



Explicit Symmetric Runge-Kutta-Nyström Methods for Parallel Computers

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(Received April 1994; revised and accepted May 1995)

Abstract—In this paper, we are concerned with parallel predictor-corrector (PC) iteration of Runge-Kutta-Nyström (RKN) methods in $P(EC)^mE$ mode for integrating initial value problems for the special second-order equation $\mathbf{y}''(t) = \mathbf{f}(\mathbf{y}(t))$. We consider symmetric Runge-Kutta-Nyström (SRKN) corrector methods based on direct collocation techniques which optimize the rate of convergence of the PC iteration process. The resulting PISRKN methods (parallel iterated SRKN methods) are shown to be much more efficient when they are compared to the PC iteration process applied to the Gauss-Legendre RKN correctors.

Keywords—Runge-Kutta-Nyström methods, Predictor-corrector methods, Parallelism.

1. INTRODUCTION

There exists in the literature a number of explicit Runge-Kutta-Nyström (RKN) methods for the numerical solution of the nonstiff second-order initial-value problem (IVP)

$$\frac{d^2\mathbf{y}(t)}{dt^2} = \mathbf{f}(\mathbf{y}(t)), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \quad \mathbf{y}'(t_0) = \mathbf{y}'_0, \quad t_0 \leq t \leq T, \quad (1.1)$$

where $\mathbf{y} : \mathbb{R} \rightarrow \mathbb{R}^N$, $\mathbf{f} : \mathbb{R}^N \rightarrow \mathbb{R}^N$. Methods up to order 10 can be found in [1-4]. In order to exploit the potential of vector and parallel computers, several classes of predictor-corrector (PC) methods based on correctors of RKN type have recently been considered in [5-7]. In the present paper, we propose a class of parallel PC methods on a new class of *Symmetric* RKN correctors (SRKN correctors). The new SRKN corrector method is constructed by optimizing the rate of convergence of the PC iteration process (see Section 3.2).

Having constructed optimal SRKN correctors, we apply the highly parallel PC iteration scheme. For a given order p , the resulting parallel iterated SRKN method (PISRKN method) has a larger number of processors, $p - 1$, when compared with $p/2$ for the PIRKN methods proposed in [5,7]. However, the rate of convergence of the PISRKN method is much better, so that its efficiency is increased. This increased efficiency is shown in Sections 4.1 and 4.2 where numerical results are presented by comparing the PISRKN methods with PIRKN methods and with sequential RKN methods available in the literature.

These investigations were partly supported by the University of Amsterdam, National Basis Research Program and Research Program B93-05-71.

The author is grateful to P.J. van der Houwen and B.P. Sommeijer for their help during the preparation of this paper. He is also grateful to the referees for their useful comments.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}\text{-T}\mathcal{E}\mathcal{X}$

2. SYMMETRIC RKN METHODS

In this section, we construct the class of SRKN correctors that will be used in the parallel PC iteration scheme. We will start with a fully implicit s -stage direct collocation-based RKN method (see, e.g., [8]). By using the Kronecker product denoted by \otimes , for the system of equations (1.1), this method assumes the form (cf., e.g., [7])

$$\mathbf{Y}_n = \mathbf{e} \otimes \mathbf{u}_n + h\mathbf{c} \otimes \mathbf{u}'_n + h^2(A \otimes I_N) \mathbf{F}(\mathbf{Y}_n), \quad (2.1a)$$

$$\begin{aligned} \mathbf{u}_{n+1} &= \mathbf{u}_n + h\mathbf{u}'_n + h^2(\mathbf{b}^\top \otimes I_N) \mathbf{F}(\mathbf{Y}_n), \\ \mathbf{u}'_{n+1} &= \mathbf{u}'_n + h(\mathbf{d}^\top \otimes I_N) \mathbf{F}(\mathbf{Y}_n). \end{aligned} \quad (2.1b)$$

Here, A is an s -by- s matrix, I_N is the identity N -by- N matrix, $\mathbf{b}, \mathbf{c}, \mathbf{d}$ and \mathbf{e} are s -dimensional vectors, \mathbf{e} is the vector with unit entries, \mathbf{Y}_n is the sN -dimensional stage vector with s vector components $\mathbf{Y}_{n,i}$ of dimension N corresponding to the n^{th} step, and $\mathbf{F}(\mathbf{Y}_n)$ is also the sN -dimensional vector ($\mathbf{f}(\mathbf{Y}_{n,i})$), $i = 1, \dots, s$. From now on, we assume that the components of the collocation vector $\mathbf{c} = (c_1, \dots, c_s)^\top$ are symmetrically distributed; that is, c_j are symmetric with respect to $1/2$. To be more precise, let c_1, \dots, c_m be given and less than $1/2$; then all the c_i greater than $1/2$ are available as free parameters and defined by

$$\begin{aligned} \text{for } s = 2m + 1: \quad c_{m+1+i} &= 1 - c_i \quad (i = 1, 2, \dots, m), \quad c_{m+1} = 1/2, \\ \text{for } s = 2m: \quad c_{m+i} &= 1 - c_i \quad (i = 1, 2, \dots, m). \end{aligned}$$

These RKN methods will be referred to as *SRKN methods*.

