CHAPTER 6

INTRODUCTION TO CONNECTED OPERATORS

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6.1 INTRODUCTION

Connectivity, in all its manifestations, has always been an important notion in the field of image processing [17]. This is even more true for methods from mathematical morphology because of their intrinsic topological and geometrical nature. A simple but extremely important instance of a morphological operation based on connectivity is the *reconstruction* of a point marker inside a set by successive dilations [21]. This reconstruction operator forms the basis for an approach in mathematical morphology which goes under the name *geodesic methods* [11, 12]. Such methods rely upon the connectivity of the underlying space; in the discrete case, one obtains a connectivity by imposing a graph structure. Another important step toward a systematic study of connected morphological operators was made by Matheron and Serra [22, Chap. 7] who discussed openings and closings and, more generally, strong filters that are connected (in a specific sense). Some years later, the first systematic studies on connected operators by Serra, Salembier, Crespo, Schafer, and others [5, 8, 13, 18, 20, 24] appeared in the literature. A major impetus to the current research on connected operators was given by the work of Vincent, providing for the first time efficient algorithms for gray-scale reconstruction [25, 26] and the area opening [27].

A connected operator is an operator that acts on the level of the flat zones of an image, rather than on the level of individual pixels. By flat zone we mean a maximal connected region where the gray-level is constant. In the binary case this means that such an operator cannot break connected components (grains) of the foreground or the background. Connected operators cannot introduce new discontinuities and as such they are eminently suited for applications where contour information is important. Image segmentation is such an application. In morphology, the main approach toward segmentation is provided by the watershed algorithm. However, this algorithm, when it is applied to an image, or rather its gradient, usually gives rise to a dramatic oversegmentation. To circumvent this problem, one might modify the image using an appropriate set of markers. Connected operators have proved to be useful for determining such markers automatically [5, 20]. Motion is another major field where connected operators have proved their usefulness. The reader is referred to [13, 19] for further details.

The goal of this chapter is to introduce the reader to the relatively new area of connected morphological operators. Apart from Section 6.6, where we treat gray-scale images, we are exclusively concerned with binary images on the two-dimensional square grid provided with 8-connectivity. In Section 6.2 we discuss the reconstruction operator and introduce the notion of *partition* associated with a binary image. This notion is used in Section 6.3 to give a formal definition of a connected operator. We present some elementary properties as well as some methods for their construction. In Section 6.3 we also introduce the concept of a *zonal graph*, also known in the literature as the *region adjacency graph*. The zonal graph concept enables a rather intuitive interpretation of connected operators on the one hand and efficient implementation of such operators on the other. An important class of connected operators is formed by the so-called *grain operators* introduced in Section 6.4. The effect of a grain operator at a given point depends exclusively on the foreground or background component to which this point belongs. In other words, grain operators use only local (or rather, regional) information. The basic examples are the area opening and closing. In Section 6.5 we show, among other things, that every grain operator is uniquely determined by two *criteria*, one for the foreground and one for the background components. As an illustration, we present a class of self-dual morphological filters which generalize the *annular filter* defined in Chapter 5. Up to this point, this chapter is exclusively concerned with binary images. In Section 6.6 we briefly discuss extensions to gray-scale images; particular attention will be given to the area opening and its implementation based on the zonal graph representation. We illustrate, by means of an example, the use of connected operators in image segmentation. We conclude with some final remarks in Section 6.7.

Some remarks about notation are in order. As we said earlier, to a large extent this paper will be concerned with binary two-dimensional images. In other words, our image space is $\mathcal{P}(\mathbb{Z}^2)$, the collection of all subsets of $\mathbb{Z}^2$. If $X \subseteq \mathbb{Z}^2$ and $h \in \mathbb{Z}^2$, then $X(h) = 1$ if $h \in X$ and $X(h) = 0$, otherwise. In other words, $X(\cdot)$ denotes the indicator function. If S is a statement then $[S]$ denotes the Boolean value (0 or 1) indicating whether S is true or false. Thus we can write $[h \in X]$ instead of $X(h)$. For other unknown notation and terminology, the reader may refer to Chapter 5.

6.2 Connectivity and Reconstruction

By $\mathcal{C}$ we denote the subcollection of all subsets of $\mathbb{Z}^2$ which are 8-connected. The family $\mathcal{C}$ is a typical example of a *connectivity class*; see also Section 6.7.

A set $X \in \mathcal{C}$ is called *connected*. Every set X is the disjoint union of the *connected components*, henceforth called *grains*, contained in X. We write $C \Subset X$ if C is a grain of X. We introduce the following notation: if $h \in \mathbb{Z}^2$, then

$$\gamma_h(X) = \begin{cases} \text{grain of } X \text{ which contains } h, & \text{if } h \in X, \\ \emptyset, & \text{if } h \notin X. \end{cases}$$

In Section 6.7 we will give a formal definition of γ_h for arbitrary connectivity classes. There, also the following result will be proved.

PROPOSITION 6–1. *γ_h is an opening, for every $h \in \mathbb{Z}^2$.*

We refer to γ_h as the *connectivity opening*. There is a simple algorithm to compute $\gamma_h(X)$ when h and X are given. In fact, it uses the notion of *reconstruction* which we describe now.

If $X, Y \subseteq \mathbb{Z}^2$, then $\rho(Y \mid X)$, the *reconstruction of Y with respect to X*, is the union of all grains of X that intersect with Y. Alternatively, we can write:

$$\rho(Y \mid X) = \bigcup_{h \in Y} \gamma_h(X). \tag{6–1}$$

In most practical cases, Y is a subset of X. As a matter of fact, it is obvious that $\rho(Y \mid X) = \rho(Y \cap X \mid X)$. If $Y \cap X = \emptyset$, then $\rho(Y \mid X) = \emptyset$. From the expression in Eq. 6–1 one easily gets the following result.

PROPOSITION 6–2. *For every collection Y_i, $i \in I$, in $\mathcal{P}(\mathbb{Z}^2)$ one has*

$$\rho\Big(\bigcup_{i \in I} Y_i \mid X\Big) = \bigcup_{i \in I} \rho(Y_i \mid X),$$

in other words, the mapping $Y \mapsto \rho(Y \mid X)$ is a dilation, for a fixed set $X \subseteq \mathbb{Z}^2$.

The reconstruction $\rho(Y \mid X)$ can be computed easily by means of the following propagation algorithm:

```
Q = ∅; R = Y ∩ X;
while Q ≠ R do {
    Q = R;
    R = (Q ⊕ B) ∩ X
}
ρ(Y | X) = R
```

Here B is the 3×3 square. The algorithm is illustrated in Fig. 6–1.

The sets X and Y in $\rho(Y \mid X)$ are called the *mask* (*image*) and *marker* (*image*), respectively. Obviously, $\gamma_h(X) = \rho(\{h\} \mid X)$, meaning that the opening γ_h can be computed with the aid of the algorithm given above.

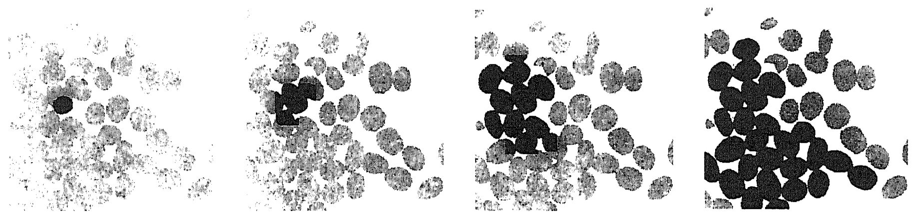

Figure 6–1. Reconstruction algorithm. From left to right: the mask image X (gray) and the marker image Y (black); 15 iterations; 50 iterations; final result $\rho(Y \mid X)$.

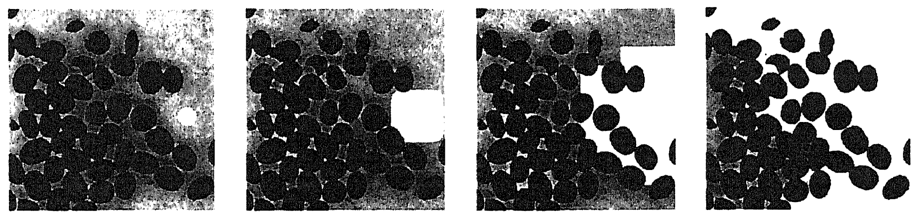

Figure 6–2. Dual reconstruction algorithm. From left to right: the mask image X (black) and the marker image Y (gray and black); 20 iterations; 75 iterations; final result $\rho^*(Y \mid X)$ (gray and black).

The reconstruction operator yields a reconstruction of the foreground. Instead, we can also perform a reconstruction of the background. We call the resulting operator the *background reconstruction* or *dual reconstruction*, and denote it by ρ^*:

$$\rho^*(Y \mid X) = \left[\rho\left(Y^c \mid X^c\right)\right]^c.$$

Now the converse of Proposition 6–2 holds, namely

$$\rho^*\left(\bigcap_{i \in I} Y_i \mid X\right) = \bigcap_{i \in I} \rho^*(Y_i \mid X),$$

that is, the mapping $Y \mapsto \rho^*(Y \mid X)$ is an erosion. The dual reconstruction is illustrated in Fig. 6–2.

Observe that

$$\rho(Y \mid X) \subseteq X \subseteq \rho^*(Y \mid X),$$

for any two sets $X, Y \subseteq \mathbb{Z}^2$.

By a *partition* of the space $\mathbb{Z}^2$ we mean a subdivision of this space into disjoint parts. A partition can be represented by a function $P : \mathbb{Z}^2 \to \mathcal{P}(\mathbb{Z}^2)$ which has the following properties:

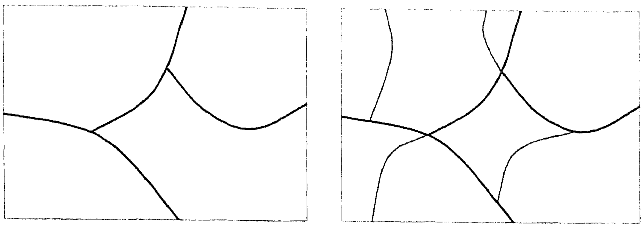

Figure 6–3. The partition at the left is coarser than the one at the right.

