i\}} \mid i \in I\setminus J\rangle$ for each subset J of I with $I\setminus J$ of cardinality at least 2.

(ii) For T in $\mathbb{T}$, take $J=t(T)$ and g in G such that $T=g\{P_i \mid i \in J\}$. Set $C_J=\{P_{J\cup\{i\}} \mid i \in I\setminus J\}$. Then the map

$$xP_i \dashrightarrow g^{-1}xP_{J\cup\{i\}} \qquad (x \in P_J)$$

from gP_JP_i/P_i to $P_J/P_{J\cup\{i\}}$ is an isomorphism of geometries from Σ_T onto $\Gamma(P_J,C_J)$ such that $\Sigma(T)$ maps onto $\Sigma(P_J,C_J)$.

PROOF.

Exercise.

2.11 PROPOSITION.

Let Sigma be a dense flag structure on Gamma. Then the following are equivalent:

(i) Sigma is rigid.

(ii) For each S,T in $\underline{\underline{T}}$ with S contained in T, the geometries

Sigma^{T} and (Sigma^{S})^{T} coincide.

If, moreover, Sigma(T) is rigid for every T in $\underline{\underline{T}}$ of rank 1, then this is equivalent to each of:

(iii) Any flag of rank 3 is in $\underline{\underline{T}}$.

(iv) Each flag X of rank 3 contains a flag T of rank 1 such that

X is a flag of Sigma^{T}.

PROOF.

The implication (i)-->(iii) is immediate. (i)-->(ii) and (i)-->(iv) follow by Lemma 2.5

and (iv)-->(iii) follows by Lemma 2.7.

Assume for the moment that Sigma(T) is rigid for each T in $\underline{\underline{T}}$ of rank 1.

and assume that (iii) holds. Let X={x1,x2,...,xs} be a flag. In proving (i), we assume that s>2 without loss of generality. Set T={x1} and Ti={x1,xi} for i>1. By (iii), we have Ti U Tj in

$\underline{\underline{T}}^{T}$ for any i,j, whence X is a flag of Sigma^{T}. But Sigma(T) is rigid

so X\T is in $\underline{\underline{T}}_{T}$, and X is in $\underline{\underline{T}}$, proving (i).

Finally, in order to prove (ii)-->(i), assume (ii) and let T be a flag of rank 1. Then (ii) also holds for the flag structure Sigma(T), so by induction on the rank of Gamma, we have that Sigma(T) is rigid. Thus we can apply the implication (iii)--(i) just proved. In view of this,

it suffices to show X in $\underline{\underline{T}}^{T}$ for any flag X of rank 3

containing T. But this follows as Sigma^{T} and Gamma^{T} coincide by (ii).

2.12 THEOREM(Aschbacher[ ]).

Let Gamma(G,C) be a coset geometry such that Sigma=Sigma(G,C) is residually connnected. Suppose that for any P in C the flag structure Sigma({P}) is rigid.

Then G is flag transitive on Gamma if and only if G is transitive on the set of flags of type J for any subset J of I of cardinality 3.

If this holds, then Gamma is residually connected.

PROOF.

Follows from Proposition 2.11, density of Sigma, see Example 2.2, and Lemma 2.5.

2.13 REFORMULATION.

Let Gamma=Gamma(G,C) satisfy $P_J = \langle P_{J \cup \{r\}} \mid r \text{ in } I\setminus J\rangle$, and

$P_i P_k$ meet $P_j P_k = P_{\{i,j\}} P_k$ for any subset J of I with $|I\setminus J|$ at least 2

and distinct i,j,k in I.

Then Gamma is residually connected and G is flag transitive on Gamma if and only if P is flag transitive on Gamma(P,C_P) for each P in C_P

where $C_P = \{P_{t(P) \cup \{j\}} \mid j \text{ in } I\setminus t(P)\}$.

PROOF.

Use Remark 1.5 and Lemma 2.10.