It is known that a direct collocation-based s -stage RKN method is of at least step point order $p = s$ and stage order $r = s$ (see [8]). These orders can be increased (beyond s) if the orthogonality relation

$$\int_0^1 \prod_{i=1}^s (x - c_i) x^{j-1} dx = 0 \quad (2.2)$$

is satisfied for $j \geq 1$ (cf., e.g., [8, Theorems 3.3 and 3.4; 9, p. 207]). For the SRKN methods, condition (2.2) is satisfied for $j = 1$ if s is odd (see also [10]). This is readily seen by setting $x = \xi + 1/2$, $s = 2m + 1$; then condition (2.2) takes the form

$$\int_{-1/2}^{1/2} \phi(\xi) d\xi = 0, \quad (2.3)$$

where $\phi(\xi)$ is an odd function defined by $\phi(\xi) = \xi \prod_{i=1}^m (\xi + 1/2 - c_i)(\xi - 1/2 + c_i)$. Thus we have the following simple theorem.

THEOREM 2.1. *Let p and r be the step point order and stage order, respectively, of an s -stage SRKN method; then $p = r = s$ if s is even and $p = r = s + 1$ if s is odd. ■*

This theorem leads us to restrict our considerations to the SRKN methods with an odd number of stages. SRKN methods are not A -stable (see [8]), but their stability intervals denoted by $(-\beta_{\text{corr}}, 0)$ are sufficiently large for nonstiff problems (see Section 3.3).

3. PARALLEL ITERATED SRKN METHODS

Using the SRKN method (2.1) as corrector and

$$\mathbf{Y}_n^{(0)} = (V \otimes I_N) \mathbf{Y}_{n-1}^{(m)} + \mathbf{w} \otimes \mathbf{y}_n \quad (3.1a)$$

as predictor, we arrive at the following PC iteration scheme (in $P(EC)^m E$ mode):

$$\mathbf{Y}_n^{(j)} = \mathbf{e} \otimes \mathbf{y}_n + h\mathbf{c} \otimes \mathbf{y}'_n + h^2(A \otimes I_N) \mathbf{F}(\mathbf{Y}_n^{(j-1)}), \quad j = 1, \dots, m, \quad (3.1b)$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{y}'_n + h^2(\mathbf{b}^\top \otimes I_N) \mathbf{F}(\mathbf{Y}_n^{(m)}), \quad (3.1c)$$

$$\mathbf{y}'_{n+1} = \mathbf{y}'_n + h(\mathbf{d}^\top \otimes I_N) \mathbf{F}(\mathbf{Y}_n^{(m)}),$$

where V is an s -by- s matrix and \mathbf{w} is an s -dimensional vector, both determined by the order conditions (see Section 3.1). Notice that the s components of dimension N of the vectors $\mathbf{F}(\mathbf{Y}_n^{(j)})$ can be computed in parallel, provided that s processors are available. Hence, the computational costs of (3.1) are measured by the number of sequential right-hand side evaluations per step which equals $m + 1$.

This parallel PC method (3.1) is of the same nature as the PIRKN methods (parallel iterated RKN methods) considered in [5,7], and only differs by its predictor (3.1a) and the underlying corrector. In analogy with the PIRKN methods, the method (3.1) will be called a *PISRKN method* (parallel iterated SRKN method).

3.1. Order Conditions for the Predictor Method

The order conditions for the predictor formula (3.1a) can be derived straightforwardly using Taylor expansions. Similar to that of the PISRK methods considered in [10], we obtain an s order predictor if

$$\frac{1}{j!} [(\mathbf{c} + \mathbf{e})^j - (V, \mathbf{w})\mathbf{a}^j] = \mathbf{0}, \quad \mathbf{a} := (\mathbf{c}^\top, 1)^\top, \quad j = 1, 2, \dots, s. \quad (3.2)$$

The parameter matrix V and vector \mathbf{w} in (3.1a) can be determined by (3.2). In order to express V and \mathbf{w} explicitly in terms of \mathbf{c} , we define the s -by- s and $(s + 1)$ -by- $(s + 1)$ matrices P and Q

$$P := (\mathbf{e}, (\mathbf{c} + \mathbf{e}), (\mathbf{c} + \mathbf{e})^2, \dots, (\mathbf{c} + \mathbf{e})^s), \quad Q := (\mathbf{e}^*, \mathbf{a}, \mathbf{a}^2, \dots, \mathbf{a}^s), \quad (3.3a)$$

where \mathbf{e}^* is $(s + 1)$ -dimensional vector with unit entries. Using (3.3a), condition (3.2) can be written in the form

$$P - (V, \mathbf{w})Q = O,$$

where O is an s -by- $(s + 1)$ matrix with zero entries. Since the abscissas c_j are assumed to be distinct, we can write

$$(V, \mathbf{w}) = PQ^{-1}. \quad (3.3b)$$

If (3.3) is satisfied, then the following order relations are obtained:

$$\begin{aligned} \mathbf{Y}_n - \mathbf{Y}_n^{(m)} &= O(h^{2m+s+1}), \\ \mathbf{u}_{n+1} - \mathbf{y}_{n+1} &= h^2 (\mathbf{b}^\top \otimes I_N)^\top [\mathbf{F}(\mathbf{Y}_n) - \mathbf{F}(\mathbf{Y}_n^{(m)})] = O(h^{2m+s+3}), \\ \mathbf{u}'_{n+1} - \mathbf{y}'_{n+1} &= h (\mathbf{d}^\top \otimes I_N) [\mathbf{F}(\mathbf{Y}_n) - \mathbf{F}(\mathbf{Y}_n^{(m)})] = O(h^{2m+s+2}). \end{aligned}$$

Thus, we have the following theorem.

