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Uniform asymptotic expansions of integrals: a selection of problems

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Abstract

On the occasion of the conference we mention examples of Stieltjes' work on asymptotics of special functions. The remaining part of the paper gives a selection of asymptotic methods for integrals, in particular on uniform approximations. We discuss several "standard" problems and examples, in which known special functions (error functions, Airy functions, Bessel functions, etc.) are needed to construct uniform approximations. Finally, we discuss the recent interest and new insights in the Stokes phenomenon. An extensive bibliography on uniform asymptotic methods for integrals is given, together with references to recent papers on the Stokes phenomenon for integrals and related topics.

Keywords: Uniform asymptotic expansion; Error function; Incomplete gamma functions; Stirling numbers; Fermi–Dirac integral; Statistical distribution functions; Stokes phenomenon

AMS Classification: 41A60; 33B20; 33C10; 33C45; 11B73; 30E15

1. Examples of Stieltjes' work in asymptotics

Stieltjes has several results on asymptotic expansions, including discussions of the remainder terms in the expansions. In this section we give examples of his interest in asymptotics of special functions.

1.1. Stirling's series for $\ln \Gamma(z)$

Stieltjes [81] has considered the well-known result for the logarithm of the Euler gamma function:

$$\ln \Gamma(z) \sim z \ln z - z + \tfrac{1}{2}\ln(2\pi/z) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} + \cdots$$

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as $z \to \infty$, $|\arg z| < \pi$. More precisely,

$$\ln \Gamma(z) = z \ln z - z + \tfrac{1}{2}\ln(2\pi/z) + \sum_{n=1}^{N-1} \frac{B_{2n}}{2n(2n-1)} \frac{1}{z^{2n-1}} + R_N(z),$$

where B_n are the Bernoulli numbers and

$$R_N(z) = \mathcal{O}\left(\frac{1}{z^{2N-1}}\right), \quad z \to \infty, |\arg z| < \pi.$$

Stieltjes showed that

$$|R_N(z)| \leqslant \frac{|B_{2N}|}{2N(2N-1)} \frac{1}{\cos^{2N}(\frac{1}{2}\theta)} \frac{1}{|z|^{2N-1}},$$

where $\theta = \arg z \in (-\pi, \pi)$. A proof of this result can also be found in [50, p. 294].

1.2. Bessel functions

Stieltjes' [80] gives several results on Bessel functions, especially the asymptotic expansions of $J_0(x)$, $Y_0(x)$, $K_0(x)$. One has for the ordinary Bessel functions, with $\chi = z - (\frac{1}{2}\nu + \frac{1}{4})\pi$,

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\,[P(\nu, z)\cos\chi - Q(\nu, z)\sin\chi],$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}}\,[P(\nu, z)\sin\chi + Q(\nu, z)\cos\chi],$$

where P and Q are the nonoscillating parts, having the expansions

$$P(\nu, z) \sim \sum_{n=0}^{\infty} (-1)^n \frac{(\nu, 2n)}{(2z)^{2n}}, \qquad Q(\nu, z) \sim \sum_{n=0}^{\infty} (-1)^n \frac{(\nu, 2n+1)}{(2z)^{2n+1}}$$

as $z \to \infty$ in $|\arg z| < \pi$. The symbols (ν, n) are given by

$$\begin{aligned}(\nu, n) &= \frac{2^{-2n}}{n!}(4\nu^2 - 1)(4\nu^2 - 3^2)\cdots[4\nu^2 - (2n-1)^2] \\ &= \frac{\Gamma(\frac{1}{2} + \nu + n)}{n!\,\Gamma(\frac{1}{2} + \nu - n)}, \quad n = 0, 1, 2, \ldots .\end{aligned} \tag{1.1}$$

Let the remainder terms in the expansions of $P(0, x)$ and $Q(0, x)$ be written in the form

$$P(0, x) = 1 - \frac{9}{2!(8x)^2} + \cdots + (-1)^{n-1} \frac{9 \cdot 25 \cdots (4n-5)^2}{(2n-2)!\,(8x)^{2n-2}} + \theta_1 (-1)^n \frac{9 \cdot 25 \cdots (4n-1)^2}{(2n)!\,(8x)^{2n}},$$

$$Q(0, x) = -\frac{1}{1!8x} + \frac{9 \cdot 25}{3!(8x)^3} - \cdots + (-1)^n \frac{9 \cdot 25 \cdots (4n-3)^2}{(2n-1)!\,(8x)^{2n-1}} + \theta_2 (-1)^{n-1} \frac{9 \cdot 25 \cdots (4n+1)^2}{(2n+1)!\,(8x)^{2n+1}}.$$

By using a quite ingenious method, Stieltjes showed that

$$0 < \theta_1 < 1, \qquad 0 < \theta_2 < 1.$$

He also showed that

$$\theta_1, \theta_2 \sim \tfrac{1}{2} \quad \text{as } x \to \infty,$$

when the expansion is terminated at the smallest term (that is, when $n \sim 2x$).

For the function $K_0(x)$, Stieltjes used the integral

$$K_0(x) = \frac{e^{-x}}{\sqrt{2}} \int_0^\infty e^{-xu} \frac{du}{\sqrt{u}\sqrt{1+\frac{1}{2}u}},$$

and he showed

$$K_0(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \left[\sum_{n=0}^{N-1} \frac{(0,n)}{(2x)^n} + \theta \frac{(0,N)}{(2x)^N} \right], \tag{1.2}$$

with $0 < \theta < 1$. His proof was based on the identity

$$\frac{1}{\sqrt{1+\frac{1}{2}u}} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\phi}{1+\frac{1}{2}u\sin^2\phi}.$$

Expansion in powers of u gives

$$\frac{1}{\sqrt{1+\frac{1}{2}u}} = \sum_{n=0}^{N-1} \binom{-\frac{1}{2}}{n} (\tfrac{1}{2}u)^n + (-1)^N \frac{2}{\pi} \int_0^{\pi/2} \frac{(\frac{1}{2}u\sin^2\phi)^N}{1+\frac{1}{2}u\sin^2\phi}\, d\phi.$$

Obviously,

$$(-1)^N \frac{2}{\pi} \int_0^{\pi/2} \frac{(\frac{1}{2}u\sin^2\phi)^N}{1+\frac{1}{2}u\sin^2\phi}\, d\phi = \theta(-1)^N \frac{2}{\pi} \int_0^{\pi/2} (\tfrac{1}{2}u\sin^2\phi)^N\, d\phi = \theta \binom{-\frac{1}{2}}{N} (\tfrac{1}{2}u)^N,$$

where θ lies between 0 and 1. Multiplying by $e^{-xu}/\sqrt{u}$ and integrating we obtain the expansion in (1.2) with remainder, and with a bound on the remainder.

Stieltjes methods for the Bessel functions are also treated in [105, Sections 7.31, 7.32].

1.3. The logarithmic integral

Stieltjes considered in [80],

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t},$$

which for $x > 1$ is a Cauchy principal value integral. He found

$$\mathrm{li}(e^x) = \frac{e^x}{x} \left[1 + \frac{1}{x} + \frac{2!}{x^2} + \cdots + \frac{(n-1)!}{x^{n-1}} + R_n \right],$$

with

$$R_n = \mathrm{e}^{-x}\sqrt{\frac{2\pi}{n}}\left[A_0 + \frac{A_1}{n} + \frac{A_3}{n^3} + \cdots\right],$$

$x = n + \eta$, and

$$A_0 = \eta - \tfrac{1}{3}, \qquad A_1 = \tfrac{1}{6}\eta^3 - \tfrac{1}{2}\eta^2 + \tfrac{1}{12}\eta + \tfrac{1}{540}.$$

We see that the remainder is exponentially small when we truncate the expansion at the right place (when $n \sim x$). Recently, re-expanding remainders at the right place has received renewed interest; we return to this topic in Section 8.

2. Asymptotic expansions of integrals: Basic steps

We are concerned with obtaining uniform asymptotics expansions of integrals of the type

$$F_\alpha(z) = \int_C f(t)\mathrm{e}^{-z\phi(t,\alpha)}\,\mathrm{d}t,$$

where C is a contour in the complex plane and z is a large parameter. For certain values of the other parameter α (the so-called uniformity parameter), the asymptotic behaviour of $F_\alpha(z)$ may change. For instance, in turning point problems, for $\alpha < 0$ the integral $F_\alpha(z)$ may show strong oscillations, whereas for $\alpha > 0$ the integral may be monotonic. For obtaining uniform expansions the following major steps can be distinguished.