- $x \in P(x)$
- $P(x) = P(y)$ or $P(x) \cap P(y) = \emptyset$, for any two points $x, y \in \mathbb{Z}^2$.

Thus, $P(x)$ is the part of the partition that contains the point x. The family of all partitions of $\mathbb{Z}^2$ forms a complete lattice [22] under the partial ordering given by

$$P \sqsubseteq P' \quad \text{if } P'(h) \subseteq P(h), \text{ for every } h \in \mathbb{Z}^2.$$

We say that P is *coarser* than P', or that P' is *finer* than P. Fig. 6–3 shows an example.

A partition P is said to be *connected* if every part $P(h)$ is a connected set. Every binary image $X \subseteq \mathbb{Z}^2$ yields a unique connected partition $P(X)$, where the parts of $P(X)$ are the grains of X and X^c. Writing $P(X, h) = P(X)(h)$, we have

$$P(X, h) = \begin{cases} \gamma_h(X), & \text{if } h \in X \\ \gamma_h(X^c), & \text{if } h \notin X^c. \end{cases}$$

It is evident that $P(X) = P(X^c)$, for every set X. Note, however, that most partitions of $\mathbb{Z}^2$ are not of the form $P(X)$, with X a subset of $\mathbb{Z}^2$. This applies in particular for the partitions depicted in Fig. 6–3.

6.3 CONNECTED OPERATORS

By $X \setminus Y$ we denote the set difference of X and Y. Furthermore $X \triangle Y$ denotes the symmetric difference.

DEFINITION 6–1. An operator ψ on $\mathcal{P}(\mathbb{Z}^2)$ is *connected* if the partition $P(\psi(X))$ is coarser than $P(X)$, for every set $X \subseteq \mathbb{Z}^2$.

Below we will present an alternative formulation of this property. We start with some simple examples. Obviously, the identity operator id as well as the complement operator $X \to X^c$ are connected. Furthermore, every connectivity opening γ_h is connected.

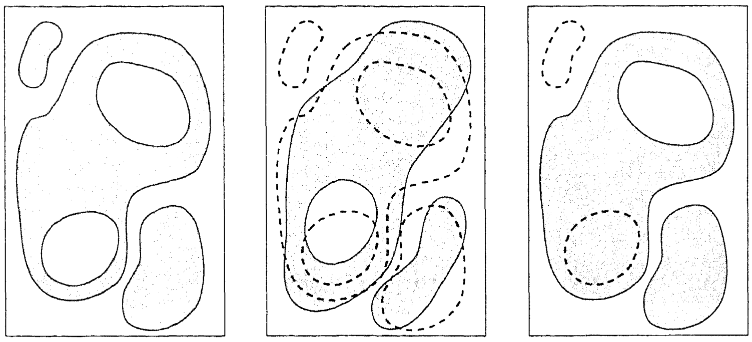

Figure 6–4. A connected operator applied to the left image can result in the image at the right but not in the one in the middle.

A connected operator acts on the grains of the foreground and background in an all-or-nothing way: either the grain is left untouched or deleted altogether. This means in particular that the borders in the image cannot be broken or changed, but only deleted. An illustration is given in Fig. 6–4: the middle image cannot be the output of a connected operator applied to the image at the left. However, the right image may result from a connected operator.

PROPOSITION 6–3. *An operator ψ is connected if and only if $X \triangle \psi(X)$ consists of grains of X and X^c, for every $X \subseteq \mathbb{Z}^2$.*

PROOF. "only if": assume that ψ is connected; then $P(\psi(X))$ is coarser than $P(X)$. We must prove that for every $h \in X \triangle \psi(X)$, the entire part $P(X, h)$ lies in $X \triangle \psi(X)$. We have to consider two cases: $h \in X$ and $h \notin X$.

$h \in X$: thus $h \notin \psi(X)$. Then $P(X, h) \subseteq P(\psi(X), h)$ leads to $\gamma_h(X) \subseteq \gamma_h(\psi(X)^c)$. But this means that $\gamma_h(X) \subseteq X \triangle \psi(X)$.

$h \notin X$: then $h \in \psi(X)$, and $P(X, h) \subseteq P(\psi(X), h)$ leads to $\gamma_h(X^c) \subseteq \gamma_h(\psi(X))$. That is, $\gamma_h(X^c) \subseteq X \triangle \psi(X)$.

"if": let $X \subseteq \mathbb{Z}^2$, we must show that $P(X, h) \subseteq P(\psi(X), h)$, for every $h \in \mathbb{Z}^2$. Again, we must distinguish between the cases $h \in X$ and $h \notin X$. We consider only the first case; the second is treated analogously. If $h \in X$, then $P(X, h) = \gamma_h(X)$. We must show that $\gamma_h(X) \subseteq P(\psi(X), h)$. Suppose $\gamma_h(X) \not\subseteq \psi(X)$; then there is a point k such that $k \in \gamma_h(X)$ and $k \notin \psi(X)$. Now $k \in X \triangle \psi(X)$, which yields that $\gamma_k(X) \subseteq X \triangle \psi(X)$. However, $\gamma_k(X) = \gamma_h(X)$, whence we conclude that $\gamma_h(X) \subseteq \psi(X)^c$, and thus $\gamma_h(X) \subseteq \gamma_h(\psi(X)^c) = P(\psi(X), h)$. ■

In fact, the condition that $X \triangle \psi(X)$ consists of grains of X and X^c, consists of two parts, namely that $X \setminus \psi(X)$ consists of grains of X, and that $\psi(X) \setminus X$ consists of grains of X^c.

PROPOSITION 6–4. *An operator ψ is connected if and only if its negative ψ^* is connected.*

PROOF. Assume that ψ is connected; then $P(\psi(X)) \sqsubseteq P(X)$, for every $X \subseteq \mathbb{Z}^2$. Substituting X^c yields that

$$P\big(\psi(X^c)\big) \sqsubseteq P\big(X^c\big).$$

Using that $P(\psi^*(X)) = P(\psi(X^c)^c) = P(\psi(X^c))$, and that $P(X^c) = P(X)$, we get that

$$P\big(\psi^*(X)\big) \sqsubseteq P(X).$$

This proves the result. ∎

We present a number of results that show how to build new connected operators from known ones using composition, supremum, infimum, as well as other "Boolean combinations."

PROPOSITION 6–5. *If ψ_1, ψ_2 are connected, then $\psi_2\psi_1$ is connected, too.*

PROOF. This result is a simple consequence of Definition 6–1:

$$P\big(\psi_2\psi_1(X)\big) \sqsubseteq P\big(\psi_1(X)\big) \sqsubseteq P(X),$$

for every $X \subseteq \mathbb{Z}^2$, if ψ_1 and ψ_2 are connected. ∎

PROPOSITION 6–6. *If ψ_i is a connected operator for every i in some index set I, then the infimum $\bigwedge_{i\in I} \psi_i$ and the supremum $\bigvee_{i\in I} \psi_i$ are connected, too.*

PROOF. We first prove the result for the infimum. Let $\psi = \bigwedge_{i\in I} \psi_i$, then $X \setminus \psi(X) = \bigcup_{i\in I}(X \setminus \psi_i(X))$ and $\psi(X) \setminus X = \bigcap_{i\in I}(\psi_i(X) \setminus X)$. As every set $X \setminus \psi_i(X)$ is a union of grains of X, $X \setminus \psi(X)$ is also a union of grains of X. Similarly, we have that $\psi(X) \setminus X$ is a union of grains of X^c, and we conclude that ψ is connected.

The result for the supremum can be obtained analogously, but it also follows from the observation that

$$\bigvee_{i\in I} \psi_i = \Big(\bigwedge_{i\in I} \psi_i^*\Big)^*,$$

in combination with Proposition 6–4. ∎

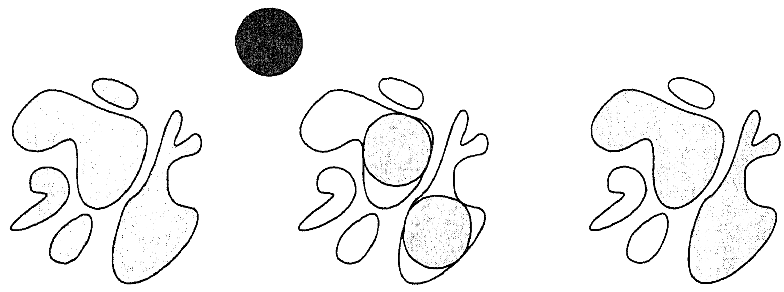

Figure 6–5. Opening by reconstruction: the original opening is an opening by a disk (in black). From left to right: $X, \alpha(X)$, and $\breve{\alpha}(X)$.

Recall that a *Boolean function* (of n variables) is a function $b: \{0, 1\}^n \to \{0, 1\}$. Given a Boolean function b and n operators $\psi_1, \ldots, \psi_n$ on $\mathcal{P}(\mathbb{Z}^2)$, we can define a new operator

$$\psi = b(\psi_1, \ldots, \psi_n)$$

as follows:

$$\psi(X)(h) = b\big(\psi_1(X)(h), \ldots, \psi_n(X)(h)\big);$$

here $X(h)$ equals 1 if $h \in X$ and 0 otherwise. For example, if $b(u_1, \ldots, u_n) = u_1 \cdot u_2 \cdot \cdots \cdot u_n$, then $b(\psi_1, \ldots, \psi_n) = \psi_1 \wedge \cdots \wedge \psi_n$.

PROPOSITION 6–7. *Given a Boolean function b of n variables and n connected operators $\psi_1, \psi_2, \ldots, \psi_n$, then the operator $\psi = b(\psi_1, \psi_2, \ldots, \psi_n)$ is connected as well.*

PROOF. The proof becomes obvious by the observation that the value of $\psi_i(X)(h)$ is constantly 0 or 1 on parts of the partition $P(X)$ (this value only depending on i). As a result, $\psi(X)(h)$ is constant on parts of $P(X)$, too, and therefore ψ is a connected operator. ■

An opening which is a connected operator is called a *connected opening* (same for closings). One can construct connected openings by starting with arbitrary openings and performing a reconstruction afterward: let α be an opening on $\mathcal{P}(\mathbb{Z}^2)$ and define

$$\breve{\alpha}(X) = \rho(\alpha(X) \mid X). \tag{6–2}$$

In Fig. 6–5 we show an example; there $\alpha(X) = X \circ B$, where B is a disk.