THEOREM 3.1. *If the corrector method (2.1) is of order p and if V and \mathbf{w} are defined by (3.3), then the PISRKN method (3.1) represents an explicit method of order $p^* = \min(p, 2m + s + 1)$ requiring $m + 1$ sequential right-hand side evaluations per step. ■*

3.2. Construction of SRKN Corrector Methods

We restrict our considerations to SRKN methods with an odd number of implicit stages ($s = 3, 5, 7, 9$) and we will construct SRKN correctors such that the corresponding PISRKN methods have an optimized rate of convergence. This rate of convergence of PISRKN methods is defined by using the model test equation $y'' = \lambda y$, where λ runs through the eigenvalues of the Jacobian matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{y}}$ (cf., e.g., [5]). For this equation, we obtain the iteration error equation

$$\mathbf{Y}_n^{(j)} - \mathbf{Y}_n = zA [\mathbf{Y}_n^{(j-1)} - \mathbf{Y}_n], \quad z := \lambda h^2, \quad j = 1, \dots, m.$$

Hence, with respect to the model test equation, the rate of convergence is determined by the spectral radius $\rho(A)$ of the matrix A . By requiring that $\rho(zA) < 1$, we are led to the convergence condition

$$|z| < \frac{1}{\rho(A)} \quad \text{or} \quad h^2 < \frac{1}{\rho(A)\rho\left(\frac{\partial f}{\partial y}\right)}. \quad (3.4)$$

We shall call $\rho(A)$ the *convergence factor* and $1/\rho(A)$ the *convergence boundary* of the PISRKN method. We exploit the freedom in the choice of the collocation vector \mathbf{c} for SRKN correctors for minimizing the convergence factor $\rho(A)$, or equivalently, for minimizing the convergence region $\{z : z < 1/\rho(A)\}$. By a numerical search, we found the collocation vectors and the corresponding convergence factors as listed in Table 1 (the specification of the parameters of the associated SRKN corrector methods can be found in the Appendix). Table 1 also lists the convergence factors for the Gauss-Legendre based indirect and direct PIRKN methods proposed in [7,10]. From this table, we see that the convergence factors of the PISRKN methods are substantially smaller than those of the PIRKN methods of the same order.

Table 1. SRKN collocation points and convergence factors of PIRKN and PISRKN methods.

Order	c_1	c_2	c_3	c_4	Convergence factors		
					Indirect PIRKN	Direct PIRKN	PISRKN
$p = 4$	0.10575846				0.083	0.048	0.025
$p = 6$	0.04282436	0.21758171			0.046	0.029	0.011
$p = 8$	0.02294808	0.11836119	0.28107352		0.027	0.018	0.006
$p = 10$	0.01532451	0.07956500	0.19035553	0.33824665	0.019	0.013	0.004

We remark that the free parameters c_i could be chosen so that the error coefficients were as small as possible. This may lead to other collocation vector \mathbf{c} . As a consequence, the collocation points found in Table 1 aim at fast convergence of the PISRKN methods.

3.3. Stability of PISRKN Methods

The stability of the PISRKN method (3.1) is investigated by again using the model test equation $y''(t) = \lambda y(t)$, where λ is assumed to be negative. Applying (3.1) to the model test equation, we obtain

$$\begin{aligned} \mathbf{Y}_n^{(m)} &= \mathbf{e}y_n + \mathbf{c}hy'_n + zA\mathbf{Y}_n^{(m-1)} = (I + zA + \dots + z^{m-1}A^{m-1})(\mathbf{e}y_n + \mathbf{c}hy'_n) + z^mA^m\mathbf{Y}_n^{(0)} \\ &= z^mA^mV\mathbf{Y}_{n-1}^{(m)} + ((I - zA)^{-1}(I - z^mA^m)\mathbf{e} + z^mA^m\mathbf{w})y_n \\ &\quad + (I - zA)^{-1}(I - z^mA^m)\mathbf{c}hy'_n, \end{aligned} \quad (3.5a)$$

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + z\mathbf{b}^\top\mathbf{Y}_n^{(m)} \\ &= z^{m+1}\mathbf{b}^\top A^mV\mathbf{Y}_{n-1}^{(m)} + (1 + z\mathbf{b}^\top(z^mA^m\mathbf{w} + (I - zA)^{-1}(I - z^mA^m)\mathbf{e}))y_n \\ &\quad + (1 + z\mathbf{b}^\top(I - zA)^{-1}(I - z^mA^m)\mathbf{c})hy'_n, \end{aligned} \quad (3.5b)$$

$$\begin{aligned} hy'_{n+1} &= hy'_n + z\mathbf{d}^\top\mathbf{Y}_n^{(m)} \\ &= z^{m+1}\mathbf{d}^\top A^mV\mathbf{Y}_{n-1}^{(m)} + z\mathbf{d}^\top(z^mA^m\mathbf{w} + (I - zA)^{-1}(I - z^mA^m)\mathbf{e})y_n \\ &\quad + (1 + z\mathbf{d}^\top(I - zA)^{-1}(I - z^mA^m)\mathbf{c})hy'_n. \end{aligned} \quad (3.5c)$$

By defining $\mathbf{X}_n^{(m)} := ((\mathbf{Y}_{n-1}^{(m)})^\top, y_n, hy'_n)^\top$, (3.5) leads us to the recursion

$$\mathbf{X}_{n+1}^{(m)} = M_m(z)\mathbf{X}_n^{(m)}, \quad (3.6)$$

$$M_m(z) =$$

$$\begin{pmatrix} z^mA^mV & (I - zA)^{-1}(I - z^mA^m)\mathbf{e} + z^mA^m\mathbf{w} & (I - zA)^{-1}(I - z^mA^m)\mathbf{c} \\ z^{m+1}\mathbf{b}^\top A^mV & 1 + z\mathbf{b}^\top(z^mA^m\mathbf{w} + (I - zA)^{-1}(I - z^mA^m)\mathbf{e}) & 1 + z\mathbf{b}^\top(I - zA)^{-1}(I - z^mA^m)\mathbf{c} \\ z^{m+1}\mathbf{d}^\top A^mV & z\mathbf{d}^\top(z^mA^m\mathbf{w} + (I - zA)^{-1}(I - z^mA^m)\mathbf{e}) & 1 + z\mathbf{d}^\top(I - zA)^{-1}(I - z^mA^m)\mathbf{c} \end{pmatrix}.$$

