Recent Progress in General Topology
M. Hušek and J. van Mill, editors

CHAPTER 19

Abstract Topological Dynamics

Jan de Vries
CWI
Postbus 4079
1009 AB Amsterdam
The Netherlands
E-mail: jandv@cwi.nl

Contents

1. Introduction

This paper is written for topologists. Therefore, I have tried to concentrate on results and problems that may be interesting from a topological point of view. Thus, sometimes the emphasis is different from what might have been expected in a paper written for topological dynamicists.

It is difficult to give a precise definition of "topological dynamics". A rough description is as follows: it is the study of the "behaviour" of points in a topological space X when they "move" under the action of a semigroup of continuous transformations or of a group of homeomorphisms of X.

What this means (i.e., what "behaviour" and "move" mean in this context) is perhaps best illustrated by the following simple example. Let X be a topological space and let $f: X \to X$ be a continuous mapping. Then one may ask whether the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ has any limit points, or even a limit, for $n \rightsquigarrow \infty$ (the "behaviour" of $f^n(x)$ for $n \rightsquigarrow \infty$). Or one may ask whether there are any points x such that $f^{n+k}(x) = f^n(x)$ for some $k \in \mathbb{N}$ and some $n \in \mathbb{N}$, hence for all $n' \geq n$ (*periodic* behaviour). In particular, is there a point x such that the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ has a periodic limit point? Or if there is an *invariant point* x_0, i.e., $f(x_0) = x_0$, what can be said about points close to x_0? Are they "attracted" by x_0, that is, will $f^n(x)$ tend to x_0 if n tends to ∞? In general this type of questions concerns the structure of the closures of the semi-orbits in X (a *semi-orbit* is a set of the form $\{f^n(x) : n \in \mathbb{Z}^+\}$).

In this example we have to do with the *semigroup* $\{f^n : n \in \mathbb{Z}^+\}$ of continuous transformations of X. If in the above f is a homeomorphism of X onto itself then we would even have a *group*, namely, $\{f^n : n \in \mathbb{Z}\}$. In that case one can also consider $f^n(x)$ for $n \rightsquigarrow -\infty$, and in general, one is interested in the structure of the *orbit closures* in X (here an *orbit* is any set of the form $\{f^n(x) : n \in \mathbb{Z}\}$ for $x \in X$).

The relevance of the above questions becomes clear after the following interpretation: X can be seen as the space of all possible *states* of some (fictitious) physical (or chemical, or biological, ...) system **S**, and for any $x \in X$ the point $f(x)$ denotes the state in which **S** will be one unit of time after the moment that **S** has state x. Here we assume that **S** is *stationary*, i.e., that $f(x)$ depends only on x and not on the particular moment that **S** is in state x. Thus, $f^n(x)$ denotes the state of **S** after n units of time, and the above questions ask for the long-time behaviour of **S**.

Of course, one can also consider systems with "continuous time". In that case, let $\pi(t, x)$ denote the state reached by the system after a time *interval* of length t when it starts at state x. Again, we assume that the system is *stationary*, i.e., that $\pi(t, x)$ does not depend on the moment that the system starts at state x. Then it is obvious that for all $x \in X$ and $s, t \in \mathbb{R}^+$:

$$\pi(0, x) = x, \quad \pi(s + t, x) = \pi(s, \pi(t, x)). \tag{1}$$

Now we have for every $t \in \mathbb{R}^+$ a transformation $\pi^t : x \mapsto \pi(t, x): X \to X$, and $\pi^s \circ \pi^t = \pi^{s+t}$ for all $s, t \in \mathbb{R}^+$. In particular, $\{\pi^t : t \in \mathbb{R}^+\}$ is a semigroup of transformations of the space X. In topological dynamics one assumes that each π^t

is a continuous mapping of X into itself; we then say that $\mathbb{R}^+$ *acts* on X as a semigroup of continuous mappings. If the system is such that time can be reversed then (1) holds for all $s, t \in \mathbb{R}$, hence each π^t for $t \in \mathbb{R}$ is a bijection of X onto itself, and $t \mapsto \pi^t$ is a homomorphism of the additive group $\mathbb{R}$ into the group of all bijections of X. Again, in topological dynamics we assume that each π^t is a continuous mapping, hence a homeomorphism (the inverse of π^t is π^{-t}, which is continuous as well), and we say that $\mathbb{R}$ acts on X as a group of homeomorphisms. In this context similar questions can be asked as in the case of "discrete" time: again, one is interested in the structure of the closures of the (semi-)orbits. (A *semi-orbit* is any set of the form $\{\pi^t(x) : t \in \mathbb{R}^+\}$, and an *orbit* is any set of the form $\{\pi^t(x) : t \in \mathbb{R}\}$ for $x \in X$.)

Historically speaking, topological dynamics is an outgrowth of the qualitative theory of differential equations. At the end of the 19th century it was observed that many important differential equations (in particular, the N-body problem for $N \geq 3$) could not be solved explicitly. For this reason H. Poincaré initiated the *qualitative* study of differential equations. His idea was to give a geometric picture of the orbits (the *phase portrait*) *without integrating the equations*; outstanding geometric features in this picture would perhaps correspond to significant physical phenomena of the system described by the differential equation.

Almost simultaneously with Poincaré, A. M. Lyapunov developed his theory of stability. Also here aspects of dynamical systems (in this case: the stability of states) are studied by methods that do not require that the differential equations describing the system are solved.

Of course, also other people contributed to this shift in attention from solving equations to studying the geometry of the phase portrait. This transition was made most explicit by G. D. Birkhoff; see BIRKHOFF [1927]. It is impossible to describe here the enormous impact he had on the development of Topological Dynamics. He was the first to discuss problems from the qualitative theory of differential equations in the context of the group $\mathbb{R}$ acting as a group of homeomorphisms on a topological space (Birkhoff mainly considered metric spaces). See also HIRSCH [1984] for more about the development of Topological Dynamics.

A standard reference for all the major developments in the theory of dynamical systems up to the middle of the 1940's is NEMYTSKIĬ and STEPANOV [1960]. See also NEMYTSKIĬ [1949] and LEFSCHETZ [1957]. For stability theory from a topological point of view, see BHATIA and SZEGÖ [1970].

In the above we have considered actions of the groups $\mathbb{Z}$ and $\mathbb{R}$ on a space X; recall that these groups are to be considered as the sets of all possible values of *time*. In the 1940's the theory of topological dynamics was generalized by admitting systems in which time was allowed to run through arbitrary topological groups. For an overview of the results obtained in this direction see, e.g., (in chronological order) GOTTSCHALK and HEDLUND [1955], R. ELLIS [1969], BRONŠTEĬN [1979], the second half of VEECH [1977], VAN DER WOUDE [1986], AUSLANDER [1988], and DE VRIES [1992]. I shall call this direction in the research of dynamical systems *Abstract Topological Dynamics*. The remainder of this paper will be devoted to this abstract direction.

This does not mean that in the more "concrete" direction (actions of $\mathbb{Z}$ and $\mathbb{R}$)

there are no interesting problems for topologists. In fact, it is a bit surprising that in this area there are so many problems of a purely topological character that have not yet been investigated untill very recently. For example, there is the classification problem of orbits: see AARTS [1988] and Chapter 3 in FOKKINK [1991]; or the problem which spaces admit a continuous action of $\mathbb{R}$ without rest points: see AARTS and MARTENS [1988] and Chapter 1 in FOKKINK [1991]. These problems turn out to have close connections with old problems in continuum theory: see AARTS and FOKKINK [1991] and Chapter 2 in FOKKINK [1991]. Another problem concerns the topological description of "strange attractors" and of Julia sets. For the latter, see, e.g., AARTS and OVERSTEEGEN [19∞]. Other interesting problems can be found in Section 32.6 and in Part VIII of VAN MILL and REED [1990], and in Chapter 15 of MORITA and NAGATA [1989]. Finally, let me mention that most of the problems that I shall mention in the next sections are still open for the special cases of $\mathbb{Z}$ and $\mathbb{R}$.

2. Topological transformation groups

In order to facilitate the exposition I shall give here some definitions concerning topological transformation groups. In what follows, T is a Hausdorff topological group with unit element e. The neighbourhood filter of a point x in a topological space will be denoted by $\mathbf{N}_x$.

2.1. DEFINITION. A *topological transformation group* (abbreviation: *ttg*) with *phase group* T is a triple $\langle T, X, \pi\rangle$ where X is a topological space and $\pi: T\times X \to X$ is a mapping satisfying the following conditions:

(a) $\forall x \in X : \pi(e,x) = x$.

(b) $\forall s,t \in T \quad \forall x \in X : \pi(s, \pi(t,x)) = \pi(st, x)$.

(c) π is continuous.

In this case X is called the *phase space* of the ttg, and π is called its *action*. Actually, any mapping $\pi: T\times X \to X$ satisfying the conditions (a) and (b) is called an action (of T on X). Thus, we speak of a ttg whenever we are dealing with a continuous action.

CONVENTION: UNLESS STATED OTHERWISE THE PHASE SPACE OF A TTG WILL ALWAYS BE ASSUMED TO BE A HAUSDORFF SPACE.

If $\langle T, X, \pi\rangle$ is a ttg then for every $t \in T$ the mapping $\pi^t : x \mapsto \pi(t,x): X \to X$ is continuous; it is called a *transition* (viz., the *t-transition*) in the ttg. Also, for every $x \in X$ the mapping $\pi : t \mapsto \pi(t,x): T \to X$ is continuous; it is called a *motion* (viz., the *x-motion*, or the *motion through x*) in the ttg. Using these notions, the conditions (a) and (b) above can be rephrased as

$$\pi^e = \mathrm{id}_X, \quad \pi^s \circ \pi^t = \pi^{st} \quad (s,\, t \in T)$$

The following Lemma implies that the transitions in a ttg together form a group of homeomorphisms of the phase space; it is often called the *transition group*.

2.2. LEMMA. *Let $\langle T, X, \pi\rangle$ be a ttg. Then each transition is a homeomorphism of X onto itself, and the mapping*

$$\bar{\pi} : t \mapsto \pi^t : T \to H(X,X)$$

is a morphism of groups from the group T into the homeomorphism group of X.

□ Straightforward. □

2.3. REMARKS.

1. The conclusion of the above Lemma is valid for any action of T on X, provided each transition is continuous. In such a case we have a *continuous* action of T_d on X; here T_d is the group T endowed with the discrete topology. Observe that every ttg with phase group T can also be considered as a ttg with phase group T_d.

2. As a synonym for the phrase "$\langle T, X, \pi\rangle$ is a ttg" also the phrase "X is a T-*space* (with action π)" will be used, or: "T acts continuously on X (by π)". In cases where the group T is understood we shall denote a ttg $\langle T, X, \pi\rangle$ just by $\langle X, \pi\rangle$. Moreover, the action of T in a T-space will in most cases be suppressed (see below). Therefore, the T-space $\langle X, \pi\rangle$ will usually be denoted by $\mathcal{X}$, *the script letter corresponding to the latin capital X by which the phase space is denoted.* (Thus, the T-spaces $\langle X, \pi\rangle$, $\langle Y, \sigma\rangle$, ..., are denoted as $\mathcal{X}, \mathcal{Y}, \ldots$.)

3. As observed above, the action in a T-space will in most cases be suppressed. So instead of $\pi(t,x)$ we shall write simply tx ($t \in T, x \in X$). Moreover, we shall use self-evident notations like tA for $\pi^t[A]$, Sx for $\pi_x[S]$, etc. Using this, the conditions (a) and (b) in the definition of a ttg read as follows:

$$ex = x, \quad s(tx) = (st)x \quad (x \in X,\ s,\, t \in T).$$

4. In the above we have defined what are usually called *left* actions and *left* ttg's. A *right* action of T on X is a mapping $\pi: X \times T \to X$ such that

$$\pi(x, e) = x, \quad \pi\big(\pi(x,t), s\big) = \pi(x, ts)$$

for all $x \in X$ and $s, t \in T$, or simply (suppressing the symbol for the action):

$$xe = x, \quad (xt)s = x(ts).$$

By writing $\tilde{\pi}(t,x) := \pi(x,t)$ every right action π defines a left action $\tilde{\pi}$ (and vice versa). Using this, all definitions and results for (left) ttg's have obvious modifications for right actions. Note that for a right action π of T on X the mapping $\underline{\pi} : t \mapsto \pi(-,t): T \to H(X,X)$ is an "*anti-morphism*" of groups, i.e., $\underline{\pi}(ts) = \underline{\pi}(s) \circ \underline{\pi}(t)$ for $t,\, s \in T$.

Many publications in Abstract Topological Dynamics deal with right actions instead of left ones. This has obvious, but sometimes slightly confusing, consequences for further terminology. Thus, in Section 3 below, in the definition of an enveloping semigroup we would use *left* semitopological semigroups (but other people would call these right semitopological!); moreover, we would have to deal with minimal right ideals, etc.