- Trace the critical points on or near $\mathscr{C}$ that significantly contribute to $F_\alpha(z)$; that is, points where $\partial\phi/\partial t$ vanishes (saddle points), poles, singularities, or end points of $\mathscr{C}$.
- Investigate the nature and significance of these critical points; which ones give the dominant contributions, and can be reached by deforming $\mathscr{C}$ without disturbing convergence, etc.?
- Transform the integral into a standard form by a conformal mapping, taking into account the mapping of the contour and the critical points.
- Construct a formal expansion by using local expansions at the relevant critical points, Laplace's method, integration by parts, etc.
- Discuss the nature and the asymptotic properties of the expansion.
- Construct error bounds for the remainders.
- Extend the results to wider domains of the parameters, e.g. by deforming $\mathscr{C}$.

The first four points are most frequently the only possibilities in practical problems (wave theory, optics, diffraction and scattering theory); often the contributions in the expansion have a physical interpretation and then just the form of the expansion is the ultimate requirement.

Standard books on asymptotic analysis are [16, 50, 108], the second one concentrating on differential equations, the other ones on integrals.

3. Why uniform asymptotics?

We give well-known examples of nonuniform expansions and indicate that a secondary parameter may disturb the nature of the expansion. The special functions in the examples all have uniform expansions in which the parameters may vary in much larger domains. Uniform expansions are useful in describing the transition of behaviour. In physical or statistical problems this may give insight in the underlying problems. In general, higher transcendental functions are needed to describe the transitions (Airy functions, error functions, parabolic cylinder functions, etc.). Uniform expansions are also useful for numerical evaluations, because more robust algorithms can be developed when uniform expansions are used. Usually, however, coefficients become more complicated in uniform expansions.

3.1. *Incomplete gamma function*

The incomplete gamma function

$$\Gamma(a, z) = \int_z^\infty t^{a-1} \mathrm{e}^{-t}\, \mathrm{d}t$$

has the well-known asymptotic expansion

$$\Gamma(a, z) \sim z^{a-1} \mathrm{e}^{-z} \left[1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \cdots \right]$$

as $z \to \infty$. This expansion can be obtained by integrating by parts. What about a? Should a be fixed, or is it possible to take $a = \mathcal{O}(\sqrt{z})$, or larger? In Section 7.4 we present a uniform expansion containing the error function that is valid in the transition area $a \sim z$ (and in a larger domain).

3.2. *Modified Bessel function*

The modified Bessel function

$$K_\alpha(z) = \int_0^\infty \mathrm{e}^{-z\cosh t} \cosh \alpha t\, \mathrm{d}t$$

has the following asymptotic expansion:

$$K_\alpha(z) \sim \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} \sum_{n=0}^\infty \frac{(\alpha, n)}{(2z)^n}, \quad z \to \infty,\ |\arg z| < \tfrac{3}{2}\pi .$$

The symbols (α, n) are defined in (1.1). What about α? Should α be fixed, can we take $\alpha = \mathcal{O}(\sqrt{z})$, or larger? A uniform expansion in terms of elementary functions is given in [50, p. 378]. The emphasis is on large values of α, but the expansion also has an asymptotic property for large z. The expansion can also be found in [1, p. 378].

3.3. *Probability distribution function*

Many statistical distribution functions can be transformed into the standard form

$$F_a(x) = \int_{-\infty}^x \mathrm{e}^{-at^2} f(t)\, \mathrm{d}t.$$

The asymptotic behaviour of $F_a(x)$ as $a \to \infty$ depends strongly on sign(x). Assume $F_a(\infty) = 1$. Then as a rule

$$F_a(x) = \begin{cases} -f(x)/(2ax)\mathrm{e}^{-ax^2}[1 + \mathcal{O}(1/a)] & \text{if } x < 0, \\ \frac{1}{2}f(0)[1 + \mathcal{O}(1/\sqrt{a})] & \text{if } x = 0, \\ 1 + \mathcal{O}(1/a) & \text{if } x > 0. \end{cases}$$

Conclusion: The asymptotic behaviour of $F_a(x)$ is completely different in the three cases distinguished and, moreover, the asymptotic forms do not pass into one another when x changes from negative values to positive ones. A uniform approximation can be given in terms of the *error function*; see Section 5 and [87].

3.4. Normalized incomplete gamma function

As a special case of the previous subsection we consider the normalized incomplete gamma function

$$Q(a, x) = \frac{1}{\Gamma(a)} \int_x^\infty t^{a-1}\,\mathrm{e}^{-t}\,\mathrm{d}t.$$

When a is large and $x > a$ the function $Q(a, x)$ is very small; when x becomes smaller and crosses the value a there is a sudden change in behaviour from 0 to 1.

We write

$$Q(a, x) = \frac{a^a \mathrm{e}^{-a}}{\Gamma(a)} \int_\lambda^\infty \mathrm{e}^{-a(t - \ln t - 1)} \frac{\mathrm{d}t}{t}, \qquad \lambda = \frac{x}{a},$$

and introduce the transformation $t \mapsto \zeta(t)$ given by $\frac{1}{2}\zeta^2 = t - \ln t - 1$, sign$(\zeta) =$ sign$(t - 1)$, such that the *gamma distribution* nearly becomes a *Gaussian distribution.* We obtain

$$Q(a, x) = \frac{a^a \mathrm{e}^{-a}}{\Gamma(a)} \int_\eta^\infty \mathrm{e}^{-(1/2)a\zeta^2} f(\zeta)\,\mathrm{d}\zeta.$$

This integral is a standard form for this kind of problem. The quantity η is given by $\eta = \zeta(\lambda)$ and $f(\zeta) = t^{-1}\mathrm{d}t/\mathrm{d}\zeta = (t - 1)/\zeta$. A uniform approximation can be given in terms of the error function, as will be explained in Sections 5 and 7.4.

4. Laplace integrals and Watson's lemma

The first step in the asymptotics of integrals is Watson's lemma. Consider

$$F_\lambda(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1}\,\mathrm{e}^{-zt} f(t)\,\mathrm{d}t.$$

Expansion at the origin $f(t) = \sum_{n=0}^{\infty} c_n t^n$ gives

$$F_\lambda(z) \sim \sum_{n=0}^{\infty} c_n \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} z^{-n-\lambda}$$

as $z \to \infty$, λ fixed; see [50, p. 113 and further] also for more general expansions of f. When λ is not fixed (say, λ is depending on z) this expansion becomes invalid. Let $\mu := \lambda/z$. When μ is not small, it is better to expand at $t = \mu$. The expansion $f(t) = \sum_{n=0}^{\infty} a_n(\mu)\,(t-\mu)^n$ gives

$$F_\lambda(z) \sim \sum_{n=0}^{\infty} a_n(\mu)\, P_n(\lambda)\, z^{-n-\lambda},$$

where $P_n(\lambda)$ are polynomials,

$$P_0(\lambda) = 1, \quad P_1(\lambda) = 0, \quad P_2(\lambda) = \lambda, \quad P_3(\lambda) = 2\lambda, \ \ldots$$

with recursion relation

$$P_{n+1}(\lambda) = n[P_n(\lambda) + \lambda P_{n-1}(\lambda)].$$

It is easily seen that

$$P_n(\lambda) = \mathcal{O}(\lambda^{[n/2]}) \quad \text{as } \lambda \to \infty.$$

Under mild conditions on $a_n(\mu)$, that is, on f, this expansion is uniformly valid with respect t $\lambda \in [0, \infty)$, and in a larger domain of the complex plane. The main condition on f is that its singularities are not too close to the point $t = \mu$. Let R_μ denote the radius of convergence of the Taylor expansion of f at $t = \mu$. Then we require

$$R_\mu^{-1} = \mathcal{O}[(1+\mu)^{-\kappa}], \qquad \mu \geqslant 0, \ k \geqslant \tfrac{1}{2}, \ \kappa \text{ fixed.}$$

For more details we refer to [88, 89].

4.1. More general Laplace integrals

A further step is to consider

$$F_\lambda(z) := \int_0^\infty q(t)^{\lambda-1}\, \mathrm{e}^{-z\,p(t)}\, \mathrm{d}t, \tag{4.1}$$

p and q analytic, and increasing functions for positive values of t, $q(0) = 0$. The problem is to find an asymptotic expansion of $F_\lambda(z)$ as $z \to \infty$, which is uniformly valid with respect to $\lambda \in [0, \infty)$. As in Watson's lemma, the basic approximant is the Euler gamma function

$$z^{-\lambda}\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}\, \mathrm{e}^{-zt}\, \mathrm{d}t, \quad \Re\lambda, \Re z > 0.$$

Examples are the following.