Similar to the stability considerations of block PIRK and PISRK methods (cf. [10,11]), the $(s + 2)$ -by- $(s + 2)$ matrix $M_m(z)$ which determines the stability of the PISRKN methods will be called the *amplification matrix*, and its spectral radius $\rho(M_m(z))$ the *stability function*. For finite, given m , the stability intervals of PISRKN methods are defined by

$$(-\beta(m), 0) := \{z : \rho(M_m(z)) < 1, z \leq 0\}.$$

From (3.6) we see that if z satisfies the convergence condition (3.4), then the stability function of the PISRKN method $\rho(M_m(z))$ converges to the stability function of the SRKN corrector method as $m \rightarrow \infty$ (cf. [8]). Hence, the *asymptotic* stability interval of the PISRKN method for $m \rightarrow \infty$, $(-\beta(\infty), 0)$, contains the intersection on the negative z -axis of the stability interval $(-\beta_{\text{corr}}, 0)$ of the SRKN corrector (see [8]) and its region of convergence defined by (3.4). Since the convergence boundary of the PISRKN method $1/\rho(A)$ is much larger than the stability boundary β_{corr} of the corresponding SRKN corrector, the asymptotic stability interval of the PISRKN method coincides with the stability interval of the SRKN corrector. Table 2 lists the stability boundaries $\beta(m)$ of the resulting PISRKN methods. We observe that these stability boundaries are sufficiently large for nonstiff problems.

Table 2. Stability boundaries for various PISRKN methods.

Order	$\beta(1)$	$\beta(2)$	$\beta(3)$	$\beta(4)$	$\beta(5)$...	$\beta(\infty) = \beta_{\text{corr}}$
$p = 4$	0.21	2.66	4.15	4.96	5.49	...	9.76
$p = 6$	0.01	0.69	3.18	7.16	10.03	...	9.86
$p = 8$	0.00	0.19	1.47	4.44	9.18	...	39.47
$p = 10$	0.00	0.05	0.68	2.64	6.24	...	79.65

4. NUMERICAL EXPERIMENTS

In this section, we present numerical experiments applied to various parallel PC methods as well as sequential methods. In order to see the efficiency of the PC methods, we applied a dynamical strategy for determining the number of iterations in the successive steps

$$\left\| \mathbf{Y}_n^{(m)} - \mathbf{Y}_n^{(m-1)} \right\|_{\infty} \leq \text{TOL} = Ch^{p-1}, \quad (4.1)$$

where p is the order of the corrector method, and C is a parameter depending on the method and on the problem (cf. [5,6,10]). Notice that by the criterion (4.2) the iteration error of the PC method is of the same order in h as the underlying corrector. For PISRKN methods, in the first integration step, we used the trivial predictor formula $\mathbf{Y}_1^{(0)} = \mathbf{e} \otimes \mathbf{y}_n + hc \otimes \mathbf{y}'_n$.

4.1. Comparison with Parallel Methods

In this section, we report numerical results obtained by the best parallel methods available in the literature, the PIRKN proposed in [7] and the PISRKN methods considered in this paper. The absolute error obtained at the end of the integration interval is presented in the form 10^{-d} (d may be interpreted as the number of correct decimal digits (NCD)). Furthermore, in the tables of results, N_{seq} denotes the total number of sequential right-hand side evaluations, and N_{steps} denotes the total number of integration steps. The following three problems possess exact solutions in closed form. Initial conditions are taken from the exact solutions.

4.1.1. Linear nonautonomous problem

As a first numerical test, we integrate the linear problem (cf. [5,12])

$$\frac{d^2 \mathbf{y}(t)}{dt^2} = \begin{pmatrix} -2\alpha(t) + 1 & -\alpha(t) + 1 \\ 2(\alpha(t) - 1) & \alpha(t) - 2 \end{pmatrix} \mathbf{y}(t), \quad \alpha(t) = \max(2 \cos^2(t), \sin^2(t)), \quad (4.2)$$

$$0 \leq t \leq 20,$$

Table 3. Values of NCD/N_{seq} for problem (4.3) obtained by various p^{th} -order parallel PC methods.

p^{th} -order PC methods	p	$N_{\text{steps}} = 80$	$N_{\text{steps}} = 160$	$N_{\text{steps}} = 320$	$N_{\text{steps}} = 640$	$N_{\text{steps}} = 1280$	C
PIRKN	4	4.0/237	5.3/477	6.5/958	7.7/1919	8.9/3836	10^{-1}
PISRKN	4	5.5/161	7.1/321	8.1/641	9.3/1281	10.5/2561	10^{-1}
PIRKN	6	7.4/320	9.2/640	11.0/1280	12.8/2559	14.6/5119	10^{-3}
PISRKN	6	9.3/232	11.0/433	12.9/704	15.0/1282	16.9/2562	10^{-3}
PIRKN	8	11.0/399	13.4/799	15.8/1600	18.2/3198	20.6/6398	10^{-4}
PISRKN	8	11.9/222	14.5/400	17.3/783	19.7/1410	23.2/2563	10^{-4}
PIRKN	10	13.3/436	18.0/921	20.9/1881	23.8/3803		10^{-4}
PISRKN	10	14.0/245	17.0/439	21.0/801	24.1/1497		10^{-4}

with exact solution $\mathbf{y}(t) = (-\sin(t), 2\sin(t))^{\top}$, by various p^{th} -order PC methods. The results listed in Table 3 clearly show that the PISRKN methods are by far superior to the PIRKN methods of the same order.