2.4. EXAMPLES.

1. Consider an autonomous differential equation in an open subset U of $\mathbb{R}^n$ of the following form:

$$\dot{x} = f(x) \qquad x \in U$$

Assume that f is such that this equation has unique solutions, depending continuously on initial conditions, and extendable to the whole real line. Thus, for every point x in the domain U of f there is a *unique* continuous mapping $\pi_x : \mathbb{R} \to U$ such that $\pi_x(0) = x$ and $\dot{\pi}_x(t) = f(\pi_x(t))$ for all $t \in \mathbb{R}$. It is easy to see that for every $s \in \mathbb{R}$ the mapping $t \mapsto \pi_x(t+s)$ is a solution of the equation with initial value $\pi_x(s)$ for $t = 0$, so it must coincide with the solution π_y with initial value $y := \pi_x(s)$. Stated otherwise, $\pi_{\pi_x(s)}(t) = \pi_x(t+s)$. If we write $\pi(t,x)$ for $\pi_x(t)$ then we get precisely condition (b) in the definition of a ttg with phase group $\mathbb{R}$ and phase space U. Note that condition (a) is fulfilled by the definition of the solutions π_x, and that condition (c) is a consequence of the assumptions on f. So an autonomous differential equation in (an open subset of) $\mathbb{R}^n$ defines (under suitable conditions) a ttg.

2. Let X be a topological space and let f be a homeomorphism of X onto itself. Then $\pi : (n,x) \mapsto f^n(x) : \mathbb{Z} \times X \to X$ defines a continuous action of the (discrete) group $\mathbb{Z}$ on the space X. Note that for this ttg we have $\pi^1 = f$. Every ttg $\langle \mathbb{Z}, X, \pi \rangle$ with phase group $\mathbb{Z}$ is obtained in this way from the homeomorphism π^1, because $(\pi^1)^n = \pi^n$ for all $n \in \mathbb{Z}$. Thus, a ttg with phase group $\mathbb{Z}$, phase space X and 1-transition $\pi^1 =: f$ may be denoted just by $\langle X, f \rangle$.

3. Let $\varphi : T \to G$ be a continuous homomorphism of topological groups. Let H be a closed subgroup of G and let $X := G/H$, the quotient space of left cosets of H in G. Define $\pi : T \times X \to X$ by $\pi^t(gH) := \varphi(t)gH$. Then $\langle X, \pi \rangle$ is a well-defined T-space. It will also be denoted by $\mathcal{G}/H$.

4. Take $G := T$ and $H := \{e\}$ in Example 3. The resulting T-space (action of T on itself by means of left translations) will be denoted by $\mathcal{T}$.

5. If X is a singleton then the obvious action of T on X (every transition leaves the unique point of X invariant) defines a (*the*) *trivial* ttg, denoted by $(\star)$.

2.5. DEFINITION. Let $\mathcal{X} = \langle X, \pi \rangle$ be a T-space. Every subset of X of the form Tx is called an *orbit* in the T-space, viz., the orbit of the point x under (the action of) T. It is easily seen that *the orbits under T form a partition of the space X.*

A subset A of X is said to be *invariant in $\mathcal{X}$* whenever $TA \subseteq A$ (equivalently: $tA = A$ for every $t \in T$). Clearly, a set is invariant iff it is a union of orbits. *If A is an invariant set in $\mathcal{X}$ then also its closure $\overline{A}$, its interior* int A, *and its complement $X \setminus A$ are invariant.* ⟦ For every subset $B \subseteq X$ and $t \in T$ we have $t\overline{B} = \overline{tB}$ and $t(X \setminus B) = X \setminus tB$. ⟧ *In particular, every orbit closure is invariant.*

2.6. DEFINITION. Let $\mathcal{X} = \langle X, \pi \rangle$ and $\mathcal{Y} = \langle Y, \sigma \rangle$ be T-spaces. A mapping $\varphi : X \to Y$ is called *equivariant* whenever $\varphi \circ \pi^t = \sigma^t \circ \varphi$ for all $t \in T$. If we suppress the symbols for the actions then the condition reads: $\varphi(tx) = t\varphi(x)$ for all $(t,x) \in T \times X$. A *morphism of T-spaces* $\varphi : \mathcal{X} \to \mathcal{Y}$ is a continuous equivariant mapping $\varphi : X \to Y$. An *isomorphism* of T-spaces is a morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ such that $\varphi: X \to Y$ is a homomeorphism of X onto Y. In the special case that $\mathcal{X} = \mathcal{Y}$ we use the terms *endomorphism* and *automorphism*, respectively. A surjective morphism of T-spaces $\varphi: \mathcal{X} \to \mathcal{Y}$ will be called an *extension* (of $\mathcal{Y}$), and also a *factor* of $\mathcal{X}$.

Clearly, the composition of two morphisms of T-spaces (if it is defined as a composition of mappings) is again a morphism of T-spaces. Also, the inverse of a bijective morphism is equivariant. So the inverse of in isomorphism is again an isomorphism.

Morphisms are ubiquitous: if $\mathcal{X} = \langle X, \pi\rangle$ is a T-space then for every $x \in X$ the motion $\pi_x : t \mapsto tx: T \to X$ is a morphism $\pi_x: \mathcal{T} \to \mathcal{X}$, where $\mathcal{T}$ is the T-space defined in Example 3 in (2.4) above. Morphisms also appear in the constructions to be presented below.

2.7. CONSTRUCTIONS.

1. Let $\mathcal{X} = \langle X, \pi\rangle$ be a T-space, and let Z be a non-empty invariant set in $\mathcal{X}$. Then $\pi|_{T\times Z}: T \times Z \to Z$ is a continuous action of T on Z; for convenience, this action will be denoted just by π. The T-space $\mathcal{Z} = \langle Z, \pi\rangle$ will be called *the T-subspace of $\mathcal{X}$ on Z*. Note that $\pi|_{T\times Z}$ is the unique action of T on Z making the inclusion mapping $Z \to X$ equivariant.

2. Let $\{\mathcal{X}_\lambda = \langle X_\lambda, \pi_\lambda\rangle : \lambda \in \Lambda\}$ be a set of T-spaces. Let $X := \prod_\lambda X_\lambda$ (with its product topology) and define $\pi: T \times X \to X$ by

$$\pi^t(x) := (\pi_\lambda^t x_\lambda)_{\lambda\in\Lambda} \quad \text{for} \quad t \in T \;\text{ and }\; x = (x_\lambda)_{\lambda\in\Lambda} \in X$$

or, suppressing the symbols for the actions, $(tx)_\lambda = tx_\lambda$. It is straightforward to show that π is a continuous action of T on X (*coordinate-wise* action). The T-space $\mathcal{X} = \langle X, \pi\rangle$ is called the *product* of the T-spaces $\mathcal{X}_\lambda$. Notation: $\mathcal{X} = \prod\{\mathcal{X}_\lambda : \lambda \in \Lambda\}$, or simply $\mathcal{X} = \prod_\lambda \mathcal{X}_\lambda$; for finite products also $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$. The coordinate-wise action of T on $\prod_\lambda \mathcal{X}_\lambda$ is the *unique* action making all projections $p_\nu: \prod_\lambda X_\lambda \to X_\nu$ equivariant ($\nu \in \Lambda$).

3. Let $\mathcal{X}$ be a T-space and let R be an *invariant equivalence relation* on X, that is, R is invariant as a set in the product T-space $\mathcal{X} \times \mathcal{X}$. If we denote the quotient map of X onto X/R by $R[-]$, then the formula

$$\sigma^t R[x] := R[tx] \quad \text{for} \quad (t,x) \in T \times X$$

unambiguously defines an action of T on X/R: it is the unique action of T on X/R making the quotient map $R[-]: X \to X/R$ equivariant. As each transition σ^t is continuous it is clear that σ is a continuous action of T_d on X/R. In a number of cases $\sigma: T \times X/R \to X/R$ is (simultaneously) continuous, the most important one for us being the following: X is a compact Hausdorff space and R is a *closed* invariant equivalence relation on X ⟦use that in this case the mapping $\mathrm{id}_T \times R[-]: T \times X \to T \times X/R$ is a quotient mapping, because $R[-]$ is a perfect mapping⟧. In this situation the space X/R is a (compact) *Hausdorff* space, hence $\langle X/R, \sigma\rangle$ is a T-space; it will be denoted by $\mathcal{X}/R$. Note that in this case $R[-]: \mathcal{X} \to \mathcal{X}/R$ is a morphism of T-spaces.

Resuming, we have:

If $\mathcal{X}$ is a compact T-space and if R is a closed invariant equivalence relation on X, then there is a unique continuous action of T on X/R, defining a T-space $\mathcal{X}/R$, such that $R[-]: \mathcal{X} \to \mathcal{X}/R$ is a morphism of T-spaces.

It is important to observe that *every extension of compact T-spaces is obtained in this way*. For consider an arbitrary extension $\varphi: \mathcal{X} \to \mathcal{Y}$ of compact T-spaces. Then the equivalence relation

$$R_\varphi := \{(x_1, x_2) \in X \times X : \varphi(x_1) = \varphi(x_2)\}$$

is a closed invariant set in $\mathcal{X} \times \mathcal{X}$, and it is easy to check that the T-spaces $\mathcal{Y}$ and $\mathcal{X}/R$ are isomorphic in such a way that the morphisms φ and $R[-]$ correspond to each other. Using these remarks it is easy to see that the following is true: If $\varphi: \mathcal{X} \to \mathcal{Y}$ and $\psi: \mathcal{X} \to \mathcal{Z}$ are two surjective morphisms of compact T-spaces, then there exists a morphism $\xi: \mathcal{Y} \to \mathcal{Z}$ such that $\psi = \xi \circ \varphi$ iff $R_\varphi \subseteq R_\psi$.

4. Let $\mathcal{X} = \langle X, \pi \rangle$ be a compact T-space (i.e., X is a compact Hausdorff space). The space of all probability measures on X will be denoted by $M_1(X)$. This is the following subset of the dual of the Banach space $C_u(X)$, the space of all continuous real-valued functions on X with the sup-norm topology:

$$M_1(X) := \{\mu \in C_u(X)^* : \mu \geq 0 \ \& \ \mu(1_X) = 1\}.$$

It is well-known that $M_1(X)$ is a subset of the unit ball of $C_u(X)^*$: $\|\mu\| \leq 1$ for all $\mu \in M_1(X)$. Endowed with the w^*-topology it is a compact Hausdorff space. Recall that the w^*-topology on $M_1(X)$ is the weakest topology making all mappings $\mu \mapsto \mu(f): M_1(X) \to \mathbb{R}$ continuous ($f \in C(X)$). By the Riesz Representation Theorem $M_1(X)$ can be identified with the set of all regular Borel measures on X, the identification being given by the formula

$$\mu(f) = \int_X f \, d\mu \quad \text{for} \quad \mu \in M_1(X) \quad \text{and} \quad f \in C(X).$$

An action of T_d on $M_1(X)$ can be defined by

$$(t\mu)(f) := \int_X f(tx) \, d\mu \quad \text{for} \quad (t, \mu) \in T \times M_1(X) \quad \text{and} \quad f \in C(X),$$

or, in terms of measures:

$$(t\mu)(A) := \mu(t^{-1}A) \quad \text{for} \quad t \in T \quad \text{and} \quad A \quad \text{a Borel set in } X.$$

It is not difficult to show that this actually defines a *continuous* action of T on $M_1(X)$. Thus, we get a new T-space, denoted by $M_1(\mathcal{X})$. In this context there appear in a natural way a number of morphisms.

First of all, there is the mapping $\underline{\delta} : x \mapsto \delta_x: X \to M_1(X)$, where for every $x \in X$ the *Dirac measure* δ_x is defined by $\delta_x(f) := f(x)$ for $f \in C(X)$, or in terms of measures,

$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

It is well-known that $\underline{\delta}$ is a closed topological embedding of X into $M_1(X)$ (we still assume that X is compact). Moreover, it is easily checked that $\underline{\delta}: \mathcal{X} \to M_1(\mathcal{X})$ is a morphism of T-spaces. Thus, $\underline{\delta}$ is an equivariant embedding of $\mathcal{X}$ into $M_1(\mathcal{X})$.
If $\varphi: \mathcal{X} \to \mathcal{Y}$ is a morphism of compact T-spaces then the formula

$$\hat{\varphi}(\mu)(f) := \int_X f \circ \varphi \, d\mu \quad \text{for} \quad \mu \in M_1(X) \text{ and } f \in C(Y)$$

defines a continuous mapping $\hat{\varphi}: M_1(X) \to M_1(Y)$, which turns out to be a morphism $\hat{\varphi}: M_1(\mathcal{X}) \to M_1(\mathcal{Y})$. In terms of measures, $\hat{\varphi}$ is given by $\hat{\varphi}(\mu)(A) := \mu(\varphi^{\leftarrow}[A])$ for Borel sets A in Y. It is well-known that if φ is injective or surjective then so is $\hat{\varphi}$. In particular, if X is a closed invariant set in the T-space $\mathcal{Y}$ and if φ is the inclusion mapping, then $\hat{\varphi}$ embeds $M_1(\mathcal{X})$ as a closed invariant set in $M_1(\mathcal{Y})$.