2.14 COROLLARY.

Let Gamma(G,C) be a coset geometry such that

Sigma=Sigma(G,C) is residually

connnected. Suppose that for any pair i,j in I, the subgroup P=

P_i of G is flag transitive on $\text{Sigma}_{\{P\}}$. Suppose, moreover, that

for any subset J of I of cardinality 3 there are i,j in J such that

with $\{k\}=J\setminus\{i,j\}$ we have either

(a) P_i is intransitive on G/P_k and $P_{\{i,j\}}$ has at most 2 orbits on each of $P_i/P_{\{i,k\}}$ and $P_j/P_{\{j,k\}}$, or

(b) $P_i = P_{\{i,j\}} P_{\{i,k\}}$.

Then G is flag transitive on Gamma.

PROOF.

Let J be a subset of I of cardinality 3.

If (a) holds for J, then there are x in P_i and y in P_j such that

$P_i P_k = P_{\{i,j\}} P_k \cup P_{\{i,j\}} x P_k$, and

$P_j P_k = P_{\{i,j\}} P_k \cup P_{\{i,j\}} y P_k$.

Therefore, either $P_i P_k$ meet $P_j P_k = P_i P_k = P_j P_k$.

or $P_i P_k$ meet $P_j P_k = P_{\{i,j\}} P_k$.

The former case leads to $P_i P_k = \langle P_i, P_j \rangle P_k = GP_k = G$,

in contradiction with the intransitivity of P_i on G/P_k,

hence $P_i P_k$ meet $P_j P_k = P_{\{i,j\}} P_k$.

If (b) holds for J, then

$P_i P_k$ meet $P_j P_k = P_{\{i,j\}} P_k$ meet $P_j P_k = P_{\{i,j\}} P_k$.

Thus in view of Remark 1.5, the group G is transistive on the set of flags of type J for any subset J of I of cardinality 3. Hence we are done by the above theorem.

2.15 EXERCISE.

Reprove the flag transitivity of the thin coset geometry Gamma(W,F) defined by means of Coxeter groups in Corollary 1.10.

Show that Gamma(W,F) is residually connected.

2.16 COROLLARY.

Let G be a group with Tits system (B,N,W,R), and let Gamma= Gamma(G,C) be its building.

Then Gamma is residually connected and flag transitive.

PROOF.

Sigma(G,C) is residually connected by Lemma 2.10 as

$\langle B W_{J \cup \{i\}} B \mid i \text{ in } R\backslash J\rangle = B \langle W_{J \cup \{i\}} \mid i \text{ in } R\backslash J\rangle B = B W_J B = P_J$ by Theorem 1.13.

By Corollary ~.15, the group G is flag transitive on Gamma. Thus, Sigma is rigid so that residual connectedness of Sigma implies residual connectedness of Gamma.

2.17 EXAMPLE (ASCHBACHER-SMITH[ ]).
Let H=Sym(7) be the linear group of the 6-dimensional vector space over F_3 permuting the basis w1,w2,w3,w4,w5,w6,w7. Then H preserves the standard inner product with respect to this basis. Let V be the orthoplement of w1+w2+w3+w4+w5+w6+w7 with respect to this form and denote by B the restriction of the form to V. Observe that V is stabilized by H. Since the rows of M=

$$\begin{matrix} - & 1 & 1 & - & 0 & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 0 & 0 \\ - & - & - & - & - & 1 & 1 \\ - & - & - & - & 1 & - & 1 \\ - & - & - & - & 1 & 1 & - \end{matrix}$$

represent an orthonormal basis of V with respect to -B, the form B has Witt index 2 and $Aut(V,B)=O^{-}(6,3)$.

Set $G=Omega^{-}(6,3)$. The reflection r=(56) of Sym(7)=H induces an outer automorphism on G. Define L to be the setwise stabilizer in H of the basis A={235,145,136,246,127,347,567},
where ijk stands for wi+wj+wk.
Clearly, L_1 is isomorphic to PSL(3,2), and L_1=<(12)(34),(1236475)>.
Consider the linear map on W sending wi to the i-th element of A. Its restriction s to V is a linear transformation of V whose square is the homothety -1. Now r and r^s are reflections on V whose eigenvectors with eigenvalue -1 are w5-w6 and w1+w2-w3-w4 respectively.
Since the latter two are perpendicular, r and r^s commute. This implies $rsr^{-1}s = s^{-1}rsr$, whence $(rs)^4=1$, in view of $s^2=-1$. Since s takes M into an orthormal basis with respect to B, it takes B to -B, so that s normalizes G.
Set S=<(12)(34),(34)(56),(14)(23)>, P_1=H'=Alt(7), and x=(246)(135).