4.1.2. Orbit problem

For the second numerical example, we consider the often-used orbit problem (or nonlinear Fehlberg problem) (cf., e.g., [1,2,13,14])

$$\frac{d^2\mathbf{y}(t)}{dt^2} = \begin{pmatrix} -4t^2 & -\frac{2}{r(t)} \\ \frac{2}{r(t)} & -4t^2 \end{pmatrix} \mathbf{y}(t), \quad r(t) = \sqrt{y_1^2(t) + y_2^2(t)}, \quad \sqrt{\frac{\pi}{2}} \leq t \leq 10. \quad (4.3)$$

The exact solution is given by $\mathbf{y}(t) = (\cos(t^2), \sin(t^2))^{\top}$. The results are reported in Table 4. Again the PISRKN methods are superior to the PIRKN methods of the same order.

Table 4. Values of NCD/N_{seq} for problem (4.4) obtained by various p^{th} -order parallel PC methods.

p^{th} -order PC methods	p	$N_{\text{steps}} = 200$	$N_{\text{steps}} = 400$	$N_{\text{steps}} = 800$	$N_{\text{steps}} = 1600$	$N_{\text{steps}} = 3200$	C
PIRKN	4	1.6/591	2.8/1197	4.0/2400	5.2/4800	6.4/9600	10^2
PISRKN	4	3.2/481	4.7/918	5.9/1693	7.0/3201	8.2/6401	10^2
PIRKN	6	4.0/775	5.8/1532	7.6/3096	9.4/6257	11.2/12648	10^3
PISRKN	6	6.8/526	8.0/1001	9.7/1887	11.5/3514	13.4/6553	10^3
PIRKN	8	6.6/1022	9.0/2032	11.5/4028	13.9/7966	16.3/15725	10^3
PISRKN	8	9.1/628	11.7/1094	14.5/2107	17.0/4076	19.4/7781	10^3
PIRKN	10	9.4/1234	12.4/2458	15.5/4893	18.5/9734	21.5/19332	10^3
PISRKN	10	12.4/699	15.4/1244	18.7/2226	22.3/4295		10^3

4.1.3. Newton's equations of motion problem

The third example is the two-body gravitational problem for Newton's equation of motion (see [15, p. 245]):

$$\begin{aligned} \frac{d^2 y_1(t)}{dt^2} &= -\frac{y_1(t)}{(r(t))^3}, & \frac{d^2 y_2(t)}{dt^2} &= -\frac{y_2(t)}{(r(t))^3}, & 0 \leq t \leq 20, \\ y_1(0) &= 1 - \varepsilon, & y_2(0) &= 0, & y_1'(0) &= 0, & y_2'(0) &= \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}, \end{aligned} \quad (4.4)$$

where $r(t) = \sqrt{y_1^2(t) + y_2^2(t)}$.

The solution components are $y_1(t) = \cos(u) - \varepsilon$, $y_2(t) = \sqrt{(1+\varepsilon)(1-\varepsilon)} \sin(u)$, where u is the solution of Kepler's equation $t = u - \varepsilon \sin(u)$ and ε denotes the eccentricity of the orbit. In this example we set $\varepsilon = 0.3$. The results are listed in Table 5. In this example, we observe the same increased efficiency of the PISRKN methods as in two previous examples.

Table 5. Values of NCD/N_{seq} for problem (4.5) obtained by various p^{th} -order parallel PC methods.

p^{th} -order PC methods	p	$N_{\text{steps}} = 100$	$N_{\text{steps}} = 200$	$N_{\text{steps}} = 400$	$N_{\text{steps}} = 800$	$N_{\text{steps}} = 1600$	C
PIRKN	4	1.9/200	3.3/400	5.0/841	6.2/1995	7.3/4800	10^1
PISRKN	4	3.0/200	4.6/400	7.0/801	8.2/1601	9.3/3201	10^1
PIRKN	6	5.1/360	6.8/800	8.6/1600	10.4/3200	12.2/6400	10^{-1}
PISRKN	6	6.6/246	8.1/443	10.3/809	12.2/1602	14.2/3202	10^{-1}
PIRKN	8	7.7/450	10.1/917	12.5/1934	14.9/4000	17.3/8000	10^{-2}
PISRKN	8	9.8/278	12.2/524	14.5/1002	16.9/1871	19.3/3487	10^{-2}
PIRKN	10	10.4/517	13.3/1050	16.2/2127	19.2/4306	22.2/8706	10^{-2}
PISRKN	10	10.5/314	14.8/558	18.1/1054	22.0/2010		10^{-2}

4.2. Comparison with Sequential Methods

In the section above the PISRKN methods were compared with PIRKN methods (the most efficient parallel methods for nonstiff problems). In this section we will compare the PISRKN methods with the sequential methods currently available.