Laplace integral (standard form):

$$\int_0^\infty t^{\lambda-1} e^{-zt} f(t)\, \mathrm{d}t.$$

Modified Bessel function:

$$\int_0^\infty [t(1+t)]^{\lambda-1}\,\mathrm{e}^{-zt}\,\mathrm{d}t.$$

Whittaker function:

$$\int_0^\infty [t(1+t)^\nu]^{\lambda-1}\,\mathrm{e}^{-zt}\,\mathrm{d}t.$$

Parabolic cylinder function:

$$\int_0^\infty t^{\lambda-1}\,\mathrm{e}^{-z[t+\alpha t^2]}\,\mathrm{d}t.$$

Beta integral:

$$\frac{\Gamma(z)\Gamma(\lambda)}{\Gamma(z+\lambda)} = \int_0^1 t^{z-1}(1-t)^{\lambda-1}\,\mathrm{d}t = \int_0^\infty (1-\mathrm{e}^{-t})^{\lambda-1}\,\mathrm{e}^{-zt}\,\mathrm{d}t.$$

In the general case,

$$F_\lambda(z) := \frac{1}{\Gamma(\lambda)}\int_0^\infty q(t)^{\lambda-1}\mathrm{e}^{-zp(t)}\,\mathrm{d}t = \frac{1}{\Gamma(\lambda)}\int_0^\infty \mathrm{e}^{-z[p(t)-\mu\ln q(t)]}\frac{\mathrm{d}t}{q(t)}, \quad \mu = \frac{\lambda}{z},$$

we determine the saddle point t_0, that is, the solution of $p'(t) - \mu q'(t)/q(t) = 0$. The transforma
to the standard form reads

$$p(t) - \mu\ln q(t) = \tau - \mu\ln\tau + A(\mu).$$

The condition on the conformal mapping is that the saddle point at $t = t_0$ maps to the saddle p
at $\tau = \mu$, and this determines $A(\mu)$. Under certain conditions on p, q this gives the standard f

$$F_\lambda(z) = \frac{e^{-zA(\mu)}}{\Gamma(\lambda)}\int_0^\infty \tau^{\lambda-1}\,\mathrm{e}^{-z\tau}f(\tau)\,\mathrm{d}\tau,$$

$$f(\tau) = \frac{\tau}{q(t)}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\tau-\mu}{q(t)\,p'(t) - \mu q'(t)}.$$

Again

$$f(t) = \sum_{n=0}^\infty a_n(\mu)(t-\mu)^n \;\Rightarrow\; F_\lambda(z) \sim \mathrm{e}^{-zA(\mu)}\sum_{n=0}^\infty a_n(\mu)\,P_n(\lambda)\,z^{-n-\lambda},$$

as $z \to \infty$, uniformly valid with respect to $\mu \in [0, \infty)$. An integration by parts procedure g
a slightly better expansion. For details we refer to [89].

5. Statistical distribution functions

Many cumulative distribution functions can be transformed into the standard form:

$$F_a(\eta) = \sqrt{\frac{a}{2\pi}}\int_{-\infty}^{\eta} \mathrm{e}^{-(1/2)a\zeta^2}f(\zeta)\,\mathrm{d}\zeta,$$

with f analytic in a neighbourhood of $\mathbb{R}$. The problem is to obtain an asymptotic expansion as $a \to \infty$, that holds uniformly with respect to $\eta \in \mathbb{R}$. The saddle point at the origin may be inside or outside the interval of integration. Especially, the transition area $\eta \sim 0$ is of interest. In terminology of asymptotics: a saddle point may cross the end-point of integration. This type of distribution function occurs frequently; in fact, the gamma distribution, the beta distribution and many other cumulative distribution functions can be transformed into the standard form (5.1).

The basic approximant is the normal distribution function (error function)

$$P(\eta\sqrt{a}) = \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\eta} e^{-(1/2)a\zeta^2}\, d\zeta = \tfrac{1}{2}\operatorname{erfc}(-\eta\sqrt{\tfrac{1}{2}a}).$$

The procedure to obtain a uniform expansion is based on integration by parts. We normalize $f(0) = 1$ and write $f(\zeta) = 1 + [f(\zeta) - 1]$. Then

$$\begin{aligned} F_a(\eta) &= P(\eta\sqrt{a}) - \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\eta} \frac{f(\zeta) - 1}{\zeta}\, d e^{-(1/2)a\zeta^2} \\ &= P(\eta\sqrt{a}) + \frac{1}{\sqrt{2\pi a}} \frac{1 - f(\eta)}{\eta} e^{-(1/2)a\eta^2} + \frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\eta} e^{-(1/2)a\zeta^2} f_1(\zeta)\, d\zeta, \end{aligned}$$

where

$$f_1(\eta) = \frac{d}{d\eta} \frac{f(\eta) - 1}{\eta}.$$

Repeating this process, we obtain

$$F_a(\eta) = P(\eta\sqrt{a})\, A(a) + \frac{e^{-(1/2)a\eta^2}}{\sqrt{2\pi a}} B_\eta(a),$$

with

$$A(a) \sim \sum_{n=0}^{\infty} \frac{A_n}{a^n}, \qquad B_\eta(a) \sim \sum_{n=0}^{\infty} \frac{B_n(\eta)}{a^n},$$

where the coefficients are given by

$$A_n = f_n(0), \quad B_n(\eta) = \frac{f_n(0) - f_n(\eta)}{\eta}, \quad f_n(\eta) = \frac{d}{d\eta} \frac{f_{n-1}(\eta) - f_{n-1}(0)}{\eta}.$$

For more details and applications to several distribution functions we refer to [87].

6. Examples of standard forms

In Table 1 we give standard forms of integrals for which well-known special functions are used as basic approximants. We give the critical points, the coalescence of which causes uniformity problems. We also give references to the literature.

Table 1
Standard forms of integrals

	Standard form	Approximant	Critical points
(1)	$\int_{-\infty}^{\infty} e^{-zt^2}\dfrac{f(t)}{t-i\alpha}\,dt$	Error function	$t=0,\ t=i\alpha$
(2)	$\int_{-\infty}^{\infty} e^{-zt^2}\dfrac{f(t)}{(t-i\alpha)^{\mu}}\,dt$	Weber function	$t=0,\ t=i\alpha$
(3)	$\int_{-\infty}^{\alpha} e^{-zt^2}f(t)\,dt$	Error function	$t=0,\ t=\alpha$
(4)	$\int_0^{\infty} t^{\beta-1}e^{-z(t^2/2-\alpha t)}f(t)\,dt$	Weber function	$t=0,\ t=\alpha$
(5)	$\int_{\mathscr{L}} e^{z(t^3/3-\alpha t)}f(t)\,dt$	Airy function	$t=\pm\sqrt{\alpha}$
(6)	$\int_0^{\infty} t^{\alpha-1}e^{-zt}f(t)\,dt$	Gamma function	$t=0,\ t=\alpha/z$
(7)	$\int_{\alpha}^{\infty} t^{\beta-1}e^{-zt}f(t)\,dt$	Incomplete gamma function	$t=0,\ t=\alpha$
(8)	$\int_0^{\infty} t^{\beta-1}e^{-z(t+\alpha/t)}f(t)\,dt$	Bessel function	$t=0,\ t=\pm\sqrt{\alpha}$
(9)	$\int_{\alpha}^{\infty} e^{-zt}(t^2-\alpha^2)^{\mu}f(t)\,dt$	Bessel function	$t=\pm\alpha$
(10)	$\int_0^{\infty}\dfrac{\sin z(t-\alpha)}{t-\alpha}f(t)\,dt$	Sine integral	$t=0,\ t=\alpha$

Remarks.

(1) We assume that the functions f are analytic in a neighbourhood of the path of integration and at the critical points.

(2) The integrals reduce to their approximants when $f=1$.

(3) Different intervals of integration are also investigated.

(4) In (5) and (8) of Table 1, two saddle points coalesce when $\alpha=0$; the cases are different, because in (8) an additional critical point at $t=0$ is present (pole of the function in the exponent and end point of integration).

(5) In all cases elementary approximations can be used for fixed values of the parameter α.

(6) Several cases need further investigations with respect to the construction of error bounds and the determination of maximal regions of validity.

References to the bibliography

We give references to the literature for the standard forms mentioned in Table 1. In many cases the integrals are different from those in the table, but the asymptotic features may be the same in the given references.