5. Let $\mathcal{X} = \langle X, \pi \rangle$ be a compact T-space. Recall that the *hyperspace* $\mathbf{2}^X$ of X is the space whose elements are the *non-empty* closed subsets of X, and which has the topology generated by all sets of the form

$$\langle V_1, \dots, V_n \rangle := \{A \in \mathbf{2}^X : A \subseteq \bigcup_{j=1}^{n} V_j \ \& \ A \cap V_i \neq \varnothing \ \text{ for } \ i = 1, \dots, n\}$$

with $n \in \mathbb{N}$ and $V_1, \dots, V_n$ non-empty open sets in X. This topology is usually called the *Vietoris topology* for $\mathbf{2}^X$; actually, the sets $\langle V_1, \dots, V_n \rangle$ form a *base* for the Vietoris topology. It is well-known that $\mathbf{2}^X$ with the Vietoris topology is a compact Hausdorff space, and that the mapping $x \mapsto \{x\}: X \to \mathbf{2}^X$ is a closed embedding. Moreover, if X is metrizable then so is $\mathbf{2}^X$; in that case a metric for $\mathbf{2}^X$ is given by the so-called *Hausdorff metric* for $\mathbf{2}^X$:

$$\begin{aligned} d^*(A, B) : &= \inf\{\epsilon \geq 0 : A \subseteq S_\epsilon[B] \ \& \ B \subseteq S_\epsilon[A]\} \\ &= \max\{\max_{a \in A} d(a, B), \ \max_{b \in B} d(b, A)\}, \end{aligned}$$

where $d(a, B) = \min\{d(a, y) : y \in B\}$, the distance of $a \in X$ to B, and $S_\epsilon[B] = \{a \in X : d(a, B) < \epsilon\}$.
The action π of T on X induces an action $\mathbf{2}^\pi$ of T on $\mathbf{2}^X$, as follows:

$$(\mathbf{2}^\pi)^t (A) := \pi^t[A] \quad \text{for} \quad (t, A) \in T \times \mathbf{2}^X.$$

If the symbols for the actions are omitted, then for $t \in T$ and a closed non-empty subset A of X the expression tA has two meanings: if A is seen as a subset of X then it means the set $\{tx : x \in A\}$, and if A is seen as an element of $\mathbf{2}^X$ then it is the same set tA as before, but now considered as an element of the space $\mathbf{2}^X$. This action of T on $\mathbf{2}^X$ turns out to be continuous. The resulting T-space is denoted as $\mathbf{2}^{\mathcal{X}}$.
There are several types of morphisms connected with actions of T on hyperspaces. First of all we have the closed embedding $x \mapsto \{x\}$ of X into $\mathbf{2}^X$; this embedding clearly is equivariant, so it is a morphism of T-spaces. In addition, if $\varphi: \mathcal{X} \to \mathcal{Y}$ is

any morphism of T-spaces, then φ can be extended in a natural way to a mapping $\mathbf{2}^{\varphi}: \mathbf{2}^X \to \mathbf{2}^Y$, namely, to the mapping defined by

$$\mathbf{2}^{\varphi}(A) := \varphi[A] \quad \text{for} \quad A \in \mathbf{2}^X.$$

It is easily checked that $\mathbf{2}^{\varphi}$ is continuous and equivariant, hence it is a morphism of T-spaces, $\mathbf{2}^{\varphi}: \mathbf{2}^{\mathcal{X}} \to \mathbf{2}^{\mathcal{Y}}$. Finally, for any morphism $\varphi: \mathcal{X} \to \mathcal{Y}$ one can define a mapping $\varphi_{ad}: Y \to \mathbf{2}^X$ by

$$\varphi_{ad}(A) := \varphi^{\leftarrow}[A] \quad \text{for} \quad A \in \mathbf{2}^Y.$$

It is obvious that φ_{ad} is injective and equivariant. In general, φ_{ad} is not continuous: it is only *upper semi-continuous*, i.e., for every open subset U of X the set $\{y \in Y : \varphi_{ad}(y) = \varphi^{\leftarrow}[y] \subseteq U\}$ is open in Y ⟦this set equals $Y \setminus \varphi[X \setminus U]$, which is open in Y because φ is a closed mapping⟧. By a straightforward argument one shows that *φ_{ad} is continuous iff φ is an open mapping* ⟦ use that $\varphi_{ad}^{\leftarrow}[\langle X, U\rangle] = \{y \in Y : \varphi^{\leftarrow}[y] \cap U \neq \varnothing\} = \varphi[U]$ ⟧. So in that case we have a morphism of T-spaces $\varphi_{ad}: \mathcal{Y} \to \mathbf{2}^{\mathcal{X}}$.

3. Minimal flows

3.1. Conventions and terminology. Unless stated otherwise, all ttg's will have as a phase group a fixed but otherwise arbitrary topological group T. In addition, we shall consider only T-spaces with a *compact* phase space. In order to stress this we shall call such a compact T-space a *T-flow* or just simply a *flow*. (A number of notions that will be defined below can also be defined for non-compact T-spaces, and some results hold in that context too, or at least in T-spaces with compact orbit closures.)

Let $\mathcal{X}$ be a flow. A subset A of X is called *a minimal set in $\mathcal{X}$* (or: a minimal subset of X) whenever A is closed, non-empty and invariant, and no proper subset of A has these properties. If X is a minimal set in $\mathcal{X}$ then we say that $\mathcal{X}$ is a *minimal flow*.

3.2. Remarks. Let $\mathcal{X}$ be a flow. The following statements are rather straightforward consequences of the definitions.

1. If A is a minimal set in $\mathcal{X}$ and B is an arbitrary invariant set in $\mathcal{X}$ then either $A \cap B = \varnothing$ or $A \subseteq B$. In particular, different minimal sets (in the same flow) are always mutually disjoint.
2. If $\varnothing \neq A \subseteq X$ then the following conditions are equivalent:
   - (i) A is a minimal set in $\mathcal{X}$;
   - (ii) $\forall x \in A : \overline{Tx} = A$;
   - (iii) A is closed and invariant, and for every non-empty open set U in X, either $A \subseteq TU$ or $A \cap TU = \varnothing$.

   It follows that every minimal set is an orbit closure; therefore, one often speaks of *minimal orbit closures* instead of minimal sets. Note that the orbit closures form a partition of the phase space iff all orbit closures are minimal (in that case the flow is sometimes called *semisimple*; for another term, see 5 below).

3. An orbit closure $\overline{Tx}$ in $\mathcal{X}$ is minimal iff

$$\forall y \in X : \quad y \in \overline{Tx} \implies x \in \overline{Ty}\,.$$

4. Every non-empty closed invariant subset of X includes a minimal set ⟦use Zorn's Lemma⟧.
5. If $\varphi : \mathcal{X} \to Y$ is a morphism and A is a minimal set in $\mathcal{X}$ then $\varphi[A]$ is a minimal set in $\mathcal{Y}$ ⟦if $B \subseteq \varphi[A]$ is non-empty, closed and invariant, then by 1 above, $A \subseteq \varphi^{\leftarrow}[B]$⟧. Conversely, if B is a minimal set in $\mathcal{Y}$ then the closed and invariant set $\varphi^{\leftarrow}[B]$ includes a minimal set C, and clearly $\varphi[C] = B$.

3.3. DEFINITION. Points situated in minimal sets (i.e., points with minimal orbit closures) can be characterized in terms of their recurrent behaviour. To this end we need some definitions. Let $\mathcal{X}$ be a flow. For any point x in X and any subset U of X we write

$$D_{\mathcal{X}}(x, U) := \{t \in T : tx \in U\}.$$

(This is the set of all values of "time" that tx dwells in U. In topological dynamics one is interested in such "dwelling" sets with U a neighbourhood of the point x: how often does x return in any one of its neighbourhoods?) A point $x \in X$ is said to be *almost periodic* whenever for every $U \in \mathbf{N}_x$ (the neighbourhood filter of x) the set $D_{\mathcal{X}}(x, U)$ is "large" in the following sense: there exists a compact (i.e., a "small") subset K of T such that $T = KD_{\mathcal{X}}(x, U)$; equivalently, every right translate of the compact set K^{-1} meets $D_{\mathcal{X}}(x, U)$. If K can always taken to be *finite* then the point x is said to be *discretely almost periodic* (i.e., almost periodic under the action of the discrete group T_d).

Using these notions we can formulate the following characterization of a minimal orbit closure:

3.4. THEOREM. *Let $\mathcal{X}$ be a flow and let $x \in X$. Then the following conditions are equivalent:*

(i) *$\overline{Tx}$ is a minimal set in $\mathcal{X}$;*

(ii) *x is an almost periodic point in $\mathcal{X}$;*

(iii) *x is a discretely almost periodic point in $\mathcal{X}$.*

□ (Outline) (i)⇒(iii): Let $U \in \mathbf{N}_x$. By (3.2)2(iii) above and compactness of $\overline{Tx}$ there is a finite set K in T such that $Tx \subseteq \overline{Tx} \subseteq KU$. Hence $T = KD_{\mathcal{X}}(x, U)$. (iii)⇒(ii): Obvious. (ii)⇒(i): Consider $y \in \overline{Tx}$; we want to show (see (3.2)3 above) that $x \in \overline{Ty}$. Let U be an arbitrary compact neighbourhood of x. There is a compact set K in T such that $T = KD_{\mathcal{X}}(x, U)$, hence $Tx \subseteq KU$. As KU is compact (image of $K \times U$ under the continuous action of T on X) it follows that $\overline{Tx} \subseteq KU$. In particular, $y \in KU$, hence $U \cap Ty \neq \varnothing$. Thus, $x \in \overline{Ty}$. □

3.5. REMARK. In view of the above, a flow in which the minimal sets form a partition of the phase space (i.e., a semisimple flow) is often said to be *pointwise almost periodic.*

3.6. ENVELOPING SEMIGROUPS. Enveloping semigroups are an important "technical" resource for the study of orbit closures, hence, in particular, for the study of minimal sets. In this survey they cannot be omitted, but we shall have hardly any occasion to illustrate their use, because almost no proofs will be given. An *enveloping semigroup* of T is a mapping $\varphi: T \to S$ of T with the following properties:

(a) S is a semigroup with a compact Hausdorff topology making all *right* translations $p \mapsto pr: S \to S$ continuous for $r \in S$.

(b) φ is a continuous morphism of semigroups (i.e., $\varphi(st) = \varphi(s)\varphi(t)$ for all $s, t \in T$) and its range $\varphi[T]$ is dense in S; note that we do *not* require that φ is a topological embedding of T in S.

(c) The mapping $(t,p) \mapsto \varphi(t)p: T \times S \to S$ is continuous. Note that this mapping is a continuous *action* of T on S; the resulting flow will be denoted by $\mathcal{S}$.

It is rather straightforward to show that there exists a *universal enveloping semigroup* $\varphi_T: T \to S_T$; it is characterized by the following property: for every enveloping semigroup $\varphi: T \to S$ there exists a unique continuous morphism of semigroups $\varphi': S_T \to S$ such that $\varphi = \varphi' \circ \varphi_T$. It can be shown that the compactification $\varphi_T: T \to S_T$ corresponds to the subalgebra $RUC^*(T)$ of all bounded right uniformly continuous functions on T. As this algebra separates points and closed subsets of T it follows that φ_T is a topological embedding of T into S_T. *Therefore, we shall consider T as a subgroup of the semigroup S_T and identify t with $\varphi(t)$ for every $t \in T$.* Thus, for example, the expression tp for $t \in T$ and $p \in S_T$ has a double interpretation (but in both interpretations it denotes the same element of S_T: firstly, it is the product of $t(= \varphi(t))$ with p in the semigroup S_T, and secondly, it is the image if p under the t-transition.

Much is known about semigroups as mentioned in (a) above (so-called *right semitopological semigroups*, because all right translations are continuous; other authors call them *left* semitopological semigroups, because multiplication is continuous in the left variable); see, e.g., RUPPERT [1984]. For topological dynamics the following properties are fundamental:

1. Every right semitopological semigroup S contains at least one *minimal left ideal*, i.e., a subset M that is minimal under inclusion with respect to the properties $M \neq \varnothing$ and $SM \subseteq M$. All minimal left ideals are closed in S.
2. Every right semitopological semigroup S contains at least one *idempotent*, i.e., an element u such that $u^2 = u$.
3. Let M be a minimal left ideal in a right semitopological semigroup. Then:

   (a) The set $J(M)$ of all idempotents in M is not empty.