We restricted our test to the comparison of our tenth-order PISRKN method with a few well-known sequential codes for the orbit problem (4.4). We selected some embedded RKN pairs presented in the form $p(p+1)$ or $(p+1)p$ constructed in [1,2,13,14] and the RKN code DOPRIN taken from [9]. We reproduced the best results obtained by these sequential methods given in the literature (cf., e.g., [7,14]) and added the results obtained by the tenth-order PISRKN method. In spite of the fact the results of the sequential methods are obtained using a stepsize strategy, whereas PISRKN method is applied with fixed stepsizes, it is the PISRKN method that is much more efficient (see Table 6).

Table 6. Comparison with the sequential methods for problem (4.4).

Methods	N_{steps}	NCD	N_{seq}
11(10)-pair (from [14])	919	20.7	15614
8(9)-pair (from [1])	1452	13.5	15973
9(10)-pair (from [2])	628	15.1	8793
	3235	21.4	45291
11(12)-pair (from [13])	876	20.3	17521
DOPRIN (from [9])	1208	12.3	9665
	4466	16.3	35729
	16667	20.3	133337
PISRKN (in this paper)	200	12.4	699
	400	15.4	1244
	800	18.7	2226
	1600	22.3	4295

5. CONCLUDING REMARKS

In this paper, we proposed a special class of s -stage SRKN methods of order $s+1$. The better performance of these SRKN methods is demonstrated when they are used as corrector methods

for generating PC methods for nonstiff problems. By three examples, we have shown that for a given order p , the resulting PISRKN method is by far superior to the PIRKN method (about a factor from 2 to 4). The price we have to pay is a larger number of processors to achieve the same order of accuracy (nearly twice as many). However, comparison of numerical results obtained by a p^{th} -order PISRKN and a p^{th} -order PIRKN method, both implemented on $p/2$ processors (the optimal number of processors for a p^{th} -order PIRKN method), also reveals that the PISRKN method is superior to the PIRKN method.

By comparing the tenth-order PISRKN method with high-order sequential method, we have shown that the PISRKN methods are much more efficient.

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APPENDIX

Here we give the parameters (c, A, b, d) in 24 decimal of the corrector RKN methods of fourth-order, sixth-order, eighth-order and tenth-order based on 'optimal' collocation vectors defined in Section 3.2. These parameters were computed on a 28-arithmetic computers.

Table A.1. Parameters (c, A, b, d) of the fourth-order corrector RKN method.

$c(1) = 1.057584600000000000000000E-01$	$a(1, 3) = 6.006821188295530530886328E-04$
$c(2) = 5.000000000000000000000000E-01$	$a(2, 3) = -2.579133516936377142451729E-03$
$c(3) = 8.942415400000000000000000E-01$	$a(3, 3) = 7.193250169095325995341243E-03$
$a(1, 1) = 7.193250169095325995341243E-03$	$b(1) = 2.397280392370675089298293E-01$
$a(2, 1) = 1.031090358897341291331342E-01$	$b(2) = 2.319202603392059946915134E-01$
$a(3, 1) = 2.119770209321705656042605E-01$	$b(3) = 2.835170042372649637865737E-02$
$a(1, 2) = -2.201506357139079048429876E-03$	$d(1) = 2.680797396607940053084866E-01$
$a(2, 2) = 2.447009762720224800931751E-02$	$d(2) = 4.638405206784119893830267E-01$
$a(3, 2) = 1.806636948295199084003982E-01$	$d(3) = 2.680797396607940053084866E-01$

Table A.2. Parameters (c, A, b, d) of the eighth-order corrector RKN method.

$c(1) = 2.294808000000000000000000E-02$	$a(1, 5) = 3.126942733774384233966542E-05$
$c(2) = 1.183611900000000000000000E-01$	$a(2, 5) = -8.043153980674912328959031E-05$
$c(3) = 2.810735200000000000000000E-01$	$a(3, 5) = 2.506506789213120465785417E-04$
$c(4) = 5.000000000000000000000000E-01$	$a(4, 5) = -1.048966242944359813066190E-03$
$c(5) = 7.189264800000000000000000E-01$	$a(5, 5) = 3.676743716035802448431083E-03$
$c(6) = 8.816388100000000000000000E-01$	$a(6, 5) = 3.148960908962287414615932E-02$
$c(7) = 9.770519200000000000000000E-01$	$a(7, 5) = 5.054761184192817014710453E-02$
$a(1, 1) = 3.541933154986615908580242E-04$	$a(1, 6) = -1.837286148408038975922562E-05$
$a(2, 1) = 5.529246407690738971594212E-03$	$a(2, 6) = 4.472358972535885599118372E-05$
$a(3, 1) = 1.513528009772205113189058E-02$	$a(3, 6) = -1.263652316005222084240929E-04$
$a(4, 1) = 2.795283215021751479627000E-02$	$a(4, 6) = 4.233557272834521915786853E-04$
$a(5, 1) = 4.098536923927815079519936E-02$	$a(5, 6) = -3.127390461158141897933397E-05$
$a(6, 1) = 5.05077297979702684826367E-05$	$a(6, 6) = 1.707891780679761069393040E-03$
$a(7, 1) = 5.614087446180754869545597E-02$	$a(7, 6) = 1.224268513283329443611598E-02$
$a(1, 2) = -1.427152046194791182509997E-04$	$a(1, 7) = 5.767864873600028482864183E-06$
$a(2, 2) = 1.707891780679761069393039E-03$	$a(2, 7) = -1.370575407018725143606739E-05$
$a(3, 2) = 2.109011441288196899556467E-02$	$a(3, 7) = 3.718306159382819680998980E-05$
$a(4, 2) = 4.996318779186110751051844E-02$	$a(4, 7) = -1.147211482494595372165563E-04$
$a(5, 2) = 7.783191058006123801491142E-02$	$a(5, 7) = -5.164032152757493669315549E-05$
$a(6, 2) = 9.912438771888066949387070E-02$	$a(6, 7) = -8.442463737599559567915652E-05$
$a(7, 2) = 1.114466916051240038024873E-01$	$a(7, 7) = 3.541933154986615908580242E-04$
$a(1, 3) = 8.094955794804809066925953E-05$	$b(1) = 5.748526667698872342684159E-02$
$a(2, 3) = -3.226299812316969239644695E-04$	$b(2) = 1.144439125282203012528527E-01$
$a(3, 3) = 3.676743716035802448431083E-03$	$b(3) = 1.405588688707730223723914E-01$
$a(4, 3) = 4.175382270796465756927991E-02$	$b(4) = 1.158442524759431999653734E-01$
$a(5, 3) = 8.585622858073934681127075E-02$	$b(5) = 5.495329096895498760769919E-02$
$a(6, 3) = 1.173373854328658567115264E-01$	$b(6) = 1.536424839906499061497320E-02$
$a(7, 3) = 1.361035096131359006634671E-01$	$b(7) = 1.350160080054774759868477E-03$
$a(1, 4) = -4.778491171129404433958801E-05$	$d(1) = 5.883542675704349818671006E-02$
$a(2, 4) = 1.395911461208244017116928E-04$	$d(2) = 1.298081609272852918678259E-01$
$a(3, 4) = -5.624449129592406108507747E-04$	$d(3) = 1.955121598397280099800906E-01$
$a(4, 4) = 6.070489013867087282635708E-03$	$d(4) = 2.316885049518863999307469E-01$
$a(5, 4) = 5.016030393261981828585988E-02$	$d(5) = 1.955121598397280099800906E-01$
$a(6, 4) = 8.856091646663785732646601E-02$	$d(6) = 1.298081609272852918678259E-01$
$a(7, 4) = 1.104796612175156206645111E-01$	$d(7) = 5.883542675704349818671006E-02$