PROPOSITION 6–8. *If α is an opening, then $\breve{\alpha}$ is a connected opening. Moreover, α is a connected opening if and only if $\alpha = \breve{\alpha}$.*

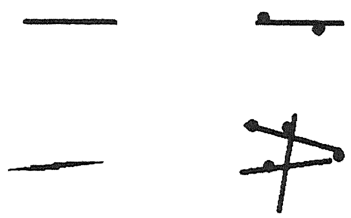

Figure 6–6. Opening by reconstruction. From left to right: the original image X (gray); the opening $\alpha(X)$ by a horizontal line segment (length is 40 pixels); the reconstruction $\breve{\alpha}(X) = \rho(\alpha(X) \mid X)$.

PROOF. Assume α is an opening; we show that $\breve{\alpha}$ is an opening, too. It is obvious that $\breve{\alpha}$ is increasing. It is also obvious that $\alpha(X) \subseteq \breve{\alpha}(X) \subseteq X$. Therefore $\breve{\alpha}^2 \leqslant \breve{\alpha}$. We must show that $\breve{\alpha}^2 \geqslant \breve{\alpha}$:

$$\begin{aligned}\breve{\alpha}^2(X) &= \rho\big(\alpha\breve{\alpha}(X) \mid \breve{\alpha}(X)\big) \supseteq \rho\big(\alpha^2(X) \mid \breve{\alpha}(X)\big)\\ &= \rho\big(\alpha(X) \mid \breve{\alpha}(X)\big) = \bigcup_{h\in\alpha(X)} \gamma_h\big(\breve{\alpha}(X)\big).\end{aligned}$$

Using that $\gamma_h(\breve{\alpha}(X)) = \gamma_h(X)$ for $h \in \alpha(X)$, we get that

$$\breve{\alpha}^2(X) \supseteq \bigcup_{h\in\alpha(X)} \gamma_h(X) = \breve{\alpha}(X).$$

This proves that $\breve{\alpha}$ is an opening. By definition, $\breve{\alpha}(X)$ is a union of grains of X, hence $\breve{\alpha}$ is a connected operator.

Conversely, assume that α is a connected opening, hence $X\Delta\alpha(X) = X \setminus \alpha(X)$ is a union of grains of X. But this yields that $\alpha(X)$ is a union of grains of X, too. Therefore, $\alpha = \breve{\alpha}$. ■

For closings β we define $\breve{\beta}(X) = \rho^*(\beta(X) \mid X)$, and we can prove the dual statement of the proposition above. Note that the following duality relations hold:

$$(\alpha^*)^{\vee} = (\breve{\alpha})^* \quad \text{and} \quad (\beta^*)^{\vee} = (\breve{\beta})^*.$$

Fig. 6–6 shows an example where α is an opening with a horizontal line segment.

To get additional insight into the way connected operators behave, we introduce the concept of a *zonal graph*, sometimes called *region adjacency graph* in the literature [1]. At this point we restrict attention to the binary case. In Section 6.6 we will briefly discuss gray-scale images.

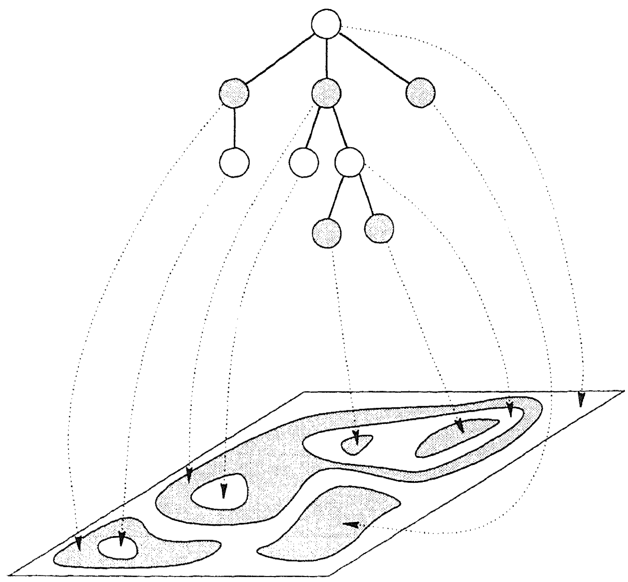

Figure 6–7. Zonal graph associated with a binary image.

As we have seen, a binary image $X \in \mathcal{P}(\mathbb{Z}^2)$ yields a colored partition (with colors 0 and 1) of the underlying space $\mathbb{Z}^2$. Now, consider the parts of $P(X)$ as the vertices of a graph. Two parts $C_1, C_2 \in P(X)$ are said to be *adjacent*, denoted by $C_1 \sim C_2$, if $C_1 \cup C_2$ is a connected set. To determine X from this graph, we have to specify for every vertex whether it is a subset of X or X^c. To this end we use the coloring $I_X : P(X) \to \{0, 1\}$:

$$I_X(C) = \begin{cases} 1, & \text{if } C \Subset X \\ 0, & \text{if } C \Subset X^c. \end{cases} \tag{6–3}$$

Note that because two adjacent vertices must have different colors, it suffices to specify the color of only one vertex; this, however, is no longer true in the gray-scale case.

The triple $(P(X), \sim, I_X)$ is called the *zonal graph* associated with X; see Fig. 6–7 for an illustration.

In the case considered here ($\mathbb{Z}^2$ with 8-connectivity), the following result holds (recall that a *tree* is a graph without cycles [2]):

PROPOSITION 6–9. *The graph $(P(X), \sim)$ is a tree, for every set $X \subseteq \mathbb{Z}^2$.*

A connected operator on $\mathcal{P}(\mathbb{Z}^2)$ amounts to a recoloring of the zonal graph associated with the set $X \subseteq \mathbb{Z}^2$. For example, the opening γ_h gives color 1 to the vertex

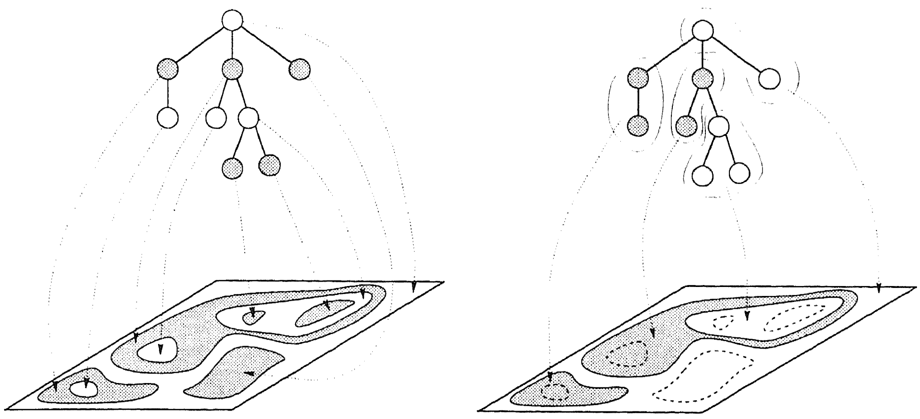

Figure 6–8. The leaves of the tree receive the color of their neighbor.

$C \in P(X)$ which satisfies $h \in C$ and $I_X(C) = 1$ (note that such a vertex exists if and only if $h \in X$), and 0 to all other vertices.

The converse is also true: every recoloring, followed by a merging of vertices with the same color, defines a connected operator. For precise statements we refer to future publications; here we only present an example.

Recall that a vertex in a tree is called a *leaf* if it possesses exactly one neighbor. For example, the tree in Fig. 6–7 contains 5 leaves. We define a recoloring as follows: the color at the leaves is flipped (from 0 to 1 and vice versa), but the colors of the other vertices are left unaltered. We apply this recoloring to the zonal graph depicted in Fig. 6–7, and merge adjacent vertices with the same color. The outcome is depicted in Fig. 6–8. The operator associated with this recoloring is connected (and self-dual).

6.4 GRAIN OPERATORS

This section discusses a particular class of connected operators, the so-called grain operators. We point out that Crespo and Schafer [7] call such operators *connected-component local operators*. Some of the basic results presented below [e.g., Proposition 6–10(a) and (c)] can also be found in their work.

DEFINITION 6–2. A connected operator $\psi : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{P}(\mathbb{Z}^2)$ is called a *grain operator* if it has the following property: if $h \in \mathbb{Z}^2$ and $X, Y \subseteq \mathbb{Z}^2$ are such that $X(h) = Y(h)$ and $P(X, h) = P(Y, h)$, then $\psi(X)(h) = \psi(Y)(h)$.

Thus given a grain operator ψ, the value $\psi(X)(h)$ is completely determined by the information provided by $X(h)$ and $P(X, h)$, which is exactly the information

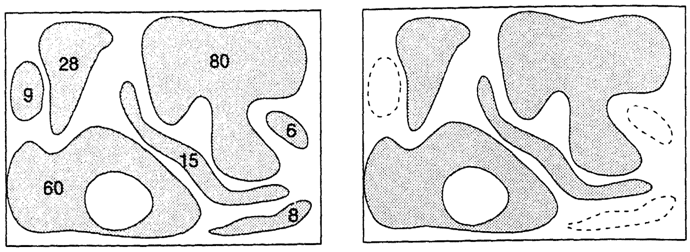

Figure 6–9. Area opening α_{10}. The numbers printed inside the grains represent their area.

"stored" at the vertex $P(X, h)$ of the associated zonal graph. In other words, the recoloring of the zonal graph corresponding with a grain operator is based entirely upon local information stored at individual vertices; all other information, e.g., about adjacent vertices, is irrelevant. For example, the recoloring considered in the last paragraph of the previous section, where the colors of the leaves are flipped, does not correspond to a grain operator. Indeed, in this case, the evaluation of a vertex being a leaf or not requires information about its neighbors: Is there one or more than one neighbor?

The identity operator, the complement operator, and the connectivity openings are grain operators. We discuss another very important example.

