Complexity of Disjoint Paths Problems in Planar Graphs

Alexander Schrijver
CWI and University, Amsterdam, The Netherlands

Let $G = (V, E)$ be a graph, and let $r_1, s_1, \ldots, r_k, s_k$ be vertices of G. The *disjoint paths problem* is the problem of finding disjoint paths $P_1, \ldots, P_k$, where P_i runs from r_i to s_i $(i = 1, \ldots, k)$.

There are several variants of this problem. "Disjoint" may mean "vertex-disjoint" or "edge-disjoint", and the graph may be directed or undirected. Each of these cases gives an NP-complete problem, as was shown by Knuth (see [2]). On the other hand, it was shown by Robertson and Seymour [7] that for each fixed k the problem is solvable in polynomial time when the graph is undirected. In the directed case, the problem is NP-complete even when fixing $k = 2$ (Fortune, Hopcroft, and Wyllie [1]).

Also when restricting ourselves to planar graphs, the problem for general k is NP-complete, as was shown by Lynch [4] and Kramer and Van Leeuwen [3]. On the other hand, there are some cases where the problem is polynomially solvable for planar graphs. Recently, Wagner and Weihe [11] showed that if all terminals are on the boundary of the infinite face and a certain parity condition is satisfied (the "Okamura-Seymour case"), the edge-disjoint undirected problem can be solved in *linear* time.

We consider some complexity results for problems in planar graphs for fixed k. With B. Reed, N. Robertson, and P.D. Seymour [5] we proved:

> For each fixed k, the vertex-disjoint undirected problem is solvable in linear time.

More generally, for each fixed k there is a linear-time algorithm for the problem of finding vertex-disjoint trees $T_1, \ldots, T_p$ in an undirected planar graph, where T_i covers a given set X_i of vertices $(i = 1, \ldots, p)$, such that $|X_1 \cup \cdots \cup X_p| \leq k$.

Our result extends a result of Suzuki, Akama, and Nishizeki [10] stating that the disjoint trees problem is solvable in linear time for planar graphs for each fixed upper bound k on $|X_1 \cup \cdots \cup X_p|$, when there exist two faces f_1 and f_2 such that each vertex in $X_1 \cup \cdots \cup X_p$ is incident with at least one of f_1 and f_2.

In fact, they showed more strongly that the problem (for nonfixed k) is solvable in time $O(k|V|)$. Indeed, recently Ripphausen, Wagner, and Weihe [6] showed that it is solvable in time $O(|V|)$.

Our proof is based on a lemma of Robertson and Seymour [8] stating that there exists a computable function $g : \mathcal{N} \longrightarrow \mathcal{N}$ with the following property:

> Let $G = (V, E)$ be an undirected plane graph, let $k \in \mathcal{N}$, let $X_1, \ldots, X_p \subseteq V$ such that $|X_1 \cup \cdots \cup X_p| \leq k$ and such that there exist vertex-disjoint trees $T_1, \ldots, T_p$ in G with $X_i \subset T_i$ for $i = 1, \ldots, p$. Moreover, let $v \in V$ be such that each closed curve C in the plane traversing v and intersecting or separating $X_1 \cup \cdots \cup X_p$ has at least $g(k)$ intersections with

G. Then there exist vertex-disjoint trees $T_1', \ldots, T_p'$ in $G - v$ such that $X_i \subseteq T_i'$ for $i = 1, \ldots, p$.

This result makes it possible to remove vertices iteratively until the graph can be decomposed into easier graphs.

For the *directed* case we showed in [9] the following:

For each fixed k there exists a polynomial-time algorithm for the vertex-disjoint directed paths problem in planar graphs.

The proof is based on cohomology over free groups with k generators. Let G be a group and let $D = (V, A)$ be a directed graph. Call two functions $\phi, \psi : A \longrightarrow G$ *cohomologous* if there exists a function $p : V \longrightarrow G$ such that for each arc $a = (u, w)$ of D one has

$$\psi(a) = p(u)^{-1}\phi(a)p(w).$$

Let Γ_k denote the free group generated by $g_1, \ldots, g_k$.

Then:

There exists a polynomial-time algorithm that for given natural number k, directed graph $D = (V, A)$ and function $\phi : A \longrightarrow \Gamma_k$, finds a function $\psi : A \longrightarrow \Gamma_k$ cohomologous to ϕ such that

$$\psi(a) \in \{1, g_1, \ldots, g_k\}$$

for each arc a (provided that such a function ψ exists).

This result is applied to an extension of the dual of the graph for which we want to solve the disjoint paths problem.

It also implies the following:

For each fixed p there exists a polynomial-time algorithm for the vertex-disjoint directed paths problem in directed planar graphs, when all terminals can be covered by the boundaries of p of the faces.

hermore, the result can be extended to finding vertex-disjoint rooted directed s with given roots and covering given sets of vertices, when all roots and given ertices can be covered by the boundaries of a fixed number of faces. We do not know if these problems are solvable in linear time.

Most of the results above can be extended to graphs embedded on any fixed surface.

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