Table A.3. Parameters (c, A, b, d) of the tenth-order corrector RKN method.

$c(1) = 1.532451000000000000000000E-02$	$a(1, 6) = -1.780067685072795180298326E-05$
$c(2) = 7.956500000000000000000000E-02$	$a(2, 6) = 3.715852039344618351538906E-05$
$c(3) = 1.903555300000000000000000E-01$	$a(3, 6) = -7.711644186027908770464365E-05$
$c(4) = 3.382466500000000000000000E-01$	$a(4, 6) = 1.491144787829555203163462E-04$
$c(5) = 5.000000000000000000000000E-01$	$a(5, 6) = -1.681047341278292245850098E-04$
$c(6) = 6.617533500000000000000000E-01$	$a(6, 6) = 2.703806284358016888223643E-03$
$c(7) = 8.096444700000000000000000E-01$	$a(7, 6) = 2.321751468771282553286217E-02$
$c(8) = 9.204350000000000000000000E-01$	$a(8, 6) = 4.123764650696515439851550E-02$
$c(9) = 9.846754900000000000000000E-01$	$a(9, 6) = 5.132944079056458641293650E-02$
$a(1, 1) = 1.601475897565237308644742E-04$	$a(1, 7) = 1.108785902903640172109336E-05$
$a(2, 1) = 2.486382917030823452822819E-03$	$a(2, 7) = -2.212787750012136523970112E-05$
$a(3, 1) = 6.874945720394802212948381E-03$	$a(3, 7) = 4.210227655932866657017992E-05$
$a(4, 1) = 1.264135385034151667873547E-02$	$a(4, 7) = -6.653893614221884175086736E-05$
$a(5, 1) = 1.901696282112353378775272E-02$	$a(5, 7) = 1.498486489321089243211681E-05$
$a(6, 1) = 2.535906250035105583973432E-02$	$a(6, 7) = -3.595811463160270341258051E-04$
$a(7, 1) = 3.117644788936442967069945E-02$	$a(7, 7) = 1.763143030701564087967156E-03$
$a(8, 1) = 3.551483704706959929795949E-02$	$a(8, 7) = 1.446189409967245622253788E-02$
$a(9, 1) = 3.804080084396757251923928E-02$	$a(9, 7) = 2.31692975809105964979794E-02$
$a(1, 2) = -7.072786167305523297453930E-05$	$a(1, 8) = -5.852542901807330962410302E-06$
$a(2, 2) = 7.964838042577941078882503E-04$	$a(2, 8) = 1.139613664490011802518314E-05$
$a(3, 2) = 9.693308808879063890577235E-03$	$a(3, 8) = -2.069935877663995913688510E-05$
$a(4, 2) = 2.298239161311529524599555E-02$	$a(4, 8) = 2.938501687852603871784536E-05$
$a(5, 2) = 3.715901359222742811958828E-02$	$a(5, 8) = 4.544471350170697280766453E-06$
$a(6, 2) = 5.147823935252965178043408E-02$	$a(6, 8) = 1.223077070119061430967624E-04$
$a(7, 2) = 6.449751421502533664588372E-02$	$a(7, 8) = -9.741585907347434901719141E-05$
$a(8, 2) = 7.432033437839941496264022E-02$	$a(8, 8) = 7.964838042577941078882565E-04$
$a(9, 2) = 7.998011378170742761703407E-02$	$a(9, 8) = 5.606300221181664870406897E-03$
$a(1, 3) = 4.786394757422258512379553E-05$	$a(1, 9) = 1.727434895380851181960374E-06$
$a(2, 3) = -1.734310363929254114335306E-04$	$a(2, 9) = -3.324158514885535438294993E-06$
$a(3, 3) = 1.763143030701564087967157E-03$	$a(3, 9) = 5.906060716062365305458302E-06$
$a(4, 3) = 1.917669714332535010411394E-02$	$a(4, 9) = -7.966167120227199810306597E-06$
$a(5, 3) = 4.091872902998243811608397E-02$	$a(5, 9) = -2.573883412562046275943545E-06$
$a(6, 3) = 6.220467110439485846731309E-02$	$a(6, 9) = -3.069089125939194977722945E-05$
$a(7, 3) = 8.184959060673778311387388E-02$	$a(7, 9) = 6.414139970977850285049937E-06$
$a(8, 3) = 9.642068558874371471603542E-02$	$a(8, 9) = -3.452928645688338183671997E-05$
$a(9, 3) = 1.049400098225438647618789E-01$	$a(9, 9) = 1.601475897565237308644713E-04$
$a(1, 4) = -3.501005871452694333679985E-05$	$b(1) = 3.864043470428443862110529E-02$
$a(2, 4) = 9.137856437248375184564131E-05$	$b(2) = 8.134021616962112677465392E-02$
$a(3, 4) = -3.062553587907659113980981E-04$	$b(3) = 1.069532753662910882219313E-01$
$a(4, 4) = 2.703806284358016888223642E-03$	$b(4) = 1.052594208016235689152302E-01$
$a(5, 4) = 2.556061115937281978761015E-02$	$b(5) = 8.122623768436601391852500E-02$
$a(6, 4) = 5.160654626578425354470667E-02$	$b(6) = 5.380198901462227089083982E-02$
$a(7, 4) = 7.490408539164461038094595E-02$	$b(7) = 2.514578703611263377462762E-02$
$a(8, 4) = 9.264085824998741485457560E-02$	$b(8) = 7.031277927866611930038891E-03$
$a(9, 4) = 1.028040819594296834288606E-01$	$b(9) = 6.013612952122469530479639E-04$
$a(1, 5) = 2.598461225500389018540929E-05$	$d(1) = 3.924179599949668557415325E-02$
$a(2, 5) = -5.862225779151530198575607E-05$	$d(2) = 8.837149409748773870469283E-02$
$a(3, 5) = 1.422791629673137348712148E-04$	$d(3) = 1.320990624024037219965588E-01$
$a(4, 5) = -4.028451654279644345416250E-04$	$d(4) = 1.590614098162458398060702E-01$
$a(5, 5) = 2.495832678590789870112953E-03$	$d(5) = 1.624524753687320278370500E-01$
$a(6, 5) = 2.587438694125692632039448E-02$	$d(6) = 1.590614098162458398060701E-01$
$a(7, 5) = 5.044478979870639706649982E-02$	$d(7) = 1.320990624024037219965589E-01$
$a(8, 5) = 6.824208422386133482168438E-02$	$d(8) = 8.837149409748773870469283E-02$
$a(9, 5) = 7.876271771330813016080123E-02$	$d(9) = 3.924179599949668557415327E-02$