(1): [21, 32, 36, 44, 82, 104].
(2): [15, 84].
(3): [32, 55, 62, 76, 85, 87, 92].
(4): [15, 16, 22, 41, 50, 56, 61, 65–67, 71, 74, 77, 106, 108].
(5): [20, 29, 30, 45, 49, 50, 66, 73, 75, 93, 98, 103, 108].
(6): [2, 69, 88, 89, 96].
(7): [68, 83, 94, 97, 109].

(8): [18, 26, 28, 29, 33, 63, 86, 93, 95].
(9): [27, 70, 101, 103, 107].
(10): [110].
Papers with other special functions as main approximants: [17, 22–25, 35, 38–40, 42, 56, 72, 73, 78, 79, 91, 99, 100, 102]
Monographs on asymptotics are [16, 50, 108].
Papers with overviews and general aspects are [60, 64, 66, 90].

7. Further examples

We consider a few recent examples of uniform expansions for particular special functions. We give a few details of the basic steps for deriving the results. More details can be found in the literature.

7.1. Laguerre polynomials

This important example is considered in [29]. The Laguerre polynomials are defined by

$$L_n^\alpha(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}$$

and have the generating function

$$(1-t)^{-\alpha-1} e^{-tx/(1-t)} = \sum_{n=0}^{\infty} L_n^\alpha(x) t^n, \quad |t| < 1.$$

Let

$$\kappa = n + \tfrac{1}{2}(\alpha + 1).$$

Then we can distinguish five x-intervals with different asymptotic behaviour for $L_n^\alpha(x)$ when n is large and α is fixed:

(i) $x < 0$: monotonic region,
(ii) x near 0: transition region,
(iii) $0 < x < 4\kappa$: oscillatory region,
(iv) x near 4κ: turning point region,
(v) $x > 4\kappa$: monotonic region.

In cases (i), (iii), (v) we can give expansions in terms of elementary functions. For the transitions from (i) to (iii), we need a Bessel function; from (iii) to (v), we need an Airy function (see Fig. 1).

When α is fixed, two uniform expansions suffice to cover the real x-axis with overlapping domains. In [29] the generating function is used to write the Laguerre polynomial as a Cauchy integral, and then the saddle point method is applied to obtain expansions in terms of Airy and Bessel functions.

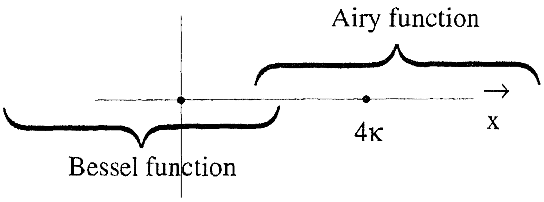

Fig. 1. The n zeros of $L_n^\alpha(x)$ occur in the interval $(0, 4\kappa)$.

Expansion in terms of Bessel functions

Let

$$A(x) = \begin{cases} \frac{1}{2}\mathrm{i}[\sqrt{x^2 - x} - \operatorname{arcsinh}\sqrt{-x}] & \text{if } x \leqslant 0, \\ \frac{1}{2}[\sqrt{x - x^2} + \arcsin\sqrt{x}] & \text{if } 0 \leqslant x < 1. \end{cases} \tag{7.1}$$

Then

$$2^\alpha \mathrm{e}^{-2\kappa x} L_n^\alpha(4\kappa x) \sim \frac{J_\alpha(\nu A)}{A^\alpha} \sum_{k=0}^{\infty} \frac{\alpha_{2k}}{(2\kappa)^{2k}} - \frac{J_{\alpha+1}(\nu A)}{A^{\alpha+1}} \sum_{k=0}^{\infty} \frac{\beta_{2k+1}}{(2\kappa)^{2k+1}}$$

as $n \to \infty$, uniformly for $x \in (-\infty, a]$, where $a \in (0, 1)$, $J_\alpha(z)$ is the familiar Bessel function. The coefficients α_{2k}, β_{2k+1} follow from recursions. Let

$$h(u) = \left[\frac{u}{\sinh z(u)}\right]^{\alpha+1} \frac{\mathrm{d}z}{\mathrm{d}u},$$

where the relation between z and u is given by the equation

$$z - x \coth z = u - \frac{A^2(x)}{u}.$$

Define a set of functions $\{h_k\}$, $\{g_k\}$ and coefficients $\{\alpha_k\}$, $\{\beta_k\}$ by writing $h_0(u) = h(u)$ and when $k \geqslant 1$:

$$h_k(u) = \alpha_k + \frac{\beta_k}{u} + \left(1 + \frac{A^2}{u^2}\right) g_k(u), \tag{7.2}$$

$$h_{k+1}(u) = g_k'(u) - \frac{\alpha + 1}{u} g_k(u).$$

The coefficients α_k, β_k are computed by substitution of $u = \pm \mathrm{i}A$ in (7.2); A is defined in (7.1).

Expansion in terms of Airy functions

Let

$$B(x) = \begin{cases} \mathrm{i}[3\beta(x)/2]^{1/3} & \text{if } 0 < x \leqslant 1, \\ [3\gamma(x)/2]^{1/3} & \text{if } x \geqslant 1, \end{cases} \tag{7.3}$$

$$\beta(x) = \tfrac{1}{4}\pi - A(x) = \tfrac{1}{2}[\arccos\sqrt{x} - \sqrt{x - x^2}],$$

$$\gamma(x) = \tfrac{1}{2}[\sqrt{x^2 - x} - \operatorname{arccosh}\sqrt{x}],$$

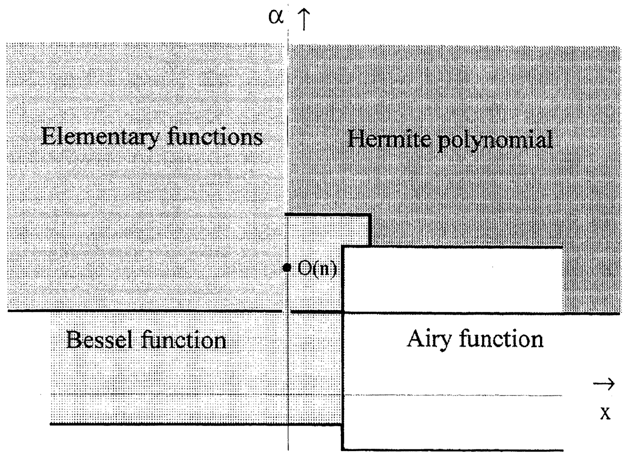

Fig. 2. When α and n of $L_n^\alpha(x)$ are both large and positive, 4 types of uniform asymptotic expansions and 4 domains are needed to cover the (x, α)-half-plane.

where for $x \in (0, 1]$ the quantity $A(x)$ is defined in (7.1). Then

$$(-1)^n 2^\alpha e^{-2\kappa x} L_n^\alpha(4\kappa x) \sim \mathrm{Ai}(\nu^{2/3} B^2) \sum_{k=0}^{\infty} \alpha_{2k} \nu^{-2k-1/3} - \mathrm{Ai}'(\nu^{2/3} B^2) \sum_{k=0}^{\infty} \beta_{2k+1} \nu^{-2k-5/3},$$

as $n \to \infty$, uniformly for $x \in (b, \infty]$, where $b \in (0, 1)$. $\mathrm{Ai}(z)$ is the Airy function. The coefficients α_{2k}, β_{2k+1} follow from the recursion

$$h_k(u) = a_k + \beta_k u + (u^2 - B^2) g_k(u), \qquad h_{k+1}(u) = g_k'(u), \tag{7.4}$$

with

$$h_0(u) = h(u) = [1 - z^2(u)]^{(\alpha-1)/2} \frac{dz}{du},$$

where the relation between z and u is given by

$$\tfrac{1}{2}[\operatorname{arctanh} z - xz] = \tfrac{1}{3}u^3 - B^2(x)u.$$

The coefficients α_k, β_k are computed by substitution of $u = \pm B$ in (7.4); B is defined in (7.3).

When α is also allowed to grow other expansions are needed (see [31] for expansions in the zeros domain in terms of elementary functions and [93] for expansions in terms of Bessel functions and Hermite polynomials); see Fig. 2.

7.2. Fermi–Dirac integral

Let

$$F_q(x) := \frac{1}{\Gamma(q+1)} \int_0^\infty \frac{t^q}{1 + e^{t-x}} \, dt, \quad q > -1.$$

These functions occur in quantum mechanics, for instance, in problems on the distribution of particles that follow the Fermi–Dirac statistics. The problem is to find an asymptotic expansion for large values of x which is uniformly valid with respect to $q \in [0, \infty)$. For details we refer to [68, 97].