   (b) $\forall u \in J(M)\, \forall p \in M : pu = p$ (so u is a *right* unit element of M).

   (c) $\forall u \in J(M) : uM$ is a group and $uM = \{p \in M : up = p\}$.

(d) $\{uM : u \in J(M)\}$ is a partition of M. In particular, M is a disjoint union of groups, and for every element p of M there are unique elements u of $J(M)$ and p^{-1} of $uM \subseteq M$ such that $up = pu = p$ and $pp^{-1} = p^{-1}p = u$.

4. The following property makes it possible to switch from one minimal left ideal to another (indispensible for proving that a number of definitions in topological dynamics is independent of the choice of such a minimal left ideal; unfortunately, I cannot illustrate this is this paper): If L and M are minimal left ideals in a right semitopological semigroup S then for every $u \in J(L)$ there exists a unique $v \in J(M)$ such that $uv = v$ and $vu = u$.
5. Let $\mathcal{S}$ be the flow defined by an enveloping semigroup. Then the minimal sets in $\mathcal{S}$ are just the minimal left ideals in the semigroup S.

3.7. THE ELLIS SEMIGROUP OF A FLOW. The use of enveloping semigroups for topological dynamics stems from the following. Let $\mathcal{X} = \langle X, \pi\rangle$ be a flow. The space X^X with its product topology is a right semitopological semigroup with composition of mappings as semigroup-operation, and the transition group $\{\pi^t : t \in T\}$ can be viewed as a subgroup of X^X. It turns out that the closure $E(\mathcal{X})$ of $\{\pi^t : t \in T\}$ is a subsemigroup of X^X; it is called the *Ellis semigroup* of $\mathcal{X}$. Moreover, $\bar{\pi} : t \mapsto \pi^t : T \to E(\mathcal{X})$ is an enveloping semigroup of T.

As observed in (3.6) above, every enveloping semigroup gives rise to a flow. In particular, this is the case for $\bar{\pi}: T \to E(\mathcal{X})$. The action of T on $E(\mathcal{X})$ is given by $t\xi := \pi^t \circ \xi$ for $t \in T$ and $\xi \in E(\mathcal{X})$, and the flow so obtained will be denoted by $\mathcal{E}(X)$. After a moment's reflection one sees that $\mathcal{E}(X)$ is a subflow of the flow $\mathcal{X}^X$, the product of $|X|$ copies of $\mathcal{X}$; in fact, $E(\mathcal{X})$ is the orbit closure of id_X in $\mathcal{X}^X$. For every point $x \in X$ the evaluation mapping $\delta_x : \xi \mapsto \xi(x): E(\mathcal{X}) \to X$ is a morphism of flows, $\delta_x: \mathcal{E}(\mathcal{X}) \to \mathcal{X}$. As $\bar{\pi}[T]$ is dense in $E(\mathcal{X})$ it follows that δ_x maps $E(\mathcal{X})$ onto the orbit closure $\overline{Tx}$ of x in the flow $\mathcal{X}$.

By the universal property of $\varphi_T : T \to S_T$ there exists a (surjective!) continuous morphism of semigroups $\pi^*: S_T \to E(\mathcal{X})$ such that $\bar{\pi} = \pi^* \circ \varphi_T$. It is now possible to define an "action" of the semigroup S_T on X by $px := \big(\pi^*(p)\big)(x)$ for $p \in S_T$ and $x \in X$. Thus, each element of S_T acts by means of its image in $E(\mathcal{X})$, which is a mapping of X into itself. As $\pi^*(t) = \pi^* \circ \varphi_T(t) = \bar{\pi}(t) = \pi^t$, this action clearly extends the original action π of T. Note that for fixed $x \in X$ the mapping $p \mapsto px: S_T \to X$ is always continuous ⟦it is the mapping $\delta_x \circ \pi^*$⟧. It follows that $p(rx) = (pr)x$ for all $p, r \in S_T$ and $x \in X$ ⟦if $p \in T$ then this equality holds if $r \in T$, and both sides of this equality depend continuously on $r \in S_T$; as T is dense in S_T, it follows that the equality holds for all $r \in S_T$; then a similar argument shows that for any fixed $r \in S_T$ the equality holds for all $p \in S_T$⟧. This equality accounts for the term "action" of S_T on X. In general, this action of S_T is not continuous: for fixed $p \in S_T$ the mapping $x \mapsto px: X \to X$ may be non-continuous (this is because of the fact that $E(\mathcal{X})$ is obtained as a subset of X^X and may, therefore, contain non-continuous mappings of X into itself as members; see also the remark just before (4.9) ahead). It should be clear from the above that the "orbit" of a point x under this action of S_T is equal to $E(\mathcal{X})$ evaluated at x, i.e., it is just the orbit closure of x: $S_T x = \overline{Tx}$. This also follows from the following observation: the mapping $p \mapsto px: S_T \to X$ is a morphism of flows $[\![(tp)x = t(px)]\!]$, hence it maps the orbit $Te = T$ of e in the flow $\mathcal{S}_T$ onto the orbit Tx of x in the flow $\mathcal{X}$; now use that T is dense in S_T.

Resuming, we have shown the following:

> *For every flow $\mathcal{X}$ the action of T on X can be extended to an action of S_T on X with the same "transition-semigroup". In general, this extended action is not continuous, but all motions are. For every point $x \in X$ the orbit under S_T coincides with the orbit closure under T, and $p \mapsto px: S_T \to \mathcal{X}$ is a morphism of flows, mapping S_T onto $\overline{Tx}$.*

The advantage of working with the action of S_T instead of working with the original action of T lies in the facility with which statements about orbit closures can be proved. Often, $E(\mathcal{X})$ can be used in exactly the same way, but S_T has the advantage that it works on *every* flow. For an example of this, see Remark 3.9 below. In that remark we shall use that the action of S_T behaves in the following way with respect to any morphism $\varphi : \mathcal{X} \to Y$ of flows:

$$\varphi(px) = p\varphi(x) \quad \text{for} \quad p \in S_T \quad \text{and} \quad x \in X$$

$[\![$the equality holds for all p in the dense subset T of S_T, and both sides of the equality depend continuously on $p]\!]$. The following is an example of the use of this action of S_T on the phase space of a flow; it is in part a reformulation of the characterization of a minimal set in (3.4) above.

3.8. THEOREM. *Let $\mathcal{X}$ be a flow and let $x \in X$. Then the following conditions are equivalent:*

(i) *x is an almost periodic point;*
(ii) *$\overline{Tx}$ is a minimal set;*
(iii) *There exists a minimal left ideal M in S_T such that $x \in Mx$;*
(iv) *For every minimal left ideal M in S_T there is an idempotent $u \in J(M)$ such that $ux = x$.*

□ (Outline) (ii)⇒(iv): If M is a minimal left ideal in S_T then by (3.6)5, (3.2)5 and the final remark in (3.7) it is clear that Mx is a minimal subset of $S_T x = \overline{Tx}$. So assuming (ii) we get $Mx = \overline{Tx}$. Then $\{p \in M : px = x\}$ is a *non-empty* closed subsemigroup, which contains by (3.6)2 an idempotent. (iv)⇒(iii): Obvious. (iii)⇒(ii): As above, one sees that Mx is a minimal subset of $\overline{Tx}$; by (iii) it contains the point x, hence it must coincide with $\overline{Tx}$. □

3.9. REMARK. This theorem gives an easy method to "construct" almost periodic points in any flow $\mathcal{X}$: for any $x \in X$ and idempotent u a minimal ideal of S_T the point ux is almost periodic $[\![u(ux) = u^2x = ux]\!]$. Now let $\varphi : \mathcal{X} \to Y$ be a morphism of flows, and let y be an almost periodic point in $\mathcal{Y}$. Then there exists an almost periodic point x in $\mathcal{X}$ such that $\varphi(x) = y$. $[\![$Let u be an idempotent in a minimal left ideal in S_T such that $uy = y$, and let $x' \in \varphi^{\leftarrow}[y]$. Then $x := ux'$ is an almost periodic point in $\mathcal{X}$, and $\varphi(x) = \varphi(ux') = u\varphi(x') = uy = y.]\!]$ (Of course, a similar but even easier argument shows that $\varphi(x)$ is an almost periodic point in $\mathcal{Y}$ if x is almost periodic in $\mathcal{X}$.)

We close this section with an important property shared by all minimal left ideals of the semigroup S_T: all flows on these ideals are mutually isomorphic, and any of them is a *universal minimal flow*. What this means is expressed by the following theorem.

3.10. THEOREM. *Let M be a minimal left ideal in S_T. Then for every minimal flow $\mathcal{X}$ there exists a morphism*[1]. *$\varphi : \mathcal{M} \to X$; thus, every minimal flow is a factor of $\mathcal{M}$. In addition, if $\psi : \mathcal{Y} \to M$ is any morphism of flows then ψ is an isomorphism. In particular, the flows on the minimal left ideals of S_T are mutually isomorphic.*

□ (Outline) Let $\mathcal{X}$ be a minimal flow and let $x \in X$. Then $p \mapsto px: \mathcal{S}_T \to \mathcal{X}$ is a morphism, mapping M *onto* X because $\mathcal{X}$ is minimal. This proves the first part of the theorem. For the second part it is sufficient to show that every endomorphism of the flow $\mathcal{M}$ is an automorphism ⟦by what has just been proved, there exists $\varphi : \mathcal{M} \to Y$ so we have the endomorphism $\psi \circ \varphi$ of $\mathcal{M}$⟧. To this end, note that every endomorphism ξ of $\mathcal{M}$ is a right translation $p \mapsto pr$ for some $r \in M$ ⟦take $r := \xi(u)$ for any idempotent u in M⟧. Then the right translation over r^{-1} (see (3.6)3(d) above) is the inverse of ξ. Finally, note that all minimal left ideals of S_T have both properties, hence are mutually isomorphic. □

3.11. REMARKS.

1. If $\mathcal{Z}$ is a minimal flow such that every minimal flow is a factor of $\mathcal{Z}$, then it follows from the second part of the theorem that $\mathcal{Z}$ is isomorphic with the flow $\mathcal{M}$ for every minimal left ideal M of S_T. Therefore, the above theorem is often expressed by saying that *$\mathcal{M}$ is the unique universal minimal flow*. Properly speaking, this is a misnomer: the term "universal" suggests that the morphism of the universal object to any other object in the category under consideration is *unique*, which in this case is certainly not true. For example, it would imply that for any minimal left ideal in S_T and any minimal flow $\mathcal{X}$ we would have $px_1 = px_2$ for all $p \in M$ and every pair $x_1, x_2 \in X$. There are many minimal flows where this is not the case (e.g., all distal flows: use the conclusion of (4.9) below).

2. Not much is known about the *topological* structure of minimal left ideals in S_T. Of course, each minimal left ideal M of S_T is a retract of S_T ⟦for $u \in J(M)$, $p \mapsto pu: S_T \to T$ is a retraction⟧. In the case that the group T is discrete it is clear that $\varphi_T: T \to S_T$ is the Čech-Stone compactification of T, so in that case each minimal left ideal M of S_T is extremally disconnected. For some other topological properties of "the" universal minimal flow, see VAN DOUWEN [1990], BALCAR and BLASZCZYK [1990] and BALCAR and DOW [1991].

3. Much of what has been said above can be done for actions of semigroups. This leads (in considering continuous self-maps of a compact space, i.e., actions of the semigroup $\mathbb{Z}^+$) to the study of minimal left ideals in $\beta\omega\backslash\omega$. See the above references, and also HINDMAN and PYM [1984] and BERGELSON and HINDMAN [1990].

[1] Of course, $\mathcal{M}$ is the subflow of S_T on M.

4. Equicontinuous and distal flows

A central problem in topological dynamics (still) is the classification of minimal flows. In full generality, this problem appears to be quite difficult. Therefore, it seems appropriate to single out certain minimal flows which are "well behaved" in some sense, and are hopefully more amenable to classification. In this paper we shall concentrate mainly on two classes of minimal flows, namely, the equicontinuous ones and the distal ones. For these classes there is a rather extensive theory, and much research in Abstract Topological Dynamics stems from attempts to generalize the results to other classes of minimal flows.