Table A.4. Parameters (c, A, b, d) of the sixth-order corrector RKN method.

$c(1) = 4.282436000000000000000000E-02$	$a(1, 4) = -1.217873863960348101558417E-04$
$c(2) = 2.175817100000000000000000E-01$	$a(2, 4) = 3.906798571783529531297708E-04$
$c(3) = 5.000000000000000000000000E-01$	$a(3, 4) = -1.590029178785210853967881E-03$
$c(4) = 7.824182900000000000000000E-01$	$a(4, 4) = 5.536940973523307750655806E-03$
$c(5) = 9.571756400000000000000000E-01$	$a(5, 4) = 4.122646570625570977975948E-02$
$a(1, 1) = 1.217324854314668862875581E-03$	$a(1, 5) = 3.789824419420387404637056E-05$
$a(2, 1) = 1.884237191394402003661308E-02$	$a(2, 5) = -1.115401574706397870239920E-04$
$a(3, 1) = 4.989776856773406065516176E-02$	$a(3, 5) = 3.686235377915825743216407E-04$
$a(4, 1) = 8.001402169931404156158724E-02$	$a(4, 5) = -9.035628915625477645592689E-05$
$a(5, 1) = 9.909618830407916003572662E-02$	$a(5, 5) = 1.217324854314668862875571E-03$
$a(1, 2) = -4.539652804281885870054783E-04$	$b(1) = 1.036977628394374000771653E-01$
$a(2, 2) = 5.536940973523307750655806E-03$	$b(2) = 1.866103573844775543383899E-01$
$a(3, 2) = 6.576803070094104585154723E-02$	$b(3) = 1.531581693715075607415999E-01$
$a(4, 2) = 1.351067996166308663641600E-01$	$b(4) = 5.189423762502504092735971E-02$
$a(5, 2) = 1.762747633597403769676393E-01$	$b(5) = 4.639472779552443915485100E-03$
$a(1, 3) = 2.374924730201506602393783E-04$	$d(1) = 1.083372356189898439926504E-01$
$a(2, 3) = -9.875523239129909533746692E-04$	$d(2) = 2.385045950095025952657497E-01$
$a(3, 3) = 1.055560637231852177293725E-02$	$d(3) = 3.063163387430151214831998E-01$
$a(4, 3) = 8.552178426295008910005288E-02$	$d(4) = 2.385045950095025952657497E-01$
$a(5, 3) = 1.402778606803148843539990E-01$	$d(5) = 1.083372356189898439926504E-01$