When $x < 0$ we have the convergent expansion

$$F_q(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,\mathrm{e}^{xn}}{n^{q+1}}.$$

When $x > 0$ we can transform into the contour integral

$$F_q(x) = \frac{1}{2\mathrm{i}} \int_{\mathscr{L}} \frac{\mathrm{e}^{xs}}{s^{q+1}\sin \pi s}\,\mathrm{d}s,$$

where, initially, $\mathscr{L}$ is a vertical line that cuts the real s-axis between 0 and 1.

The saddle point s_0 of the dominant part is the zero of the derivative of $\exp(xs - q\ln s)$, that is, $s_0 = q/x$. Translate the contour to pass through the point s_0 and sum the residues. Assuming $s_0 \neq 1, 2, 3, \ldots,$ we obtain

$$F_q(x) = \sum_{n=1}^{N-1} \frac{(-1)^{n-1}\,\mathrm{e}^{xn}}{n^{q+1}} + \frac{1}{2\mathrm{i}} \int_{\mathscr{L}} \frac{\mathrm{e}^{xs}}{s^{q+1}\sin \pi s}\,\mathrm{d}s,$$

where N is the integer satisfying $N - 1 < s_0 < N$ and $\mathscr{L}$ cuts the real positive axis at s_0, the saddle point.

To separate the pole near the saddle point write

$$h_N(s) = \frac{\pi}{\sin \pi s} - \frac{(-1)^N}{s - N}, \quad N = 0, 1, 2, \ldots .$$

Then

$$F_q(x) = \sum_{n=1}^{N-1} \frac{(-1)^{n-1}\,\mathrm{e}^{xn}}{n^{q+1}} + G_q(x) + H_q(x),$$

$$G_q(x) = \frac{(-1)^N}{2\pi\mathrm{i}} \int_{\mathscr{L}} \frac{\mathrm{e}^{xs}}{s^{q+1}(s-N)}\,\mathrm{d}s = \frac{(-1)^N\,\mathrm{e}^{xN}}{N^{q+1}}\, Q(q+1, xN),$$

$$H_q(x) = \frac{1}{2\pi\mathrm{i}} \int_{\mathscr{L}} \frac{\mathrm{e}^{xs}}{s^{q+1}}\, h_N(s)\,\mathrm{d}s,$$

where Q is the incomplete gamma function ratio:

$$Q(a, z) = \frac{1}{\Gamma(a)} \int_z^{\infty} t^{a-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

By expanding $h_N(s)$ at the saddle point:

$$h_N(s) = \sum_{n=0}^{\infty} c_n(s_0)(s - s_0)^n$$

we obtain

$$H_q(x) \sim \frac{x^q}{\Gamma(q+1)} \sum_{n=0}^{\infty} c_n(s_0)\,\phi_n(q)\,x^{-n}$$

where ϕ_n are simple polynomials defined by

$$\phi_n(q) = \frac{\Gamma(q+1)}{2\pi \mathrm{i}} \int_{\mathscr{L}} \frac{\mathrm{e}^t (t-q)^n}{t^{q+1}}\,\mathrm{d}t$$

$$\phi_0(q) = 1, \qquad \phi_1(q) = 0.$$

Higher ϕ_n's can be obtained by recursion.

7.3. Stirling numbers

The Stirling numbers of the first and second kind, denoted by $S_n^{(m)}$ and $\mathfrak{S}_n^{(m)}$, respectively, are defined as the coefficients in the expansions

$$x(x-1)\cdots(x-n+1) = \sum_{m=0}^{n} S_n^{(m)} x^m,$$

$$x^n = \sum_{m=0}^{n} \mathfrak{S}_n^{(m)} x(x-1)\cdots(x-m+1).$$

The problem is to find the asymptotic behaviour of the Stirling numbers as $n \to \infty$, uniformly with respect to $m \in [0, n]$.

Alternative generating functions:

$$\frac{[\ln(x+1)]^m}{m!} = \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!}, \qquad \frac{(\mathrm{e}^x - 1)^m}{m!} = \sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)} \frac{x^n}{n!}.$$

This gives the Cauchy-type integrals

$$\mathfrak{S}_n^{(m)} = \frac{n!}{m!} \frac{1}{2\pi \mathrm{i}} \oint \frac{(\mathrm{e}^x - 1)^m}{x^{n+1}}\,\mathrm{d}x,$$

$$(-1)^{n-m} S_{n+1}^{(m+1)} = \frac{1}{2\pi \mathrm{i}} \oint \frac{(x+1)(x+2)\cdots(x+n)}{x^{m+1}}\,\mathrm{d}x.$$

Saddle point methods, suitably modified, give the uniform expansions; see [96].

7.4. Incomplete gamma functions

The definitions are

$$\gamma(a, z) = \int_0^z t^{a-1} \mathrm{e}^{-t}\,\mathrm{d}t, \qquad \Gamma(a, z) = \int_z^{\infty} t^{a-1} \mathrm{e}^{-t}\,\mathrm{d}t$$

with normalizations

$$P(a,z)=\frac{\gamma(a,z)}{\Gamma(a)},\qquad Q(a,z)=\frac{\Gamma(a,z)}{\Gamma(a)}.$$

Then $P(a,z)+Q(a,z)=1$. Two methods discussed earlier:
- statistical distribution functions (see (3) in Table 1)
- pole near saddle point (see (1) in Table 1)

give the representations

$$Q(a,z)=\tfrac{1}{2}\operatorname{erfc}(\eta\sqrt{\tfrac{1}{2}a})+R_a(\eta),$$

$$P(a,z)=\tfrac{1}{2}\operatorname{erfc}(-\eta\sqrt{\tfrac{1}{2}a})-R_a(\eta),$$

$$\eta=(\lambda-1)\sqrt{2\frac{\lambda-1-\ln\lambda}{(\lambda-1)^2}},\quad \lambda=\frac{z}{a},$$

$$R_a(\eta)\sim\frac{e^{-(1/2)a\eta^2}}{\sqrt{2\pi a}}\sum_{n=0}^{\infty}\frac{c_n(\eta)}{a^n},\quad a\to\infty$$

uniformly with respect to $n\in\mathbb{R}$, or $z\geqslant 0$. For details we refer to [85]. This reference also admits complex values of a and z.

8. The Stokes phenomenon

The Stokes phenomenon concerns the abrupt change across certain rays in the complex plane, known as *Stokes lines*, exhibited by the coefficients multiplying exponentially subdominant terms in compound asymptotic expansions. There is much recent interest in the Stokes phenomenon, and it fits in the present paper because it has to do with sudden changes in approximations when a certain parameter (in this case the phase of the large parameter) passes critical values. A complete discussion will not be given here. For a recent survey paper on the Stokes phenomenon we refer to [59].

8.1. The Airy function

First we explain this phenomenon by using a simple example from differential equations. Consider Airy's equation

$$\frac{d^2y}{dz^2}=z\,y(z),\tag{8.1}$$

the solutions of which are entire functions. When $|z|$ is large the solutions of (8.1) are approximated by linear combinations of

$$u_{\pm}=z^{-1/4}e^{\pm\xi},\qquad \xi=\tfrac{2}{3}z^{3/2}.\tag{8.2}$$

Obviously, $u_\pm$ are multivalued functions of the complex variable z with a branch point at $z=0$. Therefore, as we go once around the origin, the solutions of (8.1) will return to their original values, but $u_\pm$ will not. It follows that the constants $c_\pm$ in the linear combination

$$y(z) \sim c_- u_-(z) + c_+ u_+(z), \quad z \to \infty$$

are domain-dependent. The constants change when we cross certain lines, the boundaries of certain sectors in the z-plane.