EXAMPLE 6–1 (Area opening). If C is a component of X, then area(C) denotes the area of C, i.e., the number of pixels. If C is unbounded, then area(C) $= +\infty$. The *area opening* α_S is the operator that deletes all grains from a set X with an area smaller than a given threshold S. Thus

$$\alpha_S(X) = \bigcup\{C \mid C \Subset X \text{ and area}(C) \geqslant S\}.$$

It is obvious that α_S is a grain operator, and that it is increasing, anti-extensive, and idempotent, hence an opening. The area opening (with $S = 10$) is illustrated in Fig. 6–9.

A grain operator which is also an opening (closing, filter) is called a *grain opening* (closing, filter). The next result states some basic properties of grain operators.

PROPOSITION 6–10.

(a) *If ψ_i is a grain operator for every $i \in I$, then $\bigvee_{i\in I} \psi_i$ and $\bigwedge_{i\in I} \psi_i$ are grain operators.*

(b) *If $\psi_1, \psi_2, \ldots, \psi_n$ are grain operators and b is a Boolean function of n variables, then $b(\psi_1, \psi_2, \ldots, \psi_n)$ is a grain operator.*

(c) *If ψ is a grain operator, then ψ^* is a grain operator.*

PROOF. (a) We show that $\psi = \bigvee_{i \in I} \psi_i$ is a grain operator. The proof for the infimum is analogous. Let h, X, Y be such that $X(h) = Y(h)$ and $P(X, h) = P(Y, h)$; we show that $\psi(X)(h) = \psi(Y)(h)$. Obviously,

$$\psi(X)(h) = \Big(\bigvee_{i \in I} \psi_i(X)\Big)(h) = \bigvee_{i \in I} \psi_i(X)(h).$$

Since every ψ_i is a grain operator, this last expression equals

$$\bigvee_{i \in I} \psi_i(Y)(h) = \Big(\bigvee_{i \in I} \psi_i(Y)\Big)(h) = \psi(Y)(h),$$

and this shows the result.

(b) Let $\psi = b(\psi_1, \psi_2, \ldots, \psi_n)$. By definition

$$\psi(X)(h) = b\big(\psi_1(X)(h), \psi_2(X)(h), \ldots, \psi_n(X)(h)\big),$$

and with this relation, the proof becomes similar to the proof of (a).

(c) This follows easily if one uses the relation $\psi^*(X)(h) = 1 - \psi(X^c)(h)$. ■

The composition of two grain operators, however, is not a grain operator, in general. In Fig. 6–10 we depict an example of a grain operator ψ for which ψ^2 is not a grain operator. The operator acts as follows: for every part of the partition $P(X)$ it switches the color from 1 to 0, or vice versa, if the area of this part is below a given threshold (20 in this specific example). The value $\psi^2(X)(h)$ is different in the upper and lower figure, although $X(h) = Y(h) = 1$ and $P(X, h) = P(Y, h)$. Therefore, ψ^2 is not a grain operator.

By the duality principle (see Chapter 5), every statement about openings has a dual version concerning closings. In what follows, we restrict ourselves to openings.

PROPOSITION 6–11. *Every anti-extensive grain operator is idempotent. In particular, an increasing grain operator is an opening if and only if it is anti-extensive.*

PROOF. Let ψ be an anti-extensive grain operator. We must show that ψ is idempotent. The nontrivial part of the proof consists of showing that $\psi^2 \geqslant \psi$. Assume that $h \in \psi(X)$; we must show that $h \in \psi^2(X)$. Put $Y = \psi(X)$, then $X(h) = Y(h) = 1$. This implies that

$$P(Y, h) = \gamma_h(Y) \subseteq \gamma_h(X) = P(X, h).$$

On the other hand,

$$P(X, h) \subseteq P(Y, h),$$

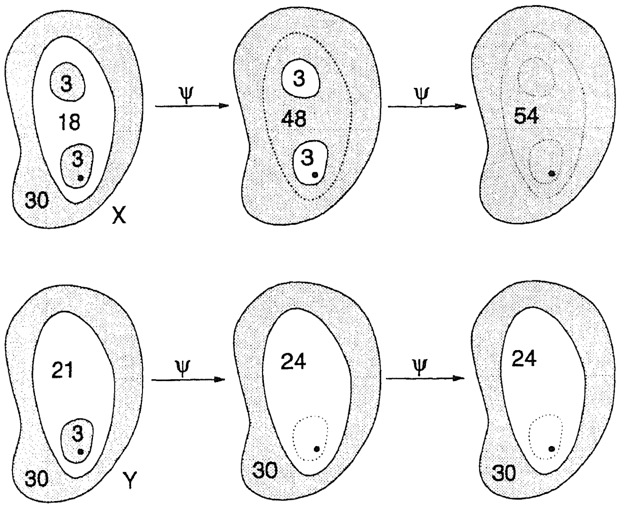

Figure 6–10. The operator ψ that changes the color of parts with areas less than 20 is a grain operator, but ψ^2 is not. The black point indicates the location of the point h.

since ψ is a connected operator, hence $P(Y)$ is coarser than $P(X)$. Thus we find that $P(X, h) = P(Y, h)$. Using that ψ is a grain operator, we get that

$$\psi(X)(h) = \psi(Y)(h) = 1.$$

This proves the result. ■

In the previous section, we defined the opening by reconstruction; see Eq. 6–2. An interesting question is the following: For which openings α is the opening by reconstruction $\breve{\alpha}$ a grain opening? Under the extra assumption that α is a structural opening [10, 16], we can give a complete characterization.

PROPOSITION 6–12. *Let α be the structural opening given by $\alpha(X) = X \circ B$. Then $\breve{\alpha}$ is a grain opening if and only if B is connected.*

PROOF. Assume first that B is not connected and let Y be a grain of B. Furthermore, pick $h \in Y$ and put $X = B$. Then $X(h) = Y(h) = 1$ and $P(X, h) = P(Y, h) = Y$. Obviously, $\alpha(X) = X$ and $\alpha(Y) = \emptyset$, hence $\breve{\alpha}(X) = X$ and $\breve{\alpha}(Y) = \emptyset$. Thus $\breve{\alpha}$ is not a grain operator, as this would imply that $\breve{\alpha}(X)(h) = \breve{\alpha}(Y)(h)$.

Assume now that B is connected; we show that $\breve{\alpha}$ is a grain operator. It is easy to see that $\breve{\alpha}(X) = \bigcup\{C \mid C \Subset X \text{ and } C \circ B \neq \emptyset\}$. From this, we derive that

$$\breve{\alpha}(X)(h) = X(h) \wedge [P(X, h) \circ B \neq \emptyset],$$

whence it follows immediately that $\breve{\alpha}$ is a grain opening. ■

PROPOSITION 6–13. *An opening α is a grain opening if and only if $\alpha = \bigvee_{h\in\mathbb{Z}^2} \alpha\gamma_h$.*

PROOF. Assume that α is a grain opening. It is trivial that $\alpha \geqslant \bigvee_{x\in\mathbb{Z}^2} \alpha\gamma_x$; therefore, we only have to show that $\alpha \leqslant \bigvee_{x\in\mathbb{Z}^2} \alpha\gamma_x$. Suppose that $h \in \alpha(X)$. Define $Y = \gamma_h(X)$, then $X(h) = \gamma_h(X)(h) = 1$. Furthermore, $P(X, h) = \gamma_h(X) = P(Y, h)$. Since α is a grain operator, we get that $\alpha(X)(h) = \alpha(Y)(h)$, and this implies that $h \in \alpha(Y) = \alpha\gamma_h(X) \subseteq \bigvee_{x\in\mathbb{Z}^2} \alpha\gamma_x(X)$.

To prove the converse, assume that $\alpha = \bigvee_{x\in\mathbb{Z}^2} \alpha\gamma_x$; we show that α is a grain operator. Suppose that $X(h) = Y(h)$ and $P(X, h) = P(Y, h)$. We must show that $\alpha(X)(h) = \alpha(Y)(h)$. If $X(h) = 0$, this is trivial; we consider the case that $X(h) = 1$. Thus $P(X, h) = \gamma_h(X) = \gamma_h(Y)$. Now

$$\begin{aligned}\alpha(X)(h) &= \Big(\bigvee_{x\in\mathbb{Z}^2} \alpha\gamma_x(X)\Big)(h) = \bigvee_{x\in\mathbb{Z}^2} \alpha\gamma_x(X)(h)\\ &= \alpha\gamma_h(X)(h) = \alpha\gamma_h(Y)(h)\\ &= \alpha(Y)(h).\end{aligned}$$

This concludes the proof. ■

Recall that a set B is *invariant* under an operator ψ if $\psi(B) = B$. We will show that any subset of grains of a set X invariant under a grain filter ψ is also invariant under ψ. Refer to [22, Chap. 7] for some related results. We start with a lemma.

LEMMA 6–1. *Let ψ be an increasing connected operator on $\mathcal{P}(\mathbb{Z}^2)$, and let $X \subseteq \mathbb{Z}^2$ satisfy $\psi(X) \subseteq X$. If Y is a union of grains of X, then $\psi(Y) \subseteq Y$.*

PROOF. Suppose that $\psi(Y) \not\subseteq Y$. Since ψ is connected, $\psi(Y) \setminus Y$ consists of grains of Y^c. Let D be a grain of Y^c contained in $\psi(Y) \setminus Y$. We show that $D \cap X^c \neq \emptyset$. Suppose that $D \subseteq X$. The grain D must be adjacent to a grain C of Y, meaning that $C \cup D$ is connected. However, $C \cup D \subseteq X$, and we conclude that C cannot be a grain of X. But this contradicts our assumption that Y consists of grains of X. Thus $D \cap X^c \neq \emptyset$.