Furthermore, set $P_{i+1}=(sr)P_i(rs)=P_i^{(rs)}$, and

$$L_{i+1}=(sr)L_i(rs)=L_i^{(rs)} \quad \text{for } i=1,2,3.$$

We are interested in the geometry Gamma=Gamma(G,C) over I={1,2,3,4}, where $C=(P_i)_{i \text{ in } I}$.

First of all we calculate some intersections.

Note that x is in P_{124}, where 124 is short for {1,2,4}.

Hence $x^{rs}=(142)(675)$ is in P_{123}

and $x^{sr}=(123)(657)$ is in P_{134}.

Moreover, S is contained in L_1 as it preserves B, and stabilized by both r and s under conjugation.

Therefore, $K=\langle S,x^{rs},x^{sr}\rangle$ is contained in P_{13}. But K is the stabilizer of {1,2,3,4} inside H=Alt(7) and hence maximal in P_{13}. Consequently, $P_{13}=K$. Applying $(rs)^2$ to K, we get P_{24}. Next we determine P_{12}.

Obviously, L_1 is contained in P_1. But L_1 is also contained in P_2, for $L_1^{sr}=\langle(12)(34),(1236475)\rangle^{sr}=\langle(12)(34)^{sr},(1236475)^{sr}\rangle=$

$=\langle(12)(34),(1345726)\rangle$ is contained in P_1.

Since L_1 is maximal in P_1, this yields that $P_{12}=L_1$. Applying rs and its powers, we get $P_{\{i,i+1\}}=L_i$.

By now it is easy to derive that $P_{124}=\langle S,x\rangle$ and that $P_{1234}=S$.

Thus, apart from images under conjugation by rs, all intersections P_J for J a subset of I have been determined.

It is now straightforward to verify the conditions of Corollary 2.14.

We conclude that Gamma is a flag transitive residually connected geometry whose rank 2 residues are generalized m(i,j)-gons with m(i,i+1)=3 and m(i,i+2)=2 for all i in I, indices modulo 4. In particular, Gamma is a Tits geometry of type the Coxeter diagram extended A_3.

2.18 DEFINITION.

A flag structure Σ on Γ is said to belong to a diagram D on I if for each subset J of I of cardinality 2 and each T in $\underline{\underline{T}}$ of cotype J we have Σ_T in D_J. The minimal diagram $D(\Sigma)$ of a dense flag structure Σ is the diagram with $D(\Sigma)_J$ a set of representatives of the isomorphism classes of all Σ_T as T as T varies over all flags T of cotype J.

Note that such flags exist as Σ is dense.

Obviously, Γ belongs to any diagram of a rigid flag structure on Γ.

2.19 PROPOSITION.

Let Σ be a residually connected dense flag structure on Γ belonging to a diagram D.

Then:

(i) The graph induced on $t^{-1}(i) \cup t^{-1}(j)$ is connected.

(ii) If I is partitioned by the subsets J and K such that the connected components of D either belong to J or to K, then for any S,T in $\underline{\underline{T}}$ of type in J,K respectively, the set $S \cup T$ is a flag in $\underline{\underline{T}}$.

PROOF.

(i) is a slight variation of Proposition 1.2.

(ii) is an extension of Proposition 1.20. Use induction on the rank of $S \cup T$.

2.20 COROLLARY.

Assume that Γ has a dense residually connected flag structure belonging to a diagram D. If i,j,k in I are such that i,k occur in distinct connected components of $D\setminus\{j\}$ and x_i,x_j,x_k are of type r with x_i*x_j and x_j*x_k, then we have x_i*x_k.

PROOF.

Apply the proposition to Σ_F where $F=\{x_j\}$.

2.21 COROLLARY.

Let Gamma(G,C) be a coset geometry over I.