In the above example one of the terms e^{ξ}, $e^{-\xi}$ maximally dominates the other one at the rays $\arg z = 0$, $\arg z = \pm\frac{2}{3}\pi$. In this example these 3 rays are the Stokes lines. At the rays $\arg z = \pm\frac{1}{3}\pi$ and the negative z-axis the quantity ξ is purely imaginary, and, hence, the terms e^{ξ}, $e^{-\xi}$ are equal in magnitude. These three rays are called the *anti-Stokes lines.*[1]

For the Airy function $\mathrm{Ai}(z)$ we have the full asymptotic expansion (see [50, p. 116])

$$\mathrm{Ai}(z) \sim c_- z^{-1/4} e^{-\xi} \sum_{n=0}^{\infty} (-1)^n c_n \xi^{-n}, \quad c_- = \tfrac{1}{2}\pi^{-1/2}, \quad |\arg z| < \pi, \tag{8.3}$$

with coefficients

$$c_n = \frac{\Gamma(3n+\frac{1}{2})}{54^n n! \Gamma(n+\frac{1}{2})}, \quad n = 0, 1, 2, \ldots .$$

On the other hand, in an other sector of the z-plane, we have

$$\mathrm{Ai}(z) \sim c_- z^{-1/4} \left[e^{-\xi} \sum_{n=0}^{\infty} (-1)^n c_n \xi^{-n} + i e^{\xi} \sum_{n=0}^{\infty} c_n \xi^{-n} \right], \tag{8.4}$$

in which exactly the same term (with the same constant c_-) is involved as in (8.3), and there is another term corresponding to u_+. We can rewrite this in a more familiar expansion

$$\mathrm{Ai}(-z) \sim \pi^{-1/2} z^{-1/4} \left[\sin(\xi + \tfrac{1}{4}\pi) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n}}{\xi^{2n}} - \cos(\xi + \tfrac{1}{4}\pi) \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n+1}}{\xi^{2n+1}} \right] \tag{8.5}$$

in the sector $|\arg z| < \frac{2}{3}\pi$. In the overlapping domain of expansions (8.3) and (8.5), that is, when $\frac{1}{3}\pi < |\arg z| < \pi$, the term with u_+ is asymptotically small compared to u_-, and it suddenly appears in the asymptotic approximation when we cross with increasing values of $|\arg z|$ the Stokes lines at $\arg z = \pm\frac{2}{3}\pi$. It seems that, when going from (8.3) to (8.4), the constant multiplying u_+ changes discontinuously from zero values (when $|\arg z| < \frac{2}{3}\pi$) to a nonzero value when we cross the Stokes line. This sudden appearance of the term u_+ does not have much influence on the asymptotic behaviour near the Stokes lines at $|\arg z| = \frac{2}{3}\pi$, because u_+ is dominated maximally by u_- at these rays. However, see Section 8.3 below.

8.2. The recent interest in the Stokes phenomenon

This phenomenon of the *discontinuity* of the constants was discovered by Stokes and was discussed by him in a series of papers (on Airy functions in 1857, on Bessel functions in 1868). It is

[1] This terminology is not the same in all branches of applied mathematics and mathematical physics: sometimes one sees a complete interchange of the names "Stokes line" and "anti-Stokes line".

a phenomenon which is not confined to Airy or Bessel functions. The discovery by Stokes was, as Watson says, apparently one of those which are made at three o'clock in the morning. Stokes wrote in a 1902 retrospective paper: "The inferior term enters as it were into a mist, is hidden for a little from view, and comes out with its coefficients changed".

In 1989 the mathematical physicist Michael Berry provided a deeper explanation. He suggested that the coefficients of the subdominant expansion should be regarded not as a discontinuous constant but, for fixed $|z|$, as a continuous function of $\arg z$. Berry's innovative and insightful approach was followed by a series of papers by himself and other writers. In particular, Olver put the formal approach by Berry on a rigorous footing in papers with applications to confluent hypergeometric functions (including Airy functions, Bessel functions, and Weber parabolic functions).

At the same time interest arose in earlier work by Stieltjes, Airey, Dingle,... to expand remainders of asymptotic expansions at optimal values of the summation variable. This resulted in *exponentially improved* asymptotic expansions, a method of improving asymptotic approximations as we have met in Section 1.3 in our introduction to Stieltjes work on asymptotics.

8.3. Exponentially small terms in the Airy expansions

We conclude this discussion by pointing out the relation between the Stokes phenomenon and the exponentially small terms in the asymptotic expansion of the Airy function. Consider the terms in the expansions in (8.3)–(8.5). They have the asymptotic form

$$c_n\xi^{-n} = \mathcal{O}[\Gamma(n)(2\xi)^{-n}], \quad n \to \infty.$$

When z is large the terms decrease at first and then increase. The least term of the first series of (8.4) is near $n = n^* = [|2\xi|]$ and its size is of order $e^{-2|\xi|}$. At the Stokes lines at $|\arg z| = \frac{2}{3}\pi$ the quantity ξ is negative and the exponential term in front of the first series in (8.4) equals $e^{|\xi|}$. Hence, the order of magnitude of $e^{-\xi}c_{n^*}\xi^{-n^*}$ is roughly of the same size as the second part in (8.5), that is, of the size of e^{ξ} that is present in front of the second series. It follows that near the Stokes lines (and of course when z turns to the negative axis) the second series in (8.5) is not at all negligible when we truncate the first series at the least term with index n^*.

At present we know, after Berry's observations, that near the Stokes lines one of the constants $c_\pm$ in the asymptotic representation in (8.2) in fact is a rapidly changing function of z. In the case of (8.4) we can write

$$\mathrm{Ai}(z) \sim c_- z^{-1/4}\left[e^{-\xi}\sum_{n=0}^{\infty}(-1)^n c_n\xi^{-n} + \mathrm{i}S(z)e^{\xi}\sum_{n=0}^{\infty}c_n\xi^{-n}\right],$$

where $S(z)$ switches rapidly but smoothly from 0 to 1 across the Stokes line at $\arg z = \frac{2}{3}\pi$. As Berry observed (see [3]), a good approximation to $S(z)$ involves the error function, which can describe the fast transition in this asymptotic problem. We have met the error function also in the cases (1) and (3) of Table 1, where it is used to describe similar fast transitions.

Many other writers have contributed recently to this field. In the References we have included recent papers concentrating on integrals, but the research is also making much progress in the area of differential equations. In [43] an introduction to the Stokes phenomenon is given from the viewpoint of differential equations; for more recent developments (also in connection with exponentially improved asymptotic expansions) we refer to [8, 48, 54]. For integrals we refer to [3–14, 19, 34, 37, 46, 47, 51–53, 57, 59].

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References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Nat. Bur. Standards Appl. Series, **55** (U.S. Government Printing Office, Washington, DC, 1964).

[2] D.E. Amos, Uniform asymptotic expansions for exponential integrals $E_n(x)$ and Bickley functions $Ki_n(x)$, *ACM Trans. Math. Software* **9** (1983) 467–479.

[3] M.V. Berry, Uniform asymptotic smoothing of Stokes' discontinuities, *Proc. Roy. Soc. Lond. Ser. A* **422** (1989) 7–21.

[4] M.V. Berry, Stokes' phenomenon; smoothing a Victorian discontinuity, *Publ. Math. IHES* **68** (1989) 211–221.

[5] M.V. Berry, Infinitely many Stokes smoothings in the gamma function, *Proc. Roy. Soc. Lond. Ser. A* **434** (1991) 465–472.

[6] M.V. Berry, Asymptotic, superasymptotics, hyperasymptotics, in: E.H. Segur et al., Eds., *Asymptotics Beyond all Orders* (Plenum Press, New York, 1991) 1–14.

[7] M.V. Berry and C.J. Howls, Stokes surfaces of diffraction catastrophes with codimension three, *Nonlinearity* **3** (1990) 281–291.

[8] M.V. Berry and C.J. Howls, Hyperasymptotics, *Proc. Roy. Soc. Lond. Ser. A* **430** (1990) 653–668.

[9] M.V. Berry and C.J. Howls, Hyperasymptotics for integrals with saddles, *Proc. Roy. Soc. Lond. Ser. A* **434** (1991) 657–675.

[10] M.V. Berry and C.J. Howls, Unfolding the high orders of asymptotic expansions with coalescing saddles: singularity theory, crossover and duality, *Proc. Roy. Soc. Lond. Ser. A* **443** (1993) 107–126.

[11] M.V. Berry and C.J. Howls, Infinity interpreted, *Physics World* (1993) 35–39.

[12] M.V. Berry and C.J. Howls, Overlapping Stokes smoothings: survival of the error function and canonical catastrophe integrals, *Proc. Roy. Soc. Lond. Ser. A* **444** (1994) 201–216.

[13] M.V. Berry and C.J. Howls, High orders of the Weyl expansion for quantum billiards: resurgence of periodic orbits and the Stokes phenomenon, submitted.

[14] M.V. Berry and J.P. Keating, A new asymptotic representation for $\zeta(1/2 + it)$ and quantum spectral determinants, *Proc. Roy. Soc. Lond. Ser. A* **437** (1992) 151–173.

[15] N. Bleistein, Uniform asymptotic expansions of integrals with stationary points and algebraic singularity, *Comm. Pure Appl. Math.* **19** (1966) 353–370.

[16] N. Bleistein and R.A. Handelsman, *Asymptotic Expansions of Integrals* (Holt, Rinehart and Winston, New York, 1975).