Since $D \subseteq \psi(Y)$ and ψ is increasing, also $D \subseteq \psi(X)$. This yields that $\psi(X) \cap X^c \neq \emptyset$, i.e., $X \cap X^c \neq \emptyset$, a contradiction. We conclude that $\psi(Y) \subseteq Y$, as asserted. ■

PROPOSITION 6–14. *Let ψ be a grain filter and $\psi(X) = X$.*

(a) *If Y is a union of grains of X, then $\psi(Y) = Y$.*

(b) *If Y is a union of grains of X^c, then $\psi(Y^c) = Y^c$.*

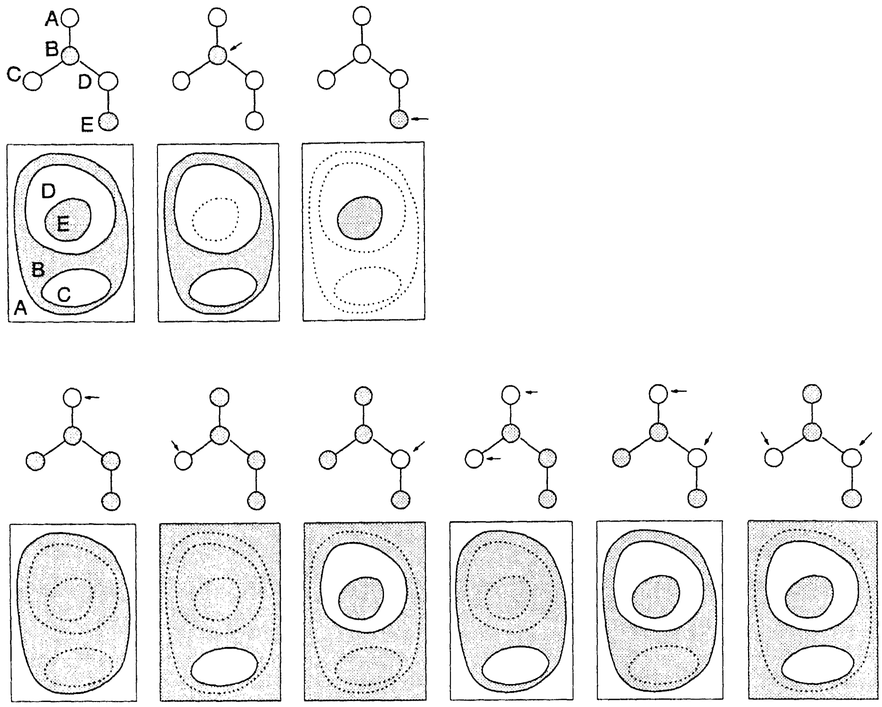

Figure 6–11. The left figure in the first row shows a set invariant with respect to some grain filter ψ. Proposition 6–14 states that the other sets (in gray) shown in this figure are invariant, too.

PROOF. (a) From the previous lemma we conclude that $\psi(Y) \subseteq Y$. We show that $Y \subseteq \psi(Y)$. Take $h \in Y$, then $X(h) = Y(h) = 1$. Furthermore, $P(X, h) = P(Y, h) = \gamma_h(X)$. From the fact that ψ is a grain operator we get that $\psi(X)(h) = \psi(Y)(h)$. But $\psi(X) = X$ and we conclude that $\psi(Y)(h) = X(h) = 1$, that is, $h \in \psi(Y)$. This shows that $Y \subseteq \psi(Y)$.

(b) If ψ is a grain filter, then ψ^* is a grain filter too. Furthermore, $\psi^*(X^c) = X^c$. If Y is a union of grains of X^c, then $\psi^*(Y) = Y$, by (a). But this means $\psi(Y^c) = Y^c$. ■

We illustrate this proposition by means of Fig. 6–11. The left figure at the first row shows a set X (along with its zonal graph) which is assumed to be invariant under a given grain filter. Our proposition gives us that the other sets depicted in this figure are invariant, too. The top row depicts the sets that are built of grains of X. The arrows in the zonal graph indicate which grains are used as building blocks. In the bottom row we consider sets that are built by using background grains, again indicated by arrows in the zonal graph.

6.5 Grain Operators and Grain Criteria

In this section we show that every grain operator is uniquely determined by two grain criteria. By a *grain criterion* we mean a mapping $u : \mathcal{C} \to \{0, 1\}$. Suppose we are given two (grain) criteria, u for the foreground and v for the background. Define an operator $\psi = \psi_{u,v}$ as follows: ψ preserves grains C of the foreground for which $u(C) = 1$ and grains C of the background for which $v(C) = 1$. In other words,

$$\psi(X) = \bigcup \{C \mid (C \Subset X \text{ and } u(C) = 1) \text{ or } (C \Subset X^c \text{ and } v(C) = 0)\}. \qquad (6\text{–}4)$$

See Fig. 6–12 for an illustration.

Note that any operator so defined acts on the level of the grains, and it follows easily that ψ is a grain operator. But the converse can also be proved: every grain operator is of the form Eq. 6–4.

Proposition 6–15. *An operator ψ is a grain operator if and only if there exist foreground and background criteria u, v such that $\psi = \psi_{u,v}$. The criteria u and v are given by*

$$u(C) = [C \subseteq \psi(C)]$$
$$v(C) = [C \subseteq \psi^*(C)].$$

Proof. Let ψ be a grain operator and define u, v as above. We show that $\psi = \psi'$, where $\psi' := \psi_{u,v}$.

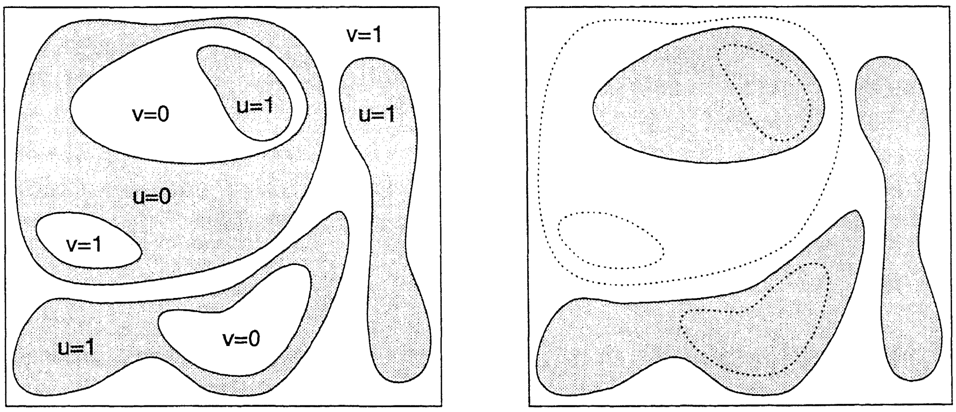

Figure 6–12. A binary image X (left) and its transform $\psi_{u,v}(X)$. In every foreground (resp. background) grain it is printed whether the grain criterion u (resp. v) equals 0 or 1.

First we show that $\psi(X) \subseteq \psi'(X)$ for every set X. Let C be a part of $P(X)$ and $C \subseteq \psi(X)$; we show that $C \subseteq \psi'(X)$. We distinguish two cases.

(1). $C \Subset X$. We show that $u(C) = 1$, for then $C \subseteq \psi'(X)$. We must prove that $C \subseteq \psi(C)$. Let $h \in C$, then $C(h) = X(h) = 1$ and $P(X, h) = P(C, h) = C$. Since ψ is a grain operator, we may conclude that $\psi(X)(h) = \psi(C)(h)$. Since $C \subseteq \psi(X)$, this expression equals 1, and we conclude that $h \in \psi(C)$. Therefore, $C \subseteq \psi(C)$.

(2). $C \Subset X^c$. We show that $C \subseteq \psi(C^c)$, which yields that $v(C) = 0$. Let $h \in C$, then $C^c(h) = X(h) = 0$. Furthermore, $P(C^c, h) = P(X, h) = C$, and using that ψ is a grain operator, we get that $\psi(X)(h) = \psi(C^c)(h)$. Since $C \subseteq \psi(X)$, this expression equals 1, and we conclude that $h \in \psi(C^c)$. Therefore, $C \subseteq \psi(C^c)$.

Next we show that $\psi'(X) \subseteq \psi(X)$. Assume that C is a part of $P(X)$ and that $C \subseteq \psi'(X)$. We show that $C \subseteq \psi(X)$. Again, we distinguish two cases.

(1). $C \Subset X$. Then $u(C) = 1$, hence $C \subseteq \psi(C)$. Let $h \in C$, then $C(h) = X(h) = 1$ and $P(X, h) = P(C, h) = C$, and, by the fact that ψ is a grain operator, we get that $\psi(X)(h) = \psi(C)(h) = 1$. This shows that $C \subseteq \psi(X)$.

(2). $C \Subset X^c$. Then $v(C) = 0$, which yields that $C \subseteq \psi(C^c)$. Let $h \in C$, then $C^c(h) = X(h) = 0$ and $P(X, h) = P(C^c, h) = C$, and we get that $\psi(X)(h) = \psi(C^c)(h)$. Since $C \subseteq \psi(C^c)$, this expression equals 1, and we conclude that $C \subseteq \psi(X)$.

This concludes our proof. ■

A criterion u is called *increasing* if $C, C' \in \mathcal{C}$ and $C \subseteq C'$ implies that $u(C) \leqslant u(C')$. One might expect that $\psi = \psi_{u,v}$ is increasing when both criteria u and v are increasing. The following example shows that this is not true in general. Let $X \subseteq Y$ be as in Fig. 6–13. Let C be a grain of X^c and $D \subseteq C$ a grain of Y. Suppose that u, v are increasing criteria (e.g., area criteria) such that $v(C) = u(D) = 0$. Then, by the increasingness of v, we have $v(C') = 0$ for the grain $C' = C \setminus D$ of Y^c, and it follows that $\psi(X) \not\subseteq \psi(Y)$.

PROPOSITION 6–16. *The grain operator $\psi_{u,v}$ is increasing if and only if both u and v are increasing criteria, and the following condition holds:*

$$u\big(\gamma_h(X \cup \{h\})\big) \vee v\big(\gamma_h(X^c \cup \{h\})\big) = 1, \tag{6–5}$$

if $X \subseteq \mathbb{Z}^2$ and $h \in \mathbb{Z}^2$.

PROOF. "if": assume that u, v satisfy the conditions above; we show that $\psi = \psi_{u,v}$ is increasing. Let $X \subseteq Y$; we must show that $\psi(X) \subseteq \psi(Y)$. Let $h \in \psi(X)$. We distinguish three cases.

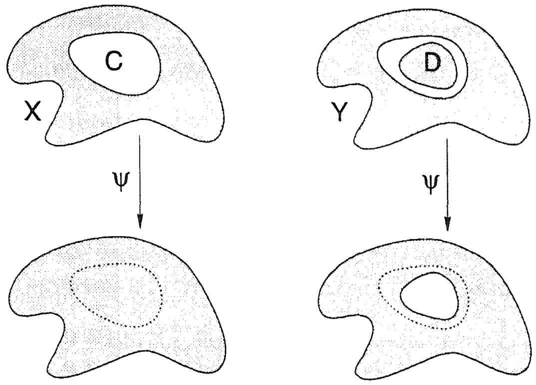

Figure 6–13. ψ is not increasing. Indeed, $X \subseteq Y$ but $\psi(X) \not\subseteq \psi(Y)$.