For the definitions of these notions, recall that the topology of a compact Hausdorff space X is generated by a unique uniformity $\mathbf{U}_X$. In what follows, expressions like $t\alpha$ or $B\alpha$ for $t \in T$ or $B \subseteq T$ and $\alpha \in \mathbf{U}_X$ are to be interpreted in the flow $\mathcal{X} \times \mathcal{X}$; thus, for example,

$$B\alpha = \{(tx_1, tx_2) : \ t \in B \ \& \ (x_1, x_2) \in \alpha\} \ .$$

4.1. EQUICONTINUITY. A flow $\mathcal{X}$ is said to be *equicontinuous* whenever the group of its transitions is an equicontinuous set of homeomorphisms of X, that is, whenever

$$\forall x \in X \ \forall \alpha \in \mathbf{U}_X \ \exists U \in \mathbf{N}_x \ : \ tU \subseteq \alpha[tx] \quad \text{for all} \quad t \in T.$$

The flow $\mathcal{X}$ is called *uniformly equicontinuous* whenever its transition group is uniformly equicontinuous, i.e.,

$$\forall \alpha \in \mathbf{U}_X \ \exists \beta \in \mathbf{U}_X \ : \ T\beta \subseteq \alpha.$$

Clearly, a uniformly equicontinuous flow is equicontinuous; as all flows have compact phase spaces, the converse is also true. Therefore, we shall use "equicontinuous" when we actually mean "uniformly equicontinuous". In the literature the term "(uniformly) almost periodic" is often used for what we call uniformly equicontinuous. This is because of the following characterization (cf. AUSLANDER [1988, Theorem 2 in Chapter 2], or DE VRIES [1992, IV(2.8)]; the proof in ELLIS [1969] is not correct): a flow $\mathcal{X}$ is uniformly equicontinuous iff for every $\alpha \in \mathbf{U}_X$ the set

$$D_{\mathcal{X}}(\alpha) := \bigcap_{x \in X} D_{\mathcal{X}}(x, \alpha[x]) = \{t \in T : (x, tx) \in \alpha \text{ for all } x \in X\}$$

is large in T in the same sense as we have met in the definition of "almost periodic point": there is a compact subset K of T (depending on α) such that $T = K D_{\mathcal{X}}(\alpha)$; actually, then there is a *finite* subset K of T with this property, so the topology of T plays no role (this was to be expected from the definition of "uniformly equicontinuous"). It follows immediately from the above that *every equicontinuous flow is pointwise almost periodic, hence the union of mutually disjoint minimal sets* (*i.e., it is semisimple*).

Examples of equicontinuous flows are, e.g., all groups that act on compact spaces by means of isometries (for example, the $\mathbb{Z}$-flow on the unit circle of $\mathbb{C}$ generated by the homeomorphism $z \mapsto az$ for $z \in Z$). Also, the class of examples described in (2.4)3 with G a compact Hausdorff group consists of equicontinuous flows; if $\varphi[T]$ is dense in G then these flows are also minimal. We shall see in (4.5) below that all equicontinuous minimal flows are of this form.

Ellis semigroups can be used to characterize equicontinuous flows:

4.2. THEOREM. *The following conditions are equivalent for a flow $\mathcal{X}$:*

(i) *$\mathcal{X}$ is equicontinuous;*

(ii) *$E(\mathcal{X})$ is a compact topological group and $\delta : (\xi, x) \mapsto \xi(x) : E(\mathcal{X}) \times X \to X$ is a continuous action of $E(\mathcal{X})$ on X;*

(iii) *$E(\mathcal{X})$ is a group of homeomorphisms of X.*

If, in addition, $\mathcal{X}$ is minimal then the above conditions are equivalent with:

(iv) *$E(\mathcal{X})$ is a topological group.*

□ (Outline) (i)⇒(ii): If (i) holds then on $E(\mathcal{X})$ the topologies of pointwise and of uniform convergence coincide, and it is well-known that now (ii) holds (cf., e.g., BOURBAKI [1966, Chapter X, §§3.4 and 3.5]).

It is obvious that (ii) implies (iii) and (iv), and that (ii) implies (i) follows from a straightforward compactness argument, using continuity of δ.

(iii)⇒(ii): As all members of $E(\mathcal{X})$ are continuous mappings, it follows that multiplication in the group $E(\mathcal{X})$ is separately continuous; also, δ is separately continuous. Now Ellis' Joint Continuity Theorem (cf. RUPPERT [1984]) implies that (ii) holds.

Finally, if $\mathcal{X}$ is minimal then for a point $x \in X$ the mapping $(\xi, \eta) \mapsto (\xi, \eta(x))$ from $E(\mathcal{X}) \times E(\mathcal{X})$ to $E(\mathcal{X}) \times X$ is surjective, hence a quotient mapping. The composition of $\delta : E(\mathcal{X}) \times X \to X$ with this quotient map is $(\xi, \eta) \mapsto (\xi \circ \eta)(x)$, which is continuous if (iv) holds. Hence δ is continuous. This shows that (iv)⇒(ii). □

4.3. REMARK. The implication (iii)⇒(ii) is highly non-trivial. Using a refinement of the proof of Ellis' Joint Continuity Theorem it is possible to prove also that if $\mathcal{X}$ is a *minimal* flow and every $\xi \in E(\mathcal{X})$ is continuous, then $\mathcal{X}$ is equicontinuous; cf. Chapter 4 of AUSLANDER [1988]. If $\mathcal{X}$ is not minimal then the inclusion $E(\mathcal{X}) \subseteq C(X, X)$ is not sufficient to imply equicontinuity. ⟦Example: Let X be the one-point compactification of $\mathbb{R}$ and let the group $\mathbb{R}$ act on X by addition, leaving the point ∞ invariant. Then $E(\mathcal{X}) \subseteq C(X, X)$, but $\mathcal{X}$ is not equicontinuous: the orbit of the point 0 is all of X, hence not minimal.⟧

Now recall the existence and defining property of the *Bohr compactification* of the topological group T: a continuous morphism of groups $\alpha_T : T \to bT$, where bT is a compact Hausdorff topological group, α_T has a dense range in bT, and characterized by the following "universal property": for every topological group compactification[2] $\psi : T \to G$ of T there exists a unique continuous morphism of groups $\psi' : bT \to G$ such that $\psi = \psi' \circ \alpha_T$. Using the previous result in combination with this "universal property" of the Bohr compactification one arrives at the following result (the proof looks much like the argument used in (3.7) above, with $\varphi_T : T \to S_T$ replaced by $\alpha_T : T \to bT$).

[2] In my terminology, a compactification need not be an embedding.

4.4. THEOREM. *Let $\mathcal{X}$ be a compact flow. The following conditions are equivalent:*

(i) *$\mathcal{X}$ is equicontinuous;*

(ii) *There is a continuous action $\tilde{\pi}$ of bT on X that "extends" the original action of T in the following sense:*

$$\tilde{\pi}(\alpha(t), x) = tx \quad \textit{for} \quad (t, x) \in T \times X.$$

Recall that for every topological group compactification $\psi: T \to G$ and every closed subgroup H of G the induced flow $\mathcal{G}/H$ (cf. (2.4)3 above) is equicontinuous and minimal. In particular, for every closed subgroup H of bT we have the equicontinuous minimal flow $b\mathcal{T}/H$.

4.5. COROLLARY. *Every equicontinuous minimal flow is isomorphic with a flow of the form $b\mathcal{T}/H$ for some closed subgroup H of bT, which is unique up to conjugacy.*

□ Select $x \in X$ and take $H := \{g \in bT : gx = x\}$. □

This corollary implies that the flow $b\mathcal{T}$ is a *universal equicontinuous minimal flow*. As such it is *unique*: any equicontinuous minimal flow $\mathcal{X}$ with the property that every equicontinuous minimal flow is a factor of $\mathcal{X}$ is isomorphic with the flow $b\mathcal{T}$. ⟦the argument is similar to that used in the second part of the proof of (3.10): every endomorphism of the flow $b\mathcal{T}$ is a right translation, hence an automorphism⟧.

4.6. QUESTIONS.

1. The equicontinuous minimal flows can be classified by the conjugacy classes of closed subgroups of the topological group $b\mathcal{T}$. Is there anything known about the set of conjugacy classes of $b\mathcal{T}$ for a general topological group T? What is its cardinallity in the case that T is a so called "maximally almost periodic group", e.g., T a locally compact abelian group? (A maximally almost periodic group is a topological group T for which $\alpha_T: T \to bT$ is injective; the opposite is a "minimally almost periodic group": here bT is the trivial one-point group; a minimally almost periodic group T clearly admids no non-trivial equicontinuous minimal flows.)

2. Another (as far as I know, still open) question is the following. It is not very difficult to show that if the group T is totally bounded (in its left, right or two-sided uniformity: these uniformities coincide in that case) then every minimal flow is equicontinuous ⟦in that case all members of $RUC(T)$ are almost periodic (they can be extended to continuous functions on the Weyl completion of T, which is a compact topological group), hence the compactifications $\varphi_T: T \to S_T$ — corresponding to $RUC(T)$ — and $\alpha_T: T \to bT$ — corresponding to the almost periodic functions — coincide; therefore, every orbit closure in any flow, being a factor of $\mathcal{S}_T = b\mathcal{T}$, is equicontinuous⟧. In GAIT [1972] it has been shown that the converse is true if T is discrete: if every minimal flow is equicontinuous then T is finite. What if T is not discrete? This question is a special case if a more general one that will be posed in (4.12)2 below.

4.7. DISTALITY. A notion closely related to equicontinuity is "distality". This notion seems to be introduced by Hilbert in order to better understand equicontinuity. For the definition we first introduce the opposite notion of "proximality".

A pair of points (x_1, x_2) in a flow $\mathcal{X}$ is called a *proximal pair* whenever for every $\alpha \in \mathbf{U}_X$ there exists $t_\alpha \in T$ such that $(t_\alpha x_1, t_\alpha x_2) \in \alpha$. Equivalently, the pair (x_1, x_2) is proximal iff $\overline{T(x_1, x_2)} \cap \Delta_X \neq \varnothing$, iff there exists $p \in S_T$ such that $px_1 = px_2$ ⟦the orbit closure $\overline{T(x_1, x_2)}$ of (x_1, x_2) in the flow $\mathcal{X} \times \mathcal{X}$ equals the set $\{(px_1, px_2) : p \in S_T\}$; to see this, show that the action of the semigroup S_T on $X \times X$ (as defined in (3.7) above) is given by $(p, (x_1, x_2)) \mapsto (px_1, px_2)$⟧. It is easy to see that the set of all proximal pairs in the flow $\mathcal{X}$ is given by

$$P_{\mathcal{X}} := \bigcap_{\alpha \in \mathbf{U}_X} \overline{T\alpha}.$$

A flow $\mathcal{X}$ is called *proximal* whenever $P_{\mathcal{X}} = X \times X$, i.e., all pairs of points are proximal. An example of a proximal flow is the $\mathbb{Z}$-flow defined by the homeomorphism $f : \exp(2\pi i\theta) \mapsto \exp(2\pi i\theta^2)$ of the unit circle $\mathbb{T}$ in $\mathbb{C}$ ⟦all points tend to 1 under the iterates of f, hence come arbitrarily close to each other⟧.

This example is not a minimal flow. It can be shown that an Abelian group T admits no non-trivial proximal minimal flows. An easy example of a proximal minimal flow is obtained as follows: let T be the group generated by the homeomorphism f of $\mathbb{T}$ defined above and a rotation of $\mathbb{T}$ of the form $z \mapsto az$ with $a \in \mathbb{T}$ not a root of unity; then $\mathbb{T}$ is proximal (due to f) and minimal (due to the rotation) under the natural action of T.

A pair of points (x_1, x_2) in a flow $\mathcal{X}$ is said to be a *distal pair* whenever either it is not proximal or $x_1 = x_2$. Equivalently, a pair (x_1, x_2) is distal iff $x_1 = x_2$ or $\overline{T(x_1, x_2)} \cap \Delta_X = \varnothing$. So the set of distal pairs is $(X \times X \setminus P_X) \cup \Delta_X$; consequently, if the pair (x_1, x_2) is both distal and proximal then $x_1 = x_2$. A flow $\mathcal{X}$ is called *distal* whenever all pairs of points are distal, that is, whenever $P_{\mathcal{X}} = \Delta_X$.

4.8. REMARKS. As to examples of distal flows, it is easy to prove directly that every equicontinuous flow is distal; see also (4.9) below. In fact, it was conjectured for some time that every distal *minimal* flow is equicontinuous. This conjecture was reinforced by the fact that many properties of equicontinuous flows also hold for distal ones. We shall mention some of those properties:

1. A distal flow is pointwise almost periodic (i.e., all orbit closures are minimal). ⟦Let $\mathcal{X}$ be a distal flow and consider a point $x \in X$. Let u be an arbitrary idempotent in a minimal left ideal of S_T. Then (x, ux) is a proximal pair: $px = pux$ for $p = u$. As all pairs in X are distal it follows that $x = ux$, i.e., x is an almost periodic point.⟧

2. Every equicontinuous flow has the following property: if for every pair of non-empty open subsets U and V of X there exists $t \in T$ such that $tU \cap V \neq \varnothing$ (in that case the flow $\mathcal{X}$ is said to be (topologically) *ergodic*) then $\mathcal{X}$ is already minimal. ⟦Let $x, y \in X$ and consider $U \in \mathbf{N}_y$. There exists $\beta \in \mathbf{U}_X$ with $(\beta^{\leftarrow} \circ \beta)[y] \subseteq U$. By equicontinuity there is $V \in \mathbf{N}_x$ such that $tV \subseteq \beta[tx]$ for all $t \in T$. Now select $t' \in T$ with $t'V \cap \beta[y] \neq \varnothing$. Then $t'x \in (\beta^{\leftarrow} \circ \beta)[y] \subseteq U$. So $Tx \cap U \neq \varnothing$. This shows that $y \in \overline{Tx}$.⟧
It is possible to show *that also for distal flows it is true that topological ergodicity implies minimality*. ⟦If X has a countable base then topological ergodicity implies,

via a Baire category argument, that there is a point with dense orbit. So by 1 above X is the orbit closure of an almost periodic point. Via a clever inverse limit argument, Ellis was able to reduce the non-metric case to the metric case; see ELLIS [1978].⟧ One may wonder whether this is important at all; it is: see (5.3) below.