[17] J. Boersma, Uniform asymptotics of a Bessel-function series occurring in a transmission-line problem, *J. Comput. Appl. Math.* **37** (1991) 143–159.

[18] Bo Rui and R. Wong, Uniform asymptotic expansion of Charlier polynomials, *Methods Appl. Analysis* **1** (1994) 294–313.

[19] W.G.C. Boyd, Stieltjes transforms and the Stokes phenomenon, *Proc. Roy. Soc. Lond. Ser. A* **429** (1990) 227–246.

[20] C. Chester, B. Friedman and F. Ursell, An extension of the method of steepest descent, *Proc. Cambridge Philos. Soc.* **53** (1957) 599–611.

[21] A. Ciarkowski, Uniform and quasi-uniform asymptotic expansions of incomplete diffraction integrals, *SIAM J. Appl. Math.* **48** (1988) 1217–1226.

[22] A. Ciarkowski, Uniform asymptotic expansion of an integral with a saddle point, a pole and a branch point, *Proc. Roy. Soc. Lond. Ser. A* **426** (1989) 273–286.

[23] J.N.L. Connor, Practical methods for the uniform asymptotic expansion of oscillating integrals with several coalescing saddle points, in: R. Wong, Ed., *Asymptotic and Computational Analysis* (Marcel Dekker, New York, 1990) 137–173.

[24] J.N.L. Connor, P.R. Curtis and D. Farrelly, The uniform asymptotic swallowtail approximation: practical methods for oscillating integrals with four coalescing saddle points, *J. Phys. A: Math. Gen.* **17** (1984) 283–310.

[25] J.N.L. Connor, P.R. Curtis and R.A.W. Young, Uniform asymptotics of oscillating integrals: applications in chemical physics, in: P.A. Martin and G.R. Wickham, Eds., *Wave Asymptotics, Proc. of the Meeting to Mark the Retirement of Fritz Ursell* (Cambridge Univ. Press, Cambridge, 1992) 283–310.

[26] H.H. Day and R. Wong, A uniform asymptotic expansion for the shear wave front layer, *Wave Motion* **19** (1994) 293–308.

[27] C.L. Frenzen, Error bounds for a uniform asymptotic expansion of the Legendre function $Q_n^{-m}(\cosh z)$, *SIAM J. Math. Anal.* **21** (1990) 523–535.

[28] C.L. Frenzen and R. Wong, A uniform asymptotic expansion of the Jacobi polynomials with error bounds, *Can. J. Math.* **37** (1985) 979–1007.

[29] C.L. Frenzen and R. Wong, Uniform asymptotic expansions of Laguerre polynomials, *SIAM J. Math. Anal.* **19** (1988) 1232–1248.

[30] B. Friedman, Stationary phase with neighboring critical points, *SIAM J.* **7** (1959) 280–289.

[31] W. Gawronski, Strong asymptotics and the asymptotic zero distributions of Laguerre polynomials $L_n^{an+\alpha}$ and Hermite polynomials $H_n^{an+\alpha}$, *Analysis* **13** (1993) 29–67.

[32] A. Guthmann, Asymptotische Entwicklungen für unvollständige Gammafunktionen, *Forum Math.* **3** (1991) 105–141.

[33] R.A. Handelsman and N. Bleistein, Uniform asymptotic expansions of integrals that arise in the analysis of precursors, *Arch. Rat. Mech. Anal.* **35** (1969) 267–283.

[34] C.J. Howls, Hyperasymptotics for integrals with finite endpoints, *Proc. Roy. Soc. Lond. Ser. A* **439** (1992) 373–396.

[35] A.J.E.M. Janssen, On the asymptotics of some Pearcey-type integrals, *J. Phys. A* **25** (1992) 823–831.

[36] D.S. Jones, A uniform expansion for a certain double integral, *Proc. Roy. Soc. Edinburgh Sect. A* **69** (1970/71) 205–226.

[37] D.S. Jones, Uniform asymptotic remainders, in: R. Wong, Ed., *Asymptotic and Computational Analysis* (Marcel Dekker, New York, 1990) 241–264.

[38] D. Kaminski, Asymptotic expansion of the Pearcey integral near the caustic, *SIAM J. Math. Anal.* **20** (1989) 987–1005.

[39] D. Kaminski, Asymptotics of the swallowtail integral near the cusp of the caustic, *SIAM J. Math. Anal.* **23** (1992) 262–285.

[40] A.S. Kryukovsky, D.S. Lukin and E.A. Palkin, Uniform asymptotics for evaluating oscillatory edge integrals by methods of catastrophy theory, *Sov. J. Numer. Anal. Math. Modelling* **2** (1987) 279–312.

[41] C. Leubner and H. Ritsch, A note on the uniform asymptotic expansion of integrals with coalescing endpoint and saddle points, *J. Phys. A* **19** (1986) 329–335.

[42] J. Martin, Integrals with a large parameter and several nearly coincident saddle points; the continuation of uniformly asymptotic expansions, *Proc. Cambridge Philos. Soc.* **76** (1974) 211–231.

[43] R.E. Meyer, A simple explanation of the Stokes phenomenon, *SIAM Rev.* **31** (1989) 435–445.

[44] C.R. Mondal, Uniform asymptotic analysis of shallow-water waves due to a periodic surface pressure, *Quart. Appl. Math.* **44** (1986) 133–140.

[45] A.B. Olde Daalhuis, Asymptotic expansions of an integral containing a phase function with three saddle points, CWI, Report AM-R8922, Amsterdam, 1989.

[46] A.B. Olde Daalhuis, Hyperasymptotic expansions of confluent hypergeometric functions, *IMA J. Appl. Math.* **49**, (1992) 203–216.

[47] A.B. Olde Daalhuis, Hyperasymptotics and the Stokes' phenomenon, *Proc. Roy. Soc. Edinburgh Sect. A* **123** (1993) 731–743.

[48] A.B. Olde Daalhuis and F.W.J. Olver, Exponentially-improved asymptotic solutions of ordinary differential equations. II, *Proc. Roy. Soc. Lond. Ser. A* **445** (1994) 39–56.

[49] A.B. Olde Daalhuis and N.M. Temme, Uniform Airy type expansions of integrals, *SIAM J. Math. Anal.* **25** (1994) 304–321.

[50] F.W.J. Olver, *Asymptotics and Special Functions* (Academic Press, New York, 1974).

[51] F.W.J. Olver, On Stokes' phenomenon and converging factors, in: R. Wong, Ed., *Asymptotic and Computational Analysis* (Marcel Dekker, New York, 1990) 329–355.

[52] F.W.J. Olver, Uniform, exponentially improved, asymptotic expansions for the generalized exponential integral, *SIAM J. Math. Anal.* **22** (1991) 1460–1474.

[53] F.W.J. Olver, Uniform, exponentially improved, asymptotic expansions for the confluent hypergeometric function and other integral transforms, *SIAM J. Math. Anal.* **22** (1991) 1475–1489.

[54] F.W.J. Olver, Exponentially-improved asymptotic solutions of ordinary differential equations. I: the confluent hypergeometric function, *SIAM J. Math. Anal.* **24** (1991) 756–767.

[55] R.B. Paris, On a generalization of a result of Ramanujan connected with the exponential series, *J. Proc. Edinburgh Math. Soc.* **24** (1989) 179–195.

[56] R.B. Paris, The asymptotic behaviour of Pearcey's integral for complex variables, *Proc. Roy. Soc. Lond. Ser. A* **432** (1991) 391–426.

[57] R.B. Paris, Smoothing of the Stokes phenomenon using Mellin–Barnes integrals, *J. Comput. Appl. Math.* **41** (1992) 117–133.

[58] R.B. Paris and A.D. Wood, Exponentially-improved asymptotics for the gamma function, *J. Comput. Appl. Math.* **41** (1992) 135–143.

[59] R.B. Paris and A.D. Wood, Stokes phenomenon demystified, *IMA Bull.* **31** (1995) 21–28.

[60] C.K. Qu and R. Wong, Transformations to canonical form for uniform asymptotic expansions, *J. Math. Anal. Appl.* **149** (1990) 210–219.

[61] V. Riekstina, Uniform asymptotic expansion of a certain class of integrals in the case of a multiple critical point, *Latv. Mat. Erzheg.* (*Latvian Math. Yearbook*) **20** (1976) 58–71 (in Russian).

[62] V. Riekstina, Uniform asymptotic expansion of a class of contour integrals, *Latv. Mat. Erzheg.* (*Latvian Math. Yearbook*) **27** (1983) 160–171 (in Russian).