1. $h \in X$: put $C = \gamma_h(X)$, then $C \Subset X$ and $C \subseteq C' = \gamma_h(Y)$. As $h \in \psi(X)$, we have $u(C) = 1$, and since u is increasing $u(C') = 1$, giving that $h \in \psi(Y)$.

2. $h \notin Y$: put $C' = \gamma_h(Y^c)$ and $C = \gamma_h(X^c)$, then $C' \subseteq C$ since $Y^c \subseteq X^c$. From the fact that $h \in \psi(X)$ we conclude that $v(C) = 0$ and thus $v(C') = 0$, yielding that $h \in \psi(Y)$.

3. $h \in Y$ and $h \notin X$: suppose $h \notin \psi(Y)$, then $u(\gamma_h(Y)) = 0$. Now Eq. 6–5 implies that $v(\gamma_h(Y^c \cup \{h\})) = 1$. Obviously, $\gamma_h(Y^c \cup \{h\}) \subseteq \gamma_h(X^c)$, and since v is increasing, we get that $v(\gamma_h(X^c)) = 1$. However, this implies that the grain $\gamma_h(X^c)$ does not lie in $\psi(X)$, contradicting $h \in \psi(X)$. Thus we conclude that $h \in \psi(Y)$.

"only if": assume that $\psi = \psi_{u,v}$ is increasing. First we show that u is an increasing grain criterion. The proof that v is increasing is analogous. Let $C \subseteq C'$ be connected, then $\psi(C) \subseteq \psi(C')$. Suppose that $u(C) = 1$, then $C \subseteq \psi(C)$, hence $C \subseteq \psi(C')$. Thus we get that $C \subseteq C' \cap \psi(C')$, and we conclude that $u(C') = 1$ since otherwise $C' \cap \psi(C') = \emptyset$. Thus it remains to show Eq. 6–5. Let $X \subseteq \mathbb{Z}^2$ and $u(\gamma_h(X \cup \{h\})) = 0$; we must show that $v(\gamma_h(X^c \cup \{h\})) = 1$. Indeed, since $h \notin \psi(X \cup \{h\})$ and ψ is increasing, it follows that $h \notin \psi(X \setminus \{h\})$. This means that $v(P(X \setminus \{h\}, h)) = 1$. Now

$$P(X \setminus \{h\}, h) = \gamma_h\big((X \setminus \{h\})^c\big) = \gamma_h\big(X^c \cup \{h\}\big).$$

This yields the result. ■

We write $u \equiv 1$ if the criterion u is identically 1, i.e., $u(C) = 1$ for every grain C. When $v \equiv 1$, we write $\psi_{u,1}$ for $\psi_{u,v}$. Similarly, $\psi_{1,v}$ denotes the grain operator for which the foreground criterion u is identically 1.

EXAMPLE 6–2. We present some examples of grain criteria.

- $u(C) = C(h)$. Now $\psi_{u,1}$ equals the connectivity opening γ_h.
- $u(C) = [\text{area}(C) \geqslant S]$. The operator $\psi_{u,1}$ is the area opening considered in Example 6–1.
- $u(C) = [\text{perimeter}(C) \geqslant S]$, where perimeter($C$) equals the number of boundary pixels in C. This criterion is not increasing.
- $u(C) = [\text{area}(C)/(\text{perimeter}(C))^2 \geqslant k]$. Note that this criterion provides a measure for the circularity of C. This criterion is not increasing.
- $u(C) = [C \ominus B \neq \emptyset]$, which gives the outcome 1 if some translate of B fits inside C. If B is connected, then $\psi_{u,1} = \breve{\alpha}$, where $\alpha(X) = X \circ B$; cf. Proposition 6–12. However, if B is not connected, then $\psi_{u,1}$ is an opening that is smaller than $\breve{\alpha}$, i.e., $\psi_{u,1} \leqslant \breve{\alpha}$.

Breen and Jones [4] discuss various other increasing and nonincreasing criteria. We state some other useful properties.

PROPOSITION 6–17.

(a) $\psi^*_{u,v} = \psi_{v,u}$.

(b) *Given grain operators* ψ_{u_i,v_i}, *for* $i \in I$, *then*

$$\bigwedge_{i\in I} \psi_{u_i,v_i} = \psi_{\bigwedge_{i\in I} u_i, \bigvee_{i\in I} v_i} \quad \text{and} \quad \bigvee_{i\in I} \psi_{u_i,v_i} = \psi_{\bigvee_{i\in I} u_i, \bigwedge_{i\in I} v_i}.$$

(c) *Let* ψ_{u_i,v_i} *be grain operators for* $i = 1, 2, \ldots, n$, *and let* b *be a Boolean function of* n *variables, then*

$$b(\psi_{u_1,v_1}, \ldots, \psi_{u_n,v_n}) = \psi_{b(u_1,\ldots,u_n), b^*(v_1,\ldots,v_n)}.$$

Here b^* *denotes the negative of* b *given by* $b^*(u_1, \ldots, u_n) = 1 - b(1 - u_1, \ldots, 1 - u_n)$.

PROOF. We prove only (c). In Proposition 6–10(b) we have seen that $\psi = b(\psi_{u_1,v_1}, \ldots, \psi_{u_n,v_n})$ is a grain operator. Therefore, ψ is of the form $\psi_{u,v}$. From Proposition 6–15 we know that the foreground criterion u is given by $u(C) = [C \subseteq \psi(C)]$. Since ψ is a grain operator, $[C \subseteq \psi(C)] = [h \in \psi(C)]$, for every $h \in C$. But this last expression equals $b(\psi_{u_1,v_1}(C)(h), \ldots, \psi_{u_n,v_n}(C)(h))$. Using that $\psi_{u_i,v_i}(C)(h) = [C \subseteq \psi_{u_i,v_i}(C)] = u_i(C)$, we finally arrive at the identity $u(C) = b(u_1(C), \ldots, u_n(C))$. In a similar way we find that $v(C) = b^*(v_1(C), \ldots, v_n(C))$, and the result is proved. ∎

The following result is obvious:

PROPOSITION 6–18. *The grain operator $\psi_{u,v}$ is extensive if and only if $u \equiv 1$. It is anti-extensive if and only if $v \equiv 1$.*

We have seen that a composition of grain operators is not a grain operator, in general. However, it is easy to see that a composition of (anti-) extensive grain operators is an (anti-) extensive grain operator. To be precise

$$\psi_{u_2,1}\psi_{u_1,1} = \psi_{u_1,1}\psi_{u_2,1} = \psi_{u_1 \wedge u_2,1} = \psi_{u_1,1} \wedge \psi_{u_2,1} \qquad (6\text{–}6)$$
$$\psi_{1,v_2}\psi_{1,v_1} = \psi_{1,v_1}\psi_{1,v_2} = \psi_{1,v_1 \wedge v_2} = \psi_{1,v_1} \vee \psi_{1,v_2}. \qquad (6\text{–}7)$$

Rather than presenting a formal proof of these relations (in fact, such a proof is rather straightforward, and we leave it as an exercise for the reader), we sketch only the underlying idea. The composition $\psi_{u_2,1}\psi_{u_1,1}$ cannot add background grains, but only delete foreground grains. A foreground grain C will be deleted in either of the two following situations: (i) $u_1(C) = 0$; (ii) $u_1(C) = 1$ but $u_2(C) = 0$. In other words, C is deleted if at least one of the criteria u_1 or u_2 is not satisfied. Therefore, the foreground grain criterion for the composition $\psi_{u_2,1}\psi_{u_1,1}$ is $u = u_1 \wedge u_2$.

Taking $u_1 = u_2 = u$ in Eq. 6–6, we find that $\psi_{u,1}^2 = \psi_{u,1}$. Note in particular that this provides an alternative proof for Proposition 6–11.

As a special case of Eq. 6–6 we mention the identity

$$\alpha\gamma_h = \gamma_h\alpha,$$

for every grain opening α. Taking the supremum over $h \in \mathbb{Z}^2$ and using that $\bigvee_{h\in\mathbb{Z}^2} \gamma_h = \mathrm{id}$, we arrive at the identity in Proposition 6–13.

In a forthcoming paper we will examine the construction of morphological filters that are connected. Here we discuss one specific example, namely the generalization of the (self-dual) annular filter on $\mathcal{P}(\mathbb{Z}^2)$ (see Chap. 5). We start with a lemma.

LEMMA 6–2. *Given $X \subseteq \mathbb{Z}^2$ and $h \in \mathbb{Z}^2$, put $C = P(X, h)$. If area$(C) \leqslant 7$ then C is a leaf of the zonal graph (tree) of X and the unique neighbor C' of C satisfies area$(C') \geqslant 8$. Moreover, $C' \cup \{h\}$ is connected.*

Verification of the validity of this lemma is just a matter of checking all possibilities, and is left as an exercise for the reader. The estimate area $(C') \geqslant 8$ is sharp only if C comprises one pixel. It is easy to verify that we can replace the value 8 by 10, 12, 12, 14, 14, 16 for area$(C) = 2, 3, 4, 5, 6, 7$, respectively. But for our purposes, the estimate in the lemma is good enough.

Consider the increasing area criterion

$$u_S(C) = [\mathrm{area}(C) \geqslant S],$$

and define

$$\omega_S = \psi_{u_S,u_S}.$$

It is evident that ω_S is a self-dual grain operator. Note that $\omega_1 = \mathrm{id}$.