3. There are examples of distal minimal flows (both in the cases $T = \mathbb{Z}$ and $T = \mathbb{R}$) that are not equicontinuous; see FURSTENBERG [1963] and AUSLANDER, GREEN and HAHN [1963].

The difference between "distal" and "equicontinuous", but also the similarity, can be clearly seen if one compares the following theorem with (4.2) above. Note that by these theorems and the examples referred to in (4.8)3 above there are minimal flows such that not all members of the Ellis semigroup are continuous maps of the phase space into itself.

4.9. THEOREM. *Let $\mathcal{X}$ be a flow. The following conditions are equivalent:*

(i) *$\mathcal{X}$ is distal;*

(ii) *$E(\mathcal{X})$ is a group.*

If these conditions are fulfilled then every element of $E(\mathcal{X})$ is a bijection of X onto itself.

□ (Outline) (i)⇒(ii): First note that the flow $\mathcal{X}^X$ is distal ⟦it is a product of distal flows⟧. In view of (4.8)1 it follows that the flow $\mathcal{E}(\mathcal{X})$ — an orbit closure in $\mathcal{X}^X$ — is minimal. So $E(\mathcal{X})$ is a minimal left ideal in the semigroup $E(\mathcal{X})$. Hence by (3.6)3(d) it is sufficient to show that $E(\mathcal{X})$ contains only one idempotent. In the proof of (4.8)1 it is shown that for every idempotent u in any minimal left ideal of S_T on has $ux = x$. As every idempotent ξ in the semigroup $E(\mathcal{X})$ is the image of such a u under the canonical morphism from S_T onto $E(\mathcal{X})$, it follows from the definition of the action of S_T on X that $\xi(x) = ux = x$ for every $x \in X$. So $\xi = \mathrm{id}_X$.

(ii)⇒(i): If $E(\mathcal{X})$ is a group then it follows from the definition of the action of S_T on X that for every $p \in S_T$ there exists $p' \in S_T$ such that $pp'x = p'px = x$ for all $x \in X$. Now let $x_1, x_2 \in X$ be an arbitrary pair of points in X. In order to show that it is a distal pair, assume that there exists $p \in S_T$ such that $px_1 = px_2$. With p' as above we now get $x_1 = p'px_1 = p'px_2 = x_2$. This shows that (x_1, x_2) is a distal pair. □

It is quite straightforward to show that if $\varphi: T \to S$ is an enveloping semigroup of T with S (algebraically) a group, then the corresponding flow $\mathcal{S}$ is distal ⟦it is an easy exercise to show that $E(\mathcal{S})$ is isomorphic to S; now apply (4.9)⟧. It is a standard result that there exists a "largest" enveloping semigroup $\varphi: T \to S$ of T with the property that S is a group; the definition is just like that of the Bohr compactification, replacing at all places "compact topological group" by "compact right semitopological *group*". Denote this compactification of T by $\gamma_T: T \to gT$. Now the following should be clear (cf. (4.4) and (4.5) above):

4.10. THEOREM. *Let $\mathcal{X}$ be a distal flow. The following conditions are equivalent:*

(i) *$\mathcal{X}$ is distal;*

(ii) *There is a continuous action* $\breve{\pi}$ *of* gT *on* X *that "extends" the action of* T*:*

$$\breve{\pi}(\gamma_T(t), x) = tx \quad \textit{for} \quad (t, x) \in T \times X.$$

4.11. COROLLARY. *Every distal minimal flow is isomorphic with a flow of the form* $g\mathcal{T}/H$ *for some closed subgroup* H *of* gT*, which is unique up to conjugacy.*

This corollary implies that the flow $b\mathcal{T}$ is a *universal distal minimal flow.* As such it is *unique*: any distal minimal flow with the property that all distal minimal flows are its factor is isomorphic with $g\mathcal{T}$ ⟦the proof is the same as in the final remark of (4.5) above).

4.12. QUESTIONS.

1. I know of no systematic study of the compactification $\gamma_T : T \to gT$ of T. How can the groups T be characterized for which the compactifications $\alpha_T : T \to bT$ and $\gamma_T : T \to gT$ are equivalent (i.e., there is a homeomorphism — automatically an isomorphism of groups — $\psi : bT \to gT$ such that $\gamma_T = \psi \circ \alpha_T$)? This are just the groups for which every distal minimal flow is equicontinuous! I can prove that such a group cannot contain a copy of $\mathbb{Z}$ or $\mathbb{R}$ as a closed cocompact subgroup. (Some aspects of this question can be found in the book BERGLUND, JUNGHENN and MILNES [1989].)

2. There are other classes of minimal flows that admit universal objects. Universal objects for certain classes of flows can most easily be treated in the context of *pointed flows* or *ambits.* An ambit $(\mathcal{X}, x)$ consists of a flow $\mathcal{X}$ and a distinguished point $x \in X$ (the *base point*) with a dense orbit in $\mathcal{X}$. If $(\mathcal{X}, x)$ and $(\mathcal{Y}, y)$ are ambits then there is at most one *base-point-preserving* (bpp) morphism from $(\mathcal{X}, x)$ to $(\mathcal{Y}, y)$, i.e., a morphism $\varphi : \mathcal{X} \to Y$ with $\varphi(x) = y$ ⟦this equality determines φ completely on the dense subset Tx of X⟧. If $\mathbf{K}$ is a class of ambits then a *universal*$\mathbf{K}$*-ambit* is an ambit $(\mathcal{A}_\mathbf{K}, a_\mathbf{K})$ of which every member of $\mathbf{K}$ is a factor under a bpp morphism. If such a universal $\mathbf{K}$-ambit exists then it is unique up to isomorphism ⟦the identity mapping is its only endomorphism⟧. For a ("the", by unicity) universal $\mathbf{K}$-ambit $(\mathcal{A}_\mathbf{K}, a_\mathbf{K})$ the mapping $t \mapsto ta_\mathbf{K} : T \to A_\mathbf{K}$ is a compactification of T. Thus, to certain properties for ambits there correspond compactifications of T, namely, the compactifications defined by the universal ambits for the classes of all ambits with those properties. One can now study the relationship between the various compactifications of T so obtained, and ask questions like the above: how can a topological group T be characterized for which the compactifications corresponding to two specific dynamical properties coincide? This question can also be put in terms of function algebras; see the paper AUSLANDER and HAHN [1967], or Sections IV.4 and IV.5 in DE VRIES [1992]; cf. also BERGLUND, JUNGHENN and MILNES [1989]. We refer the reader to these references for specific examples.

5. Extensions of minimal flows

In this section we discuss a theorem that describes "how much" a distal minimal flow is different from an equicontinuous one. The metrizable case was proved in

FURSTENBERG [1963], and the non-metrizable case in ELLIS [1978]; both publications dealt essentially with the situation that $\mathcal{Y} = (\star)$ (the so-called "absolute case"). The "relative case" (with arbitrary minimal $\mathcal{Y}$) with $\mathcal{X}$ an inverse limit of metrizable flows was proved in ELLIS [1969], and in its general form it was established in MCMAHON and WU [1981]. The theorem had an enormous influence on the further development of topological dynamics: not only people began to search for generalizations to other classes of minimal flows (e.g., the "point-distal" ones; see (5.7)2 below), but also the main attention shifted from flows to *extensions* of minimal flows (cf. the titles of the books BRONŠTEĬN [1979] and AUSLANDER [1988]).

In particular, so-called *relative* properties were introduced. In practice, a relative property is nothing but a property for morphisms; as such it has to be distinguished from an *absolute* property, which is a property for flows. The philosophy behind this usage is that when an extension $\varphi : \mathcal{X} \to Y$ has a certain property then it makes often sense (not always, but it did for the first few cases that were studied in the 60's) to say that $\mathcal{X}$ has that property *relative* $\mathcal{Y}$; usually, this means that fibers of φ have a certain property. Relative properties are applicable to flows: every flow can be seen as an extension; namely, the flow $\mathcal{X}$ is seen as the obvious extension $\mathcal{X} \longrightarrow (\star)$. If this extension has the relative property (P_{rel}) then the flow $\mathcal{X}$ is said to have the corresponding absolute property (P_{abs}). We shall illustrate this usage for the properties "distal" and "equicontinuous", the absolute versions of which have been defined already in (4.7) and (4.1), respectively.

5.1. DEFINITIONS. An extension of flows $\varphi : \mathcal{X} \to Y$ is said to be *distal* whenever every pair of points $x_1, x_2 \in X$ with $\varphi(x_1) = \varphi(x_2)$ is distal; equivalently, φ is distal iff $P_{\mathcal{X}} \cap R_\varphi = \Delta_X$.

Clearly, a flow $\mathcal{X}$ is distal iff the extension $\mathcal{X} \to (\star)$ is distal according to this definition $[\![X$ is a full fiber$]\!]$.

Similarly, an extension $\varphi : \mathcal{X} \to Y$ is called *proximal* whenever every pair of points $(x_1, x_2) \in X$ with $\varphi(x_1) = \varphi(x_2)$ is proximal, that is, whenever $R_\varphi \subseteq P_{\mathcal{X}}$. Clearly, a flow $\mathcal{X}$ is proximal iff the extension $\mathcal{X} \to (\star)$ is proximal. (In a certain "dynamical" sense, proximal extensions are close to isomorphisms: two different points in any one of its fibers can "in the long run" not well be distinguished from each other: the pair of points shrinks to one point.

If fibers shrink not pairwise but as a whole to one point then the extension is called *highly proximal*; this property turns out to be equivalent with *irreducibility* of the morphism as a continuous mapping. It is well known that irreducible mapping "behave like" homeomorphisms. Therefore, there have been made attempts to base a classification of minimal flows on the following concept of equivalence: two minimal flows are "hp-equivalent" whenever they have highly proximal extensions with a common domain; see AUSLANDER and GLASNER [1977]. The same kind of idea is used in ADLER and MARCUS [1979] for the classification of so-called subshifts of finite type (topological Markov chains).)

An extension of flows $\varphi : \mathcal{X} \to Y$ is said to be *equicontinuous* whenever the (uniform) equicontinuity condition holds only on fibers of φ:

$$\forall \alpha \in \mathbf{U}_X \; \exists \beta \in \mathbf{U}_X \; : \quad T\beta \cap R_\varphi \subseteq \alpha.$$

Clearly, a flow $\mathcal{X}$ is equicontinuous iff the extension $\mathcal{X} \to (\star)$ is equicontinuous. (Authors using the term "almost periodic flow" where we use "equicontinuous flow" usually call an equicontinuous extension an "almost periodic extension".)

For the proper formulation of the following theorem we should define what we understand by a *transfinite composition of extensions.* However, the reader who has some experience with inverse limits will have no difficulties in supplying the necessary details of the definition, and for the others the intuitive meaning of this term will no doubt suffice.

5.2. THEOREM (Furstenberg). *Every distal extension $\varphi : \mathcal{X} \to Y$ of minimal flows is a transfinite composition of equicontinuous extensions.*

□ It is rather straightforward to show that φ can be factorized as $\varphi = \zeta_\infty \circ \psi_\infty$, where ζ_∞ is an infinite composition of equicontinuous extensions and ψ_∞ has no non-trivial equicontinuous factors. (We say that an extension ζ'' is a *factor* of the extension φ' whenever $\varphi' = \zeta' \circ \psi'$ for some extension ψ'. An extension is said to be *trivial* whenever it is injective: then it is an isomorphism.) Indeed, if φ has no non-trivial equicontinuous factor then we can take $\psi_\infty = \varphi$ and $\zeta_\infty = \mathrm{id}_Y$. And if φ has such a factor, then $\varphi = \zeta_1 \circ \psi_1$ with ζ_1 an equicontinuous extension of $\mathcal{Y}$. Then apply the same procedure to ψ_1, etc. By transfinite induction, using inverse limits when one arrives at limit ordinals, one can go on untill (necessarily!) the process stops.

$$\begin{array}{ccccccccc}
\mathcal{X} & = & \cdots & = & \mathcal{X} & = & \mathcal{X} & = & \mathcal{X} \\
\psi_\infty \downarrow & & & & \downarrow \psi_2 & & \downarrow \psi_1 & & \downarrow \varphi \\
\mathcal{Z} & \longrightarrow & \cdots & \longrightarrow & \mathcal{Z}_2 & \underset{\zeta_2}{\longrightarrow} & \mathcal{Z}_1 & \underset{\zeta_1}{\longrightarrow} & \mathcal{Y}
\end{array}$$

In order to complete the proof of the theorem it is sufficient to show that the extension ψ_∞ is an isomorphism. This can be done in several ways. I shall outline a proof which consists of two main steps, each of which is a special case of a more general result. The two steps are:

(I) If an extension of minimal flows $\psi : \mathcal{X} \to Z$ has no non-trivial equicontinuous factors and if, in addition, ψ is a distal extension, then ψ is a so-called *weakly mixing* extension, that is, the subflow $\mathcal{R}_\psi$ of $\mathcal{X} \times \mathcal{X}$ is ergodic.