(i) For i,j in I we have that i,j are nonadjacent in D(G,C) if and only if $P_{I\setminus\{i,j\}} = P_{I\setminus\{i\}} P_{I\setminus\{j\}}$.

(ii) If Sigma(G,C) is residually connected and the graph of D(G,C) on I is a disjoint union of J and K, then $G = P_J P_K$.

PROOF.

(i) Take $T=P_J$, with $J=I\setminus\{i,j\}$. Since the diagram of Sigma(T) consists of two disjoint nodes, we have that $xP_{J\cup\{i\}}$ and $P_{J\cup\{j\}}$ are incident in Sigma(T) for any x,y in P_J. This implies $P_J = P_{J\cup\{j\}} P_{J\cup\{i\}}$, which comes down to the equality in statement (i). The converse is obtained by working backwards through the argument just given.

(ii) Suppose g in G. Let S,T be the subflag of C of type J,K respectively. Then S and gT are in the flag set $\underline{\underline{T}}$ of Sigma(G,C), so SUgT is a flag in $\underline{\underline{T}}$ according to Proposition 2.19. This means that there is h in G such that $hP_i = P_i$ for i in J and

$hP_i = gP_i$ for i in K.

Consequently, h in P_J meet gP_K, whence g in $P_J P_K$. This proves the corollary.

2.22 THEOREM(ASCHBACHER).

Let Sigma be a dense residually connected flag structure on Gamma belonging to a diagram whose graph consists of a disjoint union of paths. Then Sigma is rigid.

PROOF.

Let T be a flag of rank 1.

Then Sigma(T) is a dense residually connected flag structure on Sigma_T belonging to the diagram D\t(T), whose graph is again a union of paths. Hence, by induction on the rank of Gamma, Sigma(T) is rigid. By Proposition 2.11 it suffices to establish that any flag X of rank 3 is in Sigma^S for some flag S of rank 1. Let X={xi,xj,xk} be a flag of rank 3 with t(xr)=r for r=i,j,k occurring in this order along a path of D(if there is such a path at all, otherwise the order is irrelevant).

Take S={xj}. Then xi $\overset{S}{\mathrm{o}}$ xk by Corollary 2.20 applied to Sigma_S.

Hence X in $\underline{\underline{\mathbf{T}}}^S$.

This ends the proof of the theorem.

2.23 NOTATION.

If Gamma(G,C) is a coset geometry over I, then D(G,C) denotes the minimal diagram of the flag structure Sigma(G,C).

Thus, $D(G,C)_J$ ={$\mathrm{Sigma}_{\{P\}}$} for each subset J of I of cardinality 2 where $P=P_{I\backslash J}$.

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COMMAND- logout

 29/11/82  LOGGED OUT AT 12.38.15.
CPA         .487 SEC.          .150 ADJ.
IO         4.958 SEC.          .670 ADJ.
CM        48.040 KWS.          .333 ADJ.
CT        22.000 MIN.         3.299 ADJ.
SS                            4.452 ADJ.
PP         8.474 SEC.     DATE 29/11/82
           LP-BUDGET IP-BUDGET
OLD-BUDGET                386.59
ACCOUNTED                   4.45
NEW-BUDGET    4995.20     382.14
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AANKONDIGING

SEMINARIUM ALGEBRA EN MEETKUNDE

Aanvangsdatum	:	maandag 20 september 1982
Tijd	:	14.00-16.00 uur
Plaats	:	MC zaal M279
Frequentie	:	eens in de veertien dagen.

Belangstellenden zijn van harte welkom.

Hoofdthema voor aanstaand semester is de leer van Buekenhout-Tits meetkundes, die met behulp van diagrammen (als bijvoorbeeld hieronder) gedefinieerd worden.

De eerste bijeenkomsten zullen aan concrete voorbeelden gewijd worden. Daarna zal werk van Aschbacher en Timmesfeld over respectievelijk 'schoven' en transitieve groepen op dergelijke meetkunden besproken worden.

Eigen bijdragen over aanverwante of andere onderwerpen uit de Algebra en Meetkunde zijn welkom.

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Arjeh M. Cohen (tel. 5924167)