On the Uniqueness of a Certain Thin Near Octagon (or Partial 2-Geometry, or Parallelism) Derived from the Binary Golay Code

A. E. BROUWER

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Abstract—The question of the uniqueness of a certain combinatorial structure has arisen in three contexts: a) is the regular near octagon with parameters $(s, t_2, t_3, t) = (1, 1, 2, 23)$ unique [5]? b) is the partial 2-geometry with nexus three and blocksize 24 unique [2]? c) is there a unique graph such that it is the graph of a parallelism of $\binom{24}{4}$ with respect to any vertex [1]? We observe that these questions are equivalent and give an affirmative answer. In fact, we prove a more general theorem, showing the truth of a conjecture by Cameron.

I. Introduction

FOR the definition of near n-gon (see [5], [6]). We shall only use thin regular near $2d$-gons which are nothing but distance regular bipartite graphs of diameter d. If $d(x, y) = i$ then let $c_i := |\{z|z \sim y \text{ and } d(z, x) = i - 1\}|$ (where $\sim$ denotes adjacency). By distance regularity c_i does not depend on the choice of x and y. Write $k := c_d$, the valency of the graph.

A partial λ-geometry with nexus e is an incidence structure with b blocks and v points which satisfies:

a) two distinct points are on 0 or λ common blocks;
b) two distinct blocks have 0 or λ common points;
c) given a point x not on a block B, there are precisely e blocks on x meeting B;
d) no point is on all blocks and no block contains all points;
e) all blocks have size k and all points are in r blocks, where $k, r > \lambda$, (see [1], [2]).

If $\lambda > 1$ then one may weaken e) to *each block has more than* λ *points and each point is on more than* λ *blocks* and obtain the existence of k and r and moreover that $k = r$. Assume $\lambda > 1$ and $e < k$ (when $e = k$ we have a symmetric BIBD).

One immediately verifies that the bipartite graph with points and blocks as vertices and incidence as adjacency is distance regular with $(c_1, c_2, c_3, c_4) = (1, \lambda, e, k)$, i.e., we have a thin regular near octagon. The converse also being clear we see that the concepts of thin regular near 8-gon (with $c_2 > 1$) and partial λ-geometry with nexus e (with $\lambda > 1$ and $e < k$) are equivalent.

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The author is with the Mathematical Center, Kruislaan 413, Amsterdam, The Netherlands.

II. Uniqueness

Let Γ be a distance regular bipartite graph with diameter $d \geqslant 4$ and $c_i = i$ $(1 \leqslant i \leqslant d - 1)$.

Lemma: Let 2^j be a j-cube (vertices: binary vectors of length j, edges: pairs of vectors differing by a unit vector) and π be a map sending 0 to $x_0 \in \Gamma$ and the unit vectors to (distinct) neighbors of x_0 in Γ. Then π can be extended in a unique way to a map $\pi: 2^j \to \Gamma$ preserving adjacency and squares. Moreover, π is injective when restricted to $(2d - 1)$-cubes.

Proof (cf. [1, th. 5.11 (i)]): First observe that two points at distance three in Γ determine a 3-cube: each has three neighbors at distance two from the other, and on these six points we have a regular bipartite graph of valency two, hence a hexagon, completing the cube. It follows easily that a 2-claw determines a square and a 3-claw a 3-cube.

We define $\pi(\boldsymbol{v})$ by induction on the number of nonzero coordinates (the weight) of $\boldsymbol{v}$. If the weight is zero or one $\pi(\boldsymbol{v})$ is prescribed. If the weight is two so that $\boldsymbol{v} = \boldsymbol{e} + \boldsymbol{e}'$ for two unit vectors $\boldsymbol{e}, \boldsymbol{e}'$, then let $\pi(\boldsymbol{v})$ be the common neighbor of $\pi(\boldsymbol{e})$ and $\pi(\boldsymbol{e}')$ distinct from $\pi(\mathbf{0})$.