(II) An extension of minimal flows $\psi : \mathcal{X} \to Z$ which is both distal and weakly mixing is an isomorphism.

Obviously, these two results complete the proof of the theorem. □

In the following remarks I shall give some comments on the proofs of the statements (I) and (II) above. After that I shall say something about the "structure" of equicontinuous extensions, and formulate some open problems.

5.3. REMARK. Let me start with statement (II). In the case that $\mathcal{X}$ is a distal flow (then ψ certainly is a distal extension), statement (II) follows easily from (4.8)2: in that case, the flow $\mathcal{X} \times \mathcal{X}$ is distal, so its subflow $\mathcal{R}_\psi$ is distal as well; if ψ is a weakly mixing extension then $\mathcal{R}_\psi$ is ergodic, hence it is minimal by (4.8)2. As Δ_X is a closed invariant set in $\mathcal{R}_\psi$ it follows that $R_\psi = \Delta_X$, i.e., ψ is an isomorphism. The reason that this argument doesn't work for general distal extensions of minimal flows is that for a distal extension ψ it is in general not the case that $\mathcal{R}_\psi$ is a distal flow.

Fortunately, a distal extension $\psi : \mathcal{X} \to Z$ of minimal flows has the following property: R_ψ is a union of minimal sets. ⟦The mapping $\theta : (x_1, x_2) \mapsto \psi(x_1) = \psi(x_2)$: $R_\psi \to Z$ is a well-defined extension of flows, and it is not difficult to see that it is a distal extension. Now let $(x_1, x_2) \in R_\psi$, and let $z := \theta(x_1, x_2)$. By (3.8) there exists an idempotent u in a minimal left ideal of S_T such that $uz = z$. It follows that $u(x_1, x_2)$ and (x_1, x_2) have the same image under θ, so they form a distal pair in the flow $\mathcal{R}_\psi$. But they also form a proximal pair ⟦for $p := u$ we have $p\big(u(x_1, x_2)\big) = p(x_1, x_2)$⟧, so these points are equal. So by (3.8) the orbit closure of (x_1, x_2) is minimal.⟧

Now suppose that the extension $\psi : \mathcal{X} \to Z$ of minimal flows is distal and weakly mixing. If we assume, in addition, that X has a countable base, then ergodicity of the flow $\mathcal{R}_\psi$ implies that this flow has a point with dense orbit ⟦cf. the argument in (4.8)2⟧. By the above, all orbit closures in this flow are minimal. Hence $\mathcal{R}_\psi$ is minimal, and as above this implies that ψ is an isomorphism.

If X doesn't have a countable base then a generalisation of the construction used by Ellis in ELLIS [1978] (see also (4.8)2 above) settles the general case of statement (II). This generalisation was established essentially in MCMAHON and WU [1981], but for the proof of the fact that the construction works as indicated above, see DE VRIES [1991] (McMahon and Wu applied the construction in a different context). The proof of this generalization employs a construction in which morphisms of the type discussed in (2.7)5 above are used.

5.4. REMARK. The proof of statement (I) is a much more difficult task. It consists of several steps; some of these are quite straightforward, while others are rather sophisticated. We list these steps and give some indications of the proofs. (a) A *continuous invariant fibre-wise pseudometric* (CIFP) for an extension $\psi : \mathcal{X} \to Z$ is a continuous mapping ρ: $R_\psi \to \mathbb{R}^+$ satisfying the following conditions:

- $\forall z \in Z : \rho|_{\psi^{\leftarrow}[z] \times \psi^{\leftarrow}[z]}$ is a pseudometric on $\psi^{\leftarrow}[z]$.
- $\forall t \in T\ \forall (x_1, x_2) \in R_\psi$: $\rho(tx_1, tx_2) = \rho(x_1, x_2)$.

It is quite straightforward to show that if ψ has no non-trivial equicontinuous factors, then every CIFP for ψ is identically zero on R_ψ. ⟦If ρ is a CIFP then the set

$$D_\psi(\rho) := \{(x_1, x_2) \in R_\psi : \rho(x_1, x_2) = 0\}$$

is a closed invariant equivalence relation, and ψ factors over the flow $\mathcal{X}/D_\psi(\rho)$ as $\psi = \psi' \circ \eta$, where ψ' turns out to be equicontinuous. By assumption, ψ' is an isomorphism, so that $D_\psi(\rho) = R_\psi$.⟧

(b) A *relatively invariant measure* (RIM) for the extension $\psi : \mathcal{X} \to Z$ is a morphism of flows $\lambda: \mathcal{Z} \to M_1(\mathcal{X})$ such that for every $z \in Z$ the support of the measure $\lambda(z)$ is included in the fiber $\psi^{\leftarrow}[z]$ of z. Note that if $Z = (\star)$ then the fact that λ is equivariant with respect to the actions in $\mathcal{Z}$ and $M_1(\mathcal{X})$ implies that

$t\lambda(\star) = \lambda(\star)$, i.e., $\lambda(\star)$ is an invariant measure for $\mathcal{X}$. If λ is a RIM for ψ then every non-empty closed invariant subset N of R_ψ defines a CIPF, namely by

$$\rho_N(x_1, x_2) := \lambda(\psi(x_1))\Big(N[x_1]\Delta N[x_2]\Big) \quad \text{for} \quad (x_1, x_2) \in R_\psi,$$

where Δ denotes the symmetric difference operator. Thus, if there is a RIM then there is a large supply of CIFP's, and one can imagine that the condition that they are all zero implies something for the flow $\mathcal{R}_\psi$. And indeed, we have:

If ψ is an *open* extension of minimal flows which admits a *relatively invariant measure* and every CIFP for ψ is zero, then ψ is weakly mixing. A proof of this statement is essentially included in McMAHON [1978].

(c) It can be shown that a distal extension of minimal flows is an open mapping. If we knew already that a distal extension of minimal flows has a RIM (one of the consequences of Furstenbergs theorem is that it has, indeed, a RIM; an independent proof would be of great interest!) then (I) would follow directly from (a) and (b) above. This problem is, literally, circumvented by the following result. It is due to GLASNER [1975]; in the proof a construction is used that employs flows and morphisms of the type described in (2.7)4 above.

5.5. THEOREM. *Let $\psi : \mathcal{X} \to Z$ be an extension of minimal flows. Then there is diagram of the form*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\sigma} & \mathcal{X} \\ \psi' \downarrow & & \downarrow \psi \\ \mathcal{Z}' & \xrightarrow{\tau} & \mathcal{Z} \end{array}$$

where σ and τ are proximal extensions and ψ' is an extension of minimal flows with a RIM. Moreover, the flow $\mathcal{X}'$ is obtained as a minimal subflow of $\mathcal{X} \times \mathcal{Z}'$, and σ and ψ' are the restrictions to X' of the projections of $X \times Z'$ onto X and Z', respectively.

Using this result for a *distal* extension ψ one gets in the above diagram a distal extension ψ' which has the additional property that $\sigma \times \sigma$ maps the set $R_{\psi'}$ *onto* the set R_ψ. Then $\sigma \times \sigma$ is a morphism of flows, from $\mathcal{R}_{\psi'}$ onto $\mathcal{R}_\psi$, so in order to show that $\mathcal{R}_\psi$ is ergodic it is sufficient to show that $\mathcal{R}_{\psi'}$ is ergodic.

Now recall that *every distal extension of minimal flows is open*. So ψ' is an open extension with a RIM. If ψ has no non-trivial equicontinuous factors, then we know that every CIFP is zero on $\mathcal{R}_\psi$. From this it follows rather easily (using that τ is a proximal extension) that for ψ' all CIFP's are zero. Consequently, ψ' is weakly mixing, and this implies that the flow $\mathcal{R}_{\psi'}$, and therefore the flow $\mathcal{R}_\psi$ as well, is ergodic: the extension ψ is weakly mixing. This concludes the proof of (I). (For other proofs, see, e.g., VEECH [1977] and the references given there.)

5.6. REMARKS. A number of important research themes in topological dynamics show up in the above proof. For example:

- Under which additional conditions is an extension of minimal flows with no non-trivial equicontinuous factors weakly mixing?
- Which extensions can be "lifted" (in the sense of theorem (5.5) above) to extensions that are more or less well-behaved (e.g., because they belong to a class for which the previous question can be answered positively) by means of extensions that are in some sense "close to" isomorphisms (like proximal extensions)?
- A technical fact that was not mentioned explicitly in the above proof is that in the situation of a diagram as in theorem (5.5) with ψ a distal and τ a proximal extension the set

$$R_{\psi\tau} := \{(x, z') \in X \times Z' : \psi(x) = \tau(z')\}$$

is minimal. (This is often expressed by saying that ψ and τ are *disjoint*; notation: $\psi \perp \tau$.) What combinations of extensions have this property?

In this introductory survey I cannot go further into details about these topics: for each of them there is a quite extensive literature, the understanding of which requires much more technical machinery. See, e.g., VAN DER WOUDE [1986] or DE VRIES [1992] for further references.

One may wonder whether Furstenberg's theorem gives much additional insight in distal extensions: for example, it reduces questions about such extensions to questions about equicontinuous extensions. But how well-understood are the latter? Reasonably well. In fact, an equicontinuous extension of minimal flows is a *fiber bundle*; see BRONŠTEĬN [1979, 3.17.1–4].

This means the following. If $\varphi : \mathcal{X} \to Y$ is an equicontinuous extension of minimal flows then φ is a factor of an extension $\xi : \mathcal{Z} \to Y$ with the following property: there is a group G of automorphisms of $\mathcal{Z}$ which can be given a compact Hausdorff topology with which it is a topological group, acting continuously on Z in such a way that the G-orbits are just the fibers of ξ. (It is not difficult to show that this implies that ξ is what in HUSEMOLLER [1966] is called a *principal G-bundle*. In topological dynamics such an extension ξ is usually called a *group extension*, though I prefer to call it a *principal* group extension.)

One can apply the same construction to a distal extension $\varphi : \mathcal{X} \to Y$. Then one gets exactly the same as above for the equicontinuous case, except that now the topological group G is not compact. (Now ξ may be called a *weak* group extension; in ELLIS [1967] the term "group-like" extension is used. One can try to use the same topology as in the equicontinuous case, but then G turns out to be not a topological group.)

There is an interesting parallel to the theorems (4.2) and (4.9). In fact these theorems are the absolute cases of the above. Indeed, if we apply the above construction to the "absolute" case (i.e., $\mathcal{Y} = (\star)$) then it turns out that for the extension $\xi : \mathcal{Z} \to (\star)$ we get precisely $\mathcal{E}(\mathcal{X}) \to (\star)$. By (4.2) and (4.9), in the equicontinuous, respectively, distal case, $E(X)$ is a topological group, respectively, a group, so the space $E(X)$ consists of one orbit of an automorphism group of the flow $\mathcal{E}(\mathcal{X})$, namely, the group $E(X)$ acting on itself by means of right translations.

At this place I cannot go into details of the proofs of the above statements, but let me make a few remarks of a more general nature. An important role is played by *algebraic* characterizations of properties of extensions, of which (3.8), (4.2) and (4.9) are special ("absolute") cases. For the absolute case the universal enveloping semigroup $\varphi_T: T \to S_T$, the semigroup structure of S_T, and the action of S_T on the phase space of a flow play a central role. These ingredients are also used in the analysis of extensions.

First of all, it turns out to be convenient to select an arbitrary minimal left ideal M in S_T and an idempotent u in M. Every minimal flow $\mathcal{X}$ can be viewed as an ambit by selecting an arbitrary base point x'; by replacing x' by $x := ux'$ we may assume that the base points in all minimal ambits are u-invariant. For such an ambit $(\mathcal{X}, x)$ with $ux = x$, the *Ellis group* $\mathbb{G}(\mathcal{X}, x)$ is defined as

$$\mathbb{G}(\mathcal{X}, x) := \{p \in uM : px = x\}.$$

As uM is a group, it follows that $\mathbb{G}(\mathcal{X}, x)$ is a group as well. To give an idea how these Ellis groups can be used, consider an extension $\varphi : \mathcal{X} \to Y$ of minimal flows. Let $x = ux \in X$ and $y = \varphi(x)$; then $uy = y$. Put $A := \mathbb{G}(\mathcal{X}, x)$ and $F := \mathbb{G}(\mathcal{Y}, y)$. Then it is easy to see that $A \subseteq F$. The following characterizations can be obtained:

- φ is proximal iff $A = F$;
- φ is distal iff $\forall p \in M : \varphi^{\leftarrow}[py] = pFx$;
- φ is equicontinuous iff φ is distal and $H(F) \subseteq A$.