PROPOSITION 6–19. *If $S \leqslant 8$, then ω_S is a self-dual grain filter.*

PROOF. We must show that ω_S is increasing and idempotent. To show that ω_S is increasing, we apply Proposition 6–16. It is evident that u_S is increasing. We must show that condition 6–5 holds for $u = v = u_S$. Take $X \subseteq \mathbb{Z}^2$. Without loss of generality we may assume that $h \in X$. Put $C = \gamma_h(X)$, and suppose that $u_S(C) = 0$. Now, the previous lemma yields that C is a leaf of the zonal tree of X, which has a unique neighbor $C' \Subset X^c$. Furthermore, $C' \cup \{h\}$ is connected, thus $\gamma_h(X^c \cup \{h\}) = C' \cup \{h\}$, and the lemma says that the area of this component is greater than or equal to 9. Thus $u_S(\gamma_h(X^c \cup \{h\})) = 1$, which had to be shown. We conclude that ω_S is increasing.

Now we show that ω_S is idempotent. Since ω_S is self-dual, it is sufficient to show that $\omega_S \leqslant \omega_S^2$. Let C be a part of $P(X)$ and $C \subseteq \omega_S(X)$. We must show that $C \subseteq \omega_S^2(X)$. We distinguish two cases.

1. $C \Subset X$: since C is preserved by ω_S, it follows that $u_S(C) = 1$. Since u_S is increasing, the grain C' of $\omega_S(X)$ that contains C automatically obeys $u_S(C') = 1$, and therefore $C \subseteq C' \subseteq \omega_S^2(X)$.

2. $C \Subset X^c$: in view of the fact that $C \subseteq \omega_S(X)$, we have $u_S(C) = 0$. Thus $\mathrm{area}(C) < S$, in particular, $\mathrm{area}(C) \leqslant 7$. Now the previous lemma yields that C is a leaf of the zonal graph of X, and that its (unique) neighbor C' satisfies $\mathrm{area}(C') \geqslant 8$, hence $u_S(C') = 1$. This means that the connected set $C \cup C'$ lies in $\omega_S(X)$. Let C'' be the grain of $\omega_S(X)$ containing $C \cup C'$, then $u_S(C'') = 1$. Therefore $C'' \subseteq \omega_S^2(X)$, in particular $C \subseteq \omega_S^2(X)$. ■

The filter ω_2 is the annular filter discussed in Chapter 5. Note that ω_S, for $S \leqslant 8$, is the composition of the area opening $\alpha_S = \psi_{u_S,1}$ and the area closing $\beta_S = \psi_{1,u_S}$, that is,

$$\omega_S = \alpha_S \beta_S = \beta_S \alpha_S.$$

The compositions $\alpha_S \beta_S$ and $\beta_S \alpha_S$ are filters for every $S \geqslant 1$. In general, these two compositions are different. However, by our previous result, they coincide and define a self-dual grain operator if $S \leqslant 8$. An illustration is given in Fig. 6–14.

Figure 6–14. From left to right: a binary image X (see Chapter 5) and the results after filtering with ω_S for $S = 1, 4, 7$, respectively.

6.6 GRAY-SCALE IMAGES

Up to this point, we have been concerned exclusively with connected operators for binary images. In this section we describe briefly the extension to gray-scale images. Readers who want to know more details are referred to the literature [4–6, 20, 24].

In this section we restrict ourselves to images that can be modelled by numerical functions $f : \mathbb{Z}^2 \to \mathcal{T}$, where $\mathcal{T} = \{0, 1, 2, \ldots, T\}$. We denote by $\mathrm{Fun}(\mathbb{Z}^2)$ the set of all such functions. It is well known that $\mathrm{Fun}(\mathbb{Z}^2)$ defines a complete lattice (see also Chapter 5).

DEFINITION 6–3. Given a gray-scale function $f \in \mathrm{Fun}(\mathbb{Z}^2)$, a connected set $C \subseteq \mathbb{Z}^2$ is called a *flat zone* of f at level t if C is a grain of the level set $\{x \in \mathbb{Z}^2 \mid f(x) = t\}$.

In other words, a flat zone of a function is a maximal connected region where the function is constant. The flat zones of a function f yield a connected partition of the underlying space $\mathbb{Z}^2$; this partition will be denoted by $P(f)$. Observe that the partition of the indicator function of a set X coincides with the partition of the set defined in Section 6.2. We write $P(f, h) = P(f)(h)$ for $h \in \mathbb{Z}^2$. The following definition is a straightforward generalization of Definition 6–1.

DEFINITION 6–4. An operator Ψ on $\mathrm{Fun}(\mathbb{Z}^2)$ is *connected* if the partition $P(\Psi(f))$ is coarser than $P(f)$, for every function f.

Many of the results on connected operators for binary images can be extended to the gray-scale case (in particular Propositions from 6–4 to 6–7). However, we are not aiming at a comprehensive discussion of connected gray-scale operators in this section. Rather our goal is to give the reader a global impression of some aspects of such operators, e.g., their construction, their implementation, and their application in segmentation algorithms.

An important class of gray-scale operators is formed by the so-called *flat operators* [9, 10]. Given an increasing operator ψ on $\mathcal{P}(\mathbb{Z}^2)$, there exists a unique increasing operator Ψ on $\mathrm{Fun}(\mathbb{Z}^2)$ such that the following relation holds:

$$X(\Psi(f), t) = \psi(X(f, t)),$$

for $f \in \mathrm{Fun}(\mathbb{Z}^2)$ and $t \in \mathcal{T}$. Here $X(f, t) = \{x \in \mathbb{Z}^2 \mid f(x) \geqslant t\}$ is the threshold set associated with f. We say that Ψ is generated by ψ. The following result can be established (a formal proof will be given in a future publication).

PROPOSITION 6–20. *Let ψ be an increasing connected operator on $\mathcal{P}(\mathbb{Z}^2)$. Then the flat operator Ψ on $\mathrm{Fun}(\mathbb{Z}^2)$ generated by ψ is connected, too.*

In Section 6.3 we introduced the zonal graph for binary images. In fact, the same definition carries over to the gray-scale case. In this case, however, the coloring defined in Eq. 6–3 becomes a function $I_f : P(f) \to \mathcal{T}$. The zonal graph representation of a gray-scale function is tailor-made for the implementation of connected operators. This is best illustrated by means of an example. We consider the (flat extension of the) area opening with threshold S, and present an algorithm for its computation based upon the zonal graph representation of a gray-scale function f. Starting at the maximum gray-level (or color) $t = T$, we determine for each flat zone at level t if its area is greater than or equal to S. If so, then the output image receives the color t for every pixel in this zone, insofar as it hasn't been set at a previous step. If not, that is, if the area is less than S, then the color t is diminished by one. After this last step, a flat zone may have one or more neighbors with the same color. Such zones are then merged into one new flat zone. This procedure has to be repeated until the minimum color is attained. Thus we arrive at the following algorithm.

```
• initialization
  input image I;
  output image I'(x) ← 0, all x;
• find all flat zones corresponding with I;
  /* now I is defined at flat zones */
• t ← T;   /* T is maximum gray value */
• while t ≠ 0 do {
    for every flat zone C with I(C) = t do {
      if area(C) ⩾ S then
        for every x ∈ C: I'(x) ← max{I'(x), t};
      I(C) ← t − 1;
      merge C with neighbors C' with I(C') = t − 1;
      /* areas of flat zones that are merged
         can be added */
    }
    t ← t − 1;
  }
```

Our algorithm is illustrated in Fig. 6–15. A fast algorithm for the computation of the area opening for gray-scale images using the standard pixel representation has been given by Vincent [27].

A connected operator on Fun($\mathbb{Z}^2$) acts on the level of the zonal graph in the sense that it amounts to a recoloring of the vertices, followed by a merging of adjacent vertices that have obtained the same color [14]. As in the binary case, recoloring may be based on (grain) criteria. In the example given above we used a criterion

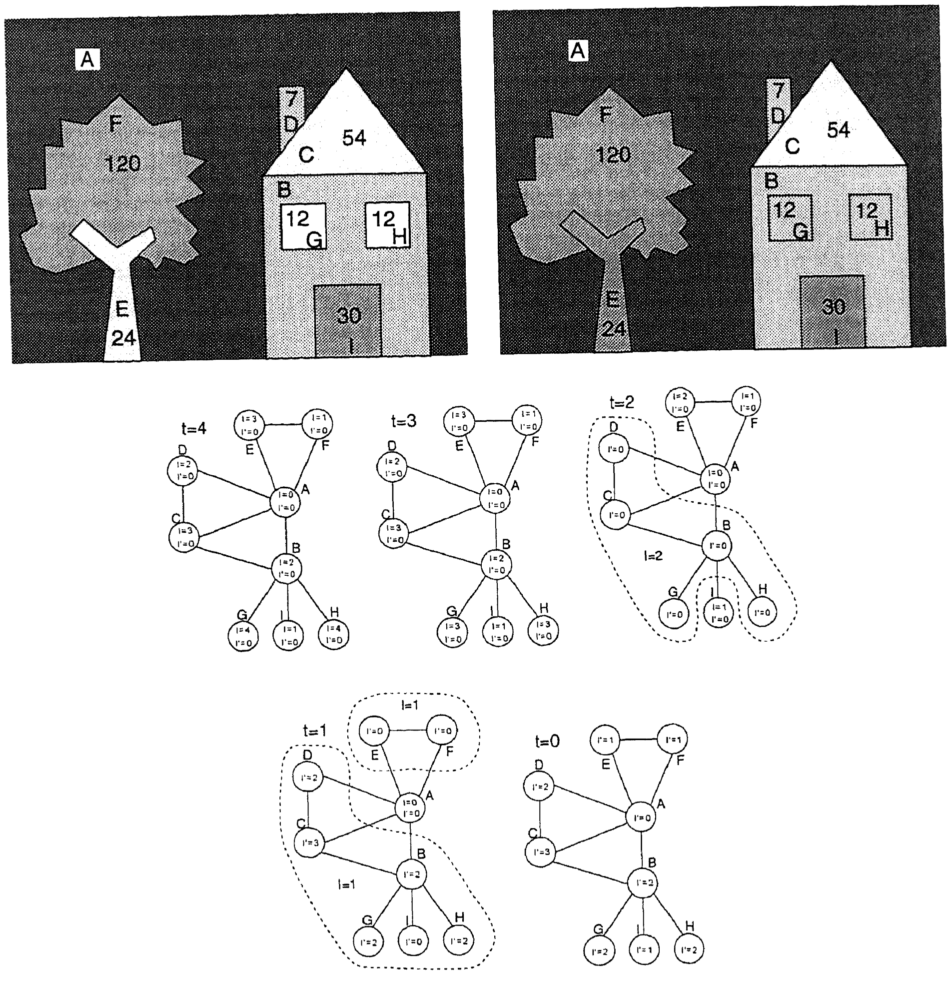

Figure 6–15. The area opening with $S = 25$ for the toy image at the top left results in the image at the right. The areas of the different flat zones are displayed. The five zonal graphs (second row, left to right) represent the successive steps in the algorithm. The input image I has gray levels between 0 and 4 (the higher the level, the brighter the image). The area opening darkens bright zones with small areas.