Suppose $\pi(\boldsymbol{v})$ defined for vectors $\boldsymbol{v}$ of weight less than i, and let $\boldsymbol{v}$ have weight $i \geqslant 3$: $\boldsymbol{v} = \sum_{h=1}^{i} \boldsymbol{e}_h$ for distinct unit vectors $\boldsymbol{e}_h$. Then we can define $\pi(\boldsymbol{v})$ as the common neighbor of $\pi(\boldsymbol{v} - \boldsymbol{e}_1)$ and $\pi(\boldsymbol{v} - \boldsymbol{e}_2)$ distinct from $\pi(\boldsymbol{v} - \boldsymbol{e}_1 - \boldsymbol{e}_2)$. For $h > 2$ the points $\pi(\boldsymbol{v})$ and $\pi(\boldsymbol{v} - \boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_h)$ have distance three, hence determine a 3-cube, and we find that $\pi(\boldsymbol{v})$ is adjacent to $\pi(\boldsymbol{v} - \boldsymbol{e}_h)$ as was required. Thus, using only $c_2 = 2$ and $c_3 = 3$ we find a unique map π satisfying all requirements and injective on 3-cubes. If we

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also have $c_i = i$ $(2 \leqslant i \leqslant h)$ then one easily sees that π is still injective on $(2h+1)$-cubes: if two vectors have the same image then by choosing the origin appropriately one may suppose that they have disjoint supports of equal weights at most h (note that Γ is bipartite and 2^j is connected so that vectors with the same image have even distance), but inside a radius of h nothing can happen.

Now let k be the valency of Γ and apply the Lemma to find a labeling of Γ with vectors from 2^k. The number of vertices of Γ is

$$|V(\Gamma)| = 1 + k + \binom{k}{2} + \cdots + \binom{k}{d-1} + \frac{d}{k}\binom{k}{d},$$

so that on the average each point gets $2^k/|V(\Gamma)|$ labels. But the collection of labels of a point x is a binary code $\mathcal{C}_x$ with word length k and minimum distance at least $2d$, and this average is just the trivial upper bound for the cardinality of such a code when one observes that spheres of radius $d-1$ around codewords are disjoint, and a vector can have distance d to at most k/d distinct codewords. Consequently each point has exactly $2^k/|V(\Gamma)|$ labels, and each $\mathcal{C}_x$ is an extended perfect code. By the classification of perfect codes (see e.g., [4, ch. 6, th. 33]) it follows that we have one of the following three possibilities:

a) $|\mathcal{C}_x| = 1$, the code is trivial. Now $d = k$, $|V(\Gamma)| = 2^k$ and we have a k-cube.

b) $|\mathcal{C}_x| = 2$, a repetition code. Now $d = \frac{1}{2}k$, $|V(\Gamma)| = 2^{k-1}$ and we have a half k-cube (i.e., a k-cube with antipodal vertices identified).

c) $|\mathcal{C}_x| = 2^{12}$, a code isomorphic to the extended binary Golay code $\mathcal{C}$. Now $k = 24$, $d = 4$ and all codes $\mathcal{C}_x$ are translates of each other (for: suppose $x \sim y$. For each $v \in \mathcal{C}_x$ there is a unique $v' \in \mathcal{C}_y$ with $d(v, v') = 1$. The map $v \to v'$ changes distances by at most two, but since all distances are multiples of four it must preserve distances. Considering triples of vectors of weight 8 u, v, w in $\mathcal{C}_x$ such that $u + v + w$ is the all-one vector one sees that this map is a translation over a vector of weight one). Our graph Γ is the graph with as vertices 2^{12} cosets of $\mathcal{C}$, and two vertices are adjacent if and only if the corresponding cosets contain vectors differing by a unit vector.

This settles question 5.7 in [1, p. 94]—there also $d = 3$ is allowed, but in this case it is given that 3-claws generate 3-cubes and our reasoning still works.

Let us formulate a theorem.

Theorem: Let Γ be a distance regular bipartite graph with diameter $d \geqslant 3$, valency k and parameters $c_i = i$ $(1 \leqslant i \leqslant d-1)$. If $d = 3$ then suppose additionally that any 3-claw generates a 3-cube (in the sense of [1, p. 91]). Then Γ is a d-cube, a half $2d$-cube, or we have $d = 4$, $k = 24$, and Γ is the graph of the sextet parallelism (or, in other words, is the thin near octagon derived from the extended binary Golay code as described above).

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