Here $H(F)$ is defined in the following way. By a certain procedure (using auxiliary flows of the type considered in (2.7)5) so-called *τ-topologies* can be defined on the u-invariant parts of all flows. In particular, uM carries such a τ-topology. It can be shown that the Ellis groups of minimal flows are just the τ-closed subgroups of uM. Every τ-closed subgroup F of uM is a compact T_1-space with a separately continuous multiplication and continuous inversion. Usually it is not a Hausdorff space, and the smallest normal subgroup of F defining a Hausdorff quotient group is denoted by $H(F)$; it turns out that $H(F) = \bigcap\{cl_\tau(V) : V \text{ a } \tau\text{-nbd of } u\}$.

It is now also easy to give a characterization of a principal group extension: i n the above notation, φ is a principal group extension iff φ is distal, A is a normal subgroup in F, and $H(F) \subseteq A$; in that case the automorphism group of $\mathcal{X}$ of which the orbits are the fibers of φ can be taken to be the compact Hausdorff topological group F/A, acting on X by the rule $[g]_A.px = pg^{-1}x$ (here $[g]_A$ is the canonical image of $g \in F$ in F/A; note that $Mx = X$ by minimality of $\mathcal{X}$, so that every element x' of X can be represented as $x' = px$ for some $p \in M$; then $pg^{-1}x$ is independent of the choice of p with $px = x'$). For details, see GLASNER [1976], Section IX.2, or DE VRIES [1992], Sections V.4 and V.5.

5.7. QUESTIONS. Let me conclude with a couple of questions related to Theorem 5.2 above.

1. If $\mathcal{X}$ is a distal minimal flow then the degree of "non-equicontinuity" of X can be expressed by the least cardinal number λ such that the extension $\mathcal{X} \to (\star)$ is a (transfinite) composition of λ equicontinuous extensions. How can distal minimal flows be characterized that have a finite degree? Or those that have degree 2? In

IHRIG and McMAHON [1984] a necessary condition for having a finite degree can be found for the case that T is an Abelian group. For the non-Abelian case, see IHRIG, McMAHON and WU [1988]. In the latter paper there is the following open problem in this connection: can a non-Abelian group have a continuous and distal *free* action on a compact manifold? ("Free" means that if $tx = x$ for some point x then $t = e$.)

2. (Related to the remarks in (5.6) above.) An extension $\varphi : \mathcal{X} \to Y$ of flows is called *point-distal* whenever there is a point $x \in X$ which has a dense orbit in $\mathcal{X}$ and which is distal from all points in $\varphi^{\leftarrow}[\varphi x]$, i.e., $P_{\mathcal{X}} \cap \varphi^{\leftarrow}[\varphi x] = \{x\}$. A flow $\mathcal{X}$ is point-distal whenever there is a point with dense orbit that is distal from all points in X (such a point is called a *distal point* in $\mathcal{X}$). A point-distal flow turns out to be minimal. In VEECH [1970] the following generalization of Furstenberg's theorem was obtained: If $\mathcal{X}$ is a point-distal minimal flow on a compact *metric* space in which the set of distal points is residual then $\mathcal{X}$ is a factor under a highly proximal extension of a minimal flow $\mathcal{X}_\infty$ such that the extension $\mathcal{X}_\infty \to (\star)$ is a transfinite composition of alternately highly proximal and equicontinuous extensions (such an infinite composition is called a *strictly HPI-extension*; here the "HP" stands, of course, for "highly proximal", and the "I" stands for "isometric", an old term for "equicontinuous"). The relative form of this result was obtained in ELLIS [1973]; there it was also shown that the additional condition that the distal points should form a residual set can be left out from the absolute (= Veech's) result. Ellis' version of Veech's theorem reads as follows:

If $\varphi : \mathcal{X} \to Y$ is a point-distal extension of minimal flows and X is metrizable, then there is a highly proximal extension $\eta_\infty : \mathcal{X}_\infty \to \mathcal{X}$ such that the composition $\varphi \circ \eta_\infty$ is a strictly HPI-extension (so φ is a factor of a strictly HPI-extension under the highly proximal extension η).

The proof needs ingredients that were mentioned in (5.6) above. In fact, one gets a diagram like that in the proof of (5.2), but with extensions $\eta_\kappa : \mathcal{X}_{\kappa+1} \to \mathcal{X}_\kappa$ instead of the equalities in the top row. In fact, the horizontal row of squares is obtained by alternately taking an equicontinuous factor (then at the top row one gets an equality, and in the bottom row an equicontinuous extension) and making a square like in theorem (5.5), but with σ and τ highly proximal and ψ' an *open* extension. This process stops at a certain stage, and one ends up with a transfinite composition of highly proximal extensions in the top row (which is again highly proximal), a strictly HPI-extension at the bottom, and an *open* extension ψ_∞ which has no non-trivial equicontinuous factors, and which turns out to be point-distal. In the case that all spaces under consideration are metrizable it can be shown that ψ_∞ is an isomorphism. Ellis could prove the above result, by a slightly different method, for the case that the flows under consideration are inverse limits of flows on metrizable phase spaces; if the group T is locally compact and σ-compact then *all* flows are such inverse limits.)

Later, in McMAHON and NACHMAN [1980], it was proved that the metrizability condition can be left out in the case that $\mathcal{Y} = (\star)$. An unpublished result by Jaap van der Woude (now a computer scientist, so he will never publish it) states that one can obtain a similar result if φ is an open point-distal extension such that there is a distal point x in X for which $\varphi(x)$ has a countable local base in Y. See DE VRIES [1992], VI(4.47). (In GLASNER [1976] there are some forms of Veech's original result where the condition that $\mathcal{X}$ is point-distal is weakened.) Conclusion: the most general form (relative, non-metric) of "Veech's structure theorem for point-distal extensions" is still open.

References

AARTS, J. M.
[1988] The structure of orbits in dynamical systems. *Fund. Math.*, **129**, 39–58.

AARTS, J. M. and R. J. FOKKINK.
[1991] The classification of solenoids. *Proc. Amer. Math. Soc.*, **111**, 1161–1163.

AARTS, J. M. and M. MARTENS.
[1988] Flows on one-dimensional spaces. *Fund. Math.*, **131**, 53–67.

AARTS, J. M. and L. G. OVERSTEEGEN.
[19∞] Flowbox manifolds. *Trans. Amer. Math. Soc.*

ADLER, R. L. and B. MARCUS.
[1979] *Topological entropy and equivalence of dynamical systems.* Memoirs Amer. Math. Soc. 219. American Mathematical Society.

AUSLANDER, J.
[1988] *Minimal flows and their extensions.* North-Holland, Amsterdam.

AUSLANDER, J. and S. GLASNER.
[1977] Distal and highly proximal extensions of minimal flows. *Indiana Math. J.*, **26**, 731–749.

AUSLANDER, J. and F. HAHN.
[1967] Point transitive flows, algebras of functions and the Bebutov system. *Fund. Math.*, **60**, 117–137.

AUSLANDER, L., L. GREEN, and F. HAHN.
[1963] *Flows on homogeneous spaces.* Annals of Mathematics Studies 53. Princeton Univ. Press.

BALCAR, B. and A. BLASZCZYK.
[1990] On minimal systems on Boolean algebras. *Comment. Math. Univ. Carolin.*, **31**, 7–11.

BALCAR, B. and A. DOW.
[1991] Dynamical systems on compact extremally disconnected spaces. Preprint, 1991.

BERGELSON, V. and N. HINDMAN.
[1990] Nonmetrizable topological dynamics and Ramsey theory. *Trans. Amer. Math. Soc.*, **320**, 293–320.

BERGLUND, J. F., H. D. JUNGHENN, and P. MILNES.
[1989] *Analysis on semigroups.* John Wiley & Sons, New York.

BHATIA, N. P. and G. P. SZEGÖ.
[1970] *Stability theory of dynamical systems.* Springer-Verlag, Berlin etc.

BIRKHOFF, G. D.
[1927] *Dynamical systems.* American Mathematical Society, New York.

BOURBAKI, N.
[1966] *Elements of mathematics: general topology.* Hermann, Paris, and Addison-Wesley, Reading (Mass.).

BRONŠTEĬN, I. U.
[1979] *Extensions of minimal transformation groups.* Sijthoff and Noordhoff, Alphen aan den Rijn.

VAN DOUWEN, E. K.
[1990] The maximally totally bounded group topology on G and the biggest minimal G-space, for Abelian groups G. *Topology Appl.*, **34**, 69–91.

ELLIS, R.
[1967] Group-like extensions of minimal sets. *Trans. Amer. Math. Soc.*, **127**, 125–135.
[1969] *Lectures in topological dynamics.* W. A. Benjamin, New York.
[1973] The Veech structure theorem. *Trans. Amer. Math. Soc.*, **186**, 203–218.
[1978] The Furstenberg structure theorem. *Pacific J. Math.*, **76**, 345–349.

FOKKINK, R.
[1991] *The structure of trajectories.* PhD thesis, Technical University Delft.

FURSTENBERG, H.
[1963] The structure of distal flows. *Amer. J. Math.*, **85**, 477–515.

GAIT, J.
[1972] Transformation groups with no equicontinuous minimal set. *Compositio Math.*, **25**, 87–92.

GLASNER, S.
[1975] Relatively invariant measures. *Pacific J. Math.*, **58**, 393–410.
[1976] *Proximal flows.* Springer-Verlag, Berlin etc.

GOTTSCHALK, W. H. and G. A. HEDLUND.
[1955] *Topological dynamics.* American Mathematical Society, Providence.

HINDMAN, N. and J. PYM.
[1984] Free groups and semigroups in βN. *Semigroup Forum*, **30**.

HIRSCH, M. W.
[1984] The dynamical systems approach to differential equations. *Bull. Amer. Math. Soc.*, **11**, 1–64.

HUSEMOLLER, K.
[1966] *Fibre bundles.* McGraw Hill, Inc. (2nd edition: Springer-Verlag, Berlin, etc.).

IHRIG, E. and D. C. MCMAHON.
[1984] On distal flows of finite codimension. *Indiana Univ. Math. J.*, **33**, 345–351.

IHRIG, E., D. C. MCMAHON, and T.-S. WU.
[1988] Distal flows of non-Abelian groups with finite codimension. In *Dynamical systems*, J. C. Alexander, editor, pages 329–341. Springer-Verlag, Berlin etc.

LEFSCHETZ, S.
[1957] *Differential equations: geometric theory.* Interscience, New York.

MCMAHON, D. C.
[1978] Relativized weak disjointness and relatively invariant measures. *Trans. Amer. Math. Soc.*, **236**, 225–237.

MCMAHON, D. C. and L. J. NACHMAN.
[1980] An intrinsic characterization for PI-flows. *Pacific J. Math.*, **89**, 391–403.

McMAHON, D. C. and T.-S. WU.

[1981] Distal homomorphisms of nonmetric minimal flows. *Proc. Amer. Math. Soc.*, **82**, 283–287.

VAN MILL, J. and G. M. REED, editors.

[1990] *Open problems in topology.* North-Holland, Amsterdam.

MORITA, K. and J. NAGATA, editors.

[1989] *Topics in general topology.* North-Holland, Amsterdam.

NEMYTSKIĬ, V. V.

[1949] Topological problems of the theory of dynamical systems. *Uspehi Mat. Nauk.*, **4**(6(34)), 91–153. (In Russian; English translation in: Amer. Math. Soc. Transl. Ser. 1, Vol.5, pp. 414–497.).

NEMYTSKIĬ, V. V. and V. V. STEPANOV.

[1960] *Qualitative theory of differential equations.* Princeton University Press, Princeton.

RUPPERT, W.

[1984] *Compact semitopological semigroups: an intrinsic theory.* Springer-Verlag, Berlin etc.

VEECH, W. A.

[1970] Point-distal flows. *Amer. J. Math*, **92**, 205–242.

[1977] Topological dynamics. *Bull. Amer. Math. Soc.*, **83**, 775–830.

DE VRIES, J.

[1991] The Furstenberg structure theorem in topological dynamics. *CWI Quarterly*, **4**, 27–44.

[1992] *Elements of topological dynamics.* Kluwer, Dordrecht.

VAN DER WOUDE, J. C. S. P.

[1986] *Topological dynamix.* CWI, Amsterdam.