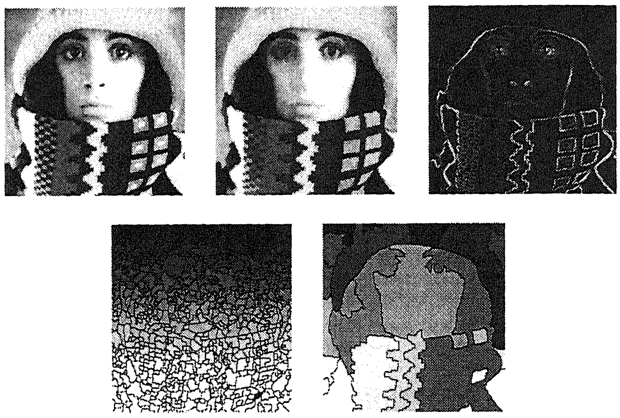

Figure 6–16. From left to right: original image, filtered image (area open-close with $S = 100$), gradient of original image, segmentation without markers, and segmentation with markers.

based on area. However, in the gray-scale case we can introduce another class of criteria, namely those based on (local) contrast; see e.g. [14, 18, 20]. In future publications we will discuss connected gray-scale operators based on contrast criteria in more detail.

We conclude this section with an application showing how to use connected operators for marker extraction in segmentation. In mathematical morphology, segmentation is often done by a watershed algorithm [3]. Usually this algorithm is not applied to the image itself, but rather to its gradient transform. Due to the noise present in the data, such a procedure often results in a huge oversegmentation; see Fig. 6–16.

This can be avoided by using markers that indicate the location of subsets of the different segments. Given a set of markers, one can then modify the gradient image, forcing the markers to become the local minima; since the watershed algorithm uses these local minima as a starting point, we have got rid of the oversegmentation in this way. Thus we are faced with the problem of marker detection. It is at this point that connected operators can be helpful. Rather than going into details, we refer again to Fig. 6–16. We have computed the area-based open-close filter with threshold $S = 100$. Then we computed the gradient of the filtered image, applied a thresholding (at level 10), and an area opening (to the binary image). The background parts can be considered as the homogeneous regions of the original image. This suggests that we can use the background grains as markers. Indeed, we use them to modify the original gradient image, and apply the watershed algorithm. As Fig. 6–16 clearly shows, we arrive at a much better segmentation.

6.7 Concluding Remarks

As the title of this chapter suggests, it contains an introduction to the theory of connected morphological operators. Our exposition is restricted to the case of binary images on a two-dimensional discrete 8-connected grid, with the exception of the previous section, which contains some results for the gray-scale case. We point out, however, that many of our results carry over to other image spaces and/or other notions of connectivity.

Serra [22] has introduced the notion of *connectivity class* for the complete Boolean lattice $\mathcal{P}(E)$, where E is an arbitrary set; see also [10, 15].

DEFINITION 6–5. A family $\mathcal{C} \subseteq \mathcal{P}(E)$ is a connectivity class if

- $\emptyset \in \mathcal{C}$ and $\{h\} \in \mathcal{C}$, for every $h \in E$;
- if $X_i \in \mathcal{C}$, $i \in I$, and $\bigcap_{i \in I} X_i \neq \emptyset$, then $\bigcup_{i \in I} X_i \in \mathcal{C}$.

This definition includes the class of 8-connected sets in $\mathbb{Z}^2$ (as well as the 4-connected sets), but one can find many other examples; see [15] for some interesting ones. Recently, Serra [23] has given an extension of the definition of connectivity class to other complete lattices than $\mathcal{P}(E)$.

Given a connectivity class $\mathcal{C}$, we can define the connectivity openings $\gamma_h : \mathcal{P}(E) \to \mathcal{P}(E)$ as follows:

$$\gamma_h(X) = \bigcup \{C \in \mathcal{C} \mid h \in C \text{ and } C \subseteq X\}.$$

To show that γ_h is indeed an opening, we observe first that γ_h is increasing and anti-extensive, hence that $\gamma_h^2 \leqslant \gamma_h$. On the other hand, $\gamma_h(X)$ is a union of sets $C \in \mathcal{C}$ with $h \in C \subseteq X$. Every C with this property satisfies $C \subseteq \gamma_h(X)$, and this yields that $\gamma_h(X) \subseteq \gamma_h^2(X)$. Now we can define the reconstruction operator ρ as in Eq. 6–1. As a matter of fact, many of the definitions and results stated in this paper carry over to this general framework. We will not elaborate on this theme here.

Another issue that we have not explored in this chapter is the theory of connected filters other than openings and closings. Readers interested in this topic are referred to [8, 13, 20, 22].

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REFERENCES

[1] Ballard, D. H., and M. Brown, *Computer Vision*, Prentice-Hall, Englewood Cliffs, NJ, 1982.

[2] Berge, C., *Graphs*, 2nd ed., North-Holland, Amsterdam, 1985.

[3] Beucher, S., and F. Meyer, "The morphological approach to segmentation: the watershed transformation," in *Mathematical Morphology in Image Processing*, E. R. Dougherty, ed., Ch. 12, pp. 433–481, Marcel Deker, New York, 1993.

[4] Breen, E., and R. Jones, "An attribute-based approach to mathematical morphology," in *Mathematical Morphology and its Applications to Image and Signal Processing*, P. Maragos, R. W. Schafer and M. A. Butt, eds., pp. 41–48, Kluwer Academic Publishers, Boston, 1996.

[5] Crespo, J., Morphological connected filters and intra-region smoothing for image segmentation, PhD thesis, Georgia Institute of Technology, Atlanta, 1993.

[6] Crespo, J., and R. W. Schafer, "The flat zone approach and color images," in *Mathematical Morphology and its Applications to Image Processing*, J. Serra and P. Soille, eds., pp. 85–92, Kluwer Academic Publishers, 1994.

[7] Crespo, J., and R. W. Schafer, "Locality and adjacency stability constraints for morphological connected operators," *J. Math. Imaging and Vision* **7** (1), 1997, pp. 85–102.

[8] Crespo, J., J. Serra, and R. W. Schafer, "Theoretical aspects of morphological filters by reconstructions," *Sign. Proc.* **47** (2), 201–225, 1995.

[9] Heijmans, H. J. A. M., "Theoretical aspects of gray-level morphology," *IEEE Trans. Patt. Anal. Mach. Intell.* **13**, 568–582, 1991.

[10] Heijmans, H. J. A. M., *Morphological Image Operators*, Academic Press, Boston, 1994.

[11] Lantuéjoul, C., and S. Beucher, "On the use of the geodesis metric in image analysis," *J. Microscopy* **121**, 29–49, 1980.

[12] Lantuéjoul, C., and F. Maisonneuve, "Geodesic methods in quantitative image analysis," *Patt. Recogn.* **17**, 177–187, 1984.

[13] Pardàs, M., J. Serra, and L. Torres, "Connectivity filters for image sequences," *SPIE Proceedings*, Vol. 1769, pp. 318–329, 1992.

[14] Potjer, F. K., "Region adjacency graphs and connected morphological operators," in *Mathematical Morphology and its Applications to Image and Signal Processing*, P. Maragos, R. W. Schafer and M. A. Butt, eds., pp. 111–118, Kluwer Academic Publishers, Boston, 1996.

[15] Ronse, C., "Set-theoretical algebraic approaches to connectivity in continuous or digital spaces," *J. Math. Imaging and Vision* **8**, 1998, pp. 41–58.

[16] Ronse, C., and H. J. A. M. Heijmans, "The algebraic basis of mathematical morphology – Part II: Openings and closings," *CVGIP: Image Understanding* **54**, 74–97, 1991.

[17] Rosenfeld, A., "Connectivity in digital pictures," *J. Assoc. Comp. Mach.* **17**, 146–160, 1970.

[18] Salembier, P., and M. Kunt, "Size-sensitive multiresolution decomposition of images with rank order based filters," *Sign. Proc.* **27** (2), 205–241, 1992.

[19] Salembier, P., and A. Oliveras, "Practical extensions of connected operators," in *Mathematical Morphology and its Applications to Image and Signal Processing*, P. Maragos, R. W. Schafer and M. A. Butt, eds., pp. 97–110, Kluwer Academic Publishers, Boston, 1996.

[20] Salembier, P., and J. Serra, "Flat zones filtering, connected operators, and filters by reconstruction," *IEEE Trans. on Image Proc.* **4** (8), 1153–1160, 1995.

[21] Serra, J., *Image Analysis and Mathematical Morphology*, Academic Press, London, 1982.

[22] Serra, J., ed., *Image Analysis and Mathematical Morphology*, Vol. II: *Theoretical Advances*, Academic Press, London, 1988.

[23] Serra, J., "Connectivity on complete lattices," in *Mathematical Morphology and its Applications to Image and Signal Processing*, P. Maragos, R. W. Schafer and M. A. Butt, eds., pp. 81–96, Kluwer Academic Publishers, Boston, 1996.

[24] Serra, J., and P. Salembier, "Connected operators and pyramids," *SPIE Proceedings*, Vol. 2030, pp. 65–76, 1993.

[25] Vincent, L., "Morphological algorithms," in *Mathematical Morphology in Image Processing*, E. R. Dougherty, ed., Ch. 8, pp. 255–288, Marcel Dekker, New York, 993.

[26] Vincent, L., "Morphological grayscale reconstruction in image analysis: efficient algorithms and applications," *IEEE Trans. Image Proc.* **2**, 176–201, 1993.

[27] Vincent, L., "Morphological area openings and closings for gray-scale images," in "*Shape in Picture*," Y.-L. O, A. Toet, D. Foster, H. J. A. M. Heijmans and P. Meer, eds., pp. 197–208, Springer, Berlin, 1994.