[63] V. Riekstina, Uniform asymptotic expansions of a class of integrals, *Latv. Mat. Erzheg.* (*Latvian Math. Yearbook*) **30** (1986) 233–244 (in Russian).

[64] E. Riekstins, On uniform and nonuniform representations of integrals, *Latv. Mat. Erzheg.* (*Latvian Math. Yearbook*) **17** (1976) 36–49 (in Russian).

[65] H.-J. Schell, Asymptotische Entwicklungen für die unvollständige Gammafunktion, *Wiss. Z. Tech. Hochsch. Karl-Marx-Stadt* **22** (1980) 477–485.

[66] H.-J. Schell, Gleichmässige asymptotische Darstellungen und Entwicklungen von Parameterintegralen mit zwei reellen Parametern, Naturwiss. Diss., Karl-Marx-Stadt, Techn. Hochsch., 1981.

[67] H.-J. Schell, Gleichmässige asymptotische Entwicklungen für unvollständige Integrale, *Z. Anal. Anwend.* **2** (1983) 427–442.

[68] H.-J. Schell, Über das asymptotische Verhalten des Fermi–Dirac-Integrals, *Z. Anal. Anwend.* **6** (1987) 421–438.

[69] H.-J. Schell, Asymptotische Entwicklungen der konfluenten hypergeometrischen Funktionen $U(a,b,z)$ und $M(a,b,z)$ für große Werte von b und z, *Z. Anal. Anwend.* **9** (1990) 361–377.

[70] P.N. Shivakumar and R. Wong, Error bounds for a uniform asymptotic expansion of the Legendre function $P_n^{-m}(\cosh z)$, *Quart. Appl. Math.* **46** (1988) 473–488.

[71] L.A. Skinner, Uniformly valid expansions for Laplace integrals, *SIAM J. Math. Anal.* **11** (1980) 1058–1067. (Erratum in same journal: **12** (1981) 487.)

[72] L.A. Skinner, Uniformly valid composite expansions for Laplace integrals, *SIAM J. Math. Anal.* **19** (1988) 918–925.

[73] L.A. Skinner, Matched asymptotic expansions of integrals, *IMA J. Appl. Math.* **50** (1993) 77–90.

[74] K. Soni and B.D. Sleeman, On uniform asymptotic expansions and associated polynomials, *J. Math. Anal. Appl.* **124** (1987) 561–583.

[75] K. Soni and R.P. Soni, A system of polynomials associated with the Chester, Friedman, and Ursell technique, in: R. Wong, Ed., *Asymptotic and Computational Analysis* (Marcel Dekker, New York, 1990) 417–440.

[76] K. Soni and R.P. Soni, An approximation connected with the exponential function, *Proc. Amer. Math. Soc.* **114** (1992) 909–918.

[77] K. Soni and N.M. Temme, On a biorthogonal system associated with uniform asymptotic expansions, *IMA J. Appl. Math.* **44** (1990) 1–25.

[78] U. Steinacker, C. Leubner and S.L. Kalla, Uniform asymptotic expansions of a class of integrals with finite endpoints of integration on the same path of steepest descent and with nearby saddle points, *J. Comput. Appl. Math.* **35** (1991) 297–302.

[79] U. Steinacker, C. Leubner and S.L. Kalla, Uniform asymptotic expansions of a class of incomplete cylindrical functions, *J. Comput. Appl. Math.* **44** (1992) 121–130.

[80] T.J. Stieltjes, Recherches sur quelques séries semi-convergents (Thèse de doctorat), *Ann. Sci. Éc. Norm. Paris Sér.* 3 **3** (1886) 201–258. *Collected Papers*, Vol. 2 (Springer, Berlin, 1993) 6–62.

[81] T.J. Stieltjes, Sur la développement de $\log\Gamma(a)$, *J. Math. Paris Sér.* **4**, **5** (1889) 425–444. *Collected Papers*, Vol. 2 (Springer, Berlin, 1993) 215–234.

[82] N.M. Temme, Uniform asymptotic expansions of the incomplete gamma functions and the incomplete beta function, *Math. Comput.* **29** (1975) 1109–1114.

[83] N.M. Temme, Remarks on a paper of A. Erdelyi, *SIAM J. Math. Anal.* **7** (1976) 767–770.

[84] N.M. Temme, Uniform asymptotic expansions of confluent hypergeometric functions, *J. Inst. Math. Appl.* **22** (1978) 215–223.

[85] N.M. Temme, The asymptotic expansions of the incomplete gamma functions, *SIAM J. Math. Anal.* **10** (1979) 757–766.

[86] N.M. Temme, On the expansion of confluent hypergeometric functions in terms of Bessel functions, *J. Comput. Appl. Math.* **7** (1981) 27–32.

[87] N.M. Temme, The uniform asymptotic expansion of a class of integrals related to cumulative distribution functions, *SIAM J. Math. Anal.* **13** (1982) 239–253.

[88] N.M. Temme, Uniform asymptotic expansions of Laplace integrals, *Analysis* **3** (1983) 221–249.

[89] N.M. Temme, Laplace integrals: transformation to standard form and uniform asymptotic expansion, *Quart. Appl. Math.* **XLIII** (1985) 103–123.

[90] N.M. Temme, Special functions as approximants in uniform asymptotic expansions of integrals; a survey, *Special Functions: Theory and Computation, Int. Conf.*, Torino, 1984, *Rend. Semin. Mat.*, Torino, Fasc. Spec. (1985) 289–317.

[91] N.M. Temme, Uniform asymptotic expansion for a class of polynomials biorthogonal on the unit circle, *Constr. Approx.* **2** (1986) 369–376.

[92] N.M. Temme, A double integral containing the modified Bessel function: asymptotics and computation, *Math. Comput.* **47** (1986) 683–691.

[93] N.M. Temme, Laguerre polynomials: asymptotics for large degree, CWI Report AM-R8610, Amsterdam, 1986.

[94] N.M. Temme, Incomplete Laplace integrals: uniform asymptotic expansions with application to the incomplete beta function, *SIAM J. Math. Anal.* **18** (1987) 1638–1663.

[95] N.M. Temme, Uniform asymptotic expansions of a class of integrals in terms of modified Bessel functions, with application to confluent hypergeometric functions, *SIAM J. Math. Anal.* **21** (1990) 241–261.

[96] N.M. Temme, Asymptotic estimates of Stirling numbers, *Stud. Appl. Math.* **89** (1993) 233–243.

[97] N.M. Temme and A.B. Olde Daalhuis, Uniform asymptotic approximation of Fermi–Dirac integrals, *J. Comput. Appl. Math.* **31** (1990) 383–387.

[98] F. Ursell, Integrals with a large parameter. The continuation of uniformly asymptotic expansions, *Proc. Cambridge Philos. Soc.* **61** (1965) 113–128.

[99] F. Ursell, Integrals with a large parameter: Several nearly coincident saddle points, *Proc. Cambridge Philos. Soc.* **72** (1980) 49–65.

[100] F. Ursell, Integrals with a large parameter: a double complex integral with four nearly coincident saddle points. *Math. Proc. Cambridge Philos. Soc.* **87** (1980) 249–273.

[101] F. Ursell, Integrals with a large parameter: Legendre functions of large degree and fixed order, *Math. Proc. Cambridge Philos. Soc.* **95** (1984) 367–380.

[102] F. Ursell, Uniformly asymptotic expansions for an integral with a large and a small parameter, *Math. Proc. Cambridge Philos. Soc.* **101** (1986) 349–362.

[103] F. Ursell, Integrals with a large parameter and the maximum-modulus principle, in R. Wong, Ed., *Asymptotic and Computational Analysis* (Marcel Dekker, New York, 1990) 477–490.

[104] B.L. Van Der Waerden, On the method of saddle points, *Appl. Sci. Res. B* **2** (1951) 33–45.
[105] G.N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge Univ. Press, London and New York, 1944).
[106] R. Wong, On uniform asymptotic expansion of definite integrals, *J. Approx. Theory* **7** (1973) 76–86.
[107] R. Wong, On a uniform asymptotic expansion of a Fourier-type integral, *Quart. Appl. Math.* **38** (1980) 225–234.
[108] R. Wong, *Asymptotic Approximations of Integrals* (Academic Press, New York, 1989).
[109] A.S. Zil'bergleit, Uniform asymptotic expansions of some definite integrals, *USSR Comput. Math. Math. Phys.* **17** (1977) 36–44.
[110] A.S. Zil'bergleit, A uniform asymptotic expansion of Dirichlet's integral, *USSR Comput. Math. Math. Phys.* **17** (1978) 237–242.